1.3 확률집합함수 ~ 1.4 조건부확률과 확률적 독립성

Thm 1.3.4 $\quad 0 \le P(C) \le 1 \qquad \forall C \in \mathcal{B}$

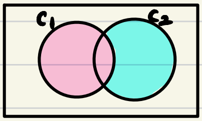
(pf) $\quad \phi \subset C \subset \mathcal{C}$

$$P(\phi) \le P(C) \le P(\mathcal{C}) \underset{\text{by (ii)}}{=} 1$$

Thm 1.3.5

$$P(C_1 \cup C_2) = P(C_1) + P(C_2) - P(C_1 \cap C_2)$$

(pf)



$$C_1 \cup C_2 = \underbrace{C_1 \cup (C_1^c}_{\text{disjoint}} \cap C_2)$$

$$P(C_1 \cup C_2) = P(C_1) + \cancel{P(C_1^c \cap C_2)} \quad \cdots (*)$$

$$C_2 = (C_1 \cap C_2) \cup (C_1^c \cap C_2)$$

$$P(C_2) = P(C_1 \cap C_2) + \cancel{P(C_1^c \cap C_2)} \quad \cdots (**)$$

$(*) - (**) \Rightarrow$

$$P(C_1 \cup C_2) - P(C_2) = P(C_1) - P(C_1 \cap C_2)$$

Remark 1.3.1 (inclusion - exclusion formula)

$$P(C_1 \cup C_2 \cup C_3) = P_1 - P_2 + P_3$$

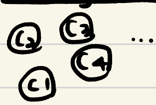
$$\begin{cases} P_1 = P(C_1) + P(C_2) + P(C_3) \\ P_2 = P(C_1 \cap C_2) + P(C_1 \cap C_3) + P(C_2 \cap C_3) \\ P_3 = P(C_1 \cap C_2 \cap C_3) \end{cases}$$

In general,

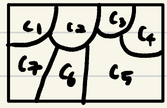
$$P(C_1 \cup C_2 \cup \cdots \cup C_k) = P_1 - P_2 + P_3 - \cdots + (-1)^{k-1} P_k$$

P_i : sum of probability of all possible intersection of i sets

$C_1, C_2, \cdots$: <u>mutually exclusive</u> if $C_i \cap C_j = \phi, \ \forall i \ne j$



Mutually exclusive sets $C_1, C_2, \cdots$ are called
<u>exhaustive</u> if $\bigcup_{i=1}^{\infty} C_i = \mathcal{C}$ $\quad \mathcal{C}$



$$\lim_{n\to\infty} C_n := \begin{cases} \bigcup_{n=1}^{\infty} C_n & C_i\text{'s : increasing seq} \quad C_1 \subset C_2 \subset C_3 \subset C_4 \subset \cdots \\ \bigcap & C_i\text{'s : decreasing 일 때} \quad C_1 \supset C_2 \supset C_3 \supset \cdots \end{cases}$$

쓸 수 있다

seq of real #'s
$\lim_{n\to\infty} a_n$

$$C_n = \{x : 0 < x < 1 - \frac{1}{n+1}\}$$
$$C_n = \{x : 0 < x < 1 + \frac{1}{n}\}$$

Thm 1.3.6

$\{C_n\}$: increasing

$\lim_{n\to\infty} P(C_n) := P(\lim_{n\to\infty} C_n)$

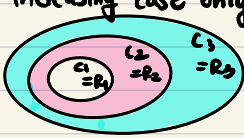
$\hookrightarrow P(\bigcup_{n=1}^{\infty} C_n)$

countable union이라고 약속함

$\{C_n\}$: decreasing

$\lim_{n\to\infty} P(C_n) = P(\lim_{n\to\infty} C_n)$

(pf) increasing case only



$\circledast \sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{j=1}^{n} a_j$

i.e. let $R_1 = C_1 \quad R_n = C_n \cap C_{n-1}^c \quad n = 2, 3 \quad R_j$'s : disjoint

