Divergence / Gauss theorem: AE 311

$$\oiiint_{CV} (\nabla \cdot V)\, d\forall = \oiint_{CS} V \cdot dS$$

$$\oiiint_{CV} \nabla\phi\, d\forall = \oiint_{CS} \phi\, d\vec{S}$$

$$\oiiint_{CV} \rho(\vec{v}\cdot\vec{\nabla})\vec{v}\, d\forall = \oiint_{CS} \rho(\vec{v}\cdot d\vec{S})\vec{v}$$

- Green's thm: $\oiint_{A} (\nabla \times V)\, d\vec{A} = \oint_{CS} V \cdot dS$

for a ~~PG~~ Calorically PG: for air

$C_p - C_v = R$

$C_p = \frac{\gamma R}{\gamma - 1}$

$C_v = \frac{R}{\gamma - 1}$

$\boxed{1\,bar = 10^5\,pa}$

at MSL (Standard sea level)

$T = 15°C$

$P = 1.01325 \times 10^5\,pa$

$\rho = 1.225\,kg/m^3$

$a = 340.17\,ms^{-1}$

- for air at standard condition:

$R = 287\,J\,kg^{-1}K^{-1}$

$1\,atm = 1.01 \times 10^5\,pa$.

$1\,mm$ of $H_2O \equiv 9.80665\,pa$

$PV^{\gamma}_{=Const} \Rightarrow T \overset{Const}{=} p^{\gamma-1/\gamma}$

SI System:

Force → N
Mass → kg
length → m
time → s
Temp → K
energy → J

English Engineering System

(lb) pound [1 lb = 4.448 N]
slug [1 slug = 14.594 kg]
foot (ft) [1 foot = 0.3048 m]
s
Rankine (°R) [$1°R = \frac{5}{9}K$]
ft-lb [1 ft-lb = 1.356 J]

universal gas constant,

$R = 8.314\,J\,mol^{-1}K^{-1}$
$= 0.0821\,L\,atm\,mol^{-1}K^{-1}$

Boltzmann constant,

$k_B = 1.381 \times 10^{-23}\,JK^{-1}$.

- $\frac{\gamma}{\gamma - 1} = 3.5$ for $\gamma = 1.4$

$\gamma_{air} = 1.4$

$R_{air} = \frac{R}{M_{air}} \approx 287\,J/(kg\cdot K)$

$M_{air} = 28.97\,g\,mol^{-1}$

- Speed of sound, a_0 at standard sea level condition $= 340.9\,ms^{-1} \equiv 1117\,ft\,s^{-1}$

Speed of sound (in air at 20°C 293K) $= 343\,ms^{-1}$
$\equiv 1235\,km/hr$.
$\equiv 767\,mph$.

1 mile = 1.61 km.

1 feet = 0.3048 m

$\boxed{60\,mi\,hr^{-1} = 88\,ft/s}$

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- High speed compressible flow ⤳ High energy flow.

↳ a study of application of Compressible flow involves a coupling of purely fluid dynamic fundamentals with the results of TD.

Compressible flow : ($Ma > 0.3$) or $\frac{d\rho}{\rho} \geq 5\%$

- Subsonic [$M_\infty < 0.8$]
- Transonic [$0.8 < M_\infty < 1.0$, $1.0 < M_\infty < 1.2$]
- Supersonic [$1.2 < M_\infty < \sim 5$]
- Hypersonic [$M_\infty > \sim 5$]

- Subsonic ($M<1$ everywhere)
- Supersonic ($M>1$ everywhere)

Sonic $M=1$

$M > 0.3$ ⤳ compressible flow.

- For most practical Aerodynamic shapes lift, L is generated mainly by surface pressure distribution. (Shear stress dist. makes a small contribution)

Lift, $L \approx y$ component of $[-\oint p\,d\vec{s}]$

Drag, D = x component of $[\underbrace{-\oint p\,d\vec{s}}_{\text{causes pressure drag.}} + \underbrace{\oint \tau_w\,d\vec{s}}_{\text{causes skin friction drag.}}]$

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Week-01 AE 311

- Kinematics: Study of motion of a body without any reference to Force & mass. exp: position, velocity, acc. etc.

- Dynamics: Study of motion & the way in which forces produce motion.

- Aerodynamics: → science concerned with the motion of air & of bodies moving through the air. one particular interest is to calculate: the forces & moments acting on the body.!
  - In fluid mechanics our interests lies on the fluid!

- compressible Aerodynamics: → Aerodynamics + TD

  ↑ We are dealing with high speed flows.

  ⇓

  large E_K

  conversion of E_K to thermal energy will be significant & therefore T & ρ will vary (We need to take into consideration Energy conservation)

  ↑

  TD comes into picture!

- In Compressible Aerodynamics we mostly deal with gases!
  - forms of matter: Solid, liquid, gas

    they differ in → intermolecular bonds, → shape & volume.

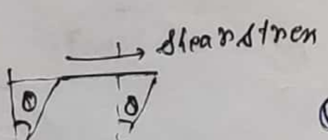
Liquid & gas: both are Fluid → a Fluid is something that flows
- they can't withstand shear forces No matter how small.
- they deform continuously under the action of shear stresses.

[However, they do resist being deformed or change in shape] ↑ viscosity.

A liquid has definite volume & is mostly incompressible

BUT

A gas acquires the volume of the container in which it is kept!



$$\mu = \frac{\text{shear}}{\text{strain rate}}$$

- <u>viscosity, μ</u> : a fluid property that represents the magnitude of fluid's resistance to deformation

(a transport property associated with momentum transfer).

<u>Consequence of viscosity:</u>

No - slip condition (Really true under continuum assumption)

- <u>viscosity:</u> → is measured by viscometer.
  → has microscopic connections. - "Transport" phenomena.

- <u>Transport of mass / Momentum / Energy :</u>

At microscopic level, Random motion of molecules transport mass, momentum & energy from one region to another.

associated transport properties: D, μ, k

- physical mechanism of viscosity is momentum transfer from high velocity region of fluid to low velocity region of fluid, due to random molecular motions!

same mechanism is responsible for mass & energy transfer.

$$\dot{m} = -D\frac{dc}{dy} \quad \text{Fick's law of diffusion.}$$

$$\tau = \mu\frac{du}{dy} \quad \text{Newton's law of friction.}$$

$$\dot{q} = -k\frac{dT}{dy} \quad \text{Fourier's law.}$$

~~A fluid~~

<u>viscous flows:</u>

effect of viscosity, thermal conduction & mass diffusion are important. } dissipative effects. → changes the entropy, S of the flow.

this is important in regions of large gradient of velocity, temp. & chemical composition.

<u>inviscid flows:</u>

dissipative effects are ignored!

| We will be dealing with Inviscid flows in this course! ★ |

- A Fluid can be described in two different ways:

i. Microscopic approach:

- Fluid is viewed as a collection of molecules separated by empty space, discontinuous at microscopic scales.
- Molecules move randomly & collide with each other & with surfaces
- Molecules are governed by the laws of dynamics or we use kinetic theory.
- In kinetic theory Boltzmann eqn or a simplified model eqn is solved to get the velocity distribution function.
- The macroscopic properties (P, T & V) can be obtained by sampling the particle - Statistical averaging!
- This approach enables microscopic interpretation of the bulk flow quantities!

ii. Continuum approach:

- on a scale much larger than the molecular distances (Mean Free path). fluid can be considered as continuous!
- rather than focusing on a particular particle we consider Continuous physical model comprising of a large no. of molecules.

↑ Fluid element

★ Associated mathematical models approximate the True physics!

- Different Method of Analysis in Gas Dynamics:

Macroscopic (continuum) method:
- NS-Fourier eqns based CFD
- Higher order model eqns e.g. Burnett eqn (extended hydrodynamics)
- Smoothed particle Hydrodynamics (SPH)

Mesoscopic (kinetic) method:
- Lattice Boltzmann method (LBM)
- Approximate Boltzmann eqn (BGK & Sakhov) solvers
- DSMC method.

Microscopic method: Molecular Dynamics (MD)

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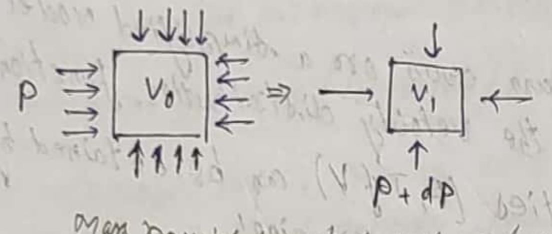
Introduction:

Incompressible flow → ρ constant (in time as well as space)

Compressible flow → ρ variable.

- Compressibility, $\tau = -\frac{dV}{V dp}$ → fractional change in the volume of a fluid element per unit change in pressure.

  ↳ is a fluid property



Mass remains same but volume decrease.

isothermal compressibility, $\tau_T = \left(-\frac{1}{V}\frac{dV}{dP}\right)_T$

[the process of compression is carried out at constant temp.]

Isentropic compressibility, $\tau_S = \left(-\frac{1}{V}\frac{dV}{dP}\right)_S$

[the compression process is carried out at constant entropy] ⇒ adiabatic process with no irreversibility

$\rho V = \text{const.}$

$$\frac{\partial \rho}{\rho} = -\frac{\partial V}{V} = -\frac{\partial \forall}{\forall}$$

$$\text{so, } \tau = -\frac{dV}{V dP} = \frac{\partial \rho}{\rho \partial P}$$

$$\Rightarrow \boxed{\partial P = \frac{1}{\tau}\frac{\partial \rho}{\rho}}$$

$$\Rightarrow \tau \partial P = \frac{\partial \rho}{\rho}$$

for steady inviscid flow with no body forces

$$\partial P = -\rho u \partial u \Rightarrow \frac{\partial \rho}{\rho} = -\tau \rho u \partial u$$

if $Ma < 0.3$ ~ Incompressible flow ← low speed flow of gases.

$Ma \geq 0.3$ ~ Compressible flow ← high speed flow of gases

- if $\frac{\partial \rho}{\rho} \geq 5\%$ ~ Compressible flow

[$\frac{d\rho}{\rho}$ becomes too large to be ignored]

- [$Ma > 0.3$ become significantly important after 1940s]

⇓ Bernoulli eqn is valid only for Incompressible Flows! *

Compressibility factor, $Z = \frac{PV}{RT}$

• Different flow regimes: (in compressible gas flows i.e. $Ma > 0.3$) with emphasis on distinguishing flow features.

i. Subsonic flow: [$M_\infty < 0.8$]

$Ma < 1$ everywhere in the flow.

- Smooth streamlines with continuously varying properties.
- The flow is forewarned of the presence of the body.

Free stream Mach number,

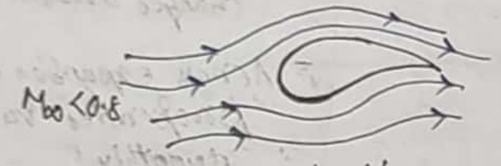
$$Ma_\infty = \frac{V_\infty}{a_\infty}$$

V_∞ → free stream velocity; a_∞ → speed of sound in uniform free stream

Local Mach number

$$Ma = \frac{V}{a}$$

V → local velocity vector at every pt along the flow



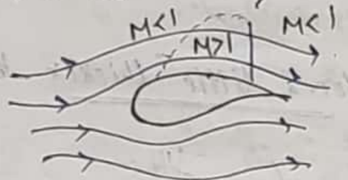
Ma<1 everywhere.

ii. Transonic flow: a. [$0.8 < M_\infty < 1.0$]

[M>1, M<1 somewhere]

($0.8 < M_\infty < 1.2$)

Shockwave.



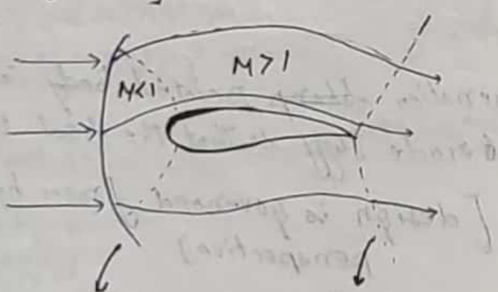
- a pocket of locally supersonic flow ($M>1$) is observed.
- Mixed flow (i.e. somewhere $Ma > 1$, somewhere $Ma \approx 1$, " $Ma < 1$)

in most cases they terminate with the formation of shockwaves across which there is discontinuous & severe change in flow properties

after the shock pressure ↑ abruptly !!

Flow separation ⇔ Shock induced stall

b. [$1.0 \le M_\infty < 1.2$]



TE shockwave + Bow shock.

Shock wave upstream of the LE

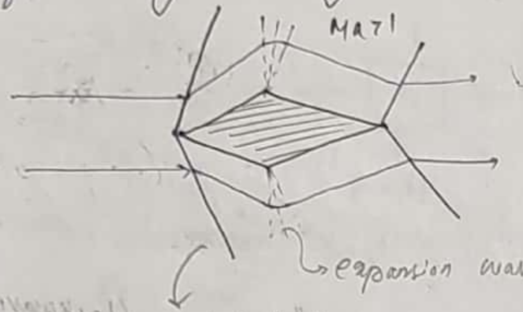
Mixed flow.

- Compressible potential eqn → Can be linearised for subsonic/ supersonic flow.
  - ↳ Can't be linearised for Transonic flow. (due to $\frac{1}{1-M_\infty^2} \to \infty$ when $M_\infty \to 1$)

iii. Supersonic flows: $[1.2 < M_\infty < \sim 5]$

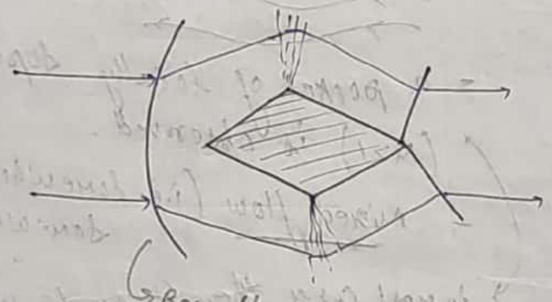
$Ma > 1$ everywhere in the flow.

- typically for supersonic flows we use diamond shaped bodies to generate lift not airfoil.

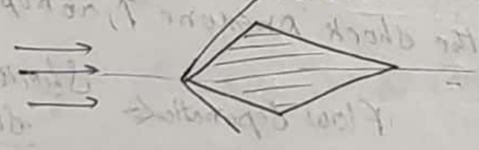


- Across oblique, Bow shock flow property changes drastically.
- Across expansion waves, properties vary smoothly!

- for a little thicker body or for smaller Ma:



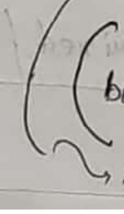
iv. Hypersonic flows: $[M_\infty > 5]$



- Flow properties changes explosively across the shock wave. $(P, T, \rho \uparrow)$
- Thin shock layer
  Thick B/L, vorticity generation - Crocco thm.

Shock appears closer to the surface and most likely it will be on attached shock

- Because of high heat generation ~~sharp~~ pointed body can melt therefore the body is made bluff so that the heat is spread out over the body. [design is governed from heat flux perspective]



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A brief review of TD:

- Ideal gas equation: $PV = nRT$ — n ~ no. of Moles, R ~ Universal Gas constant.

perfect gas (IMF forces are negligible) & volume of molecules negligible

$$\Rightarrow P\forall = RT$$ (specific volume)

$$\Rightarrow PV_M = RT$$ (Molar volume)

$$\Rightarrow P = \rho RT$$

$$\Rightarrow P = n' k_B T$$

$R = 8.314\ J/(mol.K)$

$R = \frac{\mathcal{R}}{M}$ ~ M Molecular Mass of gas.

$k_B = \frac{\mathcal{R}}{N_A} = m' R$ — m': Mass of an individual molecule; k_B: Boltzmann constant.

n' ~ ~~Boltz~~ number density (no. of molecules per unit volume)

- Gases behave "Ideally" at ~~low te~~ High Temp. & low pressure (where Temp. & pressure are wrt T_{cr} & P_{cr})

★ "Ideal gas" is a good assumption
→ at very low pressure ($P_R \ll 1$) regardless of temp.
→ at ~~very~~ high Temp. ($T_R > 2$) except when $P_R \gg 1$

$P_R = \frac{P}{P_{cr}}$, $T_R = \frac{T}{T_{cr}}$

- Vander Waals EOS:

$$\left(P + \frac{a}{V_M^2}\right)(V_M - b) = RT, \quad T_c = \frac{8a}{27Rb}$$

$$V_c = 3b$$

$$P_c = \frac{a}{27b^2}$$

$P_R = P/P_c$, $T_R = T/T_c$, $(V_M)_R = \frac{V_M}{V_C}$

Reduced form of Vanderwaals' EOS:

$$\boxed{\left(P_R + \frac{3}{V_R^2}\right)\left(V_R - \frac{1}{3}\right) = \frac{8}{3}T_R}$$

$V_R = (V_M)_R$

For O_2
$P_{cr} = 49.8\ atm$
$T_{cr} = 154.6\ K$

For N_2
$P_{cr} = 33.5\ atm$
$T_{cr} = 126.2\ K$

1st law of TD:

→ is an empirical result!

$$\Delta E_{system} = E_{in} - E_{out}$$

$\Delta E_{system} = \Delta U + \Delta E_K + \Delta PE$; $E_{in} - E_{out}$ = heat flow + work + E due to mass flow

$$E_{system} = \underline{E} + \underline{E_K + PE}$$

E: Internal energy [the sum of all microscopic form of energy]

$E_K + PE$ → Macroscopic form of energy [orderly movement of the molecules of the system]

- Random translational
- Rotational energy
- Vibrational energy
- electrons → potential, → kinetic
- Intermolecular forces.

Internal energy, $e(T, V)$ — for both real gas as well as chemically reacting mixture of perfect gas.

enthalpy per unit mass, $\underline{h} = \underline{e} + p\forall$

$h(T,p)$ $e(T,v)$

- for thermally perfect Gas
  (gas is NOT chemically reacting & IMFs are negligible)

  $\underline{h(T)}$ & $\underline{e(T)}$

  i.e. $C_p(T)$ & $C_v(T)$

- for calorically perfect Gas
  C_p, C_v assumed to be constant.
  [calorically perfect gas model is a good assumption for gas at low temp.]

- for a system ~~has~~ at rest, involving no mass flow:

$$de = \underset{\uparrow}{\delta q} + \underset{\uparrow}{\delta W_{in}}$$

heat supplied to the system. work done on the system

(for a given change in there are ∞ no. of processes (ways of heat & work addition))

If only pv work is involved then,

$$\delta w_{in} = -p d\forall$$

so,

$$\boxed{\begin{aligned} \delta q &= de + p d\forall \\ \delta q &= dh - \forall dp \end{aligned}}$$

$$\boxed{C_p = \left(\frac{\partial h}{\partial T}\right)_p \qquad C_V = \left(\frac{\partial e}{\partial T}\right)_\forall}$$

$$C_p - C_v = \left\{ p + \left(\frac{\partial e}{\partial \forall}\right)_T \right\} \left(\frac{\partial \forall}{\partial T}\right)_p$$

$$C_p - C_v = -T \left(\frac{\partial P}{\partial \forall}\right)_T \left(\frac{\partial \forall}{\partial T}\right)_p = \frac{\alpha^2 V T}{\tau}$$

- τ, Isothermal compressibility $= -\frac{1}{\forall}\left(\frac{\partial \forall}{\partial p}\right)_T$

Coeff. of Thermal expansion, $\alpha = \left(\frac{1}{\forall}\frac{\partial \forall}{\partial T}\right)_p$

[$C_p \geq C_v$
$\alpha > 0$ $\forall$ substance in all phase.
$C_p \to C_v$ as $T \to 0$]

for an ideal gas, $\underline{C_p - C_v = R}$ → valid for both calorically perfect & thermally perfect gas.
→ $\underline{\text{NOT}}$ valid for Real gas & chemically reacting gases

$C_p - C_v = R$ $\quad \frac{C_p}{C_v} = \gamma$

$C_p = \frac{\gamma R}{\gamma - 1}$ ★

$C_v = \frac{R}{\gamma - 1}$

Adiabatic, $\delta q = 0$

Reversible:
No dissipative phenomenon occurs [i.e. - No effect of viscosity
- No Thermal conductivity
- No mass diffusion]

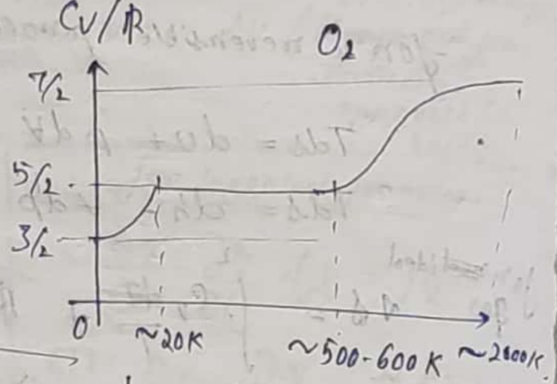
- Isentropic process is a reversible adiabatic process!

- for diatomic molecule or triatomic linear molecule:

$C_v = C_{v_{translational}} + C_{v_{rotational}} + C_{v_{vibrational}} + \cancel{C_{v_{electronic}}} \to 0$

$= 3 \cdot \frac{RT}{2} + 2 \cdot \frac{RT}{2} + f_{vib}(T)$

$$\boxed{C_v = \frac{5}{2} RT + f_{vib}(T)}$$



- for a diatomic gas like O_2, we can consider it to be calorically perfect if the working temp. $20K \leq T \leq 500K$.

- The vibrational ~~mode~~ dof gets excited only after gas temp. becomes sufficiently high!
Because of this reason the contribution of vibrational mode of energy to internal energy depends strongly on temp.

- 2nd law of TD:

energy has a quantity as well as a quality.

- Clausius inequality

$$\oint \frac{\delta q}{T} \leq 0.$$

2nd law of TD helps us to determine the direction of a particular process.

Entropy ⇝ is a measure of disorder.

- $$\Delta S_{system} = \int_1^2 \frac{\delta q}{T} + \underbrace{S_{gen}}_{\text{entropy generated!}}$$ (has to be ≥ 0)

$$\boxed{\Delta S_{total} = \Delta S_{system} + \Delta S_{surrounding}} \rightarrow \Delta S_{total} \geq 0$$

for a reversible process $S_{gen} = 0$.

- $$\left.\begin{aligned} Tds &= du + p\,dv \\ Tds &= dh - v\,dp \end{aligned}\right\}$$ 1st Tds & 2nd Tds relation. (1st & 2nd Gibbs equ^n)

for ~~real~~ ideal gas

$$\Delta s = \int_1^2 C_v \frac{dT}{T} + R \int_1^2 \frac{dv}{v}$$

[refer ESO201 notes]

$$\Delta s = \int_1^2 C_p \frac{dT}{T} - R \int_1^2 \frac{dP}{P}$$

- for isentropic process [adiabatic + reversible] $\Delta s = 0 \Rightarrow$

$$\left.\begin{aligned} &pv^{\gamma} = \text{const.} \\ &\equiv Tv^{\gamma-1} = \text{const} \\ &\equiv \frac{T^{\gamma}}{p^{\gamma-1}} = \text{const.} \end{aligned}\right]$$

$$\equiv \frac{P_2}{P_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}}$$

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Week-02

⊙ Governing eqn of Fluid Dynamics :

Fundamental physical principles:
- Mass is conserved.
- Force = mass * acc.
- Energy is conserved.

"flow model"
- the CV fixed in space with fluid moving through it. [Eulerian approach]
- the CV moving with the fluid such that the same particles are always within the CV. [Langrangian approach]
- Molecular approach.
- ∞tesimal fluid element approach.
  - fluid element of differential volume, dV (~~though~~ ~~that~~ dv is large enough for Continuum approach to hold).

NOTE:
in Book → Fixed CV approach
in lec. → ∞tesimal fluid element approach.

