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On Yu and Hoff's confidence intervals for treatment means

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ABSTRACT

Yu and Hoff constructed novel confidence intervals for the treatment means in a one-way layout. They assessed the expected lengths of these intervals using a semi-Bayesian analysis. We provide a revealing assessment of these expected lengths using a fully frequentist analysis.

1. Introduction

Suppose that we wish to find a confidence interval, with specified coverage probability, for a scalar parameter of interest. Also suppose that we have a pivotal quantity that is a continuous random variable. A confidence interval can be constructed, using this pivotal quantity, in the usual way. Now suppose that we have some prior belief, though not certainty, that this parameter takes values in some given set. This motivates us to construct a confidence interval, with the desired coverage probability, for this parameter that has relatively small expected length when the parameter belongs to this given set, possibly at the expense of having a relatively large expected length for other parameter values.

An easily-understood method for constructing a non-standard confidence interval, with the desired coverage probability, is the "tail method" using the pivotal quantity. It turns out that, by the appropriate choice of the "tail function", this confidence interval can be made to have relatively small expected length when the parameter of interest belongs to the given set (Stein, 1962; Bartholomew, 1971; Puza and O'Neill, 2006ab; Puza and Yang, 2016; Yang and Puza, 2020; Yu and Hoff, 2018; Hoff and Yu, 2019).

Pratt (1961) describes an elegant method for the construction of a confidence interval, with the desired coverage, that minimizes a specified weighted average, over the parameter space, of the expected length of this interval. For the normal mean, with known variance, Pratt (1961, 1963) constructs confidence intervals using this method that can be interpreted as "tail method" intervals (Puza and O'Neill, 2006; Yu and Hoff, 2018).

The chief advantage of the "tail method" is that the coverage constraint is effortlessly satisfied. However, this advantage comes at a price. In the situation under consideration, a highly-desirable property is that the scaled expected length of a confidence interval, where the scaling is with respect to the expected length of the usual confidence interval with the same minimum coverage, has a finite maximum value that is not too large. We do not want a small value of the scaled expected length in a specified part of the parameter space to come at the cost of extremely large values in some other parts of this space. This is in line with the pioneering work of Hodges and Lehmann (1952) who propose the minimization of a specified weighted average across the parameter space of the risk function, subject to a given upper bound on its maximum value. Unfortunately, the "tail method" confidence interval for the normal mean, with known variance, does not have this property and has a scaled expected length that has an infinite maximum value. A numerical illustration of this fact is provided by panel (c) of Figure 1 of Yu and Hoff (2018).

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Yu and Hoff (2018) and Hoff and Yu (2019) made the novel and important observation that a "tail function" that is random and independent of the pivotal quantity for the parameter of interest still leads to a confidence interval with exactly the desired coverage. Yu and Hoff (2018) apply this observation to the construction of confidence intervals for the treatment means in a one-way layout that have relatively small expected length when these means are equal or close to equal. Individually, each of these confidence intervals has exactly the desired coverage. The methodology of Yu and Hoff (2018) applies to both balanced and unbalanced one-way layouts and both homogeneous and heterogeneous random error variances. Yu and Hoff (2018) explored the properties of the expected lengths of their confidence intervals using a semi-Bayesian analysis. We explore, for the first time, the properties of these expected lengths using a fully frequentist analysis.

Our aim is to explore the properties of the scaled expected lengths of these confidence intervals, where the scaling is with respect to the expected length of the usual confidence interval with the same coverage. For conceptual and computational simplicity, we consider a balanced one-way layout, with identical random error variances across treatments in the simplified scenario that the random error variance is known. This is equivalent to assuming that the number of measurements of the response for each treatment is large.

In Section 2 we describe the confidence intervals of Yu and Hoff (2018) in the context of a balanced one-way layout, with identical random error variances across treatments. In Section 3 we describe the form that these intervals take in the simplified scenario that the random error variance is known. In Section 4 we present a convenient formula for the scaled expected length of the (Yu and Hoff, 2018) confidence interval for a given treatment in this simplified scenario. The derivation of this formula is based on the elegant results of Pratt (1963). This formula is used to derive Theorems 3 and 4, which are the main results of the paper, that describe important properties of this scaled expected length. This formula is also used to derive the computationally-convenient exact formula for the scaled expected length, stated in Theorem 2, which we use to computationally explore the properties of the scaled expected length.

