# AE-777A (Optimal Space Flight Control)

## Quiz No. 2 (Solution)

1. For the following function:

$$L(u) = \frac{1}{4}u^4 + \frac{2}{3}u^3 + \frac{1}{2}u^2$$

where $u \in \mathbb{R}$, find the stationary points (if any) and determine if they are the minimum points.

**Ans.**
Since $L(u)$ has the derivatives,

$$L'(u) = u^3 + 2u^2 + u \; ; \qquad L''(u) = 3u^2 + 4u + 1$$

which are continuous throughout the real space, we can test for the necessary and sufficient conditions. To find the stationary points, we check whether the equation

$$L'(u) = u^3 + 2u^2 + u = u(u^2 + 2u + 1) = 0$$

has any real roots. Its roots are given by

$$u = 0 \; ; \qquad u = -1$$

Hence the stationary points of $L(u)$ are $u^* = 0$ and $u^* = -1$.

To see whether the sufficient condition applies at the stationary points, we evaluate $L''(u)$ for each point as follows:

$\underline{u^* = 0}$:

$$L''(0) = 3(u^*)^2 + 4u^* + 1 = 1 > 0$$

Since the sufficient condition for minimization, $L''(u^*) > 0$, is satisfied by the stationary point, $u^* = 0$, it is a *minimum* point of $L(u)$.

$\underline{u^* = -1}$:

$$L''(-1) = 3(u^*)^2 + 4u^* + 1 = 3 - 4 + 1 = 0$$

Hence the sufficient condition for minimization is *not* satisfied by the stationary point, $u^* = -1$, since $L''(-1)$ vanishes. However, the necessary condition for minimization, $L''(u^*) \geq 0$, is satisfied, hence $u^* = -1$ is a *singular* point of $L(u)$.

2. Consider the following function:

$$L(u) = \frac{1}{3}u_1^3 - u_1u_2^2 + u_2$$

where $u = (u_1, u_2)^T \in \mathbb{R}^2$, find the stationary points (if any) and determine if they are the minimum points.

**Ans.**

Determination of the stationary points:

$$\begin{aligned} L_u &= \frac{\partial L}{\partial u} = [u_1^2 - u_2^2\ , \qquad -2u_1u_2 + 1] \\ &= [0, \quad 0] \end{aligned}$$

or

$$u_1^2 = u_2^2\ ; \qquad 2u_1u_2 = 1$$

whose solutions are

$$u_1^2 = u_2^2 = \pm\frac{1}{2}$$

However, since $(u_1, u_2)$ are real, the negative sign is discarded, and we have

$$u_1^2 = u_2^2 = \frac{1}{2}$$

or

$$u_1 = u_2 = \pm\frac{1}{\sqrt{2}}$$

<u>Necessary Conditions:</u>

If there are no constraints on the possible values of $u$, then the following *necessary conditions* must be satisfied by a set $u = (u_1, u_2)^T$ which *minimizes* $L(u)$:

$$L_u = \frac{\partial L}{\partial u} = 0 \tag{1}$$

and

$$L_{uu} = \frac{\partial^2 L}{\partial u^2} \geq 0 \tag{2}$$

A set $u = (u_1, u_2)^T$ which satisfies Eq.(1) is called a *stationary point* of $L(u)$. Hence $u^* = \pm(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T$ are the stationary points of $L(u)$.

Inequality (2) implies that $L_{uu}$ is a *positive semi-definite* matrix, i.e., all the eigenvalues of $L_{uu}$ are either greater than, or equal to zero.

<u>Sufficient Condition:</u>

A stationary point of $L(u)$ is its *minimum point*, if it satisfies the following *sufficient condition* of minimization:

$$L_{uu} = \frac{\partial^2 L}{\partial u^2} > 0 \tag{3}$$

Inequality (3) implies that $L_{uu}$ is a *positive definite* matrix, i.e., all the eigenvalues of $L_{uu}$ are greater than zero.

If at least one eigenvalue of $L_{uu}$ is zero, then the sufficient condition of minimization is not satisfied by a stationary point, and it is termed a *singular point.*

Evaluation of $L_{uu}$ at the stationary points, $u^* = \pm(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T$:

$$\begin{aligned} L_{uu} &= \frac{\partial^2 L}{\partial u^2} = \frac{\partial}{\partial u}[u_1^2 - u_2^2\,, \qquad -2u_1u_2 + 1] \\ &= \begin{pmatrix} 2u_1 & -2u_2 \\ -2u_2 & -2u_1 \end{pmatrix} \\ &= \pm \begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & -\sqrt{2} \end{pmatrix} \end{aligned}$$

The eigenvalues of $L_{uu}$ at the stationary points:

$$\det(sI - L_{uu}) = (s \mp \sqrt{2})(s \pm \sqrt{2}) - 2 = s^2 - 4 = 0$$

or $s_{1,2} = \pm 2$. Since one eigenvalue of $L_{uu}$ is positive while the other is negative at the stationary points, neither the necessary, nor the sufficient conditions of minimization are satisfied. Therefore, the stationary points, $u^* = \pm(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T$, are *not* the minimum points (they are *saddle points*) of the given function.

3. For the minimization of

$$L(x, u) = \frac{1}{2}x^2 + \frac{1}{2}u^T \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} u$$

with respect to $u \in \mathbb{R}^2$, where $x \in \mathbb{R}$, subject to the constraint

$$f(x, u) = x - [3 \,, \quad -1]u + 1 = 0$$

find the stationary points, $(x^*, u^*)$, (if any).

**Ans.**
The following Hamiltonian function:

$$H = L(x, u) + \lambda^T f(x, u)$$

is to be minimized with respect to $u = (u_1, u_2)^T$, where $\lambda \in \mathbb{R}$ is the Lagrange multiplier.

Finding the partial derivatives of $H$ w.r.t. $x$ and $u$ and then applying the necessary conditions, we have:

$$H_x = x + \lambda = 0$$

or $\lambda = -x$, and

$$H_u = u^T \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} - \lambda(3, \quad -1) = (0, \ 0)$$

which yields

$$u^T = \lambda(3, \quad -1/2) = -x(3, \quad -1/2)$$

Substituting the equality constraint,

$$x = (3 \,, \quad -1)u - 1$$

we get

$$u^T = [-(3, \quad -1)u + 1](3, \quad -1/2)$$

or

$$\begin{aligned} 10u_1 - 3u_2 &= 3 \\ u_1 - u_2 &= 1/3 \end{aligned}$$

solving which, we have the following stationary point:

$$u^* = \left\{ \begin{array}{c} 2/7 \\ \\ -1/21 \end{array} \right\}$$

$$x^* = -2/21 = -\lambda^*$$