AE-777A (Optimal Space Flight Control)

Quiz No. 4 (Solution)

1. The system of a sliding block on a horizontal, frictionless table governed by

$$\ddot{y} = u$$

where $y(t)$ is the displacement of the block measured from one end of the table, and $u(t)$ is the applied acceleration input, is to be controlled such that beginning from $y(0) = 0$ and $\dot{y}(0) = 0$ at $t = 0$, it reaches a final displacement $y(t_f) = 10$ m, and final velocity $\dot{y}(t_f) = 0$, at unspecified time t_f, while minimizing the following performance index w.r.t. $u(t)$:

$$J = 900\, t_f^2 + \frac{1}{2}\int_0^{t_f} u^2(t)\mathrm{d}t$$

Find an extremal trajectory, and determine whether it is an optimal trajectory.

Ans.
Let $x_1 = y$ and $x_2 = \dot{y}$ be the state variables of the plant. Then the state equation is given by

$$\dot{x} = Ax + Bu$$

where $x = (x_1, x_2)^T$ is the state vector and

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

are the coefficient matrices. The initial condition is given to be the rest condition at $t = 0$, i.e., $x(0) = (0, 0)^T$. Also, the terminal state at the unspecified time $t = t_f$ is a rest condition of $x(t_f) = (10\text{ m}, 0)^T$.

The state transition matrix is given by

$$e^{At} = \mathcal{L}^{-1}(sI - A)^{-1} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

The given problem is for minimizing J with $\phi = 900t^2$ and $L = u^2/2$.

Defining the Hamiltonian as follows:

$$H = u^2/2 + \lambda^T(Ax + Bu) = u^2/2 + \lambda_1 x_2 + \lambda_2 u$$

where $\lambda = (\lambda_1, \lambda_2)^T$ is the costate vector, we have the following necessary conditions:

$$H_u = u + \lambda^T B = u + \lambda_2 = 0$$

$$\dot{\lambda} = -(H_x)^T = -A^T\lambda = -\begin{pmatrix} 0 \\ \lambda_1 \end{pmatrix}$$

Let the initial costate vector be given by $\lambda(0) = (c_1, c_2)^T$, where c_1, c_2 are constants. Then the solution to the costate equation is given by

$$\lambda(t) = e^{-A^T t}\lambda(0) = \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

or

$$\lambda_1(t) = c_1, \qquad \lambda_2(t) = -c_1 t + c_2$$

The state equation is now solved for the extremal trajectory with the extremal control,

$$u^* = -\lambda_2(t) = -c_2 + c_1 t$$

to yield

$$\begin{aligned} x^*(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)} B u^*(\tau) \mathrm{d}\tau \\ &= \int_0^t \begin{pmatrix} 1 & t-\tau \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} (c_1\tau - c_2) \mathrm{d}\tau \\ &= \begin{pmatrix} c_1 \frac{t^3}{6} - c_2 \frac{t^2}{2} \\ c_1 \frac{t^2}{2} - c_2 t \end{pmatrix}, \quad (0 \le t \le t_f) \end{aligned}$$

Finally, the boundary conditions at $t = t_f$ are applied as follows:

$$
\begin{aligned}
x_1^*(t_f) &= c_1 t_f^3/6 - c_2 t_f^2/2 = 10 \\
x_2^*(t_f) &= c_1 t_f^2/2 - c_2 t_f = 0 \\
0 &= \left(H + \frac{\partial \phi}{\partial t} \right)_{t=t_f} \\
&= (u^2/2 + \lambda_1 x_2 + \lambda_2 u + 1800t)_{t=t_f} \\
&= -\frac{1}{2}\lambda_2^2(t_f) + 1800 t_f \\
&= -\frac{1}{2}(c_2 - c_1 t_f)^2 + 1800 t_f
\end{aligned}
$$

Solving these 3 equations for the constants c_1, c_2, t_f yields:

$$c_2 = c_1 t_f/2, \qquad c_1 t_f^3 = -120$$

and $t_f = 1$ s. Thus we have

$$c_1 = -120, \qquad c_2 = -60$$

Substituting these constants gives the extremal trajectory and the corresponding control history,

$$
\begin{aligned}
y^*(t) &= x_1^*(t) = -20t^3 + 30t^2 \text{ (m)} \\
\dot{y}^*(t) &= x_2^*(t) = -60t^2 + 60t \text{ (m/s)} \\
u^*(t) &= -120t + 60 \text{ (m/s}^2\text{)}, \qquad (0 \le t \le 1 \text{ s})
\end{aligned}
$$

