AE-777A (Optimal Space Flight Control)

Quiz No. 3 (Solution)

1. For the minimization of

$$L(u) = u_1^2 - u_1 u_2 + \frac{1}{3} u_2^3$$

with respect to $u = (u_1, u_2)^T \in \mathbb{R}^2$, and subject to

$$u_1 \geq 0 \, ; \quad u_2 \geq 0$$

find the minimum points, $\hat{u}$, (if any).

Ans.
Determination of the stationary points:

$$\begin{aligned} L_u &= \frac{\partial L}{\partial u} = (2u_1 - u_2 \, , \quad -u_1 + u_2^2) \\ &= [0, \quad 0] \end{aligned}$$

which is satisfied for $u_1 = u_2 = 0$ and $u_1 = 1/4$, $u_2 = 1/2$. Hence, $u^* = (0,0)^T$ and $u^* = (\frac{1}{4}, \frac{1}{2})^T$ are the stationary points of $L(u)$. Both the stationary points lie in the feasible region defined by the inequality constraints.

Evaluation of L_{uu} at the stationary point, $u^* = (0,0)^T$:

$$\begin{aligned} L_{uu} &= \frac{\partial^2 L}{\partial u^2} = \begin{pmatrix} 2 & -1 \\ -1 & 2u_2^* \end{pmatrix}_{u_2^*=0} \\ &= \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix} \end{aligned}$$

The eigenvalues of L_{uu} at the stationary point, $u^* = (0,0)^T$:

$$\det(sI - L_{uu}) = s(s-2) - 1 = s^2 - 2s - 1 = 0$$

or $s_{1,2} = 1 \pm \sqrt{2}$. Since one eigenvalue of L_{uu} is positive, while the other is negative, neither the necessary condition, $L_{uu} \geq 0$, nor the sufficient condition of minimization, $L_{uu} > 0$, is satisfied at the stationary point,

$u^* = (0,0)^T$. It is a saddle point lying on the boundary of the feasible region.

Evaluation of L_{uu} at the stationary point, $u^* = (\frac{1}{4}, \frac{1}{2})^T$:

$$L_{uu} = \frac{\partial^2 L}{\partial u^2} = \left(\begin{array}{cc} 2 & -1 \\ -1 & 2u_2^* \end{array} \right)_{u_2^*=1/2} = \left(\begin{array}{cc} 2 & -1 \\ -1 & 1 \end{array} \right)$$

The eigenvalues of L_{uu} at the stationary point, $u^* = (\frac{1}{4}, \frac{1}{2})^T$:

$$\det(sI - L_{uu}) = (s-2)(s-1) - 1 = s^2 - 3s + 1 = 0$$

or $s_{1,2} = (3 \pm \sqrt{5})/2$. Since both the eigenvalues of L_{uu} are positive, the sufficient condition of minimization, $L_{uu} > 0$, is satisfied at the stationary point, $u^* = (\frac{1}{4}, \frac{1}{2})^T$, and it is the only minimum point of $L(u)$ satisfying the inequality constraints.

The inequality constraints can be expressed as follows:

$$f(u) = \left(\begin{array}{c} -u_1 \\ -u_2 \end{array} \right) \leq \left(\begin{array}{c} 0 \\ 0 \end{array} \right) ,$$

with the necessary condition for minimization given by

$$\frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u} = 0 ,$$

where

$$\lambda \left\{ \begin{array}{ll} = (0,0)^T & f(u) < (0,0)^T \\ > (0,0)^T, & f(u) = (0,0)^T \end{array} \right.$$

and

$$\frac{\partial f}{\partial u} = \left(\begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array} \right)$$

Since the minimum point, $\hat{u} = (\frac{1}{4}, \frac{1}{2})^T$, lies strictly inside the feasible region, $f(u) < (0,0)^T$, the necessary condition is satisfied with $\lambda = (0,0)^T$.

