Week-06 Lec-09

Objective function: $\boxed{L(\underset{\sim}{x}, \underset{\sim}{u}) \in \mathbb{R}}$

State variable $\underset{\sim}{x} \in \mathbb{R}^{n\times 1}$

control variable $\underset{\sim}{u} \in \mathbb{R}^{m\times 1}$

Static equality constraint: $\boxed{f(\underset{\sim}{x}, \underset{\sim}{u}) = \underset{\sim}{0}_{n\times 1}}$

here $f : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$

Assume that both $L(\underset{\sim}{x}, \underset{\sim}{u})$ & $f(\underset{\sim}{x}, \underset{\sim}{u})$ posses continuous pds of upto ∞ order. w.r.t $\underset{\sim}{x}$ & $\underset{\sim}{u}$ (though we are interested in derivative only upto 2nd order)

- Thus far we have learnt about the "neccesary condition" for minimisation of $L(\underset{\sim}{x}, \underset{\sim}{u})$ subject to the constraint $f(\underset{\sim}{x}, \underset{\sim}{u}) = \underset{\sim}{0}_{n\times 1}$

in terms of Hamiltonian function

$H(\underset{\sim}{x}, \underset{\sim}{u}, \underset{\sim}{\lambda})$ ($\underset{\sim}{x} \in \mathbb{R}^n$, $\underset{\sim}{u} \in \mathbb{R}^m$, $\underset{\sim}{\lambda} \in \mathbb{R}^n$)

$= L(\underset{\sim}{x}, \underset{\sim}{u}) + (\underset{\sim}{\lambda})^T_{1\times n} f(\underset{\sim}{x}, \underset{\sim}{u})_{n\times 1}$

it is

$\underset{\sim}{H_x}_{1\times n} = \underset{\sim}{0}_{1\times n}$ & $\underset{\sim}{H_u}_{1\times m} = \underset{\sim}{0}_{1\times m}$

$\underset{\sim}{H_x}_{1\times n} = \underset{\sim}{L_x}_{1\times n} + (\underset{\sim}{\lambda}^T)_{1\times n} \underline{f_x}_{n\times n}$

$\underset{\sim}{H_u}_{1\times m} = \underset{\sim}{L_u}_{1\times m} + (\underset{\sim}{\lambda}^T)_{1\times n} \underline{f_u}_{n\times m}$

using $\underset{\sim}{H_x}_{1\times n} = \underset{\sim}{0}_{1\times n}$ & $\underset{\sim}{H_u}_{1\times m} = \underset{\sim}{0}_{1\times m}$ } we can find the stationary pt. $(\underset{\sim}{x}^*, \underset{\sim}{u}^*)$

only when $(\underset{\sim}{x}^*, \underset{\sim}{u}^*)$ satisfies the "sufficient condition" we can say that $(\underset{\sim}{x}^*, \underset{\sim}{u}^*)$ is minima pt

- Now we will emphasize on the "Sufficient condition" for minimisation of $L(\underset{\sim}{x}, \underset{\sim}{u})$ subject to the constraint $f(\underset{\sim}{x}, \underset{\sim}{u}) = \underset{\sim}{0}_{n\times 1}$

$$\boxed{\left(\frac{\partial^2 L}{\partial u^2}\right)_{f=0} > 0}$$

@ Stationary pts, $(\underset{\sim}{x}^*, \underset{\sim}{u}^*)$

then at $(\underset{\sim}{x}^*, \underset{\sim}{u}^*)$ $L(\underset{\sim}{x}, \underset{\sim}{u})$ achieves minima subject to constraint $f(\underset{\sim}{x}, \underset{\sim}{u}) = \underset{\sim}{0}_{n\times 1}$

$$\left(\frac{\partial^2 L}{\partial y^2}\right)_{f=0_{n\times 1}} = \left\{ -\left(\underline{\underline{f_u}}_{n\times m}\right)^T \left(\underline{\underline{f_x}}^T_{n\times n}\right)^{-1} \quad \underline{\underline{I}} \right\} \cdot \begin{pmatrix} H_{xx} & H_{xu} \\ H_{ux} & H_{uu} \end{pmatrix} \begin{Bmatrix} -\left(\underline{\underline{f_x}}_{n\times n}\right)^{-1}\left(\underline{\underline{f_u}}_{n\times m}\right) \\ \underline{\underline{I}} \end{Bmatrix} > 0.$$

$$\boxed{\left(\frac{\partial^2 L}{\partial \underset{\sim}{u}^2}\right)_{f=0} = \underline{\underline{H_{uu}}} - \underline{\underline{H_{ux}}}\,\underline{\underline{f_x}}^{-1}\underline{\underline{f_u}} - \underline{\underline{f_u}}^T\left(\underline{\underline{f_x}}^T\right)^{-1}\underline{\underline{H_{xu}}} + \underline{\underline{f_u}}^T\left(\underline{\underline{f_x}}^T\right)^{-1}\underline{\underline{H_{xx}}}\left(\underline{\underline{f_x}}^{-1}\right)\underline{\underline{f_u}} > 0}$$

Legendre-clebsche Condition!

here

$$\underline{\underline{H_{ux}}}_{m\times n} = \frac{\partial}{\partial \underset{\sim}{x}}\left(\left(\left(\frac{\partial H}{\partial \underset{\sim}{u}}\right)_{m\times 1}\right)^T\right)_{n\times m} = \frac{\partial^2 H}{\partial \underset{\sim}{u}\,\partial \underset{\sim}{x}} \in R^{m\times n}$$

$$\Rightarrow \underline{\underline{H_{xu}}}_{n\times m} = \frac{\partial^2 H}{\partial \underset{\sim}{x}\,\partial \underset{\sim}{u}} = \frac{\partial}{\partial \underset{\sim}{u}}\left(\left(\frac{\partial H}{\partial \underset{\sim}{x}}\right)^T\right) \in R^{n\times m}$$

$$\underline{\underline{H_{uu}}}_{m\times m} \qquad \underline{\underline{H_{xx}}}_{n\times n}$$

$$\boxed{\text{i.e. } \underline{\underline{H_{ux}}}_{m\times n},\ \underline{\underline{H_{uu}}}_{m\times m},\ \underline{\underline{H_{xx}}}_{n\times n},\ \underline{\underline{H_{xu}}}_{n\times m} \ \ \&\ \ \underline{\underline{f_x}}_{n\times n},\ \underline{\underline{f_u}}_{n\times m}}$$

Q. **problem:** minimize $L(\underset{\sim}{x}, \underset{\sim}{u})$ wrt $\underset{\sim}{u}$ subject to constraint

$$f(\underset{\sim}{x}, \underset{\sim}{u}) = \underset{\sim}{0}_{n\times 1}$$

First find out the stationary pt $(\underset{\sim}{x}^*, \underset{\sim}{u}^*)$

Second check whether ~~$L(\underset{\sim}{x}, \underset{\sim}{u})$ is minimized at~~ $(\underset{\sim}{x}^*, \underset{\sim}{u}^*)$ satisfies the "sufficient condition" for minimization!

↳ if it satisfies then $(\underset{\sim}{x}^*, \underset{\sim}{u}^*)$ pt. of minima

↳ ~~else $(\underset{\sim}{x}^*, \underset{\sim}{u}^*)$ is NOT the pt. of minima!~~

Step I: Objective function: $L(\underset{\sim}{x}, \underset{\sim}{u}) \in \mathbb{R}$

equality constraint: $\underset{\sim}{f}(\underset{\sim}{x}, \underset{\sim}{u}) = \underset{\sim}{0}_{n\times 1} \in \mathbb{R}^{n\times 1}$

$\underset{\sim}{x} \in \mathbb{R}^{n\times 1}$, $\underset{\sim}{u} \in \mathbb{R}^{m\times 1}$

Step II: Check whether ~~f_x~~ $\underset{\sim}{f}_{n\times 1}$ posses continuous pds wrt $\underset{\sim}{x}$ & $\underset{\sim}{u}$

Check " $L(\underset{\sim}{x}, \underset{\sim}{u})$ " " wrt $\underset{\sim}{x}$ & $\underset{\sim}{u}$

if so then proceed Else not.