The extremal trajectory is the optimal one, because the Legendre-Clebsch sufficient condition for optimality is satisfied by it:

$$H_{uu} = \left(\frac{\partial^2 H}{\partial u^2} \right)^* = 1 > 0$$

2. Suppose the sliding block in Problem 1 is to be moved from initial state $y(0) = 0$ and $\dot{y}(0) = 1$ m/s at $t = 0$, such that it reaches a final displacement, $y(t_f) = 10$ m, with a zero velocity, $\dot{y}(t_f) = 0$, in the minimum final time t_f, while having the input acceleration bounded by

$$\mid u(t) \mid \leq 1 \text{ m/s}^2$$

Solve for the optimal trajectory and control history.

Ans.
The state equations are expressed in the following form:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \end{aligned}$$

The system must be moved from the rest condition, $\mathbf{x}(0) = (x_1(0), x_2(0))^T = (0, 1 \text{ m/s})^T$, to $\mathbf{x}(t_f) = (10 \text{ m}, 0)^T$ in the minimum final time, t_f, and with the following inequality constraint on the input magnitude:

$$|u(t)| \leq 1 \text{ m/s}^2$$

The Lagrangian of this time-optimal control problem is $L = 1$, with the Hamiltonian given by

$$H = 1 + \lambda^T f = 1 + \lambda_1 x_2 + \lambda_2 u$$

The solution to the co-state equations subject to the initial condition, $\lambda(0) = (c_1, c_2)^T$, is expressed by

$$\begin{aligned} \lambda(t) &= e^{-A^T t}\lambda^*(0) = \mathcal{L}^{-1}[sI - (-A^T)]^{-1}\lambda(0) \\ &= \mathcal{L}^{-1}\frac{1}{s^2}\begin{pmatrix} s & 0 \\ -1 & s \end{pmatrix}\lambda(0) \\ &= \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}\begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} \qquad (t \geq 0) \\ &= \begin{Bmatrix} c_1 \\ -c_1 t + c_2 \end{Bmatrix} \qquad (t \geq 0) \end{aligned}$$

The terminal boundary conditions are given by

$$\begin{aligned} x_1(t_f) &= 10 \text{ m} \\ x_2(t_f) &= 0 \end{aligned}$$

where t_f is an unknown variable.

Pontryagin's minimum principle applied to this singular problem results in the following switching condition:

$$\hat{u}(t) = \begin{cases} -1 \text{ m/s}^2 , & \hat{\lambda}_2(t) > 0 \\ 1 \text{ m/s}^2 , & \hat{\lambda}_2(t) < 0 \end{cases}$$

where

$$\hat{\lambda}_2(t) = -c_1 t + c_2$$

Since $\hat{\lambda}_2(t)$ is a linear function of time, it can change sign only once. Therefore, a change in the system's state from $\mathbf{x}(0) = (0,\ 1 \text{ m/s})^T$ to $\mathbf{x}(t_f) = (10 \text{ m},\ 0)^T$ requires that a positive control, $\hat{u}(t) = 1 \text{ m/s}^2$, must be applied before switching, $t < \hat{t}$. This implies $\hat{\lambda}_2(t) < 0$ for $0 \leq t < \hat{t}$. Thus we have

$$\begin{aligned} \hat{\mathbf{x}}(t) &= e^{At}\mathbf{x}(0) + \int_0^t e^{A(t-\tau)} B u(\tau) \mathrm{d}\tau \\ &= \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} 1 & t-\tau \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathrm{d}\tau \\ &= \begin{pmatrix} t + t^2/2 \\ 1 + t \end{pmatrix} \qquad (0 \leq t < \hat{t}\,) \end{aligned}$$

