Financial math problems solutions

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2 Itô's lemma

Let $dX_t = \mu_t dt + \sigma_t dW_t$ and $f(t,x) \in C^{1,2}$. Itô's lemma:

$$df(t,X_t) = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{1}{2}\sigma_t^2 \frac{\partial f^2}{\partial x^2}\right)dt + \sigma_t \frac{\partial f}{\partial x} dW_t \tag{1}$$

Remember that this is just a short notation for the integral form.

***Problem* 2.1:** $h(\cdot)$ – is a harmonic function if:

$$\sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} = 0.$$

$h(\cdot)$ – is a subharmonic function if:

$$\sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} \geq 0.$$

Prove that for independent Wiener processes $W_1, \ldots, W_n$ and a processes X is defined by the formula: $X(t) = h(W_1(t), \ldots, W_n(t))$. Show that if h is harmonic (subharmonic) $\Rightarrow$ X is a martingale (submartingale).

Solution: Applying multidimensional version of Ito's lemma to the function h we can obtain:

$$dX = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i + \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} dt$$

Equivalently, we can rewrite the equation as:

$$h(W_1(t), \ldots, W_n(t))) = h(\mathbf{0}) + \int_0^t \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i + \int_0^t \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} dt$$

$$\mathbb{E}[X(t)|\mathcal{F}_s] = h(\mathbf{0}) + \mathbb{E}\left[\int_0^s \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i + \int_0^s \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} dt + \int_s^t \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i + \int_s^t \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} dt|\mathcal{F}_s\right] =$$
$$X(s) + \mathbb{E}\left[\int_s^t \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i + \int_s^t \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} dt|\mathcal{F}_s\right] = X(s) + \mathbb{E}\left[\int_s^t \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} dt|\mathcal{F}_s\right]$$

Looking at the last term we see that there is a sum of second order partial derivatives. If h is harmonic we have $\mathbb{E}[X(t)|\mathcal{F}_s] = X(s)$ a.s. If h is subharmonic we have $\mathbb{E}[X(t)|\mathcal{F}_s] \geq X(s)$ a.s. And thus we have proven the statement.

***Problem* 2.2:** Show that $dW_1 dW_2 = 0$ for two independent Brownian motions.

Let $\Delta W_i(t_k) = W_i(t_k) - W_i(t_{k-1})$. Define Q_n:

$$Q_n = \sum_{k=1}^{n} \Delta W_1(t_k)\Delta W_2(t_k),$$

where $0 = t_0 < t_1 < \ldots < t_n = t$. Now we just have to show that Q_n converges to 0 in L^2 as the norm of the partiotion goes to 0. We have:

$$\mathbb{E}[Q_n] = \mathbb{E}\sum_{k=1}^{n} \Delta W_1(t_k)\Delta W_2(t_k) = \sum_{k=1}^{n} \mathbb{E}[\Delta W_1(t_k)\Delta W_2(t_k)] = 0$$

$$\mathbb{E}[Q_n^2] = \mathbb{V}\mathrm{ar}[Q_n] = \mathbb{V}\mathrm{ar}\left[\sum_{k=1}^{n} \Delta W_1(t_k)\Delta W_2(t_k)\right] = \sum_{k=1}^{n} \mathbb{V}\mathrm{ar}[\Delta W_1(t_k)\Delta W_2(t_k)] =$$
$$\sum_{k=1}^{n} \mathbb{E}[\Delta W_1(t_k)^2\Delta W_2(t_k)^2] = \sum_{k=1}^{n} \mathbb{E}[\Delta W_1(t_k)^2]\mathbb{E}[\Delta W_2(t_k)^2] =$$
$$\sum_{k=1}^{n}(\Delta t_k)^2 \leq \max_k\{\Delta t_k\}\sum_{k=1}^{n}\Delta t_k = \max_{k\in\{1,\ldots n\}}\{\Delta t_k\},\ t \to 0, n \to \infty$$

3 Martingales

On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_t\}_{t=1}^{\infty}$ a process X_t is called a matrtingale iff:

1. X_t is $\{\mathcal{F}_t\}_{t=1}^{\infty}$ adapted;

2. $X_t \in L_1,\ \forall t$;

3. $\mathbb{E}[X_t|\mathcal{F}_s] = X_s, \forall t \geq s$.

