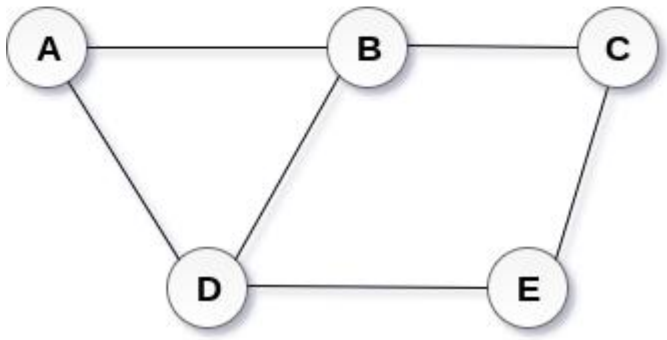
Graph

A graph can be defined as group of vertices and edges that are used to connect these vertices. A graph can be seen as a cyclic tree, where the vertices (Nodes) maintain any complex relationship among them instead of having parent child relationship.

Definition

A graph G can be defined as an ordered set G(V, E) where V(G) represents the set of vertices and E(G) represents the set of edges which are used to connect these vertices.

A Graph G(V, E) with 5 vertices (A, B, C, D, E) and six edges ((A,B), (B,C), (C,E), (E,D), (D,B), (D,A)) is shown in the following figure.



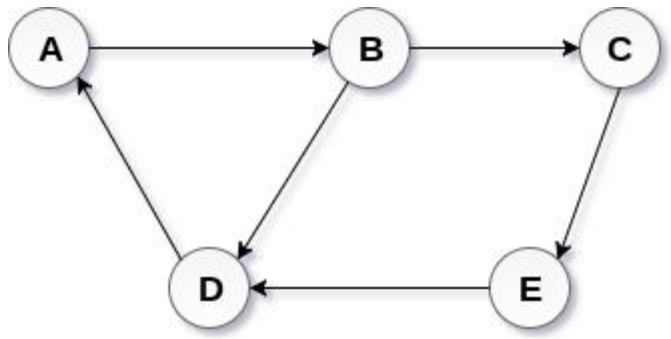
Undirected Graph

Directed and Undirected Graph

A graph can be directed or undirected. However, in an undirected graph, edges are not associated with the directions with them. An undirected graph is shown in the above figure since its edges are not attached with any of the directions. If an edge exists between vertex A and B then the vertices can be traversed from B to A as well as A to B.

In a directed graph, edges form an ordered pair. Edges represent a specific path from some vertex A to another vertex B. Node A is called initial node while node B is called terminal node.

A directed graph is shown in the following figure.



Directed Graph

Graph Terminology

Path

A path can be defined as the sequence of nodes that are followed in order to reach some terminal node V from the initial node U.

Closed Path

A path will be called as closed path if the initial node is same as terminal node. A path will be closed path if $V_0=V_N$.

Simple Path

If all the nodes of the graph are distinct with an exception $V_0=V_N$, then such path P is called as closed simple path.

Cycle

A cycle can be defined as the path which has no repeated edges or vertices except the first and last vertices.

Connected Graph

A connected graph is the one in which some path exists between every two vertices (u, v) in V. There are no isolated nodes in connected graph.

Complete Graph

A complete graph is the one in which every node is connected with all other nodes. A complete graph contain n(n-1)/2 edges where n is the number of nodes in the graph.

Weighted Graph

In a weighted graph, each edge is assigned with some data such as length or weight. The weight of an edge e can be given as w(e) which must be a positive (+) value indicating the cost of traversing the edge.

Digraph

A digraph is a directed graph in which each edge of the graph is associated with some direction and the traversing can be done only in the specified direction.

Loop

An edge that is associated with the similar end points can be called as Loop.

Adjacent Nodes

If two nodes u and v are connected via an edge e, then the nodes u and v are called as neighbours or adjacent nodes.

Degree of the Node

A degree of a node is the number of edges that are connected with that node. A node with degree 0 is called as isolated node.

Graph Representation

By Graph representation, we simply mean the technique which is to be used in order to store some graph into the computer's memory.

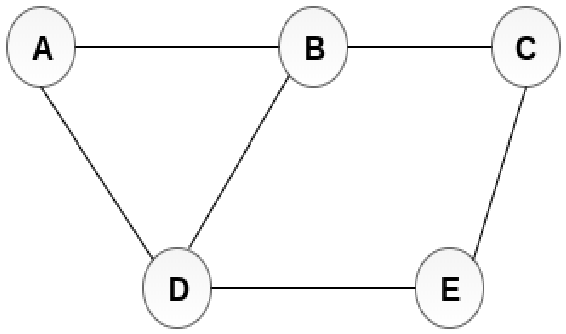
There are two ways to store Graph into the computer's memory. In this part of this tutorial, we discuss each one of them in detail.

1. Sequential Representation

In sequential representation, we use adjacency matrix to store the mapping represented by vertices and edges. In adjacency matrix, the rows and columns are represented by the graph vertices. A graph having n vertices, will have a dimension **n x n**.

An entry M_{ij} in the adjacency matrix representation of an undirected graph G will be 1 if there exists an edge between V_i and V_j.

An undirected graph and its adjacency matrix representation is shown in the following figure.



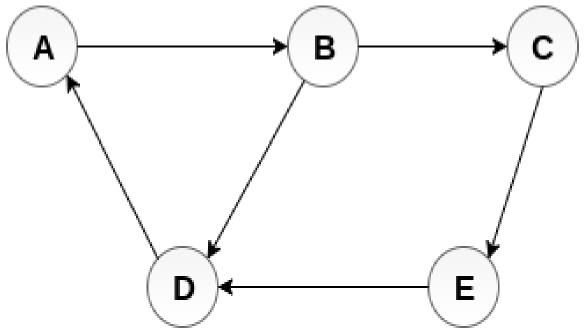
$$\begin{array}{c|ccccc|} & A & B & C & D & E \\ \hline A & 0 & 1 & 0 & 1 & 0 \\ B & 1 & 0 & 1 & 1 & 0 \\ C & 0 & 1 & 0 & 0 & 1 \\ D & 1 & 1 & 0 & 0 & 1 \\ E & 0 & 0 & 1 & 1 & 0 \end{array}$$

Undirected Graph **Adjacency Matrix**

in the above figure, we can see the mapping among the vertices (A, B, C, D, E) is represented by using the adjacency matrix which is also shown in the figure.

There exists different adjacency matrices for the directed and undirected graph. In directed graph, an entry A_{ij} will be 1 only when there is an edge directed from V_i to V_j.

A directed graph and its adjacency matrix representation is shown in the following figure.