In contrast, the necessary condition of minimization is *not* satisfied for the stationary point, $u^* = (0,0)^T$, for any non-zero value of λ, hence it cannot be a minimum point lying at the boundary of the feasible region with $f(u) = (0,0)^T$.

2. Consider a system with the following state equations $[(x_1, x_2)^T \in \mathbb{R}^2$, $u(t) \in \mathbb{R}]$:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + u\end{aligned}$$

Given the initial state vector to be $x(0) = (0, 0)^T$, determine the extremal control history, $u^*(t)$, such that the following objective function is minimized w.r.t. $u(t)$ in the control interval $0 \leq t \leq 1$:

$$J = \frac{1}{2}[x_1(1) - 1]^2 + \frac{1}{2}\int_0^1 u^2(t)\mathrm{d}t$$

Ans.

The state equations are expressed in the following form:

$$\dot{x} = Ax + Bu$$

where the coefficient matrices are the following:

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The Hamiltonian of the optimization problem is given by

$$H = L + \lambda^T(Ax + Bu) = \frac{1}{2}u^2 + \lambda_1 x_2 + \lambda_2(-x_1 + u)$$

The co-state equations of the system are next derived as follows:

$$\dot{\lambda}^* = \begin{Bmatrix} \dot{\lambda}_1^* \\ \dot{\lambda}_2^* \end{Bmatrix} = -\left(\frac{\partial H}{\partial x}\right)^{*T} = -A^T\lambda^*$$

where

$$-A^T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The solution to the co-state equations subject to the initial condition, $\lambda^*(0) = (c_1, c_2)^T$, is expressed by

$$\begin{aligned}\lambda^*(t) &= e^{-A^T t}\lambda^*(0) = \mathcal{L}^{-1}[sI - (-A^T)]^{-1}\lambda^*(0) \\ &= \mathcal{L}^{-1}\frac{1}{s^2+1}\begin{pmatrix} s & 1 \\ -1 & s \end{pmatrix}\lambda^*(0) \\ &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} \qquad (t \geq 0) \\ &= \begin{Bmatrix} c_1\cos t + c_2\sin t \\ -c_1\sin t + c_2\cos t \end{Bmatrix} \qquad (t \geq 0)\end{aligned}$$

The necessary conditions for optimality include the following stationary condition of the Hamiltonian with respect to the control history, $u(t)$:

$$H_u = \left(\frac{\partial H}{\partial u}\right)^* = u^*(t) + \lambda_2^*(t) = 0$$

which results in the following extremal control history

$$u^*(t) = -\lambda_2^*(t) = c_1 \sin t - c_2 \cos t \quad (0 \leq t \leq 1)$$

In order to determine the constants c_1, c_2, the state equations must be solved for $x(t) = (x_1(t), x_2(t))^T$, and the following boundary condition applied on the co-state, $\lambda(t) = (\lambda_1(t), \lambda_2(t))^T$:

$$\lambda^*(1) = \left\{ \begin{array}{c} c_1 \cos 1 + c_2 \sin 1 \\ \\ -c_1 \sin 1 + c_2 \cos 1 \end{array} \right\} = \left(\frac{\partial \phi}{\partial x}\right)^T_{t=1} = \left(\begin{array}{c} x_1^*(1) - 1 \\ \\ 0 \end{array} \right)$$

To apply the terminal boundary condition, the extremal trajectory must be solved from the state equations, given $x(0) = (0, 0)^T$, and it happens that $A = -A^T$ for this system:

$$\begin{aligned}
x^*(t) &= e^{At} x(0) + \int_0^t e^{A(t-\tau)} B u^*(\tau) \mathrm{d}\tau \\
&= -\int_0^t e^{A(t-\tau)} B \lambda_2^*(\tau) \mathrm{d}\tau \\
&= \int_0^t \left(\begin{array}{cc} \cos(t-\tau) & \sin(t-\tau) \\ \\ -\sin(t-\tau) & \cos(t-\tau) \end{array} \right) \left\{ \begin{array}{c} 0 \\ \\ 1 \end{array} \right\} (c_1 \sin \tau - c_2 \cos \tau) \mathrm{d}\tau \\
&= \int_0^t \left\{ \begin{array}{c} \sin(t-\tau) \\ \\ \cos(t-\tau) \end{array} \right\} (c_1 \sin \tau - c_2 \cos \tau) \mathrm{d}\tau
\end{aligned}$$

