

Financial math problems solutions

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1 Binomial model

2 Itô's lemma

Let $dX_t = \mu_t dt + \sigma_t dW_t$ and $f(t, x) \in C^{1,2}$. Itô's lemma:

$$df(t, X_t) = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{1}{2} \sigma_t^2 \frac{\partial^2 f}{\partial x^2} \right) dt + \sigma_t \frac{\partial f}{\partial x} dW_t \quad (1)$$

Remember that this is just a short notation for the integral form.

Problem 2.1: $h(\cdot)$ – is a harmonic function if:

$$\sum_{i=1}^n \frac{\partial^2 h}{\partial x_i^2} = 0.$$

$h(\cdot)$ – is a subharmonic function if:

$$\sum_{i=1}^n \frac{\partial^2 h}{\partial x_i^2} \geq 0.$$

Prove that for independent Wiener processes W_1, \dots, W_n and a processes X is defined by the formula: $X(t) = h(W_1(t), \dots, W_n(t))$. Show that if h is harmonic (subharmonic) $\Rightarrow X$ is a martingale (submartingale).

Solution: Applying multidimensional version of Ito's lemma to the function h we can obtain:

$$dX = \sum_{i=1}^n \frac{\partial h}{\partial x_i} dW_i + \frac{1}{2} \sum_{i=1}^n \frac{\partial^2 h}{\partial x_i^2} dt$$

Equivalently, we can rewrite the equation as:

$$h(W_1(t), \dots, W_n(t)) = h(\mathbf{0}) + \int_0^t \sum_{i=1}^n \frac{\partial h}{\partial x_i} dW_i + \int_0^t \frac{1}{2} \sum_{i=1}^n \frac{\partial^2 h}{\partial x_i^2} dt$$

$$\begin{aligned} \mathbb{E}[X(t)|\mathcal{F}_s] &= h(\mathbf{0}) + \mathbb{E} \left[\int_0^s \sum_{i=1}^n \frac{\partial h}{\partial x_i} dW_i + \int_0^s \frac{1}{2} \sum_{i=1}^n \frac{\partial^2 h}{\partial x_i^2} dt + \int_s^t \sum_{i=1}^n \frac{\partial h}{\partial x_i} dW_i + \int_s^t \frac{1}{2} \sum_{i=1}^n \frac{\partial^2 h}{\partial x_i^2} dt | \mathcal{F}_s \right] = \\ &= X(s) + \mathbb{E} \left[\int_s^t \sum_{i=1}^n \frac{\partial h}{\partial x_i} dW_i + \int_s^t \frac{1}{2} \sum_{i=1}^n \frac{\partial^2 h}{\partial x_i^2} dt | \mathcal{F}_s \right] = X(s) + \mathbb{E} \left[\int_s^t \frac{1}{2} \sum_{i=1}^n \frac{\partial^2 h}{\partial x_i^2} dt | \mathcal{F}_s \right] \end{aligned}$$

Looking at the last term we see that there is a sum of second order partial derivatives. If h is harmonic we have $\mathbb{E}[X(t)|\mathcal{F}_s] = X(s)$ a.s. If h is subharmonic we have $\mathbb{E}[X(t)|\mathcal{F}_s] \geq X(s)$ a.s. And thus we have proven the statement.

Problem 2.2: Show that $dW_1 dW_2 = 0$ for two independent Brownian motions.

Let $\Delta W_i(t_k) = W_i(t_k) - W_i(t_{k-1})$. Define Q_n :

$$Q_n = \sum_{k=1}^n \Delta W_1(t_k) \Delta W_2(t_k),$$

where $0 = t_0 < t_1 < \dots < t_n = t$. Now we just have to show that Q_n converges to 0 in L^2 as the norm of the partition goes to 0. We have:

$$\mathbb{E}[Q_n] = \mathbb{E} \sum_{k=1}^n \Delta W_1(t_k) \Delta W_2(t_k) = \sum_{k=1}^n \mathbb{E}[\Delta W_1(t_k) \Delta W_2(t_k)] = 0$$

$$\begin{aligned} \mathbb{E}[Q_n^2] &= \mathbb{V}\text{ar}[Q_n] = \mathbb{V}\text{ar} \left[\sum_{k=1}^n \Delta W_1(t_k) \Delta W_2(t_k) \right] = \sum_{k=1}^n \mathbb{V}\text{ar}[\Delta W_1(t_k) \Delta W_2(t_k)] = \\ &= \sum_{k=1}^n \mathbb{E}[\Delta W_1(t_k)^2 \Delta W_2(t_k)^2] = \sum_{k=1}^n \mathbb{E}[\Delta W_1(t_k)^2] \mathbb{E}[\Delta W_2(t_k)^2] = \\ &= \sum_{k=1}^n (\Delta t_k)^2 \leq \max_k \{\Delta t_k\} \sum_{k=1}^n \Delta t_k = \max_{k \in \{1, \dots, n\}} \{\Delta t_k\} t \rightarrow 0, n \rightarrow \infty \end{aligned}$$

3 Martingales

4 Partial differential equations

4.a Probabilistic representation of the Cauchy problem solutions

In this subsection, we show some simple examples on the topic of probabilistic representation of the Cauchy problem solutions.