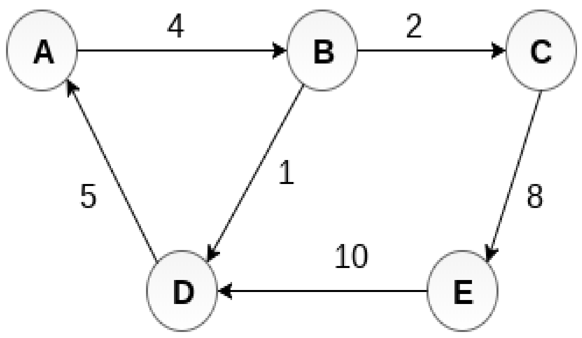


$$\begin{array}{c|ccccc|} & A & B & C & D & E \\ \hline A & 0 & 1 & 0 & 0 & 0 \\ B & 0 & 0 & 1 & 1 & 0 \\ C & 0 & 0 & 0 & 0 & 1 \\ D & 1 & 0 & 0 & 0 & 0 \\ E & 0 & 0 & 0 & 1 & 0 \end{array}$$

Directed Graph **Adjacency Matrix**

Representation of weighted directed graph is different. Instead of filling the entry by 1, the Non-zero entries of the adjacency matrix are represented by the weight of respective edges.

The weighted directed graph along with the adjacency matrix representation is shown in the following figure.



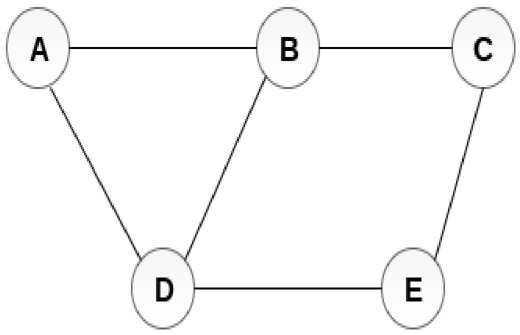
$$\begin{array}{c|ccccc} & A & B & C & D & E \\ \hline A & 0 & 4 & 0 & 0 & 0 \\ B & 0 & 0 & 2 & 1 & 0 \\ C & 0 & 0 & 0 & 0 & 8 \\ D & 5 & 0 & 0 & 0 & 0 \\ E & 0 & 0 & 0 & 10 & 0 \end{array}$$

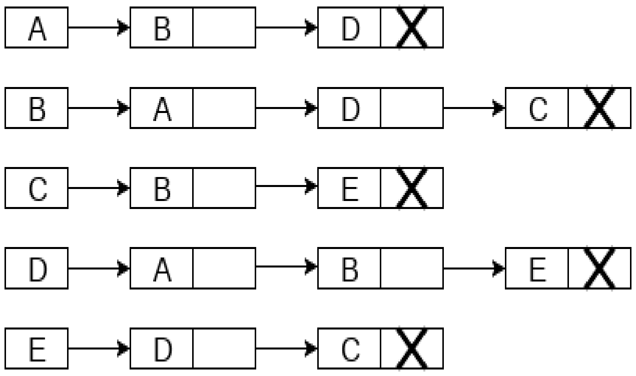
Weighted Directed Graph **Adjacency Matrix**

Linked Representation

In the linked representation, an adjacency list is used to store the Graph into the computer's memory.

Consider the undirected graph shown in the following figure and check the adjacency list representation.

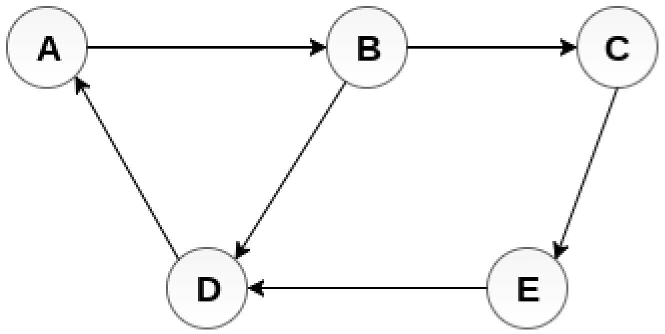




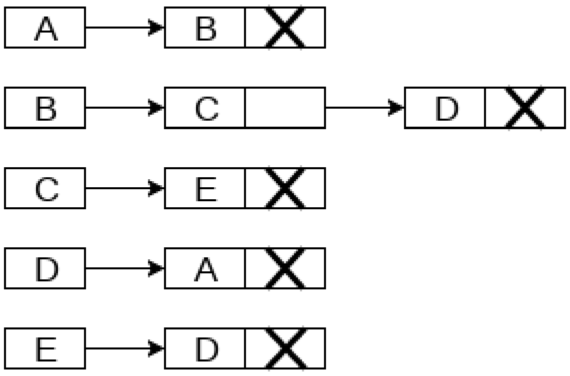
Undirected Graph **Adjacency List**

An adjacency list is maintained for each node present in the graph which stores the node value and a pointer to the next adjacent node to the respective node. If all the adjacent nodes are traversed then store the NULL in the pointer field of last node of the list. The sum of the lengths of adjacency lists is equal to the twice of the number of edges present in an undirected graph.

Consider the directed graph shown in the following figure and check the adjacency list representation of the graph.



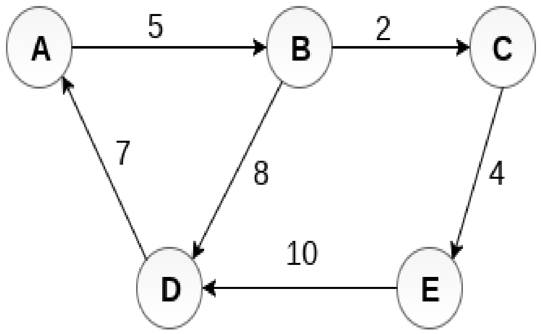
Directed Graph



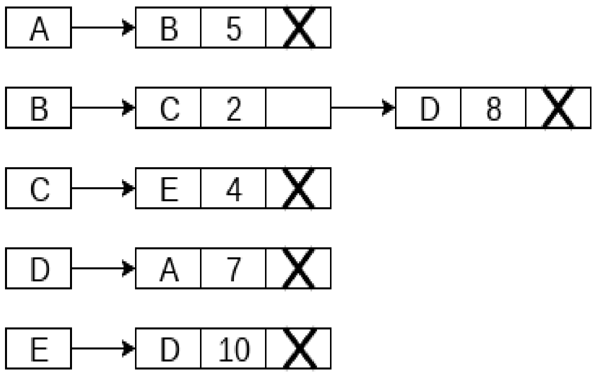
Adjacency List

In a directed graph, the sum of lengths of all the adjacency lists is equal to the number of edges present in the graph.