Few important formulas:

$$\frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t} + (\underset{\sim}{V}\cdot\underset{\sim}{\nabla})\eta$$

RTT: $$\frac{DN}{Dt} = \frac{\partial N}{\partial t} + \oiint_{CS} \rho\,\eta\,\underset{\sim}{V}\cdot d\underset{\sim}{A}$$

where $N = \iiint_{CV} \rho\,\eta\, d\forall$

if the cv is fixed in space.

divg. / Gauss theorem:

$$\iiint_{CV} \underset{\sim}{\nabla}\cdot\underset{\sim}{V}\, d\forall = \oiint_{CS} \underset{\sim}{V}\cdot d\underset{\sim}{S}$$

(I). Conservation of Mass [Mass can niether be created nor destroyed]

derivation using

i. CV fixed in space approach:

Rate of ↑ of mass inside CV = Net rate of mass flow into CV.

$$\Rightarrow \frac{\partial}{\partial t}\iiint_{CV} \rho\, d\forall = -\oiint_{CS} \rho\,\vec{V}\cdot d\vec{S}$$

$$= -\iiint_{CV} (\vec{\nabla}\cdot\rho\vec{V})\, d\forall$$

$$\Rightarrow \iiint_{CV}\left(\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\rho\vec{V}\right) d\forall = 0.$$

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V}) = 0 \iff \frac{D\rho}{Dt} + \rho(\vec{\nabla}\cdot\vec{V}) = 0}$$

ii. Using ∞tesimal fluid element approach:

$$\frac{D(\delta m)}{Dt} = 0 \Rightarrow \frac{D}{Dt}(\rho\,\delta V) = 0.$$

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$$\Rightarrow \left(\frac{D\rho}{Dt}\right)\delta V + \rho\left(\frac{D\delta V}{Dt}\right) = 0$$

$$\Rightarrow \left(\frac{D\rho}{Dt}\right) + \rho\frac{1}{\delta V}\frac{D\delta V}{Dt} = 0.$$

$\frac{1}{\delta V}\frac{D\delta V}{Dt}$: Volumetric Strain rate

$$\frac{1}{\delta V}\frac{D(\delta V)}{Dt} = \nabla\cdot \underset{\sim}{V}$$

$$\Rightarrow \frac{D\rho}{Dt} + \rho(\vec{\nabla}\cdot\vec{V}) = 0.$$

$$\Rightarrow \boxed{\frac{D\rho}{Dt} + \rho(\vec{\nabla}\cdot\vec{V}) = 0 \Leftrightarrow \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\underset{\sim}{V}) = 0}$$

Steady flow : $\frac{\partial\rho}{\partial t} = 0$

incompressible flow : $\frac{D\rho}{Dt} = 0$

(II) Conservation of Momentum : [force = mass $\times$ acc.]

derivation using :

i. material fluid element approach :

$$\frac{D}{Dt}(\vec{U}\delta m) = \vec{F}$$

$$\Rightarrow \delta m\frac{D\vec{U}}{Dt} + \vec{U}\cancel{\frac{D\delta m}{Dt}}^{\,0} = \vec{F}$$

$$\Rightarrow \rho\delta V\frac{D\vec{U}}{Dt} = \vec{F}$$

$$\Rightarrow \boxed{\rho\frac{D\vec{U}}{Dt} = \vec{f}}$$

$\vec{f} \sim$ ~~body~~ force acting on the body per unit volume

ii. Fixed CV in Space approach :

Rate of change of Momentum inside a CV = ~~Net rate of change of~~ momentum flowing into CV per unit time + ext. forces acting on the ~~cv~~ fluid elements ~~in CV~~.

$$\frac{\partial}{\partial t}\iiint_{CV}\rho U_i\,dV = -\oiint_{CS}\rho U_i\,\cancel{\vec{\nabla}\cdot\vec{U}}\,(\vec{U}\cdot d\vec{A}) + F_i$$

$$\Rightarrow \frac{\partial}{\partial t}\iiint_{CV}\rho U_i\,dV + \oiint_{CS}\rho U_i(\vec{U}\cdot d\vec{A}) = F_i$$

$$\Rightarrow \iiint_{CV}\frac{\partial}{\partial t}(\rho U_i)\,dV + \iiint_{CV}\vec{\nabla}\cdot(\rho U_i\vec{U})\,dV = F_i$$

$$\Rightarrow \oiiint_{CV} \left[\frac{\partial}{\partial t}(\rho u_i) + \vec{\nabla}\cdot(\rho u_i \vec{U}) \right] d\forall = F_i$$

$$\Rightarrow \oiiint_{CV} \left[\rho \frac{\partial u_i}{\partial t} + u_i \frac{\partial \rho}{\partial t} + u_i \vec{\nabla}\cdot(\rho \vec{U}) + \rho(\vec{U}\cdot\vec{\nabla}) u_i \right] d\forall = F_i$$

$$\cancel{\Rightarrow \oiiint_{CV} \left[-u_i(\vec{U}\cdot\vec{\nabla})\rho + \rho\frac{\partial u_i}{\partial t} + u_i \vec{\nabla}\cdot(\rho\vec{U}) + \rho(\vec{U}\cdot\vec{\nabla})u_i \right] d\forall = F_i}$$

~~applying mass conservation~~

$$\cancel{\frac{\partial \rho}{\partial t} = -}$$

$$\Rightarrow \oiiint_{CV} \left[\rho\frac{\partial u_i}{\partial t} - \underbrace{u_i \vec{\nabla}\cdot(\rho\vec{U})}_{\text{mass conservation applied}} + u_i \vec{\nabla}\cdot(\rho\vec{U}) + \rho(\vec{U}\cdot\vec{\nabla})u_i \right] d\forall = F_i$$

$$\left[\frac{\partial \rho}{\partial t} = -\nabla\cdot(\rho\vec{U}) \right]$$

$$\Rightarrow \oiiint_{CV} \left[\rho\frac{\partial u_i}{\partial t} + \rho(\vec{U}\cdot\vec{\nabla})u_i \right] d\forall = F_i$$

$$\Rightarrow \oiiint_{CV} \rho \frac{Du_i}{Dt}\, d\forall = F_i \Rightarrow \boxed{\rho\frac{Du_i}{Dt} = f_i}$$

$$\boxed{\rho\frac{D\vec{U}}{Dt} = \vec{f}}$$

- Forces acting on fluid element:
  - Body force: $\vec{f}_b$ (generally $= \rho\vec{g}$)
  - pressure force: $\vec{F}_p = -\oiint_{CS} p\, d\vec{A} = -\oiiint_{CV} (\vec{\nabla} p)\, d\forall$
  - viscous force:

$$\vec{F}_{viscous} = \oiint_{CS} \underbrace{\underline{\underline{\tau}}}_{\text{viscous stress tensor}} \cdot d\vec{A} = \oiiint_{CV} (\vec{\nabla}\cdot\underline{\underline{\tau}})\, d\forall$$

- Integral form of momentum conv. eqn

$$\boxed{\oiiint_{CV} \rho\frac{D\vec{U}}{Dt}\, d\forall = \oiiint_{CV} \vec{f}_b\, d\forall - \oiint_{CS} p\, d\vec{A} + \oiint_{CS} \underline{\underline{\tau}}\cdot d\vec{A}}$$

$$\oiiint_{CV} \left(\rho\frac{\partial \vec{U}}{\partial t}\, d\forall + \rho(\vec{U}\cdot\nabla)\vec{U} \right) d\forall = \oiiint_{CV} \vec{f}\, d\forall - \oiint p\, d\vec{A}$$

it contains body force + friction

- Differential form of momentum consv. eqn :

$$\rho \frac{D\vec{U}}{Dt} = \rho \vec{f}_b - \vec{\nabla} p + \vec{\nabla} \cdot \underline{\underline{\tau}}$$

~ Cauchy's eqn of motion [applicable to any flow]

force per unit volume

$\vec{f}_b$ ~ body force per unit mass

$\underline{\underline{\tau}}$ ~ viscous stress tensor.

- for isotropic medium & Newtonian fluid

$$\underline{\underline{\tau}} = 2\mu \underline{\underline{\varepsilon}} + \lambda (\vec{\nabla} \cdot \vec{U}) \underline{\underline{I}}$$

λ ~ represents fluid's resistant to get compressed / expanded.

$\underline{\underline{\varepsilon}}$ ~ strain rate tensor.

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

λ: second coeff. of viscosity

By invoking Stoke's hypothesis [valid only for mono atomic gases]

$$\lambda = -\frac{2}{3}\mu$$

$$\lambda + \frac{2}{3}\mu = \mu_b$$

μ_b : bulk viscosity ~ 0 in Stokes' hypothesis

~ becomes important for flow situations involving thermal non-equb [shocks, sudden expansion, sound absorption].

- Navier-Stokes' eqn :

$$\rho \frac{D\vec{U}}{Dt} = \rho \vec{f}_b - \vec{\nabla} p + \mu \nabla^2 \underset{\sim}{V}$$

for inviscid flow ($\mu = 0$) it reduces to

$$\rho \frac{D\vec{U}}{Dt} = \rho \vec{f}_b - \vec{\nabla} p$$

~ Euler eqn

III Conservation of Energy:

Specific total energy of fluid, $e_T = e + \frac{1}{2}U^2$

i. fixed CV in space approach:

Rate of rise of energy inside CV = Net rate of energy flowing into CV + Work done by pressure force, viscous force, body force

+ heat conduction term (Heat flowing into the system)

+ heat generation term (if there is any source of heat inside the CV)

Specific total energy,

$$e_T = e + \frac{1}{2}U^2$$

$$\frac{\partial}{\partial t}\iiint_{CV} \rho e_T \, dV = -\oint_{CS} \rho e_T (\vec{U}\cdot d\vec{A}) - \oint_{CS} p(\vec{U}\cdot d\vec{A}) + \oint_{CS} (\overline{\overline{\tau}}\cdot\vec{U})\cdot d\vec{A} + \iiint_{CV} (\vec{f}\cdot\vec{U})\,dV - \oint_{CS} \vec{q}\cdot d\vec{A} + \iiint_{CV} \rho q_g \, dV$$

viscous stress tensor

force per unit volume.

Heat flux vector.

heat generation per unit mass

$$\Rightarrow \frac{\partial}{\partial t}\iiint_{CV} \rho e_T \, dV + \oint_{CS} (\rho e_T + p)\,\vec{U}\cdot d\vec{A} = \oint_{CS} (\overline{\overline{\tau}}\cdot\vec{U})\cdot d\vec{A} + \iiint_{CV} (\vec{f}\cdot\vec{U})\,dV - \oint_{CS} \vec{q}\cdot d\vec{A} + \iiint_{CV} \rho q_g \, dV + \text{other}$$

$\dot{W}_{viscous}$ $\dot{W}_{shaft}$

$\dot{Q}$

Can be considered heat added

$$e_T + \frac{p}{\rho} = e + \frac{U^2}{2} + \frac{p}{\rho} = h + \frac{U^2}{2} = h_T \text{ (total enthalpy)}$$

So,

$$\boxed{\frac{\partial}{\partial t}(\rho h_T) + \vec{\nabla}\cdot \rho h_T \vec{U} = \vec{\nabla}\cdot(\overline{\overline{\tau}}\cdot\vec{U}) + \vec{f}\cdot\vec{U} - \vec{\nabla}\cdot\vec{q} + \rho q_g + \frac{\partial p}{\partial t}}$$

(Heat cond. diff...)

for inviscid flow there is no Thermal Conduction, diffusion & there is no viscous stress

$$\vec{\nabla}\cdot(P\vec{U}) = \underbrace{\vec{U}\cdot\vec{\nabla}P}_{\substack{\text{Rate of rise of}\\ \text{KE per unit}\\ \text{volume}}} + \underbrace{(\vec{\nabla}\cdot\vec{U})P}_{\substack{\text{Rate of rise of internal}\\ \text{energy per unit}\\ \text{volume}}}$$

$$\vec{\nabla}\cdot(\bar{\bar{\tau}}\cdot\vec{U}) = \underline{\vec{U}\cdot(\vec{\nabla}\cdot\bar{\bar{\tau}})} + \underbrace{(\bar{\bar{\tau}}\cdot\vec{\nabla})\vec{U}}_{\substack{\text{viscous dissipation}\\ \text{term}\\ \text{occurring due to friction}}}$$

Therefore

$$\boxed{\frac{\partial}{\partial t}(\rho h_T) + \vec{\nabla}\cdot(\rho h_T\vec{U}) = \vec{\nabla}\cdot(\bar{\bar{\tau}}\cdot\vec{U}) + \vec{f}\cdot\vec{U} - \vec{\nabla}\cdot\vec{q} + \rho q_g + \frac{\partial P}{\partial t}}$$

(the energy eqn can be split into two halves :

i. Thermal ~~eqn~~ energy eqn

$$\frac{\partial}{\partial t}(\rho h) + \vec{\nabla}\cdot(\rho h\vec{U}) = (\bar{\bar{\tau}}\cdot\vec{\nabla})\vec{U} - \vec{\nabla}\cdot\vec{q} + \rho q_g + \frac{\partial P}{\partial t}$$

ii. Kinetic energy eqn ↳ which reduces to momentum eqn

$$\rho\frac{D\vec{U}}{Dt} = \vec{\nabla}\cdot\bar{\bar{\tau}} + \vec{f}$$

in book,

$$\iiint \rho\dot{q}\,d\mathcal{V} - \oiint p\vec{V}\cdot d\vec{S} + \iiint_{\mathcal{V}} \rho(\underset{\sim}{f}\cdot\underset{\sim}{V})\,d\mathcal{V} = \iiint_{CV}\frac{\partial}{\partial t}\left[\rho\left(e+\frac{V^2}{2}\right)\right]d\mathcal{V} + \oiint_S \rho\left(e+\frac{V^2}{2}\right)\underset{\sim}{V}\cdot d\underset{\sim}{S}$$

$$\frac{\partial}{\partial t}\left[\rho\left(e+\frac{V^2}{2}\right)\right] + \underset{\sim}{\nabla}\cdot\left[\rho\left(e+\frac{V^2}{2}\right)\vec{V}\right]$$

$$= -\underset{\sim}{\nabla}\cdot(p\vec{V}) + \rho\dot{q} + \rho(\underset{\sim}{f}\cdot\underset{\sim}{V})$$

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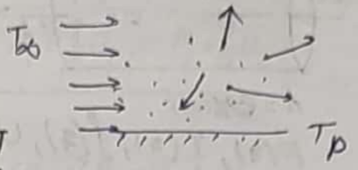
① Heat Conduction: $\boxed{\vec{q} = -K \underset{\sim}{\nabla} T}$ "(1)" fourier's relation (constitutive relation).

$$\vec{\nabla} \cdot \vec{q} = -K \nabla^2 T$$ (assuming $\underline{K}$ to be constant) thermal conductivity.

- No bulk motion
- for gases: random thermal motion of molecules

"Transport phenomenon"



$$\bar{c} = \sqrt{\frac{8RT}{\pi}}$$

$$\mu \approx \frac{1}{2} \rho \bar{c} \lambda$$

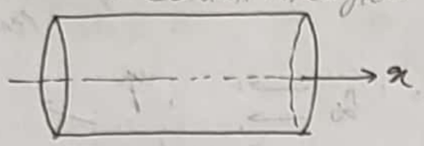
λ → Mean free path.

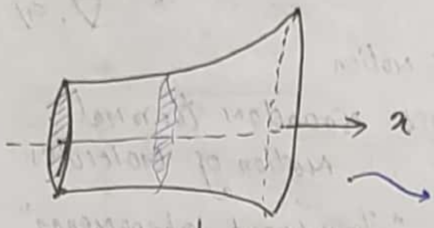
thermal conductivity of gas, $K = \varepsilon C_v \mu$

1D Flows

- 1D flows are flows in which flow property vary only with one coordinate axis.

- Often 1D flows are treated as constant area flow





A = Constant
$p(x), T(x), \rho(x), u(x)$

1D flow

A = A(x)
$p(x), T(x), \rho(x), u(x)$

Quasi-1D flow

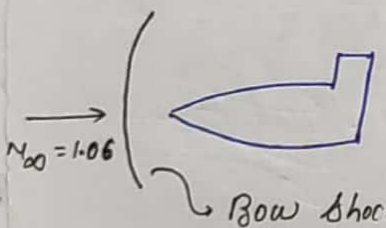
- Normal Shock waves are example of 1D flow.

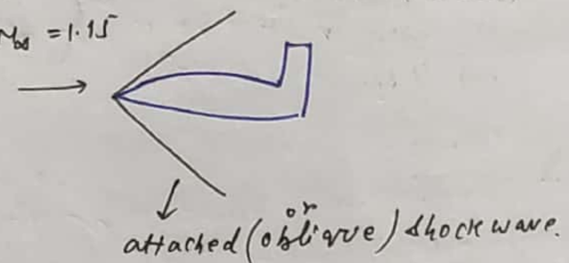
★ the portion of Shock Wave that is ⊥ to the free stream.

[Normal shock ~ 1D phenomena
oblique shock ~ 2D "]

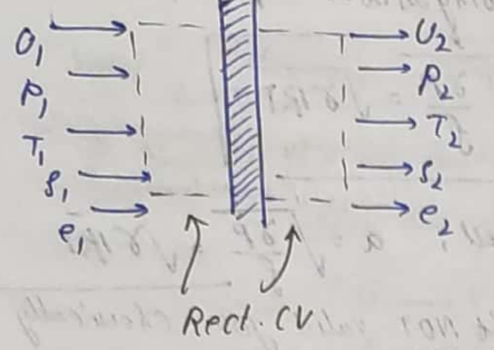
- Flow is 3D
- Flow properties are function of x, y, z

However if variation of A(x) is gradual, it is sufficiently accurate if we reject y, z flow variation & assume flow properties vary as a function of x only!





1D flow equation :



assumptions :

i. Steady state

→ $\boxed{\rho_1 U_1 = \rho_2 U_2}$ mass conv.

ii. No body force, No viscous stress

→ $\boxed{p_1 + \rho_1 U_1^2 = p_2 + \rho_2 U_2^2}$

flow is 1D

so, U_1, P_1, ρ_1, T_1 & e_1 are uniform over the LHS of the CV.

U_2, P_2, ρ_2, T_2 & e_2 " " " RHS " " CV

iii. No heat transfer due to Thermal conduction & diffusion, No change in PE.

→ $\boxed{\left(h_1 + \frac{U_1^2}{2}\right) + q = h_2 + \frac{U_2^2}{2}}$

q ~ heat added per unit mass.

$h = e + p\forall$

$$\boxed{\begin{aligned} &\rho_1 U_1 = \rho_2 U_2 \quad \text{[Mass conv.]} \\ &p_1 + \rho_1 U_1^2 = p_2 + \rho_2 U_2^2 \quad \text{[Momentum conv.]} \\ &\left(h_1 + \frac{U_1^2}{2}\right) + q = h_2 + \frac{U_2^2}{2} \quad \text{[energy conv]} \end{aligned}}$$

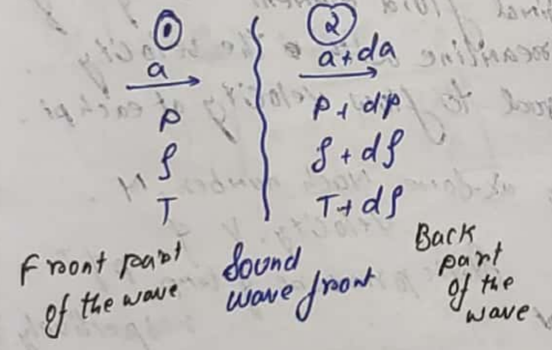
- Valid for steady state, No body force, No viscous stress & No heat transfer (due to Thermal conduction & diffusion) & No change in PE, 1D flows.

a 1D flow

Speed of sound :

a sound wave is a weak wave

the change within the wave is small i.e. flow gradients are small!



- Negligible irreversibility (ie dissipative effects of friction & thermal conduction are negligible)
- Moreover, there is No heat addition to the flow inside the wave.

rev. + adiab. process ⇒ $\boxed{\text{Isentropic process!}}$

- from mass conv & momentum conv. : $\boxed{a^2 = \frac{dp}{d\rho} = \left(\frac{\partial p}{\partial \rho}\right)\Big|_s}$

isentropic compressibility, $\tau_s = \left(\frac{1}{\rho}\frac{\partial \rho}{\partial p}\right)\Big|_s$

$$\boxed{a = \sqrt{\frac{dp}{d\rho}} = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)\Big|_s} = \sqrt{\frac{\forall}{\tau_s}}}$$

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for : i. Incompressible flow ($\tau_s = 0$) $\Rightarrow a \to \infty$

ii. Calorically PG (undergoing an isentropic process)

i.e. $pv^{\gamma} = const.$ $\boxed{a = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\gamma RT}}$

iii. For thermally PG as well, $a = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\gamma RT}$

for air at 300K $a \approx 347\ ms^{-1}$

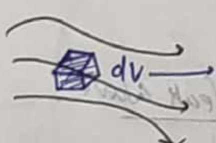
However this relation is NOT valid for chemically reacting gases or Real gases!

- Mach Number, $M = \frac{\text{speed of the flow}}{\text{speed of sound (under the same condition)}} = \frac{V}{a}$

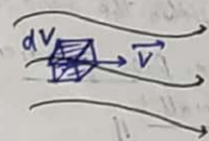
↓ is a measure of directed motion of a gas compared to Random thermal motion of the molecule.

$M^2 \propto \frac{\text{KE per unit mass of the fluid element}}{\text{Internal Energy per unit mass of the fluid element}} = M^2 \frac{\gamma(\gamma-1)}{2}$

⊙ Defining some Flow parameters :



$dv \sim$ infinitesimal fluid element fixed in space, with fluid moving through it.



$dV \sim$ infinitesimal fluid element moving along a streamline with the velocity $\vec{V}$ equal to flow velocity at each pt.

Consider the fluid element is travelling at some Mach number, M, Velocity, V & Static pressure & temp. p & T respectively

* Imagine that we take up this fluid element

& → adiabatically slow it down (if $M>1$) to $M=1$

→ adiabatically speed it up to $M=1$ (if $M<1$)

i.e. $M<1 \longrightarrow M=1$
$M>1 \longrightarrow M=1$

speed of sound, $a = \sqrt{\gamma RT}$

$a^* = \sqrt{\gamma RT^*}$

$T \longrightarrow T^*$

Original properties → properties at the imaginary state.

original Mach no. , $M = \frac{V}{a}$

Characteristic Mach no, $M^* = \frac{V}{a^*}$

"Total" / "Stagnation" pressure, Temp & density

(adia + rev.)

★ Imagine that we take up a fluid element & isentropically slow this fluid element to 0 velocity i.e. stagnate the fluid element.

i.e. Static V, P, T & ρ →

$V \longrightarrow V=0$

$P \longrightarrow P_0$

$T \longrightarrow T_0$

$\rho \longrightarrow \rho_0$

Stagnation press. temp. & density.

Stagnation speed of sound $a_0 = \sqrt{\gamma R T_0}$

$\rho_0 = \frac{P_0}{R T_0}$

NOTE: that perhaps in both of these definitions two assumptions are implicit → perfect gas → No irreversibilities (No dissipations)

- potential energy (gz) changes are extremely small as compared to KE $[\frac{1}{2}V^2]$ & enthalpy (h) changes.
- Energy Cons. (Steady flow, no shaft work, no heat transfer):

"Total" ⟶ isentropy

"*" ⟶ adiabatic

$h + \frac{1}{2}V^2 = h_0$ (Stagnation enthalpy) at Stagnation pt. V=0

$c_p T + \frac{1}{2}V^2 = c_p T_0$

$\Rightarrow \boxed{T_0 = T + \frac{U^2}{2c_p}}$

$$\boxed{\frac{T_0}{T} = 1 + \frac{\gamma-1}{2}M^2}$$

$c_p = \frac{\gamma R}{\gamma - 1}$

$M = \frac{U}{a}$

$$\boxed{\frac{a_0}{a} = \left(\frac{T_0}{T}\right)^{1/2} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{1/2}}$$

Using isentropic relation

$$\frac{P_0}{P} = \left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma-1}} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{\gamma/\gamma-1}$$

$$\frac{\rho_0}{\rho} = \left(\frac{T_0}{T}\right)^{\frac{1}{\gamma-1}} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{1/\gamma-1}$$

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⊙ for 1D Flow (under the earlier made assumptions)

$$\rho_1 U_1 = \rho_2 U_2$$
$$P_1 + \rho_1 U_1^2 = P_2 + \rho_2 U_2^2$$
$$\left(h_1 + \frac{U_1^2}{2}\right) + q = \left(h_2 + \frac{U_2^2}{2}\right)$$

- Steady State
- No body force, No viscous stress
- No heat transfer due to thermal conduction & diffusion, No change in PE

- With No heat addition: (adiabatic)

energy consv. reduces to $h_1 + \frac{U_1^2}{2} = h_2 + \frac{U_2^2}{2}$

- further for a calorically PG, with assumption that $h=0$ @ $T=0K$ (though perhaps it does not matter!) i.e. $h = C_p T$

$$C_p T_1 + \frac{U_1^2}{2} = C_p T_2 + \frac{U_2^2}{2}$$

- $C_p = \frac{\gamma R}{\gamma - 1}$ (for a calorically PG)

$$\frac{\gamma R T_1}{\gamma-1} + \frac{U_1^2}{2} = \frac{\gamma R T_2}{\gamma-1} + \frac{U_2^2}{2}$$

- Speed of sound, $a = \sqrt{\gamma R T} = \sqrt{\frac{\gamma P}{\rho}}$ (for a calorically PG & Thermally PG)

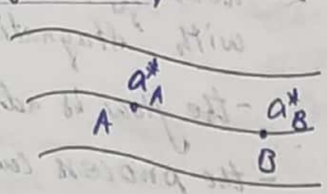
$$\boxed{\frac{a_1^2}{\gamma-1} + \frac{U_1^2}{2} = \frac{a_2^2}{\gamma-1} + \frac{U_2^2}{2}}$$

$$\Rightarrow \frac{\gamma}{\gamma-1}\cdot\frac{P_1}{\rho_1} + \frac{U_1^2}{2} = \frac{\gamma}{\gamma-1}\cdot\frac{P_2}{\rho_2} + \frac{U_2^2}{2}$$

for an adiabatic flow of calorically PG,

$a^{*2} = U_1 U_2 \Leftrightarrow 1 = M_1^* M_2^*$ — Prandtl's relation.

⊙ from defn of * quantity → obtained from energy eqn



$$\boxed{a^{*2} = \frac{2a^2 + U^2(\gamma-1)}{\gamma+1}}$$

$$\Downarrow \quad a^2 = a^{*2}\frac{(\gamma+1)}{2} - U^2\frac{(\gamma-1)}{2}$$

- if the General flowfield is adiabatic throughout then a^* has same value throughout the flowfield. (flowfield → must be 1D flow).

- if actual Flowfield (provided 1D flow assumption holds) is → adiabatic from A to B then $a^*_A = a^*_B$.

↓ NOT adiabatic from A to B then $a^*_A \neq a^*_B$

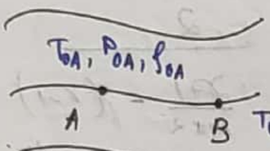
(However note that M^*_A not necessarily $= M^*_B$)

- [Many practical Aerodynamic flows are reasonably adiabatic!]

$$M^{*2} = \frac{U^2}{a^{*2}} = \cancel{(\gamma+1)^2}$$

$$\boxed{M^2 = \frac{2}{\frac{\gamma+1}{M^{*2}} - (\gamma-1)}} \Leftarrow \boxed{M^* = \frac{M\sqrt{\gamma+1}}{\sqrt{2+M^2(\gamma-1)}}}$$

from ⊙ defn of "total" / "stagnation" quantities.



$$\boxed{T_0 = T + \frac{U^2}{2C_p}}$$

is the temp. that will exist if the fluid is brought to rest adiabatically (reversible / irrev. does not matter)

$$\boxed{\frac{T_0}{T} = 1 + \frac{\gamma-1}{2}M^2}$$

$$\boxed{\frac{P_0}{P} = \left(\frac{\rho_0}{\rho}\right)^{\gamma} = \left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma-1}}}$$

Here we have ~~assumed the~~ invoked the reversibility condition on the adiabatic process.

- if actual flowfield ~~(provided 1D flow assumption holds)~~ is → adiabatic from A to B then $T_{0A} = T_{0B}$ $\left[\;\& \; \frac{\rho_{0A}}{\rho_{0B}} = \frac{P_{0A}}{P_{0B}}\right]$

↓ rev. + adiabatic " A to B then — isentropic

$T_{0A} = T_{0B}$
$P_{0A} = P_{0B}$
$\rho_{0A} = \rho_{0B}$

[all stagnation & critical properties are constant]

irrev + NOT adiab " A to B then

$T_{0A} \neq T_{0B}$
$P_{0A} \neq P_{0B}$
$\rho_{0A} \neq \rho_{0B}$

$\boxed{\text{In adiabatic, non-isentropic flows } a^*, a_0, T^*, T_0 \text{ are constant but } P^*, P_0, \rho^*, \rho_0 \text{ may vary!}}$

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⊙ $\frac{a^2}{\gamma-1}+\frac{u^2}{2}=\frac{a_0^2}{\gamma-1}=\frac{a^{*2}}{\gamma-1}+\frac{a^{*2}}{2}=\frac{\gamma+1}{2(\gamma-1)}a^{*2}$

← Relating the "*" quantities with "stagnation" quantities.
- the flow is adiabatic
- the process can be thought of as

$u_1=0$	$u_2=u^*$
$a_1=a_0$	$a_2=a^*$
$M_1=0$	$M_2=1$
$p_1=p_0$	$p_2=p^*$
$\rho_1=\rho_0$	$\rho_2=\rho^*$
$T_1=T_0$	$T_2=T^*$

So, $$\boxed{\frac{a^{*2}}{a_0^2}=\frac{T^*}{T_0}=\frac{2}{1+\gamma}}$$

So, $$\frac{p^*}{p_0}=\left(\frac{\rho^*}{\rho_0}\right)^{\gamma}=\left(\frac{T^*}{T_0}\right)^{\frac{\gamma}{\gamma-1}}$$

⊙ for air in standard condition: $(\gamma=1.4)$

$\frac{a^*}{a_0}=0.9129$ | $\frac{T^*}{T_0}=0.8333$, $\frac{p^*}{p_0}=0.5283$, $\frac{\rho^*}{\rho_0}=0.6339$

$\frac{T_0}{T^*}=1.2$, $\frac{p_0}{p^*}=1.89$, $\frac{\rho_0}{\rho^*}=1.6$

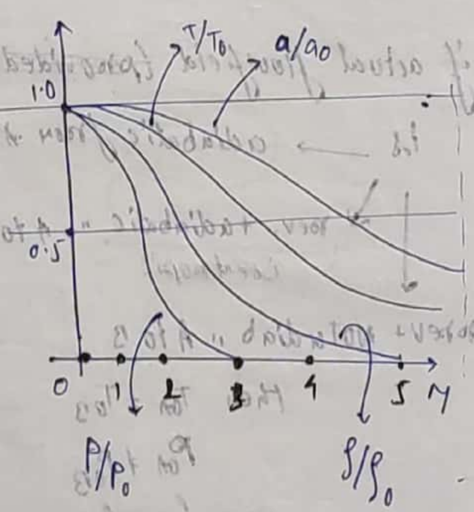
⊙ $$(M^*)^2=\frac{M^2(\gamma+1)}{2+M^2(\gamma-1)} \quad ; \quad M^2=\frac{2}{\frac{\gamma+1}{M^{*2}}-(\gamma-1)}$$

M^* & M behaves similarly

$M^*=1 \Leftrightarrow M=1$
$M^*<1 \Leftrightarrow M<1$
$M^*>1 \Leftrightarrow M>1$
$M^*\to\sqrt{\frac{\gamma+1}{\gamma-1}} \Leftrightarrow M\to\infty$

as $M\to\infty$, $M^*\to 2.45$

$$\frac{T_0}{T}=1+\frac{\gamma-1}{2}M^2$$
$$\frac{p_0}{p}=\left(1+\frac{\gamma-1}{2}M^2\right)^{\gamma/\gamma-1}$$
$$\frac{\rho_0}{\rho}=\left(1+\frac{\gamma-1}{2}M^2\right)^{1/\gamma-1}$$
$$\frac{a_0}{a}=\left(1+\frac{\gamma-1}{2}M^2\right)^{1/2}$$



• **What is Shock?** Shock is a region of finite thickness across which flow properties vary very quickly!

