제1장 확률과 확률분포

Ch1. Probability & Distributions

1.1 Introduction

- Statistical (random) experiment
  ↗ 임의
  실험 전에는 100% 확신을 가지고 예측할 수 X

- Variation (변동)
  ↕
  deterministic

- sample space 표본공간 $\mathcal{C}$ ← collection
- event : subset of samp. space
  사건 A, B, C

Ex 1.1.1. Tossing a coin

$C = \{H, T\}$
head tail

Ex 1.1.2. Tossing two dice (red & white)

$C = \{(1,1), (1,2), \cdots, (6,6)\}$

Ex 1.1.3

$C = \{\text{sum seven when tossing 2 dice}\}$
$= \{(1,6), (2,5), \cdots, (6,1)\}$

Remark 1.1.1. Two types of probability
① Relative frequency : objective
② Personal or subjective

1.2 Set theory

Def 1.2.1 subset $C_1 \subset C_2$

Def 1.2.2. null (empty) set ϕ

Def 1.2.3 union ("or")

$$C_1 \cup C_2 \cup C_3 \cup \cdots \boxed{:=} \bigcup_{k=1}^{\infty} C_k$$

Ex. 1.2.1.

$$C_k = \left\{x : \frac{1}{k+1} \le x \le 1\right\}$$

$$\Rightarrow \bigcup_{k=1}^{\infty} C_k \qquad \text{i.e. } \forall k, \frac{1}{k+1} > 0$$

$$= \{x : 0 < x \le 1\} \leftarrow \text{“}\varepsilon\text{-}\delta\text{” operation}$$

Def 1.2.4. intersection ("and")

$$C_1 \cap C_2 \cap C_3 \cap \cdots := \bigcap_{k=1}^{\infty} C_k$$

Ex. 1.2.2.

$$C_k = \{x : 0 < x < \tfrac{1}{k}\}$$

$$\Rightarrow \bigcap_{k=1}^{\infty} C_k = \phi \quad \text{because}$$

$\forall x$ ($x>0$) 임의의 x에대하여 $\exists k > 0$ (k 존재) s.t. $\frac{1}{k} < x$

Def 1.2.5. complement (여집합)

$\boxed{C}^{c}$ complement, $\bar{C}$ bar

set

$\bar{C} = \phi$

표본공간의 여집합

· function
- → point function : domain point
- → set function : set point

Ex 1.2.3

point func. $f(x) = 2x$

$x = 2 \Rightarrow f(2) = 2 \times 2 = 4$

Set func. $f(A) = \#$ of positive integers in A

$A = \{x : -\infty < x < 6\} \Rightarrow f(A) = 5$

$\int_C f(x)\,dx$: Riemann integral

$\int_{C_2}\int_{C_1} f(x,y)\,dx\,dy$

$\underbrace{\int \cdots \int}_{}$ $f(x_1, \cdots, x_k)\,dx_1 \cdots dx_k$

k-fold integration (cont.)

$\sum_{a=-\infty}^{\infty} g(a)$

$\sum_{a=-\infty}^{\infty}\sum_{b=-\infty}^{\infty} g(a,b)$

$\sum \cdots \sum g(x_1, \cdots, x_k)$ summation (disc.)

Ex. 1.2.4.

$$Q(C) = \int_C \cdots \int 1 \cdot dx_1 \cdots dx_n$$

$$C = \{(x_1, \cdots, x_n) : 0 \le x_1 \le x_2 \le \cdots \le x_n \le 1\}$$

$$Q(C) = \int_0^1 \int_0^{x_n} \cdots \int_0^{x_3} \boxed{\int_0^{x_2} 1 \, dx_1} \cdots dx_n$$

$$[x_1]_0^{x_2} = x_2 \rightarrow \left[\frac{x_2^2}{2}\right]_0^{x_3} = \frac{x_3^2}{2} \rightarrow \cdots$$

1.3. The Probability Set Function

Def 1.3.1. (σ-field)

장

$\mathcal{B}$: collection of subsets of $\mathcal{C}$ (sample space)

$\mathcal{B}$ is called a <u>σ-field</u> if

(i) $\phi \in \mathcal{B}$

(ii) $C \in \mathcal{B} \Rightarrow C^c \in \mathcal{B}$ (closed under complement)

(iii) $C_1, C_2, \cdots \in \mathcal{B} \Rightarrow \bigcup_{k=1}^{\infty} C_k \in \mathcal{B}$ (closed under countable union)

countable union

1-1 correspondence w/ integers

Ex. of σ-field

① $\mathcal{B} = \{\phi, C, C^c, \mathcal{C}\}$
(i) (ii)

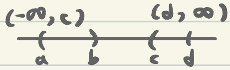
② $\mathcal{B}$: power set
= collection of all subsets of $\mathcal{C}$

③ $\mathcal{B} = \bigcap_{i=1}^{\infty} \{\mathcal{E}_i : \mathcal{D} \subset \mathcal{E}_i,\ \mathcal{E}_i \text{ is a } \sigma\text{-field}\}$

: the smallest σ-field containing $\mathcal{D}$

: <u>σ-field generated by $\mathcal{D}$</u>

④ $\mathcal{U}$: set of all open intervals in R^1



$\Rightarrow$ σ-field generated by $\mathcal{U}$: **Borel σ-field**

Def 1.3.2 probability

$\mathcal{C}$: sample space

$\mathcal{B}$: σ-field on $\mathcal{C}$

P : real-valued function

치역(range)

convenience의 개념...

$\Rightarrow$ P : probability set function if (i) $P(C) > 0$, ${}^{\forall}C \subset \mathcal{C}$ non-negativity

(ii) $P(\mathcal{C}) = 1$ normality

(iii) $C_1, C_2, \cdots \in \mathcal{B}$

$C_i \cap C_j = \emptyset$, ${}^{\forall}i \neq j$ 서로 겹치는 부분 없으면

$$P\left(\bigcup_{i=1}^{\infty} C_i\right) = \sum_{i=1}^{n} P(C_i)$$

countable additivity

Thm 1.3.1.

$P(C) = 1 - P(C^c)$, ${}^{\forall}C \in \mathcal{B}$

(pf) $\mathcal{C} = C \cup C^c$. $C \cap C^c = \emptyset$

$P(\mathcal{C}) = P(C \cup C^c)$

$\underset{\uparrow \text{ by (ii)}}{1} = P(C) + P(C^c)$ by (iii)

Thm 1.3.2. $P(\emptyset) = 0$

(pf) Let $C = \emptyset$

Thm 1.3.3. $C_1 \subset C_2 \Rightarrow P(C_1) \leq P(C_2)$

(pf)

$C_2 = C_1 \cup (C_1^c \cap C_2)$

intersection $= \emptyset$

$P(C_2) = P(C_1) + \underbrace{P(C_1^c \cap C_2)}_{\geq 0 \text{ by (i)}}$ by (iii)