AE-777A (Optimal Space Flight Control)

Practice Quiz (Solution)

For a system governed by the following differential equations:

$$\dot{\xi}_1 = \xi_1^3 + \xi_2$$

$$\dot{\xi}_2 = \eta - \xi_1 + \xi_2^2$$

where $\xi_1(t) \in \mathbb{R}$ and $\xi_2(t) \in \mathbb{R}$ are state variables, and $\eta(t) \in \mathbb{R}$ is the control input:

(a) Linearize the system about the reference solution, $\xi_1 = c = \text{const.}$, $\xi_2 = 0$, and determine the state-space coefficient matrices, A and B of the linearized system.

(b) Determine the state response, $x_1(t) = \xi_1(t) - c$, $x_2(t) = \xi_2(t)$, of the linearized system when a unit impulse function, $\eta(t) = \delta(t)$, is applied at $t = 0$, with zero initial condition, $x_1(0) = x_2(0) = 0$.

(c) Investigate the stability of the linearized system.

(d) Investigate the controllability of the linearized system.

(e) Is the linearized system observable with $y = \xi_2$ being the only output?

(f) If possible, design a state-feedback regulator for the linearized system such that the closed-loop characteristic polynomial is the following:

$$s^2 + 2s + 2 = 0$$

Solution:

(a) The state equations can be expressed in the following vector form:

$$\dot{\xi} = f(\xi, \eta) \tag{1}$$

where $\xi = (\xi_1, \xi_2)^T$, and

$$f(\xi, \eta) = \left\{ \begin{array}{c} \xi_1^3 + \xi_2 \\ \\ \eta - \xi_1 + \xi_2^2 \end{array} \right\} \tag{2}$$

To linearize the state equations about the reference solution, $\xi_r = (c, 0)^T$, $\eta_r = 0$, we carry out their following truncated Taylor series expansion about the reference solution:

$$\dot{x} = Ax + Bu \,, \tag{3}$$

where

$$x = \left\{ \begin{array}{c} x_1 \\ \\ x_2 \end{array} \right\} = \xi - \xi_r = \left\{ \begin{array}{c} \xi_1 - c \\ \\ \xi_2 \end{array} \right\} \tag{4}$$

and $u = \eta - \eta_r = \eta$, with A, B being the following Jacobian matrices:

$$\begin{aligned} A &= \frac{\partial f}{\partial \xi}(\xi_r, \eta_r) = \begin{pmatrix} 3\xi_1^2 & 1 \\ & \\ -1 & 2\xi_2 \end{pmatrix}_{\xi_1 = c, \xi_2 = 0} \\ &= \begin{pmatrix} 3c^2 & 1 \\ & \\ -1 & 0 \end{pmatrix} \end{aligned} \tag{5}$$

$$B = \frac{\partial f}{\partial \eta}(\xi_r, \eta_r) = \begin{pmatrix} 0 \\ \\ 1 \end{pmatrix} \tag{6}$$

(b) The general solution of an LTI system to an arbitrary, Laplace transformable input, which begins to act at time $t = 0$ when the system's state is $x(0) = x_0$, is given by (see Lecture 2)

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} B u(\tau) \mathrm{d}\tau \,. \tag{7}$$

Here, it is given that for the linearized system in Part (a), the unit impulse input, $u(t) = \eta(t) = \delta(t)$, is applied at $t = 0$, with zero initial condition,

$x_0 = (0,0)^T$. We first calculate the state-transition matrix using the partial fraction expansion (Lecture 3) as follows:

$$
\begin{aligned}
e^{At} &= \mathcal{L}^{-1}(sI-A)^{-1} = \mathcal{L}^{-1}\begin{pmatrix} s-3c^2 & -1 \\ 1 & s \end{pmatrix}^{-1} \\
&= \mathcal{L}^{-1}\frac{1}{s(s-3c^2)+1}\begin{pmatrix} s & 1 \\ -1 & s-3c^2 \end{pmatrix}
\end{aligned}
\tag{8}
$$

There are the following three possibilities for e^{At}, depending upon the value of the real constant, c:

(i) $\underline{9c^4 > 4}$:

$$
e^{At} = \frac{1}{\sqrt{9c^4-4}}\begin{pmatrix} p_1e^{p_1t} - p_2e^{p_2t} & e^{p_1t} - e^{p_2t} \\ e^{p_2t} - e^{p_1t} & (p_1-3c^2)e^{p_1t} - (p_2-3c^2)e^{p_2t} \end{pmatrix}
\tag{9}
$$

where

$$
p_{1,2} = \frac{3c^2 \pm \sqrt{9c^4-4}}{2}
$$

(ii) $\underline{9c^4 = 4}$:

$$
e^{At} = e^{pt}\begin{pmatrix} 1+pt & t \\ -t & 1+(p-3c^2)t \end{pmatrix}
\tag{10}
$$