Shock thickness $\sim 3.5\lambda$ ~ Mean free path.

$$\lambda \propto \frac{1}{\rho}$$

Flow across Shock wave is generally adiabatic

• At Sea level conditions

Shock thickness, $t_s \sim 10^{-7}$ m

(as you go up $t_s \uparrow$)

at these conditions shock wave can be considered as a discontinuity, across which flow properties show a jump.

Shock waves are strong pressure waves! unlike acoustic waves!

for a stationary NS for calorically or Thermally PG

Normal Shock!

P_{01}	P_{02}	$P_{01} \geq P_{02}$
T_{01}	T_{02}	$T_{01} = T_{02}$: Isoenergitic
$M_1 \geq 1$	$M_2 \leq 1$	$\rho_2 \geq \rho_1$
ρ_1	ρ_2	$T_2 \geq T_1$
T_1	T_2	$P_2 \geq P_1$
P_1	P_2	$U_2 \leq U_1$
U_1	U_2	$s_2 \geq s_1$
s_2	s_1	$A_2^* \geq A_1^*$
A_2^*	A_1^*	

$P_2, \rho_2, T_2, U_2, h_2$

5 unknowns!

⊙ **Normal Shock** → by defⁿ is ⊥r to the velocity

↑ an application of 1D flow

Shock is a discontinuity in supersonic flows
- Shock waves are very thin ($\sim 10^{-5}$ m)
- Shock waves are strong pressure waves.

Front part Upstream		Rear part Downstream
P_1		$P_2 > P_1$
T_1		$T_2 > T_1$
ρ_1		$\rho_2 > \rho_1$
U_1		$U_2 < U_1$
$M_1 > 1$		$M_2 < 1$
Flow is supersonic ($M>1$)	Normal Shock	Flow is subsonic ($M<1$)

U_∞

Stationary Shock.

$$\rho_1 U_1 = \rho_2 U_2$$
$$P_1 + \rho_1 U_1^2 = P_2 + \rho_2 U_2^2$$
$$h_1 + \frac{U_1^2}{2} = h_2 + \frac{U_2^2}{2}$$

→ these eqns are applicable across the normal shock for any type of gases & in general should be solved numerically for the properties behind the shock wave.

- for an adiabatic flow of calorically PG,

$$\boxed{M_1^* M_2^* = 1} \equiv \boxed{a^{*2} = U_1 U_2}$$ → prandtl's relation.

4 important relation across a Normal Shock (assuming gas to be calorically PG)

i. $$\boxed{M_2^2 = \frac{\frac{2}{\gamma-1} + M_1^2}{\frac{2\gamma}{\gamma-1} M_1^2 - 1}}$$

(obtained using $M_1^* M_2^* = 1$ & $M^{*2} = \frac{(\gamma+1)M^2}{2+(\gamma-1)M^2}$)

when $M_1 = 1 \Rightarrow M_2 = 1$ → a weak Normal shock → Mach wave!

as $M_1 \to \infty \Rightarrow M_2 = \sqrt{\frac{\gamma-1}{2\gamma}}$

$M_1^* > 1$	$M_2^* < 1$
Normal shock	flow behind normal shock is always subsonic (applicable to all gas not necessarily calorically PG)

ii. $$\boxed{\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2}}$$

(obtained using $\frac{\rho_2}{\rho_1} = \frac{U_1}{U_2} = M_1^{*2}$)

iii. $$\boxed{\frac{P_2}{P_1} = 1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1)}$$

iv. $$\boxed{\frac{T_2}{T_1} = \frac{h_2}{h_1} = \left[1 + \frac{2\gamma}{\gamma+1}(M_1^2-1)\right]\left[\frac{2+(\gamma-1)M_1^2}{(\gamma+1)M_1^2}\right]}$$

Scanned with CamScanner

these 4 relations ① → ④ are valid upto $M_1 = \sim 5$ (in air std. conditions)

- Beyond $M_1 = 5$, temp. becomes high enough for γ to remain constant ($\equiv$ calorically PG assumption becomes invalid).

- upto $M_1 = 2$ they are accurate.

- for $M_1 = 1 \Rightarrow M_2 = 1$, $\frac{\rho_2}{\rho_1} = 1$, $\frac{P_2}{P_1} = 1$, $\frac{T_2}{T_1} = 1$ — this is the case for an ∞ weak Normal shock degenerating to Mach wave! — this is same as Sound wave

- as $M_1 \to \infty$ (calorically PG assumption breaks.)

$$M_2 \to \sqrt{\frac{\gamma-1}{2\gamma}}, \quad \frac{\rho_2}{\rho_1} \to \frac{\gamma+1}{\gamma-1}, \quad \frac{P_2}{P_1} \to \infty, \quad \frac{T_2}{T_1} \to \infty$$

assuming $\gamma = 1.4$: 0.378, 6

- for calorically PG $M_2, \frac{\rho_2}{\rho_1}, \frac{P_2}{P_1}, \frac{T_2}{T_1}$ are all function of M_1
- for an equⁿ chemically reacting gas $M_2, \frac{\rho_2}{\rho_1}, \frac{P_2}{P_1}, \frac{T_2}{T_1}$ are all " " M_1, T_1
- non a chemically " " $M_2, \frac{\rho_2}{\rho_1}, \frac{P_2}{P_1}, \frac{T_2}{T_1}$ " " " " M_1, T_1, P_1

at high temp. cases closed form expressions are NOT possible
∴ therefore Normal shock properties must be calculated numerically!

★ observe that equⁿ ① → ④ are mathematically valid even when $M_1 < 1$

However these equⁿ ① → ④ are physically valid only when $M_1 \geq 1$
— can be proved from 2nd law of TD.

proof:
$$s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{P_2}{P_1}$$

- $s_2 - s_1 > 0$ ~ if $M_1 > 1$
- $s_2 - s_1 = 0$ ~ if $M_1 = 1$
- $s_2 - s_1 < 0$ ~ if $M_1 < 1$

only possible physical case is $s_2 - s_1 \geq 0$
$\Rightarrow M_1 \geq 1$
$\Downarrow$
- $\frac{\rho_2}{\rho_1} \geq 1$, $\frac{U_2}{U_1} \leq 1$
- $\frac{P_2}{P_1} \geq 1$,
- $\frac{T_2}{T_1} \geq 1$

$$\boxed{\frac{P_{02}}{P_{01}} = e^{-\frac{(s_2 - s_1)}{R}}}$$

$$\boxed{\frac{T_{02}}{T_{01}} = 1}$$

- The change in entropy arises because of :

Shock occurs over a very thin region $\sim 10^{-5}$ cm → the velocity & temp. gradient inside the shock structure itself are very large !

↓

in regions of large gradients the viscous effect of viscosity & thermal conduction become important.

⇓

these dissipative irrev. phenomena generated entropy.

$P_{01} \geq P_{02}$

$M_1 > 1$	$M_2 < 1$
P_1	$P_2 > P_1$
ρ_1	$\rho_2 > \rho_1$
T_1	$T_2 > T_1$
U_1	$U_2 < U_1$

- "Total" / "Stagnation" condition :

static properties show a big jump across a normal wave what happens to total/stag. props. ?

for a stationary Normal Shock the total enthalpy is always constant across the shock wave.

→ for a calorically or thermally PG it translates into a constant Total temp. across shock wave.

$$T_{02} = T_{01}$$

- for a chemically reacting gas total temp. is NOT constant across the shock !

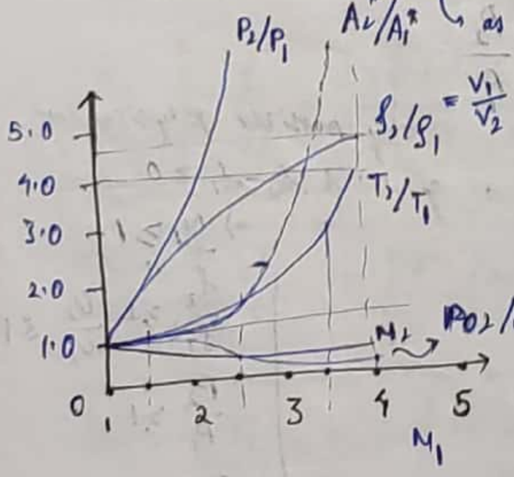
- Also if the shock wave is NOT stationary neither * total enthalpy nor total temp. are constant across the shock wave.

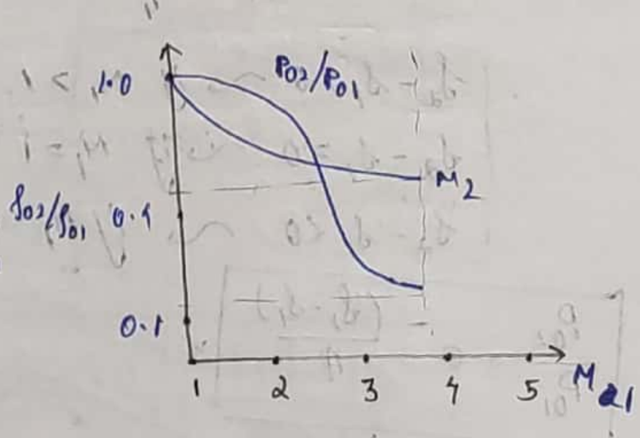
$s_{02} = s_2$, $s_{01} = s_1$ } by def^n

$T_{02} = T_{01}$ for calorically PG & stationary NS.

$$\boxed{\frac{P_{02}}{P_{01}} = e^{-\frac{(s_2 - s_1)}{R}}}$$

as $s_2 > s_1$ so $P_{02} < P_{01}$ i.e. total pressure loss across the shock wave.





- from $\rho_1 u_1 = \rho_2 u_2$
$P_1 + \rho_1 u_1^2 = P_2 + \rho_2 u_2^2$ $\Big\} \rightsquigarrow u_1^2 = \dfrac{P_2 - P_1}{\rho_2 - \rho_1}\left(\dfrac{\rho_2}{\rho_1}\right)$

$\&\ u_2^2 = \dfrac{P_2 - P_1}{\rho_2 - \rho_1}\left(\dfrac{\rho_1}{\rho_2}\right)$

from $h_1 + \dfrac{u_1^2}{2} = h_2 + \dfrac{u_2^2}{2}$ & the above two relation

$\boxed{e_2 - e_1 = \dfrac{P_1 + P_2}{2}\cdot\left(\forall_2 - \forall_1\right)}$ $\xrightarrow{\text{for calorically PG}}$ $\boxed{\dfrac{P_2}{P_1} = \dfrac{\left(\frac{\gamma+1}{\gamma-1}\right)\frac{\forall_1}{\forall_2} - 1}{\left(\frac{\gamma+1}{\gamma-1}\right) - \frac{\forall_1}{\forall_2}}}$

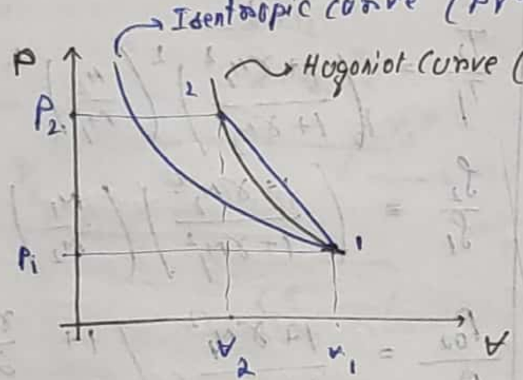
Hugoniot eqn (relates TD quantities across without any explicit mention of velocity or mach no.)

$\boxed{\Delta e = - P_{avg}\, \Delta \forall}$

$P_2 = f(P_1, \forall_1, \forall_2)$

for a given P_1 & $\forall_1$ you can plot P_2 & $\forall_2$

Note: in equil. TD any state variable can be expressed as a function of any other state variable i.e. $e(P, \forall)$ for exp.



Normal shock strength is decided just on the basis of free stream velocity (since it near Ma)

from $u_1^2 = \dfrac{P_2 - P_1}{\rho_2 - \rho_1}\left(\dfrac{\rho_2}{\rho_1}\right)$

$\Rightarrow \boxed{\dfrac{P_2 - P_1}{\forall_2 - \forall_1} = -\left(\dfrac{u_1}{\forall_1}\right)^2}$

- Each pt. on the Hugoniot curve represents a different shock (with different upstream velocity u_1)

⊙ as static pressure always rises across a shock wave, so the wave itself can be visualized as a TD device to compress the gas

for a given rise in specific volume a shock wave creates a higher pressure rise than isentropic compression!

However shock wave could more because of entropy generation & consequently total pressure loss!

⇓

i.e. Shock compression is less efficient than isentropic compression!

BUT,

- Shock wave compression is more effective than isentropic compression

Rayleigh Flow

[1D flow with heat addition. [used in turbojet / ramjet]

$$\rho_1 U_1 = \rho_2 U_2$$
$$P_1 + \rho_1 U_1^2 = P_2 + \rho_2 U_2^2$$
$$h_1 + \frac{U_1^2}{2} + q = h_2 + \frac{U_2^2}{2}$$

→ in gen. numerical solution is reqd. But for specific case of calorically PG, closed form ~~solution~~ analytical solution is possible.

– We will be dealing with calorically PG.

U_1	U_2
M_1	M_2
P_1	P_2
T_1	T_2
ρ_1	ρ_2
P_{01}	P_{02}
T_{01}	T_{02}

$$q = c_p (T_{02} - T_{01})$$

$$\frac{P_2}{P_1} = \frac{1+\gamma M_1^2}{1+\gamma M_2^2}$$

$$\frac{T_2}{T_1} = \left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)^2 \left(\frac{M_2}{M_1}\right)^2$$

$$\frac{\rho_2}{\rho_1} = \left(\frac{1+\gamma M_2^2}{1+\gamma M_1^2}\right)\left(\frac{M_1}{M_2}\right)^2$$

$$\frac{P_{02}}{P_{01}} = \frac{1+\gamma M_1^2}{1+\gamma M_2^2}\left(\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{T_{02}}{T_{01}} = \left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)^2\left(\frac{M_2}{M_1}\right)^2\left(\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}\right)$$

Solving procedure:

i. $T_{01} = T_1 + \frac{U_1^2}{2c_p}$

ii. $q = c_p (T_{02} - T_{01})$

$\Rightarrow T_{02} = \frac{q}{c_p} + T_{01}$

iii. obtain $\frac{T_{02}}{T_{01}}$

from there you can obtain M_2 (M_1 given)

↓

once you get hold of M_2 the problem is solved!

→ however M_2 ~~can be~~ is obtained from $\frac{T_{02}}{T_{01}}$ ~~from~~ by trial & error.

An alternate way: (by using sonic flow reference condition!)

suppose,	
$M_1 = 1$	$M_2 = M$
$P_1 = P^*$	$P_2 = P$
$T_1 = T^*$	$T_2 = T$
$\rho_1 = \rho^*$	$\rho_2 = \rho$
$P_{01} = P_0^*$	$P_{02} = P_0$
$T_{01} = T_0^*$	$T_{02} = T_0$

now, the formulas reduces to,

$$q = C_p (T_0 - T_0^*)$$

$$\frac{p}{p^*} = \frac{1+\gamma}{1+\gamma M^2}$$

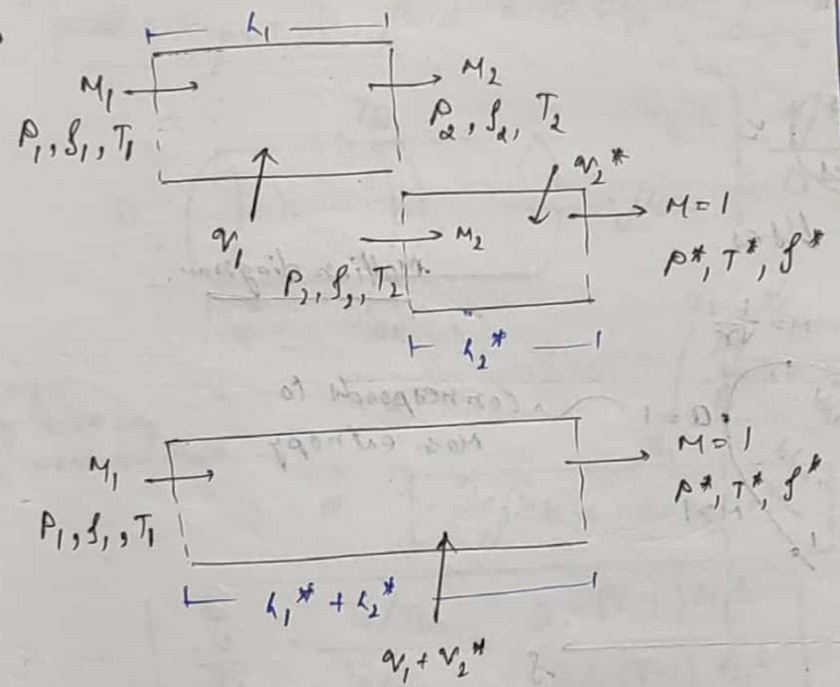
$$\frac{T}{T^*} = \left(\frac{1+\gamma}{1+\gamma M^2}\right)^2 M^2$$

$$\frac{\rho}{\rho^*} = \frac{1}{M^2}\left(\frac{1+\gamma M^2}{1+\gamma}\right)$$

$$\frac{p_0}{p_0^*} = \frac{1+\gamma}{1+\gamma M^2}\left[\frac{2+(\gamma-1)}{\gamma+1}\right]^{\frac{\gamma}{\gamma-1}}$$

$$\frac{T_0}{T_0^*} = \frac{(\gamma+1)M^2}{(1+\gamma M^2)^2}\left[2+(\gamma-1)M^2\right]$$

~~Value~~ the value of these ratio for $\gamma = 1.4$ for various M is there in table A3 pg - 615



Note: that the * condition over here is very different from earlier cases.

Here T^*, p^*, ρ^* are conditions that would exists in a 1D flow if enough heat is added to achieve Mach 1 for

Earlier * quantities ~~the process was~~ adiabatic BUT not here!

- By refering tables we can easily obtain ans. for $\gamma = 1.4$ for given Initial conditions

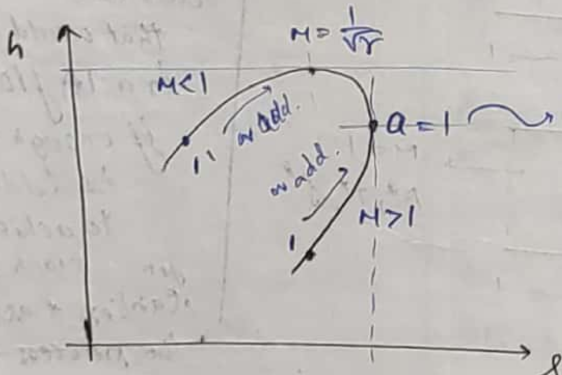
- In rayleigh flow,
  - the flow at the end of the duct tends to become sonic at the end of the duct! & when heat is added to the flow. (irrespective of the flow being supersonic/subsonic)

- for subsonic flows ($M_1 < 1$) upon heating:
  i. M ↑ses
  ii. p ↓ses
  iii. temp ↑ses upto $M_1 < \frac{1}{\sqrt{\gamma}}$
     ↓ses from $\frac{1}{\sqrt{\gamma}} < M_1 < 1$
  iv. Total temp. ↑ses.
  v. Total pres. ↓ses
  vi. velocity ↑ses.

- for supersonic flows ($M_1 > 1$) upon heating:
  i. M ↓ses
  ii. p ↑ses
  iii. T ↑ses
  iv. T_0 ↑ses
  v. P_0 ↓ses
  vi. velocity ↓ses

via. S↑ses.

Mollier diagram.



corresponds to Max. entropy.

Rayleigh curve., drawn for a given set of ICs.

Heat addition causes condition in region 2. to move towards pt. a.
→ for a certain value of q flow becomes sonic ⇒ Choked!

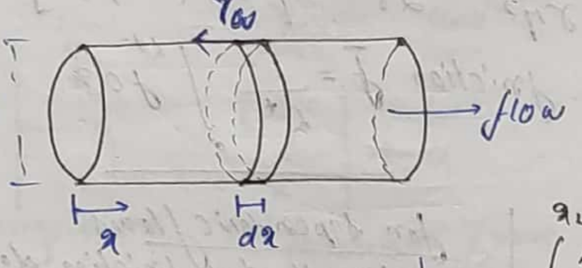
because any further ↑se in q is NOT possible without drastic revision of ICs in region 1.

for exp:
i. if the flow is subsonic: then if q ↑sed beyond "choke" then a series of pressure wave will propagate upstream & nature will adjust condition in region 1 to a lower Mach. no.

ii. if the flow is supersonic and is obtained by expansion through a ~~nozzle~~ supersonic Nozzle, then if the value of r is ↑sed beyond choke pt. then a ~~series of pressure wave~~ Normal Shock will form inside nozzle & condition in region 1 will become subsonic.

Fanno flow → 1D flow with friction

- In reality all fluids are viscous & the friction between moving fluid & stationary walls of the duct causes flow properties to change along the duct.
- Here we will be modelling fric. force as shear stress at the wall acting on the fluid with uniform properties across any CS.



$$\boxed{\frac{4f\,dx}{D} = \frac{2}{\gamma M^2}\,\frac{1-M^2}{\left(1+\frac{1}{2}(\gamma-1)M^2\right)}\,\frac{dM}{M}}$$

$$\int_{x_1}^{x_2} \frac{4f}{D}\,dx = \left[\frac{-1}{\gamma M^2} - \frac{\gamma+1}{2\gamma}\ln\left(\frac{M^2}{1+\frac{\gamma-1}{2}M^2}\right)\right]_{M_1}^{M_2}$$

changes are there only in momentum eqn

$\boxed{\cdot \text{ the flow is adiabatic!}}$ ★

$$\boxed{\frac{T_2}{T_1} = \frac{T_0/T_1}{T_0/T_2} = \frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2}}$$

$$\boxed{\frac{p_2}{p_1} = \frac{M_1}{M_2}\left[\frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2}\right]^{1/2}}$$

$$\boxed{\frac{\rho_2}{\rho_1} = \frac{M_1}{M_2}\left[\frac{2+(\gamma-1)M_2^2}{2+(\gamma-1)M_1^2}\right]^{1/2}}$$

$$\boxed{\frac{p_{02}}{p_{01}} = \frac{M_1}{M_2}\left[\frac{2+(\gamma-1)M_2^2}{2+(\gamma-1)M_1^2}\right]^{\frac{\gamma+1}{2(\gamma-1)}}}$$

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• Analagous to Rayleigh flow case: we will be defining * quantities

Here T^*, p^*, ρ^* are conditions that will exists in 1D flow if the length of the duct is sufficient (with fric.) to achieve Mach 1.

$$\frac{p}{p^*} = \frac{1}{M}\left[\frac{\gamma+1}{2+(\gamma-1)M^2}\right]^{1/2}$$

$$\frac{T}{T^*} = \frac{\gamma+1}{2+(\gamma-1)M^2}$$

$$\frac{\rho}{\rho^*} = \frac{1}{M}\left[\frac{2+(\gamma-1)M^2}{\gamma+1}\right]^{1/2}$$

$$\frac{p_0}{p_0^*} = \frac{1}{M}\left[\frac{2+(\gamma-1)M^2}{\gamma+1}\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

We define L^* as the length where $M=1$

then, $$\frac{4\bar{f}L^*}{D} = \frac{1-M^2}{\gamma M^2} + \frac{\gamma+1}{2\gamma}\ln\left[\frac{(\gamma+1)M^2}{2+(\gamma-1)M^2}\right]$$

Where the avg. friction $\bar{f} = \frac{1}{L^*}\int_0^{L^*} f\,dx$

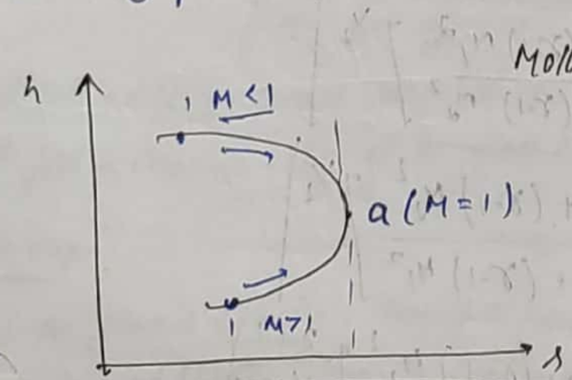
• for subsonic flows with effect of friction downstream

i. Ma ↑
ii. P ↓
iii. T ↓
iv. P_0 ↓
v. U ↑

for supersonic flows with effect of friction downstream.

i. Ma ↓
ii. P ↑
iii. T ↑
iv. P_0 ↓
v. U ↓

Mollier diagram



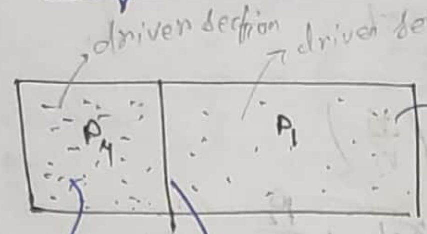
Fanno curve!

(effects of length will be similar to that of heat addition in Rayleigh Flow)

★. Unlike Rayleigh curve for flow with heating & cooling the upper & lower portion of the Fanno curve can't be traversed by the same 1D flow! ★

- Moving 1D flow : Moving Normal Shock,

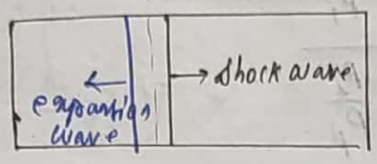
Shock Tube :



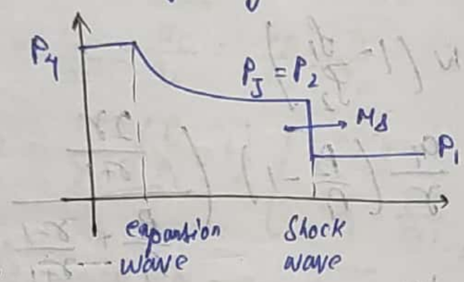
low pressure.

dense gas with high pressure.

