Malliavin Calculus

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Hilbert space

Definition

Let H be a linear space on a field K. Define an inner product
$(\cdot,\cdot) : H \times H \to K$:
1) $\forall x \in H : (x,x) \geq 0 \;\; (x,x) \iff x = 0$
2) $\forall \alpha, \beta \in K \; \forall x, y, z \in H : (\alpha x + \beta y, z) = \alpha(x,z) + \beta(y,z)$
3) $\forall x, y \in H : (x,y) = \overline{(y,x)}$.
Such space is called euclidian. The space is called Hilbert space, if the euclidean space is complete with respect to the distance function induced by the inner product.

Derivative and divergence

Definition (Derivative operator)

Let $f \in C^1(\mathbb{R})$, then the derivative operator D is such that $Df(x) = f'(x)$.

Definition (Divergence operator)

Let $f \in C^1(\mathbb{R})$, then the divergence operator δ is such that $\delta f(x) = xf(x) - f'(x)$.

Derivative and divergence

$\gamma = N(0,1)$ is the standard Gaussian probability on $\mathbb{R}$.

Lemma

The operators D and δ are adjoint with respect to the measure γ. That means, for any $f, g \in C_p^1(\mathbb{R})$, we have

$$\langle Df, g\rangle_{L^2(\mathbb{R},\gamma)} = \langle f, \delta g\rangle_{L^2(\mathbb{R},\gamma)}.$$

Proof.

Integrating by parts and using $p'(x) = -xp(x)$ we get

$$\int_{\mathbb{R}} f'(x)g(x)p(x)\,dx = -\int_{\mathbb{R}} f'(x)(g(x)p(x))'\,dx = \int_{\mathbb{R}} f(x)\delta g(x)p(x)\,dx.$$

$\square$

Finite-dimensional case

Consider the probability space $(\Omega, \mathcal{F}, P)$ is such that $\Omega = \mathbb{R}^n$, $F = \mathcal{B}(\mathbb{R}^n)$ is the Borel σ-algebra of $\mathbb{R}^n$, and P is the standard Gaussian probability. In this framework we consider, as before, two differential operators. The first is the derivative operator, which is simply the gradient of a differentiable function $F : \mathbb{R}^n \to \mathbb{R}$:

$$\nabla F = \left(\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n} \right)$$

The second differential operator is the divergence operator and is defined on differentiable vector-valued functions $u : \mathbb{R}^n \to \mathbb{R}^n$ as follows:

$$\delta(u) = \sum_{i=1}^{n} \left(u_i x_i - \frac{\partial u_i}{\partial x_i} \right)$$

Finite-dimensional case

Proposition

The operator δ is the adjoint of ∇; that is, $\mathbb{E}(\langle u, \nabla F\rangle) = \mathbb{E}(F\delta(u))$ if $F : \mathbb{R}^n \to \mathbb{R}$ and $u : \mathbb{R}^n \to \mathbb{R}^n$ are continuously differentiable functions which, together with their partial derivatives, have at most polynomial growth.

Proof.

Integrating by parts, and using $\dfrac{\partial p}{\partial x_i} = -x_i p$, we obtain

$$\int_{\mathbb{R}^n} \langle u, \nabla F\rangle p \, dx = \sum_{i=1}^{n} \int_{\mathbb{R}^n} \frac{\partial F}{\partial x_i} u_i p \, dx =$$

$$= \sum_{i=1}^{n} \left(-\int_{\mathbb{R}^n} F \frac{\partial u_i}{\partial x_i} p \, dx + \int_{\mathbb{R}^n} F u_i x_i p \, dx \right) = \int_{\mathbb{R}^n} F\delta(u) p \, dx.$$

Recap. Brownian motion

Definition

A real-valued stochastic process $B = (B_t),\ t \geq 0$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a Brownian motion if it satisfies the following conditions:

- a.s. $B_0 = 0$
- $\forall\ 0 \leq t_1 < ... < t_n$ the increments $B_{t_n} - B_{t_{n-1}}, ..., B_{t_2} - B_{t_1}$ are independent random variables
- if $0 \leq s \leq t$, the increment $B_t - B_s$ is a Gaussian random variable with mean zero and variance $t - s$
- a.s. the map $t \to B_t$ is continuous

Wiener integral

We next define the integral of square integrable functions with respect to Brownian motion.

Definition (Wiener integral of a step function)

Consider the set of step functions $\mathscr{G}_0 \subset L^2(\mathbb{R}_+)$:

$$\varphi_t = \sum_{j=0}^{n-1} a_j \, \mathbb{I}_{(t_j, t_{j+1}]}(t), \ \ t \geq 0$$

$$\textit{where } n \in \mathbb{N} \setminus \{0\}, \ a_j \in \mathbb{R} \textit{ and } 0 = t_0 < \ldots < t_n$$

The Wiener integral of a step function is defined by

$$\int_{\mathbb{R}_+} \varphi_t dB_t = \sum_{j=0}^{n-1} a_j \left(B_{t_{j+1}} - B_{t_j} \right)$$

