Flatland optics: fundamentals

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"Flatland" is the title of a 120-year-old science fiction story. It describes the life of creatures living in a two-dimensional (2D) Flatland. A superior creature living in the three-dimensional (3D) spaceland, as we do, can easily inspect, for example, the inside of a Flatland house, as well as the content of a flat man's stomach without leaving any trace. Furthermore, the 3D person has supernatural powers that enable him to change the laws of physics in Flatland. We present here the concept of a 2D Flatland optics with one transversal coordinate x and one longitudinal coordinate z. The other transversal coordinate y allows total inspection of Flatland optics, and the freedom to change the wavelength, without using something like nonlinear optics or a Doppler shift. Monochromatic 3D light can be converted reversibly into polychromatic 2D light. A large variety of 2D systems and 2D effects will be presented here and in follow-up contributions. An epilogue faces the question, how "real" is Flatland optics? © 2000 Optical Society of America [S0740-3232(00)00710-9]

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1. INTRODUCTION

"Flatland, a romance of many dimensions"[1] is the complete title of a science fiction story written by Edwin A. Abbott (1838–1926). The second edition appeared in 1884, indicating that the first edition was printed probably around 120 years ago, well before physicists proclaimed a four-dimensional universe.

Abbott's story has the format of an autobiography written by a Flatland man. He begins by explaining to ordinary three-dimensional (3D) readers like us the everyday life in Flatland. A Flatland creature, as seen from three dimensions, might have the shape of a square, a circle or a polygon. But looked at from within Flatland, every creature appears as a line of finite length. In order to recognize a particular individual, one must travel around it. If it is a circle, its angular extension will remain constant. But a very long rectangle will present a highly modulated apparent length. Three examples are illustrated in Fig. 1. With such arguments and similar ones, Abbott succeeded in presenting a credible and consistent description of the Flatland universe.

One night, the fictitious autobiographic author had a dream with severe consequences. In his dream he became aware of a one-dimensional (1D) universe, called "Lineland." Our two-dimensional (2D) man quickly realized that his capabilities to inspect and to influence the Lineland universe were far superior even to those of the King of Lineland. Next morning our 2D man remembered his dream. He contemplated an extrapolation of his 1D/2D dream to a 2D/3D situation. "On the last day of 1999" (so it reads in Abbott's book) our 2D man received the visit of a 3D creature who temporarily provided him with the capability of 3D vision. This enabled our 2D man to look into the interiors of his neighbors' 2D homes and also into the interiors of some 2D stomachs.

On January first of 2000 our autobiographic author became again an ordinary 2D man. But he wanted to share his knowledge with the intellectuals of Flatland. So he started to preach his 3D gospel, which of course contradicted the official wisdom. Hence he encountered severe opposition. Eventually, he was convicted to lifelong imprisonment. Fortunately for us, he had the opportunity to write his autobiography and to smuggle it out of the prison so that every reader of his memoirs may now ponder his own dimensionality.

Now to proper optics: Recently we noticed a conceptual similarity of two minor experimental inventions. In both cases one of the transversal coordinates (x) together with the longitudinal coordinate (z) represented the signal domain. The other transversal coordinate (y) was used as a parameter in describing the impact of the system on the signal. So we interpreted (x, z) as the coordinates of Flatland and (y) as the pathway for a 3D-spaceland citizen who wants to inspect and to influence life in Flatland. The experience would be like looking down at the ground where many ants are running around busily. The Flatland point of view became very fruitful because it enabled us to come up with many more optical inventions with Flatland flavor. We then supplemented the collection of optical Flatland experiments with an optical Flatland theory.

In this paper we start with the general theory of optics in Flatland, and then we discuss special cases, most of which are easy to implement. This Flatland optics can be viewed as the sacrifice of one dimension (the y coordinate of the admissible signals) in order to have more flexibility for designing some systems for specific applications. Sacrificing one dimension so as to gain flexibility and convenience is a very general philosophy, which has been used for example by Denisyuk in his project "Pseudodeep holography,"[2] and by Ojeda-Castaneda and his co-workers.[3] They used 2D binary masks for the synthesis of 1D complex amplitudes. The subject of "dimensional transducers"[4,5] is also partially relevant.