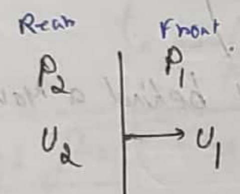
membrane / diaphragm made up of very light wt. Al. plate or so.



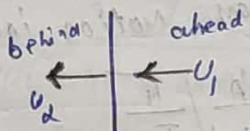
just after rupture (0^+ time)



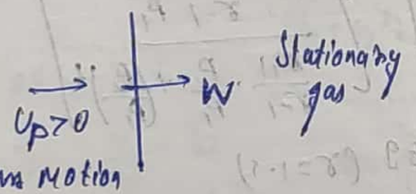
at a later time



Moving shock



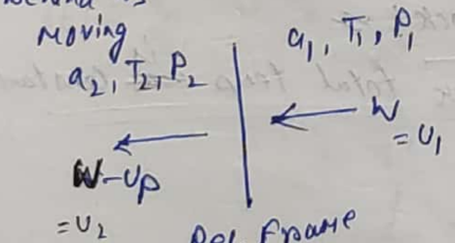
- Reference frame attached to shock.

- 

Mass motion behind NS moving

$M_s \rightarrow$ Mach no. behind moving shock

$W \sim$ shock wave moving with velocity w.



Ref. Frame attached to moving shock

Rear   Front

Equn get transformed :

$$\rho_1 W = \rho_2 (W - U_p)$$

$$P_1 + \rho_1 W^2 = P_2 + \rho_2 (W - U_p)^2$$

$$h_1 + \frac{W^2}{2} = h_2 + \frac{(W - U_p)^2}{2}$$

$$\boxed{e_2 - e_1 = -\left(\frac{P_1 + P_2}{2}\right)(V_2 - V_1)}$$

$(P_2 > P_1)$

Moving shock is unsteady flow.

for a calorically PG: $(e = C_v T;\ P = \rho R T)$

$$\boxed{M_s = \frac{W}{a_1}}$$

$$\frac{P_2}{P_1} = \frac{\left(\frac{\gamma+1}{\gamma-1}\right)\frac{V_1}{V_2} - 1}{\left(\frac{\gamma+1}{\gamma-1}\right) - \left(\frac{V_1}{V_2}\right)}$$

$$\frac{P_2}{P_1} = 1 + \frac{2\gamma}{\gamma+1}(M_s^2 - 1)$$

$$\frac{T_2}{T_1} = \frac{P_2}{P_1}\left[\frac{\frac{\gamma+1}{\gamma-1} + \frac{P_2}{P_1}}{1 + \frac{\gamma+1}{\gamma-1}\frac{P_2}{P_1}}\right]$$

$$\Rightarrow M_s = \sqrt{\frac{\gamma+1}{2\gamma}\left(\frac{P_2}{P_1} - 1\right) + 1}$$

$$\Rightarrow W = a_1\sqrt{\frac{\gamma+1}{2\gamma}\left(\frac{P_2}{P_1} - 1\right) + 1}$$

$$\frac{\rho_2}{\rho_1} = \frac{1 + \frac{\gamma+1}{\gamma-1}\frac{P_2}{P_1}}{\frac{\gamma+1}{\gamma-1} + \frac{P_2}{P_1}}$$

- from the continuity eq^n

$$U_p = W\left(1 - \frac{\rho_1}{\rho_2}\right)$$

$$= \frac{a_1}{\gamma}\left(\frac{P_2}{P_1} - 1\right)\left(\frac{\frac{2\gamma}{\gamma+1}}{\frac{P_2}{P_1} + \frac{\gamma-1}{\gamma+1}}\right)^{1/2}$$

- Mach no. associated with mass motion induced behind a moving shock

$$M_p = \frac{U_p}{a_2} = \frac{U_p}{a_1} * \frac{a_1}{a_2} = \frac{U_p}{a_1}\sqrt{\frac{T_1}{T_2}}$$

$$\Rightarrow \frac{U_p}{a_2} = \frac{1}{\gamma}\left(\frac{P_2}{P_1} - 1\right)\left[\frac{2\gamma/\gamma+1}{\frac{P_2}{P_1} + \frac{\gamma-1}{\gamma+1}}\right]^{1/2}\left[\frac{1 + \frac{\gamma+1}{\gamma-1}\frac{P_2}{P_1}}{\frac{\gamma+1}{\gamma-1}\cdot\frac{P_2}{P_1} + \left(\frac{P_2}{P_1}\right)^2}\right]^{1/2}.$$

↳ as $\frac{P_2}{P_1} \to \infty$ $M_p \to \sqrt{\frac{2}{\gamma(\gamma-1)}} = 1.89\ (\gamma = 1.4)$

★ - Total temp. ~~is~~ ~~not~~ across a moving shock! ✓✓

However in case of a stationary shock total temp. is constant.

• for a calorically PG : $(e = c_v T,\ p = \rho R T)$

$$\frac{P_2}{P_1} = \frac{\left(\frac{\gamma+1}{\gamma-1}\right)\frac{v_1}{v_2} - 1}{\left(\frac{\gamma+1}{\gamma-1}\right) - \frac{v_1}{v_2}}, \quad \frac{T_2}{T_1} = \frac{P_2}{P_1}\left[\frac{\frac{\gamma+1}{\gamma-1} + \frac{P_2}{P_1}}{1 + \frac{\gamma+1}{\gamma-1}\frac{P_2}{P_1}}\right], \quad \frac{\rho_2}{\rho_1} = \frac{1 + \frac{\gamma+1}{\gamma-1}\frac{P_2}{P_1}}{\frac{\gamma+1}{\gamma-1} + \frac{P_2}{P_1}}$$

• $$\frac{P_2}{P_1} = 1 + \frac{2\gamma}{\gamma+1}(M_s^2 - 1) \Rightarrow M_s = \sqrt{\frac{\gamma+1}{2\gamma}\left(\frac{P_2}{P_1} - 1\right) + 1}$$

$$\Rightarrow W = a_1\sqrt{\frac{\gamma+1}{2\gamma}\left(\frac{P_2}{P_1} - 1\right) + 1}$$

• from the continuity eqn :

$$U_p = W\left(1 - \frac{\rho_1}{\rho_2}\right) = \frac{a_1}{\gamma}\left(\frac{P_2}{P_1} - 1\right)\left(\frac{\frac{2\gamma}{\gamma+1}}{\frac{P_2}{P_1} + \frac{\gamma-1}{\gamma+1}}\right)^{1/2}$$

★ Mach no. associated with mass motion induced behind a moving shock :

$$\frac{U_p}{a_2} = \frac{U_p}{a_1} * \frac{a_1}{a_2} = \frac{U_p}{a_1}\sqrt{\frac{T_1}{T_2}}$$

$$\Rightarrow \frac{U_p}{a_2} = \frac{1}{\gamma}\left(\frac{P_2}{P_1} - 1\right)\left[\frac{2\gamma/\gamma+1}{\frac{P_2}{P_1} + \frac{\gamma-1}{\gamma+1}}\right]^{1/2}\left[\frac{1 + \frac{\gamma+1}{\gamma-1}\frac{P_2}{P_1}}{\frac{\gamma+1}{\gamma-1}\cdot\frac{P_2}{P_1} + \left(\frac{P_2}{P_1}\right)^2}\right]^{1/2}$$

$$\left(\text{as } \frac{P_2}{P_1} \to \infty \quad M_2 \to \sqrt{\frac{2}{\gamma(\gamma-1)}}\right.$$

Rayleigh flow : ~ 1D flow with q addition.

$$\rho_1 U_1 = \rho_2 U_2$$
$$P_1 + \rho_1 U_1^2 = P_2 + \rho_2 U_2^2$$
$$\left(h_1 + \frac{U_1^2}{2}\right) + q = \left(h_2 + \frac{U_2^2}{2}\right)$$

$$q = C_p(T_{02} - T_{01})$$

$$\frac{P_2}{P_1} = \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2}$$

$$\frac{T_2}{T_1} = \left(\frac{1 + \gamma M_1^2}{1 + \gamma M_2^2}\right)^2\left(\frac{M_2}{M_1}\right)^2$$

$$q = C_p(T_{02} - T_{01})$$

$$\Rightarrow T_{02} = \frac{q}{C_p} + T_{01}$$

obtain $\frac{T_{02}}{T_{01}} \sim M_2$

$$q = C_p(T_0 - T_0^*)$$

$$\frac{P}{P^*} = \frac{1+\gamma}{1+\gamma M^2}$$

$$\frac{T}{T^*} = \left(\frac{1+\gamma}{1+\gamma M^2}\right)^2 M^2$$

$$\frac{\rho}{\rho^*} = \frac{1}{M^2}\left(\frac{1+\gamma M^2}{1+\gamma}\right)$$

$$\frac{P_0}{P_0^*} = \frac{1+\gamma}{1+\gamma M^2}\left[\frac{2 + (\gamma-1)}{\gamma+1}\right]^{\frac{\gamma}{\gamma-1}}$$

$$\frac{T_0}{T_0^*} = \frac{(\gamma+1)M^2}{(1+\gamma M^2)^2}\left[2 + (\gamma-1)M^2\right]$$

$$\frac{\rho_2}{\rho_1} = \left(\frac{1+\gamma M_2^2}{1+\gamma M_1^2}\right)\left(\frac{M_1}{M_2}\right)^2$$

$$\frac{P_{02}}{P_{01}} = \frac{1+\gamma M_1^2}{1+\gamma M_2^2}\left(\frac{1 + \frac{\gamma-1}{2}M_2^2}{1 + \frac{\gamma-1}{2}M_1^2}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{T_{02}}{T_{01}} = \left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)^2\left(\frac{M_2}{M_1}\right)^2\left(\frac{1 + \frac{\gamma-1}{2}M_2^2}{1 + \frac{\gamma-1}{2}M_1^2}\right)$$

Fanno flow :

$$\int_{x_1}^{x_2} \frac{4f}{D}\,dx = \left[\frac{-1}{\gamma M^2} - \frac{\gamma+1}{2\gamma}\ln\left(\frac{M^2}{1+\frac{\gamma-1}{2}M^2}\right)\right]_{M_1}^{M_2}$$

- the flow is adiabatic

$$\frac{4f\,dx}{D} = \frac{2}{\gamma M^2}\,\frac{1-M^2}{1+\frac{(\gamma-1)}{2}M^2}\,\frac{dM}{M}$$

$$\frac{T_2}{T_1} = \frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2}$$

$$\frac{P_2}{P_1} = \frac{M_1}{M_2} * \left[\frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2}\right]^{1/2}$$

$$\frac{\rho_2}{\rho_1} = \frac{M_1}{M_2} * \left[\frac{2+(\gamma-1)M_2^2}{2+(\gamma-1)M_1^2}\right]^{1/2}$$

$$\frac{P_{02}}{P_{01}} = \frac{M_1}{M_2} * \left[\frac{2+(\gamma-1)M_2^2}{2+(\gamma-1)M_1^2}\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

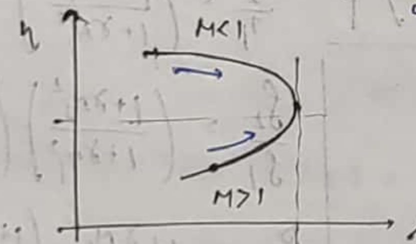
- $\frac{P}{P^*} = \frac{1}{M}\left[\frac{\gamma+1}{2+(\gamma-1)M^2}\right]^{1/2}$, $\frac{T}{T^*} = \frac{\gamma+1}{2+(\gamma-1)M^2}$, $\frac{\rho}{\rho^*} = \frac{1}{M}\left[\frac{2+(\gamma-1)M^2}{\gamma+1}\right]^{1/2}$

$$\frac{P_0}{P_0^*} = \frac{1}{M}\left[\frac{2+(\gamma-1)M^2}{\gamma+1}\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

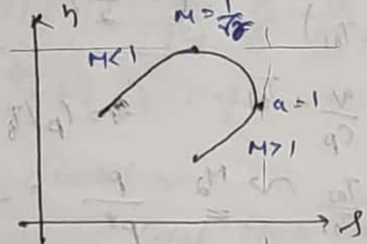
We define L^* as the length where $M=1$.

then, $\frac{4\bar{f}L^*}{D} = \frac{1-M^2}{\gamma M^2} + \frac{\gamma+1}{2\gamma}\ln\left[\frac{(\gamma+1)M^2}{2+(\gamma-1)M^2}\right]$

where $\bar{f} = \frac{1}{L^*}\int_0^{L^*} f\,dx$.



Fanno curve.



Rayleigh curve.

iii. $\frac{P_2}{P_1} = 1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1)$

iv. $\frac{T_2}{T_1} = \frac{h_2}{h_1} = \left[1 + \frac{2\gamma}{\gamma+1}(M_1^2-1)\right]\left[\frac{2+(\gamma-1)M_1^2}{(\gamma+1)M_1^2}\right]$

for $M_1 = 1 \Rightarrow M_2 = 1$, $\frac{\rho_2}{\rho_1} = 1$, $\frac{P_2}{P_1} = 1$, $\frac{T_2}{T_1} = 1$ → this is the case for an ∞nitely weak NS degenerating to Mach wave!

$\Uparrow$ this is same as sound wave.

$\Delta s = c_p \ln\frac{T_2}{T_1} - R\ln\frac{P_2}{P_1}$

- $T_{02} = T_{01}$

$\frac{P_{02}}{P_{01}} = e^{-\frac{(s_2 - s_1)}{R}}$

- Hugoniot's eqn

$\rho_1 u_1 = \rho_2 u_2$

$P_1 + \rho_1 u_1^2 = P_2 + \rho_2 u_2^2$

$\rightsquigarrow u_1^2 = \frac{P_2 - P_1}{\rho_2 - \rho_1}\left(\frac{\rho_2}{\rho_1}\right)$

$\rho u_2^2 = \frac{P_2 - P_1}{\rho_2 - \rho_1}\left(\frac{\rho_1}{\rho_2}\right)$

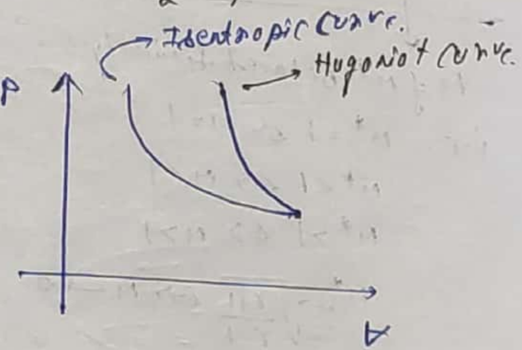
from $h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}$

$\hookrightarrow \boxed{(e_2 - e_1) = \frac{P_1 + P_2}{2} \cdot -(v_2 - v_1)}$

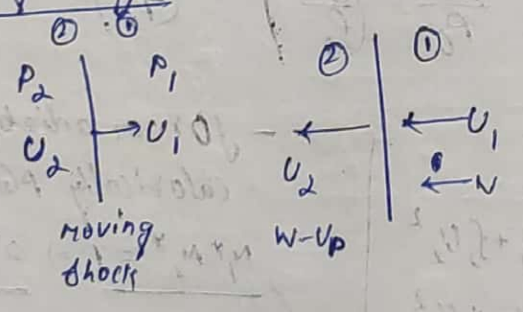
$\downarrow$ for a calorically PG

$$\boxed{\frac{P_2}{P_1} = \frac{\left(\frac{\gamma+1}{\gamma-1}\right)\frac{v_1}{v_2} - 1}{\frac{\gamma+1}{\gamma-1} - \frac{v_1}{v_2}}}$$

$\frac{P_2 - P_1}{v_2 - v_1} = -\left(\frac{u_1}{v_1}\right)^2$



- Moving 1D flow:



$\rho_1 W = \rho_2 (W - u_p)$

$P_1 + \rho_1 W^2 = P_2 + \rho_2 (W - u_p)^2$

$h_1 + \frac{W^2}{2} = h_2 + \frac{(W-u_p)^2}{2}$

$(e_2 - e_1) = \left(\frac{P_2 + P_1}{2}\right) \cdot -(v_2 - v_1)$

for an adiabatic process

$$\boxed{\frac{P_2}{P_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}}}$$

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For 1D flow:

$$\rho_1 U_1 = \rho_2 U_2$$

$$P_1 + \rho_1 U_1^2 = P_2 + \rho_2 U_2^2$$

$$\left(h_1 + \frac{U_1^2}{2}\right) + q = \left(h_2 + \frac{U_2^2}{2}\right)$$

For adiabatic flow

$$h_1 + \frac{U_1^2}{2} = h_2 + \frac{U_2^2}{2}$$

Calorically PG

$$C_p T_1 + \frac{U_1^2}{2} = C_p T_2 + \frac{U_2^2}{2}$$

$$\Rightarrow \frac{\gamma R T_1}{\gamma-1} + \frac{U_1^2}{2} = \frac{\gamma R T_2}{\gamma-1} + \frac{U_2^2}{2}$$

$$\Rightarrow \boxed{\frac{a_1^2}{\gamma-1} + \frac{U_1^2}{2} = \frac{a_2^2}{\gamma-1} + \frac{U_2^2}{2}}$$

* props.

$$a^{*2} = \frac{2}{\gamma+1} a^2 + \frac{\gamma-1}{\gamma+1} U^2$$

$$a^2 = \frac{\gamma+1}{2} a^{*2} - \frac{\gamma-1}{2} U^2$$

$$M^{*2} = \frac{M^2(\gamma+1)}{2 + M^2(\gamma-1)}$$

$$M^2 = \frac{2}{\frac{(\gamma+1)}{M^{*2}} - (\gamma-1)}$$

"total" prop.

$$T_0 = T + \frac{U^2}{2C_p}$$

$$\frac{T_0}{T} = 1 + \frac{\gamma-1}{2} M^2$$

$$\frac{P_0}{P} = \left(\frac{\rho_0}{\rho}\right)^{\gamma} = \left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma-1}}$$

M & M^* behaved similarly

i.e. $M^* = 1 \Leftrightarrow M = 1$

$M^* < 1 \Leftrightarrow M < 1$

$M^* > 1 \Leftrightarrow M > 1$

$M^* \rightarrow \sqrt{\frac{\gamma+1}{\gamma-1}} \Leftrightarrow M \rightarrow \infty$

$$\frac{a^2}{\gamma-1} + \frac{U^2}{2} = \frac{a_0^2}{\gamma-1} = \frac{a^{*2}}{\gamma-1} + \frac{a^{*2}}{2}$$

$$= \frac{\gamma+1}{2(\gamma-1)} a^{*2}$$

$$\boxed{\frac{T^*}{T_0} = \left(\frac{a^*}{a_0}\right)^2 = \frac{2}{1+\gamma}}$$

$$\frac{P^*}{P_0} = \left(\frac{\rho^*}{\rho_0}\right)^{\gamma} = \left(\frac{T^*}{T_0}\right)^{\frac{\gamma}{\gamma-1}}$$

- ~~1D flow~~ Normal Shock:

P_1, T_1, ρ_1, U_1, $M_1 > 1$	$P_2 > P_1$, $T_2 > T_1$, $\rho_2 > \rho_1$, $U_2 < U_1$, $M_2 < 1$

$$\rho_1 U_1 = \rho_2 U_2$$

$$P_1 + \rho_1 U_1^2 = P_2 + \rho_2 U_2^2$$

$$h_1 + \frac{U_1^2}{2} = h_2 + \frac{U_2^2}{2}$$

- for an adiab flow of calorically PG:

$M_1^* M_2^* = 1 \Rightarrow a^{*2} = U_1 U_2$

Prandtl's relation.

4 important relations across a NS

i. $$M_2^2 = \frac{\frac{2}{\gamma-1} + M_1^2}{\frac{2\gamma}{\gamma-1} M_1^2 - 1}$$

ii. $$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1) M_1^2}{2 + (\gamma-1) M_1^2}$$

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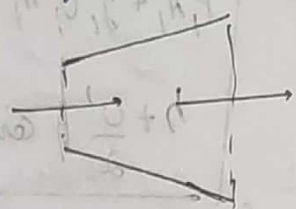
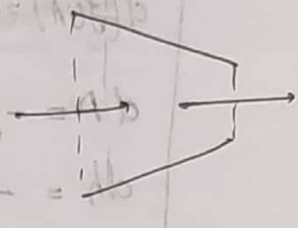
Quasi 1D Flow ↝ Flow through ducts with changing CS



- Flow property do not change normal to the length of the duct.
  i.e. all flow properties are uniform across a given CS & hence are function of x (& time, if it is unsteady) only!

- Under what ~~assumptions~~ conditions Quasi 1D assumption holds?
  i. CS area should change smoothly
  ii. Change in CS Area should be small
  ↳ (~~because~~ ~~if~~ else there can be flow separation)
  ⇕
  Quasi-1D framework assumes NO flow separation!



- In reality, flow in a variable area streamtube is a 3D flow.
  ↳ Quasi-1D flow is just an approximation which is widely used in → Flows through wind tunnels
  → Rocket engines.

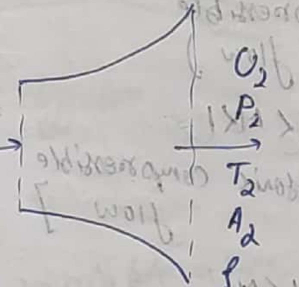
- for a stream tube CV:

assumption:

i. Steady state:

Mass Consv.

$$\boxed{\rho_1 U_1 A_1 = \rho_2 U_2 A_2}$$



ii. No body force, No viscous stress:

Momentum eqn

$$P_1 A_1 + \rho_1 U_1^2 A_1 + \int_{A_1}^{A_2} p\,dA = P_2 A_2 + \rho_2 U_2^2 A_2$$

iii. No heat transfer due to thermal conduction & diffusion, No change in PE

energy ~~eqn~~ Consv

$$\boxed{\left(h_1 + \frac{U_1^2}{2}\right) + q = h_2 + \frac{U_2^2}{2}}$$

q ↝ heat added per unit mass

for an adiabatic flow $q = 0 \Rightarrow$ $h + \frac{U^2}{2} = \text{Const}.$

$\Rightarrow h_0 = \text{Const}.$

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$$\rho U A = \text{Const.}$$

$$P_1 A_1 + \rho_1 U_1^2 A_1 + \int_{A_1}^{A_2} p\,dA = P_2 A_2 + \rho_2 U_2^2 A_2$$

$$h + \frac{U^2}{2} = \text{Const.}$$ adiabatic flow.

(differential form

$$d(\rho U A) = 0 \Leftrightarrow \frac{d\rho}{\rho} + \frac{dU}{U} + \frac{dA}{A} = 0$$

$$dP = -\rho U dU$$ Euler eqn

$$dh = -U dU$$

- For a steady, adiabatic + inviscid flow: isentropic flow

[Typically all the nozzle flows will be considered "isentropic" flow]

["isentropic" flow] entropy remains the same along any given streamtube.

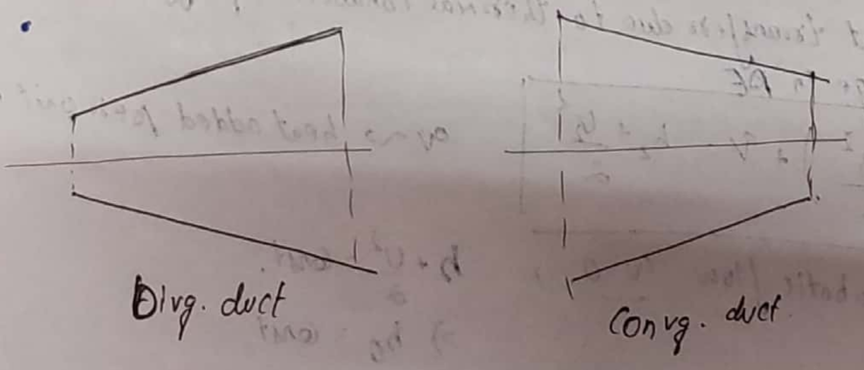
$$\boxed{\frac{dA}{A} = (M^2-1)\frac{dU}{U}}$$ Area - Velocity relationship.

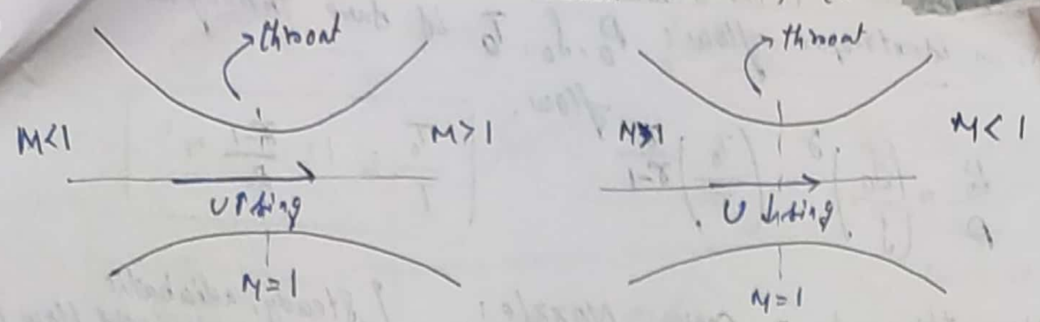
i. $M \to 0 \Rightarrow AU = \text{Const.}$ $A\uparrow \Leftrightarrow U\downarrow$
(Incompressible flow)

ii. $0 < M < 1 \Rightarrow A\uparrow \Leftrightarrow U\downarrow$
[Subsonic compressible flow]

iii. $1 < M \Rightarrow \boxed{A\uparrow \Leftrightarrow U\uparrow}$ *
[Supersonic flow]

* iv. $M = 1 \Rightarrow \frac{dA}{A} = 0 \Rightarrow$ Minima in Area (Physically realistic)



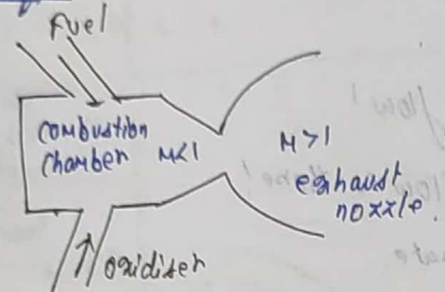


subsonic → supersonic
acc.

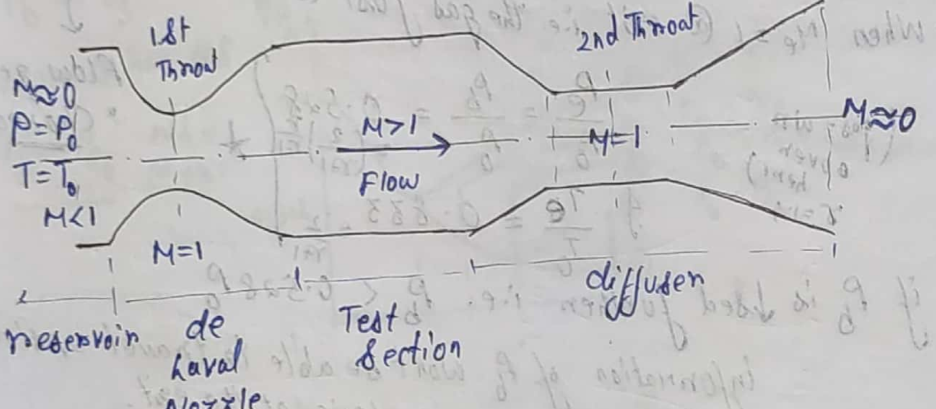
supersonic → subsonic
deceleration

- Conv. - Div. Nozzle (CD Nozzle) / de-Laval nozzle / Laval nozzle.

