Week-01

AE777 (Optimal Space Flight Control)

Lec-01

$\underset{\sim}{\eta}(t) \in \mathbb{R}^n$
$\underset{\sim}{\xi}(t) \in \mathbb{R}^n$
$\underset{\sim}{\zeta}(t) \in \mathbb{R}^p$

$\underset{\sim}{\eta}(t) \in \mathbb{R}^n \longrightarrow$ $\boxed{\text{System} \quad \underset{\sim}{\xi}(t) \in \mathbb{R}^n}$ $\longrightarrow \underset{\sim}{\zeta}(t) \in \mathbb{R}^p$

I/P vector to the System

State vector of the system

O/P vector of the system

- The system is governed by

vector State eqn

$$\dot{\underset{\sim}{\xi}} = f(\underset{\sim}{\xi}(t), \underset{\sim}{\eta}(t))$$

$$I/C: \underset{\sim}{\xi}(t_i) = \underset{\sim}{\xi}_i$$

$$\underset{\sim}{\zeta} = h(\underset{\sim}{\xi}(t), \underset{\sim}{\eta}(t))$$

O/P eqn

⊙ Determining whether the system is Linear or **NOT**!

⇒ the system is said to be linear if

for I/P vector: $\eta(t) = c_1 \eta_1(t) + c_2 \eta_2(t)$

the O/P vector: $\zeta(t) = c_1 \zeta_1(t) + c_2 \zeta_2(t)$

where $\zeta_1(t)$ is the O/P for I/P $\eta_1(t)$

$\zeta_2(t)$ is " O/P " I/P $\eta_2(t)$

- ⇒ If the functional $f(\cdot)$ & $h(\cdot)$ are continuous in some time intv. $t_i \le t \le t_f$ & do **NOT** contain Non-linear functions of the state vector $\underset{\sim}{\xi}(t)$ & I/P vector $\underset{\sim}{\eta}(t)$, then the system is linear in the given time intv. $t_i \le t \le t_f$.

Linearization :

$\underset{\sim}{\xi}_r(t) \rightarrow$ a reference state vector

$\underset{\sim}{\eta}_r(t) \rightarrow$ corresponding reference I/P vector

$$\dot{\underset{\sim}{\xi}}_r(t) = f(\underset{\sim}{\xi}_r(t), \underset{\sim}{\eta}_r(t))$$

$$\underset{\sim}{\xi}_r(t_i) = \underset{\sim}{\xi}_{ri} \quad : I/C$$

for $t_i \leq t \leq t_f$

then $\underset{\sim}{\xi}_r(t)$ is the trajectory of the system begining from I/C $\underset{\sim}{\xi}_{ri}$ & $\underset{\sim}{\eta}_r(t)$ is the corresponding control history in the time intv. $t_i \leq t \leq t_f$

$$\begin{cases} \underset{\sim}{\xi}(t) = \underset{\sim}{\xi}_r(t) + \underset{\sim}{z}(t) \\ \underset{\sim}{\eta}(t) = \underset{\sim}{\eta}_r(t) + \underset{\sim}{u}(t) \end{cases}$$

arbitrarily small state & control deviations

perturbed solution

Subject to I/C : $\underset{\sim}{\xi}_r(t_i) = \underset{\sim}{\xi}_{ri}$ & $\underset{\sim}{\xi}(t_i) = \underset{\sim}{\xi}_i$ $\Rightarrow$ $\boxed{\underset{\sim}{z}(t_i) = \underset{\sim}{\xi}_i - \underset{\sim}{\xi}_{ri}}$

Now, assume that the vector functional $f(\cdot)$ posseses continuous derivatives upto an ∞ order at the reference w.r.t. state & control variables, we expand the state equation about the reference solution $(\underset{\sim}{\xi}_r, \underset{\sim}{\eta}_r)$, we

solution .

$$\dot{\underset{\sim}{\xi}}(t) - \dot{\underset{\sim}{\xi}}_r(t) = f(\underset{\sim}{\xi}(t), \underset{\sim}{\eta}(t)) - f(\underset{\sim}{\xi}_r(t), \underset{\sim}{\eta}_r(t))$$

$$= f(\underset{\sim}{\xi}_r(t) + \underset{\sim}{z}(t), \underset{\sim}{\eta}_r(t) + \underset{\sim}{u}(t)) - f(\underset{\sim}{\xi}_r(t), \underset{\sim}{\eta}_r(t))$$

$$= f(\underset{\sim}{\xi}_r(t), \underset{\sim}{\eta}_r(t)) + \left[\frac{\partial f}{\partial \xi}\bigg|_{(\underset{\sim}{\xi}_r, \underset{\sim}{\eta}_r)} \cdot \underset{\sim}{z}(t) + \frac{\partial f}{\partial \eta}\bigg|_{(\underset{\sim}{\xi}_r, \underset{\sim}{\eta}_r)} \cdot \underset{\sim}{u}(t) \right]$$

$$+ O(HOTs)$$

$$- f(\underset{\sim}{\xi}_r(t), \underset{\sim}{\eta}_r(t))$$

Neglect the HOTs! (Good when $\underset{\sim}{z}(t)$ & $\underset{\sim}{y}(t)$ are ~~very~~ small; as is assumed)

So, $$\dot{\underset{\sim}{\xi}}(t) - \dot{\underset{\sim}{\xi}}_n(t) = \left.\frac{\partial f}{\partial \xi}\right|_{(\underset{\sim}{\xi}_n, \underset{\sim}{\eta}_n)} \cdot \underset{\sim}{z}(t) + \left.\frac{\partial f}{\partial \eta}\right|_{(\underset{\sim}{\xi}_n, \underset{\sim}{\eta}_n)} \cdot \underset{\sim}{y}(t)$$

$$\Rightarrow \boxed{\dot{\underset{\sim}{z}}(t) = \underset{\text{state or dynamic matrix}}{\underline{\underline{A}}} \cdot \underset{\sim}{z}(t) + \underset{\text{control or I/p matrix}}{\underline{\underline{B}}} \cdot \underset{\sim}{y}(t)}$$ Subject to IC: $\underset{\sim}{z}(t_i) = \underset{\sim}{z}_i$

Linearized State Equation (Linearized about the reference solution)

where $$\underline{\underline{A}} = \left.\frac{\partial f}{\partial \underset{\sim}{\xi}}\right|_{(\underset{\sim}{\xi}_n, \underset{\sim}{\eta}_n)} = \left[\frac{\partial f}{\partial \xi_1} \;\; \frac{\partial f}{\partial \xi_2} \cdots\right]_{n\times n}$$ Jacobian matrix