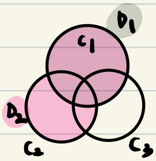
$P(\lim_{n\to\infty} C_n) = P(\bigcup_{n=1}^{\infty} C_n) = P(\bigcup_{n=1}^{\infty} R_n) = \sum_{n=1}^{\infty} P(R_n) \leftarrow$ countable additivity

$= \lim_{n\to\infty} \sum_{j=1}^{n} P(R_j) = \lim_{n\to\infty} \{P(R_1) + \sum_{j=2}^{n} P(R_j)\}$

$= \lim_{n\to\infty} \{P(C_1) + \sum_{j=2}^{n} (P(C_j) - P(C_{j-1}))\}$

$= \lim_{n\to\infty} \{\cancel{P(C_1)} + \cancel{P(C_2)} - \cancel{P(C_1)} + P(C_3) - \cancel{P(C_2)} + \cancel{\cdots} + P(C_n) - \cancel{P(C_{n-1})}\}$

$= \lim_{n\to\infty} P(C_n)$



Thm 1.3.7 (Boole's inequality)

$\{C_n\}$: arbitrary seq of sets

$$P\left(\bigcup_{n=1}^{\infty} C_n\right) \le \sum_{n=1}^{\infty} P(C_n)$$

(pf) Let $D_n := \bigcup_{i=1}^{n} C_i$ then $\{D_n\}$: increasing

Since $D_j = D_{j-1} \cup C_j \quad j = 2, 3, \cdots$

$$P(D_j) = P(D_{j-1}) + P(C_j) - \underbrace{P(D_{j-1} \cap C_j)}_{\ge 0 \text{ by (i)}}$$

$$\le P(D_{j-1}) + P(C_j)$$

i.e. $P(D_j) - P(D_{j-1}) \le P(C_j) \quad \cdots (*)$

$$P\left(\bigcup_{n=1}^{\infty} C_n\right) = P\left(\bigcup_{n=1}^{\infty} D_n\right) = \lim_{n\to\infty} P(D_n)$$

$$= \lim_{n\to\infty} \left[P(D_1) + \sum_{j=2}^{n} \{P(D_j) - P(D_{j-1})\}\right]$$

$$\le \lim_{n\to\infty} \left[P(D_1) + \sum_{j=2}^{n} P(C_j)\right] \text{ by } (*)$$

$$= \lim \left[\sum_{j=1}^{n} P(C_j)\right] = \sum_{n=1}^{\infty} P(C_n)$$

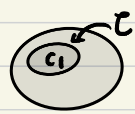
1.4 Conditional Prob & Independence

조건부확률 독립성

$C_1, C_2 \subset \mathcal{C}$

The conditional probability of C_2 given C_1

$$P(C_2 \mid \underset{\substack{\uparrow \\ \text{given}}}{C_1}) := \frac{P(C_2 \cap C_1)}{P(C_1)} \quad \text{if } P(C_1) \neq 0$$



(i) $P(C_2 \mid C_1) \ge 0$: non-neg

(ii) $P(C_1 \mid C_1) = 1$: normality

(iii) $C_2, C_3, C_4 \cdots$: mutually exclusive

$$P\left(\bigcup_{i=2}^{\infty} C_i \mid C_1\right) = \sum_{i=2}^{\infty} P(C_i \mid C_1)$$

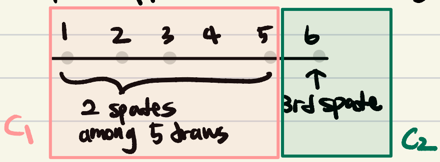
: countable add

Ex 1.4.1. (clover, spade, ◇, ♡) 1~13 (1, 2, ⋯, 10, J, Q, K)

draw cards from a deck at random w/o rep (비복원)

$13 \times 4 = 52$

P(3rd spade appear at the 6th draw)