Step III: form the Hamiltonian,

$$H(\underset{\sim}{x}, \underset{\sim}{u}, \underset{\sim}{\lambda}) = \underbrace{L(\underset{\sim}{x}, \underset{\sim}{u})}_{\text{Langrangian}} + \underset{\sim}{\lambda}^T \cdot f(\underset{\sim}{x}, \underset{\sim}{u})$$

$\underset{\sim}{\lambda} \in \mathbb{R}^{n\times 1}$

Step IV: at stationary pt. $(\underset{\sim}{x}^*, \underset{\sim}{u}^*)$

$$\underset{\sim}{H_x}_{1\times n} = \underset{\sim}{0}_{1\times n} \; \& \; \underset{\sim}{H_u}_{m\times 1} = \underset{\sim}{0}_{m\times 1} \; \& \; f(\underset{\sim}{x}^*, \underset{\sim}{u}^*) = \underset{\sim}{0}_{n\times 1}$$

$$\underset{\sim}{H_x} \equiv \underset{\sim}{L_x}_{1\times n} + \underset{\sim}{\lambda}^T \cdot \underset{=}{f_x}_{n\times n} = \underset{\sim}{0}_{1\times n}$$

$$\underset{\sim}{H_u} \equiv \underset{\sim}{L_u}_{1\times m} + \underset{\sim}{\lambda}^T \cdot \underset{=}{f_u}_{n\times m} = \underset{\sim}{0}_{1\times m}$$

$$\underset{\sim}{\lambda}^T = -\underset{\sim}{L_x}_{1\times n} \, \underset{=}{f_x}^{-1}_{n\times n}$$

Use these 3 eqns to find out $\underset{\sim}{x}^*$ & $\underset{\sim}{u}^*$.

$(n+m+n)$

Step V: Now ~~apply the~~ check for the legendre-clebsche condition at $(\underset{\sim}{z}^*, \underset{\sim}{y}^*)$.

i.e. $(\underset{\sim}{z}^*, \underset{\sim}{y}^*)$ will be the pt. of minima if

$$\left(\frac{\partial^2 L}{\partial u^2}\right)_{\underset{\sim}{L} = \underset{\sim}{0}} > 0.$$

i.e.
$$\underline{\underline{H}}_{uu} - \underline{\underline{H}}_{uz}\, \underline{\underline{f}}_z^{-1}\, \underline{\underline{f}}_u - \underline{\underline{f}}_u^T \left(\underline{\underline{f}}_z^T\right)^{-1} \underline{\underline{H}}_{zu} + \underline{\underline{f}}_u^T \left(\underline{\underline{f}}_z^T\right)^{-1} \underline{\underline{H}}_{zz} \left(\underline{\underline{f}}_z^{-1}\right) \underline{\underline{f}}_u > 0$$

$\underline{\underline{H}}_{uu\, m\times m}$, $\underline{\underline{H}}_{uz\, m\times n}$, $\underline{\underline{H}}_{zu\, n\times m}$, $\underline{\underline{H}}_{zz\, n\times n}$ & $\underline{\underline{f}}_{z\, n\times n}$, $\underline{\underline{f}}_{u\, n\times m}$ But $\underset{\sim}{h}_{u\, 1\times m}$, $\underset{\sim}{h}_{z\, 1\times n}$ & $\underset{\sim}{H}_{z\, 1\times n}$, $\underset{\sim}{H}_{u\, 1\times m}$

& $\underset{\sim}{\lambda} \in \mathbb{R}^{n\times 1}$

✓ if legendre-clebsche condition is satisfied @ $(\underset{\sim}{z}^*, \underset{\sim}{y}^*)$ the pt. we have is the minima pt. $(\underset{\sim}{z}^*, \underset{\sim}{y}^*)$!!

Inequality Constraint :

Objective function : $L(\underset{\sim}{x}, \underset{\sim}{u}) \in \mathbb{R}$, $\underset{\sim}{x} \in \mathbb{R}^n$, $\underset{\sim}{u} \in \mathbb{R}^m$

inequality constraint : $f(\underset{\sim}{x}, \underset{\sim}{u}) \leq \underset{\sim}{0}_{p \times 1}$ — NOT $\underset{\sim}{0}_{n \times 1}$

p may or may not be equal to n!!

↳ defines a feasible region in which the search for any minimum points is to be performed!

Such search requires a numerical procedure called Non linear programming!

- if a minimum pt. $(\hat{\underset{\sim}{x}}, \hat{\underset{\sim}{u}})$ exists then it has two possibility

either $f(\hat{\underset{\sim}{x}}, \hat{\underset{\sim}{u}}) < \underset{\sim}{0}_{p \times 1}$ or $f(\hat{\underset{\sim}{x}}, \hat{\underset{\sim}{u}}) = \underset{\sim}{0}_{p \times 1}$

in such case use unconstrained optimization!

- Consider the Augmented objective function:

$$J(\underset{\sim}{x}, \underset{\sim}{u}) = L(\underset{\sim}{x}, \underset{\sim}{u}) + \underset{\sim}{\lambda}^T_{1 \times p} f(\underset{\sim}{x}, \underset{\sim}{u})$$

Neccessary condition for minimization:

$$\boxed{\begin{gathered} \frac{\partial L}{\partial \underset{\sim}{u}} + \underset{\sim}{\lambda}^T \frac{\partial f}{\partial \underset{\sim}{u}} = 0_{1 \times m} \\ \text{where } \underset{\sim}{\lambda} \begin{cases} = 0, & f(\underset{\sim}{x}, \underset{\sim}{u}) < 0 \\ > 0, & f(\underset{\sim}{x}, \underset{\sim}{u}) = 0 \end{cases} \end{gathered}}$$

- ~~The Search~~

- The Search for minimum in the presence of inequality constraints ~~could yield~~ will result in any of the following:

i. minima pt. lies inside the feasible region } Both necessary + Sufficient condition of minimization is Satisfied!

ii. minima pt. lies on the boundary

Satisfies "necessary" But not "Sufficient" Condition of minimization

does not Satisfy "necessary condition! of minimization

i.e. a pt. on the boundary of feasible region can be a minima pt. even if it does NOT Satisfy the necessary conditions!!

Problem: minimize $l(\underset{\sim}{x}, \underset{\sim}{u})$ w.r.t $\underset{\sim}{u}$ subject to constraint $f(\underset{\sim}{x}, \underset{\sim}{u}) \le \underset{\sim}{0}_{p\times 1}$

Step I: - First find out the stationary pt. of $l(\underset{\sim}{x}, \underset{\sim}{u})$

$$\cancel{l(\underset{\sim}{x}^*, \underset{\sim}{u}^*)}$$

$$\underset{\sim}{l_u}{}_{1\times m} = \frac{\partial l}{\partial \underset{\sim}{u}} = 0 \rightsquigarrow (\underset{\sim}{x}^*, \underset{\sim}{u}^*)$$

- Then ~~after~~ consider only those stationary pts. which lies in the feasible region.

- find $\underset{\approx}{L_{uu}}{}_{m\times m}$ $\rightsquigarrow$ if at $(\underset{\sim}{x}^*, \underset{\sim}{u}^*)$ $L_{uu} > 0 \Rightarrow (\underset{\sim}{x}^*, \underset{\sim}{u}^*)$ is a pt. of minima.

  else you need to solve by inspection!

if there is No stationary pts which lies in feasible region. Then minima must occur on the boundaries $\Rightarrow$ Solve by inspection!

Lec-10

- Consider a plant $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x}, \underset{\sim}{u}, t)$ with $\underset{\sim}{x} \in \mathbb{R}^n$, $\underset{\sim}{u} \in \mathbb{R}^m$, $t \in \mathbb{R}$, $\dot{\underset{\sim}{x}} \in \mathbb{R}^n$.