In Section 2 we present the fundamentals of coherent Flatland optics. In Section 3 we describe an elementary Flatland experiment, whereby the light changes its wavelength while propagating in the z direction. This capa-

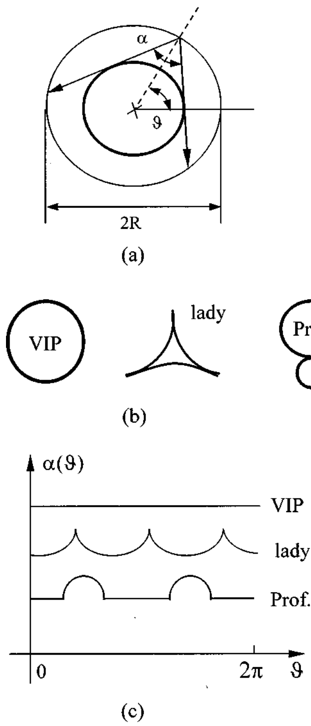

Fig. 1. Identification of a creature in Flatland by observation of its angular size (α) from all directions (ϑ). The parameter R is the standard distance from the center of the Flatland creature while it is tested. (a) Measuring $\alpha(\vartheta)$ of a circle, (b) three Flatland individuals, (c) angular signatures of the three Flatland individuals.

bility can be useful in connection with various developments in Fourier optics (Section 4). In Section 5 monochromatic light is converted into polychromatic light and vice versa. Some conclusions (Section 6) are followed by an epilogue (Section 7), in which we attempt to answer the questions of skeptical readers.

In a follow-up paper we shall demonstrate the reality of Flatland wave optics by performing the equivalent of those elementary experiments that did establish the wave nature of 3D-optics.[6] Next we expect to demonstrate the usefulness of Flatland optics by using it for perfect achromatization of diffraction in white light.[7]

2. FUNDAMENTALS OF COHERENT FLATLAND OPTICS

The generic setup is shown in Fig. 2. The Flatland universe is a plane $y = \text{constant}$ at $z \geqslant 0$. A monochromatic point source in plane $z = -2f$ at $y = f \tan\alpha$ emits a spherical wave, which is converted by a lens (focal length f) into a tilted plane wave,

$$V(y, z) = \exp\left[\frac{2\pi i}{\lambda}(y \sin\alpha + z \cos\alpha)\right]. \tag{1}$$

A 1D object at $z = 0$, with complex transmission $u_0(x)$, modifies the complex amplitude into

$$V(x, y, z = 0) = u_0(x)\exp\left(\frac{2\pi i}{\lambda} y \sin\alpha\right). \tag{2}$$

We replace $u_0(x)$ with its Fourier integral

$$u_0(x) = \int \tilde{u}_0(\nu)\exp(2\pi i \nu x)\mathrm{d}\nu. \tag{3}$$

Hence the substance of Eq. (2) is now described by

$$V(x, y, 0) = \int \tilde{u}_0(\nu)\exp\left[\frac{2\pi i}{\lambda}(x\lambda\nu + y \sin\alpha)\right]\mathrm{d}\nu. \tag{4}$$

Of course, the complex amplitude $V(x, y, z)$ to the right of the object at $z \geqslant 0$ has to obey the wave equation, in accordance with

$$\Delta_3 V(x, y, z) + k^2 V(x, y, z) = 0, \tag{5}$$

where

$$\Delta_3 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \qquad k = 2\pi/\lambda. \tag{6}$$

This is accomplished by adding a z-dependent term in the exponent of Eq. (4), to yield

$$V(x, y, z) = \int \tilde{u}_0(\nu)\exp\left(\frac{2\pi i}{\lambda}\{x\lambda\nu + y\sin\alpha + z[1 - (\lambda\nu)^2 - \sin^2\alpha]^{1/2}\}\right)\mathrm{d}\nu. \tag{7}$$

Now we replace in the root the $1 - \sin^2\alpha$ with $\cos^2\alpha$, pull the factor $\cos\alpha$ out of the root, and place the y-dependent term in front of the integral, to yield

$$V(x, y, z) = \exp\left(\frac{2\pi i y \sin\alpha}{\lambda}\right)\int \tilde{u}_0(\nu) \times \exp\left(2\pi i\left\{x\nu + \frac{z\cos\alpha}{\lambda} \times \left[1 - \left(\frac{\lambda\nu}{\cos\alpha}\right)^2\right]^{1/2}\right\}\right)\mathrm{d}\nu. \tag{8}$$

The integral represents a 2D wave $u(x, z)$ of the form

$$u(x, z) = \int \tilde{u}_0(\nu) \times \exp\left(\frac{2\pi i}{\Lambda}\{x\Lambda\nu + z[1 - (\Lambda\nu)^2]^{1/2}\}\right)\mathrm{d}\nu = V(x, 0, z), \tag{9}$$

where Λ is the effective wavelength, given by

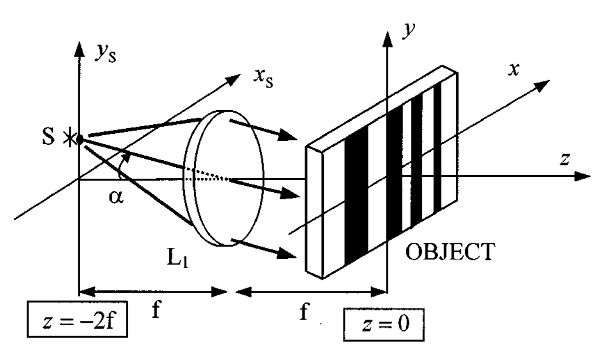

Fig. 2. Generic optical configuration of coherent Flatland optics.

