AE-641A (Space Dynamics-I)

Quiz No. 3 (Solution)

1. The orientation of a frame $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ relative to a reference frame $(\mathbf{I}, \mathbf{J}, \mathbf{K})$ is described by a positive elementary rotation by 30° about the first axis $\mathbf{I}$, followed by a positive elementary rotation by 330° about the second axis of the intermediate frame $(\mathbf{I}', \mathbf{J}', \mathbf{K}')$. Determine the Euler angles Ω, i, ω corresponding to the sequence $(\Omega_3, i_1, \omega_3)$ for the given orientation.

   Solution:
   A positive rotation of the frame, $(\mathbf{I}, \mathbf{J}, \mathbf{K})$, about $\mathbf{I}$ (i.e., the first axis of the frame) by the angle 30° is described by the following rotation matrix (see Lecture 8):

$$\mathbf{C_1}(30^\circ) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(30^\circ) & \sin(30^\circ) \\ 0 & -\sin(30^\circ) & \cos(30^\circ) \end{pmatrix}$$

   which results in the following intermediate orientation of the frame:

$$\begin{Bmatrix} \mathbf{I}' \\ \mathbf{J}' \\ \mathbf{K}' \end{Bmatrix} = \mathbf{C_1}(30^\circ) \begin{Bmatrix} \mathbf{I} \\ \mathbf{J} \\ \mathbf{K} \end{Bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & 1/2 \\ 0 & -1/2 & \sqrt{3}/2 \end{pmatrix} \begin{Bmatrix} \mathbf{I} \\ \mathbf{J} \\ \mathbf{K} \end{Bmatrix}$$

   A positive rotation about $\mathbf{J}'$ (the second axis of the intermediate frame) by an angle 330° is the following (Lecture 8):

$$\mathbf{C_2}(330^\circ) = \mathbf{C_2}(-30^\circ) \begin{pmatrix} \cos(-30^\circ) & 0 & -\sin(-30^\circ) \\ 0 & 1 & 0 \\ \sin(-30^\circ) & 0 & \cos(-30^\circ) \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & \sqrt{3}/2 \end{pmatrix}$$

   which results in the following final orientation of the frame $(\mathbf{i}, \mathbf{j}, \mathbf{k})$:

$$\begin{Bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{Bmatrix} = \mathbf{C_2}(330^\circ) \begin{Bmatrix} \mathbf{I}' \\ \mathbf{J}' \\ \mathbf{K}' \end{Bmatrix} = \mathbf{C_2}(330^\circ)\mathbf{C_1}(30^\circ) \begin{Bmatrix} \mathbf{I} \\ \mathbf{J} \\ \mathbf{K} \end{Bmatrix}$$

Hence the orientation of $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ relative to $(\mathbf{I}, \mathbf{J}, \mathbf{K})$ is given by the following rotation matrix:

$$\begin{aligned}
\mathbf{C} &= \mathbf{C_2}(330°)\mathbf{C_1}(30°) \\
&= \begin{pmatrix} \sqrt{3}/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & 1/2 \\ 0 & -1/2 & \sqrt{3}/2 \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{3}/2 & -1/4 & \sqrt{3}/4 \\ 0 & \sqrt{3}/2 & 1/2 \\ -1/2 & -\sqrt{3}/4 & 3/4 \end{pmatrix}
\end{aligned}$$

As derived in Lecture 8, the rotation matrix corresponding to the Euler-angle representation $(\Omega_3, i_1, \omega_3)$ is the following:

$$\mathbf{C} = \mathbf{C_3}(\omega)\mathbf{C_1}(i)\mathbf{C_3}(\Omega) = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix},$$

where the elements, c_{ij}, are the following:

$$\begin{aligned}
c_{11} &= \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i \\
c_{12} &= \sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i \\
c_{13} &= \sin\omega\sin i \\
c_{21} &= -\cos\Omega\sin\omega - \sin\Omega\cos\omega\cos i \\
c_{22} &= -\sin\Omega\sin\omega + \cos\Omega\cos\omega\cos i \\
c_{23} &= \cos\omega\sin i \\
c_{31} &= \sin\Omega\sin i \\
c_{32} &= -\cos\Omega\sin i \\
c_{33} &= \cos i
\end{aligned}$$