2. Balanced one-way layout and homogeneous error variances: the confidence intervals of Yu and Hoff for treatment means

Consider a balanced one-way layout for the comparison of p treatments. Suppose that for each treatment j, we have n independent and identically $N(\theta_j, \sigma^2)$ responses denoted by $Y_{1j}, \ldots, Y_{nj}$. The aim is to find confidence intervals, with specified coverage $1-\alpha$, for each of the θ_j and the following expected length property. The expected lengths of these confidence intervals are relatively small when θ_i are equal or close to equal and not too large otherwise. Let $\overline{Y}_j = \sum_{i=1}^{n} Y_{ij} \big/ n$.

For the sake of concreteness, we focus on finding such a confidence interval for θ_1 using a pivotal quantity based on $\left(\overline{Y}_1, S_1^2\right)$, where $S_1^2 = \sum_{i=1}^{n} \left(Y_{i1} - \overline{Y}_1\right)^2 \big/ (n-1)$. Let t_a denote the ath quantile of a t_{n-1} distribution. The confidence interval of Yu and Hoff (2018) for θ_1 is found as follows. We begin with the confidence interval

$$C\left(\overline{Y}_1, S_1^2; w\right) = \left\{ \theta : \overline{Y}_1 + n^{-1/2}\, S_1\, t_{\alpha(1-w(\theta))} < \theta < \overline{Y}_1 + n^{-1/2}\, S_1\, t_{1-\alpha w(\theta)} \right\} \tag{1}$$

for θ_1, with coverage $1-\alpha$, where $w : \mathbb{R} \to [0,1]$ is a continuous nondecreasing function. This function, called a "tail function" by Puza and O'Neill (2006) and a "spending function" by Hoff and Yu (2019), is chosen using the following two-step procedure. In the first step, the parameters μ and τ^2 determine a weight function. In the second step, these parameters determine prior distributions that are used in an empirical Bayes analysis. However, we will ultimately assess the resulting confidence interval for θ_1 using the purely frequentist criteria of coverage probability and expected length.

Step 1: Let $\psi = (\mu, \tau^2, \sigma^2)$ and suppose that ψ is given. Also let $\phi(x; \mu, \tau^2)$ denote the $N(\mu, \tau^2)$ probability density function, evaluated at x. Define w_ψ to be the function w that minimizes the following weighted average of the expected length of the confidence interval $C\left(\overline{Y}_1, S_1^2; w\right)$ for θ_1:

$$\int E_{\theta_1}\left(\text{length of } C\left(\overline{Y}_1, S_1^2; w\right)\right) \phi(\theta_1; \mu, \tau^2)\, d\theta_1 \tag{2}$$

with respect to the function w.

Step 2: We use a dash subscript (boldface hyphen subscript in the Supplementary Material) to denote an estimator based solely on $Y_{12}, \ldots, Y_{n2}, \ldots, Y_{1p}, \ldots, Y_{np}$. Let $\hat{\psi}_-$ denote the empirical Bayes estimator of ψ based solely on $Y_{12}, \ldots, Y_{n2}, \ldots, Y_{1p}, \ldots, Y_{np}$, assuming that $\theta_2, \ldots, \theta_p$ have independent prior $N(\mu, \tau^2)$ distributions. We then estimate the function w_ψ by $w_{\hat{\psi}_-}$. The final confidence interval for θ_1 is $C\left(\overline{Y}_1, S_1^2; w_{\hat{\psi}_-}\right)$.

As pointed out by Yu and Hoff (2018), although in Step 2 we assume that $\theta_2, \ldots, \theta_p$ have prior distributions, this assumption is not necessary for the exploration of the coverage probability and expected length properties of the confidence interval $C\left(\overline{Y}_1, S_1^2; w_{\hat{\psi}_-}\right)$ for θ_1. We can carry out this exploration in purely frequentist terms. The estimator $\hat{\psi}_-$ found in Step 2 is independent of $\left(\overline{Y}_1, S_1^2\right)$. By Proposition 7 of Yu and Hoff (2018), the confidence interval $C\left(\overline{Y}_1, S_1^2; w_{\hat{\psi}_-}\right)$ for θ_1 has coverage probability $1-\alpha$.

To evaluate the expected length properties of this confidence interval is both conceptually and computationally difficult because there is no simple formula for the function w_ψ. To carry out this evaluation, we use the *simplified scenario* described in the next section.

