Midterm Exam

Discrete Structures 2

9 March 2023, 9am

Name: _____
(please write legibly)

Question	Topic	Value
1	(modulus)	/20
2	(modulus)	/20
3	(counting)	/60
4	(counting)	/70
5	(induction)	/120
6	(induction)	/60
6	(induction)	/80
7	(proof)	/50
	bonus	
Total		/480

There are a few rules:

(1) You are not allowed to use outside online resources. No outside help (e.g., from a classmate, a friend, online search, documents on your own laptop, etc.) will be tolerated. Any attempt to obtain help or information about the exam will be reported.

(2) **You are not allowed to use headphones during the exam.** Your phone should be in your backpack.

(3) You should only have a pencil, an eraser, and a pencil sharpener with you on the table at the time of the exam. Everything else should be safely packed in your backpack and not to be used at any time during the exam.

(4) You were told to attend to the bathroom before the exam starts: you will not be allowed to leave the room during the first hour of the exam (unless you have a doctor's note to indicate otherwise).

A few pieces of advice:

(1) Read the questions carefully and try the tracing exercises on draft paper before you answer on your exam copy.

(2) Pay careful attention to the instructions written in the exam.

Please write legibly and in a structured manner: keep in mind that what you write needs to be read. Answers that are unreadable or hard to follow will not receive full credit.

1. For a given $x \in \mathbb{Z}^{\geq 1}$, what is minimum value of $\underline{x \bmod 20}$? What is the maximum?

2. For a given $\ell \in \mathbb{Z}^{\geq 1}$ and $y \in \mathbb{Z}^{\geq 0}$, what is the relationship between the values of $\underline{\ell \bmod y}$ and $\underline{(\ell + y) \bmod y}$?

3. For the first decade or so of Twitter's existence, a tweet was a sequence of at most 140 characters. (This length restriction was loosened in 2017.) Assuming there are 256 valid characters that can appear in each position, how many distinct tweets are possible? (no need for a calculator, you can leave the number unreduced)

4. One of your relatives was given a piece of paper with the password to a wireless access point that was written as follows: a154bc0401011. But they couldn't tell from this handwriting whether each "1" was 1 (one), ℓ (ell), or `I` (eye); or whether "0" was 0 (zero) or `O` (oh). How many possible passwords would she have to try before having exhausted all of the possibilities? (again, no need for a calculator, you can leave the number unreduced)

5. For the following questions, use the following task:

   We want to prove by **strong induction** on n that, for every integer $n \geq 4$, it is possible to make n dollars using only two- and five-dollar bills. (That is, prove that any integer $n \geq 4$ can be written as $n = 2a + 5b$ for some integer $a \geq 0$ and some integer $b \geq 0$.)

   (a) For what value(s) of n would we need to prove as the base case(s):

   $n =$

   (b) Provide the proof(s) of the base case(s) noted in part 5a:

   (c) What would be the inductive hypothesis (remember this is **strong** induction)?

6. We want to prove by induction that $2^n \leq n!$ for large enough $n \in \mathbb{Z}^{\geq 1}$, find the base case for this problem.[1] (That is, find the smallest value such that the inequality starts to become true.)

7. Complete the proof below: We will prove by *weak* induction on n that

$$\sum_{i=1}^{n} f_i = f_{n+2} - 1$$

(where f_k is the k-th fibonacci number).

Base cases ($n = 1$ and $n = 2$):

$$\sum_{i=1}^{1} f_i = f_1 = 1 = 2 - 1 = f_3 - 1$$

and

$$\sum_{i=1}^{2} f_i = f_1 + f_2 = 1 + 1 = 2 = 3 - 1 = f_4 - 1.$$

Inductive case ($n \geq 3$):

We will assume the inductive hypothesis holds for $n - 1$, that is $\sum_{i=1}^{n-1} f_i = f_{n+1} - 1$.
(note you will likely use direct prove to complete the rest of the inductive case)

8. Prove or disprove the following: if xy and x are rational, y must also be rational.

[1] Remember that $n! = 1 \cdot 2 \cdot 3 \cdot ... \cdot n$

9. BONUS:

   (a) $|\mathbb{Z}| =$

   (b) $|\mathbb{Q}| =$

   (c) $|\mathbb{R}| =$

   (d) True or False: $|\mathbb{Z}| = |\mathbb{R}|$

   (e) True or False: $|\mathbb{Q}| \leq |\mathbb{R}|$