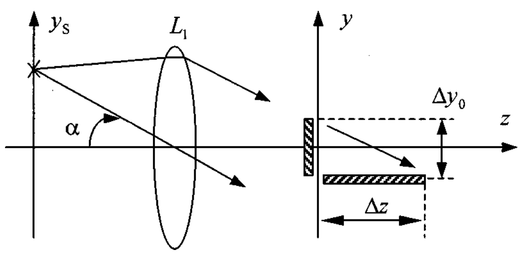

Fig. 3. Size of the Flatland experiment.

$$\Lambda = \lambda/\cos\alpha. \tag{10}$$

The complex amplitude $u(x, z)$ of Eq. (8) obeys the 2D wave equation

$$\Delta_2 u(x, z) + k_2{}^2 u(x, z) = 0, \tag{11}$$

where

$$\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}, \qquad k_2 = \frac{2\pi}{\Lambda} = \frac{2\pi\cos\alpha}{\lambda}. \tag{12}$$

The wavelength Λ in this 2D wave equation depends of course on the wavelength λ of the monochromatic point source but also on the angle α, which is controlled by the y coordinate of the point source. Hence it is apparently easy to manipulate the effective wavelength Λ. A vertical shift of the source S will do it.

Some additional remarks: We may write $u(x, z)$ as a 2D Fourier integral:

$$u(x, z) = \int\int \tilde{\tilde{u}}(\nu, \rho)\exp[2\pi i(\nu x + \rho z)]\mathrm{d}\nu\mathrm{d}\rho. \tag{13}$$

The 2D Fourier transform $\tilde{\tilde{u}}(\nu, \rho)$ is nonzero only on the Ewald circle:

$$\nu^2 + \rho^2 = (1/\Lambda)^2 = \cos^2\alpha/\lambda^2. \tag{14}$$

This follows directly on insertion of Eq. (13) into the wave equation [Eq. (11)]. The limiting spatial frequency at the boundary between plane waves and evanescent waves is

$$\nu_L = 1/\Lambda = \cos\alpha/\lambda \leqslant 1/\lambda. \tag{15}$$

The intensity distribution $I(x, z) = |u(x, z)|^2$ can be recorded by a photographic plate located, for example, at the plane $y = -\Delta y_0/2$, which touches the lower edge of the object $u_0(x)\mathrm{rect}(y/\Delta y_0)$ (see Figs. 2 and 3). The maximal length $\Delta z = \Delta y_0/\tan\alpha$ of the photographic plate follows from simple geometrical considerations, as illustrated in Fig. 3. A holographic reference wave has to be added if one wants to record the complex amplitude $u(x, z)$. This inductive derivation of the central equations [Eqs. (10) and (11)] is supplemented by a deductive derivation in the Conclusions.

3. SIMPLE PROCEDURE FOR VARYING THE FLATLAND WAVELENGTH WITHOUT CHANGING THE LIGHT SOURCE

As we have seen, the effective wavelength Λ in Flatland depends on the angle α and hence on the y coordinate y_S of the point source. We may vary the effective wavelength simply by moving the light source up and down along the y axis (Fig. 2). The relevant relations are collected in Eq. (16) and illustrated in Fig. 2 as

$$\Lambda = \frac{\lambda}{\cos\alpha}; \qquad \frac{1}{\cos\alpha} = (1 + \tan^2\alpha)^{1/2}; \qquad \tan\alpha = \frac{y_s}{f}. \tag{16}$$

Together this yields

$$\Lambda/\lambda = [1 + (y_S/f)^2]^{1/2} \approx 1 + y_S{}^2/(2f^2). \tag{17}$$

A red shift of 1% requires that $y_S/f = 1/7$. The illumination angle α ought to be 60° if one desires to increase the effective wavelength by a factor of 2. The corresponding source location would be $y_S = f\sqrt{3}$, which represents a fairly wide angle (see Fig. 4). Fortunately, a wide-angle lens is not needed. A collimator followed by a prism will suffice, as shown in Fig. 5. This is equivalent to bending the z axis of the configuration somewhere between the collimating lens L_1 and the object.

