# 1 Algebra

Let $F : C \to C$ be an endofunctor on category C. An F-algebra is a pair $(A, \varphi)$ , where A is an object and $\varphi : FA \to A$ is an arrow in the category C. The object A is the *carrier* and the functor F is the *signature* of the algebra.

# 2 Algebra Homomorphism

Let $(C, \varphi)$ and $(D, \psi)$ be two F-algebras. An F-homomorphism from $(C, \varphi)$ to $(D, \psi)$ is an arrow $f : C \to D$ in the category C, such that $f \circ \varphi = \psi \circ Ff$. This means that the following diagram commutes.

$$
\begin{tikzcd}
FC \arrow[r, "\varphi"] \arrow[d, "Ff"'] & C \arrow[d, "f"] \\
FD \arrow[r, "\psi"'] & D
\end{tikzcd}
$$

Figure 1: $f$ is the algebra homomorphism

## 2.1 What does it mean for a diagram to commute

In the above diagram, each path is an arrow. If for every pair of vertices (A and B) in a diagram for a particular category C, all paths between A and B are equal, then the diagram commutes. Path equality means equality of arrows, since every path is an arrow. In the diagram above, we have 2 paths between vertices FC and D. They are equal means $f \circ \varphi$ (the top path) is equal to $\psi \circ Ff$ (the bottom one), which is precisely the criterion for homomorphism of the 2 algebras defined above.

# 3 Category of Algebras

If F is a functor $F : C \to C$, then category of F-algebras over C, denoted by Alg(F) is defined as follows :-

- Objects as F-algebras
- Arrows as F-homomorphisms

- Identities as in C
- Composition as in C

## 3.1 Properties of Alg(F)

For this definition to be valid, we need to have the following conditions satisfied:

- Identities must be F-homomorphisms, i.e. the following diagram should commute for each $\phi$.

$$\begin{tikzcd}
FA \arrow[r, "\varphi"] \arrow[d, "F id_A"'] & A \arrow[d, "id_A"] \\
FA \arrow[r, "\psi"'] & A
\end{tikzcd}$$

Figure 2: Identities as F-homomorphisms

  This should hold if $id_A \circ \varphi = \phi \circ F(id_A)$, which is true since F is a functor and $F(id_A) = id_{FA}$.

- Composition of F-algebras must preserve F-homomorphisms. Consider the following diagram, where we have
  - 3 F-algebras, $\varphi_1$, $\varphi_2$ and $\varphi_3$ as the signatures
  - f and g are the F-homomorphisms between $\varphi_1 - \varphi_2$ and $\varphi_2 - \varphi_3$ respectively

  In order to prove that the composition of the F-algebras is a homomorphism, we need to show that the outer diagram commutes.

  *Proof.* $f$ is an F-homomorphism $\Rightarrow f \circ \varphi_1 = \varphi_2 \circ F(f)$ ... (1)
  $g$ is an F-homomorphism $\Rightarrow g \circ \varphi_2 = \varphi_3 \circ F(g)$ ... (2)

  $(g \circ f) \circ \varphi_1$
  $= g \circ f \circ \varphi_1$
  $= g \circ \varphi_2 \circ F(f)$ (from (1))
  $= \varphi_3 \circ F(g) \circ F(f)$ (from (2))
  $= \varphi_3 \circ F(g \circ f)$ ($F$ is a functor)

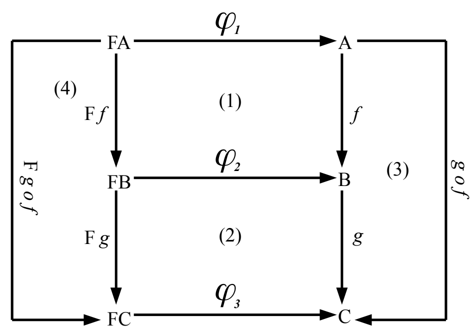


Figure 3: F-homomorpisms preserved under composition

Hence the outer diagram commutes. Also intuitively:
- diagram (1) commutes ($f$ is an F-homomorphism)
- diagram (2) commutes ($g$ is an F-homomorphism)
- diagram (3) commutes (by composition)
- diagram (4) commutes (F is a functor)

Hence the outer diagram commutes and $g \circ f$ is also an F-homomorphism.

$\square$

# 4 Initial object

An initial object of a category C is an object I in C such that for every object X in C, there exists precisely one *unique* arrow of type $I \to X$. Here are examples of initial objects in some categories:

- The empty set is the unique initial object in the category of sets since $\{\} \to A$ is the empty function for any object A in the set
- In the category of semigroups, the empty semigroup is the unique initial object

It's not mandatory that every category has an initial object. For example, a non-empty set does not have an initial object.

## 4.1 Uniqueness of initial objects

If I1 and I2 are both initial objects in the category C, then there's exactly one unique arrow $I1 \to I2$ and that arrow is an isomorphism. i.e. the initial objects are *uniquely isomorphic*.