$\underset{\sim}{x} \sim$ state variable

$\underset{\sim}{u} \sim$ control "

$$IC: \quad \underset{\sim}{x}(t_i) = x_i \sim \text{Initial condition}$$

problem: determine the control history $\underset{\sim}{u}(t)$, $t_i \le t \le t_f$ (t_f: given terminal time (fixed) Const.)
such that the following objective function
J is "minimised" w.r.t. $\underset{\sim}{u}(t)$

$$J = \underbrace{\phi[\underset{\sim}{x}(t_f), t_f]}_{\in \mathbb{R}} + \int_{t_i}^{t_f} \cancel{[H(\underset{\sim}{x}, \underset{\sim}{u}, \underset{\sim}{\lambda}, t) - \underset{\sim}{\lambda}^T \cdot \dot{\underset{\sim}{x}}]} \, \underbrace{L(\underset{\sim}{x}, \underset{\sim}{u}, t)}_{L \in \mathbb{R}} dt$$

- Consider the Hamiltonian,

$$H(\underset{\sim}{x}, \underset{\sim}{u}, t, \underset{\sim}{\lambda}) = L(\underset{\sim}{x}, \underset{\sim}{u}, t) + \underset{\sim}{\lambda}^T \cdot f(\underset{\sim}{x}, \underset{\sim}{u}, t)$$

$\underset{\sim}{\lambda} \in \mathbb{R}^{n \times 1}$ $\cancel{H - \underset{\sim}{\lambda}^T}$ $\dot{\underset{\sim}{x}}$

($\underset{\sim}{\lambda}(t)$ is an additional variable called costate variable)

- Necessary condition for minimisation of J w.r.t. $\underset{\sim}{u}(t)$

Step I $\leftarrow$ $\dot{\underset{\sim}{\lambda}}^T = -\dfrac{\partial H}{\partial \underset{\sim}{x}}$

Step III (will use this relation after step IV) $\leftarrow$ $\underset{\sim}{\lambda}^T(t_f) = \left(\dfrac{\partial \phi}{\partial \underset{\sim}{x}}\right)_{t=t_f}$

the 1st order differential (costate) eqn & the BC to be solved for costate vector $\underset{\sim}{\lambda}(t)$

Step II $\leftarrow$ $\dfrac{\partial H}{\partial \underset{\sim}{u}} = \underset{\sim}{0}_{1 \times m}$

Step IV $\leftarrow$ $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x}, \underset{\sim}{u}, t)$ & $\underset{\sim}{x}(t_i) = \underset{\sim}{x}_i$

for a generalized LTI system.

$$\underset{\sim}{x}(t) = e^{\underset{=}{A}t} \underset{\sim}{x}(t=0) + \int_0^t e^{\underset{=}{A}(t-\tau)} \underset{=}{B} \cdot \underset{\sim}{u}(\tau) d\tau$$

NOTE: No Constraint on control variable $\underset{\sim}{y}(t)$

No other constraint on control interval $t_i \le t \le t_f$

No constraint on t_f (apart from the fact that it is fixed)

t_f ← terminal time.

- Suppose $\exists$ a trajectory $\underset{\sim}{x}^*(t)$ with corresponding control history $\underset{\sim}{y}^*(t)$, $t_i \le t \le t_f$ which satisfies the necessary Conditions for minimization.

  Such a trajectory is called an extremal trajectory corresponding to the extremal control history.

⊙ For a generalized LTI system:

$$\underset{\sim}{x}(t) = e^{\underset{=}{A}(t-t_i)}\,\underset{\sim}{x}(t=t_i) + \int_{t_i}^{t} e^{\underset{=}{A}(t-\tau)} \cdot \underset{=}{B} \cdot \underset{\sim}{y}(\tau)\, d\tau$$

Where $e^{\underset{=}{A}t} = \mathcal{L}^{-1}\left\{ \left(s\underset{=}{I} - \underset{=}{A}\right)^{-1} \right\}$

Lec-11 week-07

① Consider a dynamic plant governed by :

$$\dot{\underset{\sim}{\xi}} = f(\underset{\sim}{\xi}, \underset{\sim}{\eta}, t) \quad \text{—— (i)}$$

State vector $\in \mathbb{R}^{n\times 1}$

control or I/P vector $\in \mathbb{R}^{m}$

Let $\underset{\sim}{\xi}^*(t) \longrightarrow$ extremal trajectory

$\underset{\sim}{\eta}^*(t) \sim\!\!\rightarrow$ corresponding control history

$$\dot{\underset{\sim}{\xi}}^*(t) = f(\underset{\sim}{\xi}^*(t), \underset{\sim}{\eta}^*(t), t)$$

available from necessary condition for minimisation of $J = \phi(\underset{\sim}{\xi}(t_f), t_f)$

(ii) $+ \int_{t_0}^{t_f} L(\underset{\sim}{\xi}, \underset{\sim}{\eta}, t)\, dt$ wrt $\underset{\sim}{u}(t)$

subject to IC :

$$\underset{\sim}{\xi}(t_i) = \underset{\sim}{\xi}_i$$

$t_i \leq t \leq t_f$

✓ Consider small control & state deviation

$$\begin{cases} \underset{\sim}{u}(t) = \underset{\sim}{\eta}(t) - \underset{\sim}{\eta}^*(t) \;; & t_0 \leq t \leq t_f \\ \underset{\sim}{x}(t) = \underset{\sim}{\xi}(t) - \underset{\sim}{\xi}^*(t) \;; & t_0 \leq t \leq t_f \end{cases}$$

Linearizing ① about $\underset{\sim}{\xi}^*(t)$ & $\underset{\sim}{\eta}^*(t)$ [assuming $f(\cdot)$ to be ∞ differentiable]

$$\boxed{\dot{\underset{\sim}{x}}(t) = \overbrace{\underset{=}{A}(t)\cdot \underset{\sim}{x}(t) + \underset{=}{B}(t)\cdot \underset{\sim}{u}(t)}^{f(\underset{\sim}{x}, \underset{\sim}{u}, t)} \quad \& \quad \underset{\sim}{x}(t_0) = \underset{\sim}{x}_0} \longleftarrow (b)$$

here $\underset{n\times n}{\underset{=}{A}(t)} = \left.\dfrac{\partial f}{\partial \underset{\sim}{\xi}}\right|_{(\underset{\sim}{\xi}^*, \eta^*)}$

$\underset{n\times m}{\underset{=}{B}(t)} = \left.\dfrac{\partial f}{\partial \underset{\sim}{\eta}}\right|_{(\underset{\sim}{\xi}^*, \eta^*)}$

Linearizing (ii) about $\xi^*(t)$ & $\eta^*(t)$

$$J(\xi,\eta) - J(\xi^*,\eta^*) \approx \delta^2 J(z,y) \equiv \hat{J}(z,y)$$

(a) $\rightarrow$

$$\hat{J}(z,y) = \underbrace{\frac{1}{2} z^T(t_f) \cdot Q_f \cdot z(t_f)}_{\phi(z(t_f),t_f)}$$

$$+ \underbrace{\int_{t_0}^{t_f} \frac{1}{2} \{ z^T(t) \quad y^T(t) \} \begin{bmatrix} Q(t) & S(t) \\ S^T(t) & R(t) \end{bmatrix} \begin{Bmatrix} z(t) \\ y(t) \end{Bmatrix}}_{L(z,y,t)}$$

where $Q_f = \frac{\partial^2 \phi}{\partial \xi^2}\Big|_{\xi^*(t_f)}$

$Q(t) = \frac{\partial^2 L}{\partial \xi^2}\Big|_{(\xi^*,\eta^*)}$ $\quad$ $S(t) = \frac{\partial^2 L}{\partial \xi \partial \eta}\Big|_{(\xi^*,\eta^*)}$

$R(t) = \frac{\partial^2 L}{\partial \eta^2}\Big|_{(\xi^*,\eta^*)}$

$Q_f(t)$, $Q(t)$ & $R(t)$ are Symmetric, Real, Sq. matrix.

$H(z,y,\lambda,t) = L(z,y,t) + \lambda^T f(z,y,t)$

: our target is to minimise $\hat{J}(z,y)$ in (a) w.r.t $y(t)$ where z & y satisfies (b)

on applying "necessary condition" for minimisation of $\hat{J}(z,y)$ w.r.t $y(t)$

We obtain Ricatti eqn

$$\dot{P} = -Q - (A - BR^{-1}S^T)^T P - P(A - BR^{-1}S^T) + PBR^{-1}B^TP + SR^{-1}S^T$$

$P(t_f) = Q_f$: BC

Linear feedback law, $y(t) = -R^{-1}(t)\,[B^T(t) P(t) + S^T(t)]\, z(t)$

- Ricatti eqn" reflects the "necessary condition" for existence of an optimal control law for linear system with quadratic performance index.

(b) "Sufficient condition" for minimisation of $\hat{J}(x, u)$ wrt $u(t)$ (in (a)) (subject to (b))

$$H(x, u, \lambda, t) = l(x, u, t) + \lambda^T f(x, u, t)$$

if $H_{uu} \Big|_{(x^*(t), u^*(t))} > 0$ ~ +ve definite

then the HE is minimised on the extremal trajectory $(x^*(t), u^*(t))$

if $H_{uu} \Big|_{(x^*(t), u^*(t))}$ → "Singular"

requires additional constraints on $x(t)$ & $u(t)$ to determine optimal trajectory!