Now we want a particular wavelength Λ within the range between $z = 0$ and $z = z_P$, and another wavelength $\Lambda' = \lambda/\cos\alpha'$ at $z = z_P$. This can be accomplished by inserting a prism at $z > z_P$ (Fig. 6). As shown in Fig. 6, the tilt angle α' behind the prism is larger than the original angle α. Hence the new wavelength Λ' behind the prism is larger than the original Λ. One can produce the opposite effect, $\Lambda' < \Lambda$, simply by inverting the wedge direction of the prism. We conclude that a passive 3D component such as a prism can change the wavelength Λ of Flatland.

Thus instant second-order harmonic generation as well as subharmonic generation can be performed effectively without employing nonlinear optics. These statements refer to wave optics but not to quantum optics: $\Lambda' = \Lambda/2$; $\omega' = \omega$.

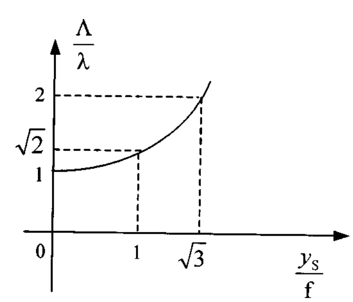

Fig. 4. Dependence of the effective wavelength Λ on the y coordinate y_S.

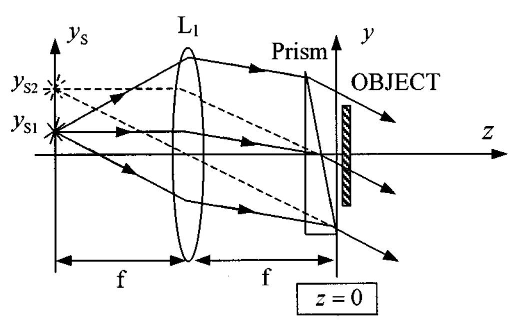

Fig. 5. Increasing the illumination angle by use of a prism. With the help of the prism, the practical point source at any y_{S1} is equivalent to a virtual source at y_{S2}.

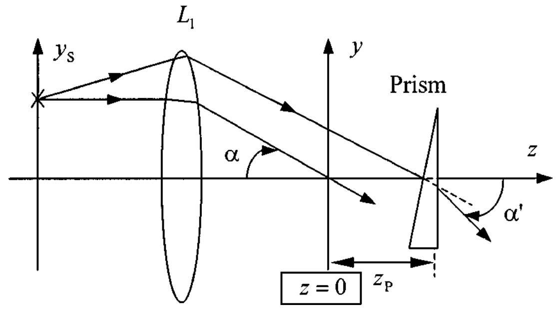

Fig. 6. Prism arrangement for obtaining different wavelengths Λ within the different ranges $z < z_P$ and $z > z_P$.

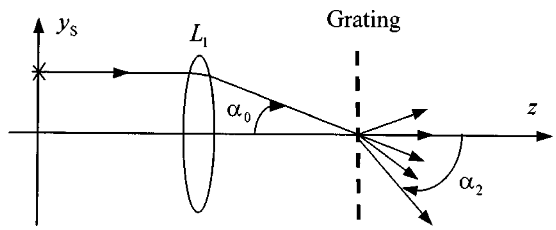

Fig. 7. Deflection of an incoming plane wave by a grating.

So far, we have used a prism for deflecting an incoming plane wave, either before the object as shown in Fig. 5 or somewhere behind the object as shown in Fig. 6. Now we replace the prism with a grating whose grooves are oriented parallel to the x axis, as shown in Fig. 7. The former tilt angle α_0 is now converted into a multitude of angles, as

$$\sin \alpha_m = \sin \alpha_0 + m \frac{\lambda}{D}. \tag{18}$$

The corresponding Flatland wavelengths are

$$\Lambda_m = \frac{\lambda}{(1 - \sin^2 \alpha_m)^{1/2}} \approx \lambda_0 \left(1 + \frac{m^2 \lambda^2}{2D^2} \right), \tag{19}$$

or, expressed in reciprocal wavelengths, as

$$\frac{1}{\Lambda_m} = \frac{1}{\lambda}(1 - \sin^2 \alpha_m)^{1/2} = \frac{1}{\lambda} \cos \alpha_m . \tag{20}$$

Now, the grating as a tool of the 3D man did convert the Flatland wavelength λ into a discrete set of coexisting 2D waves. Wavelength shifts that are even more chaotic would occur if the grating $GR(y)$ were to be replaced by an aperiodic mask $M(y)$.