*Proof.* I1 is an initial object in C
$\Rightarrow$ for each object X in C, there's a unique arrow $I1 \rightarrow X$
$\Rightarrow$ For $X = I2$ (substituting I2 for X), we have the unique arrow $f : I1 \rightarrow I2$

I2 is also initial
$\Rightarrow$ for each object X in C, there's a unique arrow $I2 \rightarrow X$
$\Rightarrow$ For X = I1 (substituting I1 for X), we have the unique arrow $g : I2 \rightarrow I1$

$f \circ g = I2 \rightarrow I2$ (by composition)
$g \circ f = I1 \rightarrow I1$ (by composition)

$I2 \rightarrow I2 = id_{I2}$ and $I1 \rightarrow I1 = id_{I1}$ (by initiality)
$\Rightarrow f \circ g = id_{I2}$ and $g \circ f = id_{I1}$

$\Rightarrow$ There's a bidirectional inverse between f and g since the arrow $I1 \rightarrow I2$ has an inverse arrow $I2 \rightarrow I1$. Hence the unique arrows f and g establish the unique isomorphism.

But there's something more. When we say that 2 objects are isomorphic, there may be many isomorphisms establishing that fact between the two objects. But for initial objects, there's only one - hence all initial objects are indistinguishable. So we call *the* initial object rather than *an* initial object of a category. The initial object is usually denoted by 0 and the unique arrow $0 \rightarrow X$ is denoted by !(rev)X (gnab). □

# 5 Terminal object

A terminal object of a category C is an object T in C if for every object X in C there exists a unique arrow $X \rightarrow T$. A category may have more than one terminal objects, but all are isomorphic (like initial objects) and hence we call *the* terminal object. The terminal object is usually denoted by 1 and the unique arrow $X \rightarrow 1$ is denoted by !X (bang).

Here are examples of terminal objects in some categories:

- Every one-element set (singleton) is a terminal object in this category
- In the category of semigroups, any singleton semigroup is a terminal object

It's not mandatory that every category has a terminal object. For example, The category of simple graphs [1] does not have a terminal object.

[1] A simple graph is an undirected graph that has no loops and no more than one edge between any two different vertices

# 6 Initial Algebra

An initial F-algebra is an initial object in the category of F-algebras (Alg(F) defined above). Let's look at it in a bit more detail. In Alg(F) if there exists an F-algebra $(\mu F, in)$ such that for any F-algebra $(C, \varphi)$ in that category, there exists a unique arrow $\{\varphi\} : \mu F- > C$ making the following diagram commute.[2]

$$\begin{tikzcd}
F\,\mu F \arrow[r, "in"] \arrow[d, "F\{\varphi\}"'] & \mu F \arrow[d, "\{\varphi\}"] \\
FC \arrow[r, "\varphi"] & C
\end{tikzcd}$$

Figure 4: $\{\varphi\}$ is the catamorphism

In the diagram, the unique arrow $\{\varphi\}$ is known as *Catamorphism*. We say that the initial algebra $(\mu F, in)$ is an initial object in the category Alg(F), and the catamorphism $\{\varphi\}$ is the mediating arrow out of it.

## 6.1 Properties of initial F-algebra

Let $(\mu F, in)$ be an initial F-algebra. Here are some of the laws that apply to initial algebras.

1. **Cancellation:** For any F-algebra $\varphi : FC \to C$, $\{\varphi\} \circ in = \varphi \circ F\{\varphi\}$ (follows straight from the diagram)

2. **Reflection:** $id = \{in\}$. Follows from the diagram below, since $\mu F \to \mu F$ is the id of the algebra.

$$\begin{tikzcd}
F\,\mu F \arrow[r, "in"] \arrow[d, "F\{in\}"'] & \mu F \arrow[d, "\{in\}"] \\
F\,\mu F \arrow[r, "in"] & \mu F
\end{tikzcd}$$

Figure 5: Reflection in initial F-algebra

[2] Isn't this the precise definition of an initial object?

3. **Universality:** The diagram for initial F-algebra commutes and we get the following equivalence:

$$f \circ in = \varphi \circ Ff \Leftrightarrow f = \{\varphi\}$$

This is known as the *Universal Property* and finds extensive use in proving various properties of catamorphism.

4. **Fusion:** For any two F-algebras $f : FC \to C$ and $g : FD \to D$ and an arrow $h : C \to D$, we have:

$$h \circ f = g \circ Fh \Rightarrow h \circ \{f\} = \{g\}$$

*Proof.* Consider the following diagram:

$$\begin{tikzcd}
\mathrm{F}\,\mu\mathrm{F} \arrow[r, "\mathrm{F}\{f\}"] \arrow[d, "\mathrm{in}"'] & \mathrm{FC} \arrow[r, "\mathrm{F}\,h"] \arrow[d, "f"] & \mathrm{FD} \arrow[d, "g"] \\
\mu\mathrm{F} \arrow[r, "\{f\}"'] & \mathrm{C} \arrow[r, "h"'] & \mathrm{D}
\end{tikzcd}$$

Figure 6: Fusion

Here

- $(\mu F, in)$ is an initial F-algebra
- $(C, f)$ and $(D, g)$ are F-algebras
- $h : (C, f) \to (D, g)$ is an F-homomorphism