- **Rocket Engine:**

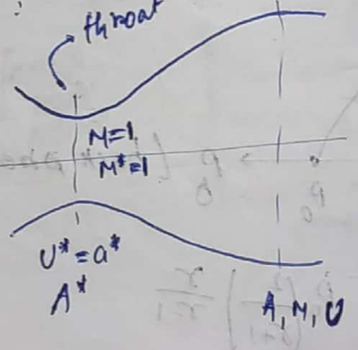


- **Supersonic wind Tunnel:**



- The analysis of flow through variable area ducts in general sense requires numerical solutions. However, for a calorically PG a closed form solution is possible.

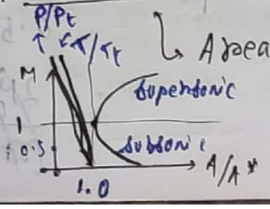
- Isentropic flow of a calorically PG through variable area ducts:



$M = f\left(\frac{A}{A^*}\right)$

$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{\gamma-1}}$$

↳ Area-Mach no. relationship.

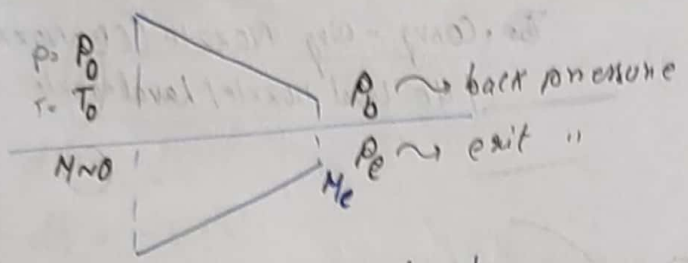


For an isentropic flow: P_0, ρ_0, T_0 is same throughout the flow.

- $$\frac{P_0}{P} = \left(\frac{\rho_0}{\rho}\right)^{\gamma} = \left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\boxed{\frac{T_0}{T} = 1 + \frac{\gamma-1}{2} M^2}$$

⊙ Flow through a conv. Nozzle: [Steady, adiabatic inviscid flow] → isentropic flow



When $P_b = P_0 \Rightarrow$ No flow!

$P_b < P_0 \Rightarrow$ Flow is there!

(gas will accelerate.

- $\boxed{P_e = P_b \text{ as long as } M_e \le 1}$

$\Rightarrow$ When $M_e = 1$ @ exit, i.e. the gas just turns sonic.

(gas = air over here) $\gamma = 1.4$

$$\left.\begin{aligned} \frac{P_e}{P_0} &= \frac{P_b}{P_0} = 0.528 = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} \\ \& \ \frac{T_e}{T_0} &= 0.833 = \frac{2}{\gamma+1} \end{aligned}\right\} \star$$

Flow gets "Choked"!

- if P_b is used further i.e. $P_b < 0.528 P_0$

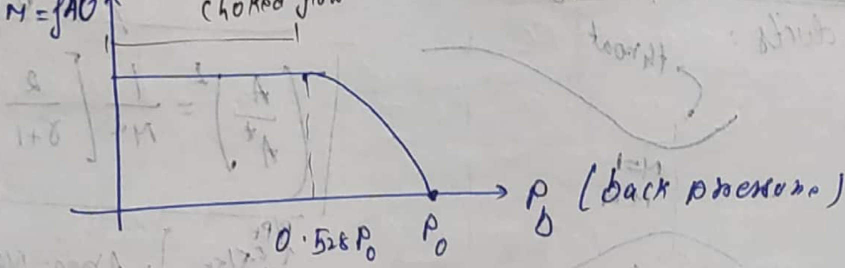
information of P_b won't be able to travel upstream as the flow becomes sonic at the exit.

Flow gets "Choked" $\Rightarrow$ No further Ise in Mass flow rate possible!

Solution through the Nozzle remains unchanged.

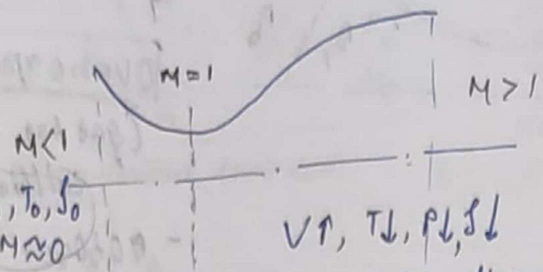
i.e. $\frac{P_e}{P_0} = 0.528$ but $P_b < P_e$

$\& \ M_e = 1$



$$\frac{2 P_0}{\gamma+1} \quad P_0 \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}$$

⊙ Flow through a CD Nozzle : [Steady, isentropic flow]



$V\uparrow,\ T\downarrow,\ P\downarrow,\ \rho\downarrow$

expanding flow.

M_e ↳ sometimes called "design Mach no."

$\boxed{\text{back pressure is the guiding factor!}}$

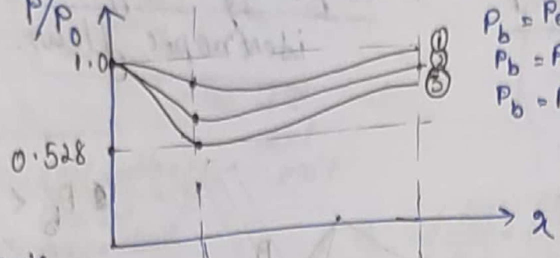
i. $P_b = P_0 \Rightarrow$ No flow

ii. $P_b < P_0 \Rightarrow$ Flow

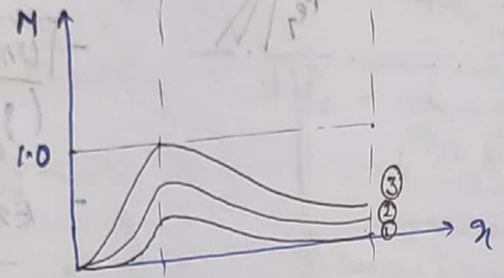
↳ $P_{e_3} \leq P_b < P_0$

Flow is subsonic throughout

at $P_b = P_{e_3}$ $M=1$ @ throat



$P_b = P_{e_1}$
$P_b = P_{e_2}$
$P_b = P_{e_3}$

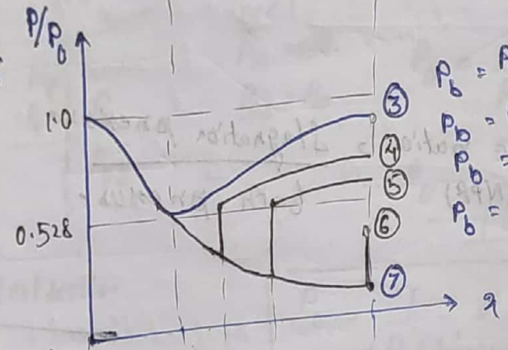


$\boxed{\text{at ③ Nozzle is Choked!}}$

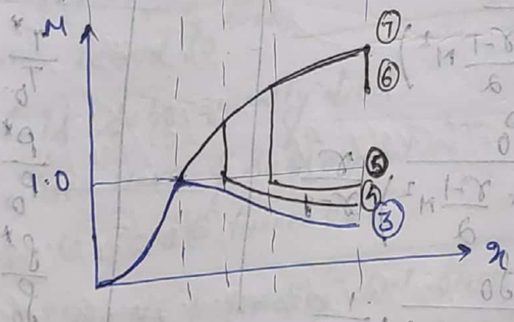
$P_{e_6} \leq P_b < P_{e_3}$

Flow has NS

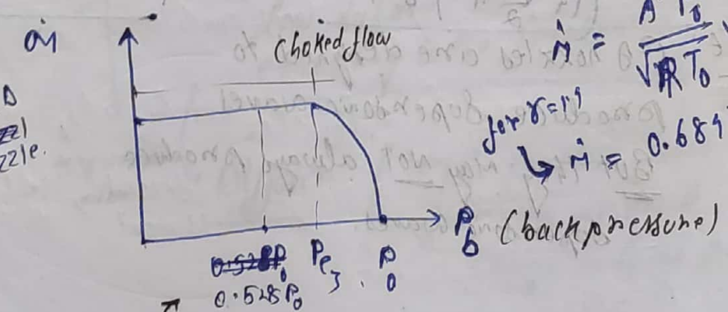
at $P_b = P_{e_6}$ NS is @ exit



$P_b = P_{e_3}$
$P_b = P_{e_4}$
$P_b = P_{e_5}$
$P_b = P_{e_6}$



for CD ~~nozzle~~ NOZZLE

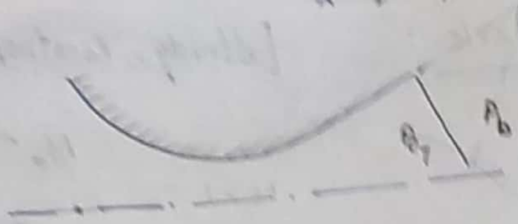


↳ $\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} P_0$

$$\dot{m} = \frac{A^* P_0}{\sqrt{RT_0}} \sqrt{\gamma\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}$$

for $\gamma = 1.4$

↳ $\dot{m} = 0.6847 \dfrac{A^* P_0}{\sqrt{RT_0}}$



$P_{e_7} < P_b < P_{e_6}$

- **Overexpanded flow** (gas has expanded more within the nozzle $P_{e_7} < P_b$)
- oblique shock @ nozzle end.

• only at $P_b = P_{e_7}$ ~ completely shock free supersonic isentropic flow.

Correctly expanded.

• $P_b < P_{e_7}$

- **Underexpanded flow** (gas will further expand outside the duct)
- Expansion fan.

Nozzle pressure ratio (NPR) = $\dfrac{\text{Stagnation pressure}}{\text{back pressure}}$

$$T = \frac{T_0}{\left(1+\frac{\gamma-1}{2}M^2\right)}$$

$$P = \frac{P_0}{\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}}$$

$$\rho = \frac{\rho_0}{\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{1}{\gamma-1}}}$$

for air ($\gamma = 1.4$)

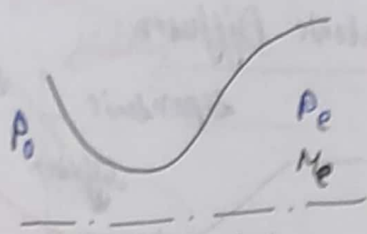
$$\frac{T^*}{T_0} = 0.833$$

$$\frac{p^*}{p_0} = 0.528$$

$$\frac{\rho^*}{\rho_0} = 0.633$$

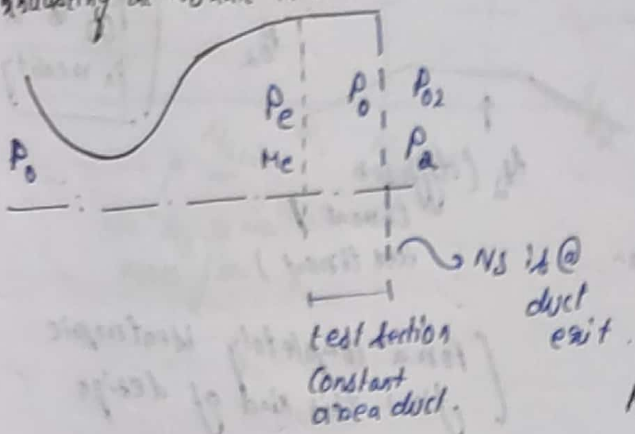
NOTE: CD nozzles are designed to produce supersonic waves. But they may NOT always produce supersonic waves.

Diffuser :



In order to avoid shock / expansion wave in the test region downstream of the exit, $P_e = P_b$ ~~= P_{atmosphere}~~

- exhausting the nozzle into a constant area duct :



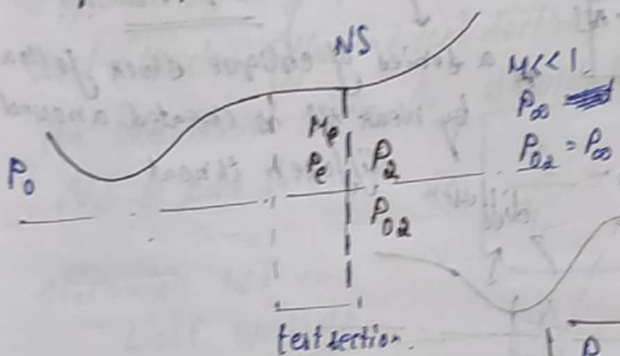
$$P_0 = \left(\frac{P_0}{P_e}\right)_{@M_e} P_e$$

$$\boxed{P_0 = \frac{1}{(P_e/P_0)^{isentropic}_{@M_e}} P_b}$$

$$P_0 = \left(\frac{P_0}{P_e}\right)_{@M_e} \cdot \left(\frac{P_e}{P_2}\right)_{@M_e} \cdot P_2$$

$$\Rightarrow \boxed{P_0 = \frac{1}{(P_e/P_0)^{isentropic}_{@M_e}} \cdot \frac{1}{(P_2/P_e)^{NS}_{@M_e}} \cdot P_2}$$

- adding a div. nozzle.



$$P_0 = \frac{P_0}{P_e} \cdot \frac{P_e}{P_2} \cdot \frac{P_2}{P_\infty} \cdot P_\infty$$

$$\Rightarrow \cancel{P_0 = \frac{1}{(P_e/P_0)_{@M_e}} \cdot \frac{1}{(P_2/P_e)_{@M_e}} \cdot \frac{1}{(P_2/P_{02})_{@M_e}}}$$

$$\boxed{P_0 = \frac{1}{(P_e/P_0)^{isentropic}_{@M_e}} \cdot \frac{1}{(P_2/P_e)^{NS}_{@M_e}} \cdot \left(\frac{P_2}{P_{02}}\right)_{@M_e} \cdot P_{\infty}}$$

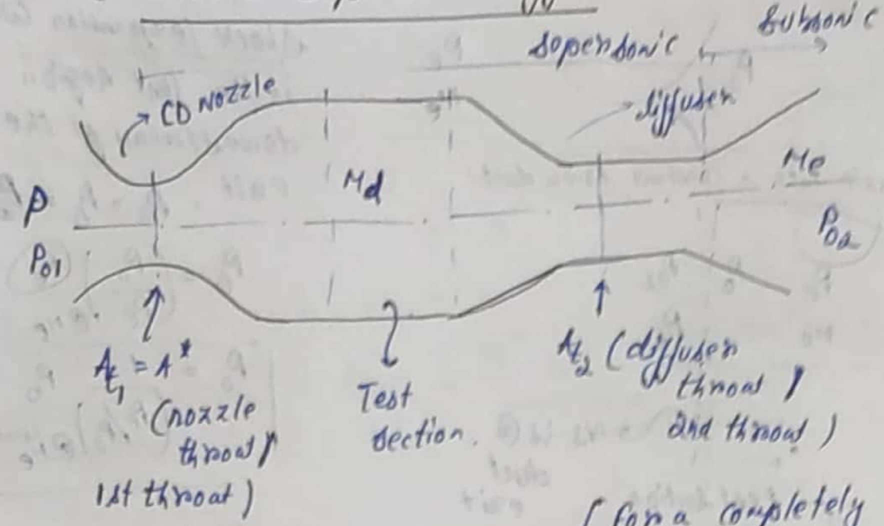
We infer that : (& hence power required to operate)

the reservoir pressure required to drive the wind tunnel is subsequently led by the creation of shock wave & subsequent isentropic diffusion to $M \simeq 0$ @ tunnel exit.

The NS & the div. duct slows the air to low subsonic speeds before exhausting to the atmosphere.

this mechanism → **Diffuser** [Target: to slow down the flow with as small change in P_0 as possible]

· Conventional Supersonic Diffuser:



$P_{02} \neq P_{01}$ in reality

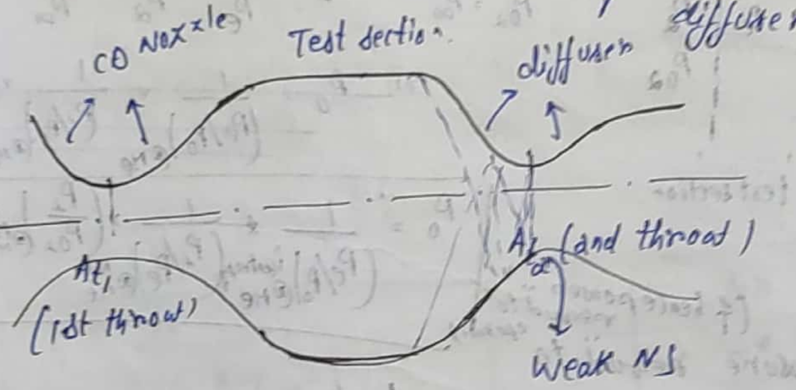
$M_d \rightarrow$ design Mach no.
$M_e \rightarrow$ exit Mach no.

P_0 drop across a NS
> P_0 drop across a series of oblique shock followed by weak NS

[For a completely isentropic flow this kind of design should have $P_{02} = P_{01}$
i.e. No total pressure loss]

However in reality this is NOT possible.!!

★ a series of oblique shock followed by weak NS is created around the diffuser throat

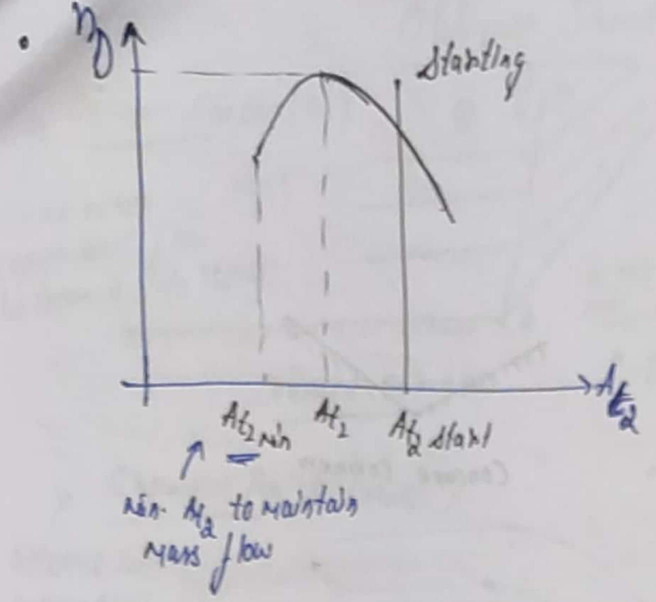


Diffuser efficiency, $\eta_D = \dfrac{(P_{02}/P_{01})_{actual}}{(P_{02}/P_{01})_{NS\,@\,Test\ Section\,@\,M_d}}$

★ { $\eta_D > 1 \sim$ for low supersonic ~~flow~~ test section Mach no. M_e
$\eta_D < 1 \sim$ " Hypersonic condition.

· $\boxed{\dfrac{A_{t_2}}{A_{t_1}} = \dfrac{P_{01}}{P_{02}}}$ $P_{01} > P_{02} \Rightarrow A_{t_2} > A_{t_1}$
second throat > 1st throat.

$A_{t_2} > A_{t_1}$!



$$\boxed{\frac{(A_{t_2})_{\min}}{A_{t_1}} = \frac{P_{01}}{P_{02}}\bigg|_{actual}}$$

- At a large value of A_{t_2}, gas is <u>NOT</u> sufficiently compressed in the diffuser.
- Flow may remain supersonic past the 2nd diffuser throat & ~~the~~ a NS can be produced downstream of the 2nd throat which ↓ efficiency!

Starting problem: (related to operation of Wind Tunnel)

When valves (separating settling chamber from the duct) is ~~opened~~ suddenly opened, then a NS passes through the duct (Nozzle + diffuser + Test section)

↳ The 2nd throat area, A_{t_2} should be large enough to allow NS to pass through i.e. $\boxed{\frac{A_{t_2}}{A_{t_1}} = \frac{P_{01}}{P_{02}}\bigg|_{NS}}$

otherwise WT wont operate.

Therefore, In the starting (transient phase) 2nd throat area > ~~that~~ of in the most efficient output.

↳ i.e. efficiency at $A_{t_2}|_{start}$ is less.

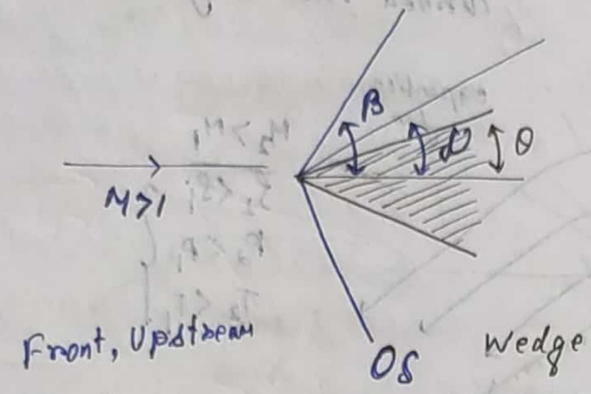
↳ Variable Geometry diffusers should be employed to resolve the problem!!

Oblique Shock :

★ **NOTE:** ~~OS &~~ Mach wave & NS are the two extreme cases of an OS.

- Mechanism of OS formation:

Mach angle, $\nu = \sin^{-1}\left(\frac{1}{M}\right)$



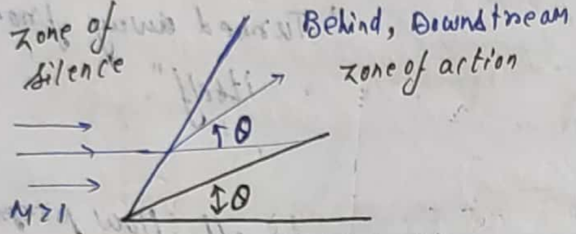
The strong disturbances coalesce into an OS wave at an angle β to the free stream where $\beta > \nu$.

$\beta \rightarrow$ OS angle

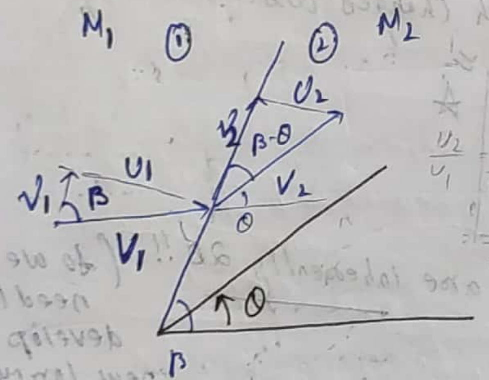
$\theta \rightarrow$ half angle of wedge. = Flow deflection angle.

$\nu \sim$ Mach angle.

$M \rightarrow$ preshock Mach no.



- Oblique Shock relations :



assumptions:

i. Steady state:

Mass Consv. $\rightarrow$ $\boxed{\rho_1 U_1 = \rho_2 U_2}$

ii. No body force, No viscous stress

Momentum Consv. $\rightarrow$ $P_1 + \rho_1 U_1^2 = P_2 + \rho_2 U_2^2$

$\rightarrow$ $v_1 = v_2$

iii. adiabatic flow

$$h_1 + \frac{1}{2}V_1^2 = h_2 + \frac{1}{2}V_2^2$$

$$\equiv h_1 + \frac{1}{2}U_1^2 = h_2 + \frac{1}{2}U_2^2$$

$$\left.\begin{aligned} &\rho_1 U_1 = \rho_2 U_2 \\ &P_1 + \rho_1 U_1^2 = P_2 + \rho_2 U_2^2 \\ &v_1 = v_2 \end{aligned}\right\}$$

$$h_1 + \frac{1}{2}U_1^2 = h_2 + \frac{1}{2}U_2^2.$$

$$M_{n_1} = M_1 \sin\beta$$

$$M_{n_2} = M_2 \sin(\beta-\theta)$$

★★ **NOTE: vvimp** All the static properties ratio across the OS wave can be obtained by using M_{n_1}; however to obtain "total" / "stagnation" property you need to use the M_1 & M_2 NOT M_{n_1} or M_{n_2}

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★ NOTE: Once we obtain the static properties, the "total"/ "stagnation" properties are determined from the isentropic flow relations: (v.imp.)

$$T = \frac{T_0}{\left(1+\frac{\gamma-1}{2}M^2\right)}, \quad p = \frac{p_0}{\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}}, \quad \rho = \frac{\rho_0}{\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{1}{\gamma-1}}}$$

- as the flow is adiabatic throughout so, a^* is same throughout the flow!

$$\boxed{\frac{\tan(\beta-\theta)}{\tan\beta} = \frac{u_2}{u_1} = \frac{\rho_1}{\rho_2} = \frac{2+(\gamma-1)M_{n_1}^2}{(\gamma+1)M_{n_1}^2}}$$ → Calorically PG assumed

$M_{n_1} = M_1 \sin\beta$

$$\boxed{\tan\theta = 2\cot\beta\left[\frac{M_1^2\sin^2\beta - 1}{M_1^2(\gamma+\cos 2\beta)+2}\right]}$$

★ θ-β-M relation!!

For calorically PG:

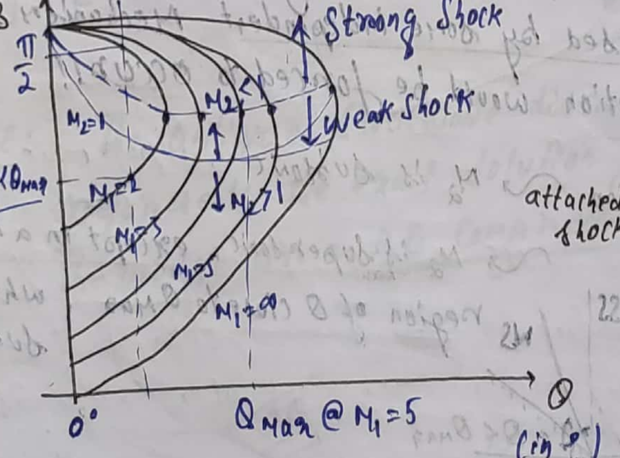
$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_{n_1}^2}{2+(\gamma-1)M_{n_1}^2}$$

the relations for NS @ M_{n_1} (for a calorically PG)

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}(M_{n_1}^2 - 1)$$

$$\frac{T_2}{T_1} = \frac{h_2}{h_1} = \left[1+\frac{2\gamma}{\gamma+1}(M_{n_1}^2-1)\right] \times \left[\frac{2+(\gamma-1)M_{n_1}^2}{(\gamma+1)M_{n_1}^2}\right]$$

$$M_{n_2}^2 = \frac{\frac{2}{\gamma-1} + M_{n_1}^2}{\frac{2\gamma}{\gamma-1}M_{n_1}^2 - 1}$$

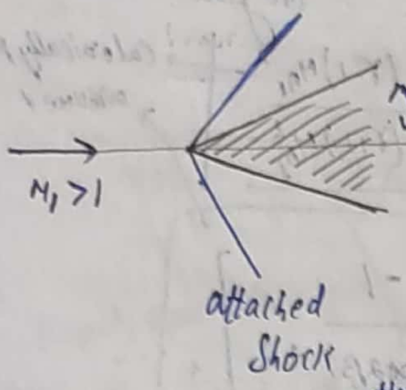


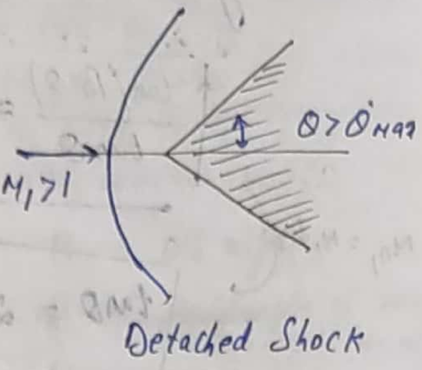
$M_1 \geq 1$, $\theta < \theta_{max}$ for OS to exist

The plot is for attached shocks / Oblique shocks (for calorically PG)

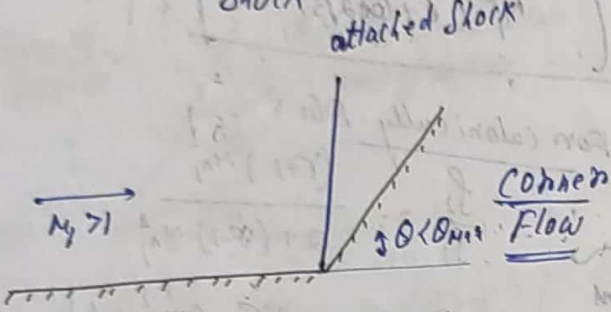
Observations: i. for any given M_1, there is a Max deflection angle, θ_{max}

if $\theta > \theta_{max}$ there is No OS possible. instead the shock will be curved & detached ~ Bow Shock

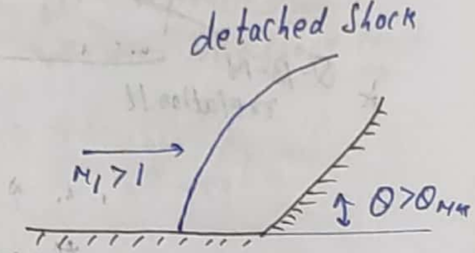




Wedge



Corner Flow



ii. For a given Mach no. M, for $\theta < \theta_{max}$ there are 2 valid βs. → higher value of β ~ corresponds to Strong Shock solution

↓

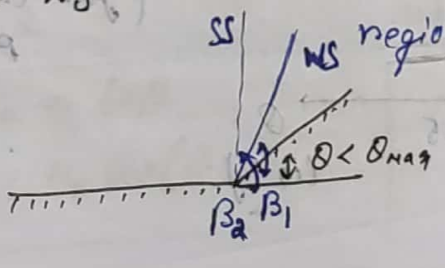
lower value of β ~ " " Weak Shock "

NOTE: nature favours weak shock solution

→ however whether WS or SS will be observed depends on the back pressure i.e. if the downstream pressure were ↑sed by some independent mechanism, then the SS solution would be forced to occur!!