In the case of weighted directed graph, each node contains an extra field that is called the weight of the node. The adjacency list representation of a directed graph is shown in the following figure.



Weighted Directed Graph



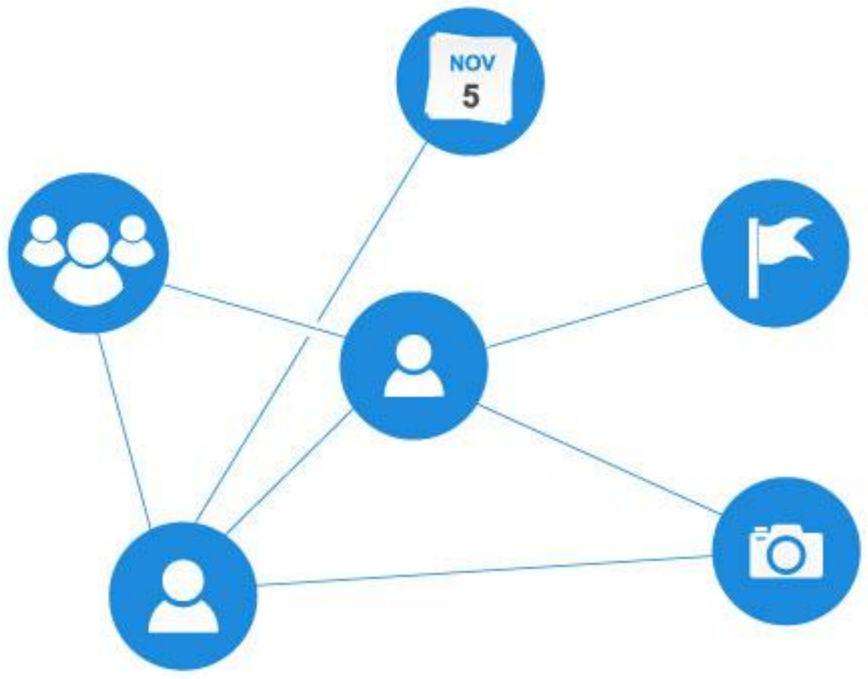
Adjacency List

Graph Data Stucture

A graph data structure is a collection of nodes that have data and are connected to other nodes.

Let's try to understand this by means of an example. On facebook, everything is a node. That includes User, Photo, Album, Event, Group, Page, Comment, Story, Video, Link, Note...anything that has data is a node.

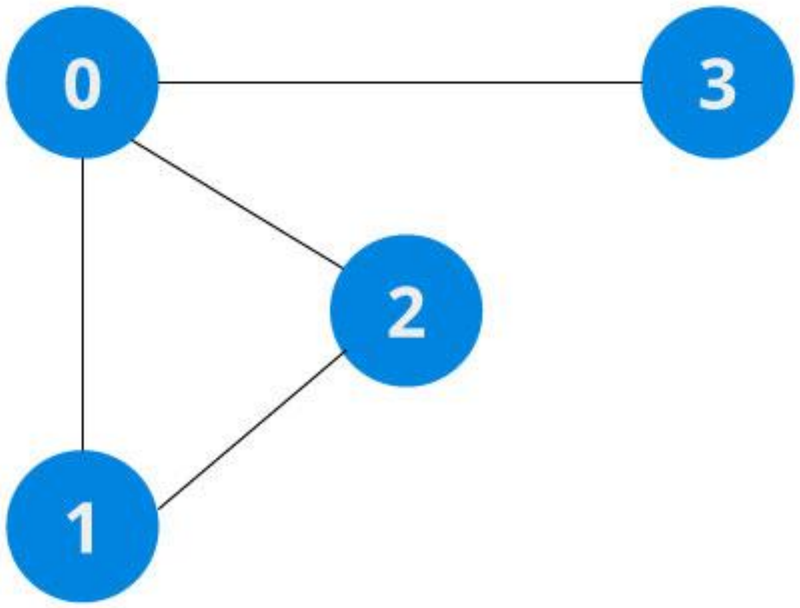
Every relationship is an edge from one node to another. Whether you post a photo, join a group, like a page etc., a new edge is created for that relationship.



All of facebook is then, a collection of these nodes and edges. This is because facebook uses a graph data structure to store its data.

More precisely, a graph is a data structure (V,E) that consists of

- A collection of vertices V
- A collection of edges E, represented as ordered pairs of vertices (u,v)



In the graph,

```
V = {0, 1, 2, 3}
E = {(0,1), (0,2), (0,3), (1,2)}
G = {V, E}
```

Graph Terminology

- **Adjacency**: A vertex is said to be adjacent to another vertex if there is an edge connecting them. Vertices 2 and 3 are not adjacent because there is no edge between them.
- **Path**: A sequence of edges that allows you to go from vertex A to vertex B is called a path. 0-1, 1-2 and 0-2 are paths from vertex 0 to vertex 2.
- **Directed Graph**: A graph in which an edge (u,v) doesn't necessary mean that there is an edge (v, u) as well. The edges in such a graph are represented by arrows to show the direction of the edge.

Graph Representation

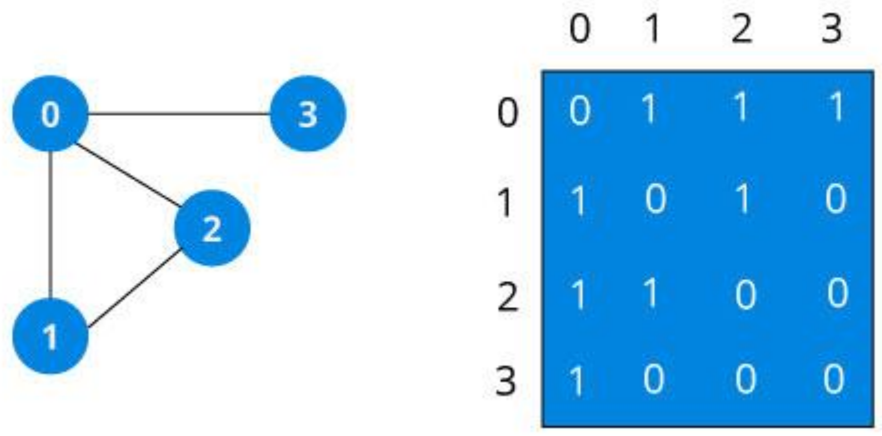
Graphs are commonly represented in two ways:

1. Adjacency Matrix

An adjacency matrix is 2D array of V x V vertices. Each row and column represent a vertex.