Wiener integral

Proposition

The mapping $\varphi \to \int_{\mathbb{R}_+} \varphi_t dB_t$ from $\mathscr{G}_0$ to $L^2(\Omega)$ is linear and isometric (distance preserving)

Proof.

$$\mathbb{E}\left(\int_{\mathbb{R}_+} \varphi_t dB_t\right) = \sum_{j=0}^{n-1} a_j \, \mathbb{E}(B_{t_{j+1}} - B_{t_j}) = 0$$

$$\mathbb{E}\left(\left(\int_{\mathbb{R}_+} \varphi_t dB_t\right)^2\right) = \sum_{j=0}^{n-1} a_j^2 \, (t_{j+1} - t_j) = \int_{\mathbb{R}_+} \varphi_t^2 dt = \|\varphi\|_{L^2(\mathbb{R}_+)}^2$$

$\square$

Wiener integral

Proposition

The space $\mathscr{G}_0$ is a dense subspace of $L^2(\mathbb{R}_+)$. Therefore the mapping can be extended to a linear isometry between $L^2(\mathbb{R}_+)$ and the Gaussian subspace of $L^2(\Omega)$ spanned by the Brownian motion.

Definition

Consider a sequence φ_n of step functions from $\mathscr{G}_0$ such that $\varphi_n \to \varphi$ in $L^2(\mathbb{R}_+)$ i.e. $\int_{\mathbb{R}_+}(\varphi_n(t) - \varphi(t))^2 dt \to 0$
Then we define the integral of a function $\varphi \in L^2(\mathbb{R}_+)$ as

$$\int_{\mathbb{R}_+} \varphi(t) dB_t := \lim_{n\to\infty} \int_{\mathbb{R}_+} \varphi_n(t) dB_t$$

It is a Gaussian random variable with mean zero and variance $\|\varphi\|^2_{L^2(\mathbb{R}_+)}$

Malliavin derrivative

Let $B = (B_t)_{t \geq 0}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ such that $\mathcal{F}$ is the σ-algebra generated by B. Set $H = L^2(\mathbb{R}_+)$, and for any $h \in H$, consider the Wiener integral

$$B(h) = \int_0^\infty h(t) dB_t.$$

Let now $\mathscr{W} \subset L^2(\Omega, P)$ denote the set of all random variables F such that there exists $N \geq 0$, a function $f : \mathbb{R}^N \to \mathbb{R}$ which, together with its derivatives, grows at most polynomially at infinity, and elements $h_i \in H$ such that

$$F = f(B(h_1), \ldots, B(h_n)).$$

Malliavin derivative

- We want some natural properties from a thing which is called a derivative
- $D_t(B(h)) = h(t)$
- Satisfying a chain rule

Definition (Malliavin derivative)

If $F \in \mathscr{W}$ the derivative DF is the H-valued random variable defined by:

$$D_t F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(B(h_1), \ldots, B(h_n))h_i(t)$$

For instance, $D(B(h)) = h$.

Malliavin directional derivative

Definition (Cameron-Martin space)

Cameron-Martin space $H^1 \subset \Omega$ is the set of functions of the form

$$\psi(t) = \int_0^t h_s \, ds$$

where $h \in H$.

Definition (Malliavin directional derivative)

Consider the Cameron-Martin space, then for $h \in H$, $\langle DF, h \rangle_H$ is the derivative of F in the direction of $\psi(t)$:

$$\langle DF, h \rangle_H = \int_0^T h_t D_t F \, dt = \frac{d}{d\varepsilon} F(\omega + \varepsilon \int_0^T h_s ds)|_{\varepsilon=0}$$

Malliavin directional derivative

Definition (Divergence)

Denote by S_H the class of stochastic processes $u = (u_t)_{t \geq 0}$ of the form

$$u_t = \sum_{j=1}^{n} F_j h_j(t),$$

where $F_j \in \mathscr{W}$ and $h_j \in H$. Then the divergence of an element u_t is a random variable given by

$$\delta(u) = \sum_{j=1}^{n} F_j B(h_j) - \sum_{j=1}^{n} \langle DF_j, h_j \rangle_H$$

Malliavin directional derivative

Proposition

Let $F \in \mathscr{W}$ and $u \in S_H$, then:

$$\mathbb{E}(F\delta(u)) = \mathbb{E}(\langle DF, u\rangle_H).$$

Which is obvious from the proposition for operator δ.

Martingale representation theorem

Theorem

Let B be a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\{\mathcal{F}_t\}_{t\geq 0}$ be its natural filtration. Then, every $\{\mathcal{F}_t\}$—local martingale M can be written as

$$M = \mathbb{E}(M) + \int \xi \, dB,$$

for a predictable, B-integrable, process ξ.

Clark-Ocone formula

The next result expresses the integrand of the integral representation theorem of a square integrable random variable in terms of the conditional expectation of its Malliavin derivative.

Definition (Clark-Ocone formula)

Let $F \in D^{1,2} \cap L^2(\Omega, \mathcal{F}_T, P)$. Then F admits the following representation:

$$F = \mathbb{E}(F) + \int_0^T \mathbb{E}(D_t F | \mathcal{F}_t) dB_t$$

Clark-Ocone formula. Proof

Proof.

