

Financial math problems solutions

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1 Binomial model

2 Itô's lemma

Problem 2.1: $h(\cdot)$ – is a harmonic function if:

$$\sum_{i=1}^n \frac{\partial^2 h}{\partial x_i^2} = 0.$$

$h(\cdot)$ – is a subharmonic function if:

$$\sum_{i=1}^n \frac{\partial^2 h}{\partial x_i^2} \geq 0.$$

Prove that for independent Wiener processes W_1, \dots, W_n and a processes X is defined by the formula: $X(t) = h(W_1(t), \dots, W_n(t))$. Show that if h is harmonic (subharmonic) $\Rightarrow X$ is a martingale (submartingale).

Solution: Applying multidimensional version of Ito's lemma to the function h we can obtain:

$$dX = \sum_{i=1}^n \frac{\partial h}{\partial x_i} dW_i + \frac{1}{2} \sum_{i=1}^n \frac{\partial^2 h}{\partial x_i^2} dt$$

Equivalently, we can rewrite the equation as:

$$h(W_1(t), \dots, W_n(t)) = h(\mathbf{0}) + \int_0^t \sum_{i=1}^n \frac{\partial h}{\partial x_i} dW_i + \int_0^t \frac{1}{2} \sum_{i=1}^n \frac{\partial^2 h}{\partial x_i^2} dt$$

$$\begin{aligned} \mathbb{E}[X(t)|\mathcal{F}_s] &= h(\mathbf{0}) + \mathbb{E} \left[\int_0^s \sum_{i=1}^n \frac{\partial h}{\partial x_i} dW_i + \int_0^s \frac{1}{2} \sum_{i=1}^n \frac{\partial^2 h}{\partial x_i^2} dt + \int_s^t \sum_{i=1}^n \frac{\partial h}{\partial x_i} dW_i + \int_s^t \frac{1}{2} \sum_{i=1}^n \frac{\partial^2 h}{\partial x_i^2} dt | \mathcal{F}_s \right] = \\ &= X(s) + \mathbb{E} \left[\int_s^t \sum_{i=1}^n \frac{\partial h}{\partial x_i} dW_i + \int_s^t \frac{1}{2} \sum_{i=1}^n \frac{\partial^2 h}{\partial x_i^2} dt | \mathcal{F}_s \right] = X(s) + \mathbb{E} \left[\int_s^t \frac{1}{2} \sum_{i=1}^n \frac{\partial^2 h}{\partial x_i^2} dt | \mathcal{F}_s \right] \end{aligned}$$

Looking at the last term we see that there is a sum of second order partial derivatives. If h is harmonic we have $\mathbb{E}[X(t)|\mathcal{F}_s] = X(s)$ a.s. If h is subharmonic we have $\mathbb{E}[X(t)|\mathcal{F}_s] \geq X(s)$ a.s. And thus we have proven the statement.

Problem 2.2: Show that $dW_1 dW_2 = 0$ for two independent Brownian motions.

Let $\Delta W_i(t_k) = W_i(t_k) - W_i(t_{k-1})$. Define Q_n :

$$Q_n = \sum_{k=1}^n \Delta W_1(t_k) \Delta W_2(t_k),$$

where $0 = t_0 < t_1 < \dots < t_n = t$. Now we just have to show that Q_n converges to 0 in L^2 as the

norm of the partition goes to 0. We have:

$$\begin{aligned}\mathbb{E}[Q_n] &= \mathbb{E} \sum_{k=1}^n \Delta W_1(t_k) \Delta W_2(t_k) = \sum_{k=1}^n \mathbb{E}[\Delta W_1(t_k) \Delta W_2(t_k)] = 0 \\ \mathbb{E}[Q_n^2] &= \mathbb{V}\text{ar}[Q_n] = \mathbb{V}\text{ar} \left[\sum_{k=1}^n \Delta W_1(t_k) \Delta W_2(t_k) \right] = \sum_{k=1}^n \mathbb{V}\text{ar}[\Delta W_1(t_k) \Delta W_2(t_k)] = \\ &= \sum_{k=1}^n \mathbb{E}[\Delta W_1(t_k)^2 \Delta W_2(t_k)^2] = \sum_{k=1}^n \mathbb{E}[\Delta W_1(t_k)^2] \mathbb{E}[\Delta W_2(t_k)^2] = \\ &= \sum_{k=1}^n (\Delta t_k)^2 \leq \max_k \{\Delta t_k\} \sum_{k=1}^n \Delta t_k = \max_{k \in \{1, \dots, n\}} \{\Delta t_k\} t \rightarrow 0, n \rightarrow \infty\end{aligned}$$

3 Martingales

4 Partial differential equations

4.a Probabilistic representation of the Cauchy problem solutions

In this subsection, we show some simple examples on the topic of probabilistic representation of the Cauchy problem solutions.

