Bayesian Prediction of Failure Probability of a Space Shuttle Launch

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Abstract

In this project we consider the famous Challenger Crash data based on a historical accident. Dalal et al performed a staistical analysis (frequentist) on the O-Ring dataset with a goal to provide information to decision makers prior to the launch. They tried to estimate the probability of shuttle failure due to O-Ring failure at low temperatures but since their analysis was frequentist in nature, probability distributions representing epistemic uncertainty in the input parameters were not available, and the authors had to resort to an approximate approach based on bootstrap confidence intervals.Later Kelly and Smith re-evaluated the analyses of Dalal et al from a Bayesian perspective. Markov chain Monte Carlo (MCMC) sampling was used to sample from the joint posterior distribution of the model parameters, and to sample from the posterior predictive distributions at the estimated launch temperature, a temperature that had not been observed in prior launches of the space shuttle. Uncertainties, which are represented by probability distributions in the Bayesian approach, were propagated through the model to obtain a probability distribution for O-ring failure, and subsequently for shuttle failure as a result of O-ring failure. No approximations were required in the Bayesian approach and the resulting distributions can be input to a decision analysis to obtain expected utility for the decision to launch. We will do our analysis following the approches of the papers mentioned above, but with our own modifications.

Contents

1 Data Description

On January 20, 1986 the space shuttle **Challenger** broke apart, 73 seconds into its flight. All of its crew members died. The cause of the disaster was the failure of an O-ring on the right solid rocket booster. O-ring is a device that help seal the joints of dierent segments of the solid rocket boosters. It is now known that a leading factor in the O-ring failure was the exceptionally low temperature (about 31 degress farenheit) at the time of the launch. The Challenger Data contains data from the performance of O-rings in 23 U.S. space shuttle flights prior to the Challenger disaster of January 20, 1986. We will work with only a subset of it, namely the **O-Ring** data. We extract temperature and pressure of the O-rings at the time of launch (continuous explanatory variables) and whether at least one O-ring failed (binary response variable). Although number of O-rings that failed among 6 O-rings can be found in the bigger dataset, here an implicit assumption in our analysis is that, failure of only one O-ring can cause the space shuttle to break apart. The data can be found in the library **alr4** in R as a dataframe, just by calling the keyword **Challeng**.

2 Objective

We illustrate how to obtain Bayesian estimations of the probability of failure of a space shuttle launch as a function of the two explanatory variables : temperature and pressure. This allows us to introduce Bayesian methods for logistic regression models which are included in the class of Generalized Linear Models (GLM). We start with the simple logistic regression model in a frequentist approach, and getting a cue from there, continue to fit two different Bayesian Models and compare between them.

3 Logistic Regression (Frequentist Approach)

3.1 Model Fitting

At first we start with a simple logistic regression model where y is the 0-1 response variable where $y_i = 1$ if there was atleast one O-ring failure in the $i^{th}(i = 1,2,..23)$ lauch and 0 otherwise. $y_i \sim$Bernoulli(p_i) where p_i denoting the failure probability of the i^{th} launch such that

$$log(\frac{p_i}{1-p_i}) = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i}$$

where x_{1i} and x_{2i} denote the temperature and pressure of the i^{th} launch respectively.

The fitted model is summarized in the table below:

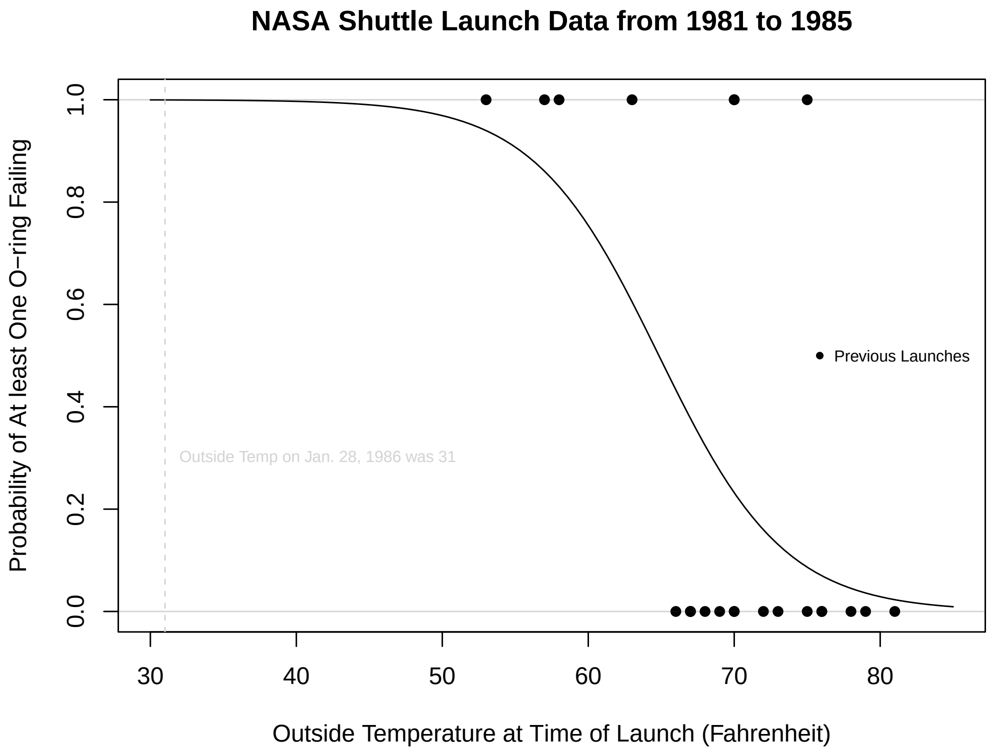
	Estimate	Standard Error	Z-Value (observed)	P-Value
Intercept	14.36	7.443	1.929	0.05369
Temperature	-0.2415	0.1097	-2.201	0.02771
Pressure	0.009534	0.008703	1.095	0.2733

It can be noted from the table that (by looking at the p-values) the variable pressure does not seem to play a significant role in determining whether atleast one O-ring will fail or not, temperature is clearly the key factor. Thus we propose the model with only one covariate (temperature) given by:

$$log(\frac{p_i}{1-p_i}) = \beta_0 + \beta_1 x_{1i}$$

3.2 Visualization

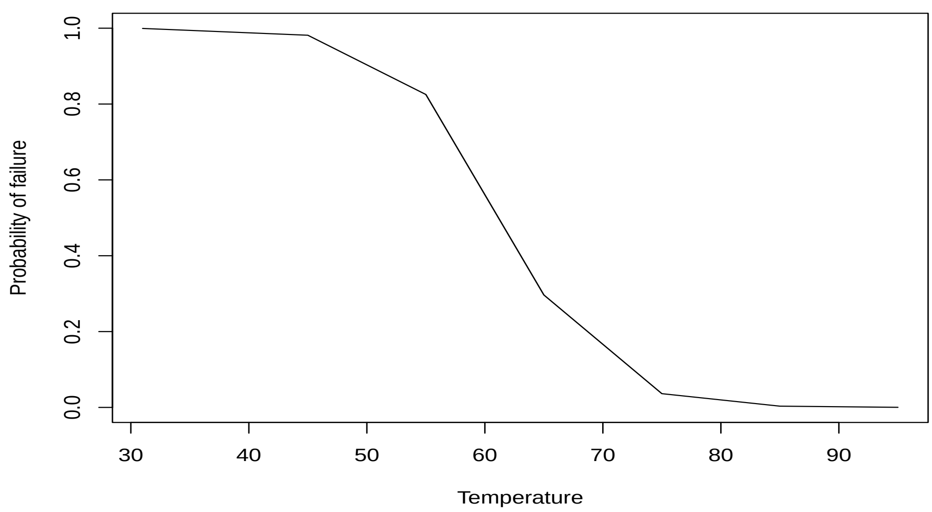
So to get a visual inspection, we look at the following plot:



From this curve it is easy to see that the probability of an O-ring failure decreases with increase in temperature, which is similar to apriori scientific reasoning.

3.3 Prediction

We can easily use the fitted model to predict the probability of O-ring failure at an unknown temperature, values for some choices are summarised in the plot below:



4 Bayesian Approach

As mentioned earlier, since in the frequentist approach the probability distribution of the parameter of interest is not available and one has to rely on some approximations and bootstrap based methods for confidence intervals, thus we opt for a Bayesian Method. We will consider two different models here, namely logit and probit.