- By the integral representation theorem:

$$F = \mathbb{E}(F) + \int_0^T u_t dB_t$$

- Consider a process $v \in L^2_T(P)$. On the one hand, the isometry property yields (exercise)

$$\mathbb{E}(\delta(v)F) = \int_0^T \mathbb{E}(v_s u_s) ds$$

□

Proof.

- On the other hand, by the duality relationship (Proposition on slide 15), and taking into account that v is progressively measurable,

$$\mathbb{E}(\delta(v)F) = \mathbb{E}\left(\int_0^T v_t D_t F dt\right) = \int_0^T \mathbb{E}(v_t \mathbb{E}(D_t F | \mathcal{F}_t))dt$$

Therefore $u_t = \mathbb{E}(D_t F | \mathcal{F}_t)$ for almost all $(t, \omega) \in [0, T] \times \Omega$ which concludes the proof.

$\square$

Applications. Clark-Ocone formula to Black-Scholes

Proposition

- Consider a market consisting of one stock (risky asset) and one bond (risk-less asset). Assume that the price process $(S_t)_{t \geq 0}$ follows a Black-Scholes model with constant coefficients $\sigma > 0$ and μ, that is,

$$S_t = S_0 exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B_t\right)$$

  where $B = (B_t)_{t \in [0,T]}$ is a Brownian motion difined in a complete probability space $(\Omega, \mathcal{F}, P)$

- $(\mathcal{F}_t)_{t \in [0,T]}$ the filtration generated by the Brownian motion and completed by the P-null sets. By Ito formula we obtain that S_t satisfies a linear stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

Applications. Clark-Ocone formula to Black-Scholes

Proposition

- The coefficient μ is the mean return rate and σ is the volatility. The price of the bond at time t is e^{rt}, where r is the interest rate.
- Consider an investor who starts with some initial endowement $x \geq 0$ and invests in the assets described above. Let α_t be the number of non-risky assets and β_t the number of stocks owned by the investor at time t. The couple $\phi_t = (\alpha_t, \beta_t), t \in [0, T]$ is called a portfolio, and we assume that the process α_t and β_t are measurable and adapted processes such that

$$\int_0^T \beta_t^2 dt < \infty, \int_0^T |\alpha_t| dt < \infty$$

Applications. Clark-Ocone formula to Black-Scholes

Proposition

- Then the value of the portfolio at time t is

$$V_t(\phi) = \alpha_t e^{-rt} + \beta_t S_t$$

We say that the portfolio ϕ is self-financing if

$$V_t(\phi) = x + r \int_0^t \alpha_s e^{-rs} ds + \int_0^t \beta_s dS_s$$

From now on we will consider only self-financing portfolios. Discounted value of a self-financing portfolio

$$\tilde{V}_t(\phi) = e^{-rt} V_t(\phi) = x + \int_0^t \beta_s d\tilde{S}_s$$

where $\tilde{S}_t = e^{-rt} S_t$

Proposition

- Notice that
$$d\tilde{S}_t = (\mu - r)\tilde{S}_t dt + \sigma \tilde{S}_t dB_t$$
- Set $\theta = \dfrac{\mu - r}{\sigma}$ Consider the martingale measure defined on $\mathcal{F}_T$, by
$$\frac{dQ}{dP} = exp\left(-B_t - \frac{\theta^2}{2}\right)$$
- Under Q $W_t = B_t + \theta t$ is a Brownian motion and the discounted price process $\tilde{S}_t$ is a martingale because
$$d\tilde{S}_t = \sigma \tilde{S}_t dW_t$$

Applications. Clark-Ocone formula to Black-Scholes

Proposition

- Suppose that $F \geq 0$ is an $\mathcal{F}_t$ - measurable such that $E_Q(F^2) < \infty$. It represents the payoff of some derivative. We say that F can be replicated if there exists a self-financing portfolio ϕ such that $V_T(\phi) = F$. The Ito integral representaion theorem implies that any derivative is replicable, and this means that the Black-Scholes market is complete. It sufficies to write

$$e^{-rT}F = E_Q(e^{-rT}F) + \int_0^T u_s dW_s$$

and take the self-financing portfolio $\phi_t = (\alpha_t, \beta_t)$, where $\beta_t = \dfrac{u_t}{\sigma \tilde{S}_t}$

- The price of a derivative with payoff F at time $t \leq T$ is given by the value at time t of a self-financing portfolio which replicates F. Then $V_t(\phi) = e^{-r(T-t)}E_Q(F|\mathcal{F}_t)$

Applications. Clark-Ocone formula to Black-Scholes

Proposition

- In this section we discuss the application of Clark-Ocone formula to find explicit formulas for a replicating portfolio in the Black-Scholes model. Suppose that $F \in \mathbb{D}^{1,2}$. Then, applying Clark-Ocone's formula we obtain

$$\beta_t = \frac{e^{-r(T-t)}}{\sigma S_t} E_Q(D_t F | \mathcal{F}_t)$$

- We can consider different derivatives with their payoffs and obtain explicit formulas