3. Balanced one-way layout and homogeneous error variances: the simplified scenario

To assess the expected length performance of a confidence interval, with coverage $1-\alpha$, we use the scaled expected length, defined to be (expected length of this confidence interval)/(expected length of the usual confidence interval with coverage $1-\alpha$). To evaluate the scaled expected length of the (Yu and Hoff, 2018) confidence intervals, we introduce the following simplified scenario. Suppose that n is large, so that S_1 estimates σ with great accuracy and t_a is well-approximated by z_a, the ath quantile of the $N(0,1)$ distribution. We expect the scaled expected length of the confidence interval $C\big(\overline{Y}_1, S_1^2; w_{\hat{\psi}_-}\big)$ for θ_1 to be well-approximated by the scaled expected length of the final confidence interval for θ_1 found in the following simplified scenario.

Suppose that σ^2 is known. The conceptual and computational advantage of this scenario is that the endpoints of the confidence interval resulting from the application of the analogues of Steps 1 and 2 (described in the previous section) are given by relatively simple formulae. A further simplification results from the fact that we do not need to estimate σ^2. Therefore we can carry out an initial data reduction to $\overline{Y}_1, \overline{Y}_2, \dots, \overline{Y}_p$.

To carry out the analogue of Step 1 (described in the previous section), we use the consequences of the results of Pratt (1963) described in Section S1 of the Supplementary Material. Let $R(\overline{Y}_1)$ be a confidence interval for θ_1 with coverage $1-\alpha$. For given μ and τ^2, let $R^*(\overline{Y}_1)$ denote the confidence interval $R(\overline{Y}_1)$ that minimizes the following weighted average of the expected length of $R(\overline{Y}_1)$:

$$\int E_{\theta_1,\sigma}\big(\text{length of } R(\overline{Y}_1)\big)\, \phi(\theta_1;\mu,\tau^2)\, d\theta_1.$$

The confidence interval $R^*(\overline{Y}_1)$ has endpoints

$$\mu + n^{-1/2}\sigma\, \underline{\theta}_\alpha\left(\frac{\overline{Y}_1-\mu}{n^{-1/2}\sigma}, \varphi\right) \quad \text{and} \quad \mu + n^{-1/2}\sigma\, \bar{\theta}_\alpha\left(\frac{\overline{Y}_1-\mu}{n^{-1/2}\sigma}, \varphi\right), \tag{3}$$

where $\varphi = n\tau^2/\sigma^2$ and the functions $\underline{\theta}_\alpha$ and $\bar{\theta}_\alpha$ are defined in Appendix A.

Consider the "tail method" interval

$$C\big(\overline{Y}_1; w\big) = \left\{\theta : \overline{Y}_1 + n^{-1/2}\,\sigma\, z_{\alpha(1-w(\theta))} < \theta < \overline{Y}_1 + n^{-1/2}\,\sigma\, z_{1-\alpha w(\theta)}\right\}.$$

Let $\psi = (\mu, \varphi)$, where $\varphi = n\tau^2/\sigma^2$. By Proposition 3 of Yu and Hoff (Yu and Hoff, 2018), $C\big(\overline{Y}_1; w_\psi\big) = R^*\big(\overline{Y}_1\big)$ when $w_\psi(\theta) = g^{-1}\big(2n^{1/2}(\theta-\mu)/(\sigma\varphi)\big)$, where $g(x) = \Phi^{-1}(\alpha x) - \Phi^{-1}(\alpha(1-x))$. In other words, the endpoints of the confidence interval $C\big(\overline{Y}_1; w_\psi\big)$ for θ_1 are given by (3)