Problem 4.1: Use a stochastic representation result in order to solve the following boundary value problem in the domain $[0, T] \times \mathbb{R}$:

$$\frac{\partial F}{\partial t} + \mu x \frac{\partial F}{\partial x} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 F}{\partial x^2} = 0$$

$$F(T, x) = \ln(x^2)$$

Here μ and σ are assumed to be known constants.

Solution: Applying Feynman–Kac formula we have

$$F(t, x) = \mathbb{E}[\ln X_T^2 | X_t = x]$$

where

$$dX_s = \mu X_s ds + \sigma X_s dW_s$$

$$X_t = x$$

Applying Ito formula, we can investigate the process $Z_s = \ln X_s^2$

$$dZ_s = \frac{2}{X_s} dX_s - \sigma^2 ds = \frac{2\mu X_s ds + 2\sigma X_s dW_s}{X_s} - \sigma^2 ds = (2\mu - \sigma^2) ds + 2\sigma dW_s$$

Thus we have the equation

$$d \ln X_s^2 = (2\mu - \sigma^2)ds + 2\sigma dW_s$$

We can rewrite it in the following form

$$\ln X_T^2 = \ln X_t^2 + \int_t^T (2\mu - \sigma^2)ds + \int_t^T 2\sigma dW_s = \ln X_t^2 + (2\mu - \sigma^2)(T - t) + 2\sigma(W_T - W_t)$$

Thus we have the solution:

$$F(t, x) = \mathbb{E}[\ln X_T^2 | X_t = x] = \ln(x^2) + (2\mu - \sigma^2)(T - t)$$

Problem 4.2: Consider the following boundary value problem in the domain $[0, T] \times R$.

$$\begin{aligned} \frac{\partial F}{\partial t} + \mu(t, x) \frac{\partial F}{\partial x} + \frac{1}{2} \sigma^2(t, x) \frac{\partial^2 F}{\partial x^2} + k(t, x) &= 0 \\ F(T, x) &= \Phi(x) \end{aligned}$$

Here μ, σ, k and Φ are assumed to be known functions.

Prove that this problem has the stochastic representation formula.

$$F(t, x) = \mathbb{E}[\Phi(X_T) | X_t = x] + \int_t^T E[k(s, X_s) | X_t = x] ds$$

where

$$dX_s = \mu(s, X_s)ds + \sigma(s, X_s)dW_s$$

$$X_t = x$$

Solution: Considering the process $Z_s = F(s, X_s)$ and applying Ito formula, we have

$$\begin{aligned} dZ_s &= \frac{\partial F}{\partial s}(s, X_s)ds + \frac{\partial F}{\partial x}(s, X_s)dX_s + \frac{1}{2}\sigma^2(s, X_s)\frac{\partial^2 F}{\partial x^2}(s, X_s)ds = \\ &= \underbrace{\left(\frac{\partial F}{\partial s}(s, X_s) + \mu(s, X_s)\frac{\partial F}{\partial x}(s, X_s) + \frac{1}{2}\sigma^2(s, X_s)\frac{\partial^2 F}{\partial x^2}(s, X_s) \right)}_{=-k(s, X_s), \text{ assuming that } F \text{ actually solves PDE}} ds + \sigma(s, X_s)\frac{\partial F}{\partial x}(s, X_s)dW_s \end{aligned}$$

We can rewrite this equation in the following form

$$F(t, X_t) = F(T, X_T) + \int_t^T k(s, X_s)ds - \int_t^T \sigma(s, X_s)\frac{\partial F}{\partial x}(s, X_s)dW_s$$

If the process $\sigma(s, X_s) \frac{\partial F}{\partial x}(s, X_s)$ is in \mathcal{L}^2 , taking expected values we have the proof (remember that initial value $X_t = x$, $F(T, x) = \Phi(x)$ and expected value of Wiener integral is equal to zero)

$$F(t, x) = \mathbb{E}[\Phi(X_T) | X_t = x] + \int_t^T \mathbb{E}[k(s, X_s) | X_t = x] ds$$

5 Stochastic differential equations

6 Black-Scholes