- for an LQR system (ⓐ + ⓑ) the "sufficient condition" becomes

$\dot{P}(t) \geq 0.$

$\Downarrow$

$\boxed{P(t) > 0 \ \forall \ t_0 \leq t \leq t_f.}$

① Steady State Riccati Solution:

Consider an asymptotically stable ($x(t) \to 0$ as $t \to \infty$) LTI plant : $\dot{x}(t) = A \cdot x(t) + B \cdot u(t)$

$x(t=0) = x_0$

A, B are const coeff matrix

$= \int_0^\infty \frac{1}{2}\left[x^T Q x + u^T S^T x + x^T S u + u^T R u\right]$

then

Quadratic objective funcⁿ with a control intr.

$$J = \int_0^\infty \frac{1}{2} \left\{ x^T(t) \ \ u^T(t) \right\} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{Bmatrix} x(t) \\ u(t) \end{Bmatrix}$$

with Q, R, S ~ constant coeff. matrices.
symmetric ~~Symm~~

requires $\dot{P} = 0$

P ~ can be obtained from Riccati eqnⁿ

$$u(t) = -R^{-1}\left[B^T \cdot P + S^T\right] \cdot x(t)$$

$\boxed{P(t) > 0 \text{ ~~}\Rightarrow\text{~~ "sufficient condition" for LQR system.}}$

Lec-12 General formulation for bounded I/P.

- Consider a general plant with

vector state eqn: $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x}, \underset{\sim}{u}, t) \longrightarrow (1)$

subject to IC: $\underset{\sim}{x}(t_i) = \underset{\sim}{x}_i \longrightarrow (2)$ here $\underset{\sim}{x}_i$ & t_i are both constant fixed.

terminal BC: $g(\underset{\sim}{x}(t_f), t_f) = 0 \longrightarrow (3)$

Ineq. constraint: $h(\underset{\sim}{x}, \underset{\sim}{u}, t) \geq 0 \longrightarrow (4)$

- A problem where t_f is not mentioned in advance is called an open control intv. problem.

- The optimal control problem is the problem of determining a control history $\underset{\sim}{u}^*(t)$, $t_i \leq t \leq t_f$, such that the following objective function is minimized w.r.t. control I/P $\underset{\sim}{u}(t)$.

$$\boxed{J' = \phi[\underset{\sim}{x}(t_f), t_f] + \int_{t_i}^{t_f} L(\underset{\sim}{x}, \underset{\sim}{u}, t)\,dt} \longrightarrow (5)$$

NOTE: $\underset{\sim}{x} \in \mathbb{R}^{n\times1}$, $\underset{\sim}{u} \in \mathbb{R}^{m\times1}$

so, $f : \mathbb{R}^{n\times1} \times \mathbb{R}^{m\times1} \times \mathbb{R} \longrightarrow \mathbb{R}^{n\times1}$

$g : \mathbb{R}^{n\times1} \times \mathbb{R} \longrightarrow \mathbb{R}^{k\times1}$

$\phi : \mathbb{R}^{n\times1} \times \mathbb{R} \longrightarrow \mathbb{R}$

$L : \mathbb{R}^{n\times1} \times \mathbb{R}^{m\times1} \times \mathbb{R} \longrightarrow \mathbb{R}$

v.imp.

$\boxed{f, g, \phi \text{ \& } h \text{ are continuously diff'ble w.r.t. their arguments.}}$

- The control history $\hat{u}(t)$, which minimizes J' subject to the state equation (1), initial & terminal BCs (2 & 3), & inequality constraint (3) is called an optimal control history of the

& the corresponding trajectory, $\hat{x}(t)$, $t_i \le t \le t_f$ is called optimal trajectory of the system.

Optimal Control with Unbounded I/P : (i.e. there is no ineq. Constraint ③)

control variable $\underset{\sim}{u}(t)$ is completely unrestricted in the optimization process.

$$\underset{\sim}{\dot{x}} = f(\underset{\sim}{x}, \underset{\sim}{u}, t) \quad \text{vector state eqn}$$

$$\underset{\sim}{x}(t_i) = \underset{\sim}{x}_i \quad \text{IC}$$

$$J' = \phi[\underset{\sim}{x}(t_f), t_f] + \int_{t_i}^{t_f} L(\underset{\sim}{x}, \underset{\sim}{u}, t)\, dt$$

objective function to be minimized.

- ~~suppose~~ $\exists$ a trajectory $\underset{\sim}{x}^*(t)$ with corresponding control history $\underset{\sim}{u}^*(t)$, $t_i \le t \le t_f$ which satisfies the "necessary condition" for minimization of J' wrt $\underset{\sim}{u}(t)$ subject to constraint ① & ②

Such a trajectory $\underset{\sim}{x}^*(t)$ will be called extremal trajectory corresponding to the extremal control history, $\underset{\sim}{u}^*(t)$

✓ The "necessary condition", for an extremal trajectory $\underset{\sim}{x}^*(t)$ are : to exists in the control intv. $t \in [t_i, t_f]$

[$\underset{\sim}{x}^*(t)$ ~ extremal trajectory $\in \mathbb{R}^{n\times1}$

$\underset{\sim}{u}^*(t)$ ~ corresponding control history $\in \mathbb{R}^{m\times1}$

$\underset{\sim}{\lambda}^*(t)$ ~ corresponding costate vector)] $\in \mathbb{R}^{n\times1}$

$$\boxed{\begin{aligned} (\dot{\lambda}^*)^T &= -\frac{\partial L}{\partial \underset{\sim}{x}} - (\underset{\sim}{\lambda}^*)^T \frac{\partial f}{\partial \underset{\sim}{x}} \\ \frac{\partial L}{\partial \underset{\sim}{u}} &+ (\underset{\sim}{\lambda}^*)^T \frac{\partial f}{\partial \underset{\sim}{u}} = \underset{\sim}{0} \\ \underset{\sim}{\dot{x}} &= f(\underset{\sim}{x}, \underset{\sim}{u}, t) \end{aligned}}$$

$$+\boxed{\left(\frac{\partial L_a}{\partial \dot{x}}\right)_{t=t_f} \delta x_f + \left(L_a - \frac{\partial L_a}{\partial \dot{x}} \dot{x}^*\right)_{t=t_f} \delta t_f = 0}$$

$\hookrightarrow$ transversality condition

where $L_a = L + \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x} \cdot \dot{x} + \lambda^T \cdot (f - \dot{x})$

augmented Langrangian $L_a(x, u, \lambda, \dot{x}, t)$

$$L_a = H + \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x} \cdot \dot{x} - \lambda^T \cdot \dot{x}$$

where $H = L + \lambda^T \cdot f$

- In terms of Hamiltonian, $H(x, u, \lambda, t)$
$= L(x, u, t) + \lambda^T \cdot f(x, u, t)$

you may need to assume $x(0) = \begin{cases} c_1 \\ c_2 \end{cases}$

$\dot{x} = A \cdot x + B \cdot u$

$x(t) = e^{A(t-t_i)} x(t=t_i) + \int_{t_i}^{t} e^{A(t-\tau)} B u(\tau) d\tau$

here $e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$

the "necessary conditions" becomes:

Step I $\rightarrow$ $\dot{\lambda}^* = -\left(\frac{\partial H}{\partial x}\right)^T \Big|_{(x^*, u^*, \lambda^*, t)}$

Step II $\rightarrow$ $\frac{\partial H}{\partial u}\Big|_{(x^*, u^*, \lambda^*, t)} = 0_{1\times n}$

Step III $\rightarrow$ $\dot{x}^* = \left(\frac{\partial H}{\partial \lambda}\right)^T \Big|_{(x^*, u^*, \lambda^*, t)} = f(x^*, u^*, t)$

$\rightarrow$ "Euler Langrange equations" ★

$x^*(t) = e^{A(t-t_i)} x^*(t=t_i) + \int_{t_i}^{t} e^{A(t-\tau)} B u^*(\tau) d\tau$

$$+\left[\left(\frac{\partial \phi}{\partial x}\right)^T_{t=t_f} - \lambda^*(t_f)\right]^T \delta x_f + \left(H + \frac{\partial \phi}{\partial t}\right)_{t=t_f} \delta t_f = 0 \quad \rightarrow (1)$$

NOTE: We need $(2n+1)$ BCs to solve the Euler-Langrangian equation.

of these n BCs are specified by ICs $\underset{\sim}{x}(t_i) = \underset{\sim}{x}_{i\,n\times 1}$

We need $(n+1)$ more BCs.