To carry out the analogue of Step 2 (described in the previous section) we first find an estimator $\hat{\psi}_- = (\hat{\mu}_-, \hat{\varphi}_-)$ of $\psi = (\mu, \varphi)$, based on $\overline{Y}_2, \dots, \overline{Y}_p$. For the purpose of deriving this estimator, assume that $\theta_2, \dots, \theta_p$ have independent $N(\mu, \tau^2)$ distributions. When $\tau^2 = 0$, we interpret the $N(\mu, \tau^2)$ probability density function, evaluated at x, as $\delta(x-\mu)$, where δ denotes the Dirac delta function. Considering their marginal distributions, $\overline{Y}_2, \dots, \overline{Y}_n$ are independent and identically $N(\mu, \upsilon)$ distributed, where $\upsilon = \sigma^2(1+\varphi)/n$. Unbiased estimators of μ and υ are $\hat{\mu}_- = \sum_{j=2}^p \overline{Y}_j/(p-1)$ and $\hat{\upsilon}_- = \sum_{j=2}^p \big(\overline{Y}_j - \hat{\mu}_-\big)^2/(p-2)$, respectively. Since $\varphi = (n\upsilon/\sigma^2) - 1$ and $\varphi \ge 0$, we estimate φ by $\hat{\varphi}_- = \max\big(\Delta, (n\hat{\upsilon}_-/\sigma^2) - 1\big)$, where Δ is a given nonnegative number (usually chosen to be zero). Having motivated the estimators $\hat{\mu}_-$ and $\hat{\upsilon}_-$, we drop the assumption that $\theta_2, \dots, \theta_p$ have a joint prior distribution.

We conclude that the confidence interval $C\big(\overline{Y}_1; w_{\hat{\psi}_-}\big)$ for θ_1, resulting from the application of the analogues of Steps 1 and 2, has endpoints

$$\hat{\mu}_- + n^{-1/2}\sigma\, \underline{\theta}_\alpha\left(\frac{\overline{Y}_1-\hat{\mu}_-}{n^{-1/2}\sigma}, \hat{\varphi}_-\right) \quad \text{and} \quad \hat{\mu}_- + n^{-1/2}\sigma\, \bar{\theta}_\alpha\left(\frac{\overline{Y}_1-\hat{\mu}_-}{n^{-1/2}\sigma}, \hat{\varphi}_-\right). \tag{4}$$

4. Scaled expected lengths of the Yu and Hoff confidence intervals: main results

We examine the scaled expected length properties of the Yu and Hoff confidence interval $C\big(\overline{Y}_1; w_{\hat{\psi}_-}\big)$ for θ_1, in the simplified scenario described in Section 3. This scaled expected length is defined to be

$$\frac{E_{\theta,\sigma}\Big(\text{length of } C\big(\overline{Y}_1; w_{\hat{\psi}_-}\big)\Big)}{\text{length of the usual } 1-\alpha \text{ confidence interval for } \theta_1} = \frac{E_{\theta,\sigma}\Big(\text{length of } C\big(\overline{Y}_1; w_{\hat{\psi}_-}\big)\Big)}{2\, n^{-1/2}\,\sigma\, z_{1-\alpha/2}}.$$

Let $\gamma_i = n^{1/2}\theta_i/\sigma$ $(i = 1, \dots, p)$ and $\bar{\gamma}_- = \sum_{j=2}^p \gamma_j/(p-1)$. Now let $\xi = \gamma_1 - \bar{\gamma}_-$ and $\eta = \sum_{j=2}^p (\gamma_j - \bar{\gamma}_-)^2$. The parameter ξ is a scaled measure of the difference between θ_1 and the average of $\theta_2, \dots, \theta_p$. The parameter η is a scaled sample variance of $\theta_2, \dots, \theta_p$. We will show that the scaled expected length is a function of (ξ, η).

Suppose that ξ is fixed. When η is large, we expect the confidence interval $C\big(\overline{Y}_1; w_{\hat{\psi}_-}\big)$ for θ_1 to be close to the usual confidence interval for θ_1, with coverage $1-\alpha$. Consequently, we anticipate that the scaled expected length of the confidence interval $C\big(\overline{Y}_1; w_{\hat{\psi}_-}\big)$ will converge to 1 as $\eta \to \infty$. The first of our overall findings is that this scaled expected length does, indeed, converge to 1 as $\eta \to \infty$.

Now suppose that η is fixed. The second of our overall findings is that the scaled expected length of the confidence interval $C\big(\overline{Y}_1; w_{\hat{\psi}_-}\big)$ diverges to ∞ as either $\xi \to \infty$ or $\xi \to -\infty$. The source of this divergence seems to be traceable to the requirement, underlying the construction of the confidence interval $C\big(\overline{Y}_1; w_{\hat{\psi}_-}\big)$ for θ_1, that the random "tail function" must be independent of the pivotal quantity for the parameter of interest θ_1.

