Homework 5

Discrete Structures 1

due: 13 April 2023, 8:00am

Your task for this homework will be to answer the following questions without using any calculating resources. Your responses should be submitted via blackboard by the due date above as a PDF (submissions in any other format will be returned to the user and a resubmissions will be requested). You are free to use whatever tools you would like to generate the response document: scanned hand-written paper, tablet generated hand-written, microsoft word (with this option, please use the equation editor to correctly format your responses), LaTeX, etc. Your TA, IA, and Instructor are available to help during their designated office hours or via email (note that emails sent during non-business hours may not be responded to until the next working day).

1. Let F denote the set of all functions $f : \mathbb{R} \to \mathbb{R}$ taking real numbers as input and producing real numbers as output. (For one example, $plusOne(x) = x{+}1$ is a function $plusOne : \mathbb{R} \to \mathbb{R}$, so $plusOne \in F$.) Determine the truth of the following propositions, and justify your answer.
   (a) $\forall c \in \mathbb{R}\,[\exists f \in F : f(0) = c]$
   (b) $\exists f \in F\,[\forall c \in \mathbb{R} : f(0) = c]$
   (c) $\forall c \in \mathbb{R}\,[\exists f \in F : f(c) = 0]$
   (d) $\exists f \in F\,[\forall c \in \mathbb{R} : f(c) = 0]$

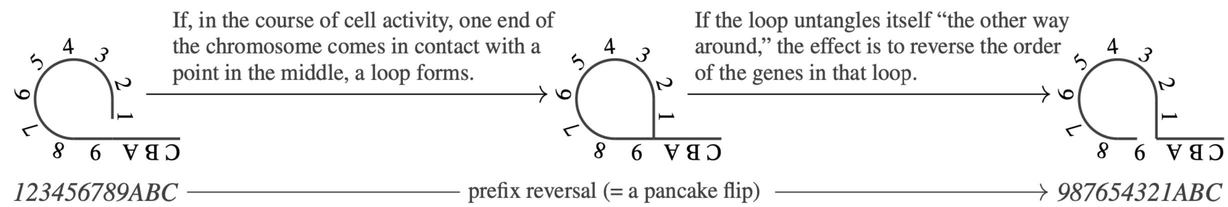
2. Let $P \in \{0,1\}^{n \times m}$ be a 2-dimensional array of the pixels of a black-and-white image: for every x and y, the value of $P[x,y] = 0$ if the $\langle x, y \rangle$-th pixel is black, and $P[x,y] = 1$ if it's white. Translate these statements into predicate logic:
   (a) Every pixel in the image is black
   (b) There is at least one white pixel
   (c) Every row has at least one white pixel
   (d) There are never two consecutive white pixels in the same column

3. According to Definition 2.32 from the book, a partition of a set S is a set $\{A_1, A_2, ..., A_k\}$ of sets such that (i) $A_1, A_2, ..., A_k$ are all nonempty; (ii) $A_1 \cup A_2 \cup \cdots \cup A_k = S$; and (iii) for any i and $j \neq i$, the sets A_i and A_j are disjoint. Formalize this definition using nested quantifiers and basic set notation.

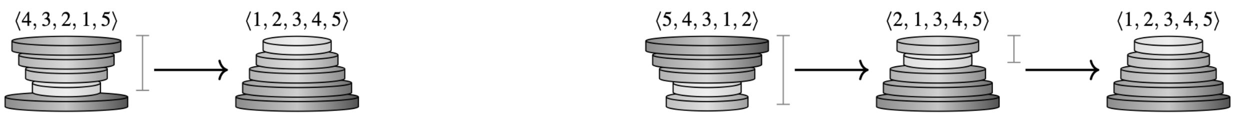
4. Consider the "maximum" problem: given a vector of numbers, return the maximum element of that vector:

   Input: A vector $A = \langle a_1, a_2, ...a_n \rangle$, where each $a_i \in \mathbb{Z}$.
   Output: An integer $x \in \mathbb{Z}$ such that ...

   Complete the formal specification for this problem by finishing the specification for the output.



(a) A prefix reversal, in which some prefix of the genome is reversed, as in $\langle \underline{3, 2, 1}, 4, 5\rangle \to \langle \underline{1, 2, 3}, 4, 5\rangle$.



(b) The pancake-flipping problem: you are given a stack of pancakes, which you want to arrange in order of size by repeatedly flipping some number of pancakes at the top of the pile, using as few flips as possible.

Figure 1: Genome rearrangements: prefix reversals and the pancake-flipping problem.

5. A classic topic of study for computational biologists is genomic distance measures: given two genomes, we'd like to report a single number that represents how different those two genomes are. These distance computations are useful in, for example, reconstructing the evolutionary tree of a collection of species. Consider two genomes A and B of bacterium. Label the n genes that appear in A's chromosome, in order, as $\pi_A = \langle 1, 2, ..., n\rangle$. The same genes appear in a different order in B—say, in the order $\pi_B = \langle r_1, r_2, ...r_n\rangle$. A particular model of genomic distance will define a specific way in which this list of numbers can mutate; the question is to find a minimum-length sequence of mutations to explain the difference between the orders π_A and π_B. One type of biologically motivated mutation is the prefix reversal, shown in Figure 1a. This model defines what's called the pancake-flipping problem (coincidentally, the subject of the lone academic paper with Bill Gates as an author). See Figure 1b.

   (a) You are given a sequence of pancake radii $\langle r_1, r_2, ..., r_n\rangle$, listed from top to bottom, where $\{_r1, r_2, ..., r_n\} = \{1, 2, ..., n\}$ (but not necessarily in order). Give a fully quantified logical expression expressing the condition that the given pancakes are sorted.

   (b) Again, you are given a sequence of pancake radii $\langle r_1, r_2, ..., r_n\rangle$, listed from top to bottom. Give a fully quantified logical expression expressing the condition that the given pancakes can be sorted with exactly one flip.

   > It may be useful to think about two parts separately, one part is sorted in increasing order, and one in decreasing order (before and after the point where it is flipped)

6. Let S be an arbitrary nonempty set and let P be an arbitrary binary predicate. Decide whether the following statements are always true (for any P and S), or whether they can be false. Prove your answers.

   (a) $[\exists y \in S : \forall x \in S : P(x, y)] \to [\forall x \in S : \exists y \in S : P(x, y)]$

   (b) $[\forall x \in S : \exists y \in S : P(x, y)] \to [\exists y \in S : \forall x \in S : P(x, y)]$