The important point to remember is that the wave optics of Flatland (x, z) can be easily manipulated by a y-dependent action, initiated by an (x, y, z)-spaceland person, such as us. The Flatland wavelength Λ can be changed by passive optical components from 3D spaceland. The Flatland wavelength Λ is always longer than the spaceland wavelength λ, but the process of enlarging Λ is reversible.

4. FOURIER OPTICS IN FLATLAND

A. The Exact and the Paraxial Wave Equation

In Section 2 we found the exact wave equation for Flatland optics, as

$$\Delta_2 u(x, z) + {k_2}^2 u(x, z) = 0, \tag{21}$$

with

$$\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}, \qquad k_2 = \frac{2\pi}{\Lambda} = k \cos \alpha. \tag{22}$$

A particular primitive solution is a plane wave propagating in the z direction $\exp(ik_2 z)$. It is something like a spatial carrier frequency among the other tilted plane waves,

$$\exp\{ik_2[x\Lambda\nu + z\sqrt{1 - (\Lambda\nu)^2}]\}, \tag{23}$$

which also belong to the wave field $u(x, z)$ in Eq. (9). Hence it makes sense to separate the complex amplitude $u(x, z)$ from the carrier. That leads us to the modified complex amplitude

$$v(x, z) = u(x, z)\exp(-ik_2 z). \tag{24}$$

With such a substitution, the wave equation [Eq. (21)] assumes the form given by

$$\Delta_2 v(x, z) + 2ik_2 \frac{\partial v(x, z)}{\partial z} = 0. \tag{25}$$

This modified wave equation is still exact.

Now we introduce the paraxial approximation

$$\frac{\partial^2 v(x, z)}{\partial z^2} \ll \frac{\partial^2 v(x, z)}{\partial x^2}. \tag{26}$$

The result is the paraxial wave equation, of the form

$$\frac{\partial^2 v(x, z)}{\partial x^2} + 2ik_2 \frac{\partial v(x, z)}{\partial z} = 0; \qquad k_2 = \frac{2\pi}{\Lambda}. \tag{27}$$

This is sometimes called the optical Schrödinger equation.

B. Light Propagation in Free Plane

To be consistent with the Flatland concept, we use the term free plane (2D) instead of free space (3D). But the laws of free propagation are very similar in Flatland, except for the nonexisting y coordinate and for the modified wavelength $\Lambda = \lambda/\cos \alpha$. We begin by expressing free propagation in the Flatland frequency domain. From Eq. (9) we deduce

$$\int u(x, z)\exp(-2\pi i \nu x)\mathrm{d}x = \tilde{u}(\nu, z) = \tilde{u}_0(\nu)\exp[ik_2 z\sqrt{1 - (\Lambda\nu)^2}]. \tag{28}$$

The corresponding statement for the modified complex amplitude $v = u \exp(-ik_2 z)$ is

$$\tilde{v}(\nu, z) = \tilde{v}_0(\nu)\exp\{ik_2 z[\sqrt{1 - (\Lambda\nu)^2} - 1]\}. \tag{29}$$

The paraxial approximation thereof is based on $\sqrt{1 - (\Lambda\nu)^2} - 1 \approx -(\Lambda\nu)^2/2$, given by

$$\tilde{v}(\nu, z) = \tilde{v}_0(\nu)\exp(-i\pi\Lambda z\nu^2). \tag{30}$$

Free-plane propagation can apparently be described as a frequency filtering operation. The filter function $\exp\{ik_2 z[\sqrt{1-(\Lambda\nu)^2}-1]\}$ is a pure phase factor. Hence it is possible to refer to this effect as planar frequency dispersion.

Filtering in the frequency domain corresponds to a convolution in the Flatland domain, such as

$$v(x,z) = v_0(x) * p(x;z). \tag{31}$$

To find the point-spread function $p(x;z)$, which describes free-plane propagation, we insert

$$\tilde{v}_0(\nu) = \tilde{u}_0(\nu) = \int v_0(x')\exp(-2\pi i\nu x')\mathrm{d}x'$$

into

$$v(x,z) = \int \tilde{v}_0(\nu)\exp(ik_2\{x\Lambda\nu + z[\sqrt{1-(\Lambda\nu)^2}-1]\})\mathrm{d}\nu. \tag{32}$$

Equation (32) allows us to extract the point-spread function as

$$p(x;z) = \int \exp(ik_2\{x\Lambda\nu + z[\sqrt{1-(\Lambda\nu)^2}-1]\})\mathrm{d}\nu. \tag{33}$$

The integral is easy to solve for the case of the paraxial approximation, which is characterized by

$$\sqrt{1-(\Lambda\nu)^2}-1 \approx -(\Lambda\nu)^2/2, \tag{34}$$

to yield

$$p(x;z) \approx \exp\left(-i\frac{\pi}{4}\right)\exp\left(i\pi\frac{x^2}{\Lambda z}\right) \Big/ \sqrt{\Lambda z}. \tag{35}$$

As is evident, the point-spread function has the form of a cylindrical wave in 3D space.