Then the *fusion* law states that the composition of a homomorphism and catamorphism is again a catamorphism. Here's how we can derive it:

$h \circ \{f\} = \{g\}$

$\Leftrightarrow$ {universal property}
$h \circ \{f\} \circ in = \{g\} \circ F(h \circ \{f\})$

$\Leftrightarrow$ {F is a functor}
$h \circ \{f\} \circ in = \{g\} \circ Fh \circ F\{f\}$

$\Leftrightarrow$ {{f} is an F-homomorphism}
$h \circ f \circ F\{f\} = \{g\} \circ Fh \circ F\{f\}$

$\Leftrightarrow$ {cancellation}
$h \circ f = \{g\} \circ Fh$

$\Leftrightarrow$ {h is an F-homomorphism}
true $\square$

5. **Isomorphism:** The initial algebra $(\mu F, in)$ for an endofunctor F in category C defined as $in : F\mu F \to \mu F$ is an isomorphism. i.e $\mu F$ is isomorphic to $F\mu F$ via $in$, with the inverse defined as $in^{-1} : \{F\,in\}$. This is the Lambek's theorem.

   *Proof.* We need to show that in is the pre and post inverse of $in^{-1}$.

   $\Longrightarrow$ pre
   $in \circ in^{-1}$
   $= in \circ \{F\,in\}$

   $=$ {fusion law}
   $\{in\}$

   $=$ {reflection}
   $id$

   $\Longrightarrow$ post
   $in^{-1} \circ in$
   $= \{F\,in\} \circ in$

   $=$ {cancellation}
   $F\,in \circ F\{F\,in\}$

   $=$ {F is a functor}
   $F(in \circ \{F\,in\})$

   $=$ from pre
   $F\,id$

   $=$ F is a functor
   $id$

   $\square$

   In the above initial algebra we have $\mu F$ as the carrier and it defines an isomorphism. This means the carrier of the initial algebra is (up to isomorphism) a fixed point of the functor.

# 7 Initial Algebra and Recursive Data Types

An initial algebra generalizes the notion of a recursive data type. Consider a List data type which can be represented by the following Sum type:

// nil takes no arguments and returns a List data type
$nil : 1 \to List[A]$

// cons takes 2 arguments and returns a List data type
$cons : (A \times List[A]) \to List[A]$

Combining the 2 functions we get:

$in = [nil, cons] : 1 + (A \times List[A]) \to List[A]$, which translates to the functor $LA(X) = 1+(A\times X)$. So the data type of lists over the set A can be represented as an initial F-algebra $(\mu LA, in)$ over the functor LA. Here we write $\mu LA$ for List[A]. Let's prove it.

In order that $(\mu LA, in)$ is an initial algebra, we need to show that for an arbitrary F-algebra $(C, \varphi)$, where $\varphi$ is an arrow out of the sum type given by :

$c : 1 \to C$
$h : (A \times C) \to C$

and the join gives : $[c, h] : 1 + (A \times C) \to C$

Here's the category diagram In order for $(\mu LA, in)$ to be an initial F-algebra,

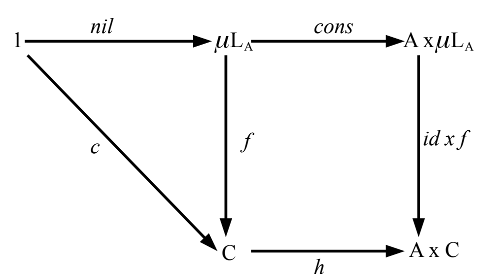


Figure 7: List as initial F-algebra

we need to find a homomorphism $f : \mu LA \to C$ and show that it is unique. Doing this we will be able to find the initial object for the initial algebra. f will be a homomorphism if the above diagram commutes, for which we need :

$f \circ nil = c$ and
$f \circ cons = h \circ (idxf)$

From the Universal property of initial F-algebras discussed above, it's easy to see that this system of equations has a unique solution which is fold(c, h) [try it as an exercise]. It's the catamorphism represented by:

$$f : \{[c, h]\} : List[A] \rightarrow C$$

This proves our initial hypothesis that $\mu LA$ is an initial F-algebra over the endofunctor $F : LA(X) = 1 + (A \times X)$.

Treating List as an initial F-algebra lets us define many of the list properties algebraically in terms of catamorphism. Consider length of a List, which is defined as $length : List[A] \rightarrow Nat$, where Nat is the set of natural numbers defined with its zero and successor functions as follows:

$zero : 1 \rightarrow Nat$
$succ : Nat \rightarrow Nat$

We can define *length* as a catamorphism : $length = \{[zero, \lambda(a, n).succ(n)]\}$.

Similarly for $concat : List[A] \times List[A] \rightarrow List[A]$, we can define it in terms of catamorphism as:

$$concat(xs, ys) = \{[\lambda(x).ys, cons]\}(xs)$$

When we talk about algebraic data types, we need to understand the algebra that goes with it. We can consider any algebraic data type as an initial F-algebra on the endofunctor F.