Financial math problems solutions

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***Problem* 2.1:** $h(\cdot)$ – is a harmonic function if:

$$\sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} = 0.$$

$h(\cdot)$ – is a subharmonic function if:

$$\sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} \geq 0.$$

Prove that for independent Wiener processes $W_1, \ldots, W_n$ and a processes X is defined by the formula: $X(t) = h(W_1(t), \ldots, W_n(t))$. Show that if h is harmonic (subharmonic) $\Rightarrow$ X is a martingale (submartingale).

Solution: Applying multidimensional version of Ito's lemma to the function h we can obtain:

$$dX = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} dt$$

Equivalently, we can rewrite the equation as:

$$h(W_1(t), \ldots, W_n(t))) = h(\mathbf{0}) + \int_0^t \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i + \int_0^t \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} dt$$

$$\mathbb{E}[X(t)|\mathcal{F}_s] = h(\mathbf{0}) + \mathbb{E}\bigg[\int_0^s \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i + \int_0^s \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} dt + \int_s^t \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i + \int_s^t \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} dt |\mathcal{F}_s\bigg] =$$

$$X(s) + \mathbb{E}\bigg[\int_s^t \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i + \int_s^t \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} dt |\mathcal{F}_s\bigg] = X(s) + \mathbb{E}\bigg[\int_s^t \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} dt |\mathcal{F}_s\bigg]$$

Looking at the last term we see that there is a sum of second order partial derivatives. If h is harmonic we have $\mathbb{E}[X(t)|\mathcal{F}_s] = X(s)$ a.s. If h is subharmonic we have $\mathbb{E}[X(t)|\mathcal{F}_s] \geq X(s)$ a.s. And thus we have proven the statement.

3 Martingales

4 Partial differential equations

5 Stochastic differential equations

6 Black-Scholes