AE-641A (Space Dynamics-I)

Quiz No. 4 (Solution)

1. Calculate the velocity impulse magnitude required to change the orbital speed from 7.5 km/s to 8.5 km/s while also changing the orbital plane by 20° at a given point in an orbit.

   Solution:
   The combined maneuver of changing the orbital speed from v_i to v_f, as well as a change of orbital plane by angle α results in a velocity triangle at the given point. Hence, the cosine law of trigonometry (Lecture 10) requires that the velocity impulse magnitude is given by

   $$\Delta v = \sqrt{v_i^2 + v_f^2 - 2v_i v_f \cos\alpha}$$

   Here, $v_i = 7.5$ km/s, $v_f = 8.5$ km/s, and $\alpha = 20°$. Thus, we have

   $$\Delta v = \sqrt{(7.5)^2 + (8.5)^2 - 2(7.5)(8.5)\cos(20°)} = 2.94774 \text{ km/s}$$

2. Calculate the total velocity change, waiting time, and total propellant mass required for a Hohmann transfer around the Earth from a circular orbit of radius 8000 km, for rendezvous with a target in a coplanar circular orbit of radius 9000 km, assuming that the target is currently leading the maneuvering spacecraft by 30°, the specific impulse of the rocket engine is 300 s, and the initial spacecraft mass is 1500 kg.

   Solution:
   Given the initial and final radii of $r_1 = 8000$ km and $r_2 = 9000$ km, we have the following correct phase angle required for the rendezvous by Hohmann transfer:

   $$\theta_H = \pi \left[1 - \left(\frac{1 + r_1/r_2}{2}\right)^{3/2}\right] = 0.258129 \text{ rad. } (14.7897°)$$

   The target leads the maneuvering spacecraft by a phase angle of 30°. The error in the phase angle is thus given by

   $$\Delta\theta = 0.26547 \text{ rad. (or } 15.21°)$$

The waiting time is therefore calculated to be the following:

$$
\begin{aligned}
t_w &= \frac{\Delta\theta}{\sqrt{\mu/r_1^3} - \sqrt{\mu/r_2^3}} \\
&= = 1857.835 \text{ s}
\end{aligned}
$$

Hence, the first impulse is applied at $t = t_w$, and the second at $t = t_w + t_H$, where t_H, the flight time during Hohmann transfer, is calculated as follows:

$$
t_H = \frac{\pi - \theta_H}{\sqrt{\mu/r_2^3}} = 3899.504 \text{ s}
$$

which is the same as the time spent during the transfer orbit of semi-major axis $a = (r_1 + r_2)/2 = 8500$ km

$$
t_H = \frac{\pi}{\sqrt{\mu/a^3}} = 3899.504 \text{ s}
$$

The two velocity impulses required for the Hohmann transfer are calculated by

$$
\Delta v_1 = \sqrt{\frac{2\mu}{r_1} - \frac{2\mu}{r_1 + r_2}} - \sqrt{\frac{\mu}{r_1}} = 0.204642 \text{ km/s}
$$

$$
\Delta v_2 = \sqrt{\frac{\mu}{r_2}} - \sqrt{\frac{2\mu}{r_2} - \frac{2\mu}{r_1 + r_2}} = 0.198702 \text{ km/s}
$$

Hence, the total velocity change for rendezvous is $\Delta v = \Delta v_1 + \Delta v_2 = 0.403344$ km/s.

Finally, we calculate the total propellant mass required for the maneuver from the rocket equation (Lecture 11) as follows:

$$
\begin{aligned}
\Delta m &= m_i \left[1 - e^{-\Delta v/(g_0 I_{sp})}\right] \\
&= 1500 \left[1 - e^{(-0.403344 \times 1000)/(9.81 \times 300)}\right] \\
&= 192.1124 \text{ kg}
\end{aligned}
$$

3. A spacecraft is in a circular orbit of period 90 min. around the Earth. An astronaut performing spacewalk has an out-of-plane distance of 100 m from the spacecraft when his out-of-plane velocity is zero. His radial and in-track displacement and velocity components are also zero at this point. Determine the position and the velocity of the astronaut relative to the spacecraft after 30 min.

   Solution:
   Since the relative separation of the astronaut is small, one can employ the CW model for this calculation. The orbital frequency is $n = \frac{2\pi}{90\times 60} =$ 0.00116355 rad/s, which makes the final angular dispacement of the target $nt = 2\pi/3$. The initial condition is given by $z(0) = 100$ m, $\dot{z}(0) = 0$, with all the remaining motion variables being zero at $t = 0$. Since the out-of-plane motion is decoupled in the CW equations from the coplanar motion, we have the following:

$$z(t) = z(0)\cos nt + \frac{\dot{z}(0)}{n}\sin nt = -50 \text{ m}$$

$$\begin{aligned}\dot{z}(t) &= -nz(0)\sin nt + \dot{z}(0)\cos nt \\ &= -0.1 \text{ m/s}\end{aligned}$$

   The coordinates of coplanar motion are zero, $x(t) = y(t) = 0$, as well as the coplanar velocity components, $\dot{x}(t) = \dot{y}(t) = 0$.