If the value of any element `a[i][j]` is 1, it represents that there is an edge connecting vertex i and vertex j.

The adjacency matrix for the graph we created above is



$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \end{array}$$

Since it is an undirected graph, for edge (0,2), we also need to mark edge (2,0); making the adjacency matrix symmetric about the diagonal.

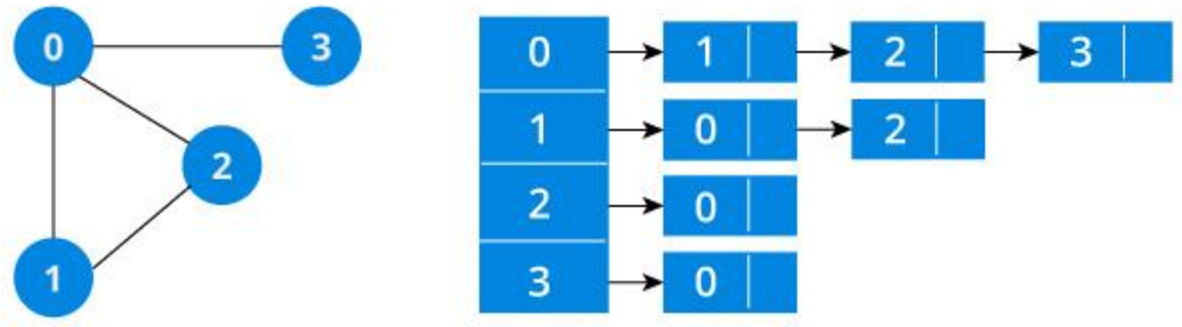
Edge lookup(checking if an edge exists between vertex A and vertex B) is extremely fast in adjacency matrix representation but we have to reserve space for every possible link between all vertices(V x V), so it requires more space.

2. Adjacency List

An adjacency list represents a graph as an array of linked list.

The index of the array represents a vertex and each element in its linked list represents the other vertices that form an edge with the vertex.

The adjacency list for the graph we made in the first example is as follows:



```
0 → 1 → 2 → 3
1 → 0 → 2
2 → 0
3 → 0
```

An adjacency list is efficient in terms of storage because we only need to store the values for the edges. For a graph with millions of vertices, this can mean a lot of saved space.

Graph Operations

The most common graph operations are:

- *Check if element is present in graph*
- *Graph Traversal*
- *Add elements(vertex, edges) to graph*
- *Finding path from one vertex to another*

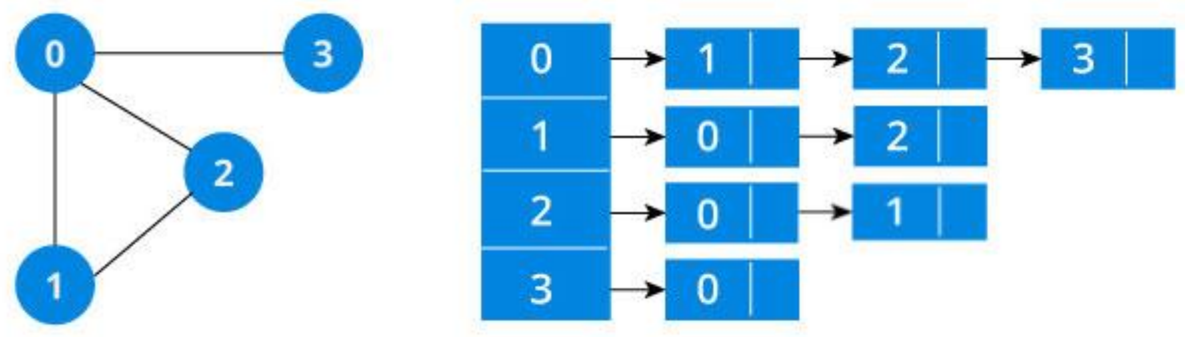
Adjacency List

An adjacency list represents a graph as an array of linked list.

The index of the array represents a vertex and each element in its linked list represents the other vertices that form an edge with the vertex.

Adjacency List representation

A graph and its equivalent adjacency list representation is shown below.



An adjacency list is efficient in terms of storage because we only need to store the values for the edges. For a sparse graph with millions of vertices and edges, this can mean a lot of saved space.

Adjacency List Structure

The simplest adjacency list needs a node data structure to store a vertex and a graph data structure to organize the nodes.

We stay close to the basic definition of graph - a collection of vertices and edges `{V, E}`. For simplicity we use an unlabeled graph as opposed to a labeled one i.e. the vertexes are identified by their indices 0,1,2,3.

Let's dig into the data structures at play here.

```
struct node
{
    int vertex;
    struct node* next;
};

struct Graph
{
    int numVertices;
    struct node** adjLists;
};
```

Don't let the `struct node** adjLists` overwhelm you.

All we are saying is we want to store a pointer to `struct node*`. This is because we don't know how many vertices the graph will have and so we cannot create an array of Linked Lists at compile time.

Adjacency Matrix

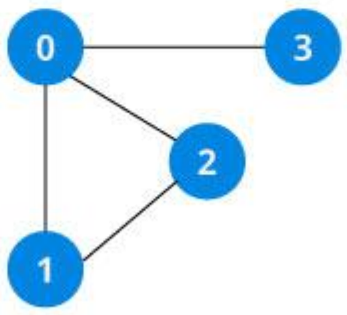
An adjacency matrix is a way of representing a graph G = {V, E} as a matrix of booleans.

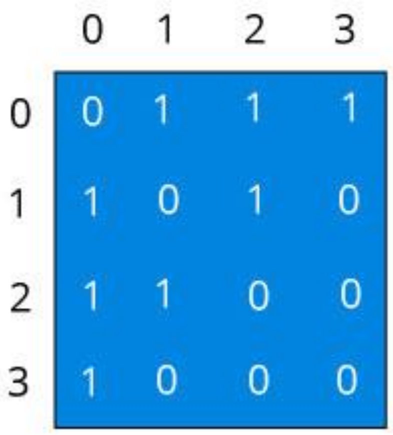
Adjacency matrix representation

The size of the matrix is `VxV` where `V` is the number of vertices in the graph and the value of an entry `Aij` is either 1 or 0 depending on whether there is an edge from vertex i to vertex j.

Adjacency Matrix Example

The image below shows a graph and its equivalent adjacency matrix.





	0	1	2	3
0	0	1	1	1
1	1	0	1	0
2	1	1	0	0
3	1	0	0	0

In case of undirected graph, the matrix is symmetric about the diagonal because of every edge `(i,j)`, there is also an edge `(j,i)`.