Some theory. Given the conditions:

$$\frac{\partial F}{\partial t}(t, x) + \mu(t, x) \frac{\partial F}{\partial x}(t, x) + \frac{1}{2} \sigma^2(t, x) \frac{\partial^2 F}{\partial x^2}(t, x) - rF(t, x) = 0, \quad F(T, x) = \Phi(x)$$

$$dX_s = \mu(s, X_s) ds + \sigma(s, X_s) dW_s, \quad X_t = x$$

We can write down the Feynman-Kac formula for the solution:

$$F(t, x) = e^{-r(T-t)} \mathbb{E}_{t,x}[\Phi(X_T)]$$

Problem 4.1: Use a stochastic representation result in order to solve the following boundary value problem in the domain $[0, T] \times \mathbb{R}$:

$$\frac{\partial F}{\partial t} + \mu x \frac{\partial F}{\partial x} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 F}{\partial x^2} = 0$$

$$F(T, x) = \ln(x^2)$$

Here μ and σ are assumed to be known constants.

Solution: Applying Feynman–Kac formula we have

$$F(t, x) = \mathbb{E}[\ln X_T^2 | X_t = x]$$

where

$$dX_s = \mu X_s ds + \sigma X_s dW_s$$

$$X_t = x$$

Applying Ito formula, we can investigate the process $Z_s = \ln X_s^2$

$$dZ_s = \frac{2}{X_s} dX_s - \sigma^2 ds = \frac{2\mu X_s ds + 2\sigma X_s dW_s}{X_s} - \sigma^2 ds = (2\mu - \sigma^2) ds + 2\sigma dW_s$$

Thus we have the equation

$$d \ln X_s^2 = (2\mu - \sigma^2) ds + 2\sigma dW_s$$

We can rewrite it in the following form

$$\ln X_T^2 = \ln X_t^2 + \int_t^T (2\mu - \sigma^2) ds + \int_t^T 2\sigma dW_s = \ln X_t^2 + (2\mu - \sigma^2)(T - t) + 2\sigma(W_T - W_t)$$

Thus we have the solution:

$$F(t, x) = \mathbb{E}[\ln X_T^2 | X_t = x] = \ln(x^2) + (2\mu - \sigma^2)(T - t)$$

Problem 4.2: Consider the following boundary value problem in the domain $[0, T] \times R$.

$$\begin{aligned} \frac{\partial F}{\partial t} + \mu(t, x) \frac{\partial F}{\partial x} + \frac{1}{2} \sigma^2(t, x) \frac{\partial^2 F}{\partial x^2} + k(t, x) &= 0 \\ F(T, x) &= \Phi(x) \end{aligned}$$

Here μ, σ, k and Φ are assumed to be known functions.

Prove that this problem has the stochastic representation formula.

$$F(t, x) = \mathbb{E}[\Phi(X_T) | X_t = x] + \int_t^T \mathbb{E}[k(s, X_s) | X_t = x] ds$$

where

$$dX_s = \mu(s, X_s)ds + \sigma(s, X_s)dW_s$$

$$X_t = x$$

Solution: Considering the process $Z_s = F(s, X_s)$ and applying Ito formula, we have

$$\begin{aligned} dZ_s &= \frac{\partial F}{\partial s}(s, X_s)ds + \frac{\partial F}{\partial x}(s, X_s)dX_s + \frac{1}{2}\sigma^2(s, X_s)\frac{\partial^2 F}{\partial x^2}(s, X_s)ds = \\ &= \underbrace{\left(\frac{\partial F}{\partial s}(s, X_s) + \mu(s, X_s)\frac{\partial F}{\partial x}(s, X_s) + \frac{1}{2}\sigma^2(s, X_s)\frac{\partial^2 F}{\partial x^2}(s, X_s) \right)}_{=-k(s, X_s), \text{ assuming that } F \text{ actually solves PDE}} ds + \sigma(s, X_s)\frac{\partial F}{\partial x}(s, X_s)dW_s \end{aligned}$$

We can rewrite this equation in the following form

$$F(t, X_t) = F(T, X_T) + \int_t^T k(s, X_s)ds - \int_t^T \sigma(s, X_s)\frac{\partial F}{\partial x}(s, X_s)dW_s$$

If the process $\sigma(s, X_s)\frac{\partial F}{\partial x}(s, X_s)$ is in \mathcal{L}^2 , taking expected values we have the proof (remember that initial value $X_t = x$, $F(T, x) = \Phi(x)$ and expected value of Wiener integral is equal to zero)

$$F(t, x) = \mathbb{E}[\Phi(X_T)|X_t = x] + \int_t^T \mathbb{E}[k(s, X_s)|X_t = x]ds$$

5 Stochastic differential equations

6 Black-Scholes model

Suppose that we have an underlying asset with price S_t . Its dynamics follows geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t d\tilde{W}_t$$