& $$\underline{\underline{B}} = \left.\frac{\partial f}{\partial \underset{\sim}{\eta}}\right|_{(\underset{\sim}{\xi}_n, \underset{\sim}{\eta}_n)} = \left[\frac{\partial f}{\partial \eta_1} \;\; \frac{\partial f}{\partial \eta_2} \cdots\right]_{n\times m}$$

$\underset{\sim}{\xi}(t)$ ⤳ state variable

$\underset{\sim}{\eta}(t)$ ⤳ I/p variable

$\underset{\sim}{\zeta}(t)$ ⤳ O/p variable

- $$\boxed{\underset{\sim}{\zeta}_n(t) = h\left(\underset{\sim}{\xi}_n(t), \underset{\sim}{\eta}_n(t)\right) \text{ for } t_i \le t \le t_f}$$

$$\underset{\sim}{\zeta}(t) = h\left(\underset{\sim}{\xi}(t), \underset{\sim}{\eta}(t)\right)$$
$$= h\left(\underset{\sim}{\xi}_n(t) + \underset{\sim}{z}(t),\; \underset{\sim}{\eta}_n(t) + \underset{\sim}{y}(t)\right)$$

assuming that the vector functional $h(\cdot)$ posses continuous derivative wrt state & control variables upto an ∞ order at the reference solution $(\underset{\sim}{\xi}_n(t), \underset{\sim}{\eta}_n(t))$, we expand the O/P eqn about the reference solution.

$$\underset{\sim}{\zeta}(t) - \underset{\sim}{\zeta}_n(t) = h\left(\underset{\sim}{\xi}_n(t) + \underset{\sim}{z}(t),\; \underset{\sim}{\eta}_n(t) + \underset{\sim}{y}(t)\right) - h\left(\underset{\sim}{\xi}_n(t), \underset{\sim}{\eta}_n(t)\right)$$

$$= h\left(\underset{\sim}{\xi}_n(t), \underset{\sim}{\eta}_n(t)\right) + \left(\left.\frac{\partial h}{\partial \xi}\right|_{(\xi_n, \eta_n)} \underset{\sim}{z}(t) + \left.\frac{\partial h}{\partial \eta}\right|_{(\xi_n, \eta_n)} \underset{\sim}{y}(t)\right) + \text{HOTs} - h\left(\underset{\sim}{\xi}_n(t), \underset{\sim}{\eta}_n(t)\right)$$

e) $\tilde{\zeta}(t) - \tilde{\zeta}_r(t) = \left.\frac{\partial h}{\partial \xi}\right|_{(\xi_r, \eta_r)} \tilde{x}(t) + \left.\frac{\partial h}{\partial \eta}\right|_{(\xi_r, \eta_r)} \tilde{u}(t)$

~~$\Rightarrow \tilde{y}(t) = \partial$~~

$$\boxed{\tilde{y}(t) = \underline{C} \cdot \tilde{x}(t) + \underline{D} \cdot \tilde{u}(t)}$$

where $\underline{C} = \left.\frac{\partial h}{\partial \xi}\right|_{(\xi_r, \eta_r)}$ & $\underline{D} = \left.\frac{\partial h}{\partial \eta}\right|_{(\xi_r, \eta_r)}$

Linearized O/P eqn (Linearized about the reference solu^n)

$$\boxed{\begin{aligned} \tilde{x}(t) &= \xi(t) - \xi_r(t) \\ \tilde{u}(t) &= \eta(t) - \eta_r(t) \\ \tilde{y}(t) &= \zeta(t) - \zeta_r(t) \end{aligned}}$$

$\underline{C}$ ~ is the O/P coeff. Matrix

$\underline{D}$ ~ direct transmission "

if $\boxed{\underline{D}(t) = \underline{0}}$ $\Rightarrow$ System is "proper"

⊙ Given:

$\dot{\xi} = f(\xi, \eta, t)$ — ξ: State vector, η: I/P "

$\xi(t_i) = \xi_i$: IC

$\zeta = h(\xi, \eta, t)$ — ζ: O/P vector.

$\rightarrow$ ⊛

Consider ~~the~~ a reference solution: ξ_r, η_r & ζ_r such that:

$\dot{\xi}_r = f(\xi_r, \eta_r, t)$

$\xi_r(t_i) = \xi_{ri}$

$\zeta_r = h(\xi_r, \eta_r, t)$

On linearizing ⊛ about the reference solution

$$\boxed{\begin{aligned} \dot{\tilde{x}}(t) &= \underline{A}(t) \cdot \tilde{x}(t) + \underline{B}(t) \cdot \tilde{u}(t) \\ \tilde{y}(t) &= \underline{C}(t) \cdot \tilde{x}(t) + \underline{D}(t) \cdot \tilde{u}(t) \end{aligned}}$$

where $\underline{A} = \left.\frac{\partial f}{\partial \xi}\right|_{(\xi_r, \eta_r)}$, $\underline{B} = \left.\frac{\partial h}{\partial \eta}\right|_{(\xi_r, \eta_r)}$, $\underline{C} = \left.\frac{\partial h}{\partial \xi}\right|_{(\xi_r, \eta_r)}$, $\underline{D} = \left.\frac{\partial h}{\partial \eta}\right|_{(\xi_r, \eta_r)}$

Solution to Linear State Equations:

$$\boxed{\begin{aligned}&\dot{\underset{\sim}{x}}(t) = \underline{\underline{A}}(t)\cdot \underset{\sim}{x}(t) + \underline{\underline{B}}(t)\cdot \underset{\sim}{u}(t)\\ &\text{subject to IC: } \underset{\sim}{x}(t_i) = \underset{\sim}{x}_i\end{aligned}} \longrightarrow (i)$$

is expressed as $x(t)$ = homogenous solution + particular solution

$$x(t) = \Phi(t,t_i)\,\underset{\sim}{x}_i + \int_{t_i}^{t} \Phi(t,\tau)\,\underline{B}(\tau)\,\underline{u}(\tau)\,d\tau$$

where $t \geq t_i$

- **Homogenous Solution:**

$$\dot{\underset{\sim}{x}}(t) = \underline{\underline{A}}(t)\cdot \underset{\sim}{x}(t)$$

$$\& \quad \underset{\sim}{x}(t) = \underbrace{\Phi(t,t_i)}_{\text{State Transition Matrix (STM)}}\,\underset{\sim}{x}_i$$

has the following properties.