⊙ Boundary Conditions:

1. Fixed terminal time: t_f is fixed (t_f has a specified value.)

$\Downarrow$

$\delta t_f = 0$

We need n more BCs.

a. Fixed terminal state: $\underset{\sim}{x}(t_f) = \underset{\sim}{x}_f \Rightarrow \delta \underset{\sim}{x}_f = \underset{\sim}{0}$

(i) is trivially satisfied.

b. Free terminal state: $\delta \underset{\sim}{x}_f \neq 0$

$$\underset{\sim}{\lambda}^*(t_f) = \left(\frac{\partial \phi}{\partial \underset{\sim}{x}}\right)^T_{t=t_f}$$ n BCs

c. Terminal state on a hypersurface: terminal state lying anywhere in the region specified by

$$F[\underset{\sim}{x}(t_f)] = \underset{\sim}{0}_{K\times 1}$$

$$F: \mathbb{R}^{n\times 1} \to \mathbb{R}^{K\times 1}$$

with $1 \le K \le n-1$

$$\underset{\sim}{\lambda}^*(t_f) = \left(\frac{\partial \Phi}{\partial \underset{\sim}{x}}\right)^T_{t=t_f}$$

where $\Phi = \phi + \underset{\sim}{c}^T F[\underset{\sim}{x}(t_f)]$

here $\underset{\sim}{c} = (c_1\ c_2\ c_3, \dots, c_K)^T$

are constant Langrange multipliers to be determined.

$\boxed{\text{2. Free terminal time:}}$ t_f is NOT specified

$\Downarrow$

$\delta t_f \neq 0$.

① provides BC to determine t_f.

a. Fixed terminal state: $x(t_f) = x_f \rightarrow \delta x_f = 0$

$\Downarrow$

solving jointly $\sim$ $\left(H + \frac{\partial \phi}{\partial t}\right)_{t=t_f} = 0$

b. Free terminal state:

$$\left.\begin{aligned} \lambda^*(t_f) &= \left(\frac{\partial \phi}{\partial x}\right)^T_{t=t_f} \\ \left(H + \frac{\partial \phi}{\partial t}\right)_{t=t_f} &= 0 \end{aligned}\right\}$$

c. time varying terminal state: terminal BC is a moving point in the state space described by time varying function

$h(t) : \mathbb{R} \longrightarrow \mathbb{R}^{n\times 1}$

$$\boxed{\delta x_f = \left(\frac{dh}{dt}\right)_{t=t_f} \delta t_f}$$ $\rightsquigarrow$ put this in ①

d. terminal state on a moving hypersurface: The terminal state lies anywhere in a region of the state space ~~defined~~ specified by:

$$\boxed{F[x(t_f), t_f] = 0}$$

$F : \mathbb{R}^{n\times 1} \times \mathbb{R} \longrightarrow \mathbb{R}^{k\times 1}$, $1 \leq k \leq n-1$

$$\boxed{\lambda^*(t_f) = \left(\frac{\partial \Phi}{\partial x}\right)^T_{t=t_f}}$$

$\&$

$$\boxed{\left(H + \frac{\partial \Phi}{\partial t}\right)_{t=t_f} = 0}$$

where $\Phi = \phi + c^T \cdot F[x(t_f), t_f]$

here $c = (c_1, c_2, \dots, c_k)^T$

$c_1, c_2, \dots, c_k$ are time varying coefficients to be treated as additional ~~co~~state variable.

e. Terminal state partially fixed:

Some state variables are fixed.

or

lies on a hypersurface.

& some state variables are free.

Sufficient Condition: for the existence of an extremal trajectory $\underset{\sim}{x}^*(t)$ & corresponding control history $\underset{\sim}{u}^*(t)$, which minimizes J w.r.t $\underset{\sim}{u}(t)$ subject to constraint $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x}, \underset{\sim}{u}, t)$

$$\boxed{H_{uu} > 0 \ @ \ (\underset{\sim}{x}^*, \underset{\sim}{u}^*, \underset{\sim}{\lambda}^*, t)}$$

Legendre clebsch condition.

H_{uu} ↝ Hessian.

- When $H_{uu} = 0$ on $(\underset{\sim}{x}^*, \underset{\sim}{u}^*, \underset{\sim}{\lambda}^*, t)$ ↝ Such a problem is said to be Singular in control

& requires additional constraints (such as ineq. constraints on $\underset{\sim}{x}(t)$ & $\underset{\sim}{u}(t)$) to determine optimal trajectory.

NOTE: if H is a linear function of $\underset{\sim}{u}$ then $H_{uu} = 0$

Lec-13

⊙ Consider a general plant with

vector state eqn $\dot{\underset{\sim}{z}} = f(\underset{\sim}{z}, \underset{\sim}{u}, t)$ ⟶ ② , $\underset{\sim}{z} \in R^{n\times1}$, $\underset{\sim}{u} \in R^{m\times1}$

Subject to IC : $\underset{\sim}{z}(t_i) = \underset{\sim}{z}_i$ ⟶ ③

Ineq. constraint : $h(\underset{\sim}{z}, \underset{\sim}{u}, t) \geq 0$ ⟶ ④

Consider the minimization of $J = \phi[\underset{\sim}{z}(t_f), t_f] + \int_{t_i}^{t_f} L(\underset{\sim}{z}, \underset{\sim}{u}, t)\, dt$

wrt $\underset{\sim}{u}(t) \in R^{m\times1}$

$$\boxed{H(\underset{\sim}{z}, \underset{\sim}{u}, \underset{\sim}{\lambda}, t) = L(\underset{\sim}{z}, \underset{\sim}{u}, t) + \underset{\sim}{\lambda}^T f(\underset{\sim}{z}, \underset{\sim}{u}, t)}$$

✓ Pontryagin's minimum principle states that if $\hat{\underset{\sim}{u}}(t)$ minimizes J wrt $\underset{\sim}{u}(t)$ subject to constraint ② & ④ then a variation $\delta\hat{\underset{\sim}{u}}$ in the control $\hat{\underset{\sim}{u}}(t)$ applied on the optimal trajectory $\hat{\underset{\sim}{z}}(t)$ produces either an +ve or an invariance of Hamiltonian, H

i.e. $H(\hat{\underset{\sim}{z}}, \hat{\underset{\sim}{u}} + \delta\hat{\underset{\sim}{u}}, \hat{\underset{\sim}{\lambda}}, t) \geq H(\hat{\underset{\sim}{z}}, \hat{\underset{\sim}{u}}, \hat{\underset{\sim}{\lambda}}, t)$

$\forall\, \underset{\sim}{u} \in \underset{=}{U},\ t \in [t_i, t_f]$

$U \subset R^{m\times1}$ is the set of all admissible controls $\underset{\sim}{u}(t)$ which satisfies ④.

⇓

> an optimal control $\hat{\underset{\sim}{u}}(t)$ is the one which minimises $H(\underset{\sim}{z}, \underset{\sim}{u}, \underset{\sim}{\lambda}, t)$ on optimal trajectory $\hat{\underset{\sim}{z}}(t)$

*NOTE:
- Pontryagin's minimum principle is a more general necessary condition than Euler-Lagrange eqn
- The minimum principle doesnot require H to be a smooth function of control variable $\underset{\sim}{u}(t)$

- The Euler Lagrangian equⁿ leads to the minimum principle if the Hamiltonian possesses continuous pds upto 2nd order wrt $\underset{\sim}{u}(t)$

⊙ the Pontryagin's minimum principle is used in minimization of J in Singular control problem

$H_{uu} = 0$, so euler lagrange eqⁿ is of no use to determine the optimal control profile.

the "sufficient condition" is NOT satisfied

Legendre Clebsch condition.

additional condition needs to be specified.