We also have a riskless asset (bond) with the following dynamics (r is the interest rate):

$$dB_t = rB_t dt$$

Some assumptions about the market are also necessary (infinite liquidity etc). Suppose we now want to price some derivative with payment obligation $\mathcal{X} = \Phi(S_T)$ at time moment T . Then under

risk-neutral measure \mathbb{Q} we have no-arbitrage price:

$$\Pi_t = F(t, S_t) = e^{-r(T-t)} \mathbb{E}_{t,s}^{\mathbb{Q}}[\Phi(S_T)]$$

For a price of a european call option the formula is:

$$F(t, s) = s \cdot N[d_1(t, s)] - e^{-r(T-t)} K \cdot N[d_2(t, s)] \quad (2)$$

$$d_1(t, s) = \frac{1}{\sigma\sqrt{T-t}} \left(\log\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t) \right) \quad (3)$$

$$d_2(t, s) = d_1(t, s) - \sigma\sqrt{T-t} \quad (4)$$

$$N(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \quad (5)$$

Problem 6.1: Consider basic Black-Scholes model with T -payment obligation $\mathcal{X} = \Phi(S_T)$. Let Π_t be a no-arbitrage price.

a) Show that under martingale measure \mathbb{Q} :

$$d\Pi_t = r\Pi_t dt + g(t)dW_t$$

b) Show that $\frac{\Pi_t}{B_t}$ is a martingale.

Solution:

a) Under measure \mathbb{Q} we know that S_t is a geometric Brownian motion with drift r . We also know that F is an actual solution for the equation:

$$\frac{\partial F}{\partial t} + r\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial s^2} - rF = 0 \quad F(T, s) = \Phi(s)$$

Using these facts and having $g(t) \equiv \sigma\frac{\partial F}{\partial s}$:

$$d\Pi_t = dF = \left(\frac{\partial F}{\partial t} + r\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial s^2} \right) dt + \sigma\frac{\partial F}{\partial s} dW_t = rF dt + \sigma\frac{\partial F}{\partial s} dW_t = r\Pi_t dt + g(t)dW_t \quad (6)$$

b) We are going to apply Itô's lemma to the function $e^{-rt}B_0^{-1}\Pi_t$:

$$\begin{aligned} d(e^{-rt}B_0^{-1}\Pi_t) &= -re^{-rt}B_0^{-1}\Pi_t dt + e^{-rt}B_0^{-1}d\Pi_t = \\ &= -re^{-rt}B_0^{-1}\Pi_t dt + e^{-rt}B_0^{-1}(r\Pi_t dt + g(t)dW_t) = e^{-rt}B_0^{-1}\sigma\frac{\partial F}{\partial s}dW_t \quad (7) \end{aligned}$$

This stochastic differential contains zero drift component $\Rightarrow \frac{\Pi_t}{B_t}$ is a martingale.

Problem 6.2: Consider a derivative with payoff $\Phi(S_T) = \log(S_T)$. Find its no-arbitrage price.

Solution: For simplicity let us assume $S_0 \equiv 1$ (it does not affect the answer). In fact we only have to compute conditional expectation with respect to a risk-neutral measure \mathbb{Q}

$$\begin{aligned}\mathbb{E}_{t,s}^{\mathbb{Q}}[\log S_T] &= \mathbb{E}^{\mathbb{Q}}\left[\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T \middle| \left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t = \log(s)\right] = \\ &\left(r - \frac{\sigma^2}{2}\right)T + \mathbb{E}^{\mathbb{Q}}\left[\sigma W_T \middle| \sigma W_t = \log(s) - \left(r - \frac{\sigma^2}{2}\right)t\right] = \left(r - \frac{\sigma^2}{2}\right)(T - t) + \log(s)\end{aligned}\quad (8)$$

Here we used the fact that σW_t is a martingale. Then we can write down the price:

$$\Pi_t = e^{-r(T-t)}\mathbb{E}_{t,s}^{\mathbb{Q}}[\log S_T] = e^{-r(T-t)}\left(\left(r - \frac{\sigma^2}{2}\right)(T - t) + \log(s)\right)\quad (9)$$

Problem 6.3: Price a european put-option (knowing the Black-Scholes formula for the call).

Solution: We know that if we buy call (denote its price by C) and sell a put (denote its price by P) at time T , we get $\max\{S - K, 0\}$ for call and $-\max\{K - S, 0\}$ for put. Formally:

$$C - P = \max\{S - K, 0\} - \max\{K - S, 0\} = S - K\quad (10)$$

For a more time moment $t < T$ we should also discount the strike price. Now let's rearrange the terms and get the formula:

$$\begin{aligned}P(t, s) &= C(t, s) - s + e^{-r(T-t)}K = s \cdot N[d_1(t, s)] - e^{-r(T-t)}K \cdot N[d_2(t, s)] - s + e^{-r(T-t)}K = \\ &= e^{-r(T-t)}K \cdot N[-d_2(t, s)] - s \cdot N[-d_1(t, s)]\end{aligned}\quad (11)$$