$$\Phi(t_i,t_i) = \underline{\underline{I}}$$

$$\Phi(t_i,t) = \Phi(t,t_i)^{-1}$$

$$\Phi(t_i,t) = \Phi(t_i,t_0)\,\Phi(t_0,t)$$

where $t_i \leq t_0 \leq t$

$$\frac{d\Phi(t,t_i)}{dt} = \underline{\underline{A}}(t).\underline{\underline{\Phi}}(t,t_i)$$

- General Solution: to eqn (i)

$$\boxed{\underset{\sim}{x}(t) = \underline{\Phi}(t,t_i)\,\underset{\sim}{x}_i + \int_{t_i}^{t} \underline{\Phi}(t,\tau)\,\underline{B}(\tau)\,\underline{u}(\tau)\,d\tau, \quad t \geq t_i}$$

O/P : $\underset{\sim}{y}(t) = \underline{C}(t)\,\underset{\sim}{x}(t) + \underline{D}(t)\,\underline{u}(t)$

$$\underset{\sim}{y}(t) = \underline{C}\cdot\underline{\Phi}(t,t_i)\,\underset{\sim}{x}_i + \int_{t_i}^{t}\left(\underline{C}\cdot\underline{\Phi}(t,\tau)\,\underline{B}(\tau)\,\underset{\sim}{u}(\tau) + \underset{u(\tau)}{\cancel{\;}}\,\delta(t-\tau)\,\underline{D}(t)\right)d\tau$$

$$\text{So}\quad \boxed{\underset{\sim}{y}(t) = \underline{C}(t).\underline{\Phi}(t,t_i).\underset{\sim}{x}_i + \int_{t_i}^{t}\left[\underline{C}(t)\,\underline{\Phi}(t,\tau)\,\underline{B}(\tau) + \underline{D}(t)\cdot\delta(t-\tau)\right]\underset{\sim}{u}(\tau)\,d\tau}$$

$t_i \leq \tau \leq t$

So, $$\boxed{\underset{\sim}{x}(t) = \underline{\underline{\Phi}}(t, t_i)\, \underset{\sim}{x_i} + \int_{t_i}^{t} \underline{\underline{\Phi}}(t, \tau)\, \underline{\underline{B}}(\tau) \cdot \underset{\sim}{u}(\tau)\, d\tau}\, , \quad t_i \leq \tau \leq t$$

$$\boxed{\underset{\sim}{y}(t) = \underbrace{\underline{\underline{C}} \cdot \underline{\underline{\Phi}}(t, t_i) \cdot \underset{\sim}{x_i}}_{\text{initial response}} + \underbrace{\int_{t_i}^{t} \left[\underline{\underline{C}}(t)\, \underline{\underline{\Phi}}(t, \tau)\, \underline{\underline{B}}(\tau)\, \underset{\sim}{u}(\tau) + \underline{\underline{D}}(t) \cdot \delta(t-\tau) \cdot \underset{\sim}{u}(\tau) \right] d\tau}_{\text{Convolution integral of the linear system}}} \quad t_i \leq \tau \leq t$$

$\delta(t-\tau)$ ↳ is Dirac Delta function representing a unit impulse applied at $t = \tau$

© When IC $\underset{\sim}{x_i} = \underset{\sim}{0}$, then $$\boxed{\underset{\sim}{y}(t) = \int_{t_i}^{t} \underline{\underline{G}}(t, \tau)\, \underset{\sim}{u}(\tau)\, d\tau}$$

↳ Impulse response Matrix

$$\boxed{\underline{\underline{G}}(t, \tau) = \left[\underline{\underline{C}}(t)\, \underline{\underline{\Phi}}(t, \tau)\, \underline{\underline{B}}(\tau) + \underline{\underline{D}}(t) \cdot \delta(t-\tau) \right]} \quad \tau \leq t$$

When $\underset{\sim}{u}(\tau) = \delta(t-\tau)$ i.e. when the input is a unit impulse

then, $$\underset{\sim}{y}(t) = \int_{t_i}^{t} \underline{\underline{G}}(t, \tau)\, \delta(t-\tau)\, d\tau = \underline{\underline{G}}(t, t)$$

impulse response Matrix

- $\underline{\underline{G}}(t, \tau)_{ij}$ = is the value of the i^{th} o/p variable at time t, when the j^{th} I/P variable is a unit impulse function applied at time τ, subject to zero Initial condition, $\underset{\sim}{x_i} = \underset{\sim}{0}$

Step response Matrix

- $$\boxed{\underset{\sim}{S}(t, t_i) = \int_{t_i}^{t} \underline{\underline{G}}(t, \tau)\, d\tau}\, , \quad t_i \leq \tau \leq t$$

$\underline{G}(t,\tau) \rightsquigarrow$ Impulse response matrix.

$$\underset{\sim}{S}(t,t_i) = \int_{t_i}^{t} \underline{G}(t,\tau)\,d\tau$$

Step response matrix!

$$\underset{\sim}{S}(t,t_i) = \int_{t_i}^{t} \underline{G}(t,\tau)\,d\tau \quad t_i \le \tau \le t$$

$\underset{\sim}{S}(t,t_i)_{ij}$ = is the value of ith o/p at time t when the j^{th} I/P variable is a unit step applied at time t_i, subject to 0 IC, $\underset{\sim}{x}_i = \underset{\sim}{0}$

Linear Time Invariant System: (LTI system) — the linear systems' properties are invariant wrt time, i.e. $\underline{A}, \underline{B}, \underline{C}, \underline{D}$ are independent of time, t.

$$\boxed{\begin{aligned}\dot{\underset{\sim}{x}}(t) &= \underline{A}\cdot\underset{\sim}{x}(t) + \underline{B}\cdot\underset{\sim}{u}(t)\\ \underset{\sim}{y}(t) &= \underline{C}\cdot\underset{\sim}{x}(t) + \underline{D}\cdot\underset{\sim}{u}(t)\end{aligned}}$$

$\underset{\sim}{x}(t_i) = \underset{\sim}{x}_i$

- for LTI system, the STM $\underline{\Phi}(t,t_i) = e^{\underline{A}(t-t_i)}$

where

$$e^{\underline{A}(t-t_i)} = \underline{I} + \underline{A}(t-t_i) + \frac{1}{2!}\underline{A}^2(t-t_i)^2 + \frac{1}{3!}\underline{A}^3(t-t_i)^3 + \cdots$$