C. Talbot Effect

Talbot[8] found in 1836 that if a diffracting object $v_0(x)$ is laterally periodic in x, then the wave field $v(x,z)$ behind the object is longitudinally periodic in z. Some recent advances of Talbot applications are reported by Klaus *et al.*,[9] whose paper also provides a convenient survey of the Talbot literature.

We now want to show the existence of the Talbot effect in Flatland, with $\Lambda = \lambda/\cos\alpha$ as the effective wavelength. The derivation is based on the paraxial wave equation and its free-plane solution

$$v(x,z) = \int \tilde{v}_0(\nu)\exp\left\{ik_2\left[x\Lambda\nu - \frac{z(\Lambda\nu)^2}{2}\right]\right\}\mathrm{d}\nu. \tag{36}$$

Talbot's assumption about the object was periodicity,

$$v_0(x) = \sum_{(m)} A_m\exp(2\pi i m\nu_0 x); \qquad \nu_0 = \frac{1}{D}, \tag{37}$$

where D is the lateral period. The equivalent statement in the frequency domain is

$$\tilde{v}_0(\nu) = \sum_m A_m\delta(\nu - m\nu_0). \tag{38}$$

We insert Eq. (38) into Eq. (36) to obtain

$$v(x,z) = \sum_{(m)} A_m\exp(2\pi i m\nu_0 x)\exp[-i\pi z(\Lambda m\nu_0)^2]. \tag{39}$$

The z-dependent factor can be written as

$$\exp(-2\pi i m^2 z/z_P). \tag{40}$$

This factor is apparently periodic in z, with a period z_P

$$z_P = \frac{2}{\Lambda{\nu_0}^2} = \frac{2D^2}{\Lambda} = \frac{2D^2}{\lambda}\cos\alpha = z_T\cos\alpha, \tag{41}$$

where $z_T = 2D^2/\lambda$ is the classical Talbot length in spaceland. The period z_P can be varied simply by changing the illumination angle.

D. Montgomery Effect

Montgomery[10] wanted to show that lateral periodicity is sufficient but not necessary for the longitudinal periodicity of the wave field $v(x,z)$. Hence he started with the assumption of a longitudinal periodic wave field:

$$v(x,z) = \sum_{(n)} B_n(x)\exp\left(2\pi i n\frac{z}{z_P}\right). \tag{42}$$

We insert this assumption into the paraxial Flatland wave equation [Eq. (27)], which yields

$$\sum_{(n)} \exp\left(2\pi i n\frac{z}{z_P}\right)\left[\frac{\mathrm{d}^2B_n(x)}{\mathrm{d}x^2} - B_n(x)\frac{4\pi n k_2}{z_P}\right] = 0. \tag{43}$$

The ordinary differential equation is satisfied by

$$B_n(x) = B_n(0)\exp[2\pi i x\nu_0\sqrt{|n|}\,\mathrm{sgn}(n)]. \tag{44}$$

Hence the set of all objects that generate longitudinal periodicity of the wave field $v(x,z)$ is

$$v(x,0) = \sum_{(n)} B_n(x) = v_0(x). \tag{45}$$

The input signal $v_0(x)$ would be strictly periodic if $B_n(0)$ were nonzero only if $|n|$ were a quadratic integer 0, 1, 4, 9,... such that $\sqrt{|n|}$ integer. However, Eq. (44) also allows frequencies such as $v_0\sqrt{2}$, $v_0\sqrt{3}$, $v_0\sqrt{5}$,... . Again, the longitudinal period z_P is shorter than the classical Talbot length by a factor $\cos\alpha$.

5. POLYCHROMATIC OPTICS IN FLATLAND

A. Achromatization of the Wave Equation

So far, we have used a monochromatic point-source illumination, whose location is defined by the angle α. This illumination generated the Flatland wavelength Λ, given by

$$\Lambda = \frac{\lambda}{\cos\alpha} = \Lambda(\lambda,\alpha). \tag{46}$$

Now we want many source wavelengths λ, each starting at a different source location α, such that all λ generate the same (fixed) Flatland wavelength $\bar{\Lambda}$, as

$$\Lambda(\lambda,\alpha) = \bar{\Lambda}. \tag{47}$$

This will be accomplished if the source distribution is

$$S(\lambda, \alpha) = S_0(\lambda)\delta[\lambda - \bar{\Lambda}\cos\alpha(\lambda)]. \tag{48}$$