After switching at $t = \hat{t}$, the optimal control changes to $\hat{u}(t) = -1 \text{ m/s}^2$, resulting in the following state response:

$$\begin{aligned} \hat{\mathbf{x}}(t) &= e^{A(t-\hat{t}\,)}\mathbf{x}(\hat{t}\,) + \int_{\hat{t}}^t e^{A(t-\tau)} B u(\tau) \mathrm{d}\tau \\ &= \begin{pmatrix} 1 & t-\hat{t} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{t} + \hat{t}^2/2 \\ 1 + \hat{t} \end{pmatrix} \\ &+ \int_{\hat{t}}^t \begin{pmatrix} 1 & t-\tau \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} (-1) \mathrm{d}\tau \\ &= \begin{pmatrix} -t^2/2 + t + 2\hat{t}t - \hat{t}^2 \\ -t + 2\hat{t} + 1 \end{pmatrix} \qquad (\hat{t} < t \leq t_f) \end{aligned}$$

To satisfy the terminal boundary conditions, we must have

$$\begin{aligned} x_1(t_f) &= -t_f^2/2 + t_f + 2\hat{t}t_f - \hat{t}^2 = 10 \\ x_2(t_f) &= -t_f + 2\hat{t} + 1 = 0 \end{aligned}$$

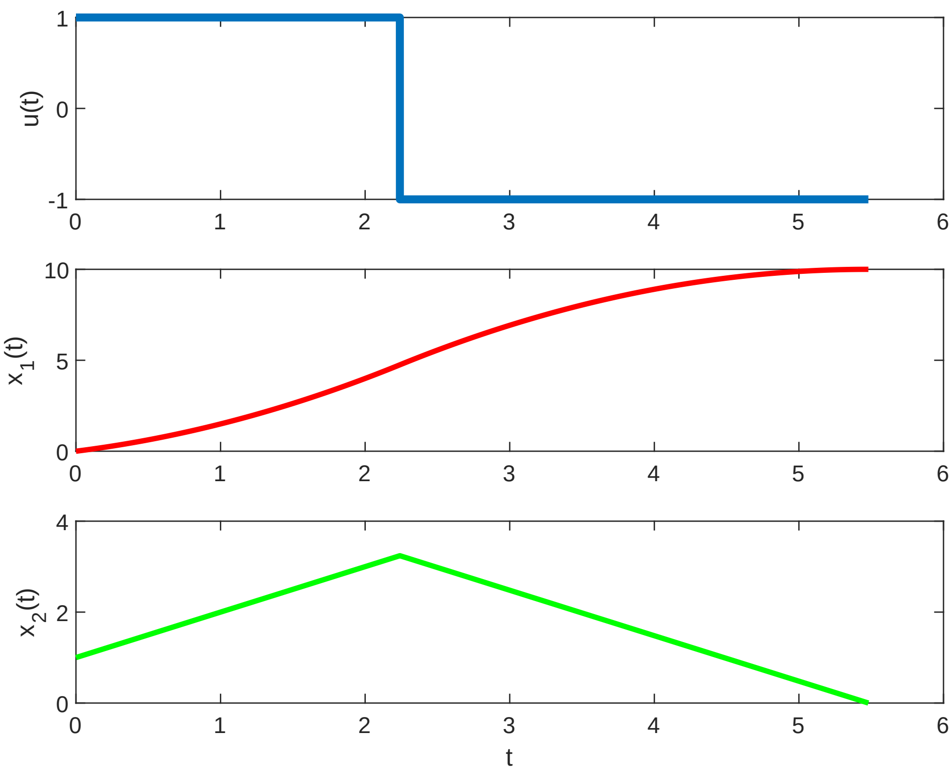
Solving the second condition yields

$$\hat{t} = \frac{t_f - 1}{2}$$

which substituted into the first gives

$$t_f = \sqrt{42} - 1$$

hence $\hat{t} = \sqrt{42}/2 - 1$. The resulting bang-bang optimal trajectory is plotted in the figure on the next slide.



1
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3
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10
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3. Write either "True" or "False" against each of the following statements:

   (a) *Orbital dynamics* refers to the translational motion of the spacecraft's centre of mass. (**Ans.**: *True*).

   (b) *Space navigation* is the control of the rotational dynamics of the spacecraft about its centre of mass. (**Ans.**: *False*).

   (c) The navigational *feedforward controller* compares the actual trajectory with the specific waypoints, and generates corrective inputs. (**Ans.**: *False*).

   (d) The attitude control system acts as a slave to the navigational control system. (**Ans.**: *True*).

   (e) The idealized navigational control system neglects the time scale of the attitude control system. (**Ans.**: *True*).