To apply the terminal boundary condition, only the first state variable needs to be solved for:

$$x_1^*(t) = \int_0^t \left[c_1 \sin(t-\tau) \sin \tau - c_2 \sin(t-\tau) \cos \tau \right] \mathrm{d}\tau$$

To carry out the integration, the following trigonometric identities are employed:

$$\begin{aligned}
\sin(t-\tau) \sin \tau &= \frac{1}{2}[\cos(t-2\tau) - \cos t] \\
\sin(t-\tau) \cos \tau &= \frac{1}{2}[\sin(t-2\tau) + \sin t]
\end{aligned}$$

which yields

$$
\begin{aligned}
x_1^*(t) &= \frac{c_1}{2}\int_0^t \cos(t-2\tau)\mathrm{d}\tau - \frac{c_1}{2}t\cos t - \frac{c_2}{2}t\sin t - \frac{c_2}{2}\int_0^t \sin(t-2\tau)\mathrm{d}\tau \\
&= \frac{c_1}{4}\int_{-t}^t \cos y\mathrm{d}y - \frac{c_1}{2}t\cos t - \frac{c_2}{2}t\sin t - \frac{c_2}{4}\int_{-t}^t \sin y\mathrm{d}y \\
&= \frac{c_1}{2}(\sin t - t\cos t) - \frac{c_2}{2}t\sin t
\end{aligned}
$$

Finally, the boundary condition is applied to produce the following linear equations:

$$
\begin{pmatrix} x_1^*(1)-1 \\ \\ 0 \end{pmatrix} = \begin{Bmatrix} c_1\cos 1 + c_2\sin 1 \\ \\ -c_1\sin 1 + c_2\cos 1 \end{Bmatrix}
$$