Hence the angle i being in the range $0 \leq i \leq \pi$ due to the singularities at $i = 0, \pi$, is given by

$$i = \cos^{-1}(c_{33}) = \cos^{-1}(3/4) = 0.722734 \text{ rad.} = 41.4096°$$

The sine and cosine of the angle ω are next determined as follows:

$$\cos\omega = \frac{c_{23}}{\sin i} = 0.755929\ ; \qquad \sin\omega = \frac{c_{13}}{\sin i} = 0.654654$$

from which the angle ω is uniquely determined to be

$$\omega = 0.713724 \text{ rad.} = 40.8934°$$

Finally, the angle Ω is uniquely determined as follows:

$$\cos\Omega = -\frac{c_{32}}{\sin i} = 0.654654\ ; \qquad \sin\Omega = \frac{c_{31}}{\sin i} = -0.755929$$

$$\Omega = -0.857072 \text{ rad.} = -49.1066° = 310.8934°$$

2. For an orbit around the Earth ($\mu = 398600.4$ km^3/s^2) with $a = 7500$ km, $e = 0.1$, $\Omega = 30°$, $\omega = 180°$, $i = 90°$, determine the position and velocity vectors in the geocentric celestial reference frame 10 min after reaching the radius of 8250 km.

   Solution:
   The perifocal velocity and position are calculated by first solving Kepler's equation (Lecture 7) as follows. The given radius is identified to be the apogee radius of the orbit, $r_a = a(1+e) = 8250$ km. The mean anomaly at the apogee is π rad., therefore the mean anomaly 10 min. after the apogee is calculated as follows:

$$n = \sqrt{\frac{\mu}{a^3}} = 0.000972024 \text{ rad/s}$$

$$M = n(t - t_0) = \pi + n \times 60 \times 10 = 3.724807029 \text{ rad.}$$

   The solution to Kepler's equation is obtained using Newton's method, starting with the initial guess, $E = M$, and the iteration steps are tabulated in the following table.

E	$f(E)$	$f'(E)$	ΔE
3.724807029	0.055070981	1.083469678	-0.050828354
3.673978675	-6.9297$\times 10^{-5}$	1.086159840	6.37997$\times 10^{-5}$
3.674042475	-1.03309$\times 10^{-10}$	1.086156602	9.5114$\times 10^{-11}$

   Thus a tolerance of $\delta = 10^{-10}$ rad. is met in 2 iterations, with final result $E = 3.674042475$ rad. (210.5071°).

   The perifocal velocity and position vectors are next computed from the eccentric anomaly as follows:

$$\begin{aligned}
\mathbf{r} &= a(\cos E - e)\mathbf{i_e} + a\sqrt{1-e^2}\sin E\mathbf{i_p} \\
&= -7211.745\mathbf{i_e} - 3788.257\mathbf{i_p} \text{ km} \\
\mathbf{v} &= \frac{an(-\sin E\mathbf{i_e} + \sqrt{1-e^2}\cos E\mathbf{i_p})}{1 - e\cos E} \\
&= 3.407269\mathbf{i_e} - 5.753763\mathbf{i_p} \text{ km/s}
\end{aligned}$$

   The $(\Omega_3, i_1, \omega_3)$ inverse rotation matrix giving the transformation from the perifocal frame to the celestial frame is computed according to Lecture 8 as follows:

$$\begin{Bmatrix} \mathbf{I} \\ \mathbf{J} \\ \mathbf{K} \end{Bmatrix} = \mathbf{C}^* \begin{Bmatrix} \mathbf{i_e} \\ \mathbf{i_p} \\ \mathbf{i_h} \end{Bmatrix}$$

   where $\mathbf{C}^* = \mathbf{C}^T = \mathbf{C_3}^T(\Omega)\mathbf{C_1}^T(i)\mathbf{C_3}^T(\omega)$ for the given Euler angles, $\Omega = 30°$, $\omega = 180°$, $i = 90°$ is the following:

