

Final Exam

Discrete Structures 2

8 May 2023

Name: _____
(please write legibly)

Question	Topic	Value
1	(functions)	/24
2	(induction)	/25
3	(induction)	/24
4	(graphs)	/25
5	(counting)	/25
6	(counting)	/25
7	(graphs/counting)	/25
8	(graphs)	/25
9	(counting)	/25
10	(asymptotics)	/30
11	(proofs)	/25
Total		/275

There are a few rules:

- (1) You are not allowed to use outside online resources. No outside help (e.g., from a classmate, a friend, online search, documents on your own laptop, etc.) will be tolerated. Any attempt to obtain help or information about the exam will be reported.
- (2) **You are not allowed to use headphones during the exam.** Your phone should be in your backpack.
- (3) You should only have a pencil (or pen), an eraser, and a pencil sharpener with you on the table at the time of the exam. Everything else should be safely packed in your backpack and not to be used at any time during the exam.
- (4) You were told to attend to the bathroom before the exam starts: you will not be allowed to leave the room during the first hour of the exam (unless you have a doctor's note to indicate otherwise).

A few pieces of advice:

- (1) Read the questions carefully and try the tracing exercises on draft paper before you answer on your exam copy.
- (2) Pay careful attention to the instructions written in the exam.

Please write legibly and in a structured manner: keep in mind that what you write needs to be read. Answers that are unreadable or hard to follow will not receive full credit.

1. Select the smallest size set that represents the domain and range of each of the following functions:

Function	Domain	Range		
$f(x) = x \bmod 3$	_____	_____	(A) \mathbb{Z}	(H) $\{0\}$
$g(x) = 3 \bmod ((x \bmod 2) + 4)$	_____	_____	(B) $\mathbb{Z}^{\geq 1}$	(K) $\{1\}$
$h(x) = 3x^2 + \pi x + \sqrt{2}$	_____	_____	(C) $\mathbb{Z}^{\geq 0}$	(M) $\{2\}$
$\ell(x) = \lfloor x \rfloor$	_____	_____	(D) \mathbb{R}	(N) $\{3\}$
$e(x) = x $	_____	_____	(E) $\mathbb{R}^{\geq 1}$	(O) $\{0, 1\}$
$r(x) = x \bmod 4 + \frac{3}{2}$	_____	_____	(F) $\mathbb{R}^{\geq 0}$	(P) $\{0, 1, 2\}$
			(G) \emptyset	(Q) $\{0, 1, 2, 3\}$

2. Complete the proof below: We will prove by *weak* induction on n that

$$\sum_{i=1}^n f_i = f_{n+2} - 1$$

(where f_k is the k -th fibonacci number).

Base cases ($n = 1$ and $n = 2$):

$$\sum_{i=1}^1 f_i = f_1 = 1 = 2 - 1 = f_3 - 1$$

and

$$\sum_{i=1}^2 f_i = f_1 + f_2 = 1 + 1 = 2 = 3 - 1 = f_4 - 1.$$

Inductive case ($n \geq 3$):

We will assume the inductive hypothesis holds for $n - 1$, that is $\sum_{i=1}^{n-1} f_i = f_{n+1} - 1$.
(note you will likely use direct prove to complete the rest of the inductive case)

3. We want to prove the following by some type of induction, answer the questions below.

*For $x \in \mathbb{Z}^{>0}$ that are powers of two,
the closed form solution of the recurrence relation $T(x) = 3T(\frac{n}{2}) + 4$ is $\Theta(3^{\log n})$*

(a) What type of induction will we use? (i.e. “weak”, strong, structural, etc) Why?

(b) What value of x will be the base case?

(c) What is the *inductive hypothesis*? (not the whole inductive step)

4. We know that a tree is a connected acyclic graph. Prove or disprove:

Any graph $G = \langle V, E \rangle$ with $|E| = |V| - 1$ must be a tree.

5. How many permutations are there of SUMMER?

6. You’re taking a classics class this semester that required you to buy y books, but only x of them fit in your backpack at once. What formula describes the number of combinations you must consider when choosing what to bring to campus on any given day?

7. Thinking about the graphs quiz, assume we have a bipartite graph $G = \langle L \cup R, V \rangle$, where $|L| = |R| = |V| = n$, meaning each node has exactly one connection.

How many different graphs are there that satisfy this condition? (Another way to think of it, given fixed L and R how many different sets V are there such that $|V| = n$.)

8. Prove or disprove the following statement:

Given a tree $T = \langle V, E \rangle$, there must exist a root $r \in V$ which has at most two children.

9. You're writing a method to find the pair of elements in an array of size k that have the highest product. How many products do you need to calculate (in the naive implication that just computes all pair's products)? (Note, this answer should be an exact number, **not** in O, Θ, Ω notation).

10. You're working in a team to construct an program for an assignment. Your two colleagues made modules that you need to use and they reported the following to you:

- Method A has a worst-case running time of $O(n)$
- Method B has a worst-case running time of $\Theta(\log n)$

You wrote a method that runs Method A $\frac{n}{2}$ times on inputs of size $2 \cdot n$, then runs Method B 10 times on inputs of size n . What can you conclude about the worst-case running time of your program as a whole?

11. Identify and describe the flaw in the following direct proof.

False Claim: $1=0$

Proof. Let $x, y \in \mathbb{Z}^{\geq 1}$ where $x = y$.

$x = y$	by assumption
$x^2 = x \cdot y$	multiply both sides by x
$x^2 y^2 = x \cdot y y^2$	subtract y^2 from both sides
$(x - y)(x + y) = y(x - y)$	algebra
$x + y = y$	divide both sides by $(x - y)$
$2y = y$	because $x = y$
$2 = 1$	divide by y , which is non-zero
$1 = 0$	subtract 1 from both sides

□