Consider the following Singular control problem:

State eqⁿ of the system: $\underset{\sim}{\dot{y}}_{n\times 1} = \underset{\sim}{f}_{n\times 1}(\underset{\sim}{y}, t) + \underset{=}{B}(\underset{\sim}{y}, t)_{n\times m} \underset{\sim}{u}(t)_{m\times 1}$

control affine system [a system linear in $\underset{\sim}{u}(t)_{m\times 1}$]

Lagrangian to be minimized ; $L(\underset{\sim}{y}, \underset{\sim}{u}, t) = l_1(\underset{\sim}{y}, t) + \underset{\sim}{l_2}(t)_{1\times m} \underset{\sim}{u}(t)_{m\times 1}$

so, $H(\underset{\sim}{y}, \underset{\sim}{u}, \underset{\sim}{\lambda}, t) = (l_1 + l_2 \underset{\sim}{u}) + \underset{\sim}{\lambda}^T (\underset{\sim}{f} + \underset{=}{B} \underset{\sim}{u})$

$H_{uu} = 0$ so we need more conditions to find the optimal trajectory $\hat{\underset{\sim}{y}}(t)$ & corresponding optimal control $\hat{\underset{\sim}{u}}(t)$

• When $|u(t)| \le u_M$ a practical solution can be obtained using Pontryagin's min. principle. ⓐ

PMP yields

$$\left[\underset{1\times m}{l_2(t)} + \underset{1\times n}{\hat{\lambda}^T}\, \underset{n\times m}{B(\hat{x},t)}\right] \underset{m\times 1}{\hat{u}(t)} \le \left[\underset{1\times m}{l_2(t)} + \underset{1\times n}{\hat{\lambda}^T} \cdot \underset{n\times m}{B(\hat{x},t)}\right] \underset{m\times 1}{u(t)}$$

$$\hat{u}_i(t) = \begin{cases} -u_M, & (l_2 + \hat{\lambda}^T B)_i > 0 \\ 0, & (l_2 + \hat{\lambda}^T B)_i = 0 \\ u_M, & (\underset{1\times m}{l_2} + \underset{1\times n}{\hat{\lambda}^T}\, \underset{n\times m}{B})_i < 0 \end{cases}, \quad i = 1, \dots, m$$

$$\dot{x} = f(x,t) + B \cdot u(t)$$

$$L = l_1(x,t) + \underset{1\times m}{l_2(x,t)}\, \underset{m\times 1}{u(t)}$$

Switching condition for optimal control $\hat{u}(t)$

$(l_2 + \hat{\lambda}^T B)_i$ ~ Switching Function

$$\hat{u}_i(t) = -u_M \operatorname{sgn}(l_2 + \hat{\lambda}^T B)_i \quad i = 1, \dots, m$$

NOTE: ⓐ $\Rightarrow$ $\hat{u}_i = -u_M$ then $\hat{u}_j = 0 \;\forall\, i \ne j$

or $\hat{u}_i = u_M$ then $\hat{u}_j = 0 \;\forall\, i \ne j$

Optimality requires the norm of the control I/P vector to be on the boundary which is either maximum or the minimum value.

Such an optimal control law switching between the two extreme I/P values is typical of singular problems & is called Bang Bang control.

Lagrangian of a time Optimal Control problem
, $L \equiv 1$

Time optimal transfer, $(J = t_f)$

Lec-14 Intro to Spaceflight control

⊙ **Space Flight:** primarily governed by gravity field.

generated by one or more bodies (which are themselves moving under mutual gravitational attraction)

unless thrust is applied by rocket engines, a spacecraft moves in well defined trajectories.

is perturbed by the presence of small force & torque disturbances due to environment (atmosphere, solar radiation pressure, geo-magnetism etc.)

though small could cause an appreciable change in long period of time!!

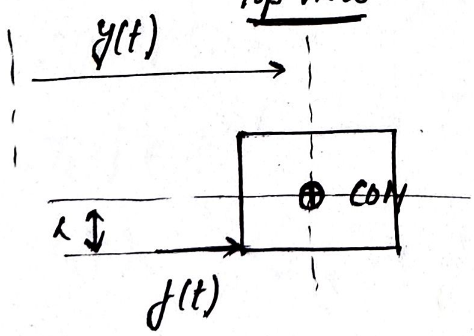
⊙ **Motion of a 'Spacecraft':** can be divided into two parts

(by assuming that it is a rigid body)

- Translation of COM [orbital dynamics]
- Rotation of spacecraft about its COM [attitude dynamics]

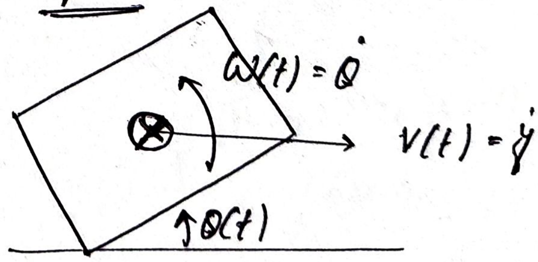
⊙ **Consider a Sliding block on a Horizontal, Frictionless Table:**

Top View



Translation

Top View



Rotation

translational dynamics: $m \dfrac{d^2 y}{dt^2} = f$

Rotational dynamics: $J \dfrac{d^2\theta}{dt^2} = fL$

System parameters: m, J, L, f

Linear System
Order of System = 4

- Let us select the state variables:

$$\left.\begin{aligned} x_1 &= y \\ x_2 &= \frac{dy}{dt} \end{aligned}\right\} \qquad \left.\begin{aligned} x_3 &= \theta \\ x_4 &= \frac{d\theta}{dt} \end{aligned}\right\}$$

State Eqn

Translational Kinematics → $\dot{x}_1 = x_2$

" Kinetics → $\dot{x}_2 = f/m$

Rotational Kinematics → $\dot{x}_3 = x_4$

" Kinetics → $\dot{x}_4 = fL/J$

Space Flight Control:

is classified into

Navigation

Consisting of a control of position, $r(t) \in \mathbb{R}^{3\times1}$ & linear velocity, $v(t) \in \mathbb{R}^{3\times1}$ (i.e. translation) of the Spacecraft's COM relative to a "fixed inertial reference frame."

has its origin taken to be at the centre of a large body (such as a planet, the moon, or the Sun) assumed to be at rest during the time of flight.

$r(t) \in \mathbb{R}^{3\times1}$
$v(t) \in \mathbb{R}^{3\times1}$
$q(t) \in \mathbb{R}^{4\times1}$
$\omega(t) \in \mathbb{R}^{3\times1}$

Attitude control

consists of control of rigid vehicle's angular orientation (or attitude), $q(t) \in \mathbb{R}^{4\times1}$ (called quaternion) & angular velocity vector, $\omega(t) \in \mathbb{R}^{3\times1}$ about the COM relative to an "inertial ~~fixed~~ ref. frame."

has its origin at the COM & its axes are fixed wrt distant stars. The $\omega(t)$ & $q(t)$ of the body frame (a frame rigidly attached with the spacecrafts) are measured relative to this ref. frame.

The attitude is controlled by ext. applied torque I/Ps &/or internally variable angular momentum via rotors.

Translational Motion of COM:

$\dot{r} = v$ ⟶ kinematical eqn

$\dot{v} = g(r,t) + T/m$ ⟶ kinetics eqn

$g(r,t)$ ⟶ acc. due to gravity.

$T(t) \in \mathbb{R}^{3\times1}$ ⟶ Thrust by rocket engines.

$m(t)$ ⟶ mass of the vehicle.

Rotational Motion of the body :

$$\dot{q} = A(\omega) \sim \text{attitude kinematics}$$

$$\dot{\omega} = \underline{J}^{-1}\tau - \underline{J}^{-1}(\omega \times \underline{J}\omega) \sim \text{attitude kinetics}$$

$A(\cdot) : \mathbb{R}^{3\times1} \longrightarrow \mathbb{R}^{4\times1}$

Non linear attitude kinematics functional.