★ in the SS solution ~ M_2 is subsonic

in the WS " ~ M_2 is supersonic, except in a narrow region of θ close to θ_{max}, where it is subsonic.



⊙ at $\theta = 0°$, $\rightarrow \beta = \pi/2$ (NS ~ corresponding to SS)

$\searrow \beta = \sin^{-1}\left(\frac{1}{M}\right)$ (~~MS~~ Mach wave ~ corresponding to WS)

- as $M_1 \downarrow$ $\theta_{max} \downarrow$

- at $M_1 = 1$, both the solutions conv. i.e. $\beta = \mu = \frac{\pi}{2}$

• $\frac{P_2 - P_1}{P_1} = \frac{\Delta P}{P_1}$ ~ is a measure of Shock Strength!

$(\Delta P)_{\text{Strong shock}} > (\Delta P)_{\text{weak shock}}$

$\Rightarrow (P_2)_{SS} > (P_2)_{WS}$

★ For Weak Shock Solution: $M_{n_1} = M_1 \sin\beta$

Stronger Shock wave $\Downarrow$ $P_2, \rho_2, T_2 \uparrow$

i. as $\theta \uparrow$ses (holding M_1 = Const.) Shock wave becomes Stronger & $\beta \uparrow$ses.

ii. as $M_1 \uparrow$ses (holding θ = Const.) Shock wave becomes Stronger & $\beta \downarrow$ses.

↳ observe that even though $\beta \downarrow$ but $M_1 \sin\beta \geq 1$ (as $\beta \geq \mu$)

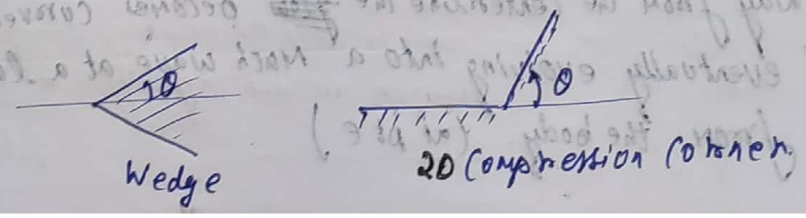
For Strong Shock Solutions:

[this is to be confirmed from a verified source.]

i. as $\theta \uparrow$ses (holding M_1 = Const.) Shock wave becomes weaker & $\beta \downarrow$ses.

ii. as $M_1 \uparrow$ses (holding θ = Const.) Shock wave becomes ~~Stronger~~ weaker & $\beta \uparrow$ses.

NOTE: The oblique shock wave discussed so far represents the exact solution for flow over the wedge or a 2D Compression corner.



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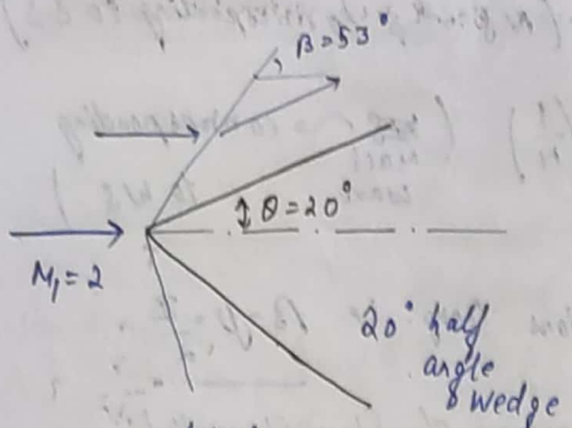
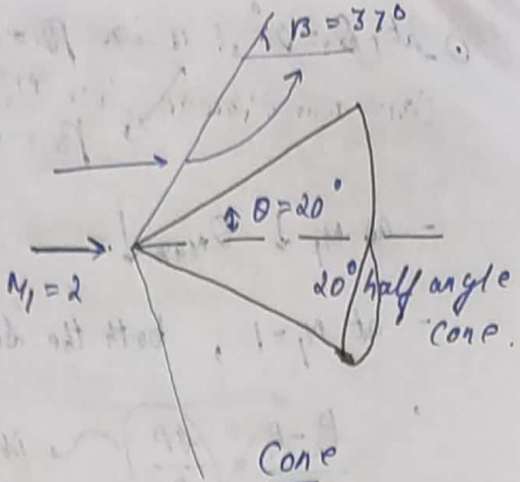


Wedge

Pressure on the surface of the wedge is const.



Cone

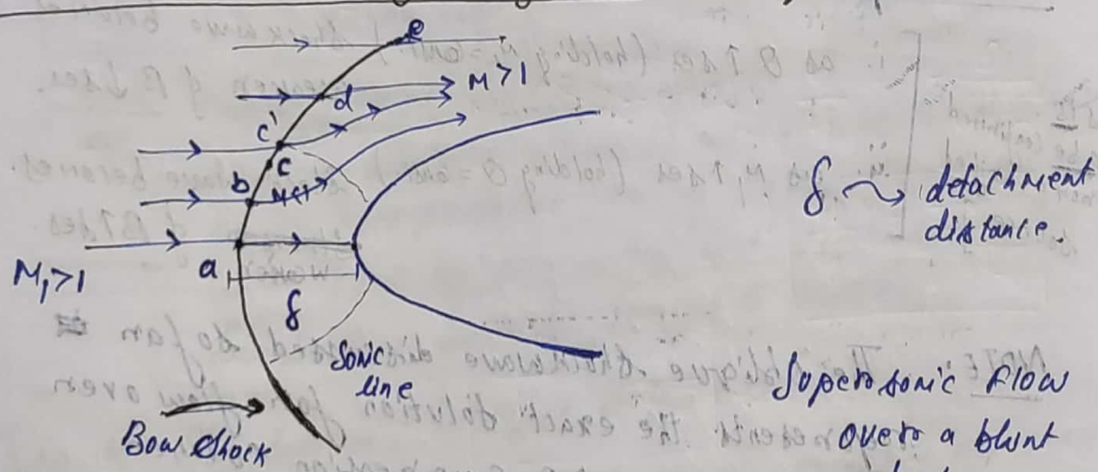
Unlike the wedge flow over a cone is inherently 3D & hence is NOT uniform.

[Note: the properties immediately behind the cone are given by the OS]

- The "3D relieving effect" of the flow over the cone results in a weaker shock wave than for a wedge of same angle.

lower P_2, ρ_2 & T_2 behind the shock than that of wedge of same half angle θ

⊙ Detached Shock wave in front of a blunt body:

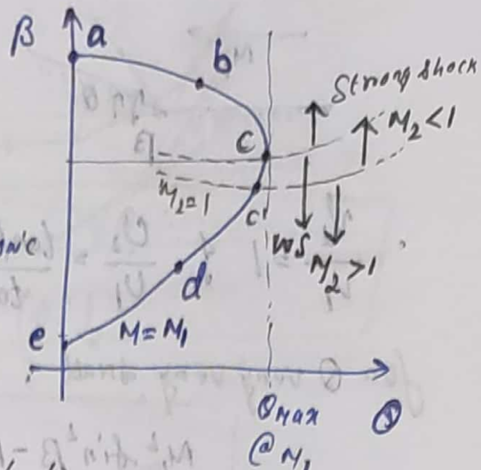


δ ~ detachment distance.

Supersonic flow over a blunt body.

- pt. a ~ corresponds to NS as upstream flow is ⊥ to the wave.
- Away from the centerline the ~~flow~~ shockwave becomes curved & weaker, & eventually evolving into a Mach wave at a large distance from the body. (at pt e)

①. between pt. a & e the curved shock goes through all possible conditions allowed for OS for an upstream Mach no. of M_1.



- From pt. a to pt. c', the flow after the shock is subsonic

- ~~From~~ Above pt. c' the flow is supersonic (imp) behind the shock.

  . So the flowfield between the curved bowshock & the blunt-body is a mixture of subsonic - supersonic flow & the imaginary curved line dividing the two flows in these region is called Sonic line.

: from pt a to pt c. ⟶ Strong shock

the streamline through pt. c experiences Max deviation!

from pt c to pt e. ⟶ weak shock.

② .The shape of the detached shock ⟶ standoff distance

. Detachment distance δ

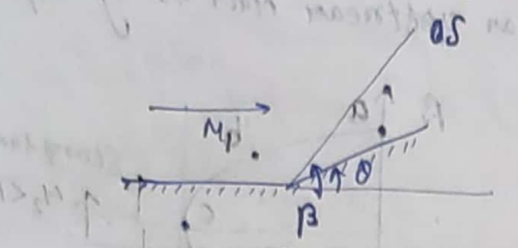
. The complete ~~streamline~~ flowfield (with curved streamline) between the shock & the body

⟶ depends on M_1 & the (shape & size) of the body.

. properties of a NS depends on ⟶ M_1 only.

. properties of an OS depends on ⟶ two parameters $(M_1, \theta \,\&\, \beta)$ any 2 can be selected.

① Weak Oblique Shock:



Weak oblique shocks appear for small deflection angle θ.

at small deflection angle, θ

Shock angle, $\beta \approx \mu$

- $\frac{V_2}{V_1} \approx 1$ & $\frac{u_2}{u_1} = \frac{\tan(\beta-\theta)}{\tan\theta} = \frac{2+(\gamma-1)M_1^2\sin^2\beta}{(\gamma+1)M_1^2\sin^2\beta}$

for θ very very small;

$$\boxed{M_1^2\sin^2\beta - 1 = \left(\frac{\gamma+1}{2}M_1^2\tan\beta\right)\theta}$$

now, $\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}\left(M_1^2\sin^2\beta - 1\right)$

i.e. $\boxed{\frac{p_2}{p_1} = 1+\gamma M_1^2\tan\beta\,\theta}$ ~ for θ very very small!

$$\boxed{\frac{\Delta p}{p_1} = \left(\gamma M_1^2\tan\beta\right)\Delta\theta}$$ [here $\Delta\theta = \theta - 0$]

$\frac{\Delta p}{p_1} \propto \Delta\theta$

Strength of shock wave.

- $\beta = \varepsilon + \mu$, $\varepsilon \to 0$ for weak OS

$$\boxed{\varepsilon = \frac{\gamma+1}{4}\,\frac{M_1^2}{M_1^2-1}\,\Delta\theta}$$ $\varepsilon \propto (\Delta\theta)$

- Change of speed across ~~shock~~ weak OS, $\frac{V_2}{V_1} = 1 - \frac{\theta}{\sqrt{M_1^2-1}}$

$\Rightarrow \boxed{\frac{\Delta V}{V_1} = \frac{-\Delta\theta}{\sqrt{M_1^2-1}}}$

$$\boxed{\frac{dV}{V} = \frac{-d\theta}{\sqrt{M^2-1}}}$$

- Change of entropy for weak OS, is $\Delta s \propto (\Delta\theta)^3$

therefore, if we want to keep the entropy change minimum we have to go for stepwise turn / smooth turn!

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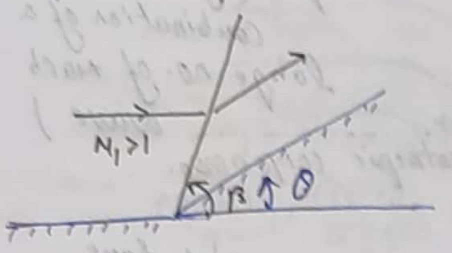
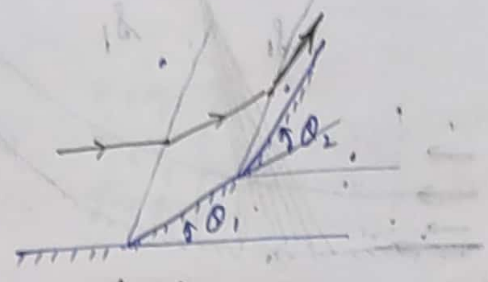
Supersonic compression by Stepwise / Smooth turning :

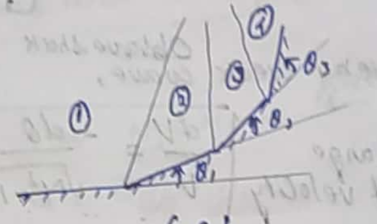


$(\Delta S)_1$



$(\Delta S)_2$

$(\Delta S_1) > (\Delta S)_2 > (\Delta S)_3$



$(\Delta S)_3$

• Assumption made :

each region of the flow is ~~[illegible]~~ independent of the following one & therefore the flow can be assumed to be constructed step by step.

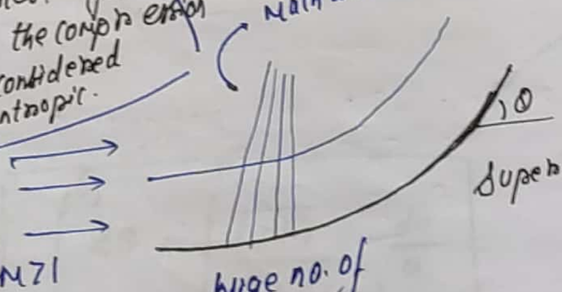
this is due to limited upstream influence i.e. info. doesnot travel upstream. → true as long as the flow after the shock is supersonic.

• as $(\Delta S) \propto (\Delta \theta)^3$ for weak OS

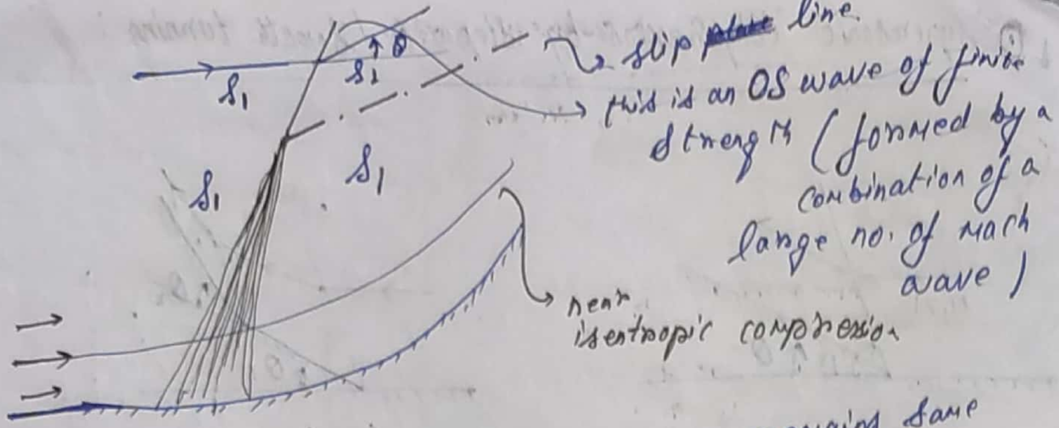
the total entropy change if the compression is obtained using a large no. of weak waves is $\ll$ than the single shock giving a net deflection $\Delta\theta$.

ΔS can be made contentionally small & the compression can be considered isentropic.

Mach wave (isentropic compression wave)



Supersonic compression by smooth turning.

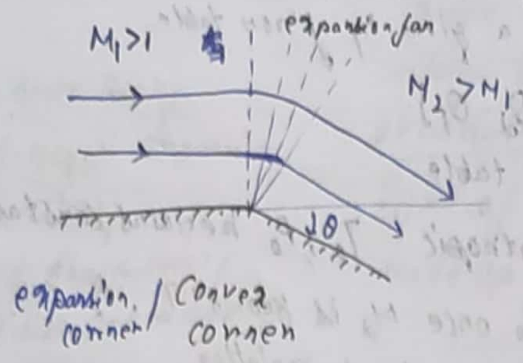


Across a slip line:
- pressure remains same
- magnitude of velocity } changes abruptly
- entropy } changes abruptly

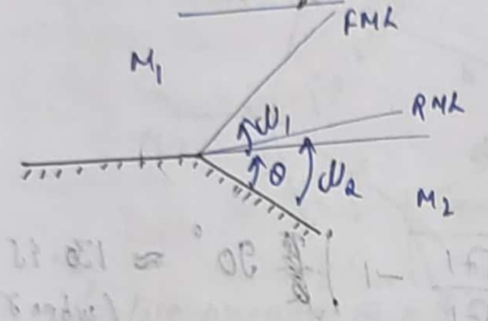
- for very weak Oblique shock wave,

Change of velocity $\boxed{\dfrac{dV}{V} = \dfrac{-d\theta}{\sqrt{M^2-1}}}$

⊙ Prandtl Meyer Expansion Fan : / Supersonic expansion by turning.



- The expansion fan itself is a continuous expansion region consisting of an ∞ no. of Mach waves bounded by upstream Mach angle μ_1 & downstream Mach angle μ_2.



as $M_2 > M_1$
$\Rightarrow \mu_2 < \mu_1$

FML → Forward Mach line

RML → Rearward Mach line.

- Streamlines through expansion waves are smooth curved lines [No discontinuity in properties]
- Flow is isentropic throughout.

$$\boxed{\frac{dV}{V} = \frac{d\theta}{\sqrt{M^2-1}}}$$

Governing differential eqn for Prandtl Meyer flow.

This eqn is derived from geometry & hence is valid for any type of gas.

$$\int_{V_1}^{V_2} \sqrt{M^2-1}\,\frac{dV}{V} = \int_{\theta_1}^{\theta_2} d\theta$$

- from $V = Ma$

$$\frac{dV}{V} = \frac{dM}{M} + \frac{da}{a}$$

assuming calorically PG & flow to be adiab.

we get. $\frac{dV}{V} = \frac{1}{\left(1+\frac{\gamma-1}{2}M^2\right)}\frac{dM}{M}$.

so, $$\boxed{\int_{\theta_1}^{\theta_2} d\theta = \int_{M_1}^{M_2} \frac{\sqrt{M^2-1}}{\left(1+\frac{\gamma-1}{2}M^2\right)}\frac{dM}{M}}$$

$$\boxed{\theta_2-\theta_1 = \nu(M_2) - \nu(M_1)}$$

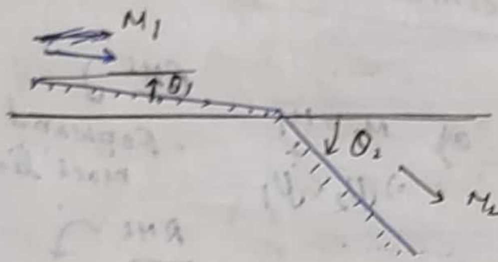
* Prandtl Meyer function!

$$\nu(M) = \int \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M}$$

$$\Rightarrow \nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left(\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)}\right) - \tan^{-1}\left(\sqrt{M^2-1}\right) + C \quad = 0 \text{ (assumed)}$$

· **Algorithm for Prandtl Meyer Expansion Fan:**

1. Obtain $\nu(M_1)$ for a given M_1 from table.
2. $\nu(M_2) = \nu(M_1) + (\theta_2 - \theta_1)$
3. Obtain M_2 from the table.
4. Since flow is isentropic T_0, P_0 remains constant therefore once M_2 is known T_2, P_2 can be determined from the use of isentropic relation.



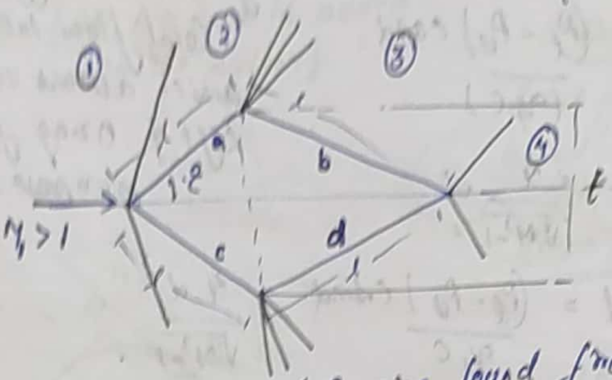
$\theta_2 \Rightarrow, \theta_1$ angle with horizontal

· as $M \to \infty$ $\quad \nu(M) \to \left(\sqrt{\frac{\gamma+1}{\gamma-1}} - 1\right) 90^\circ \approx 130.45^\circ$ (when $\gamma = 1.4$)

① **Shock-Expansion Theory:** The oblique shock & expansion wave discussed so far allows the exact calculation of aerodynamic forces (lift & drag) of many type of 2D supersonic airfoils made up of straight line segments.

- Oblique shock theory
- Prandtl-Meyer expansion wave theory
- detached shock theory (not discussed in detail)

ex. • Symmetrical diamond wedge airfoil:



$\theta \sim$ semiangle of the symmetric diamond wedge airfoil.

AoA = 0°.

$D = (P_2 - P_3) t S$

$S \rightarrow$ span

Surface pressure @ a & c are found from OS theory.
" " @ b & d " " EF "

• a supersonic inviscid flow over an ∞ span wing gives a finite drag. → This new source of drag in supersonic flow is called Wave Drag!

~ this is inherently related to the loss of total pressure & ↑se in entropy across the OS waves created by the airfoil.

- **Thin airfoil Theory:** for thin airfoil at small AOA, we can make use of approximate relations for weak expansion waves, which leads to simple theoretical expressions for L & D.

For weak shocks, $\frac{\Delta P}{P} = \frac{\gamma M^2}{\sqrt{M^2-1}}\,\theta$

$\hookrightarrow \boxed{C_P = \frac{2}{\sqrt{M^2-1}}\,\theta}$

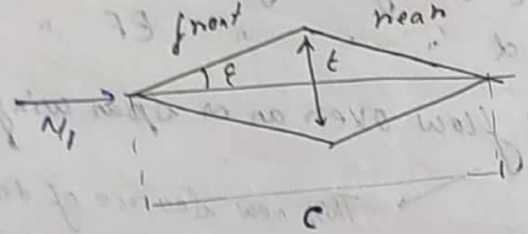
$C_P = \pm \frac{2\alpha}{\sqrt{M^2-1}}$

$\hookrightarrow$ pressure coeff. is $\propto$ local flow inclinations, which allows us to find Lift & Drag for various purpose.

Lift coeff. : $C_l = \frac{(P_l - P_U)\cos\alpha}{(q_1 c)}$

$= \frac{4}{\sqrt{M^2-1}}\,\alpha$

Drag coeff., $C_d = \frac{(P_l - P_U)\,c\sin\alpha}{q_1 c} = \frac{4\alpha^2}{\sqrt{M^2-1}}$

- 

[Thin airfoil]

thickness, t

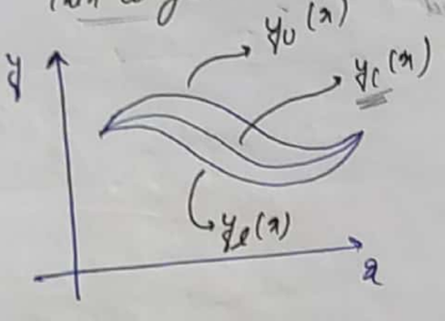
$\alpha = 0°$

$C_P = \frac{2}{\sqrt{M_1^2-1}}\,\varepsilon$ (front)

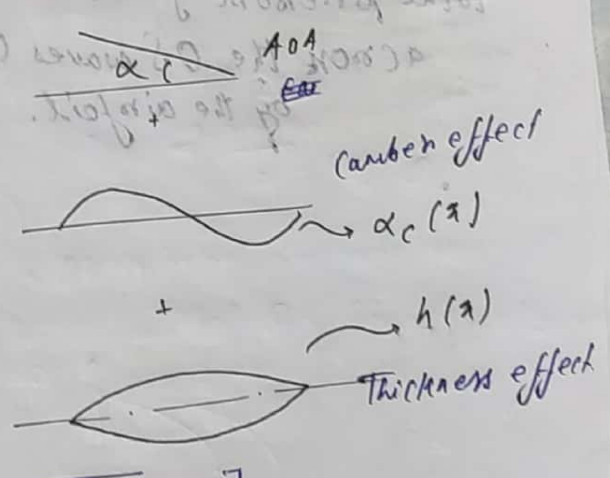
$C_P = \frac{-2}{\sqrt{M_1^2-1}}\,\varepsilon$ (rear)

$C_d = \frac{4\varepsilon^2}{\sqrt{M_1^2-1}} = \frac{4}{\sqrt{M_1^2-1}}\left(\frac{t}{c}\right)^2$

- A generic result for any arbitrary thin airfoil:



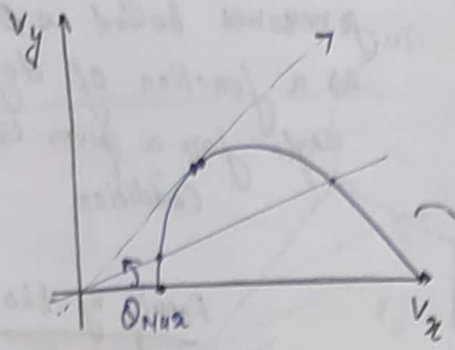
$\equiv$



~~the lift~~ $C_l = \frac{4\alpha}{\sqrt{M_1^2-1}}$

$$C_d = \frac{4}{\sqrt{M_1^2-1}}\left[\underset{\text{due to lift}}{\alpha^2} + \underset{\text{due to camber}}{\alpha_c^2(x)} + \underset{\text{due to thickness}}{\left(\frac{d\,h(x)}{dx}\right)^2}\right]$$

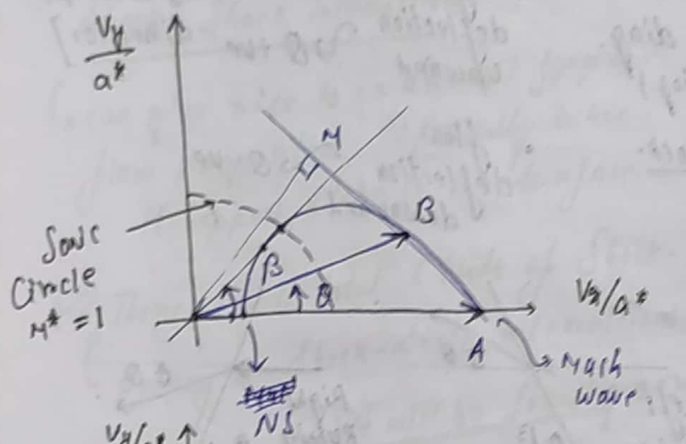
Hodograph plane:



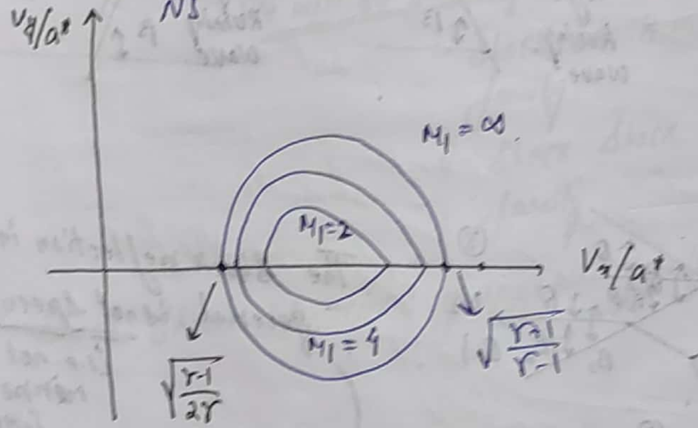
→ Shock polar.

if θ is varied ~~for~~ through all possible θ for which there is an attached shock i.e. $\theta \le \theta_{max}$, then the locus of all possible velocities behind the shockwave for a given M_1 is called the shock polar.