A convenient formula for the scaled expected length of the confidence interval $C\big(\overline{Y}_1; w_{\hat{\psi}_-}\big)$ for θ_1 is given in the following theorem, proved in Section S2 of the Supplementary Material.

Theorem 1. *Let U and V_η be independent random variables with $U \sim N\left(0, 1+(p-1)^{-1}\right)$ and V_η having a noncentral χ^2 distribution with $p-2$ degrees of freedom and noncentrality parameter η. Let $d(x,\omega^2) = \bar{\theta}_\alpha(x,\omega^2) - \underline{\theta}_\alpha(x,\omega^2)$, where $\bar{\theta}_\alpha(x,\omega^2)$ and $\underline{\theta}_\alpha(x,\omega^2)$ are defined in Appendix A. The scaled expected length of the confidence interval $C\left(\overline{Y}_1; w_{\hat{\psi}_-}\right)$ for θ_1 is equal to*

$$\frac{1}{2\, z_{1-\alpha/2}} E\left(d\left(\xi - U,\ \max\left(\Delta, \frac{V_\eta}{p-2} - 1\right)\right)\right). \tag{5}$$

This theorem shows that the scaled expected length of the confidence interval $C\left(\overline{Y}_1; w_{\hat{\psi}_-}\right)$ for θ_1 is a function of (ξ,η), which we denote by $\mathrm{SEL}(\xi,\eta)$. The condition $(\xi,\eta)=(0,0)$ is equivalent to $\gamma_1=\gamma_2=\cdots=\gamma_p$ *i.e.* the treatment means are equal. It may be shown that $\mathrm{SEL}(\xi,\eta)$ is an even function of ξ. Therefore, in determining the properties of $\mathrm{SEL}(\xi,\eta)$ it is sufficient to consider $(\xi,\eta)\in[0,\infty)^2$.

For any given (ξ,η), $\mathrm{SEL}(\xi,\eta)$ can be evaluated using (5) via Monte Carlo simulation. The evaluation of $d(x,\omega^2)$ for any given (x,ω^2) requires the evaluation of $\underline{\theta}_\alpha(x,\omega^2)$ and $\bar{\theta}_\alpha(x,\omega^2)$, which are found using a numerical root-finding algorithm. This numerical root-finding is enhanced through the use of the following lemma, proved in Section S2 of the Supplementary Material. This lemma will also be used later in the proofs of our main results Theorems 3 and 4.

Lemma 1. *The following results hold for all $\omega^2>0$.*
(a) If $x > z_{1-\alpha/2}$ then $\underline{\theta}_\alpha(x,\omega^2) \in \left[(x-z_{1-\alpha/2})/(1+2\omega^{-2}),\ x-z_{1-\alpha/2}\right]$.
(b) If $x = z_{1-\alpha/2}$ then $\underline{\theta}_\alpha(x,\omega^2) = 0$.
(c) If $x < z_{1-\alpha/2}$ then $\underline{\theta}_\alpha(x,\omega^2) \in \left[x-z_{1-\alpha/2},\ (x-z_{1-\alpha/2})/(1+2\omega^{-2})\right]$.

To accurately compute $\mathrm{SEL}(\xi,\eta)$ directly using (5) via numerical integration is problematic due to the lack of smoothness of the max function. However, the max function can be removed by the appropriate decomposition of the integrals, resulting in the following theorem, proved in Section S2 of the Supplementary Material, which provides a computationally-convenient exact formula for $\mathrm{SEL}(\xi,\eta)$.

Theorem 2. *Let the random variables U and V_η and the function $d(x,\omega^2)$ be as defined in the statement of Theorem 1. Then*

$$\begin{aligned}\mathrm{SEL}(\xi,\eta) = \frac{1}{2\, z_{1-\alpha/2}} \Bigg(&\int_{-\infty}^{\infty} d(\xi-u,\Delta) f_U(u)\, du\ P\big(V_\eta \le (1+\Delta)(p-2)\big) \\ &+ \int_{(1+\Delta)(p-2)}^{\infty}\int_{-\infty}^{\infty} d\left(\xi-u, \frac{v}{p-2}-1\right) f_U(u)\, du\, f_{V_\eta}(v) dv \Bigg),\end{aligned} \tag{6}$$

where f_U and f_{V_η} denote the probability density functions of U and V_η, respectively.

