Semimartingales and Girsanov's theorem

Nikita Orlov, Nikolai Averianov

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Motivation

- Want to extend stochastic integration on the most general class (which semimartingales happen to be).
- Semimartingales have significant importance for financial mathematics, because if want a model a price of an asset on an arbitrage-free market by a stochastic process, we can only do that using semimartingales.
- During this lecture we will cover some general facts about semimartingales, discover Girsanov's theorem and show some of its applications in pricing.

Recap

Definition

Filtration $\mathbb{F}$ on probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a collection of increasing sub-σ-algebras of $\mathcal{F}$: $\mathbb{F} := (\mathcal{F}_t)_{t \in T}$; $\mathcal{F}_s \subset \mathcal{F}_t, \forall t > s$. Space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is called filtered.

Definition

A martingale X_t on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is an $\mathbb{F}$-adapted stochastic process, which satisfies: $\mathbb{E}[X_{t+h} | \mathcal{F}_t] = X_t, \forall t, h \geq 0$.

Definition

A stopping time τ is a random variable on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ if $\{\tau \leq t\} \in \mathcal{F}_t, \forall t \in T$.

Recap

Definition

A stochastic process X_t is a Lévy process if the following holds:

1. $X_0 = 0$ a.s.
2. For any set of time moments $0 \leq t_1 < \ldots < t_n$, increments $X_{t_1} - X_0, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent.
3. $\forall t_1, t_2, h \geq 0$, $X_{t_2+h} - X_{t_1+h} \overset{d}{=} X_{t_2} - X_{t_1}$ – increments are stationary.
4. $X_{t+h} \xrightarrow{\mathbb{P}} X_t, \;\; h \to 0$ – the process is continous in probability.

Definition

A function f is a càdlàg function if it is right-continous everywhere and has left limits everywhere.

One can show that any Lévy process has a modification with a.s. càdlàg trajectories.

Local martingales

Definition

Let M be a stochastic process, τ - stopping time. M^τ denotes a process M stopped at time τ:

$$M_t^\tau = M_{t\wedge\tau}$$

Definition

Let S be a class of stochastic processes. A process M is locally in S if there exists a sequence of stopping times $\tau_n \uparrow \infty$ such that $\mathbb{I}\{\tau_n > 0\}M^{\tau_n}$ are in S. The sequence $(\tau_n)_{n\geq 1}$ is called a localizing sequence.

Definition

An $\mathbb{F}$-adapted process M_t is called a local martingale if it is càdlàg and locally in the class of martingales.

Local martingales

Denote $M_t^* = \sup_{s \le t} |M_s|$, $M^* = \sup_{s \ge 0} |M_s|$

Theorem

Let M be a local martingale.

1. If $\mathbb{E}[M_t^*] < \infty \, \forall t$ then M is a martingale
2. If $\mathbb{E}[M^*] < \infty$ then M is a u.i. (uniformly integrable) martingale

Proof.

Let $(\tau_n)_{n \ge 1}$ be a localizing sequence of M. Then

$$\forall s \le t : \mathbb{E}[M_{\tau_n \wedge t} | \mathcal{F}_s] = M_{\tau_n \wedge s}$$

Applying dominated convergence theorem (hence $M_{\tau_n \wedge t}$ is dominated by the integrable r.v. M_t^*) gives that M is a martingale. If $\mathbb{E}[M^*] < \infty$ the family $(M_t)_{t \ge 0}$ is dominated by M^* thus it is u.i. $\square$

Local martingales

Corollary

- If M is a bounded local martingale then M is a u.i. martingale
- If M is a local martingale and a Lévy process then M is a martingale
- If $(M_n)_{n\geq 1}$ is a discrete time local martingale and $\mathbb{E}|M_n| < \infty \, \forall n$ then M is a martingale

Example

Let W_t be a Brownian motion and $\tau = \inf\{t : W_t = -1\}$. The stopped process $W_{t\wedge\tau}$ is a martingale, $\mathbb{E}[W_t] = 0$ however its limit is equal to -1. Rescaling the time index $X_t = W^{\tau}_{t/(1-t)} \cdot \mathbb{I}\{t < 1\} - 1 \cdot \mathbb{I}\{t \geq 1\}$ defines a local martingale with respect to its natural filtration, however $\mathbb{E}[X_1] = -1 \neq \mathbb{E}[X_0] = 0$, so X is not a martingale. A localizing sequence can be chosen $\tau_k = \min\{t : X_t = k\}$ if there is such t, otherwize $\tau_k = k$.

