Coding Exercise 4

1. Consider the system of linear equations of the form $Ax = b$. Given the matrix A (of size $m \times n$) and the vector b (of length n), write a code that finds whether the system has a unique solution, infinite solutions, or no solutions. Furthermore, the code must compute the solution if it is unique, characterize the complete solution if there are infinitely many, and provide the best fit solution along with the squared error if there are no solutions.

2. Write a code to determine whether a given square symmetric matrix A is positive (semi)definite, negative (semi)definite, or indefinite. Furthermore, the code should find an x such that $x^T Ax = 0$ and a y such that $y^T Ay \neq 0$ if A is semidefinite; and should find an x such that $x^T Ax > 0$ and a y such that $y^T Ay < 0$ if A is indefinite.

3. Assume that you are given a matrix $A = [a_1, a_2, \ldots, a_m]$ where $a_i \in \mathbb{R}^n$ are n-length vectors. Consider $m < n$, and that A is an orthonormal matrix. Write a code to construct another $(n-m)$ vectors $a_{m+1}, a_{m+2}, \ldots, a_n$ such that the matrix $B = [a_1, \ldots, a_n]$ is still an orthonormal matrix.

4. Consider that you are given a set of vectors $a_1, a_2, \ldots, a_l$ with $a_i \in \mathbb{R}^m$, and another set of vectors $b_1, b_2, \ldots, b_l$ with $b_i \in \mathbb{R}^n$. Write a code to construct a matrix A such that $Col(A) = span(a_1, \ldots, a_n)$ and $Row(A) = span(b_1, b_2, \ldots, b_m)$. Your code must print "Not possible" if such a matrix cannot be constructed.

5. Write a code to decompose a given matrix into the product of a lower and upper triangular matrix using recursion.

6. Write a code to compute the maximal set of linearly independent columns of any given matrix using Gram-Schmidt orthogonalization process. Use the code to find the orthogonal basis of the column space of the matrix.