- $\dot{\underset{\sim}{x}} = \underline{\underline{A}} \cdot \underset{\sim}{x} + \underline{\underline{B}} \cdot \underset{\sim}{u}$

$s \to$ is laplace variable. sec^{-1}

$\downarrow \mathcal{L}\{\}$

here $X(s) = \mathcal{L}\{x(t)\}(s)$

$U(s) = \mathcal{L}\{u(t)\}(s)$

$$s\underset{\sim}{X}(s) - \underset{\sim}{x}(t=0) = \underline{\underline{A}} \cdot \underset{\sim}{X}(s) + \underline{\underline{B}} \cdot \underset{\sim}{U}(s)$$

$$\Rightarrow s\underset{\sim}{X}(s) - \underset{\sim}{x_0} = \underline{\underline{A}} \cdot \underset{\sim}{X}(s) + \underline{\underline{B}} \cdot \underset{\sim}{U}(s)$$

$$\Rightarrow (s\underline{\underline{I}} - \underline{\underline{A}}) \cdot \underset{\sim}{X}(s) = \underset{\sim}{x_0} + \underline{\underline{B}} \cdot \underset{\sim}{U}(s)$$

$$\Rightarrow \boxed{\underset{\sim}{X}(s) = (s\underline{\underline{I}} - \underline{\underline{A}})^{-1} \underset{\sim}{x_0} + (s\underline{\underline{I}} - \underline{\underline{A}})^{-1} \cdot \underline{\underline{B}} \cdot \underset{\sim}{U}(s)}$$

$$\underset{\sim}{x}(t_i=0) = \underset{\sim}{x_0}$$

- For homogenous LTI system: $\underset{\sim}{U}(s) = \underset{\sim}{0}$

$$\Downarrow$$

$$\underset{\sim}{X}(s) = (s\underline{\underline{I}} - \underline{\underline{A}})^{-1} \underset{\sim}{x_0}$$

$$\underset{\sim}{x}(t) = \mathcal{L}^{-1}\{(s\underline{\underline{I}} - \underline{\underline{A}})^{-1}\} \underset{\sim}{x_0}$$

where $t \geq t_i$

so, $e^{\underline{\underline{A}}(t-t_i)} = \mathcal{L}^{-1}\{(s\underline{\underline{I}} - \underline{\underline{A}})^{-1}\}$, where $t_i = 0$

i.e. $\boxed{e^{\underline{\underline{A}}t} = \mathcal{L}^{-1}\{(s\underline{\underline{I}} - \underline{\underline{A}})^{-1}\}}$

here $t_i = 0$

- For a generalized LTI system:

$$\underset{\sim}{x}(t) = e^{\underline{\underline{A}}(t-t_i)} \underset{\sim}{x_i} + \int_{t_i}^{t} e^{\underline{\underline{A}}(t-\tau)} \cdot \underline{\underline{B}} \cdot \underset{\sim}{u}(\tau)\, d\tau$$

~~when $t_i = 0$~~, if $\underset{\sim}{x}(t_i=0) = \underset{\sim}{x_0}$

then, $\boxed{\underset{\sim}{x}(t) = e^{\underline{\underline{A}}t} \underset{\sim}{x_0} + \int_0^t e^{\underline{\underline{A}}(t-\tau)} \cdot \underline{\underline{B}} \cdot \underset{\sim}{u}(\tau)\, d\tau}$

$\to 0$ as $t \to \infty$ for an asymptotically stable system.

$\to$ finite value (steady state) as $t \to \infty$ or assumes same functional form as that of the I/P (forced response)

• $\underset{\sim}{y}(t) = \underline{\underline{C}} \cdot \underset{\sim}{x}(t) + \underline{\underline{D}} \cdot \underset{\sim}{u}(t)$

$\downarrow \mathcal{L}\{\}$

$\underset{\sim}{Y}(s) = \underline{\underline{C}} \cdot \underset{\sim}{X}(s) + \underline{\underline{D}} \cdot \underset{\sim}{U}(s)$

for LTI system, with $\underset{\sim}{x}(t=0) = \underset{\sim}{x}_0$

$\underset{\sim}{Y}(s) = \underline{\underline{C}}(s\underline{\underline{I}} - \underline{\underline{A}})^{-1} \underset{\sim}{x}_0 + [\underline{\underline{C}}(s\underline{\underline{I}} - \underline{\underline{A}})^{-1} \cdot \underline{\underline{B}} + \underline{\underline{D}}] \underset{\sim}{U}(s)$

for LTI system, subject to 0 ICs

i.e. $y(0) = \dot{y}(0) = \ddot{y}(0) = \dots = 0$

$$\boxed{\underset{\sim}{Y}(s) = \underline{G}(s) \cdot \underset{\sim}{U}(s)}$$

Transfer Matrix.

• $\underline{G}(s) = \mathcal{L}\{$ Impulse Response Matrix subjected to 0 ICs $\}$

$= \dfrac{\text{numerator}}{\text{Denom}}$

numerator → zeros of the system.

Denom → poles of the system. $\equiv$ eigen values of $\underline{\underline{A}}$

$$\star \boxed{\underline{\underline{G}}(s) = \underline{\underline{C}} \cdot (s\underline{\underline{I}} - \underline{\underline{A}})^{-1} \cdot \underline{\underline{B}} + \underline{\underline{D}}}$$

for LTI system with 0 ICs

i.e. $\underset{\sim}{x}(t_i = 0) = \underset{\sim}{x}_0 \equiv 0$

⊙ Stability criteria for ~~a~~ a LTI system with OICs

~~for~~ $\det(sI - A) = 0$ $(sI - A)\cdot X(s) = 0$

↳ eigenvalues   ↳ eigenvectors

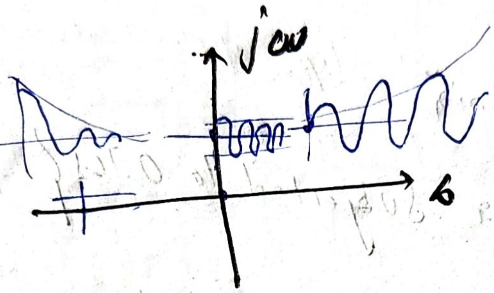
Characteristic eqn of the system.

- the eigenvalue of the linear system is obtained from $\det(sI - A) = 0$

$s + j\omega$

s: growth (or decay) of amplitude of the characteristic vector w.r.t time.