4.1 Model fitting

The setup remains same as in the frequentist approach, the only difference being we consider prior distributions of the model parameters $\beta = (\beta_0, \beta_1)^T$. Since there is no conjugate prior, we consider a proper but non informative prior on β such as $N_2(0, \lambda^{-1} I_2)$ for small values of λ. From the conclusions of the frequentist logistic regression, we do not take into account the variable pressure anymore, temperature is our only explanatory variable now onwards.

4.1.1 Model 1 : Bayesian Logit Model

We can write the logistic model as :-

$$P(Y_i = 1|x_i) = \frac{e^{\beta_0+\beta_1 x_i}}{1+e^{\beta_0+\beta_1 x_i}} = p_i, i = 1, \ldots, 23$$

so, the likelihood function can be written as :-

$$\begin{aligned} f(y_i|\beta, x_i) &= p_i^{y_i} (1-p_i)^{1-y_i} \\ &= \left[\frac{e^{\beta_0+\beta_1 x_i}}{1+e^{\beta_0+\beta_1 x_i}}\right]^{y_i} \left[\frac{1}{1+e^{\beta_0+\beta_1 x_i}}\right]^{1-y_i} \end{aligned}$$

now, if we assume flat normal prior for $\beta \sim N_2\left(0, \lambda^{-1} I_2\right)$ for large value of λ :-

$$\pi(\beta) \propto \exp\left(-\frac{\lambda}{2}\beta^T \beta\right)$$

Hence, the posterior of β can be written as :-

$$\begin{aligned} \pi(\beta|x,y) &\propto \pi(\beta) f(y|\beta, x) \\ &\propto \exp\left(-\frac{\lambda}{2}\beta^T\beta\right) \prod_{i=1}^{23} \left[\frac{e^{\beta_0+\beta_1 x_i}}{1+e^{\beta_0+\beta_1 x_i}}\right]^{y_i} \left[\frac{1}{1+e^{\beta_0+\beta_1 x_i}}\right]^{1-y_i} \\ &\propto \exp\left(-\frac{\lambda}{2}\beta^T\beta\right) \prod_{i=1}^{23} \left[\frac{e^{y_i \beta^T x_i}}{1+e^{\beta^T x_i}}\right] \end{aligned}$$

4.1.2 Model 2 : Bayesian Probit Model

Here, we take the Probit link function to model the probability of faliure/not faliure i.e.,

$$P(Y_i = 1|x_i) = \Phi(\beta_0 + \beta_1 x_i) = p_i, i = 1, \ldots, 23$$

here, the posterior density of β is :-

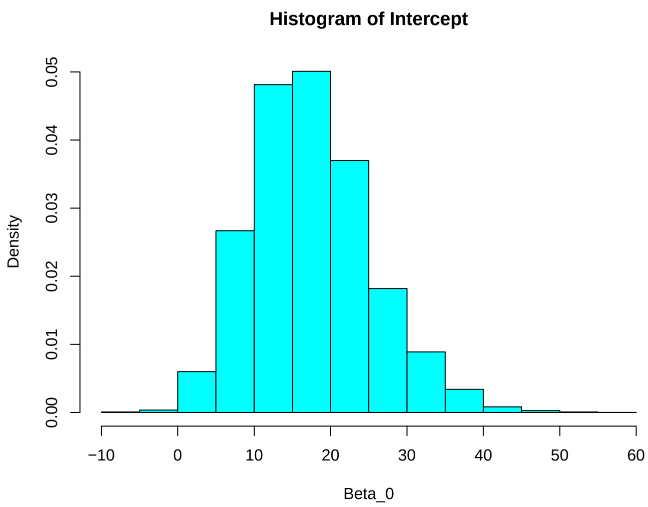
$$\begin{aligned} \pi(\beta|x,y) &\propto \pi(\beta) f(y|\beta, x) \\ &\propto \exp\left(-\frac{\lambda}{2}\beta^T\beta\right) \prod_{i=1}^{23} \left\{\Phi\left(\beta^T x_i\right)\right\}^{y_i} \left\{1-\Phi\left(\beta^T x_i\right)\right\}^{1-y_i} \end{aligned}$$