$$\mathbf{C}^* = \mathbf{C}^T = \begin{pmatrix} -\sqrt{3}/2 & 0 & 1/2 \\ -1/2 & 0 & -\sqrt{3}/2 \\ 0 & -1 & 0 \end{pmatrix}$$

from which the celestial position and velocity are obtained to be

$$\begin{aligned}
\mathbf{r} &= \mathbf{C}^* \left\{ \begin{array}{c} -7211.745 \\ -3788.257 \\ 0 \end{array} \right\} \\
&= 6245.555\mathbf{I} + 3605.873\mathbf{J} + 3788.257\mathbf{K} \text{ (km)}
\end{aligned}$$

$$\mathbf{v} = \mathbf{C}^* \left\{ \begin{array}{c} 3.407269 \\ -5.753763 \\ 0 \end{array} \right\} = -2.950781\mathbf{I} - 1.703634\mathbf{J} + 5.753763\mathbf{K} \text{ (km/s)}$$

3. An Earth-orbiting spacecraft is observed to have the following geocentric position and velocity vectors in a celestial reference frame:

$$\begin{aligned}
\mathbf{r} &= -6000\mathbf{I} + 4000\mathbf{J} \text{ (km)} \\
\mathbf{v} &= 2\mathbf{I} + 7.5\mathbf{K} \text{ (km/s)}
\end{aligned}$$

Determine the following at the given point:

(a) Declination.
(b) Right ascension.
(c) Flight-path angle.
(d) Velocity azimuth.

Solution:
The orbital radius is $r = \mid \mathbf{r} \mid = 7211.1026$ km, and the orbital speed is $v = \mid \mathbf{v} \mid = 7.762087$ km/s. From the spherical coordinates of radius vector, we have

$$\delta = \sin^{-1} \frac{0}{7211.1026} = 0$$

$$\begin{aligned}
\sin\lambda &= \frac{4000}{7211.1026\cos(0)} = 0.55470 \\
\cos\lambda &= \frac{-6000}{7211.1026\cos(0)} = -0.83205
\end{aligned}$$

which yield $\lambda = 2.55359$ rad. $= 146.3099°$. The rotation matrix of the local horizon frame is the following, according to Lecture 8:

$$\left\{ \begin{array}{c} \mathbf{i_r} \\ \mathbf{i}_\lambda \\ \mathbf{i}_\delta \end{array} \right\} = \mathbf{C_{LH}} \left\{ \begin{array}{c} \mathbf{I} \\ \mathbf{J} \\ \mathbf{K} \end{array} \right\}$$

where

$$\begin{aligned}
\mathbf{C_{LH}} &= \mathbf{C_2}(\frac{-\pi}{2})\mathbf{C_2}(\frac{\pi}{2} - \delta)\mathbf{C_3}(\lambda) \\
&= \begin{pmatrix} \cos\delta\cos\lambda & \cos\delta\sin\lambda & \sin\delta \\ -\sin\lambda & \cos\lambda & 0 \\ -\sin\delta\cos\lambda & -\sin\delta\sin\lambda & \cos\delta \end{pmatrix}
\end{aligned}$$

$$\mathbf{C_{LH}} = \begin{pmatrix} -0.83205 & 0.5547002 & 0 \\ -0.5547002 & -0.83205 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The velocity components in the local horizon frame are thus the following:

$$\begin{aligned} \mathbf{v} &= \mathbf{C_{LH}} \begin{Bmatrix} 2 \\ 0 \\ 7.5 \end{Bmatrix} \\ &= -1.6641\mathbf{i_r} - 1.1094\mathbf{i}_\lambda + 7.5\mathbf{i}_\delta \text{ (km/s)} \end{aligned}$$

Finally, by employing the spherical coordinates of the velocity vector (Lecture 8) the flight-path angle and the velocity azimuth are obtained as follows:

$$\phi = \sin^{-1} \frac{-1.6641}{7.762087} = -12.3796°$$

$$\begin{aligned} \sin\psi &= \frac{-1.1094}{7.762087\cos(-12.3796°)} = -0.146328 \\ \cos\psi &= \frac{7.5}{7.762087\cos(-12.3796°)} = 0.989236 \end{aligned}$$

which yield $\psi = 351.5858°$.