where

$$
p = \frac{3c^2}{2}
$$

(iii) $\underline{9c^4 < 4}$:

$$
e^{At} = e^{at}\begin{pmatrix} \cos bt + \frac{a}{b}\sin bt & \frac{1}{b}\sin bt \\ -\frac{1}{b}\sin bt & \cos bt + \frac{a-3c^2}{b}\sin bt \end{pmatrix}
\tag{11}
$$

where

$$
a = \frac{3c^2}{2}\,; \qquad b = \sqrt{1-\frac{9c^4}{4}}
$$

Then, substituting e^{At} into Eq.(7), we have the following state solution of the linearized system:

$$
\begin{aligned}
x(t) &= \int_0^t e^{A(t-\tau)}B\delta(\tau)\mathrm{d}\tau \\
&= \int_0^t e^{A\tau}B\delta(t-\tau)\mathrm{d}\tau \\
&= e^{At}B, \quad (t \geq 0)
\end{aligned}
\tag{12}
$$

or,

(i) $\underline{9c^4 > 4}$:

$$x(t) = \frac{1}{\sqrt{9c^4 - 4}} \left\{ \begin{array}{c} e^{p_1 t} - e^{p_2 t} \\ \\ (p_1 - 3c^2)e^{p_1 t} - (p_2 - 3c^2)e^{p_2 t} \end{array} \right\}, \quad (t \geq 0) \tag{13}$$

where

$$p_{1,2} = \frac{3c^2 \pm \sqrt{9c^4 - 4}}{2}$$

(ii) $\underline{9c^4 = 4}$:

$$x(t) = e^{pt} \left\{ \begin{array}{c} t \\ \\ 1 + (p - 3c^2)t \end{array} \right\}, \quad (t \geq 0) \tag{14}$$

where

$$p = \frac{3c^2}{2}$$

(iii) $\underline{9c^4 < 4}$:

$$x(t) = e^{at} \left\{ \begin{array}{c} \frac{1}{b} \sin bt \\ \\ (1 - 6c^2) \cos bt + \frac{a}{b} \sin bt \end{array} \right\}, \quad (t \geq 0) \tag{15}$$

where

$$a = \frac{3c^2}{2}; \qquad b = \sqrt{1 - \frac{9c^4}{4}}$$

(c) Stability analysis of the linearized system reveals the following characteristic equation:

$$\det(sI - A) = \det \begin{pmatrix} s - 3c^2 & -1 \\ \\ 1 & s \end{pmatrix} = s^2 - 3c^2 s + 1 = 0 \,, \tag{16}$$

whose roots have a positive real part, irrespective of the value of the real constant c (as seen in Part (b)). Hence, the system is unstable.

(d) Controllability is analyzed using the following test matrix for controllability:

$$\begin{aligned} P &= (B, \ AB) \\ &= \begin{pmatrix} 0 & 1 \\ \\ 1 & 0 \end{pmatrix} \end{aligned} \tag{17}$$

whose rank equals two, the order of the system. Hence, the system is *controllable*.

(e) Observability with only the state variable, $\xi_2 = x_2 = y$, being measured as the output yields $C = (0, 1)$, and is analyzed using the following test matrix for observability:

$$
\begin{aligned}
N &= (C^T,\ A^T C^T) \\
&= \begin{pmatrix} 0 & -1 \\ \\ 1 & 0 \end{pmatrix}
\end{aligned}
\tag{18}
$$

whose rank equals two, the order of the system. Hence, the system is *observable*.

(f) The linear state-feedback control law is given by:

$$u = -Kx \tag{19}$$

where $x = (x_1, x_2)^T$ is the state vector and $K = (k_1, k_2)$ is the regulator gain matrix. The linearized state equations are obtained in Part (a) to be the following:

$$\dot{x} = Ax + Bu\ , \tag{20}$$

where the state coefficient matrices are the following:

$$A = \begin{pmatrix} 3c^2 & 1 \\ -1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{21}$$

Substituting Eq.(14) into Eq.(15) we have the following closed-loop state equation:

$$\dot{x} = (A - BK)x\ , \tag{22}$$

whose characteristic polynomial is given by

$$\begin{aligned} \det(sI - A + BK) &= \det \begin{pmatrix} s - 3c^2 & -1 \\ 1 + k_1 & s + k_2 \end{pmatrix} \\ &= s^2 + (k_2 - 3c^2)s + 1 + k_1 \end{aligned} \tag{23}$$

Comparing Eq.(18) with the given characteristic polynomial, $s^2 + 2s + 2$, we get

$$k_1 = 1 \ , \qquad k_2 = 3c^2 + 2 \ , \tag{24}$$

or $K = (2,\ 3c^2 + 2)$.

Note: The same result as Eq.(19) is obtained by Ackermann's formula (see Lecture 4):

$$K = (\hat{a} - \mathbf{a})(PW)^{-1} \tag{25}$$