4.2 Drawing Samples from Posterior Distribution

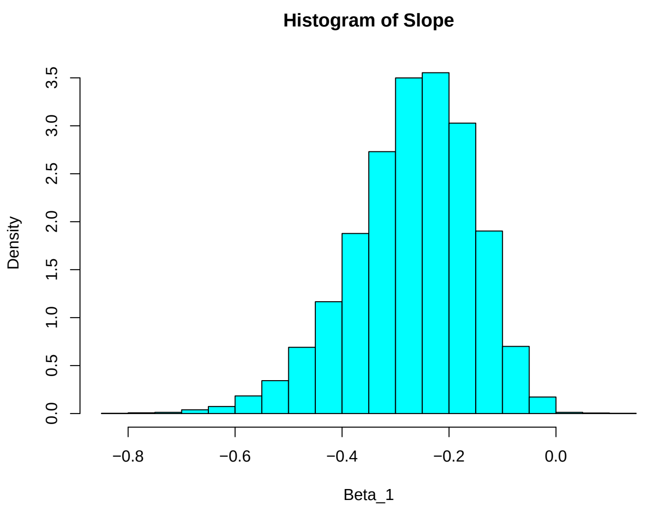
In either of the two models,since the posterior distribution has no closed form, we will use Markov Chain Monte Carlo (MCMC) method to sample from the posterior distribution of β. In particular, we use the Random Walk Metropolis Hastings algorithm. we use the proposal distribution as $q(z'|z) \sim N(z, \Sigma)$ for a suitably chosen Σ.We are not able to use the Gibbs Sampler since finding all the conditional distributions are too much tedious.Thus in our model we obtain the simulated posterior distributions of $\beta = (\beta_0, \beta_1)^T$, we can very easily get the posterior distribution of the failure probability p_i for fixed values of temperature.We represent the posterior densities of the failure probability for some fixed values of temperature of the O-Ring. We run our MH algorithm for 21000 iterations discarding the first 1000 as burn in iterations.

4.2.1 Posterior densities of Model Parameters for logit model

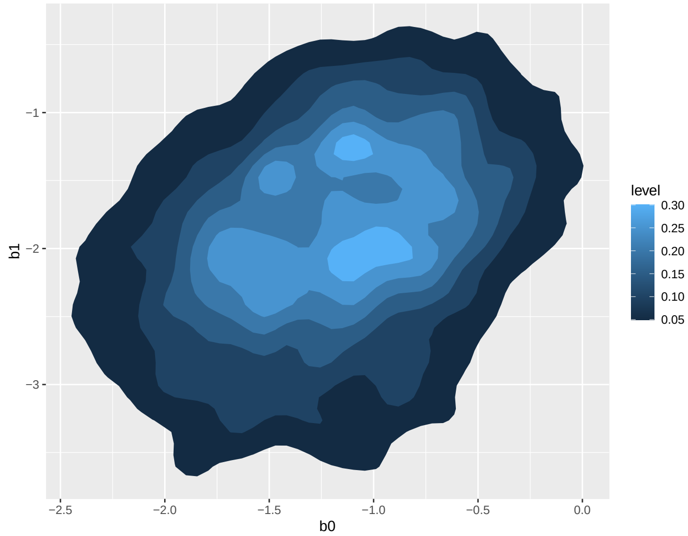
Posterior Density of β_0:



Posterior Density of β_1:

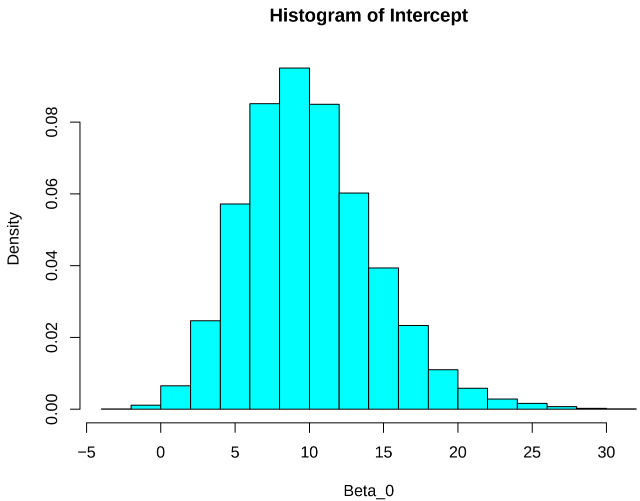


Joint Posterior Density of (β_0, β_1):