Semimartingales

Definition

A real-valued process X on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is called a semimartingale if it decomposes

$$X_t = A_t + M_t$$

where M is a local martingale and A is an adapted càdlàg process with finite variation (the total variation of each path $t \longmapsto A_t(\omega)$ is bounded over each finite interval $[0, t]$).

Intuitively, a process A plays the role of a drift term (as $\int_0^t b_s \, ds$) and a local martingale M is a pure randomness part (as $\int_0^t \sigma_s \, dW_s$)

Definition

A stochastic integral with respect to a semimartingale X can be defined

$$\int_0^t H_s \, dX_s = \sum_{i \geq 1} H_{t_{i-1}} (X_{t \wedge t_i} - X_{t \wedge t_{i-1}})$$

where $0 = t_0 < t_1 < \ldots$ and $t_n \xrightarrow{n \to \infty} \infty$, H is predictable, and locally bounded: $|H_t(\omega)| \leq n \, \forall 0 < t < \tau_n(\omega)$, where (τ_n) is a sequence of stopping times increasing to ∞.

Theorem (Lévy-Ito decomposition)

Let X_t be a Lévy process, $b \in \mathbb{R}, c \in \mathbb{R}_+, \nu$ is a Lévy measure s.t.

$$\int_{|x| \leq 1} x^2 \nu(dx) < \infty, \int_{|x|>1} \nu(dx) < \infty$$

Then X_t can be decomposed

$$X_t = bt + cW_t + J_t$$

where $bt + cW_t$ is a continuous part and J_t is a jump part.

Therefore, any Lévy process is a semimartingale as it the sum of a square integrable martingale and a finite variation process.

Quadratic variation

The existence of quadratic (co)variations is a huge difference between standard calculus and stochastic calculus.

Definition

Let X_t be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and time index t ranging over non-negative real numbers. The quadratic variation of X_t written as $[X]_t$ is defined as:

$$[X]_t = \lim_{||P|| \to 0} \sum_{k=1}^{n} (X_{t_k} - X_{t_{k-1}})^2$$

where P ranges over partitions $0 = t_0 < t_1 < \cdots < t_n = t$ and the norm is the mesh $\max(t_i - t_{i-1})$. This limit is defined using convergence in probability.

Quadratic Variation

Definition

Let X, Y be semimartingales. The quadratic covariation of X and Y is defined by

$$[X, Y]_t = X_t Y_t - \int_0^t X_{s-} dY_s - \int Y_{s-} dX_s$$

The quadratic variation $[X]$ is equal to $[X, X]$. The polarization identity:

$$[X, Y]_t = \frac{1}{2}([X + Y]_t - [X]_t - [Y]_t)$$

Quadratic Variation

Definition

Let X be a semimartingale. The continuous part of quadratic variation $[X]^c$ is defined

$$[X]_t = [X]_t^c + X_0^2 + \sum_{0<s\leq t} (\Delta X_s)^2$$

If $[X]^c \equiv 0$, X is called a quadratic pure jump semimartingale.

Example

If N is a Poisson process then $[N] = N$, so it is a quadratic pure jump semimartingale.

Quadratic Variation

Theorem

If X is an adapted, càdlàg process of finite variation then it is a quadratic pure jump semimartingale.

Proof.

The integration by parts formula and definition of $[X]$ give us:

$$X_t^2 = \int_0^t X_{s-} dX_s + \int_0^t X_s dX_s, \quad X_t^2 = 2\int_0^s X_{s-} dX + [X]_t$$

Simply rewriting one of integrals:

$$\int_0^t X_s dX_s = \int_0^t (X_{s-} + \Delta X_s) dX_s = \int_0^t X_{s-} dX_s + \sum_{0<s\leq t} (\Delta X_s)^2$$

Therefore $[X]_t = \sum_{s\leq t} (\Delta X_s)^2$ $\square$

Generalized Ito's formula

Theorem

If X is a semimartingale and $f \in \mathcal{C}^2(\mathbb{R})$, then $(f(X_t))_{t \geq 0}$ is also a semimartingale given by

$$f(X_t) = f(X_0) + \int_0^t f'(X_{s-})\, dX_s + \frac{1}{2} \int_0^t f''(X_{s-})\, d[X]_s^c + \sum_{0<s\leq t} \Big(f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s \Big)$$

In particular, if X has continuous paths then the last term cancels out. If X is of finite variation then the second term cancels out.