Pros of adjacency matrix

The basic operations like adding an edge, removing an edge and checking whether there is an edge from vertex i to vertex j are extremely time efficient, constant time operations.

If the graph is dense and the number of edges is large, adjacency matrix should be the first choice. Even if the graph and the adjacency matrix is sparse, we can represent it using data structures for sparse matrix.

The biggest advantage however, comes from the use of matrices. The recent advances in hardware enable us to perform even expensive matrix operations on the GPU.

By performing operations on the adjacent matrix, we can get important insights into the nature of the graph and the relationship between its vertices.

Cons of adjacency matrix

The `VxV` space requirement of adjacency matrix makes it a memory hog. Graphs out in the wild usually don't have too many connections and this is the major reason why adjacency lists are the better choice for most tasks.

While basic operations are easy, operations like `inEdges` and `outEdges` are expensive when using the adjacency matrix representation.

Adjacency Matrix code

If you know how to create two dimensional arrays, you also know how to create adjacency matrix.

Graph Traversal Algorithm

In this part of the tutorial we will discuss the techniques by using which, we can traverse all the vertices of the graph.

Traversing the graph means examining all the nodes and vertices of the graph. There are two standard methods by using which, we can traverse the graphs. Lets discuss each one of them in detail.

- Breadth First Search
- Depth First Search

Breadth First Search (BFS) Algorithm

Breadth first search is a graph traversal algorithm that starts traversing the graph from root node and explores all the neighbouring nodes. Then, it selects the nearest node and explore all the unexplored nodes. The algorithm follows the same process for each of the nearest node until it finds the goal.

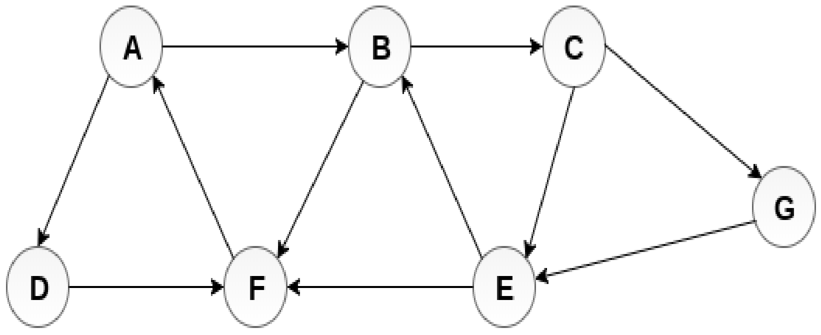
The algorithm of breadth first search is given below. The algorithm starts with examining the node A and all of its neighbours. In the next step, the neighbours of the nearest node of A are explored and process continues in the further steps. The algorithm explores all neighbours of all the nodes and ensures that each node is visited exactly once and no node is visited twice.

Algorithm

- **Step 1:** SET STATUS = 1 (ready state)
  for each node in G
- **Step 2:** Enqueue the starting node A
  and set its STATUS = 2
  (waiting state)
- **Step 3:** Repeat Steps 4 and 5 until
  QUEUE is empty
- **Step 4:** Dequeue a node N. Process it
  and set its STATUS = 3
  (processed state).
- **Step 5:** Enqueue all the neighbours of
  N that are in the ready state
  (whose STATUS = 1) and set
  their STATUS = 2
  (waiting state)
  [END OF LOOP]
- **Step 6:** EXIT

Example

Consider the graph G shown in the following image, calculate the minimum path p from node A to node E. Given that each edge has a length of 1.



Adjacency Lists

A : B, D

B : C, F

C : E, G

G : E

E : B, F

F : A

D : F

Solution:

Minimum Path P can be found by applying breadth first search algorithm that will begin at node A and will end at E. the algorithm uses two queues, namely **QUEUE1** and **QUEUE2**. **QUEUE1** holds all the nodes that are to be processed while **QUEUE2** holds all the nodes that are processed and deleted from **QUEUE1**.

Lets start examining the graph from Node A.

1. Add A to QUEUE1 and NULL to QUEUE2.

    1. QUEUE1 = {A}
    2. QUEUE2 = {NULL}

2. Delete the Node A from QUEUE1 and insert all its neighbours. Insert Node A into QUEUE2

    1. QUEUE1 = {B, D}
    2. QUEUE2 = {A}

3. Delete the node B from QUEUE1 and insert all its neighbours. Insert node B into QUEUE2.

    1. QUEUE1 = {D, C, F}
    2. QUEUE2 = {A, B}

4. Delete the node D from QUEUE1 and insert all its neighbours. Since F is the only neighbour of it which has been inserted, we will not insert it again. Insert node D into QUEUE2.

1. QUEUE1 = {C, F}
2. QUEUE2 = { A, B, D}

5. Delete the node C from QUEUE1 and insert all its neighbours. Add node C to QUEUE2.

1. QUEUE1 = {F, E, G}
2. QUEUE2 = {A, B, D, C}

6. Remove F from QUEUE1 and add all its neighbours. Since all of its neighbours has already been added, we will not add them again. Add node F to QUEUE2.

1. QUEUE1 = {E, G}
2. QUEUE2 = {A, B, D, C, F}

7. Remove E from QUEUE1, all of E's neighbours has already been added to QUEUE1 therefore we will not add them again. All the nodes are visited and the target node i.e. E is encountered into QUEUE2.

1. QUEUE1 = {G}
2. QUEUE2 = {A, B, D, C, F, E}

Now, backtrack from E to A, using the nodes available in QUEUE2.

The minimum path will be **A → B → C → E**.

Depth First Search (DFS) Algorithm

Depth first search (DFS) algorithm starts with the initial node of the graph G, and then goes to deeper and deeper until we find the goal node or the node which has no children. The algorithm, then backtracks from the dead end towards the most recent node that is yet to be completely unexplored.

The data structure which is being used in DFS is stack. The process is similar to BFS algorithm. In DFS, the edges that leads to an unvisited node are called discovery edges while the edges that leads to an already visited node are called block edges.