- a^* remains the same before & after the shock. (as No heat is added to the system)



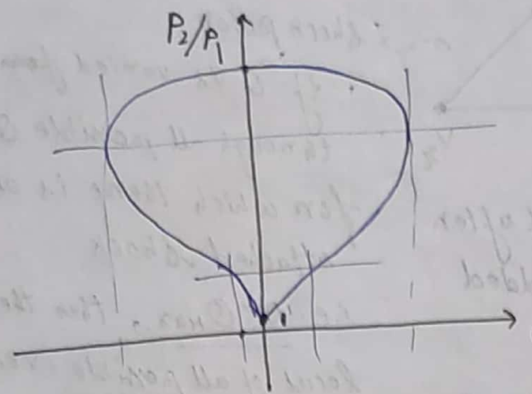
[convenience of using M^* is that as $M \to \infty$, $M^* \to 2.45$]



- The analytic equation for the shock polar:

$$\left(\frac{V_y}{a^*}\right)^2 = \frac{\left(M_1^* - V_x/a^*\right)^2\left[\left(\frac{V_x}{a^*}\right)M_1^* - 1\right]}{\frac{2}{\gamma+1}\left(M_1^*\right)^2 - \left(\frac{V_x}{a^*}\right)M_1^* + 1}$$

- Pressure Deflection Diagram: ↳ locus of all possible static pressures behind an OS wave as a function of deflection angle for a given upstream condition.



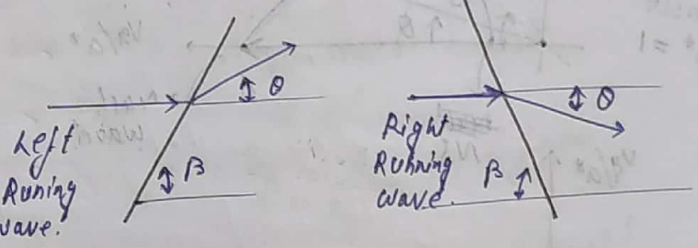
↑ pressure deflection diag. (Pθ diag.)

[we use Pθ diag. in Shock-Shock interactions]

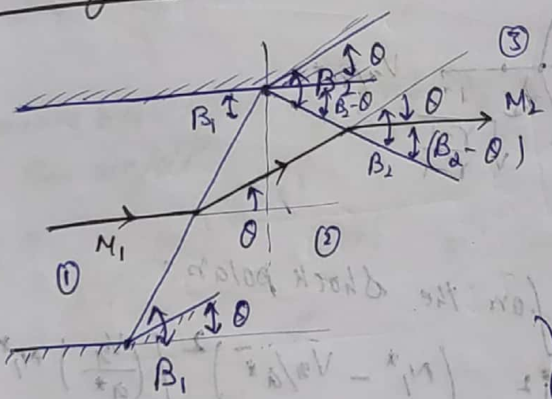
Family of shock
- Left running
- Right running.

[Left & Right is wrt upstream direction]

- flow deflection upward ↝ θ +ve.
- flow deflection downward ↝ θ -ve.



- Shock Reflection:

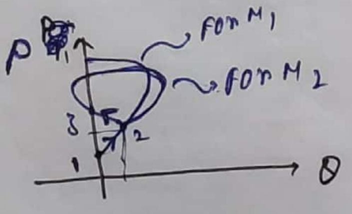


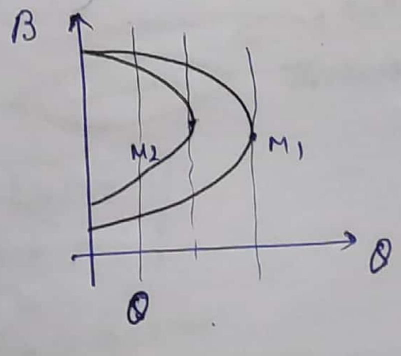
The shock reflection in general is not specular!! (i.e. not mirror like)

↳ in general $\beta_1 \neq (\beta_2 - \theta)$

↳ over here we have assumed that $\theta < \theta_{max}$ @ $M = M_2$

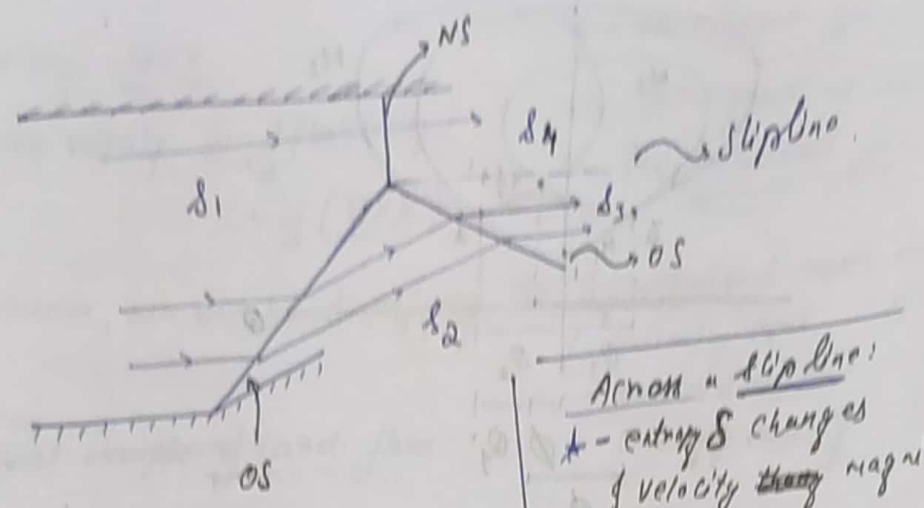
[in shock reflection boundary condition is imposed by the horizontal surface]





- Irregular Reflection / Mach Reflection :

$\theta_{max} @ M_2 < \theta < \theta_{max} @ M_1$



Across a slip line :
- entropy s changes
& velocity magnitude may change.

However pressure across a slip line remains the same!

⊙ Shock - Shock interactions :

↳ can give rise to an abrupt jump in flow properties especially when it happens close to the surface.

✓ There are in total 6 kinds of Shock-Shock interactions.

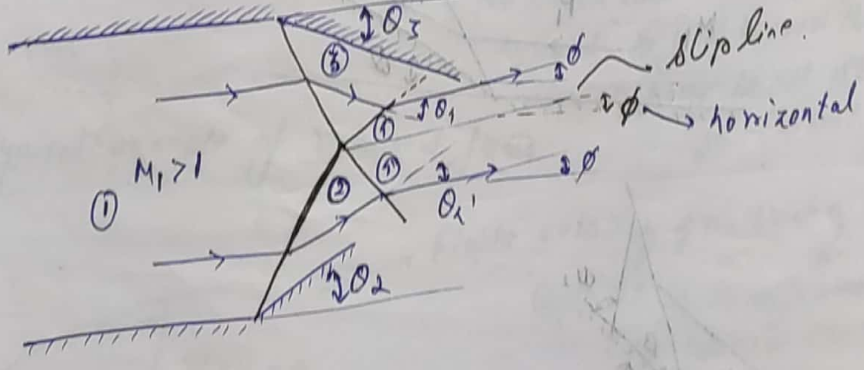
↑ Edney's Shock-Shock interactions of various kinds

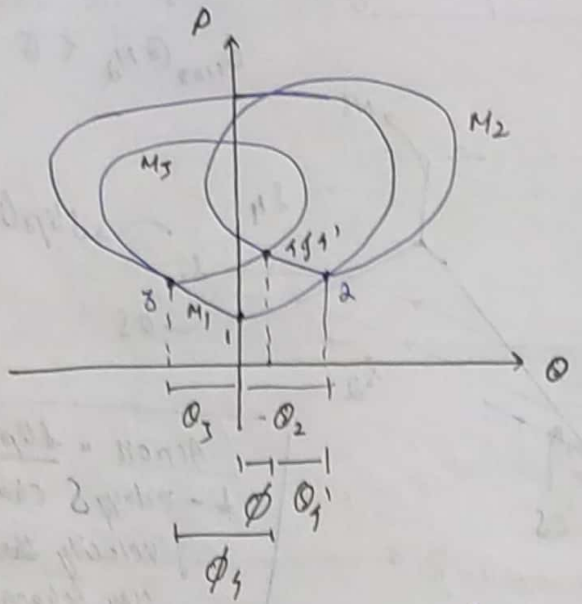
We will be focusing on :

i. Shock-Shock interactions of opposite family.

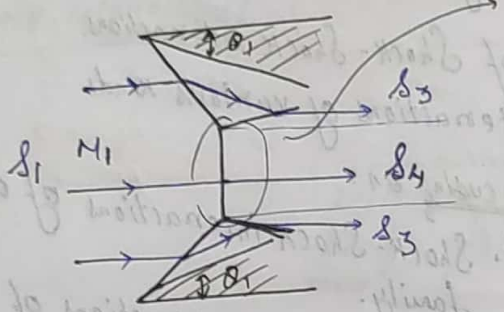
ii. Shock-Shock interactions of same family.

i. Shock-Shock interactions of opposite family :





① When combination of M_2 & θ is such that we can't get a regular shock-shock interaction, then in that case a <u>Mach stem</u> / <u>Mach disk</u> appears!

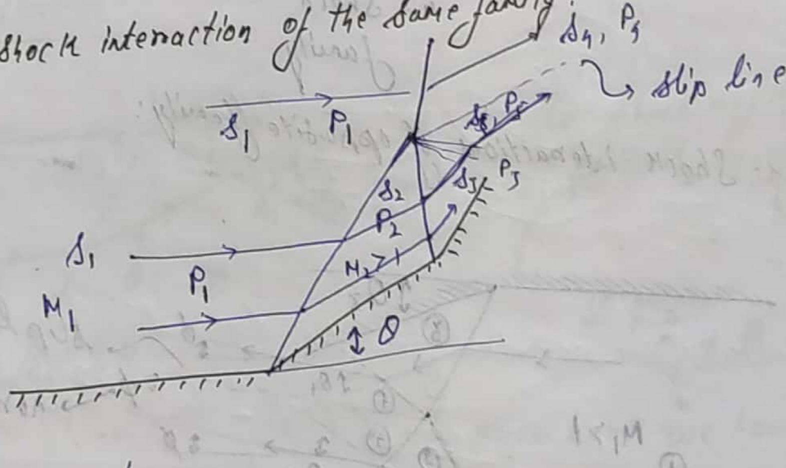


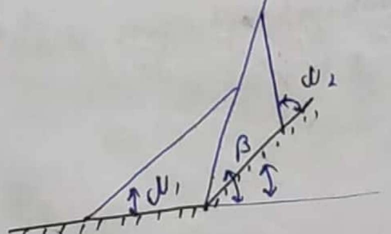
$S_3 \neq S_4$

& $|V_3| \neq |V_4|$

but $P_3 = P_4$

ii. Shock - Shock interaction of the same family!





a Mach wave at any pt. behind the shock wave must intersect the shock wave.

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① **Irrotational Flow:** ~ $\nabla \times \underset{\sim}{V} = 0$ everywhere in the flow.

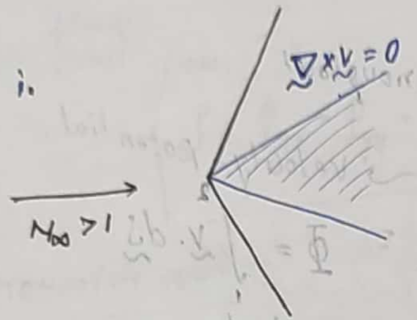
vorticity $= \nabla \times \underset{\sim}{V}$

angular velocity, $\underset{\sim}{\omega} = \frac{1}{2}(\text{vorticity})$
$= \frac{1}{2}(\nabla \times \underset{\sim}{V})$

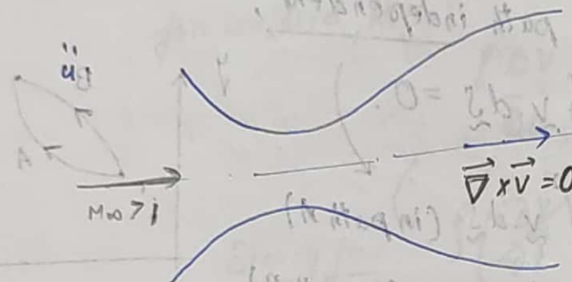
· a Flow where $\nabla \times \underset{\sim}{V} \neq 0$ throughout it's called the rotational flow.

Over here, we will be simplifying the 3 generalized eqn when flow is irrot.!
[- Continuity eqn
- Newton's 2nd law
- Energy consv.]
$(\nabla \times \underset{\sim}{V} = 0)$

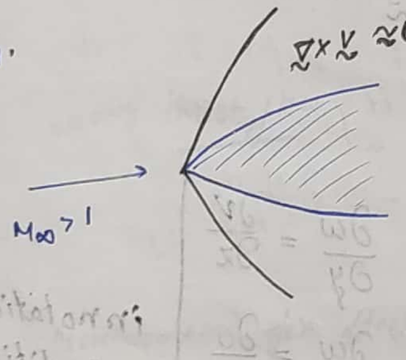
- some typical example of irrot. flows:

i. 

Flowfield over a sharp wedge or cone.

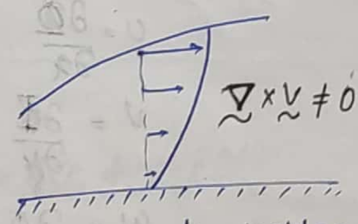
ii. 

QD or axisymmetric nozzle flows
(as long as there is no shock inside the nozzle $\nabla \times \underset{\sim}{V} = 0$)

iii. 

Flowfield behind the shock wave on a slender, sharpnosed body is almost irrot.
[the shock wave is slightly curved so flow over the slender body is not strictly irrot.]

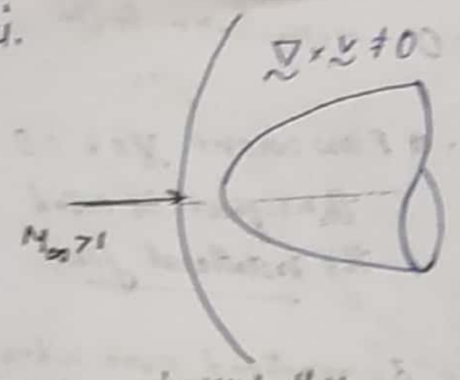
- some typical example of rotational flows:

i. 

viscous flow inside a boundary layer.

finite velocity gradient → Bulk KE is transformed into internal energy
⇓
- Temp. rises ↑
- Entropy is generated.
- KE ↓ses (effectively it is 0 on the surface)

ii.



inviscid flow behind a curved shock wave.

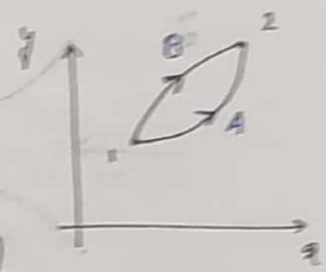
- The study of irrot. flows. ($\nabla \times \mathbf{V} = 0$) is of immense practical use.
- irrot. flows $\Leftrightarrow$ $\nabla \times \mathbf{V} = 0$ throughout

$$\Updownarrow$$

$$\mathbf{V} = \nabla \Phi \quad \text{velocity potential.} \qquad \Rightarrow \Phi = \int_1^2 \mathbf{V}\cdot d\mathbf{s}$$

- the integral, $\int_1^2 \mathbf{V}\cdot d\mathbf{s}$ is path independent,

or Circulation, $\Gamma = \oint \mathbf{V}\cdot d\mathbf{s} = 0.$



$$\Phi = \int_1^2 \mathbf{V}\cdot d\mathbf{s} \text{ (in path A)}$$

$$= \int_1^2 \mathbf{V}\cdot d\mathbf{s} \text{ (in path B)}$$

✓ irrotationality condition:

$$\nabla \times \mathbf{V} = 0$$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = 0 \quad \Rightarrow \quad \left.\begin{aligned} \frac{\partial w}{\partial y} &= \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} &= \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} &= \frac{\partial u}{\partial y} \end{aligned}\right| \text{ irrotationality condition}$$

$$\equiv \left\{\begin{aligned} u &= \frac{\partial \Phi}{\partial x} \\ v &= \frac{\partial \Phi}{\partial y} \\ w &= \frac{\partial \Phi}{\partial z} \end{aligned}\right.$$

We will be using these irrotationality conditions to make simplifications.

i. Continuity eqn :

for steady + ~~inviscid~~ flow : $\cancelto{0}{\frac{\partial \rho}{\partial t}} + \nabla \cdot (\rho \underset{\sim}{V}) = 0$

$\left(\frac{\partial \rho}{\partial t} = 0\right)$

$\Rightarrow \underset{\sim}{\nabla} \cdot (\rho \underset{\sim}{V}) = 0$

$\Rightarrow \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$

$\Rightarrow \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$

This term would vanish if the flow is incompressible.

$+\left(u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}\right) = 0.$

continuity eqn for steady + irrot. flow,

$$\rho\left(\Phi_{xx} + \Phi_{yy} + \Phi_{zz}\right) + \left(\Phi_x \frac{\partial \rho}{\partial x} + \Phi_y \frac{\partial \rho}{\partial y} + \Phi_z \frac{\partial \rho}{\partial z}\right) = 0 \quad \text{— (i)}$$

ii. Momentum eqn :

for inviscid flow with no body force :

$\rho \frac{D\underset{\sim}{V}}{Dt} = -\underset{\sim}{\nabla} p$ ~ Euler eqn

$\rho\left(\cancelto{0}{\frac{\partial \underset{\sim}{V}}{\partial t}} + (\underset{\sim}{V}\cdot\underset{\sim}{\nabla})\underset{\sim}{V}\right) = -\underset{\sim}{\nabla} p$

Steady flow

$\Rightarrow \rho(\underset{\sim}{V}\cdot\underset{\sim}{\nabla})\underset{\sim}{V} = -\underset{\sim}{\nabla} p$

assuming irrot flow $\Rightarrow \boxed{dp = -\rho V dV}$ where $V = |\vec{V}| = \sqrt{u^2+v^2+w^2}$

Momentum eqn for steady + inviscid + no body force + irrot flow
or
Euler eqn

- holds for any direction in an in an irrot + inviscid + no body force + steady flow (not necessarily along streamline)

- it holds ~~for~~ along any streamline for a rot. + inviscid + no body force + steady flow.

• for ther simplification...

for steady + inviscid + no body force + irrot. flow

$dp = -\rho V dV = -\rho\, d\left(\frac{V^2}{2}\right)$

$$\Rightarrow \boxed{dp = -\frac{\rho}{2}\, d(\Phi_x^2 + \Phi_y^2 + \Phi_z^2)} \quad \text{— (ii)}$$

- further if the ~~#~~ no heat is ~~added to~~ transferred to the fluid (isenthalpic), then ~~g~~ the flow is going to be isentropic (by Crocos thm.) [flow is already assumed to be steady]

then, $\frac{dp}{d\rho} = \left(\frac{dp}{d\rho}\right)_s = a^2$ ← speed of sound

[assumptions : - steady
- inviscid
- no body force
- irrot.
- isentropic flow / = isenthalpic flow]

$$\Rightarrow \boxed{d\rho = \frac{dp}{a^2}} \quad \text{— (iii)}$$

now,

$$\left.\begin{aligned} \frac{\partial \rho}{\partial x} &= \frac{-\rho}{a^2}\left(\Phi_x \Phi_{xx} + \Phi_y \Phi_{yx} + \Phi_z \Phi_{zx}\right) \\ \frac{\partial \rho}{\partial y} &= -\frac{\rho}{a^2}\left(\Phi_x \Phi_{xy} + \Phi_y \Phi_{yy} + \Phi_z \Phi_{zy}\right) \\ \frac{\partial \rho}{\partial z} &= \frac{-\rho}{a^2}\left(\Phi_x \Phi_{xz} + \Phi_y \Phi_{yz} + \Phi_z \Phi_{zz}\right) \end{aligned}\right\} \text{obtained using (ii) \& (iii)} \quad \text{— (iv)}$$

plugging (iv) in eqn (i) we have,

Momentum + continuity underlying assumptions in flow : - steady - inviscid - irrot - no body force - isenthalpic ≡ isentropic

$$\boxed{\left(1 - \frac{\Phi_x^2}{a^2}\right)\Phi_{xx} + \left(1 - \frac{\Phi_y^2}{a^2}\right)\Phi_{yy} + \left(1 - \frac{\Phi_z^2}{a^2}\right)\Phi_{zz} - \frac{2\Phi_x\Phi_y}{a^2}\Phi_{xy} - \frac{2\Phi_x\Phi_z}{a^2}\Phi_{xz} - \frac{2\Phi_y\Phi_z}{a^2}\Phi_{yz} = 0.} \quad \text{— (v)}$$

2nd order non linear pde

↳ Velocity Potential Equation. a is still unknown.

iii. Energy Consv. $h_0 = \text{const.}$ (isenthalpic process)

$$\Rightarrow \boxed{a^2 = a_0^2 - \frac{\gamma - 1}{2}\left(\Phi_x^2 + \Phi_y^2 + \Phi_z^2\right)} \quad \text{— (vi)}$$

calorically PG assumed. a_0 is a known const. for the flow.

- eqn (v) + (vi) combined represents a combination of continuity, momentum & energy equations. for irrot, isentropic steady, inviscid flow of a calorically PG, with no body force.

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assumptions in eqn (v) : i. Steady
ii. inviscid
iii. no body force
iv. isentropic = isenthalpic
v. irrot.

assumptions in eqn (vi) : i. adiabatic
ii. Steady
iii. inviscid
iv. no body force.
v. irrot.
vi. Calorically PG.

- i. Solve for (v) + (vi), for specified boundary conditions of the given problem.

ii. Calculate. $u = \frac{\partial \phi}{\partial x}$; $v = \frac{\partial \phi}{\partial y}$ & $w = \frac{\partial \phi}{\partial z}$

obtain, $V = \sqrt{u^2+v^2+w^2}$

iii. Calculate a, from eqn (vi).

iv. Calculate $M = V/a$.

v. use, $T = \frac{T_0}{\left(1+\frac{\gamma-1}{2}M^2\right)}$, $p = \frac{p_0}{\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}}$

[isentropic table] , $\rho = \frac{\rho_0}{\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{1}{\gamma-1}}}$

to calculate, T, p & ρ.

- (v)+(vi) → non linear PDE
  → applies to steady, irrot, isentropic inviscid flow with no body force. [Subsonic, transonic, supersonic, hypersonic doesnot matter]
  → exact

reduces to laplace eqn $\nabla^2 \Phi = 0$ for incompressible, isentropic, irrot flow ($a \to \infty$)

NOTE: There is no general closed form solution to (v).

hence we approach the solution by 3 ways:

Exact Numerical Solution: The results are raw numbers which have to be analysed.

Transformation of variables: in order to make (v) linear but still exact.

Linearized Solution: we will find linear eqns which are approximations to the exact Non-linear eqns, but which lend themselves to closed form analytic solution.

Crocos' Theorem : relation between TD & fluid kinematics of a compressible flow.

Fluid element's movement is Translational + Rotational.
$\vec{V}$ $\vec{\omega}$

- for inviscid flow, with no body force:

$\vec{\omega} = \frac{1}{a}$ (vorticity) $= \frac{1}{a}(\vec{\nabla} \times \vec{V})$

$$\rho \frac{DV}{Dt} = -\nabla P \qquad \left[T\nabla s = \nabla h - v\nabla P \right]$$

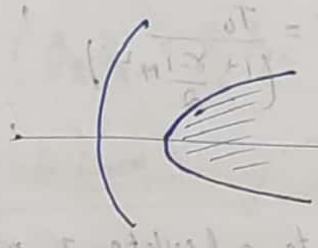
$$\Rightarrow \boxed{T\nabla s = \nabla h_T - V \times (\nabla \times V) + \frac{\partial V}{\partial t}} \rightarrow \text{crocos' theorem}$$

for steady inviscid flow with no body force:

$$\boxed{\underbrace{V \times (\nabla \times V)}_{\text{vorticity}} = \underbrace{\nabla h_T}_{\text{total enthalpy gradient}} - T\underbrace{\nabla s}_{\text{entropy gradient}}}$$

When a steady, inviscid flow with no body force has gradients of total enthalpy &/or entropy then that flow is ~~irro~~ rotational!

Crocos' Theorem $\Rightarrow$



Flow behind a curved shock is rotational.

doesnot hold for supersonic flow.

D'Alembert proved that for incompressible, inviscid, irrot flow Drag force is 0, on a body moving with constant velocity wrt fluid. ← D'Alembert's paradox.

holds for subsonic flow as well.

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Linearized Flow solution

ⓥ (vi)

- We will be linearizing the exact non-linear governing equns to extract information about the flow!, by making certain simplifying assumptions!

↓

Small perturbation assumption.

[Linearized ~~flow~~ solutions in compressible flow always contain the assumption of small perturbation]

However small perturbation donot always guarantee that governing equns can be linearized!

- Uptill 1950s (prior to computer age) linearized solns to flow dominated the aerodynamics & gas dynamics.
- In modern days whenever accuracy is desired the full non-linear equns are solved numerically in computer.

