AE-777A (Optimal Space Flight Control)

Quiz No. 1 (Solution)

1. Consider the planar orbital dynamics around a spherical body of gravitational constant, μ, governed by the following differential equations:

$$\ddot{r} - \frac{h^2}{r^3} + \frac{\mu}{r^2} = a_r$$

$$\dot{h} = r a_\theta$$

   Here r gives the radial location of the spacecraft from the centre of the spherical body, h is its angular momentum, and a_r and a_θ are the acceleration inputs applied to the spacecraft in the radial and circumferential directions, respectively.

   (a) Derive the state equations when only the radial acceleration input, $u = a_r$, is applied, and $a_\theta = 0$. (*Hint*: Use r and $\dot{r}$ as state variables).

   (b) Linearize the system about the circular orbit, $r = c =$ const., $h = \sqrt{\mu c}$, and determine the state-space coefficient matrices, A and B.

2. Derive the state-transition matrix of a system whose state equations are the following:

$$\begin{aligned} \dot{x}_1 &= 2x_2 \\ \dot{x}_2 &= -2x_1 + u \end{aligned}$$

3. A system has the following state equations:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \end{aligned}$$

   (a) Investigate the stability of the system.

   (b) Investigate the controllability of the system.

   (c) Is the system observable with $y = x_2$ being the only output?

   (d) If possible, design a state-feedback regulator for the system such that the closed-loop characteristic polynomial is the following:

$$s^2 + s + 1 = 0$$

Solution:

1. (a) To derive the state equations of a satellite in a planar motion around a spherical planet, we begin with the given governing equations of the system:

$$\ddot{r} - \frac{h^2}{r^3} + \frac{\mu}{r^2} = a_r$$
$$\dot{h} = r a_\theta$$

When only the radial thrust acceleration input is applied, we have $a_\theta = 0$, hence

$$\ddot{r} - \frac{h^2}{r^3} + \frac{\mu}{r^2} = a_r \tag{1}$$

and

$$\dot{h} = 0 \tag{2}$$

which implies that the orbital angular momentum, h, is constant and Eq. (2) must be dropped from the state equations of the system.

Then a possible choice of state variables for this second-order system is the following:

$$\begin{aligned} \xi_1 &= r \\ \xi_2 &= \dot{r} \end{aligned} \tag{3}$$

with the control input, $\eta = a_r$, and we write the state equations in the following vector form:

$$\dot{\xi} = f(\xi, \eta) \tag{4}$$

where $\xi = (\xi_1, \xi_2)^T$, and

$$f(\xi, \eta) = \begin{Bmatrix} \xi_2 \\ \\ \eta + \frac{h^2}{\xi_1^3} - \frac{\mu}{\xi_1^2} \end{Bmatrix} \tag{5}$$

(b) The given reference solution to the unforced system ($\eta = 0$) for a constant radius, $r = c$, representing a circular orbit, is given by

$$\begin{aligned} \xi_{1r}(t) &= c \\ \xi_{2r}(t) &= 0 \end{aligned} \tag{6}$$

Equation (1) for the circular orbit yields the constant orbital angular momentum, $h = \sqrt{\mu c}$.

To linearize the state equations about the reference solution, $\xi_r = (\xi_{1r}, \xi_{2r})^T$, $\eta_r = 0$, we carry out their following truncated Taylor series expansion about the reference solution:

$$\dot{x} = Ax + Bu\ , \tag{7}$$

where $u = \eta - \eta_r = \eta$,

$$x = \xi - \xi_r = \left\{ \begin{array}{c} \xi_1 - c \\ \\ \xi_2 \end{array} \right\} \tag{8}$$

and

$$\begin{aligned} A &= \frac{\partial f}{\partial \xi}(\xi_r, \eta_r) \\ &= \left(\begin{array}{cc} 0 & 1 \\ \\ -3\frac{h^2}{\xi_1^4} + 2\frac{\mu}{\xi_1^3} & 0 \end{array} \right)_{\xi_1 = c, \eta = 0} \\ &= \left(\begin{array}{cc} 0 & 1 \\ \\ -\frac{\mu}{c^3} & 0 \end{array} \right) \end{aligned}$$

$$\begin{aligned} B &= \frac{\partial f}{\partial \eta}(\xi_r, \eta_r) \\ &= \left(\begin{array}{c} 0 \\ \\ 1 \end{array} \right)_{\xi_1 = c, \eta = 0} \\ &= \left(\begin{array}{c} 0 \\ \\ 1 \end{array} \right) \end{aligned} \tag{9}$$