The needed distribution is illustrated in Fig. 8(a). The different λ contributions [Fig. 8(b)] do not interact in any way, of course. But they all implement the same wave equation [Eq. (11)] in Flatland with the common Flatland wavelength $\bar{\Lambda}$. Hence the intensity contributions are the same for all wavelengths of the source, given by

$$|v(x, z;\ \lambda)|^2 = |v(x, z)|^2. \tag{49}$$

The overall polychromatic intensity distribution takes the spectral distribution of the source into account by

$$I(x, z) = |v(x, z)|^2 \int S_0(\lambda)\mathrm{d}\lambda. \tag{50}$$

This is obviously an incoherent summation. Yet all different monochromatic contributions (λ, at $\cos\alpha = \lambda/\bar{\Lambda}$) obey exactly the same Flatland wave equation [Eqs. (11) and (21)] or the modified wave equation [Eq. (25)],

$$\Delta_2 v(x, z) + 2i\bar{k}_2 \frac{\partial v(x, z)}{\partial z} = 0, \tag{51}$$

where $\bar{k}_2 = 2\pi/\bar{\Lambda}$.

B. Achromatic Talbot Effect

As we have seen, Talbot's effect occurs when light from a periodic object propagates in free space (x, y, z) or in free plane (x, z). In the latter case, the longitudinal period is $z_P = 2D^2/\Lambda$. In Subsection 5.A we saw how the Flatland wavelength Λ could be made to be the same for all wavelengths λ of the source.

It is possible to employ a combination of diffractive and refractive dispersion, as indicated in Fig. 9, where the light is emitted by a polychromatic point source. The

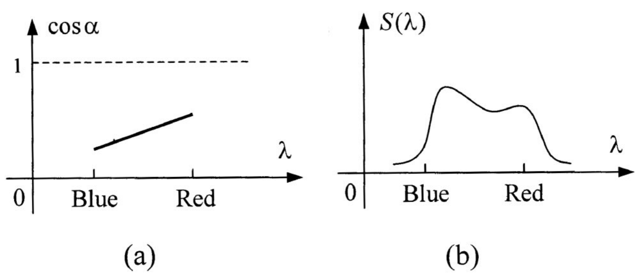

Fig. 8. Source distributions for achromatization: (a) cosine of the deflecting angle as a function of wavelength, (b) spectral distribution.

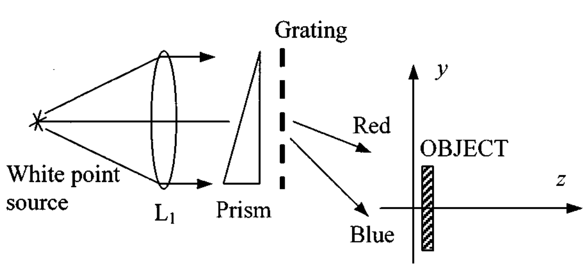

Fig. 9. Combination of diffractive and refractive dispersions for approximating the condition for Flatland achromatization.

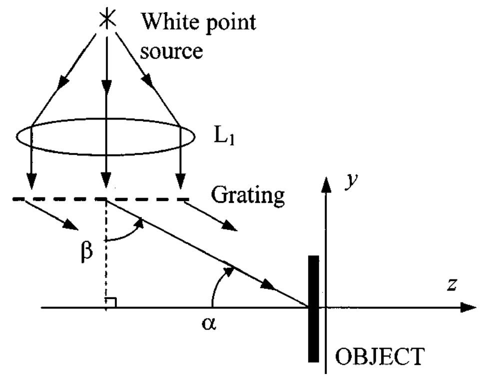

Fig. 10. Possible configuration for an exact Flatland achromatization.

grating period D, together with the wedge angle and the inherent refractive dispersion, probably provide enough free parameters for approximating the condition for Flatland achromatization [Eq. (47)]. An exact achromatization is possible if the main direction is bent by 90°, as shown in Fig. 10. Such bending leads to

$$\sin\beta = \lambda/D, \tag{52a}$$

$$\alpha + \beta = \pi/2, \tag{52b}$$

$$\cos\alpha = \cos(\pi/2 - \beta) = \sin\beta = \lambda/D. \tag{52c}$$

According to Eqs. (52) the cosine α is directly proportional to the 3D wavelength λ. Hence the 2D wavelength is constant $\lambda/\cos\alpha = \bar{\Lambda} = D$. In other words, the grating period D is the achromatic Flatland wavelength. The fundamentals of the achromatic Talbot effect are described elsewhere in more detail in research that we intend to publish.[7]