***Problem* 3.1:** Let Y is a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_t\}_{t=1}^{\infty}$. Show that a process X_t defined as:

$$X_t = \mathbb{E}[Y|\mathcal{F}_t] \tag{2}$$

Solution: For this problem we need to use the following property of a conditional expectation. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two sigma-algebras such that $\mathcal{F}_1 \subset \mathcal{F}_2$. Then the following property holds:

$$\mathbb{E}[\mathbb{E}[Z|\mathcal{F}_1]|\mathcal{F}_2] = \mathbb{E}[\mathbb{E}[Z|\mathcal{F}_2]|\mathcal{F}_1] = \mathbb{E}[Z|\mathcal{F}_1] \tag{3}$$

Let's apply this property to X_t and prove that this process a martingale:

$$\mathbb{E}[X_t|\mathcal{F}_s] = \mathbb{E}[\mathbb{E}[Y|\mathcal{F}_t]|\mathcal{F}_s] = \mathbb{E}[Y|\mathcal{F}_s] = X_s \tag{4}$$

4 Partial differential equations

4.a Probabilistic representation of the Cauchy problem solutions

In this subsection, we show some simple examples on the topic of probabilistic representation of the Cauchy problem solutions.

Some theory. Given the conditions:

$$\frac{\partial F}{\partial t}(t,x) + \mu(t,x)\frac{\partial F}{\partial x}(t,x) + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x) - rF(t,x) = 0, \quad F(T,x) = \Phi(x)$$

$$dX_s = \mu(s,X_s)ds + \sigma(s,X_s)dW_s, \quad X_t = x$$

We can write down the Feynman-Kac formula for the solution:

$$F(t,x) = e^{-r(T-t)}\mathbb{E}_{t,x}[\Phi(X_T)]$$

***Problem* 4.1:** Use a stochastic representation result in order to solve the following boundary value problem in the domain $[0,T]\times\mathbb{R}$:

$$\frac{\partial F}{\partial t} + \mu x\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 F}{\partial x^2} = 0$$

$$F(T,x) = \ln(x^2)$$

Here μ and σ are assumed to be known constants.

Solution: Applying Feynman–Kac formula we have

$$F(t,x) = \mathbb{E}[\ln X_T^2 | X_t = x]$$

where

$$dX_s = \mu X_s ds + \sigma X_s dW_s$$

$$X_t = x$$

Applying Ito formula, we can investigate the process $Z_s = \ln X_s^2$

$$dZ_s = \frac{2}{X_s}dX_s - \sigma^2 ds = \frac{2\mu X_s ds + 2\sigma X_s dW_s}{X_s} - \sigma^2 ds = (2\mu - \sigma^2)ds + 2\sigma dW_s$$

Thus we have the equation

$$d\ln X_s^2 = (2\mu - \sigma^2)ds + 2\sigma dW_s$$

We can rewrite it in the following form

$$\ln X_T^2 = \ln X_t^2 + \int_t^T (2\mu - \sigma^2)ds + \int_t^T 2\sigma dW_s = \ln X_t^2 + (2\mu - \sigma^2)(T-t) + 2\sigma(W_T - W_t)$$

Thus we have the solution:

$$F(t,x) = \mathbb{E}[\ln X_T^2 | X_t = x] = \ln(x^2) + (2\mu - \sigma^2)(T-t)$$

***Problem* 4.2:** Consider the following boundary value problem in the domain $[0,T] \times R$.

$$\frac{\partial F}{\partial t} + \mu(t,x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2} + k(t,x) = 0$$
$$F(T,x) = \Phi(x)$$

Here μ, σ, k and Φ are assumed to be known functions.
Prove that this problem has the stochastic representation formula.

$$F(t,x) = \mathbb{E}[\Phi(X_T)|X_t = x] + \int_t^T E[k(s,X_s)|X_t = x]ds$$

where

$$dX_s = \mu(s,X_s)ds + \sigma(s,X_s)dW_s$$

$$X_t = x$$

Solution: Considering the process $Z_s = F(s,X_s)$ and applying Ito formula, we have

$$\begin{aligned} dZ_s &= \frac{\partial F}{\partial s}(s,X_s)ds + \frac{\partial F}{\partial x}(s,X_s)dX_s + \frac{1}{2}\sigma^2(s,X_s)\frac{\partial^2 F}{\partial x^2}(s,X_s)ds = \\ &= \underbrace{\Big(\frac{\partial F}{\partial s}(s,X_s) + \mu(s,X_s)\frac{\partial F}{\partial x}(s,X_s) + \frac{1}{2}\sigma^2(s,X_s)\frac{\partial^2 F}{\partial x^2}(s,X_s)\Big)}_{=-k(s,X_s),\text{ assuming that } F \text{ actually solves PDE}} ds + \sigma(s,X_s)\frac{\partial F}{\partial x}(s,X_s)dW_S \end{aligned}$$

We can rewrite this equation in the following form

$$F(t,X_t) = F(T,X_T) + \int_t^T k(s,X_s)ds - \int_t^T \sigma(s,X_s)\frac{\partial F}{\partial x}(s,X_s)dW_S$$