$\underline{J} \in \mathbb{R}^{3\times3} \rightarrow$ Inertia tensor (assumed to be const. for rigid b/c)

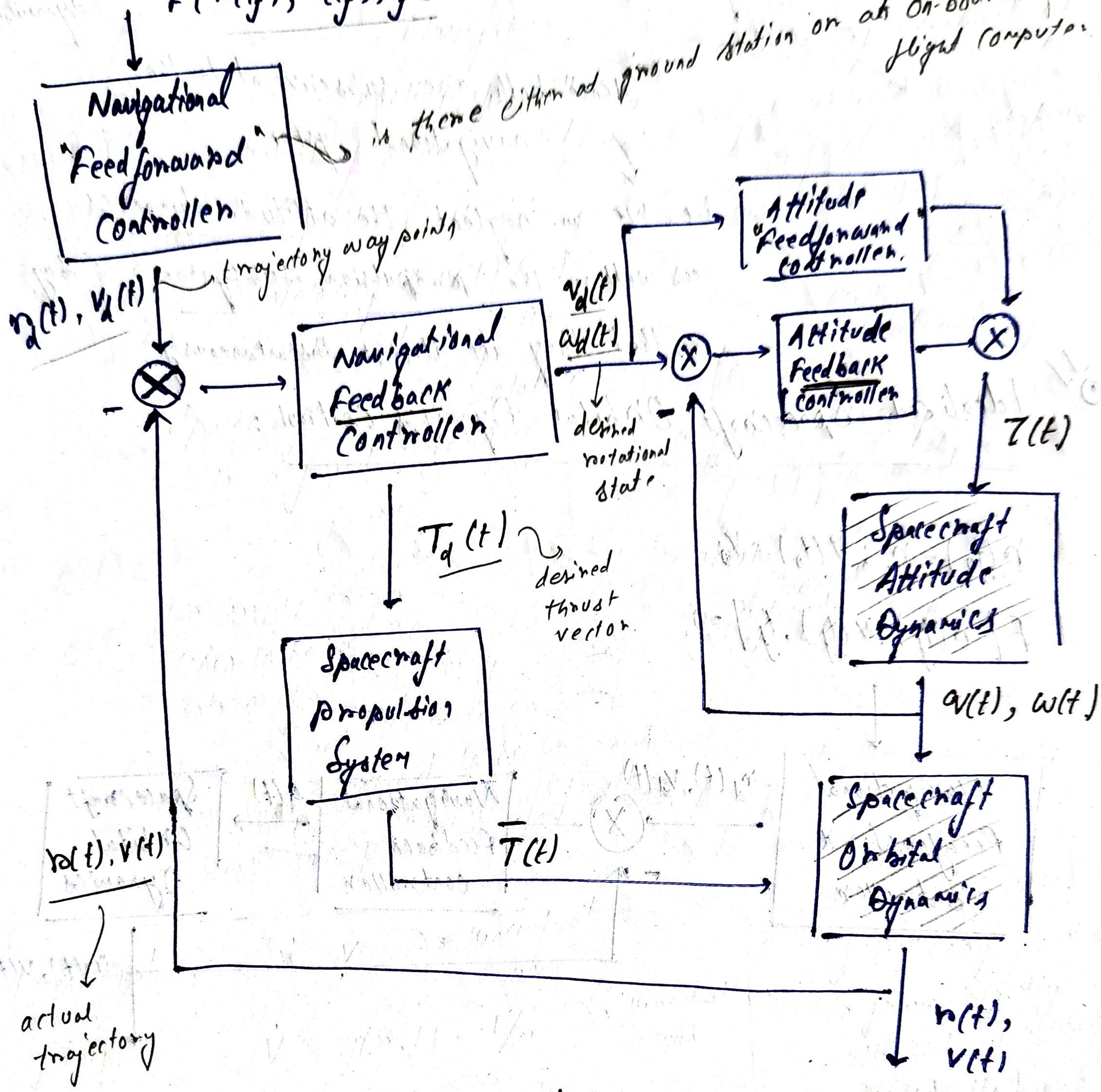
$\tau(t) \in \mathbb{R}^{3\times1}$ } applied torque I/p.

General Spacecraft Control System:

$F(\cdot) : \mathbb{R}^{3\times1} \times \mathbb{R}^{3\times1} \rightarrow \mathbb{R}$

$r(t_0) = r_0,\ v(t_0) = V_0 \rightarrow$ I/Cs

$F[r(t_f), v(t_f), t_f] = 0 \rightarrow$ terminal condition



- Navigational control I/Ps are the commands issued to propulsion subsystem for producing a desired thrust vector $T_d(t)$ & to attitude control system to achieve desired rotational state : $q_d(t), \omega_d(t)$

① The attitude controller ⟶ has smaller timescale ⟶ wrt that of Navigation System (or Orbital dynamics)

The propulsion system ⟶ has even " "

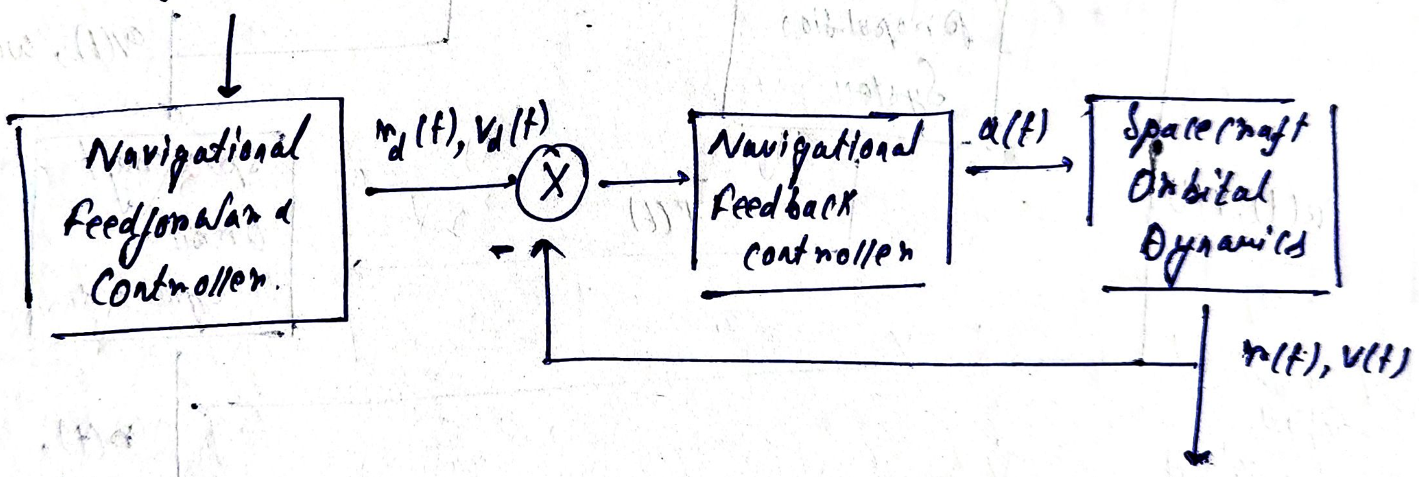
Hence both are subservient to the navigational System.

i.e. we can neglect the attitude dynamics as well as the propulsion subsystem & say that they act almost instantaneously.

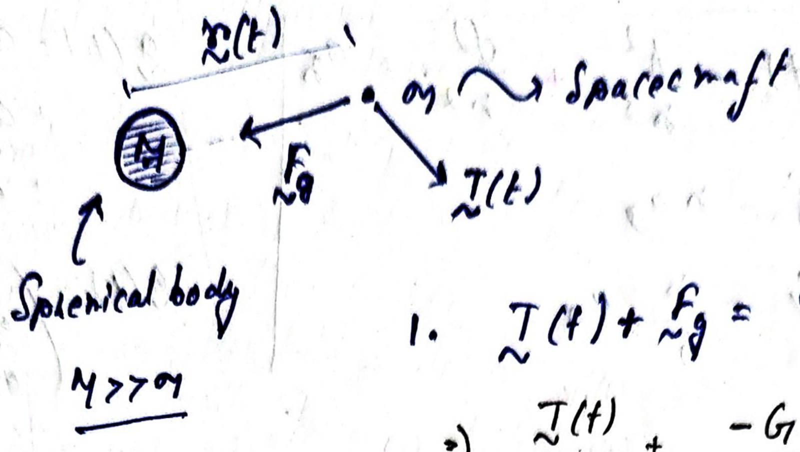
② Idealized Spacecraft Orbital ~~Dynamics~~ Control :

$r(t_0) = r_0 \,;\, v(t_0) = v_0$

$F[r(t_f), v(t_f), t_f] = 0$



① Consider Orbiting around a Spherical body :



1. $\underset{\sim}{T}(t) + \underset{\sim}{F}_g = m(t)\, \dot{\underset{\sim}{V}}$

$$\Rightarrow \frac{\underset{\sim}{T}(t)}{m(t)} + \frac{-GMm}{r^2_{(t)}\, m(t)} \cdot \hat{r}_{(t)} = \dot{\underset{\sim}{V}}$$

$$\Rightarrow \dot{\underset{\sim}{V}} = \underset{\sim}{a}(t) - GM \frac{\underset{\sim}{r}(t)}{r^3(t)}$$

$$\Rightarrow \boxed{\dot{\underset{\sim}{V}} = \underset{\sim}{a}(t) - \mu \frac{\underset{\sim}{r}(t)}{r^3(t)}} \rightarrow (i)$$

where $\mu = GM$ — Grav. Constant.