C_1: two spades in the first five draws

C_2: 3rd spade in the 6th draw

Need to compute $P(C_1 \cap C_2)$

$$P(C_1 \cap C_2) = \underline{P(C_2 \mid C_1)} \cdot P(C_1)$$

$$= \frac{(13-2)}{(52-5)} \cdot \frac{\binom{13}{2}\binom{39}{3}}{\binom{52}{5}} \quad \text{총 가짓수}$$

$$= \underline{0.064}$$

$$P(C_1 \cap C_2) = P(C_1)\, P(C_2 \mid C_1)$$

Joint를 marginal과 conditional의 곱으로 표현한다

$$P(\underbrace{C_3}_{A} \mid \underbrace{C_i \cap C_2}_{B}) = \frac{P(\underbrace{C_3}_{A} \cap \underbrace{C_1 \cap C_2}_{B})}{P(\underbrace{C_1 \cap C_2}_{B})}$$

typo

$$\Rightarrow P(C_1 \cap C_2 \cap C_3) = P(C_1 \cap C_2) \cdot P(C_3 \mid C_1 \cap C_2)$$

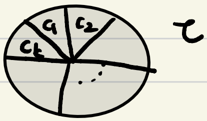
$$= P(C_1)\, P(C_2 \mid C_1)\, P(C_3 \mid C_1 \cap C_2)$$

In general,

$$P(C_1 \cap C_2 \cap \cdots \cap \cdots) = P(C_1)\, P(C_2 \mid C_1)\, P(C_3 \mid C_1 \cap C_2)\, P(C_4 \mid C_1 \cap C_2 \cap C_3) \cdots$$

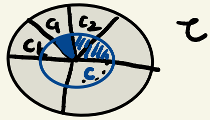
Bayes Theorem

- $C_1, C_2, \cdots C_k$: mutually exclusive & exhaustive



- $P(C_i) > 0 \quad \forall i = 1, \cdots, k$

$$\underbrace{P(C_j \mid C)}_{\text{difficult, unrealistic}} = \frac{\overbrace{P(C_j)\, P(C \mid C_j)}^{\text{easy, realistic}}}{\sum_{i=1}^{k} P(C_i)\, P(C \mid C_i)}, \quad j = 1, \cdots, k$$



Note that

$$C = (C\cap C_1)\cup(C\cap C_2)\cup\cdots\cup(C\cap C_k)$$

disjoint

$$\begin{aligned}P(C) &= P(C\cap C_1)+\cdots+P(C\cap C_k)\\ &= P(C_1)P(C\mid C_1)+\cdots+P(C_k)P(C\mid C_k)\\ &= \sum_{i=1}^{k} P(C_i)P(C\mid C_i)\end{aligned}$$

Now, $$P(C_j\mid C) = \frac{P(C_j\cap C)}{P(C)} = \frac{P(C_j)\cdot P(C\mid C_j)}{\sum_{i=1}^{n} P(C_i)P(C\mid C_i)}$$

Remark 1.4.1.

관측 observation 하는 행위
before event

- $P(C_j)$: prior probability (사전 확률)
- $P(C_j\mid C)$: posterior (사후)

Def 1.4.1

C_1, C_2 : (Statistical) independent if

$$P(C_1\mid C_2) = P(C_1)$$

In general,

$C_1, C_2, \cdots, C_n$: independent

iff every collection of k events $(2\le k\le n)$

$$P(C_{i1}\cap\cdots\cap C_{ik}) = \prod_{j=1}^{k} P(C_{ij})$$

cf.

C_i and C_j : pairwise indep if

$$P(C_i\cap C_j) = P(C_i)P(C_j)$$

mutual indep $\Longrightarrow$ pairwise indep
mutual indep $\not\Longleftarrow$ pairwise indep

$P(A\cap B) = P(A)P(B)$
$P(A\cap C) = P(A)P(C)$
$P(B\cap C) = P(B)P(C)$
but $P(A\cap B\cap C) \neq P(A)P(B)P(C)$