Financial math problems solutions

Nick Averyanov, Sasha Plakhin

June 12, 2022

Contents

1 Binomial model

2 Itô's lemma

Let $dX_t = \mu_t dt + \sigma_t dW_t$ and $f(t,x) \in C^{1,2}$. Itô's lemma:

$$df(t,X_t) = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{1}{2}\sigma_t^2 \frac{\partial f^2}{\partial x^2}\right) dt + \sigma_t \frac{\partial f}{\partial x} dW_t \tag{1}$$

Remember that this is just a short notation for the integral form.

***Problem* 2.1:** $h(\cdot)$ – is a harmonic function if:

$$\sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} = 0.$$

$h(\cdot)$ – is a subharmonic function if:

$$\sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} \geq 0.$$

Prove that for independent Wiener processes $W_1, \dots, W_n$ and a processes X is defined by the formula: $X(t) = h(W_1(t), \dots, W_n(t))$. Show that if h is harmonic (subharmonic) $\Rightarrow X$ is a martingale (submartingale).

Solution: Applying multidimensional version of Ito's lemma to the function h we can obtain:

$$dX = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} dt$$

Equivalently, we can rewrite the equation as:

$$h(W_1(t), \dots, W_n(t))) = h(\mathbf{0}) + \int_0^t \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i + \int_0^t \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} dt$$

$$\mathbb{E}[X(t)|\mathcal{F}_s] = h(\mathbf{0}) + \mathbb{E}\left[\int_0^s \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i + \int_0^s \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} dt + \int_s^t \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i + \int_s^t \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} dt | \mathcal{F}_s\right] =$$
$$X(s) + \mathbb{E}\left[\int_s^t \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i + \int_s^t \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} dt | \mathcal{F}_s\right] = X(s) + \mathbb{E}\left[\int_s^t \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} dt | \mathcal{F}_s\right]$$

Looking at the last term we see that there is a sum of second order partial derivatives. If h is harmonic we have $\mathbb{E}[X(t)|\mathcal{F}_s] = X(s)$ a.s. If h is subharmonic we have $\mathbb{E}[X(t)|\mathcal{F}_s] \geq X(s)$ a.s. And thus we have proven the statement.

***Problem* 2.2:** Show that $dW_1 dW_2 = 0$ for two independent Brownian motions.

Let $\Delta W_i(t_k) = W_i(t_k) - W_i(t_{k-1})$. Define Q_n:

$$Q_n = \sum_{k=1}^{n} \Delta W_1(t_k) \Delta W_2(t_k),$$

where $0 = t_0 < t_1 < \ldots < t_n = t$. Now we just have to show that Q_n converges to 0 in L^2 as the norm of the partiotion goes to 0. We have:

$$\mathbb{E}[Q_n] = \mathbb{E} \sum_{k=1}^{n} \Delta W_1(t_k) \Delta W_2(t_k) = \sum_{k=1}^{n} \mathbb{E}[\Delta W_1(t_k) \Delta W_2(t_k)] = 0$$

$$\mathbb{E}[Q_n^2] = \mathbb{V}\mathrm{ar}[Q_n] = \mathbb{V}\mathrm{ar}\left[\sum_{k=1}^{n} \Delta W_1(t_k) \Delta W_2(t_k)\right] = \sum_{k=1}^{n} \mathbb{V}\mathrm{ar}[\Delta W_1(t_k) \Delta W_2(t_k)] =$$
$$\sum_{k=1}^{n} \mathbb{E}[\Delta W_1(t_k)^2 \Delta W_2(t_k)^2] = \sum_{k=1}^{n} \mathbb{E}[\Delta W_1(t_k)^2] \mathbb{E}[\Delta W_2(t_k)^2] =$$
$$\sum_{k=1}^{n} (\Delta t_k)^2 \leq \max_k \{\Delta t_k\} \sum_{k=1}^{n} \Delta t_k = \max_{k \in \{,\ldots n\}} \{\Delta t_k\} t \to 0, n \to \infty$$

3 Martingales

4 Partial differential equations

4.a Probabilistic representation of the Cauchy problem solutions

In this subsection, we show some simple examples on the topic of probabilistic representation of the Cauchy problem solutions.

Some theory. Given the conditions:

$$\frac{\partial F}{\partial t}(t,x) + \mu(t,x) \frac{\partial F}{\partial x}(t,x) + \frac{1}{2}\sigma^2(t,x) \frac{\partial^2 F}{\partial x^2}(t,x) - rF(t,x) = 0, \quad F(T,x) = \Phi(x)$$

$$dX_s = \mu(s, X_s) ds + \sigma(s, X_s) dW_s, \quad X_t = x$$

We can write down the Feynman-Kac formula for the solution:

$$F(t,x) = e^{-r(T-t)} \mathbb{E}_{t,x}[\Phi(X_T)]$$

***Problem* 4.1:** Use a stochastic representation result in order to solve the following boundary value problem in the domain $[0,T] \times \mathbb{R}$:

$$\frac{\partial F}{\partial t} + \mu x \frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2} = 0$$

$$F(T,x) = \ln(x^2)$$

Here μ and σ are assumed to be known constants.