Stochastic exponential

Definition

A stochastic exponential $\mathcal{E}(X)$ of a semimartingale X is a process U that solves the following SDE:

$$U_t = 1 + \int_0^t U_{s-} dX_s$$

After applying Ito's lemma we get the closed-form solution to the equaiton given by:

$$\mathcal{E}(X)_t = \exp\left(X_t - X_0 - \frac{1}{2}[X]_t^c\right) \prod_{s \leq t} e^{-\Delta X_s}(1 + \Delta X_s).$$

Radon-Nikodym derivative

To get to Girsanov's theorem we should remember what a Radon-Nikodym derivative is.

Reminder

A measure $\mathbb{Q}$ is absolutely continuous w.r.t measure $\mathbb{P}$ ($\mathbb{Q} \ll \mathbb{P}$) if the following identity holds:

$$\mathbb{Q}(A) = \int_A Z d\mathbb{P}$$

Z is called a Radon-Nikodym derivative denoted by $\dfrac{d\mathbb{Q}}{d\mathbb{P}}$

If $\mathbb{Q} \ll \mathbb{P}$ and $\mathbb{P} \ll \mathbb{Q}$ we say that the measures are equivalent. In fact it happens iff the measures "agree" on what sets they assign a zero measure.

Girsanov's theorem

Now we are going to move to Girsanov's theorem.

Proposition

Let $\mathbb{Q}$ be a probability a probability measure and $\mathbb{Q} \ll \mathbb{P}$ on $(\Omega, \mathcal{F})$. Then $Z_t = \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t}$ is unique and called density process. Z_t is a $\mathbb{P}$ martingale with a càdlàg modification.

Theorem (Girsanov)

Assume $\mathbb{Q} \sim \mathbb{P}$ on $(\Omega, \mathcal{F})$, L_t is a (unique) continuous local martingale such that $Z_t = \mathcal{E}(L)_t$ and M_t a local martingale on $\mathbb{P}$. Then

$$\tilde{M}_t \equiv M_t - [M, L]_t$$

is a $\mathbb{Q}$ continuous local martingale.

Girsanov and semimartingales

- In order to apply Girsanov's theorem to semimartingales $X_t = X_0 + M_t + A_t$ we should hope that there exists a certain representation for A_t: $A_t = J_t \langle M \rangle_t$, where J_t is a càglàd process (left continuous with right limits).
- Then the density process is given by:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}(-J \cdot M)_t - Z_t, \quad dZ_t = -J_t Z_{t-} dM_t$$

Applications of Girsanov's theorem

Assume the simplest model, where a stock price S_t has the following log-normal dynamics:

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t,$$

where W_t is a $\mathbb{P}$-Brownian motion and μ_t and σ_t are (good) adapted processes. If we "discount"the process we get that it has the dynamics given by:

$$dS_t = (\mu_t - r_t) S_t dt + \sigma_t S_t dW_t,$$

In a (risk-neutral) $\mathbb{Q}$-world we want the discounted price be a martingale $\Rightarrow$ we need to remove the drift.

Applications of Girsanov's theorem

From Girsanov's theorem it follows that we with the following density process:

$$Z_t = \mathcal{E}\left(-\int_0^t \frac{\mu_s - r_s}{\sigma_s} dW_S\right) = exp\left(-\int_0^t \frac{\mu_s - r_s}{\sigma_s} dW_s - \frac{1}{2}\int_0^t \left(\frac{\mu_s - r_s}{\sigma_s}\right)^2 ds\right)$$

we get $\tilde{W}_t$ which a $\mathbb{Q}$-Brownian motion

$$d\tilde{W}_t = dW_t + \frac{\mu_s - r_s}{\sigma_s} ds$$

If we plug our $\mathbb{Q}$ Brownian motion into price SDE we get that:

$$dS_t = \sigma_t S_t d\tilde{W}_t$$

Sources

1. From Diffusions to Semimartingales
2. Martingales and Local Martingales
3. Semimartingales and stochastic integration
4. Girsanov's Theorem
5. Almost Sure