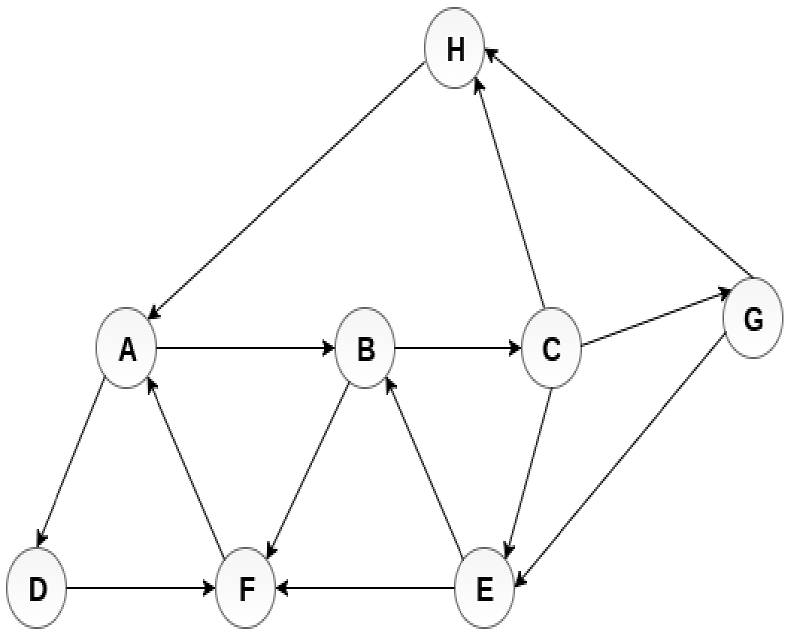
Algorithm

- **Step 1:** SET STATUS = 1 (ready state) for each node in G
- **Step 2:** Push the starting node A on the stack and set its STATUS = 2 (waiting state)

- **Step 3:** Repeat Steps 4 and 5 until STACK is empty
- **Step 4:** Pop the top node N. Process it and set its STATUS = 3 (processed state)
- **Step 5:** Push on the stack all the neighbours of N that are in the ready state (whose STATUS = 1) and set their
  STATUS = 2 (waiting state)
  [END OF LOOP]
- **Step 6:** EXIT

Example :

Consider the graph G along with its adjacency list, given in the figure below. Calculate the order to print all the nodes of the graph starting from node H, by using depth first search (DFS) algorithm.



Adjacency Lists

A : B, D

B : C, F

C : E, G, H

G : E, H

E : B, F

F : A

D : F

H : A

Solution :

Push H onto the stack

1. STACK : H

POP the top element of the stack i.e. H, print it and push all the neighbours of H onto the stack that are is ready state.

1. Print H
2. STACK : A

Pop the top element of the stack i.e. A, print it and push all the neighbours of A onto the stack that are in ready state.

1. Print A
2. Stack : B, D

Pop the top element of the stack i.e. D, print it and push all the neighbours of D onto the stack that are in ready state.

1. Print D
2. Stack : B, F

Pop the top element of the stack i.e. F, print it and push all the neighbours of F onto the stack that are in ready state.

1. Print F
2. Stack : B

Pop the top of the stack i.e. B and push all the neighbours

1. Print B
2. Stack : C

Pop the top of the stack i.e. C and push all the neighbours.

1. Print C
2. Stack : E, G

Pop the top of the stack i.e. G and push all its neighbours.

1. Print G
2. Stack : E

Pop the top of the stack i.e. E and push all its neighbours.

1. Print E
2. Stack :

Hence, the stack now becomes empty and all the nodes of the graph have been traversed.

The printing sequence of the graph will be :

1. H → A → D → F → B → C → G → E

DFS algorithm

Traversal means visiting all the nodes of a graph. Depth first traversal or Depth first Search is a recursive algorithm for searching all the vertices of a graph or tree data structure. In this article, you will learn with the help of examples the DFS algorithm, DFS pseudocode and the code of the depth first search algorithm with implementation in C++, C, Java and Python programs.

DFS algorithm

A standard DFS implementation puts each vertex of the graph into one of two categories:

1. Visited
2. Not Visited

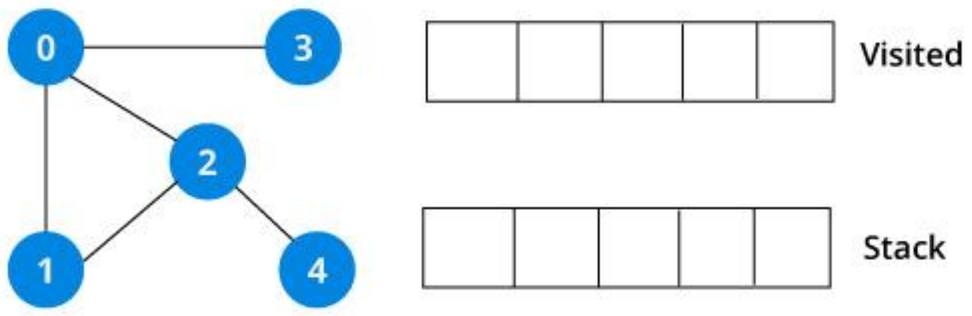
The purpose of the algorithm is to mark each vertex as visited while avoiding cycles.

The DFS algorithm works as follows:

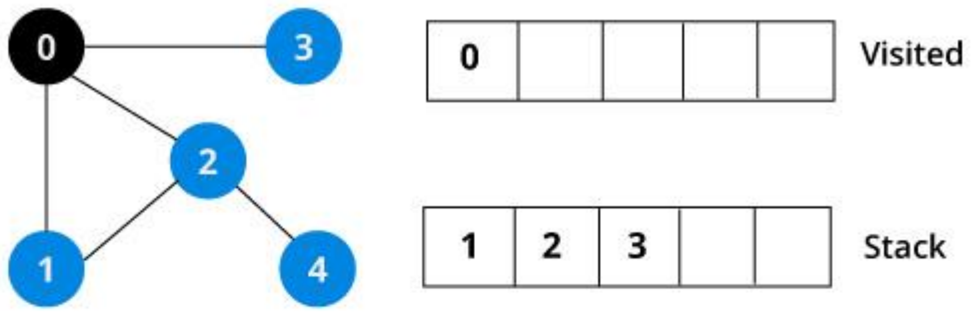
1. Start by putting any one of the graph's vertices on top of a stack.
2. Take the top item of the stack and add it to the visited list.
3. Create a list of that vertex's adjacent nodes. Add the ones which aren't in the visited list to the top of stack.
4. Keep repeating steps 2 and 3 until the stack is empty.

DFS example

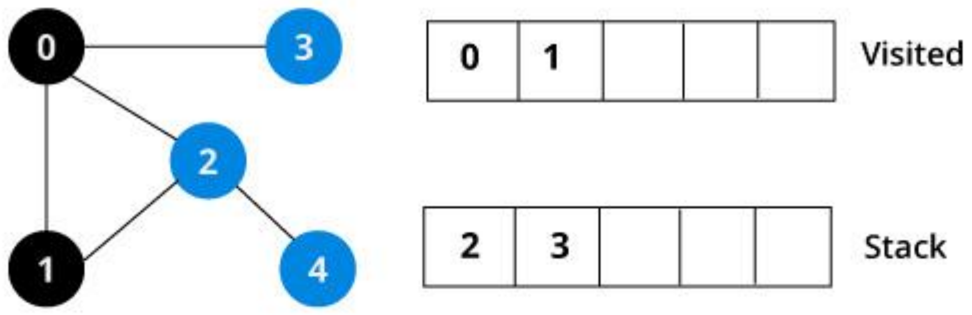
Let's see how the Depth First Search algorithm works with an example. We use an undirected graph with 5 vertices.