If the process $\sigma(s,X_s)\frac{\partial F}{\partial x}(s,X_s)$ is in $\mathcal{L}^2$, taking expected values we have the proof(remember that intitial value $X_t = x$, $F(T,x) = \Phi(x)$ and expected value of Wiener integral is equal to zero)

$$F(t,x) = \mathbb{E}[\Phi(X_T)|X_t = x] + \int_t^T \mathbb{E}[k(s, X_s)|X_t = x]ds$$

5 Stochastic differential equations

6 Black-Scholes model

Suppose that we have an underlying asset with price S_t. Its dynamics follows geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t d\bar{W}_t$$

We also have a riskless asset (bond) with the following dynamics (r is the interest rate):

$$dB_t = rB_t dt$$

Some assumptions about the market are also necessary (inifinite liquidity etc). Suppose we now want to price some derivative with payment obligation $\mathcal{X} = \Phi(S_T)$ at time moment T. Then under risk-neutral measure $\mathbb{Q}$ we have no-arbitrage price:

$$\Pi_t = F(t, S_t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}_{t,s}[\Phi(S_T)]$$

For a price of a european call option the formula is:

$$\begin{aligned}
F(t,s) &= s \cdot N[d_1(t,s)] - e^{-r(T-t)}K \cdot N[d_2(t,s)] && (5)\\
d_1(t,s) &= \frac{1}{\sigma\sqrt{T-t}}\left(\log\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right) && (6)\\
d_2(t,s) &= d_1(t,s) - \sigma\sqrt{T-t} && (7)\\
N(x) &= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt && (8)
\end{aligned}$$

***Problem* 6.1:** Consider basic Black-Scholes model with T-payment obligation $\mathcal{X} = \Phi(S_T)$. Let Π_t be a no-arbitrage price.

a) Show that under martingale measure $\mathbb{Q}$:

$$d\Pi_t = r\Pi_t dt + g(t)dW_t$$

b) Show that $\dfrac{\Pi_t}{B_t}$ is a martingale.

Solution:

a) Under measure $\mathbb{Q}$ we know that S_t is a geometric Brownian motion with drift r. We also know that F is an actual solution for the equation:

$$\frac{\partial F}{\partial t} + rs\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 F}{\partial s^2} - rF = 0 \quad F(T,s) = \Phi(s)$$

Using these facts and having $g(t) \equiv \sigma \dfrac{\partial F}{\partial s}$:

$$d\Pi_t = dF = \left(\frac{\partial F}{\partial t} + rs\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 F}{\partial s^2}\right)dt + \sigma\frac{\partial F}{\partial s}dW_t = rFdt + \sigma\frac{\partial F}{\partial s}dW_t = r\Pi_t dt + g(t)dW_t \tag{9}$$

b) We are going to apply Itô's lemma to the function $e^{-rt}B_0^{-1}\Pi_t$:

$$\begin{aligned} d(e^{-rt}B_0^{-1}\Pi_t) &= -re^{-rt}B_0^{-1}\Pi_t dt + e^{-rt}B_0^{-1}d\Pi_t = \\ &- re^{-rt}B_0^{-1}\Pi_t dt + e^{-rt}B_0^{-1}(r\Pi_t dt + g(t)dW_t) = e^{-rt}B_0^{-1}\sigma\frac{\partial F}{\partial s}dW_t \end{aligned} \tag{10}$$

This stochastic differntial contains zero drift component $\Rightarrow \dfrac{\Pi_t}{B_t}$ is a martingale.

***Problem* 6.2:** Consider a derivative with payoff $\Phi(S_T) = \log(S_T)$. Find its no-arbitragee price.

Solution: For simplicity let us assume $S_0 \equiv 1$ (it does not affect the answer). In fact we only have to compute conditional expectation with respect to a risk-neutral measure $\mathbb{Q}$

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}_{t,s}[\log S_T] &= \mathbb{E}^{\mathbb{Q}}\left[\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T \middle| \left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t = \log(s)\right] = \\ &\left(r - \frac{\sigma^2}{2}\right)T + \mathbb{E}^{\mathbb{Q}}\left[\sigma W_T \middle| \sigma W_t = \log(s) - \left(r - \frac{\sigma^2}{2}\right)t\right] = \left(r - \frac{\sigma^2}{2}\right)(T-t) + \log(s) \end{aligned} \tag{11}$$

Here we used the fact that σW_t is a martingale. Then we can write down the price:

$$\Pi_t = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}_{t,s}[\log S_T] = e^{-r(T-t)}\left(\left(r - \frac{\sigma^2}{2}\right)(T-t) + \log(s)\right) \tag{12}$$

