Coding Exercise 6

1. Generate a 5×5 matrix X where each element is a random integer in the range $\{-9, \dots, 9\}$. Generate a positive definite matrix A as $A = XX^T + I_5$. Also generate a 5-length vector c. Now solve for the minimum of $(\frac{1}{2}x^T Hx + c^T x)$ by using conjugate gradient descent method with exact line search. Start with any arbitrary initial point. Verify if the solution converges to $(-H^{-1}c)$ in five iterations for all random matrices generated.

2. Let $H = \begin{pmatrix} 2 & a \\ a & 2 \end{pmatrix}$. Solve for the minimum of $(\frac{1}{2}x^T Hx)$ for every a in the set $\{-1.9, -1.8, \dots, 1.8, 1.9\}$ using gradient descent method with exact line search. Start with the same initial point for each value of a. Plot the number of iterations for convergence as a function of a.

3. Let $f(x) = x_1^2 e^{x_2} + x_2^2 e^{x_1}$. Solve for its minimum using gradient descent and conjugate gradient methods but with inexact line search. Also solve it using Newton's method. Let $x^0 = (1, 1)$. Print the number of iterations for convergence in each case.

4. Solve for the minimum of the Rosenbrock function $(100(x_2 - x_1^2)^2 - (1 - x_1)^2)$ using gradient descent algorithm with inexact line search. Start with $(0.5, 0.5)$. Print the number of iterations needed for converging to the solution $(1, 1)$.

5. Consider the problem of minimizing $4x_1^2 - 2x_1x_2 + x_2^2$ subject to the constraint $x_1 + x_2 \geq 1$. Solve this using the penalty method with an appropriate value of ρ. Print the number of iterations taken for three different algorithms used to solve the unconstrained variant of the above problem.

6. Repeat the above problem when we
   (a) maximize $x_1 + x_2$ s.t. $x_1^2 + x_2^2 \leq 1$.
   (b) minimize $(x_1 - 3)^2 + (x_2 - 2)^2$ s.t. $x_1 \geq x_2$, $x_1 + x_2 \leq 1$, $x_2 \leq 0$.

7. Solve the following LP's using simplex method.
   (a) $\min(3x_1 - 5x_2 + x_3)$ s.t. $x_1 - 2x_3 \geq 4$, $2x_1 - x_2 + x_3 \geq 2$, $x \geq 0$.
   (b) $\min(2x_1 + 15x_2 + 5x_3 + 6x_4)$ s.t. $x_1 + 6x_2 + 3x_3 + x_4 \geq 2$, $2x_1 - 5x_2 + x_3 - 3x_4 \geq 3$, $x \geq 0$.
   (c) $\max(2y_1 - 3y_2)$ s.t. $y_1 - 2y_2 \leq 2$, $6y_1 + 5y_2 \leq 15$, $3y_1 - y_2 \leq 5$, $y_1 + 3y_2 \leq 6$, $y \geq 0$.

8. Consider the optimal transport problem where we have

- two production units producing a quantity of 10 and 15 units of a specific good respectively,
- two consumption units consuming a quantity of 20 and 5 units of the good respectively,
- the cost of transporting a unit of the good from production unit i to consumption unit j is

$$c(i,j) = \begin{cases} 20 & \text{if } i=1, j=1, \\ 5 & \text{if } i=1, j=2, \\ 10 & \text{if } i=2, j=1, \\ 15 & \text{if } i=2, j=2. \end{cases}$$

Every unit of good produced must be transported out of the production unit, and every unit of good demanded must be transported to the consumption unit, in such a way that the total cost of transportation is minimized. Formulate this as a linear program and solve it using simplex method.

9. Consider the following LP with n inequality constraints $C_1, C_2, \ldots, C_n$:

$$\begin{aligned} & \max(2^{n-1}x_1 + 2^{n-2}x_2 + \ldots + 2x_{n-1} + x_n) \\ \text{s.t. } & (C_1)\, x_1 \le 5,\ (C_2)\, 4x_1 + x_2 \le 25,\ (C_3)\, 8x_1 + 4x_2 + x_3 \le 125,\ \ldots, \\ & (C_n)\, 2^n x_1 + 2^{n-1}x_2 + \ldots + 4x_{n-1} + x_n \le 5^n, \\ & (C_{n+1})\, x \ge 0. \end{aligned}$$

Fix $n = 3$. Start at the point $(0,0,0)$, and then show that the simplex algorithm takes 8 iterations to reach the solution. Repeat this for $n = 4$ and $n = 5$, and show that the algorithm takes 16 and 32 iterations respectively. This is a pathological example that shows that simplex algorithm can take 2^n iterations in the worst case.

10. Consider the data points in *data1.csv*. Each column represents a random draw from either *Pareto*(α) distribution or *exp*(λ) distribution, both with a support set $[1, \infty)$. In other words, the pdf of *exp*(λ) distribution is $f(x) = \lambda e^{-\lambda(x-1)}$ for $x \ge 1$. Use Kolmogorov-Smirnov test and find the distribution from which the data points in column i were drawn. You can use ML estimation to estimate the parameter of the distribution before applying the KS test.

11. Repeat the above problem on *data2.csv*, where each column represents a random draw from either *Log-normal*$(0, \sigma^2)$ or *Gamma*$(3, \beta)$.

12. Repeat the above problem on *data3.csv*, where each column represents a random draw from either *Binomial*$(9, p)$ or *Negative Binomial*$(4, p)$.