AE-641A (Space Dynamics-I)

Quiz No. 5 (Solution)

1. Write "True" or "False" against each of the following statements (5 marks for each correct answer):

   (a) An *osculating orbit* is the trajectory followed by a spacecraft under the influence of a perturbing acceleration. (Ans. **False**).

   (b) An arbitrarily shaped central body causes a non-conservative orbital perturbation. (Ans. **False**).

   (c) Oblateness effect causes a variation in the inclination of the orbital plane. (Ans. **False**).

   (d) Third-body gravity causes a change in the semi-major axis and the eccentricity of the orbit. (Ans. **False**).

   (e) A *Molniya* orbit has a fixed argument of perigee. (Ans. **True**).

2. A spacecraft is powered by an ion engine, which applies a radial acceleration,

$$\mathbf{u} = a\mathbf{r}/r$$

   with a being constant. Derive the expressions for the following due to the applied acceleration:

   (a) The perturbation potential, U.

   (b) The change in the orbital angular momentum.

   Solution:

   (a) According to Lecture 14, the perturbation potential, $U(\mathbf{r})$, must satisfy

$$\mathbf{u} = \left(\frac{\partial U}{\partial \mathbf{r}}\right)^T$$

   Let the perturbation potential be given by

$$U(\mathbf{r}) = ar$$

where the position vector resolved in a Cartesian inertial frame is

$$\mathbf{r} = X\mathbf{I} + Y\mathbf{J} + Z\mathbf{K} = \left\{ \begin{array}{c} X \\ Y \\ Z \end{array} \right\}$$

with the magnitude

$$r = \sqrt{X^2 + Y^2 + Z^2}$$

Taking the gradient of the perturbation potential, we have

$$\begin{aligned}
\frac{\partial U}{\partial \mathbf{r}} &= a\frac{\partial r}{\partial \mathbf{r}} \\
&= a\left(\frac{\partial r}{\partial X}, \frac{\partial r}{\partial Y}, \frac{\partial r}{\partial Z}\right) \\
&= a\,(X,\ Y,\ Z)\,/\sqrt{X^2 + Y^2 + Z^2} \\
&= a\mathbf{r}^T/r
\end{aligned}$$

Hence $U = ar$ satisfies

$$\mathbf{u} = \left(\frac{\partial U}{\partial \mathbf{r}}\right)^T = a\mathbf{r}/r$$

(b) The equations of perturbed motion are the following (Lecture 14):

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \mu\frac{\mathbf{r}}{r^3} = \mathbf{u}$$

and

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{v}$$

Taking the time derivative of the angular momentum vector, $\mathbf{h}$, we have

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t} &= \frac{\mathrm{d}(\mathbf{r}\times\mathbf{v})}{\mathrm{d}t} \\
&= \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\times\mathbf{v} + \mathbf{r}\times\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \\
&= \mathbf{v}\times\mathbf{v} + \mathbf{r}\times\mathbf{u} \\
&= a\mathbf{r}\times\mathbf{r}/r \\
&= \mathbf{0}
\end{aligned}$$

3. The orbit of a spacecraft around the Earth has the elements, $i = 28.8°$, $e = 0.732$, and period, $T = 10.6$ hr. Find the rotation of apsides and regression of nodes due to Earth's oblateness.

<u>**Solution**</u>:
This is Exercise 1 of Lecture 16, which was solved in Lecture 17. From Lecture 16, we have the following results for the mean variation per orbit of the orbital elements Ω and ω due to the oblateness effect:

$$
\begin{aligned}
\frac{\mathrm{d}\bar{\Omega}}{\mathrm{d}t} &= \frac{-3}{2} n J_2 \left(\frac{r_0}{p}\right)^2 \cos i \\
\frac{\mathrm{d}\bar{\omega}}{\mathrm{d}t} &= \frac{3}{4} n J_2 \left(\frac{r_0}{p}\right)^2 (5\cos^2 i - 1)
\end{aligned}
$$

The oblateness parameter for the Earth is given in Lecture 16 to be $J_2 = 0.00108263$. From the given data, we calculate the mean motion of the spacecraft as follows:

$$n = \frac{2\pi}{T} = 0.0001646537 \text{ rad/s}$$

its semi-major axis to be

$$a = \left(\frac{\mu}{n^2}\right)^{1/3} = 24498.0533 \text{ km}$$

and its orbital parameter to be

$$p = a(1 - e^2) = 11371.4084 \text{ km}$$

The mean equatorial radius of Earth is $r_0 = 6378.14$ km. Substituting the given data and these values into the mean variations, we get the regression of nodes to be the following:

$$
\begin{aligned}
\frac{\mathrm{d}\bar{\Omega}}{\mathrm{d}t} &= -7.37155 \times 10^{-8} \text{ rad/s} \\
&= -0.3649°/\text{day}
\end{aligned}
$$

and the rotation of apsides as follows:

$$
\begin{aligned}
\frac{\mathrm{d}\bar{\omega}}{\mathrm{d}t} &= 1.19433 \times 10^{-7} \text{ rad/s} \\
&= 0.5912°/\text{day}
\end{aligned}
$$