2. The coefficient matrix A for the given LTI system is the following:

$$A = \left(\begin{array}{cc} 0 & 2 \\ \\ -2 & 0 \end{array} \right) \tag{10}$$

The first step is to form the *resolvent*, $sI - A$, as follows:

$$sI - A = \left(\begin{array}{cc} s & -2 \\ \\ 2 & s \end{array} \right) \tag{11}$$

whose inverse is then calculated as follows:

$$\begin{aligned} (sI - A)^{-1} &= \frac{\mathrm{adj}(sI - A)}{\det(sI - A)} \\ &= \frac{1}{s^2 + 4} \left(\begin{array}{cc} s & 2 \\ \\ -2 & s \end{array} \right) \end{aligned} \tag{12}$$

Each element in $(sI - A)^{-1}$ has the same denominator polynomial, $D(s) = s^2 + 4$, which indicates complex conjugate poles, $s_{1,2} = \pm 2i$. The corresponding residues are also complex conjugates. For example, the element $s/(s^2 + 4)$ is expanded as follows:

$$\frac{s}{s^2+4} = \frac{r_1}{s-2i} + \frac{r_2}{s+2i} \tag{13}$$

where

$$\begin{aligned} r_1 &= \lim_{s\to 2i}(s-2i)\frac{s}{s^2+4} \\ &= \lim_{s\to 2i}\frac{s}{s+2i} = \frac{1}{2} \end{aligned} \tag{14}$$

$$\begin{aligned} r_2 &= \lim_{s\to -2i}(s+2i)\frac{s}{s^2+4} \\ &= \lim_{s\to -2i}\frac{s}{s-2i} = \frac{1}{2} \end{aligned} \tag{15}$$

Hence we have

$$\frac{s}{s^2+4} = \frac{1}{2(s-2i)} + \frac{1}{2(s+2i)} \tag{16}$$

whose inverse Laplace transform is

$$\begin{aligned} \mathcal{L}^{-1}\frac{s}{s^2+4} &= \mathcal{L}^{-1}\frac{1}{2(s-2i)} + \mathcal{L}^{-1}\frac{1}{2(s+2i)} \\ &= \frac{1}{2}\left(e^{2it} + e^{-2it}\right) = \cos 2t\ . \end{aligned} \tag{17}$$

Similarly,

$$\begin{aligned} \mathcal{L}^{-1}\frac{2}{s^2+4} &= \mathcal{L}^{-1}\frac{1}{2i(s-2i)} - \mathcal{L}^{-1}\frac{1}{2i(s+2i)} \\ &= \frac{1}{2i}\left(e^{2it} - e^{-2it}\right) = \sin 2t\ . \end{aligned} \tag{18}$$

The matrix exponential is thus derived to be the following:

$$\begin{aligned} e^{At} &= \mathcal{L}^{-1}\frac{1}{s^2+4}\begin{pmatrix} s & 2 \\ -2 & s \end{pmatrix} \\ &= \begin{pmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{pmatrix} \end{aligned} \tag{19}$$

where $t \geq 0$.

3. For the given LTI system, we have the following state-coefficient matrices:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\ , \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{20}$$

(a) Stability analysis of the given system reveals the following characteristic equation:

$$\det(sI - A) = \det \begin{pmatrix} s & -1 \\ 0 & s \end{pmatrix} = s^2 = 0 \; , \tag{21}$$

whose roots are

$$s_{1,2} = 0 \; . \tag{22}$$

Since both the eigenvalues of A are zeros, the system is unstable.

(b) Controllability is analyzed using the following test matrix for controllability:

$$\begin{aligned} P &= (B, \; AB) \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned} \tag{23}$$

whose rank equals two, the order of the system. Hence, the system is *controllable*.

(c) Observability with only the state variable, $x_2 = y$, being measured as the output yields $C = (0, 1)$, and is analyzed using the following test matrix for observability:

$$\begin{aligned} N &= (C^T, \; A^T C^T) \\ &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{aligned} \tag{24}$$

whose rank equals one, which is less than the order of the system. Hence, the system is *unobservable*.

(d) The linear state-feedback control law is given by:

$$u = -Kx \tag{25}$$

where $x = (x_1, x_2)^T$ is the state vector and $K = (k_1, k_2)$ is the regulator gain matrix. The linearized state equations are obtained in Part (a) to be the following:

$$\dot{x} = Ax + Bu \; , \tag{26}$$

Substituting Eq.(27) into Eq.(28), we have the following closed-loop state equation:

$$\dot{x} = (A - BK)x \; , \tag{27}$$

whose characteristic polynomial is given by

$$\begin{aligned} \det(sI - A + BK) &= \det \begin{pmatrix} s & -1 \\ k_1 & s + k_2 \end{pmatrix} \\ &= s^2 + k_2 s + k_1 \end{aligned} \tag{28}$$

Comparing Eq.(30) with the given characteristic polynomial, s^2+s+1, we get

$$k_1 = k_2 = 1 \ , \tag{29}$$

or $K = (1, 1)$.

Note: The same result as Eq.(31) is obtained by Ackermann's formula (see Lecture 4):

$$K = (\hat{a} - a)(PW)^{-1} \tag{30}$$