***Problem* 6.3:** Price a european put-option (knowing the Black-Scholes formula for the call).

Solution: We know that if we buy call (denote its price by C) and sell a put (denote its price by P) at time T, we get $\max\{S-K,0\}$ for call and $-\max\{K-S,0\}$ for put. Formally:

$$C - P = \max\{S-K,0\} - \max\{K-S,0\} = S - K \tag{13}$$

For a more time moment $t < T$ we should also discount the strike price. Now let's rearrange the terms and get the formula:

$$P(t,s) = C(t,s) - s + e^{-r(T-t)}K = s \cdot N[d_1(t,s)] - e^{-r(T-t)}K \cdot N[d_2(t,s)] - s + e^{-r(T-t)}K = \\ = e^{-r(T-t)}K \cdot N[-d_2(t,s)] - s \cdot N[-d_1(t,s)] \quad (14)$$

***Problem* 6.4:** Dynamics of the stock price in dollars follows geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t d\bar{W}_t^1. \quad (15)$$

While the euro-dollar exchange rate also follows geometric Brownian motion:

$$dY_t = \beta Y_t dt + \delta Y_t d\bar{W}_t^2. \quad (16)$$

Both Brownian motions are independent of each other. There is also a derivative which pays $\ln(Z_T^2)$ at time T, where Z_t is a price of a stock in euro. A risk-free rate in euros is equal to r. Find the arbitrage free price for the derivative.

Solution: In fact we can rewrite Z_t as $Y_t \cdot S_t$. Le'ts now find dZ_t using Ito's product rule:

$$dZ_t = d(Y_t \cdot S_t) = Y_t dS_t + S_t dY_t + dY_t dS_t = Y_t dS_t + S_t dY_t = \\ (\mu+\beta)Z_t dt + \sigma Z_t d\bar{W}_t^1 + \delta Z_t d\bar{W}_t^2 = (\mu+\beta)Z_t dt + \sqrt{\sigma^2+\delta^2} Z_t d\tilde{W}_t \quad (17)$$

Here $\tilde{W}_t$ is also a Brownian motion. Now we can use a result from one of the previous problems and get:

$$\Pi_t = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[\log(Z_t^2)] = e^{-r(T-t)}((2r - (\sigma^2+\delta^2))(T-t) + 2\log(z)) \quad (18)$$

7 Duality and delta-hedging

Here the linearity of the price function is a useful fact:

$$\Pi(t; \alpha\Phi(S_T) + \beta\Psi(S_T)) = \alpha\Pi(t; \Phi(S_T)) + \beta\Pi(t; \Psi(S_T)) \quad (19)$$

For any continuous contract function we can find a portfolio consisting of bonds, stock and European call-options that can replicate it arbitrarily accurately.

By "Greeks" we understand the following set of the parameters (V is value):

$$\Delta = \frac{\partial V}{\partial S} \tag{20}$$

$$\Gamma = \frac{\partial^2 V}{\partial S^2} \tag{21}$$

$$\rho = \frac{\partial V}{\partial r} \tag{22}$$

$$\Theta = \frac{\partial V}{\partial t} \tag{23}$$

$$\mathcal{V} = \frac{\partial V}{\partial r} \tag{24}$$

$$\tag{25}$$

These parameters measure sensitivity of the portfolio to changes of different parameters and values.

***Problem* 7.1:** Consider a Straddle contract:

$$\mathcal{X} = \begin{cases} K - S_T, 0 < S_T \leq K, \\ S_T - K, S_T > K \end{cases} \tag{26}$$

Replicate it and find arbitrage free price for it.

Solution: It is easy to see that this contract is simply equivalent to buying both call and put options. So in fact we have just to buy a call and a put in order to replicate it. From the put-call duality we can derive:

$$\Pi = P + C = C + Ke^{-r(T-t)} - S + C = 2C + Ke^{-r(T-t)} - S \tag{27}$$

So we have to buy two calls with strike K, buy K amount of bonds and sell a single stock. In order to find the price, we can substitute C with the Black-Scholes formula for option price.

***Problem* 7.2:** Consider a bull spread contract:

$$\mathcal{X} = \begin{cases} A, S_T < A, \\ S_T, A \leq S_T \leq B, \\ B, S_T > B \end{cases} \tag{28}$$

Replicate it and find arbitrage free price for it.

Solution: It is easy if we consider the case when $A = 0$. Then buying a call with strike 0 and selling a call with strike 0 and selling a call with strike B leads us to a correct formula. Now we have to buy A amount of bonds. So the price for the bull spread is:

$$\Pi = C(t, S, A) - C(t, S, B) + Ae^{-r(T-t)} \tag{29}$$