4. Calculate the maximum possible radius of a circular, Sun-synchronous Earth orbit.

   Solution:
   For a sun-synchronous earth orbit, we have from Lecture 17 the following relationship:

$$\begin{aligned}\frac{\mathrm{d}\bar{\Omega}}{\mathrm{d}t} &= \frac{-3}{2} n J_2 \left(\frac{r_0}{p}\right)^2 \cos i \\ &= 0.98565^\circ/\mathrm{day} \\ &= \frac{2\pi}{(365.242)(24)(3600)} \mathrm{~rad/s}\end{aligned}$$

   or

$$\cos i = -\frac{2\left(\frac{p}{r_0}\right)^2}{3nJ_2} \frac{2\pi}{(365.242)(24)(3600)}$$

   where $n = \sqrt{\mu/a^3}$. For a circular orbit, $p = a$. A substitution of these gives

$$\cos i = -\frac{2\frac{a^{7/2}}{r_0^2}}{3\sqrt{\mu}J_2} \frac{2\pi}{(365.242)(24)(3600)}$$

   For the largest possible value of a which satisfies this relationship, we must have $i = \pi$, or

$$1 = \frac{2\frac{a^{7/2}}{r_0^2}}{3\sqrt{\mu}J_2} \frac{2\pi}{(365.242)(24)(3600)}$$

   solving which for a with $r_0 = 6378.14$ km and $J_2 = 0.00108263$ for Earth, we get

$$a = 12352.507 \mathrm{~km}$$

5. Calculate the regression of nodes and the rotation of apsides on an Earth satellite in a circular orbit of 12 hr period, inclined at 60° to the equator, due to the Moon's gravity.

   Solution:
   For the given spacecraft orbit, we have $e = 0$, $i = 60^\circ$, and

$$n = 2\pi/(12 \times 3600) = 1.4544 \times 10^{-4} \mathrm{~rad/s}$$

   The Moon's orbit around the Earth is inclined at $i_3 = 28.582^\circ$ relative to the Earth's equatorial plane, with a mean motion of

$$n_3 = 2\pi/(27.3 \times 24 \times 3600) = 2.6638 \times 10^{-6} \mathrm{~rad/s}$$

From Lecture 16, we have the following average values for the perturbations caused by the Moon on the spacecraft orbit:

$$\begin{aligned}
\frac{\partial \bar{\Omega}}{\partial t} &= -\frac{3}{8}\frac{n_3^2}{n}\frac{\left(1+\frac{3}{2}e^2\right)}{\sqrt{1-e^2}}\cos(i-i_3)\left(3\cos^2 i_3 - 1\right) \\
&= -2.0505 \times 10^{-8} \text{ rad/s } (-0.1015°/\text{day}) \\
\frac{\partial \bar{\omega}}{\partial t} &= \frac{3}{4}\frac{n_3^2}{n}\frac{\left(1-\frac{3}{2}\sin^2 i_3\right)}{\sqrt{1-e^2}}\left[2 - \frac{5}{2}\sin^2(i-i_3) + \frac{e^2}{2}\right] \\
&= 3.1733 \times 10^{-8} \text{ rad/s } (0.15709°/\text{day})
\end{aligned}$$

6. Estimate the life of a satellite of frontal cross-sectional area, $A = 50$ m^2, drag coefficient based on the frontal area, $C_D = 2.2$, mass, $m = 200$ kg., initially placed in a circular orbit of altitude 250 km around the Earth ($\mu = 398600.4$ km^3/s^2, $r_0 = 6378.14$ km), assuming an exponential atmosphere with $\rho_0 = 1.752$ kg/m^3 and $H = 6.7$ km.

Solution:
Here we will apply the approximation obtained in Eq.(15) of Lecture 18 for a low circular orbit with an exponential atmospheric density model:

$$t_d = \frac{H}{\beta\rho_0\sqrt{\mu r_0}}\left[\exp(z_0/H) - 1\right]$$

The ballistic coefficient and the constant, $\sqrt{\mu r_0}$ are calculated to be the following:

$$\beta = \frac{C_D A}{m} = 0.55\ ; \quad \sqrt{\mu r_0} = 50421514805.25 \text{ m}^2/\text{s}$$

which is substituted into the equation to give

$$\begin{aligned}
t_d &= \frac{6700\left[\exp(250/6.7) - 1\right]}{1.752 \times 0.55 \times 50421514805.25} \\
&= 2210956020.612 \text{ s } (70.06 \text{ yr.})
\end{aligned}$$

Thus the life of this satellite is estimated to be approximately 70 yr.