$j\omega$: frequency of oscillation of the characteristic vector about the equb pt.

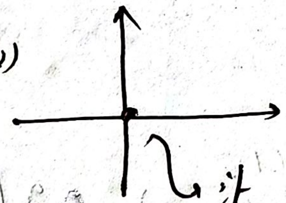


~~Stable~~ asymptotically stable

Marginally " → oscillatory behaviour of const. amplitude ~~or const. signal no oscillation~~

Unstable.

in notes AE777 (lec2) they are called 'stable'



if p no. of poles are at origin

then $f(t) = k_1 + k_2 t^1 + k_3 t^2 + \dots + k_p t^{p-1}$ + ... other terms.

for $p > 1$ system is unstable

$p = 1$ system ~~is marginally~~ may be stable (but ~~Not~~ asymptotically stable!)

Calculation of State Transition Matrix (LTI)

Homogenous state eqn of LTI system : $\dot{\underline{x}} = \underline{A} \cdot \underline{x}$

State Transition Matrix, $\phi(t, t_i) = e^{\underline{A}(t-t_i)}$ $t \geq t_i$

where, $e^{\underline{A}t} = \mathcal{L}^{-1}\{(s\underline{I} - \underline{A})^{-1}\}$

$s \in \mathbb{C}$ is the Laplace variable

Partial fraction Expansion:

Consider a rational function, $F(s) = \dfrac{N(s)}{D(s)}$

Whose poles are $\underbrace{p_1, p_2, \ldots, p_{M-1}}_{\text{of multiplicity 1 i.e. simple poles}}, \underbrace{p_M}_{\text{multiplicity } K}$

then $F(s) = \left[\dfrac{r_1}{s-p_1} + \dfrac{r_2}{s-p_2} + \cdots + \dfrac{r_{M-1}}{s-p_{M-1}}\right]$

$+ \left[\dfrac{r_{M1}}{s-p_M} + \dfrac{r_{M2}}{(s-p_M)^2} + \cdots + \dfrac{r_{MK}}{(s-p_M)^K}\right]$

Where $\underbrace{r_1, r_2, \ldots, r_{M-1}}_{M-1 \text{ residues}}, \underbrace{r_{M1}, r_{M2}, \ldots, r_{MK}}_{K \text{ residues}}$

total = $K + M - 1$ residues.

$$\boxed{\begin{aligned} \text{Where, } r_i &= \lim_{s \to p_i} (s-p_i) F(s) \; ; \; 1 \leq i \leq M-1 \\ \& \; r_{Mi} &= \lim_{s \to p_M} \frac{d^{(K-i)}}{ds^{(K-i)}} (s-p_M)^K F(s) \; ; \; 1 \leq i \leq K \end{aligned}}$$

Controllability : (of a LTI system)

↳ property of a system where it is possible to move the system from "any" initial state $x(t_i)$ to any given final state $x(t_f)$ solely by the application of I/p vector $u(t)$ acting in "finite" control interval $t_i \leq t \leq t_f$

✓ In an uncontrollable system
- → it may be possible to move between some specific states by applying control I/p.
- → ∞ time may be required to change between arbitary states!

For a system to be "controllable" all its state variables, must be influenced either directly or indirectly by the control I/P, ~~u(t)~~

↳ if there is a subsystem that is unaffected by control I/p then the whole system is uncontrollable!

★ an LTI system is Controllable $\Leftrightarrow$ $\underline{P} = (\underline{B} \; \underline{A}\underline{B} \; \underline{A}^2\underline{B} \; \dots \; \underline{A}^{n-1}\underline{B})$ must have rank n, $\underline{x} \in \mathbb{R}^n$
$(\underline{A}, \underline{B})$

$\dot{\underline{x}} = \underline{A} \cdot \underline{x} + \underline{B} \cdot \underline{u}$, $\underline{A}$: $n \times n$, $\underline{B}$: $n \times p$, $\underline{u}$: $p \times 1$, $\underline{x} \in \mathbb{R}^n$

Controllability test Matrix ($\underline{P}$) — else the system is uncontrollable.

★ For an uncontrollable system: → we can isolate controllable part from the uncontrollable part using LT such that the transformed system's coeff. matrix is partitioned into controllable & uncontrollable subsystem.

↓

if the uncontrollable part is stable Then the system is said to be $\boxed{\text{Stabilizable}}$ ★

⊙ A controllable system which is unstable, can be stabilized using a feedback control system.

Observability : (of a LTI System)

it is defined as the ability of an arbitrary state $x(\tau)$, of a linear system to be determined from a finite record of the O/P $y(t)$, $t \geq \tau$, when NO input acts on the system! (Unforced LTI System)

for a forced linear system $u(t)$ should be known in the period of observation $t \geq t_i$

For a system to be "Observable", the O/P variables must be either directly or indirectly be affected by the State variables!

an unforced LTI System is observable $\Leftrightarrow$ $N = [C^T \; A^T \cdot C^T \; (A^T)^2 \cdot C^T \; \cdots \; (A^T)^{n-1} \cdot C^T]$
$(u(t) \equiv 0)$

Observability test matrix.

Must have rank n else Unobservable.

$\dot{x} = A \cdot x + B \cdot u$

$y = C \cdot x + D \cdot u \leftarrow (u = 0)$

If there is subsystem that leaves the O/P vector unaffected then the entire system is Unobservable!

For an unobservable system: we can isolate unobservable part from observable part

if the unobservable part of the system is stable then the system is $\boxed{\text{Detectable}}$ ★

The Observability (or atleast detectibility) of a plant allows the design of an Observer to $\boxed{\text{estimate}}$ the actual state of the plant by an estimated state $\hat{x}(t)$ from measurement of the O/P $y(t)$ & knowledge of applied I/P $u(t)$

therefore Observer is also called State estimator for the plant!