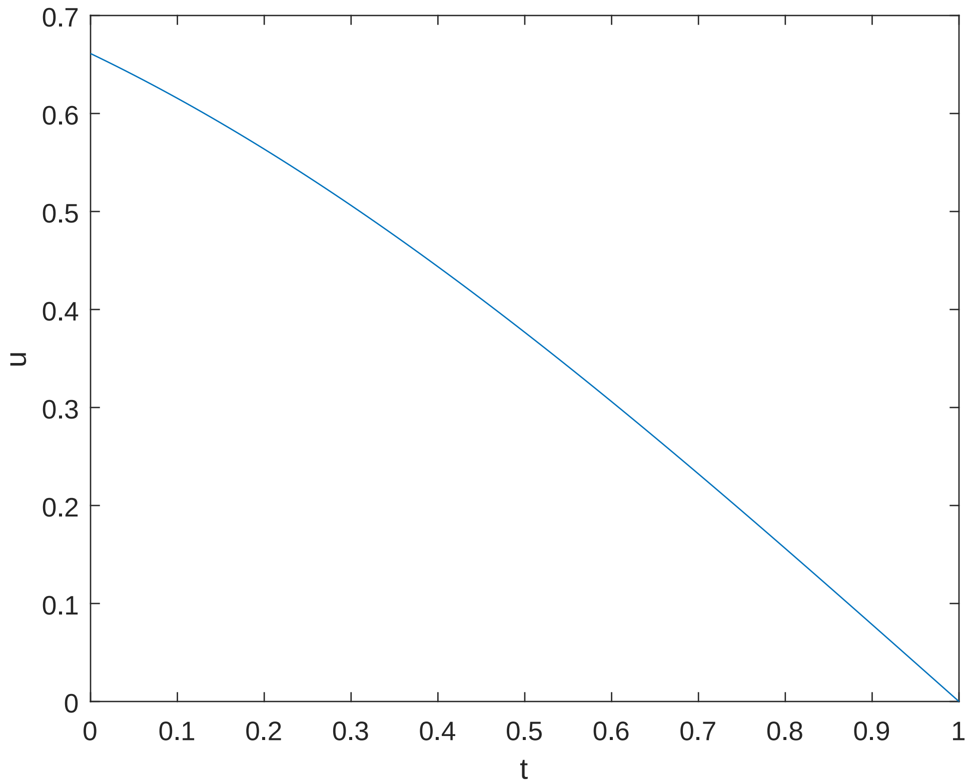
or,

$$
\begin{aligned}
c_2 &= c_1\tan 1 \\
c_1\left(\frac{3}{2}\cos 1 - \frac{1}{2}\sin 1\right) &+ \frac{3}{2}c_2\sin 1 = -1
\end{aligned}
$$

which are solved to give $c_1 = -0.42454$ and $c_2 = -0.6611826$. Therefore, the extremal control is the following:

$$
u^*(t) = -0.42454\sin t + 0.6611826\cos t \quad (0 \le t \le 1)
$$

which is plotted in the figure on the next page.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3. For the system of Problem 2, find a state-feedback regulator (if any) such that the following objective function is minimized w.r.t. $u(t)$:

$$J = \frac{1}{2}\int_0^\infty \left\{x_1^2(t) + u^2(t)\right\} \mathrm{d}t$$

Ans.
The system in Problem 2 has the following state-space coefficients:

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

For minimizing the following quadratic objective function w.r.t. the control input, $u(t)$,

$$J = \frac{1}{2}\int_0^\infty \left\{x_1^2(t) + u^2(t)\right\} \mathrm{d}t \tag{1}$$

the following cost-coefficient matrices are selected (see Lecture 11 for details):

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad R = 1\,, \quad S = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2}$$

The optimal LQR feedback law is sought

$$u(t) = -R^{-1}[B^T P + S^T]x(t) = -\left(p_{12},\ p_{22}\right)x(t) \tag{3}$$

where the constant, real matrix

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}, \tag{4}$$

is the solution to the algebraic Riccati equation:

$$0 = Q + A^T P + PA - PBR^{-1}B^T P \tag{5}$$

or

$$\begin{aligned} &\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 - p_{12}^2 - 2p_{12} & p_{11} - p_{22} - p_{12}p_{22} \\ p_{11} - p_{22} - p_{12}p_{22} & 2p_{12} - p_{22}^2 \end{pmatrix} \end{aligned} \tag{6}$$

We look for a positive-definite matrix, P, which satisfies the algebraic Riccati equation. For this purpose, we first solve the quadratic equation for p_{12} to obtain

$$p_{12} = -1 \pm \sqrt{2} \tag{7}$$

Then we express p_{22} in terms of p_{12} as follows:

$$p_{22} = \sqrt{2p_{12}} \tag{8}$$

Since p_{22} must be real, we require $p_{12} \geq 0$, thus the negative sign in Eq.(7) must be rejected. Therefore, we have

$$p_{12} = -1 + \sqrt{2} \tag{9}$$

and

$$p_{22} = \sqrt{2\sqrt{2} - 2} \tag{10}$$

Solving for p_{11} we get

$$p_{11} = \sqrt{2p_{12}} + \sqrt{2}p_{12}^{3/2} \tag{11}$$

where p_{12} is given by Eq.(9). Hence, P is unique and positive definite.

Therefore, the required LQR feedback law is the following:

$$u(t) = -\left(-1 + \sqrt{2},\ \sqrt{2\sqrt{2} - 2}\right) x(t) \tag{12}$$