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Quiz 3

Ans. 1) We have

~~$d(Q)$~~

$L(u) = u_1^2 - 2u_1u_2 + 4u_2^3$

According to the constraints

$u_1 \geq 0$ ; $u_2 \geq 0$

So, to find the stationary point,

$L_u = [2u_1 - 2u_2 \quad -2u_1 + 12u_2^2]$

So, For necessary condition of stationary points,

$L_u = 0$

$\Rightarrow u_1 = u_2 = 0$ & $u_1 = u_2 = \frac{1}{6}$

$\therefore u^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a stationary point

and $u^* = \begin{pmatrix} 1/6 \\ 1/6 \end{pmatrix}$

$L_{uu} = \begin{bmatrix} 2 & -2 \\ -2 & 24u_2 \end{bmatrix}$

$L_{uu}(u^*) = \underbrace{\begin{bmatrix} 2 & -2 \\ -2 & 0 \end{bmatrix}}_{\times}$ at $u^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

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Now, we find the eigen value of $L_{uu}(u^*)$

$$|\lambda I - X| = 0$$

$$X = \begin{bmatrix} 2 & -2 \\ -2 & 0 \end{bmatrix}$$

$$\begin{vmatrix} \lambda - 2 & 2 \\ 2 & \lambda \end{vmatrix} = 0$$

$$\lambda^2 - 2\lambda - 4 = 0$$

$$\Rightarrow \lambda = \frac{2 \pm \sqrt{4+16}}{2}$$

$$= \frac{2 \pm 2\sqrt{5}}{2}$$

Is it also a minimum point?

So, there is one $\lambda$ which is negative. Thus, it is a stationary point ?

$$L(u^*_0)_1 = 0$$

Now, for $(u^*)_2 = \begin{pmatrix} 1/6 \\ 1/6 \end{pmatrix}$

$$L_{uu}(u^*)_2 = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}$$

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Now finding the eigen value,

$$\left| \begin{bmatrix} \lambda-2 & 2 \\ 2 & \lambda-4 \end{bmatrix} \right| = 0$$

$$\therefore \quad \lambda^2 - 6\lambda + 8 - 4 = 0$$
$$= \lambda^2 - 6\lambda + 4 = 0$$

$$\Rightarrow \quad \lambda = \frac{6 \pm \sqrt{36-16}}{2}$$
$$= \frac{6 \pm \sqrt{20}}{2}$$

As both of the eigen values are positive, the point Explain why!

$u^* = \begin{pmatrix} 1/6 \\ 1/6 \end{pmatrix}$ is the minimum point.

And Also, $u_1^* \geq 0$ & $u_2^* \geq 0$ hence the point lies in the feasible region. Thus the minimum point is

$$\hat{u} = \begin{pmatrix} 1/6 \\ 1/6 \end{pmatrix}$$

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Q.2)

$\hat{x} = u + 2x \quad (x(t) \in \mathbb{R} ; u(t) \in \mathbb{R})$

Objective function:-

$$J = \frac{1}{2}[x_1(1) - 1]^2 + \frac{1}{2}\int_0^1 u^2(t)\,dt$$

(for $t \in [0,1]$)

Intial condition :-

$$x(0) = 0$$

$$\dot{x} = Ax + Bu$$
$$\Rightarrow A = 2, \quad B = 1$$

Now, Defining Hamiltonian :-

$$H = \frac{1}{2}u^2 + \lambda(Ax + Bu)$$

($\lambda$ = costate factor)

