Financial math problems solutions

Nick Averyanov, Sasha Plakhin

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***Problem* 2.1:** $h(\cdot)$ – is a harmonic function if:

$$\sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} = 0.$$

$h(\cdot)$ – is a subharmonic function if:

$$\sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} \geq 0.$$

Prove that for independent Wiener processes $W_1, \ldots, W_n$ and a processes X is defined by the formula: $X(t) = h(W_1(t), \ldots, W_n(t))$. Show that if h is harmonic (subharmonic) $\Rightarrow$ X is a martingale (submartingale).

Solution: Applying multidimensional version of Ito's lemma to the function h we can obtain:

$$dX = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} dt$$

Equivalently, we can rewrite the equation as:

$$h(W_1(t), \ldots, W_n(t))) = h(\mathbf{0}) + \int_0^t \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i + \int_0^t \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} dt$$

$$\mathbb{E}[X(t)|\mathcal{F}_s] = h(\mathbf{0}) + \mathbb{E}\left[\int_0^s \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i + \int_0^s \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} dt + \int_s^t \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i + \int_s^t \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} dt | \mathcal{F}_s\right] =$$
$$X(s) + \mathbb{E}\left[\int_s^t \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i + \int_s^t \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} dt | \mathcal{F}_s\right] = X(s) + \mathbb{E}\left[\int_s^t \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} dt | \mathcal{F}_s\right]$$

Looking at the last term we see that there is a sum of second order partial derivatives. If h is harmonic we have $\mathbb{E}[X(t)|\mathcal{F}_s] = X(s)$ a.s. If h is subharmonic we have $\mathbb{E}[X(t)|\mathcal{F}_s] \geq X(s)$ a.s. And thus we have proven the statement.

***Problem* 2.2:** Show that $dW_1 dW_2 = 0$ for two independent Brownian motions.

Let $\Delta W_i(t_k) = W_i(t_k) - W_i(t_{k-1})$. Define Q_n:

$$Q_n = \sum_{k=1}^{n} \Delta W_1(t_k) \Delta W_2(t_k),$$

where $0 = t_0 < t_1 < \ldots < t_n = t$. Now we just have to show that Q_n converges to 0 in L^2 as the norm of the partiotion goes to 0. We have:

$$\mathbb{E}[Q_n] = \mathbb{E}\sum_{k=1}^{n} \Delta W_1(t_k)\Delta W_2(t_k) = \sum_{k=1}^{n} \mathbb{E}[\Delta W_1(t_k)\Delta W_2(t_k)] = 0$$

$$\mathbb{E}[Q_n^2] = \mathbb{V}\text{ar}[Q_n] = \mathbb{V}\text{ar}\left[\sum_{k=1}^{n} \Delta W_1(t_k)\Delta W_2(t_k)\right] = \sum_{k=1}^{n} \mathbb{V}\text{ar}[\Delta W_1(t_k)\Delta W_2(t_k)] =$$
$$\sum_{k=1}^{n} \mathbb{E}[\Delta W_1(t_k)^2 \Delta W_2(t_k)^2] = \sum_{k=1}^{n} \mathbb{E}[\Delta W_1(t_k)^2]\mathbb{E}[\Delta W_2(t_k)^2] =$$
$$\sum_{k=1}^{n} (\Delta t_k)^2 \leq \max_k\{\Delta t_k\} \sum_{k=1}^{n} \Delta t_k = \max_{k\in\{,\dots n\}}\{\Delta t_k\} t \to 0, n \to \infty$$

3 Martingales

4 Partial differential equations

5 Stochastic differential equations

6 Black-Scholes