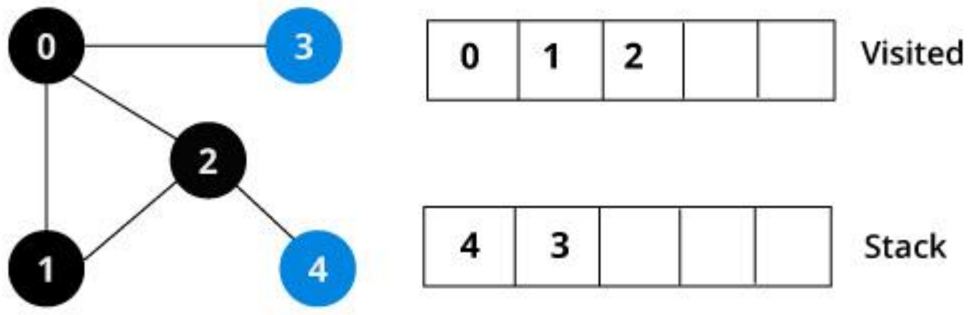
We start from vertex 0, the DFS algorithm starts by putting it in the Visited list and putting all its adjacent vertices in the stack.

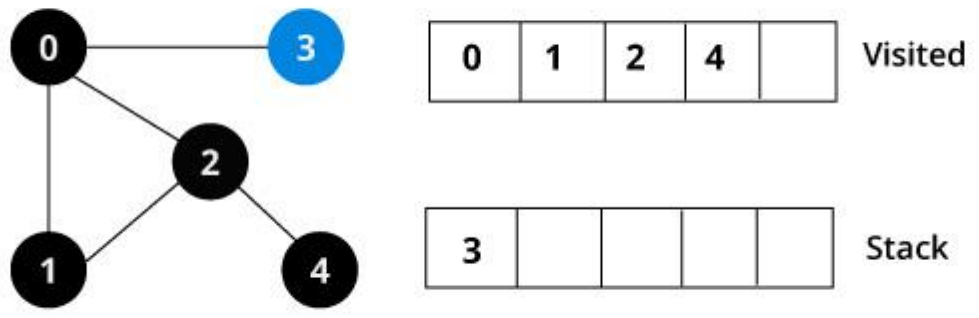


Next, we visit the element at the top of stack i.e. 1 and go to its adjacent nodes. Since 0 has already been visited, we visit 2 instead.

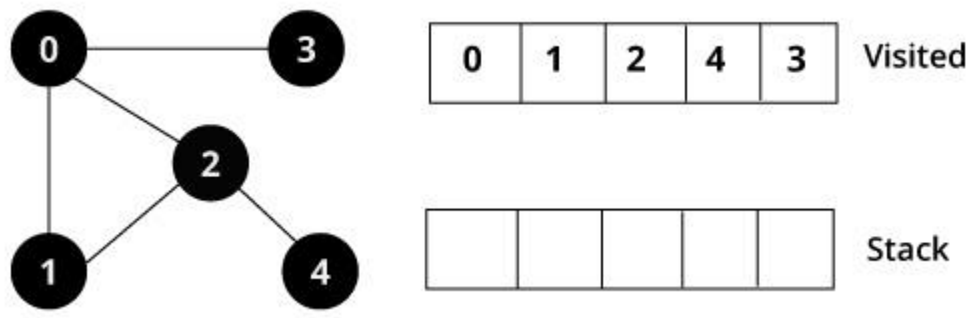


Vertex 2 has an unvisited adjacent vertex in 4, so we add that to the top of the stack and visit it.





After we visit the last element 3, it doesn't have any unvisited adjacent nodes, so we have completed the Depth First Traversal of the graph.



DFS pseudocode (recursive implementation)

The pseudocode for DFS is shown below. In the init() function, notice that we run the DFS function on every node. This is because the graph might have two different disconnected parts so to make sure that we cover every vertex, we can also run the DFS algorithm on every node.

```
DFS(G, u)
    u.visited = true
    for each v ∈ G.Adj[u]
        if v.visited == false
            DFS(G,v)

init() {
    For each u ∈ G
        u.visited = false
     For each u ∈ G
       DFS(G, u)
}
```

Breadth first search

Traversal means visiting all the nodes of a graph. Breadth first traversal or Breadth first Search is a recursive algorithm for searching all the vertices of a graph or tree data structure. In this article, you will learn with the help of examples the BFS algorithm, BFS pseudocode and the code of the breadth first search algorithm with implementation in C++, C, Java and Python programs.

BFS algorithm

A standard DFS implementation puts each vertex of the graph into one of two categories:

1. Visited
2. Not Visited

The purpose of the algorithm is to mark each vertex as visited while avoiding cycles.

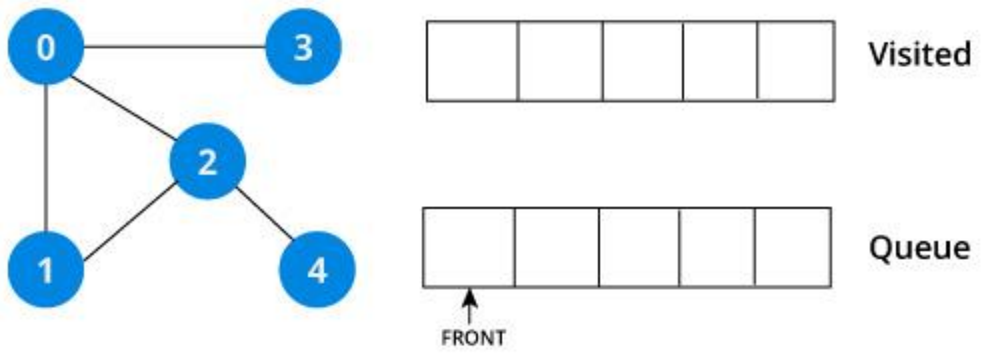
The algorithm works as follows:

1. Start by putting any one of the graph's vertices at the back of a queue.
2. Take the front item of the queue and add it to the visited list.
3. Create a list of that vertex's adjacent nodes. Add the ones which aren't in the visited list to the back of the queue.
4. Keep repeating steps 2 and 3 until the queue is empty.