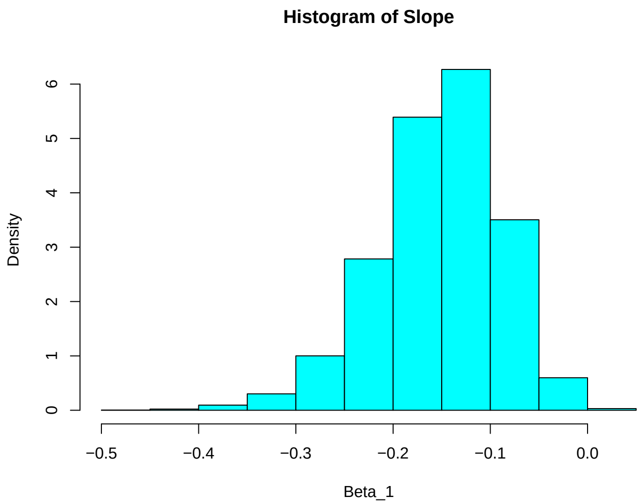


4.2.2 Posterior Densities of Model Parameters for Probit Model

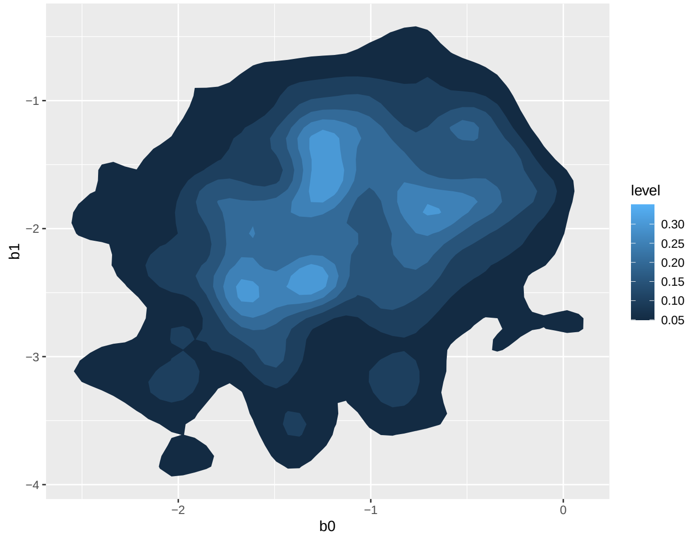
Posterior Density of β_0:



Posterior Density of β_1:

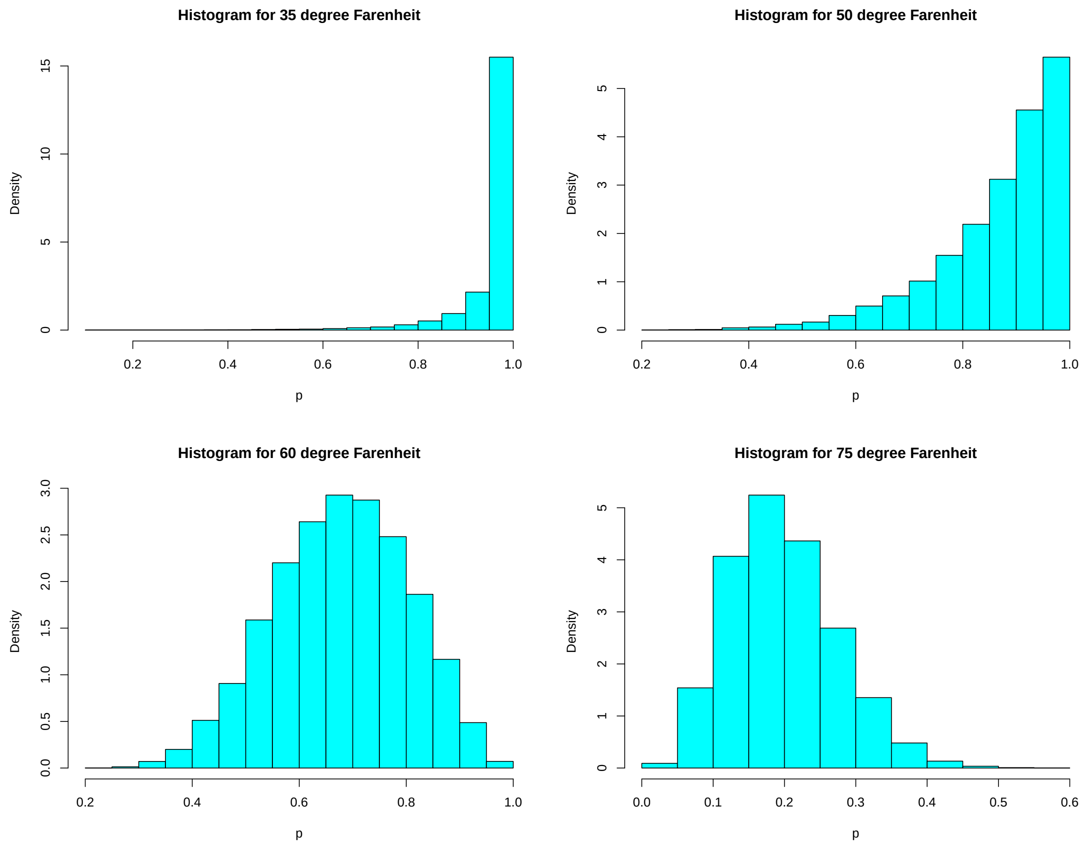


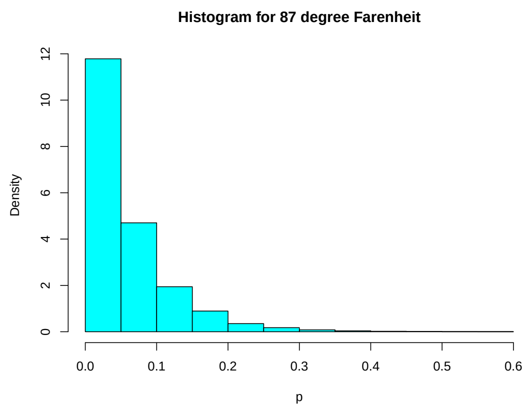
Joint Posterior Density of (β_0, β_1):



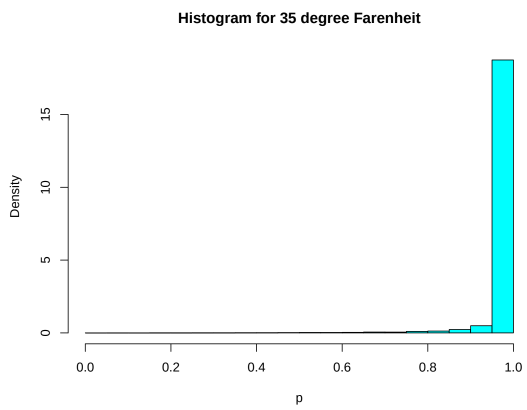
Once we have obtained the simulated posterior distributions of $\beta = (\beta_0, \beta_1)^T$, we can very easily get the posterior distribution of the failure probability p_i for fixed values of temperature.We represent the posterior densities of the failure probability for some fixed values of temperature of the O-Ring.

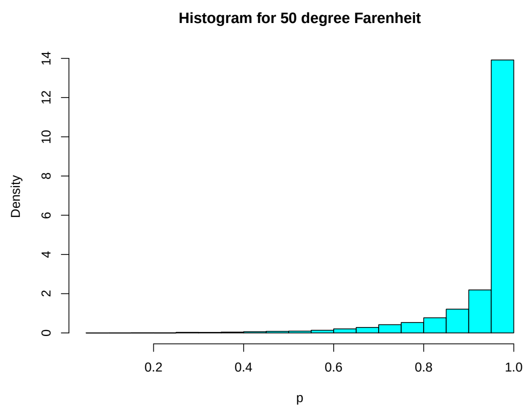
4.2.3 Posterior Distribution of Failure Probability for logit model