& $r(t) = |\underset{\sim}{r}(t)|$

2. $\boxed{\dot{\underset{\sim}{r}}(t) = \underset{\sim}{V}(t)} \rightarrow (ii)$

$\underset{\sim}{r}(t) \in \mathbb{R}^{3\times1}$

$\underset{\sim}{V}(t) \in \mathbb{R}^{3\times1}$

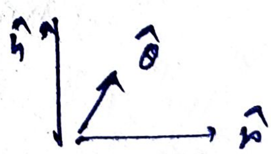
When $\underset{\sim}{a}(t) = \underset{\sim}{0}$: Unforced Case we will have Orbit ★

• define $\underset{\sim}{h}(t) = \underset{\sim}{r}(t) \times \underset{\sim}{V}(t)$ — angular momentum vector with magnitude $h(t) = |\underset{\sim}{h}(t)|$

$\underset{\sim}{h}(t)$ ⊥r to $\underset{\sim}{r}(t)$ & $\underset{\sim}{V}(t)$

lets define, $\hat{i}_r = \dfrac{\underset{\sim}{r}(t)}{r(t)}$ — unit vector in radial, $\quad \hat{i}_h = \dfrac{\underset{\sim}{h}(t)}{h(t)}$ — normal ~~circumferential~~

$\hat{i}_\theta = \hat{i}_h \times \hat{i}_r$ — circumferential direction



resolving eqn (i) in $\hat{i}_r$, $\hat{i}_\theta$ & $\hat{i}_h$ directions we have :

$$\boxed{\begin{aligned} \ddot{r}(t) - \frac{h^2}{r^3} + \frac{\mu}{r^2} &= a_r \\ \dot{h} &= r a_\theta \\ \dot{\alpha} &= \frac{r}{h} a_h \end{aligned}}$$

$\mathbf{a}(t) = a_r \hat{i}_r + a_\theta \hat{i}_\theta + a_h \hat{i}_h$

$\mathbf{a}(t) = \begin{pmatrix} a_r \\ a_\theta \\ a_h \end{pmatrix}$

Change in $|\mathbf{r}(t)|$

change in $|\mathbf{h}(t)|$

Change in the orbital plane's orientation α

if $a_h = 0 \rightsquigarrow \dot{\alpha} = 0 \Rightarrow \underline{\alpha = \text{Const.}}$

if $a_\theta = 0 \rightsquigarrow \dot{h} = 0 \Rightarrow \underline{|\mathbf{h}(t)| = \text{Const.}}$

if $a_h = 0 = a_\theta \rightsquigarrow \underline{\mathbf{h}(t) = \text{Const.}}$

$\underline{n = h/c^2}$ | $\underline{n = \sqrt{\frac{\mu}{c^3}}}$ $\quad \underline{\mu = GM}$

Frequency.

① Bang Bang control problem:

Given : $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x}, \underset{\sim}{u}, t)$

$\underset{\sim}{x}(t_i) = \underset{\sim}{x}_i$

$\underset{\sim}{x}(t_f) = \underset{\sim}{x}_f$

Bound on I/P $|\underset{\sim}{u}(t)| \leq u_M$

Objective function, $J = \phi[\underset{\sim}{x}(t_f), t_f] + \int_{t_i}^{t_f} L(\underset{\sim}{x}, \underset{\sim}{u}, t)\,dt$

Step I :
Form the Hamiltonian, $\underset{1\times1}{H} = \underset{1\times1}{L}(\underset{\sim}{x}, \underset{\sim}{u}, t) + \underset{\sim}{\lambda}^T_{n\times1} \cdot \underset{n\times1}{f}(\underset{\sim}{x}, \underset{\sim}{u}, t)$

Check whether $H_{uu} = 0$ (Singular problem)

if so then proceed.

Step II: apply PMP: $H(\hat{\underset{\sim}{x}}, \underset{\sim}{u}_0, \hat{\underset{\sim}{\lambda}}, t) \geq H(\hat{\underset{\sim}{x}}, \hat{\underset{\sim}{u}}, \hat{\underset{\sim}{\lambda}}, t) \sim (-)\,\hat{u}(t) \leq (-)\,u(t)$

$u(t) = \{-u_M$ ..., 0 ..., u_M ...

compare ① $\dot{\underset{\sim}{x}} = \underline{f(\underset{\sim}{x}, \underset{\sim}{u}, t)}$ are different

with $\dot{\underset{\sim}{x}} = \underline{f(\underset{\sim}{x}, t)} + \underset{=}{B}(\underset{\sim}{x}, t) \cdot \underset{\sim}{u}(t)$ [control affine system]

↳ a system linear in $\underset{\sim}{u}(t)$

① $L(\underset{\sim}{x}, \underset{\sim}{u}, t)$ with

$L(\underset{\sim}{x}, \underset{\sim}{u}, t) = \underset{1\times1}{l_1}(\underset{\sim}{x}, t) + \underset{1\times m}{l_2}(t)\, \underset{\sim}{u}_{m\times1}(t)$

Find $\underset{=}{B}$ & $l_2(t)$, $\hat{\lambda}$ is needed to determine ~~whether swi~~ no of switches.

expression for

Step III: Find the costate variable $\hat{\lambda}(t)$ corresponding to the extremal control $\hat{u}(t)$ & extremal trajectory $\hat{x}(t)$

here $e^{At} = L^{-1}\{(sI - A)^{-1}\}$; $\dot{\underset{\sim}{x}} = A\underset{\sim}{x} + \underset{\sim}{B}\cdot\underset{\sim}{u}$

$\underset{\sim}{x}(t) = e^{A(t-t_i)}\underset{\sim}{x}(t = t_i) + \int_{t_i}^{t} e^{A(t-\tau)} B\, \underset{\sim}{u}(\tau)\, d\tau$

$\dot{\underset{\sim}{\lambda}}^* = -\left(\frac{\partial H}{\partial \underset{\sim}{x}}\right)^T \Big|_{(\underset{\sim}{x}^*, \underset{\sim}{u}^*, \underset{\sim}{\lambda}^*, t)}$

in this process you may need to assume $\hat{\lambda}_K(t_i) = c_K$, $1 \leq K \leq n$

Step IV: Now apply what Pontryagin's minimum principle yields under the bound $|u_i(t)| \leq u_M$

So, $\hat{u}_i(t) = \begin{cases} -u_M, & (l_2 + \hat{\lambda}^T B)_i > 0 \\ 0 & (l_2 + \hat{\lambda}^T B)_i = 0 \\ u_M & (l_2 + \hat{\lambda}^T B)_i < 0 \end{cases}$, $i = 1, \dots, M$

switching condition

Step V: decide the bounds on c_K, $K = 1 \dots n$ such that the BCs are satisfied based on $x(t_i)$ & $x(t_f)$. after switching is applied at t_s, decide whether you need +ve $\hat{u}_i(t)$ or -ve $\hat{u}_i(t)$ in min. possible t_f

Also you need to determine the switching time $\hat{t}$.

this is decided from $\hat{\lambda}(t)$

Step vi: ~~$\hat{x}(t) = e$~~

Find $\hat{x}(t)$ from $t_i \leq t \leq \hat{t}$ & for $\hat{t} \leq t \leq t_f$

Step vii: $\hat{t}$ can be found out by imposing $x(t_f) = x_f$ & t_f BC. on $\hat{x}(t)$

$$\boxed{\begin{aligned} \dot{x} &= A \cdot x + B \cdot u \\ x &= e^{A(t-t_i)} x(t=t_i) + \int_{t_i}^{t_f} e^{A(t-\tau)} B\, u(\tau)\, d\tau \end{aligned}}$$

- plot graphs of $\hat{x}(t)$, $\hat{\lambda}(t)$ & $u(t)$