The following theorems, proved in Section S2 of the Supplementary Material, show that $\mathrm{SEL}(\xi,\eta)$ is (a) poorly-behaved as either $\xi\to\infty$ or $\xi\to-\infty$ and (b) well-behaved as $\eta\to\infty$.

Theorem 3. *For every given η and nonnegative Δ, $\mathrm{SEL}(\xi,\eta)$ diverges to ∞ as either $\xi\to\infty$ or $\xi\to-\infty$.*

Theorem 4. *For every given ξ and nonnegative Δ, $\mathrm{SEL}(\xi,\eta)\to 1$ as $\eta\to\infty$.*

The contour plot shown in Fig. 1 is of $\mathrm{SEL}(\xi,\eta)$ as a function of $(\xi,\eta)\in[0,5]\times[0,30]$ for the case that $p=5$, $\alpha=0.05$ and $\Delta=0$. The minimum value of $\mathrm{SEL}(\xi,\eta)$ is 0.858, which is achieved at $(\xi,\eta)=(0,0)$ *i.e.* when the treatment means are equal. This contour plot was computed using the expression (6) for $\mathrm{SEL}(\xi,\eta)$ and the methods of numerical integration described in Section S3 of the Supplementary Material. This plot provides a numerical illustration of Theorems 3 and 4.

In addition, $\mathrm{SEL}(\xi,\eta)$ was computed by simulation using (5), for several selected values of $(\xi,\eta)\in[0,5]\times[0,30]$, for $p=5$, $\alpha=0.05$ and $\Delta=0$. Of course, the results computed in this way match (to within simulation sampling error) the results found by numerical integration. All of the computations presented in this paper were carried out using R (R Core Team, 2024).

5. Discussion

We have considered a balanced one-way layout, with identical random error variances across treatments in the simplified scenario that the random error variance is known. This is equivalent to assuming that the number of measurements of the response for each treatment is large. We have shown that, irrespective of the number of treatments, the scaled expected lengths of the confidence intervals of Yu and Hoff (2018) are functions of two scalar parameters ξ and η. We note that η is nonnegative and that the scaled expected length is an even function of ξ. Our main results are that (a) for each fixed η, the scaled expected length diverges to infinity as ξ diverges to infinity and (b) for each fixed ξ, the scaled expected length converges to 1 as η diverges to infinity. Property (a) appears to be caused by the requirement that the "tail function" is random and independent of the pivotal quantity for the parameter of interest. It is this requirement that leads to the coverage constraint being effortlessly satisfied. In other words, it appears that effortless satisfaction of the coverage constraint comes at the price of property (a).

Our results also suggest that if the treatment population means are such that only one of these is an outlier then the (Yu and Hoff, 2018) confidence interval for the outlying treatment population mean will have a very large expected length, while the (Yu and Hoff, 2018) confidence intervals for the remaining treatment population means will be close to their usual confidence intervals