6. CONCLUSIONS

We have presented a two-dimensional version of scalar wave optics, which we called Flatland optics because of its conceptual kinship with the science fiction story "Flatland." Let us briefly summarize the fundamentals in a more compact and deductive manner. The basic assumption was separability of the 3D wave field $V(x, y, z)$:

$$V(x, y, z) = u(x, z)w(y), \tag{53}$$

with

$$w(y) = \exp\left(\frac{2\pi i}{\lambda} y \sin\alpha\right) \tag{54}$$

such that

$$\frac{\mathrm{d}^2 w}{\mathrm{d}y^2} = -\left(\frac{2\pi\sin\alpha}{\lambda}\right)^2 w(y). \tag{55}$$

With these assumptions one obtains a 2D wave equation

$$\Delta_2 u(x, z) + {k_2}^2 u(x, z) = 0, \tag{56}$$

with

$$\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}, \qquad k_2 = \frac{2\pi}{\Lambda} = k\cos\alpha. \tag{57}$$

The Flatland wave equation [Eq. (56)] is a factor of the 3D wave equation

$$\Delta_3 V(x, y, z) + k^2 V(x, y, z) = w(y)[\Delta_2 u(x, z) + k_2{}^2 u(x, z)] = 0. \quad (58)$$

The effective wavelength Λ is related to the genuine wavelength λ as

$$\Lambda = \frac{2\pi}{k_2} = \frac{\lambda}{\cos\alpha}. \quad (59)$$

The remarkable feature of this equation is the ability to modify the effective wavelength Λ simply by changing an illumination angle α. A corresponding effect would be much more complicated in 3D optics.

The Fourier optics in Flatland contains a large variety of effects, for example, those named after Talbot and Montgomery. We saw that it is possible to let the 2D optics behave as if it were perfectly monochromatic, although the genuine light source emits white light. We showed also that monochromatic 3D light can behave like polychromatic light in Flatland.

7. EPILOGUE

Readers of this paper may be amused, amazed, or skeptical. For example, in the context of Figs. 11 and 12, where monochromatic light is converted into polychromatic light, reversibly, one might wonder, how *real* is this wavelength Λ? Three answers and an Einstein quotation will settle this issue, we hope.

If light moves from air into glass, it changes its wavelength, reversibly. The temporal frequency is not changed. Our factor $\cos\alpha$ is what the refractive index n is for optics within glass.

"Reality" in physics can be attested, if something is observed, or even measured. A wavelength is measured by the deflection angle β, caused by a grating with a period D. Hence the reality of our Flatland wavelength Λ is ascertained if a diffraction experiment yields

$$D \sin\beta = \Lambda. \quad (60)$$

Another experiment would consist of the interference between two tilted waves. Suppose that we know the tilted angle 2γ and that we observe a fringe period F; then the wavelength follows as

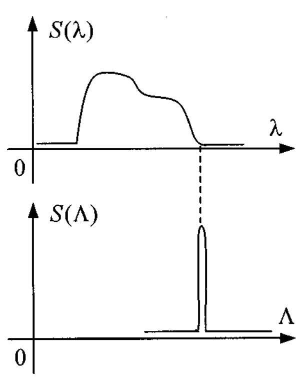

Fig. 11. Conversion of polychromatic 3D light into monochromatic Flatland light.

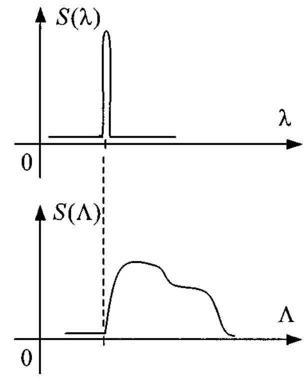

Fig. 12. Conversion of monochromatic 3D light into polychromatic Flatland light.

$$2F \sin\gamma = \Lambda. \quad (61)$$

As theoreticians we know that a wavelength is something that appears at a particular place in a wave equation. If that wave equation stands on solid ground, and if a certain factor is $(k_2)^2$, then

$$\frac{2\pi}{k_2} = \Lambda. \quad (62)$$

So far the "reality" of our Flatland optics is based only on the wave nature of light. Does the quantum nature cause any difficulty? We quote Einstein, who responded to the question "What is light?" by saying, "Light can be described by different models, as photons, as waves, or as rays. Light behaves like Voltaire, the French genius, 250 years ago. He was born as a conservative Catholic, he lived as a liberal Protestant, and he returned to Catholicism when he faced the end of his life. Light is born as photons. It travels through space as a wave, and it dies finally as a photon." This quotation, which is not verbatim, will, we hope, support the acceptance of Flatland optics.

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