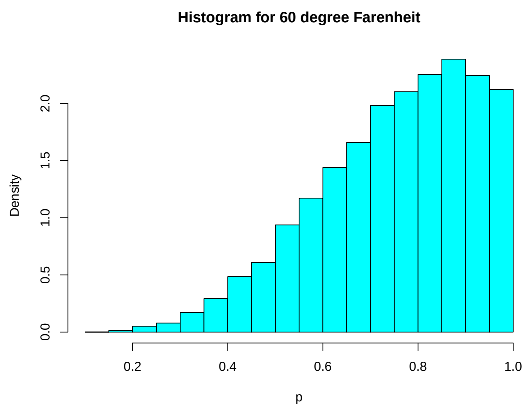


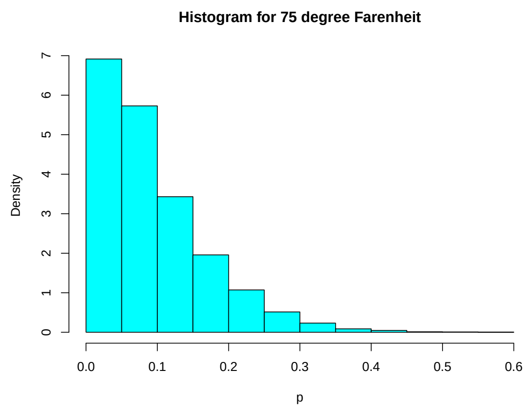


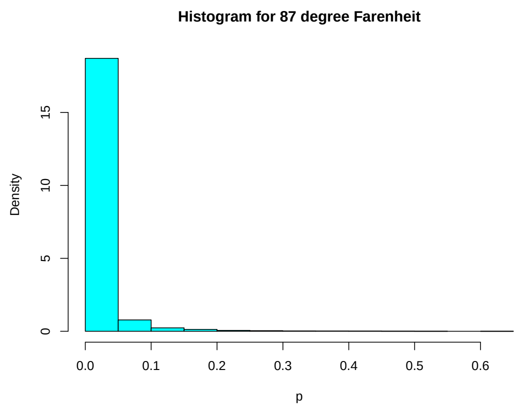
4.2.4 Posterior Distribution of Failure Probability for probit model











4.3 Comments

In both the models, we can see that the probability of failure (p) tends to take high values when it is given that the temperature is low and vice versa. Hence, if the temperature is substantially low, the launch should be avoided. A firm decision should not be made about launching a space shuttle unless the temperature is high enough, because even in a temperature around 55-65 degress farenheit, a failure is not at all less likely to occur.

4.4 Model Comparison

Now, we will use Bayes Factor to compare between the two Bayesian models. The Bayes Fcator is defined as the ration of the marginal densities of the data under the two different models i.e.

$$B_{12} = \frac{f(y,x|M_1)}{f(y,x|M_2)} = \frac{\exp\left(-\frac{\lambda}{2}\beta^T\beta\right)\prod_{i=1}^{23}\left[\frac{e^{y_i\beta^T x_i}}{1+e^{\beta^T x_i}}\right]}{\exp\left(-\frac{\lambda}{2}\beta^T\beta\right)\prod_{i=1}^{23}\left\{\Phi\left(\beta^T x_i\right)\right\}^{y_i}\left\{1-\Phi\left(\beta^T x_i\right)\right\}^{1-y_i}}$$

where forms of both the marginals under both the models (logit and probit) are known to us and the priors are also known which are easy to simulate hence computing the Bayes Factor is straight forward. We compute the Bayes Factor and it comes out to be **0.99** which implies that both the models are almost equivalent, one cannot be given significant preference over the other.

5 Acknowledgements

I express my sincere gratitude to our professor Dr.Sourabh Bhattacharya Sir for giving us the opportunity to explore some interesting stuff outside the conventional coursework. It was really a nice experience doing this project.

References

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[3] Risk Analysis of the Space Shuttle: Pre- Challenger Bayesian Prediction of Failure by Kelly and Smith.

[4] Risk Analysis of the Space Shuttle: Pre- Challenger Prediction of Failure by Dalal et al.

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