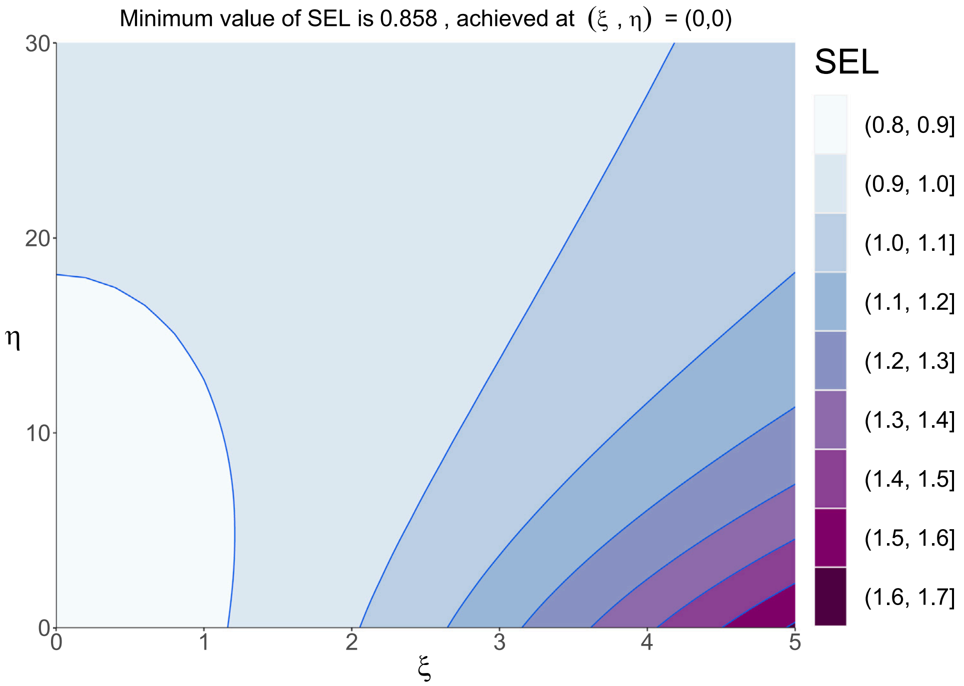

Fig. 1. Contour plot of $\mathrm{SEL}(\xi,\eta)$ as a function of $(\xi,\eta) \in [0,5] \times [0,30]$, for the case that $p = 5$, $\alpha = 0.05$ and $\Delta = 0$.

(with the desired coverage). Our results also suggest that if the treatment population means are such that at least two of these are outliers then the (Yu and Hoff, 2018) confidence intervals will all be close to the usual confidence intervals (with the desired coverage).

The simple computationally-convenient approximate confidence intervals described in Section 3.2 of Yu and Hoff (2018) do not satisfy the requirement that the "tail function" is random and independent of the pivotal quantity for the parameter of interest. This suggests that these approximate confidence intervals will have the attractive property that their scaled expected lengths are bounded. As noted by Yu and Hoff (2018), these approximate confidence intervals have the desired coverage in the limit as the number of treatments approaches infinity. It would be interesting to assess the speed of this convergence and the properties of the scaled expected lengths of these confidence intervals. However, such an assessment is outside the scope of the present paper.

Data availability

No data was used for the research described in the article.

Appendix A. Some results of Pratt (1963)

In this appendix, we describe some results of Pratt (1963) and define $\bar{\theta}_\alpha(x,\omega^2)$ and $\underline{\theta}_\alpha(x,\omega^2)$. Suppose that $X \sim N(\theta,\sigma^2)$, where θ is unknown and σ^2 is known. Let $R(X)$ be a confidence interval for θ with coverage $1-\alpha$. For given θ_0 and ω^2, let $R^*(X)$ denote the confidence interval $R(X)$ that minimizes the following weighted average of the expected length of $R(X)$:

$$\int E_{\theta'}\big(\text{length of } R(X)\big)\, \phi(\theta';\theta_0,\omega^2)\, d\theta', \tag{7}$$

where $\phi(x;\theta_0,\omega^2)$ denotes the $N(\theta_0,\omega^2)$ probability density function, evaluated at x. When $\omega^2 = 0$, we interpret the $N(\theta_0,\omega^2)$ probability density function, evaluated at x, as $\delta(x-\theta_0)$, where δ denotes the Dirac delta function.

Suppose that $\omega^2 > 0$. Let

$$r\big(\theta,x,\omega^2\big) = \Phi(x-\theta) - \Phi\big((1+2\,\omega^{-2})\,\theta - x\big).$$

This is a decreasing continuous function of θ that approaches 1 as $\theta \to -\infty$ and approaches 0 as $\theta \to \infty$. Define $\underline{\theta}_\alpha(x,\omega^2)$ as the solution for θ of

$$r\big(\theta,x,\omega^2\big) = 1-\alpha$$

and $\bar{\theta}_\alpha(x,\omega^2) = -\underline{\theta}_\alpha(-x,\omega^2)$. Also we define $\underline{\theta}_\alpha(x,0) = \min\big(0, x - z_{1-\alpha}\big)$ and $\bar{\theta}_\alpha(x,0) = \max\big(0, x + z_{1-\alpha}\big)$.

The confidence interval $R^*(X)$ has endpoints

$$\theta_0 + \sigma\,\underline{\theta}_\alpha\left(\frac{X-\theta_0}{\sigma}, \frac{\omega^2}{\sigma^2}\right) \quad \text{and} \quad \theta_0 + \sigma\,\bar{\theta}_\alpha\left(\frac{X-\theta_0}{\sigma}, \frac{\omega^2}{\sigma^2}\right). \tag{8}$$

For the particular case that $\theta_0 = 0$, $\sigma^2 = 1$ and $\omega^2 = 0$, this is the confidence interval of Pratt (1961, Section 5).

Appendix B. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.spl.2024.110170.

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