4. Spacecraft A is in a circular orbit of frequency, n, and radius, c, around a spherical body. Another spacecraft B has an initial separation and relative velocity with respect to A at $t = 0$ given by

$$\begin{aligned}x(0) &= -0.001c\,; \quad \dot{x}(0) = 0.005nc \\ y(0) &= 0.002c\,; \quad \dot{y}(0) = -0.001nc\end{aligned}$$

   and $z(0) = \dot{z}(0) = 0$. Estimate the two velocity impulses applied on B such that it makes a rendezvous with A at $t_f = \frac{\pi}{2n}$.

   Solution:
   Since the out-of-plane motion is decoupled in the CW equations from the coplanar motion, there will be no change in the out-of-plane displacement and velocity. Hence we have $z(t) = \dot{z}(t) = 0$, and the out-of-plane components of the velocity impulses are zero. Therefore, maneuvering for the rendezvous merely involves the application of impulses to control the coplanar motion, $x(t), y(t)$, such that the relative position and velocity become zero at the given rendezvous time, $t_f = \frac{\pi}{2n}$, at which we have $\cos nt_f = 0$ and $\sin nt_f = 1$. Substituting these into Eq.(32) of Lecture 12 yields the following two-dimensional matrix inverse, ${\boldsymbol{\Phi}_{\boldsymbol{\rho\nu}}}^{-1}(t_f)$, where the last row and column involving out-of-plane motion are removed:

$${\boldsymbol{\Phi}_{\boldsymbol{\rho\nu}}}^{-1}(t) = \frac{n}{\Delta}\begin{bmatrix} (4-3\pi/2) & -2 \\ 2 & 1 \end{bmatrix}$$

where $\Delta = 8 - 3\pi/2$. Furthermore, we have

$$\mathbf{\Phi}_{\boldsymbol{\rho\rho}} = \begin{bmatrix} 4 & 0 \\ 6(1-\pi/2) & 1 \end{bmatrix}$$

and $\boldsymbol{\rho}(0) = (-0.001c,\ 0.002c)^T$. Therefore, the required initial relative velocity is the following:

$$\begin{aligned} \boldsymbol{\nu}_r(0) &= -\mathbf{\Phi}_{\boldsymbol{\rho\nu}}{}^{-1}(t_f)\mathbf{\Phi}_{\boldsymbol{\rho\rho}}(t_f)\boldsymbol{\rho}(0) \\ &= \frac{0.001nc}{\Delta}\left\{ \begin{array}{c} 8 \\ 12-3\pi \end{array} \right\} \end{aligned}$$

Since the initial relative velocity is

$$\boldsymbol{\nu}(0) = (0.005nc,\ -0.001nc)^T$$

the first velocity impulse is given by

$$\begin{aligned} \Delta\mathbf{v}_1 &= \boldsymbol{\nu}_r(0) - \boldsymbol{\nu}(0) = 0.001nc\left\{ \begin{array}{c} \frac{8}{\Delta} - 5 \\ \\ \frac{12-3\pi}{\Delta} + 1 \end{array} \right\} \\ &= \frac{0.001nc}{8-3\pi/2}\left\{ \begin{array}{c} -32 + 15\pi/2 \\ \\ 20 - 9\pi/2 \end{array} \right\} \end{aligned}$$

To calculate the second velocity impulse, the following coplanar transition matrices are required:

$$\mathbf{\Phi}_{\boldsymbol{\nu\rho}}(t_f) = \begin{pmatrix} 3n & 0 \\ -6n & 0 \end{pmatrix}$$

$$\mathbf{\Phi}_{\boldsymbol{\nu\nu}}(t_f) = \begin{pmatrix} 0 & 2 \\ -2 & -3 \end{pmatrix}$$

Substituting these into Eq.(31) of Lecture 12, we have

$$\begin{aligned} \Delta\mathbf{v}_2 &= -\mathbf{\Phi}_{\boldsymbol{\nu\rho}}(t_f)\boldsymbol{\rho}(0) - \mathbf{\Phi}_{\boldsymbol{\nu\nu}}(t_f)\boldsymbol{\nu}_r(0) \\ &= 0.001nc\left\{ \begin{array}{c} 3 + \frac{(6\pi-24)}{\Delta} \\ \\ -6 + \frac{(52-9\pi)}{\Delta} \end{array} \right\} \\ &= \frac{0.001nc}{8-3\pi/2}\left\{ \begin{array}{c} 3\pi/2 \\ \\ 4 \end{array} \right\} \end{aligned}$$