State Feedback control of LTI system : [Design of "regulator"]

a LTI system $\begin{cases} \dot{\underset{\sim}{x}}_{n\times 1} = \underline{\underline{A}}_{n\times n}\,\underset{\sim}{x}_{n\times 1} + \underline{\underline{B}}_{n\times m}\,\underset{\sim}{u}_{m\times 1} \\ \underset{\sim}{y}_{p\times 1} = \underline{\underline{C}}_{p\times n}\,\underset{\sim}{x}_{n\times 1} + \underline{\underline{D}}_{p\times m}\,\underset{\sim}{u}_{m\times 1} \end{cases}$ $\underset{\sim}{x} \in R^n$, $\underset{\sim}{u} \in R^m$, $\underset{\sim}{y} \in R^p$

$= 0 \Rightarrow$ System is 'proper'

if the uncontrollable part is stable

- for a stabilizable plant, the overall closed loop system is made stable via Linear Static Feedback

Control law $\boxed{\underset{\sim}{u}_{m\times 1} = -\underline{\underline{K}}_{m\times n}\,\underset{\sim}{x}_{n\times 1}}$ $\Rightarrow$ ~~"Regulator"~~

$\underline{\underline{K}}$: regular Gain Matrix

- $\boxed{\dot{\underset{\sim}{x}} = (\underline{\underline{A}} - \underline{\underline{B}}\cdot\underline{\underline{K}})\,\underset{\sim}{x}}$

"regulator"

regulator dynamic matrix or the dynamic matrix

we select $\underline{K}$ such that $(\underline{A} - \underline{B}\cdot\underline{K})$ has all its EVs on left hand plane

$\Downarrow$

dynamic matrix is "Hurwitz"!

(System is asymptotically stable)

Method for selecting $\underline{\underline{K}}$:

Method ① Auckerman's formula: ★ [applicable when $\underset{\sim}{u} \in R^{1}$]

$$\boxed{\underline{K}_{1\times n} = (\hat{\underset{\sim}{a}} - \underset{\sim}{a})\,(\underline{\underline{P}}\cdot\underline{\underline{W}})^{-1}}$$

where

$\underset{\sim}{a} = (a_n \;\; a_{n-1} \;\; a_{n-2} \cdots a_2 \;\; a_1)$ ~ $\det(s\underline{\underline{I}} - \underline{\underline{A}}) = s^n + a_n s^{n-1} + a_{n-1} s^{n-2} + \cdots + a_2 s + a_1$

$\hat{\underset{\sim}{a}} = (\hat{a}_n \;\; \hat{a}_{n-1} \;\; \hat{a}_{n-2} \cdots \hat{a}_2 \;\; \hat{a}_1)$ ~ $\det(s\underline{\underline{I}} - \overline{\underline{\underline{A}} - \underline{\underline{B}}\cdot\underline{\underline{K}}})$

$= s^n + \hat{a}_n s^{n-1} + \hat{a}_{n-1} s^{n-2} + \cdots + \hat{a}_2 s + \hat{a}_1$

controllability test matrix

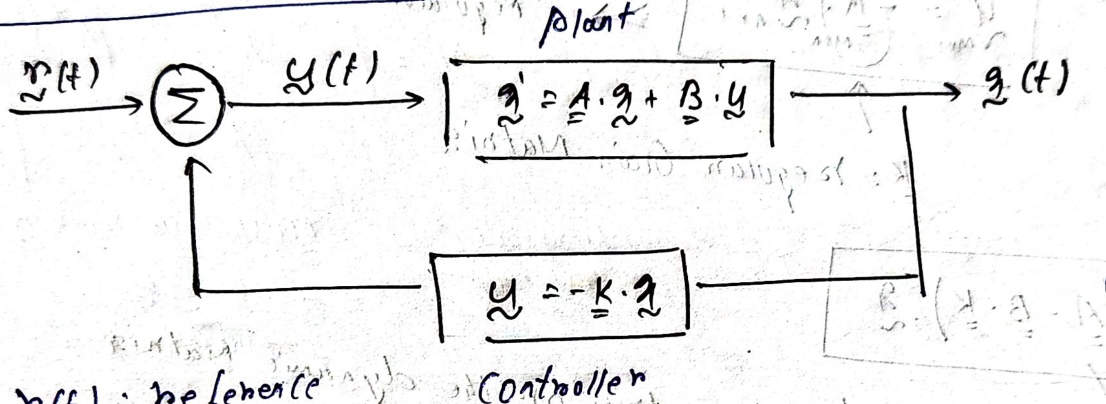
$\underline{\underline{P}} = (\underline{\underline{B}} \;\; \underline{\underline{A}}\cdot\underline{\underline{B}} \;\; \underline{\underline{A}}^2\cdot\underline{\underline{B}} \cdots \underline{\underline{A}}^{n-1}\cdot\underline{\underline{B}})$

$$\underline{\underline{W}} = \begin{pmatrix} 1 & a_n & a_{n-1} & \cdots & a_4 & a_3 & a_2 \\ 0 & 1 & a_n & \cdots & a_5 & a_4 & a_3 \\ 0 & 0 & 1 & \cdots & a_6 & a_5 & a_4 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & a_n & a_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & 1 & a_n \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}_{n \times n}$$

Upper triangular, Toeplitz matrix

Method

(ii) Knowing the roots on comparing with the desired characteristic eqn. $\det(s\underline{\underline{I}} - \overline{\underline{\underline{A}} - \underline{\underline{B}}\cdot\underline{\underline{K}}})$



LTI State feedback Regulator.

$\underset{\sim}{r}(t)$: reference Signal

$\underset{\sim}{u}(t) = \underset{\sim}{r}(t) - \underline{\underline{K}}\cdot\underset{\sim}{x}(t)$

O/P feedback Control of LTI System : [Design of "Observer"]

The State vector $\underset{\sim}{x}(t)$ is often **NOT** measurable (because some state variables may not be any physical quantities)

even if they are physically meaningful variable, sometimes measuring them won't be of any help!

O/P feedback control law :

where only the physically measurable variables → O/P vector, $\underset{\sim}{y}(t)$ are used for feedback!

a General Non-linear time varying system / plant :

State eqn : $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x}(t), \underset{\sim}{u}(t))$

O/P eqn : $\underset{\sim}{y} = h(\underset{\sim}{x}(t), \underset{\sim}{u}(t))$

The O/P feedback control

$\underset{\sim}{u} = g(\underset{\sim}{y}(t))$: Static control law.

or

$\underset{\sim}{u} = g(\underset{\sim}{y}(t), \underset{\sim}{z}(t))$: dynamic " "

where $\dot{\underset{\sim}{z}} = T(\underset{\sim}{y}(t), \underset{\sim}{z}(t))$

- an exp: (Dynamic Feedback Control)
  - Observer based control where $\underset{\sim}{z}(t) = \hat{\underset{\sim}{x}}(t)$ → Function$(\underset{\sim}{u}(t), \underset{\sim}{y}(t))$

★ If the system is observable (or atleast detectable)

⇓

We can build **Observer**!