- → For steady, inviscid, irrot., isentropic ($\equiv$ isenthalpic) flow with No body force:

velocity potential eqn.

$$\left(1-\frac{\Phi_x^2}{a^2}\right)\Phi_{xx}+\left(1-\frac{\Phi_y^2}{a^2}\right)\Phi_{yy}+\left(1-\frac{\Phi_z^2}{a^2}\right)\Phi_{zz}$$
$$-\frac{2\Phi_x\Phi_y}{a^2}\Phi_{xy}-\frac{2\Phi_x\Phi_z}{a^2}\Phi_{xz}-\frac{2\Phi_y\Phi_z}{a^2}\Phi_{yz}=0.$$

Momentum + Continuity.

→ further if we assume ~~the~~ Calorically PG:

$$\boxed{a^2=a_0^2-\frac{\gamma-1}{2}\left(\Phi_x^2+\Phi_y^2+\Phi_z^2\right)}$$ ~ energy

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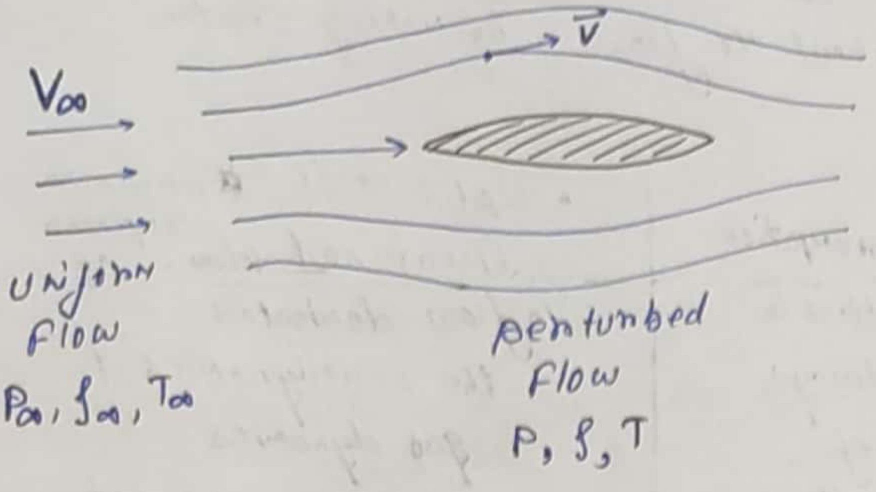

in perturbed flow,

$$\left.\begin{aligned} V_x &= V_\infty + u' \\ V_y &= v' \\ V_z &= w' \end{aligned}\right\}$$

in uniform flow,

$$\left.\begin{aligned} V_x &= V_\infty \\ V_y &= 0 = V_z \end{aligned}\right\}$$

- in "perturbed flow" region, flow is irrot.

so, $\underset{\sim}{\nabla}\Phi = (V_\infty + u')\hat{i} + v'\hat{j} + w'\hat{k}$

Φ – total velocity potential

, we define $\underset{\sim}{\nabla}\phi = u'\hat{i} + v'\hat{j} + w'\hat{k}$

ϕ – perturbation velocity potential

i.e.
$$\left.\begin{aligned} \Phi_x &= V_\infty + \phi_x \\ \Phi_y &= \phi_y \\ \Phi_z &= \phi_z \end{aligned}\right\} \Rightarrow \boxed{\Phi = V_\infty x + \phi}$$

$$\left.\begin{aligned} \Phi_{xx} &= \phi_{xx} \\ \Phi_{yy} &= \phi_{yy} \\ \Phi_{zz} &= \phi_{zz} \end{aligned}\right\}$$

- Now, Generalized velocity potential eqn $\xrightarrow{\Phi = V_\infty x + \phi}$ Perturbation velocity potential eqn

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$$\left[a^2-\left(V_\infty+\frac{\partial\phi}{\partial x}\right)^2\right]\frac{\partial^2\phi}{\partial x^2}+\left[a^2-\left(\frac{\partial\phi}{\partial y}\right)^2\right]\frac{\partial^2\phi}{\partial y^2}$$

$$+\left[a^2-\left(\frac{\partial\phi}{\partial z}\right)^2\right]\frac{\partial^2\phi}{\partial z^2}-2\left(V_\infty+\frac{\partial\phi}{\partial x}\right)\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\partial y}-2\left(V_\infty+\frac{\partial\phi}{\partial x}\right)\frac{\partial\phi}{\partial z}\frac{\partial^2\phi}{\partial x\partial z}$$

perturbation velocity potential equⁿ

$$-2\frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial z}\frac{\partial^2\phi}{\partial y\partial z}=0$$

$$\equiv\left[a^2-(V_\infty+u')^2\right]\frac{\partial u'}{\partial x}+\left[a^2-v'^2\right]\frac{\partial v'}{\partial y}$$

$$+\left[a^2-w'^2\right]\frac{\partial w'}{\partial z}-2(V_\infty+u')v'\frac{\partial u'}{\partial y}$$

$$-2(V_\infty+u')w'\frac{\partial u'}{\partial z}-2v'w'\frac{\partial v'}{\partial z}=0 \quad \text{—— (i)}$$

∴ energy equⁿ $a^2=a_\infty^2-\frac{\gamma-1}{2}\left(2u'V_\infty+u'^2+v'^2+w'^2\right)$ —— (ii)

Combining (i) & (ii) we have,

LHS linear

$$(1-M_\infty^2)\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}$$

$$=M_\infty^2\left[(\gamma+1)\frac{u'}{V_\infty}+\left(\frac{\gamma+1}{2}\right)\frac{u'^2}{V_\infty^2}+\left(\frac{\gamma-1}{2}\right)\left(\frac{v'^2+w'^2}{V_\infty^2}\right)\right]\frac{\partial u'}{\partial x}$$

1st

$$+M_\infty^2\left[(\gamma-1)\frac{u'}{V_\infty}+\left(\frac{\gamma+1}{2}\right)\frac{v'^2}{V_\infty^2}+\left(\frac{\gamma-1}{2}\right)\left(\frac{w'^2+u'^2}{V_\infty^2}\right)\right]\frac{\partial v'}{\partial y}$$

2nd

$$+M_\infty^2\left[(\gamma-1)\frac{u'}{V_\infty}+\left(\frac{\gamma+1}{2}\right)\frac{w'^2}{V_\infty^2}+\left(\frac{\gamma-1}{2}\right)\left(\frac{u'^2+v'^2}{V_\infty^2}\right)\right]\frac{\partial w'}{\partial z}$$

3rd

$$+M_\infty^2\left[\frac{v'}{V_\infty}\left(1+\frac{u'}{V_\infty}\right)\left(\frac{\partial u'}{\partial y}+\frac{\partial v'}{\partial x}\right)+\frac{w'}{V_\infty}\left(1+\frac{u'}{V_\infty}\right)\left(\frac{\partial u'}{\partial z}+\frac{\partial w'}{\partial x}\right)+\frac{v'w'}{V_\infty^2}\left(\frac{\partial w'}{\partial y}+\frac{\partial v'}{\partial z}\right)\right]$$

4th term

- So far we havenot invoked any approximation. the above equⁿ holds true for any steady, inviscid, irrot, isentropic (≡ isenthalpic) flow of calorically PG with no body force.

- Now, we are introducing small perturbation approximation:

$$\left| \frac{u'}{V_\infty} \ll 1 ,\ \frac{v'}{V_\infty} \ll 1 ,\ \frac{w'}{V_\infty} \ll 1 \right| \Rightarrow \left| \frac{u'}{V_\infty} \lll 1 ,\ \frac{v'}{V_\infty} \lll 1 \ \&\ \frac{w'}{V_\infty} \lll 1 \right|$$

NOTE: Small perturbation approximation ~~holds~~ is valid for a slender bodies only! exp: flow over a bumpy, or wavy wall (refer exp. in copy)

- under small perturbation assumption:

if (flow == subsonic $M_\infty \leq 0.8$) → 1st term, 2nd, 3rd term, 4th term of RHS } can be neglected.

else if (flow == transonic $0.8 \leq M_\infty \leq 1.2$) → 2nd, 3rd term, 4th term of RHS } can be neglected

else if (flow == supersonic $1.2 \leq M_\infty < \sim 5$) → 1st term, 2nd, 3rd term, 4th term of RHS } can be neglected.

else (flow == hypersonic $\sim 5 < M_\infty$) → 1st term → can be neglected.

NOTE: 2nd, 3rd term gain priority much after $M_\infty = 5$

- So under small perturbation approximation, [provided the flow is - steady,
  - inviscid,
  - irrot.,
  - no body force,
  - isentropic & isenthalpic
  - calorically PG]

→ for subsonic & supersonic flow we have,

$$\boxed{(1 - M_\infty^2)\,\phi_{xx} + \phi_{yy} + \phi_{zz} = 0.}$$

→ elliptic for subsonic
→ hyperbolic for supersonic
→ it is an approximate equi. it no longer represents the exact physics of the problem.

this is a linear equi.

- Linearization of transonic & hypersonic flows is not possible.

this makes there analysis more complex!

⊙ Linearized pressure coefficients : $\left[C_p = -\frac{2u'}{V_\infty} \right]$

pressure coeff., $C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2}$ •

for calorically PG, $\longrightarrow$ $C_p = \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_\infty} - 1\right)$

- for a steady, inviscid, adiabatic flow with (no s transfer due to thermal conduction / diffusion, No change in PE) → isentropic flow,

with no body force,

we have, $h + \frac{V^2}{2} = h_\infty + \frac{V_\infty^2}{2}$

$\Rightarrow T + \frac{V^2}{2c_p} = T_\infty + \frac{V_\infty^2}{2c_p}$

$\Rightarrow T - T_\infty = \frac{(V_\infty^2 - V^2)}{2c_p} = \frac{V_\infty^2 - V^2}{2\frac{\gamma R}{\gamma - 1}}$

$\Rightarrow \left(\frac{T}{T_\infty} - 1\right) = \frac{\gamma - 1}{2}\frac{V_\infty^2 - V^2}{a_\infty^2}$

for isentropic flow, $\frac{p}{p_\infty} = \left(\frac{T}{T_\infty}\right)^{\frac{\gamma}{\gamma-1}}$

$\Rightarrow \frac{p}{p_\infty} = \left(1 + \frac{\gamma - 1}{2}\frac{V_\infty^2 - V^2}{a_\infty^2}\right)^{\frac{\gamma}{\gamma-1}}$

$\Rightarrow \frac{p}{p_\infty} = \left[1 - \frac{\gamma - 1}{2}M_\infty^2\left(\frac{2u'}{V_\infty} + \frac{u'^2 + v'^2 + w'^2}{V_\infty^2}\right)\right]^{\frac{\gamma}{\gamma-1}}$

NOTE: the diagram / physical situation is exactly same as the slender fig drawn in this module.

∴ for small perturbation: $\frac{p}{p_\infty} = (1 - \varepsilon)^{\frac{\gamma}{\gamma-1}}$ ε is very very small

$\Rightarrow \frac{p}{p_\infty} = 1 - \frac{\gamma \varepsilon}{\gamma - 1} + \cdots$

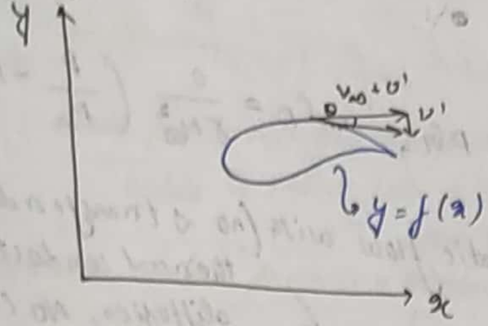
$= 1 - \frac{\gamma}{2}M_\infty^2\left(\frac{2u'}{V_\infty} + \frac{u'^2 + v'^2 + w'^2}{V_\infty^2}\right) + \cdots$

so, $C_p = -\frac{2u'}{V_\infty} - \left(\frac{u'^2 + v'^2 + w'^2}{V_\infty^2}\right) + \cdots$

$\boxed{C_p = -\frac{2u'}{V_\infty}}$ as $\left(\frac{u'}{V_\infty}\right)^2 \ll 1 \ldots$

i.e. C_p depends only on the x component of the perturbation velocity !!

Subsonic Flow: ($M_\infty \leq 0.8$)



for inviscid flow:

BC: NO "No slip"

therefore, $\dfrac{dy}{dx} = \dfrac{V'}{U' + V_\infty} = \tan\theta$

$\Rightarrow \dfrac{df}{dx} = \dfrac{V'}{U' + V_\infty}$

for small perturbation it reduces to,

$$\frac{df}{dx} = \frac{V'}{V_\infty} = \tan\theta \approx \theta.$$

$$\Rightarrow \boxed{V' = \phi_y = V_\infty \frac{df}{dx}}$$

BC on the surface for inviscid flow, with small perturbation.

NOTE: the geometry has to be such that small perturbation approx. holds [exp: thin airfoil at small AoA]

- for a 2D, subsonic flow,

$$(1 - M_\infty^2)\phi_{xx} + \phi_{yy} = 0$$

$$\Rightarrow \beta^2 \phi_{xx} + \phi_{yy} = 0. \quad \text{elliptic eqn}$$

- we carry out CS transformation,

$$\left.\begin{aligned} \xi &= x \\ \eta &= \beta y \end{aligned}\right\}$$

in transformed space we define a new velocity potential

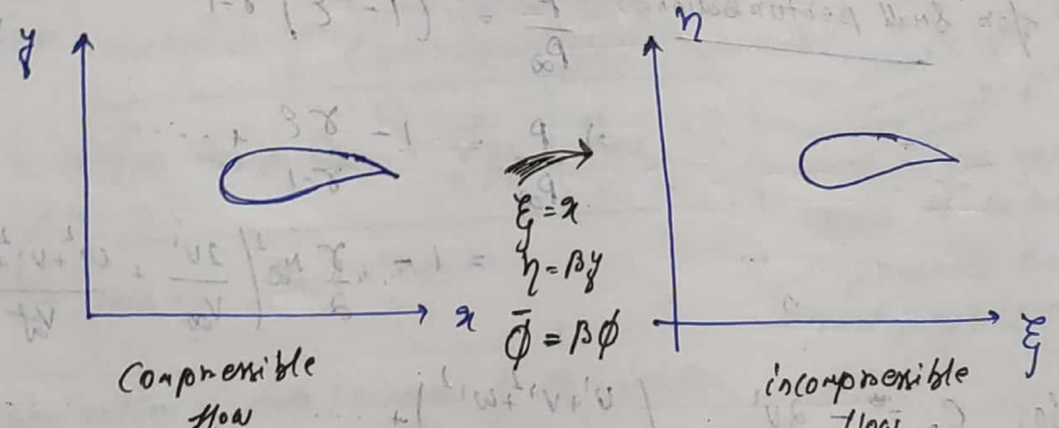
$$\bar{\phi}(\xi, \eta) = \beta \phi(x, y)$$

reduces to

$$\bar{\phi}_{\xi\xi} + \bar{\phi}_{\eta\eta} = 0 \quad \text{laplace eqn in the transformed space.}$$

- NOTE: the shape of the body remains same in the transformed space.

$$\left.\begin{aligned} \xi &= x \\ \eta &= \beta y \end{aligned}\right\}$$



Compressible flow | incompressible flow.

$$C_p = -\frac{2U'}{V_\infty} \qquad \boxed{C_p = \frac{C_{p_0}}{\beta}} \qquad C_{p_0} = -\frac{2\bar{U}'}{V_\infty}$$

$U' = \dfrac{\bar{U}'}{\beta}$

C_{p_0} = incompressible pressure coeff.

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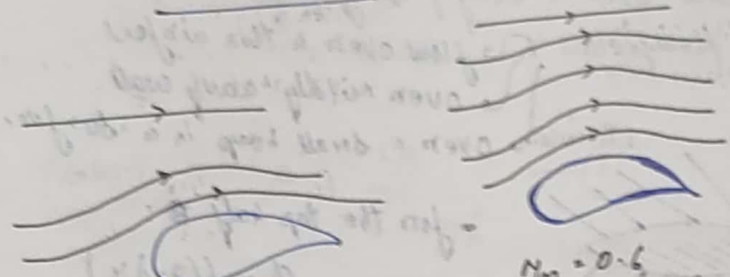
i.e. $$\boxed{C_p = \frac{C_{p_0}}{\sqrt{1-M_\infty^2}}}$$ ~ we can use this formula to relate incompressible flow database with compressible subsonic flow.

- on a similar note,

$$\boxed{\begin{aligned} C_L &= \frac{C_{L_0}}{\sqrt{1-M_\infty^2}} \\ C_M &= \frac{C_{M_0}}{\sqrt{1-M_\infty^2}} \end{aligned}}$$ ~ viscous effect completely neglected



$M_\infty < 0.3$ incompressible regime

$M_\infty = 0.6$ compressible regime.

For a given body, compressibility effects strengthens the perturbations which would then be able to reach further away from the surface.

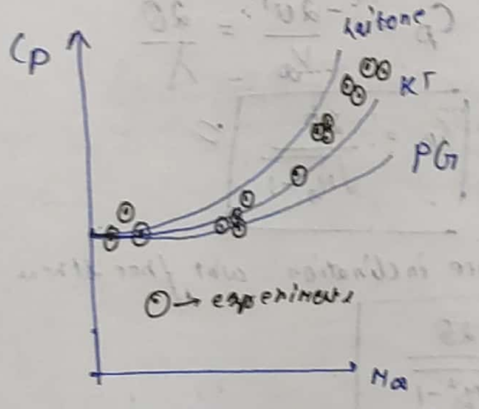
① Compressibility correction:

$$\boxed{C_p = \frac{C_{p_0}}{\sqrt{1-M_\infty^2}}}$$ ~ prandtl - Glauert rule. (valid upto $M_\infty \leq 0.7$)

Laitone's Rule.

$$\boxed{C_p = \frac{C_{p_0}}{\sqrt{1-M_\infty^2} + \frac{M_\infty^2\left(1+\frac{\gamma-1}{2}M_\infty^2\right)}{\sqrt{1-M_\infty^2}} \cdot \frac{C_{p_0}}{2}}}$$

$$\boxed{C_p = \frac{C_{p_0}}{\sqrt{1-M_\infty^2} + \left(\frac{M_\infty^2}{1+\sqrt{1-M_\infty^2}}\right)\frac{C_{p_0}}{2}}}$$ ~ Karman - Tsien rule. (valid upto $M_\infty \sim 0.7$ or before transonic regime)



NACA 4412, $\alpha = 1°53'$

- PG underpredicts the experimental value
- Both Laitone & KT rules consider non-linear aspect of flow & hence are more accurate

• Linearised Supersonic Flow:

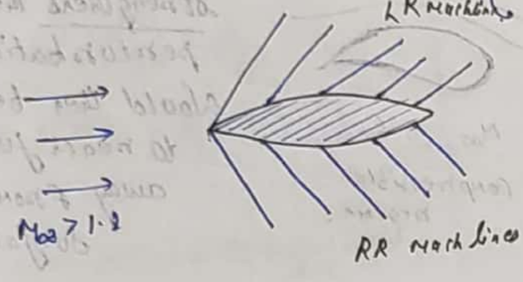
For 2D, supersonic flow:

$$(M_\infty^2 - 1)\,\phi_{xx} - \phi_{yy} = 0 \quad \text{— a 2D, 2nd order wave eqns}$$

$$\Rightarrow \lambda^2 \phi_{xx} - \phi_{yy} = 0 \;\rightsquigarrow \text{hyperbolic pde}$$

↳ gene. sol. $\phi = f(x - \lambda y) + g(x + \lambda y)$

- Consider supersonic flow over a body / surface, which introduces small changes in the flowfield:
  - ↳ flow over a thin airfoil.
  - ↳ over mildly wavy wall.
  - ↳ over a small hump in a surface.



- for the top half:

$$\phi = f(x - \lambda y)$$

slope of the LR or $C^{\ominus}$ charac. lines.

$$\boxed{\frac{dy}{dx} = \frac{1}{\lambda} = \frac{1}{\sqrt{M_\infty^2 - 1}} = \tan\mu}$$

perturbation velocity:

$$\left.\begin{aligned} u' &= f' \\ v' &= -\lambda f' \end{aligned}\right\} \rightsquigarrow \boxed{u' = \frac{-v'}{\lambda}}$$

• BC on the surface $\dfrac{dy}{dx} = \tan\theta = \dfrac{v'}{V_\infty + u'}$

↳ $\theta \approx \dfrac{v'}{V_\infty}$ (for small pertub)

$$\Rightarrow \boxed{v' = V_\infty \theta} \quad \text{on the boundary surface}$$

so $\boxed{u' = -\dfrac{V_\infty \theta}{\lambda}}$ on the boundary surface.

. the pressure coeff. on the top surface: $C_p = \dfrac{-2u'}{V_\infty} = \dfrac{2\theta}{\lambda}$

for supersonic flow $\Rightarrow \boxed{C_p = \dfrac{2\theta}{\sqrt{M_\infty^2 - 1}}}$

$C_p \propto$ local surface inclination wrt free stream

- for the lower surface: (RR or $C^{\ominus}$ charac.) $\boxed{C_p = \dfrac{-2\theta}{\sqrt{M_\infty^2 - 1}}}$

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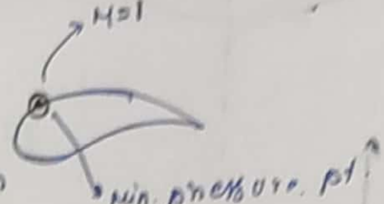
① Critical Mach no: is the free stream Mach no., M_{cr} where sonic flow is first encountered on an airfoil.

(M_{cr}) ↗ depends on the geometry of the airfoil.



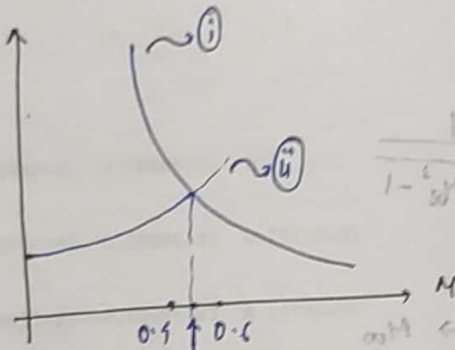
- finding M_{cr} for a given airfoil:

$$C_{p_{cr}} = \frac{2}{\gamma M_{cr}^2}\left[\left(\frac{1+\frac{\gamma-1}{2}M_{cr}^2}{\frac{\gamma+1}{2}}\right)^{\frac{\gamma}{\gamma-1}} - 1\right] \quad (C_{p_{cr}} = f(M_{cr})) \quad \text{— (i)}$$

(subsonic compressible flow)

$$C_p = \frac{C_{p_0}}{\sqrt{1-M_{cr}^2}} \quad \text{— (ii)}$$

Solve for M_{cr} from (i) & (ii).



Airfoil designed for higher M_{cr} must have thin profile.

$M_{cr} \downarrow$ as thickness ↑

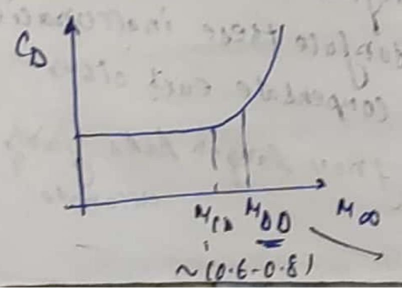
$M_{cr} \sim (0.6-0.8)$ lec. notes.

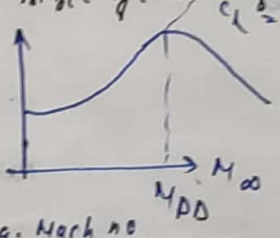
- When free stream Mach no. exceeds M_{cr}, a finite region of supersonic flow exists on top of the airfoil. At a high enough Mach no. this embedded supersonic region will be terminated by a weak shock wave!! → flow separation actually take place

- The free stream Mach no. at which large drag rise begins is defined as the Drag-Divergence Mach no.

$$\boxed{M_{DD} \underset{\text{slightly}}{>} M_{cr}}$$ ~ at M_{DD} there is a massive rise in drag.

- Significance of M_{cr}: for viscous flows, the C_D value at low Mach no. obtained from experiment would vary in the compressible flow regime.





→ Drag. Divg. Mach no.

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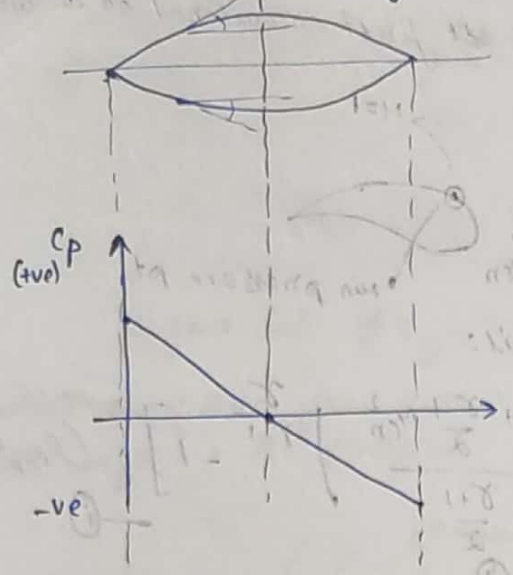
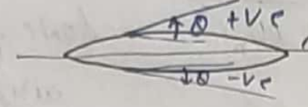
• 6 Biconvex Airfoil





the body has to be slender

$$C_p = \frac{2\theta}{\sqrt{M_\infty^2 - 1}} \quad \text{(on upper surf.)}$$

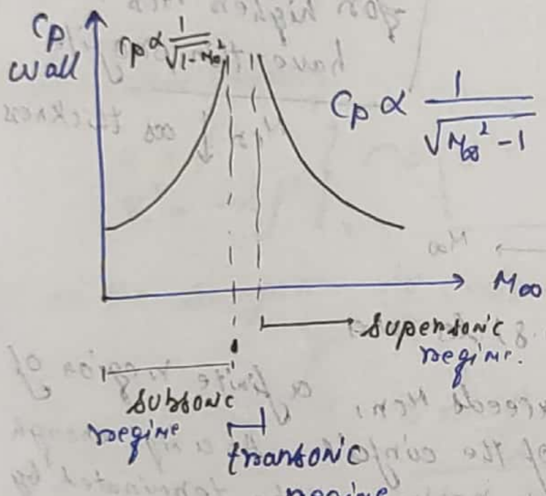
$$C_p = \frac{-2\theta}{\sqrt{M_\infty^2 - 1}} \quad \text{(on lower surf.)}$$

[Note: unlike subsonic flow, where there is no drag, supersonic flow generates wave drag!]

[a key difference of linearized subsonic flow with linearized supersonic flow]

[NOTE: we have obtained similar results in thin airfoil theory]

•



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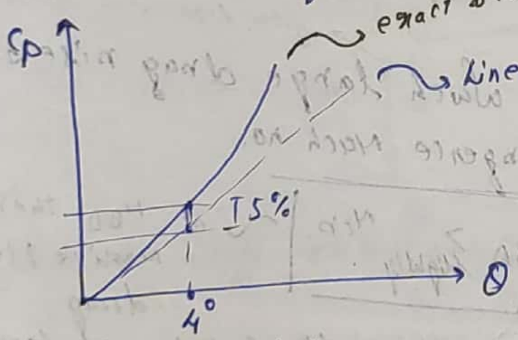

exact shock theory (from OS concepts)

linearized theory!!

• For $M_\infty = 2$.

- linearized theory Cp is reasonably accurate when $\theta < 4°$

- beyond $\theta = 4°$, Cp from linearized theory is fairly inaccurate!

(when integrated over the top & bottom surface these inaccuracies tend to compensate each other making C_L, C_D from larger AoAs fairly accurate!

flow over a wedge