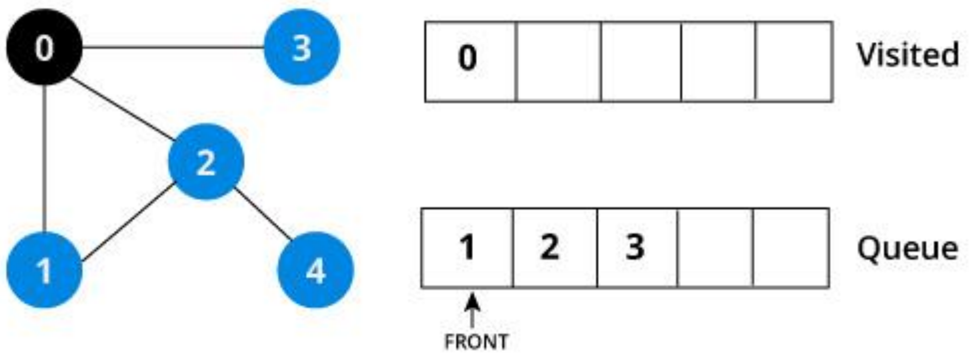
The graph might have two different disconnected parts so to make sure that we cover every vertex, we can also run the BFS algorithm on every node

BFS example

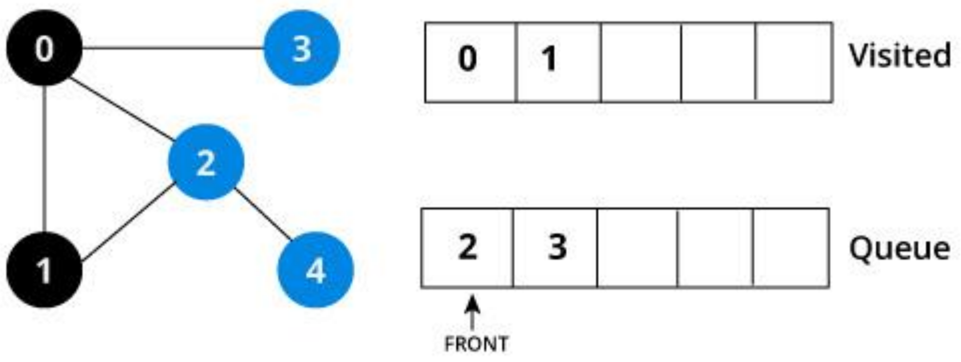
Let's see how the Breadth First Search algorithm works with an example. We use an undirected graph with 5 vertices.



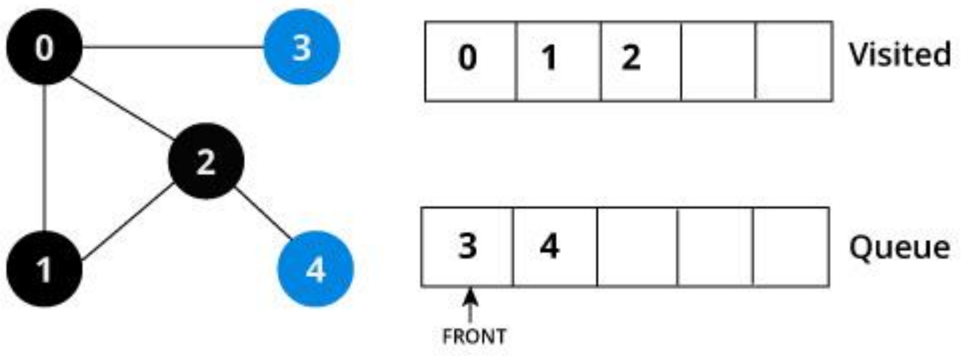
We start from vertex 0, the BFS algorithm starts by putting it in the Visited list and putting all its adjacent vertices in the stack.

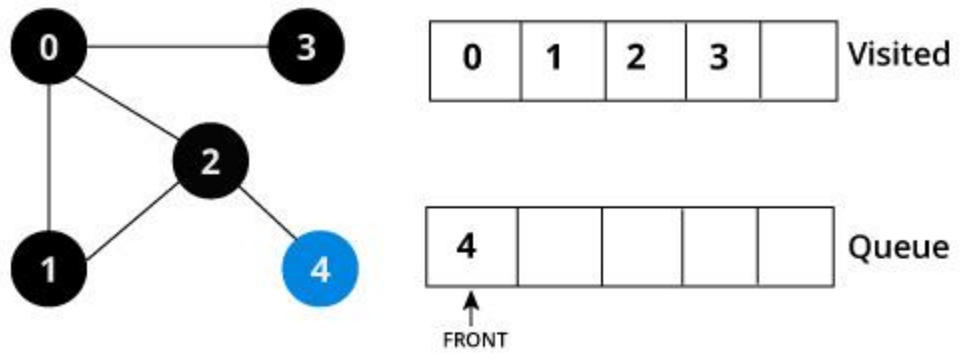


Next, we visit the element at the front of queue i.e. 1 and go to its adjacent nodes. Since 0 has already been visited, we visit 2 instead.

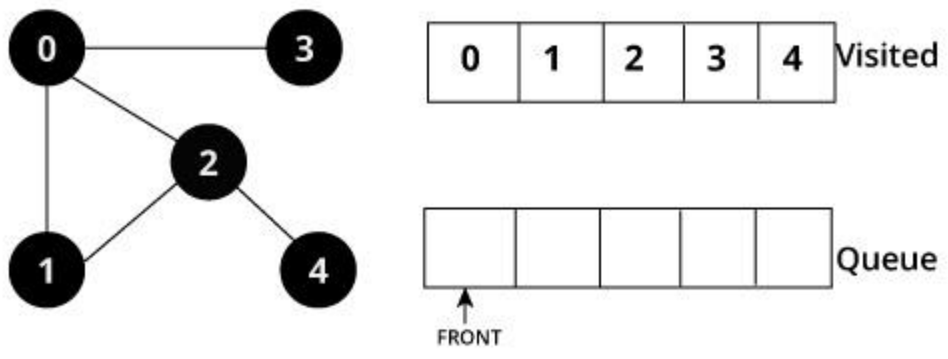


Vertex 2 has an unvisited adjacent vertex in 4, so we add that to the back of the queue and visit 3, which is at the front of the queue.





Only 4 remains in the queue since the only adjacent node of 3 i.e. 0 is already visited. We visit it.



Since the queue is empty, we have completed the Depth First Traversal of the graph.

BFS pseudocode

```
create a queue Q
mark v as visited and put v into Q
while Q is non-empty
    remove the head u of Q
    mark and enqueue all (unvisited) neighbours of u
```