Now, the following ~~n~~ necessary conditions are found :-

$$H_u = 0 \Rightarrow u + \lambda B = 0$$
$$= u + \lambda = 0 \quad - \quad (1)$$

$$\dot{\lambda} = -(H_x) \Rightarrow \dot{\lambda} = -\lambda A$$
$$\Rightarrow -2\lambda \quad - \quad (2)$$

$$\lambda(1) = \left(\frac{\partial \phi}{\partial x}\right)_{z=1}$$

$$\phi = \frac{1}{2}(x-1)^2 \Rightarrow \frac{\partial \phi}{\partial x} = (x-1)$$

$$\therefore \quad \lambda(1) = x(1) - 1 \quad - (3)$$

Let $\lambda(0) = c$

From (2) :-

$$\underline{\lambda^* = ce^{-2t}}$$

From (1)

$$u^* = -\lambda^* = -ce^{-2t}$$

Solving the state equation gives

$$x^*(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} Bu^*(\tau)d\tau$$

$$= e^{2t} \times 0 + \int_0^t e^{2(t-\tau)} \times 1 \times (-ce^{-2\tau})d\tau$$

$$= \int_0^t -ce^{2t-4\tau}\, d\tau$$

$$= -c\left[\frac{e^{2t-4\tau}}{-4}\right]_0^t = \frac{c}{4}\left[e^{-2t} - e^{2t}\right]$$

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$$\therefore \quad x^*(t) = \frac{c}{4}\left[e^{-2t} - e^{2t}\right]$$

Using (3)

$$\lambda(1) = x(1) - 1$$

$$\Rightarrow \quad c = \frac{c}{4}\left[e^{-2} - e^{2}\right] - 1$$

$$= \quad 1 = c\left[\frac{e^{-2} - e^{2}}{4} - 1\right]$$

$$c = \frac{4}{e^{-2} - e^{2} - 4} = -0.355$$

$$\therefore \quad \boxed{x^*(t) = \frac{-0.355}{4}\left(e^{-2t} - e^{2t}\right)}$$

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$$\boxed{u^*(t) = 0.355\, e^{-2t}}$$

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Ans 3)

State equation :-

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_1 + u$$

Co-state vector $\lambda(0) = (c_1, c_2)^T$

$$x(0) = (0, 0)^T$$
$$0 \le t \le 5$$

$$J = \frac{1}{2}[x_1(5) - 1]^2 + \frac{1}{2}x_2^2(5) + \frac{1}{2}\int_0^5 u^2(t)\,dt$$

Vector function

$$= \begin{Bmatrix} x_2 \\ x_1 + u \end{Bmatrix}, \quad x = (x_1, x_2)^t \text{ state vector}$$

$$A = \frac{\partial f}{\partial x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$B = \frac{\partial f}{\partial u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Now, defining Hamiltonian :

$$H = \frac{1}{2}u^2 + \lambda^T (Ax + Bu)$$

Necessary conditions:-

$$H_u = u + \lambda^T B = u + (\lambda_1, \lambda_2)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

$$\Rightarrow \quad u + \lambda_2 = 0 \quad - \quad (1)$$

$$\dot{\lambda} = -(H_x)^T = -A^T \lambda$$

$$= -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

$$= -\begin{pmatrix} \lambda_2 \\ \lambda_1 \end{pmatrix}$$

&

$$\lambda(5) = \left(\frac{\partial \phi}{\partial x}\right)^T_{t=5}$$

$$= \begin{pmatrix} x_1(5) - 1 \\ x_2(5) \end{pmatrix}$$

$$\lambda(1) = (c_1, c_2)^T$$

Now, for solution to co-state equation:-

$$\lambda^*(t) = e^{-A^T t} \lambda(0) = ?$$

$$= \mathcal{L}^{-1}\left(sI - (AT)\right)^{-1} \lambda(0).$$

Derivation steps?

$$= \begin{pmatrix} \frac{e^t + e^{-t}}{2} & \frac{-e^t - e^{-t}}{2} \\ \frac{-e^t - e^{-t}}{2} & \frac{e^t + e^{-t}}{2} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$n = ?$
$a$

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$$= \begin{pmatrix} \cos nt & -\sin nt \\ -\sin nt & \cos nt \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\lambda_1^*(t) = c_1 \cos n(t) - c_2 \sin(t)$$
$$\lambda_2^*(t) = -c_1 \sin n(t) + c_2 \cos n(t)$$

0 $$u^* = -\lambda_2^*(t)$$

$$u^* = c_1 \sin n(t) - c_2 \cos n(t)$$