AE-777A (Optimal Space Flight Control)

Quiz No. 5 (Solution)

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1. Show that for a two-body orbit, both the magnitude and the direction of the orbital angular momentum are constants.

   Ans.
   Recall from Lecture 15 that for a system of two bodies (assumed spherical) of masses m_1 and m_2, Newton's law of gravitation and the second law of motion yield the following equations of motion:

$$m_1\ddot{\mathbf{R}}_1 = G\frac{m_1 m_2}{r^3}(\mathbf{R}_2 - \mathbf{R}_1) \tag{1}$$

$$m_2\ddot{\mathbf{R}}_2 = G\frac{m_1 m_2}{r^3}(\mathbf{R}_1 - \mathbf{R}_2) \tag{2}$$

   where G is the universal gravitational constant, overdot represents the time derivative, $\mathbf{R}_i \in \mathbb{R}^3$, $i = 1, 2$, denote the position vector of the mass m_i in an inertial reference frame,

$$\mathbf{r} = \mathbf{R}_2 - \mathbf{R}_1 \tag{3}$$

   is the relative position vector (*radius vector*) of m_2 w.r.t. m_1, and the Euclidean norm (magnitude) of $\mathbf{r}$ is denoted r:

$$r = \mid \mathbf{r} \mid \tag{4}$$

   called the *radius.*

   Subtracting Eq.(1)/m_1 from Eq.(2)/m_2, we write

$$\ddot{\mathbf{R}}_2 - \ddot{\mathbf{R}}_1 = \frac{G(m_1 + m_2)}{r^3}(\mathbf{R}_1 - \mathbf{R}_2) \tag{5}$$

   or

$$\ddot{\mathbf{r}} + \frac{\mu}{r^3}\mathbf{r} = 0 \tag{6}$$

   where

$$\mu = G(m_1 + m_2) \tag{7}$$

   is the gravitational constant of the two-body system. Equation (6) is the governing differential equation of the motion of the centre of mass m_2, relative to the centre of mass m_1 (called the *orbital motion*).

By defining the relative velocity vector,

$$\mathbf{v} = \dot{\mathbf{r}} \tag{8}$$

and taking the vector product of Eq.(6) with $\mathbf{r}$, we have

$$\begin{aligned}
\mathbf{r}\times\ddot{\mathbf{r}} \quad &+ \quad \mathbf{r}\times\frac{\mu}{r^3}\mathbf{r} = \mathbf{0} \\
\Rightarrow \quad \mathbf{r}\times\ddot{\mathbf{r}} \quad &= \quad \mathbf{0} \\
\Rightarrow \quad \dot{\mathbf{r}}\times\dot{\mathbf{r}}+\mathbf{r}\times\ddot{\mathbf{r}} \quad &= \quad \mathbf{0} \\
\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}\times\dot{\mathbf{r}}) \quad &= \quad \mathbf{0} \qquad (9)
\end{aligned}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}\times\mathbf{v}) = \mathbf{0} \tag{10}$$

which implies that the *orbital angular momentum* vector,

$$\mathbf{h} = \mathbf{r}\times\mathbf{v} = \text{const.} \tag{11}$$

is constant. Since $\mathbf{h}$ is a constant vector, it implies that both the magnitude and the direction of $\mathbf{h}$ are constants.

2. A spacecraft is detected by radar to be moving at a speed of 8 km/s around the Earth with a flight-path angle of 20°, when its radius is 10,000 km. Calculate the following:

   (a) Semi-major axis of the orbit.

   (b) Orbital eccentricity.

   (c) Orbital speed at the minimum radius point.

   (d) True anomaly when the radar detection is made.

   Ans.

   (a) From the given data, the semi-major axis of the orbit is determined as follows:

$$\begin{aligned}
\epsilon = \frac{v^2}{2} - \frac{\mu}{r} \quad &= \quad -\frac{\mu}{2a} = -7.86004 \text{ km}^2/\text{s}^2 \\
a \quad &= \quad -\frac{\mu}{2\epsilon} = 25356.1305 \text{ km}
\end{aligned}$$

   (b) Next, the orbital eccentricity is computed as follows:

$$\begin{aligned}
h &= rv\cos\phi = (10,000)(8)\cos(20^\circ) = 75175.4097 \text{ km}^2/\text{s} \\
p &= h^2/\mu = 14177.964 \text{ km} \\
e &= \sqrt{1-\frac{p}{a}} = 0.66396
\end{aligned}$$

(c) The minimum radius and the corresponding speed are now calculated as follows:

$$\begin{aligned} r_p &= a(1-e) = 8520.6013 \text{ km} \\ v_p &= \sqrt{\frac{2\mu}{r_p} - \frac{\mu}{a}} = 8.8228 \text{ km/s} \end{aligned}$$

(d) Then the orbit equation is used to determine the true anomaly at the given point ($r = 10,000$ km, $v = 8$ km/s, $\phi = 20°$):

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{p}{er} - \frac{1}{e}\right) \\ &= \pm 51.005° \end{aligned}$$

Since the given point has $\phi > 0$, it follows that $0 < \theta < \pi$, hence $\theta = 51.005°$.

3. For an Earth orbit with a semi-major axis of 10,000 km and an eccentricity of 0.2, what is the position (radius, true anomaly) 15 min. after crossing the perigee? (Your answer for the angles should be correct to within 10^{-6} rad.)

Ans.
We begin by calculating the mean anomaly 15 min. after perigee as follows:

$$n = \sqrt{\frac{\mu}{a^3}} = 0.00063135 \text{ rad/s}$$

$$M = n(t - t_0) = n \times 15 \times 60 = 0.56821327 \text{ rad.}$$

The solution to Kepler's equation is next obtained using Newton's method for achieving a tolerance, $\delta = 10^{-6}$ rad., starting with the initial guess, $E = M$, and the iteration steps are tabulated in the following table (all angles in rad.):

E	$f(E)$	$f'(E)$	ΔE
0.56821327	-0.10762539	0.83142724	0.12944655
0.69765982	0.00096134	0.84673047	-0.00113535
0.69652447	8.2773×10^{-8}	0.84658469	-9.777×10^{-8}

Thus the specified tolerance, $\delta = 10^{-6}$ rad., is reached in 2 iterations, with the final result $E = 0.69652447$ rad. (39.908°). The true anomaly is calculated by the following relationship (see Lecture 17):

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$$

to be $\theta = 0.8368025$ rad. (47.945°).
Finally, the required orbital radius is obtained as follows:

$$r = \frac{a(1-e^2)}{1+e\cos\theta} = a(1 - e\cos E) = 8465.847 \text{ km}$$