Solution: Applying Feynman–Kac formula we have

$$F(t,x) = \mathbb{E}[\ln X_T^2 | X_t = x]$$

where

$$dX_s = \mu X_s ds + \sigma X_s dW_s$$

$$X_t = x$$

Applying Ito formula, we can investigate the process $Z_s = \ln X_s^2$

$$dZ_s = \frac{2}{X_s} dX_s - \sigma^2 ds = \frac{2\mu X_s ds + 2\sigma X_s dW_s}{X_s} - \sigma^2 ds = (2\mu - \sigma^2)ds + 2\sigma dW_s$$

Thus we have the equation

$$d \ln X_s^2 = (2\mu - \sigma^2)ds + 2\sigma dW_s$$

We can rewrite it in the following form

$$\ln X_T^2 = \ln X_t^2 + \int_t^T (2\mu - \sigma^2)ds + \int_t^T 2\sigma dW_s = \ln X_t^2 + (2\mu - \sigma^2)(T-t) + 2\sigma(W_T - W_t)$$

Thus we have the solution:

$$F(t,x) = \mathbb{E}[\ln X_T^2 | X_t = x] = \ln(x^2) + (2\mu - \sigma^2)(T-t)$$

***Problem* 4.2:** Consider the following boundary value problem in the domain $[0,T] \times R$.

$$\frac{\partial F}{\partial t} + \mu(t,x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2} + k(t,x) = 0$$
$$F(T,x) = \Phi(x)$$

Here μ, σ, k and Φ are assumed to be known functions.
Prove that this problem has the stochastic representation formula.

$$F(t,x) = \mathbb{E}[\Phi(X_T)|X_t = x] + \int_t^T E[k(s,X_s)|X_t = x]ds$$

where

$$dX_s = \mu(s, X_s)ds + \sigma(s, X_s)dW_s$$

$$X_t = x$$

Solution: Considering the process $Z_s = F(s, X_s)$ and applying Ito formula, we have

$$\begin{aligned} dZ_s &= \frac{\partial F}{\partial s}(s, X_s)ds + \frac{\partial F}{\partial x}(s, X_s)dX_s + \frac{1}{2}\sigma^2(s, X_s)\frac{\partial^2 F}{\partial x^2}(s, X_s)ds = \\ &= \underbrace{\left(\frac{\partial F}{\partial s}(s, X_s) + \mu(s, X_s)\frac{\partial F}{\partial x}(s, X_s) + \frac{1}{2}\sigma^2(s, X_s)\frac{\partial^2 F}{\partial x^2}(s, X_s)\right)}_{=-k(s,X_s),\text{ assuming that } F \text{ actually solves PDE}} ds + \sigma(s, X_s)\frac{\partial F}{\partial x}(s, X_s)dW_S \end{aligned}$$

We can rewrite this equation in the following form

$$F(t, X_t) = F(T, X_T) + \int_t^T k(s, X_s)ds - \int_t^T \sigma(s, X_s)\frac{\partial F}{\partial x}(s, X_s)dW_S$$

If the process $\sigma(s, X_s)\frac{\partial F}{\partial x}(s, X_s)$ is in $\mathcal{L}^2$, taking expected values we have the proof(remember that intitial value $X_t = x$, $F(T, x) = \Phi(x)$ and expected value of Wiener integral is equal to zero)

$$F(t, x) = \mathbb{E}[\Phi(X_T)|X_t = x] + \int_t^T \mathbb{E}[k(s, X_s)|X_t = x]ds$$

5 Stochastic differential equations

6 Black-Scholes model

Suppose that we have an underlying asset with price S_t. Its dynamics follows geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t d\bar{W}_t$$

We also have a riskless asset (bond) with the interest rate r. Some assumptions about the market are also necessary (inifinite liquidity etc).

Suppose we now want to price some derivative with payment obligation $\mathcal{X} = \Phi(S_T)$ at time moment T. Then under risk-neutral measure $\mathbb{Q}$ we have no-arbitrage price:

$$\Pi_t = F(t, S_t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}_{t,s}[\Phi(S_T)]$$

***Problem* 6.1:** Consider basic Black-Scholes model with T-payment obligation $\mathcal{X} = \Phi(S_T)$. Let Π_t

be a no-arbitrage price.

a) Show that under martingale measure $\mathbb{Q}$:

$$d\Pi_t = r\Pi_t dt + g(t)dW_t$$

b) Show that $\dfrac{\Pi_t}{B_t}$ is a martingale.

Solution:

a) Under measure $\mathbb{Q}$ we know that S_t is a geometric Brownian motion with drift r. We also know that F is an actual solution for the equation:

$$\frac{\partial F}{\partial t} + r\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial s^2} - rF = 0 \quad F(T,s) = \Phi(s)$$

Using these facts and having $g(t) \equiv \sigma\dfrac{\partial F}{\partial s}$:

$$d\Pi_t = dF = \left(\frac{\partial F}{\partial t} + r\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial s^2}\right)dt + \sigma\frac{\partial F}{\partial s}dW_t = rFdt + \sigma\frac{\partial F}{\partial s}dW_t = r\Pi_t dt + g(t)dW_t \tag{2}$$

b) Applying Itô's lemma to the function $Z_t \equiv \dfrac{\Pi_t}{B_t} = e^{rt}\Pi_t B_0^{-1}$ we get:

$$dZ_t = \left(re^{rt}\Pi_t B_0^{-1} + re^{rt}B_0^{-1}\right)dt + \sigma\frac{\partial F}{\partial s}dW_t \tag{3}$$