✓ Observer is an essential tool for building practical systems with the help of O/P feedback!

- The state equⁿ of a LTI ~~system~~ observer for a plant :

$$\begin{cases} \dot{\underset{\sim}{x}}(t) = \underset{n\times n}{\underset{=}{A}} \cdot \underset{n\times 1}{\underset{\sim}{x}(t)} + \underset{n\times m}{\underset{=}{B}} \cdot \underset{m\times 1}{\underset{\sim}{u}(t)} \\ \underset{p\times 1}{\underset{\sim}{y}(t)} = \underset{p\times n}{\underset{=}{C}} \cdot \underset{n\times 1}{\underset{\sim}{x}(t)} + \underset{p\times m}{\underset{=}{D}} \cdot \underset{m\times 1}{\underset{\sim}{u}(t)} \end{cases}$$

$$\dot{\hat{\underset{\sim}{x}}} = \hat{\underset{=}{A}}\,\hat{\underset{\sim}{x}} + \underset{=}{B}\cdot\underset{\sim}{u} + \underset{=}{L}\,\underset{\sim}{y}$$

observer's dynamic Matrix

Observer Gain Matrix

$$\boxed{\hat{\underset{=}{A}} = \underset{=}{A} - \underset{=}{L}\cdot\underset{=}{C}}$$

In order to ensure that

$\hat{x}(t) \longrightarrow \underset{\sim}{x}(t)$ as $t \to \infty$

we should select $\underset{=}{L}$ in such a fashion that $(\underset{=}{A} - \underset{=}{L}\cdot\underset{=}{C})$ is Hurwitz!

i.e. all EVs of $(\underset{=}{A} - \underset{=}{L}\cdot\underset{=}{C})$ are on Left Hand Side.

For stabilizable & detectable plant we can build $\left\{ \begin{array}{l} \rightarrow \text{regulator} \\ \rightarrow \text{observer} \end{array} \right\}$ separately

on putting together $\left. \begin{array}{c} \text{Regulator} \\ + \\ \text{Observer} \end{array} \right\} \Rightarrow \boxed{\text{Compensator}}$ ~ a state o/p feedback system

with dynamic feedback law

designed separately.

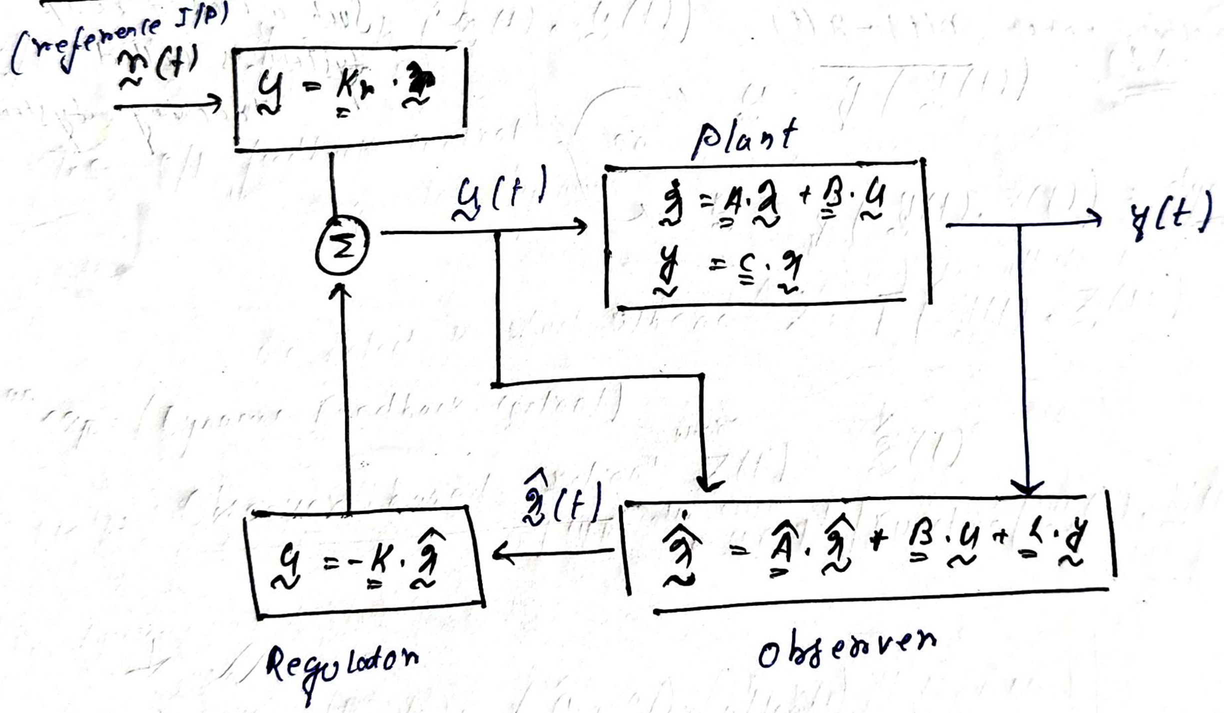
Separation principle for the design of a stable, observer based compensator

$$\underset{\sim}{u} = -\underset{=}{K} \cdot \hat{\underset{\sim}{x}}$$

$$\dot{\hat{\underset{\sim}{x}}} = (\hat{\underset{=}{A}} - \underset{=}{B} \cdot \underset{=}{K}) \cdot \hat{\underset{\sim}{x}} + \underset{=}{L} \cdot \underset{\sim}{y}$$

where $\hat{\underset{=}{A}} = \underset{=}{A} - \underset{=}{L} \cdot \underset{=}{C}$

- LTI O/P feedback Compensator:



- State eqn of LTI Compensator:

$$\left. \begin{array}{l} \dot{\underset{\sim}{x}}_{n\times1} = \underset{=}{A}_{n\times n} \cdot \underset{\sim}{x}_{n\times1} - \underset{=}{B}_{n\times m} \cdot \underset{=}{K}_{m\times n} \cdot \hat{\underset{\sim}{x}}_{n\times1} \\ \dot{\hat{\underset{\sim}{x}}}_{n\times1} = (\underset{=}{A}_{n\times n} - \underset{=}{B}_{n\times m} \underset{=}{K}_{m\times n} - \underset{=}{L}_{n\times p} \cdot \underset{=}{C}_{p\times n}) \cdot \hat{\underset{\sim}{x}}_{n\times1} + \underset{=}{L}_{n\times p} \cdot \underset{=}{C}_{p\times n} \cdot \underset{\sim}{x}_{n\times1} \end{array} \right\} \sim \dot{\underset{\sim}{X}} = \bar{\underset{=}{A}} \cdot \underset{\sim}{X}$$

closed loop dynamic matrix

compensator state

where

$$\bar{\underset{=}{A}} = \begin{pmatrix} \underset{=}{A} & -\underset{=}{B}\cdot\underset{=}{K} \\ \underset{=}{A} - \underset{=}{B}\cdot\underset{=}{K} - \underset{=}{L}\cdot\underset{=}{C} & \underset{=}{L}\cdot\underset{=}{C} \end{pmatrix}$$

$$\underset{\sim}{X} = (\underset{\sim}{x}^T, \hat{\underset{\sim}{x}}^T)^T$$

○ Suppose a reference signal $\underset{\sim}{r}(t) \in \mathbb{R}^m$ acts on the compensated system; then $\underset{\sim}{u}(t) = \underset{\sim}{r}(t) - \underline{K}\cdot\hat{\underset{\sim}{x}}$

$$\text{So, } \dot{\underset{\sim}{X}} = \bar{\underline{A}}\cdot\underset{\sim}{X} + \bar{\underline{B}}\cdot\underset{\sim}{r}$$

$$\text{where } \bar{\underline{A}} = \begin{pmatrix} \underline{A} & -\underline{B}\cdot\underline{K} \\ \underline{A}-\underline{B}\cdot\underline{K}-\underline{L}\cdot\underline{C} & \underline{L}\cdot\underline{C} \end{pmatrix} \;\&\; \bar{\underline{B}} = \begin{pmatrix} \underline{B} \\ \underline{B} \end{pmatrix}$$

to make the plant's state $\underset{\sim}{x}(t) \rightarrow \underset{\sim}{r}(t)$ we introduce a feedforward gain $\underline{K}_r$.

$$\underset{\sim}{u}(t) = \underline{K}_r\cdot\underset{\sim}{r} - \underline{K}\cdot\hat{\underset{\sim}{x}}$$

~~the~~ $\;\; r(t) \longrightarrow x(t)$ as $t \to \infty$ irrespective of $\hat{x}(t)$ & $r(t)$

tracking error $\underline{r(t) - x(t)}$

Such a closed loop system is called Tracking system.

Week-04 Lec-06 Static Optimization

Single variable Optimization:

Consider a function $L(u) \in \mathbb{R}$ of a scalar variable $u \in \mathbb{R}$;

(assume that $L(u)$ possesses continuous derivatives of upto 2nd order

i.e. consider $L(u) \in \mathbb{R}$, $u \in \mathbb{R}$ such that $L''(u)$ is continuous

[This assumption is there in both "sufficient" & "necessary" condition]

Necessary Condition for a pt. to be a minima provided $u \in \mathbb{R}$ & not any particular subset of u

Suppose $u = u^*$ is the point where $L(u)$ achieves minimum

$\Downarrow$ provided $u \in \mathbb{R}$ (i.e. $u \in$ whole of the $\mathbb{R}$ & not to any particular subset of $\mathbb{R}$)

($\equiv$ there is no constraint on u.

$$\left.\frac{dL}{du}\right|_{u=u^*} = 0$$

$$\left.\frac{d^2L}{du^2}\right|_{u=u^*} \geq 0$$

Sufficient Condition for a pt. to be minima provided $u \in \mathbb{R}$ & no constraint on u.

if at $u = u^*$; $\left.\frac{dL}{du}\right|_{u=u^*} = 0$

$\left.\frac{d^2L}{du^2}\right|_{u=u^*} > 0$

$\Downarrow$ provided $u \in \mathbb{R}$ (i.e. no constraint placed on u)

$u = u^*$ is a point of minima

if at $u = u^*$ {necessary condition is satisfied, sufficient " NOT "} then $u = u^*$ is Stationary pt. !!

Stationary pts: all those pts. where $\frac{dL(u)}{du} = 0$

---

if at $u = u^*$; $\left.\frac{dL}{du}\right|_{u=u^*} = 0$ & (u^* becomes stationary pt.)

[provided no constraint is placed on u]

- $\left.\frac{d^2L}{du^2}\right|_{u=u^*} < 0$: $u = u^*$ is pt. of Maxima
- $\left.\frac{d^2L}{du^2}\right|_{u=u^*} > 0$: $u = u^*$ is pt. of minima.
- $\left.\frac{d^2L}{du^2}\right|_{u=u^*} = 0$: We can't decide! $u = u^*$ is singular point

In order to determine whether a singular point is a minimum point, an additional condition needs to be specified (called an equation of constraint)

- prior applying the "necessary" & "sufficient" conditions you should make sure that $h(\underset{\sim}{y})$ posses continuous pd of upto 2nd order wrt $\underset{\sim}{y}$

  if $h(\underset{\sim}{y})$ does NOT posses continuous pd of upto 2nd order wrt $\underset{\sim}{y}$ then you can neither apply the "necessary" condition nor the "sufficient" condition!

- All the EVs of a symmetric matrix are real!

- Let $\underset{=}{A}$ be a symmetric matrix & $Q(\underset{\sim}{z}) = \underset{\sim}{z}^T \cdot \underset{=}{A} \cdot \underset{\sim}{z}$ be the corresponding quadratic form.

  $\underset{=}{A}$ is called +ve semidefinite if $Q(\underset{\sim}{z}) \geq 0 \;\forall \underset{\sim}{z}$

  $\underset{=}{A}$ is called +ve definite if $Q(\underset{\sim}{z}) > 0 \;\forall \underset{\sim}{z} \neq 0$

  if all its EVs are ≥ 0

  $\Leftrightarrow$ EVs of $\underset{=}{A}$ are +ve

⊙ Multivariable Optimization :

Consider a function $L(\underset{\sim}{u}) \in \mathbb{R}$, $\underset{\sim}{u} \in \mathbb{R}^m$

Such that $L(\underset{\sim}{u})$ posses continuous partial derivative of upto 2nd order w.r.t. $\underset{\sim}{u}$

Necessary condition for opt. to be minima!

① Suppose $\underset{\sim}{u} = \underset{\sim}{u}^*$ is the point where $L(\underset{\sim}{u})$ achieves minima provided that there is No constraint on $\underset{\sim}{u}$

⇓

$$\left.\frac{\partial L}{\partial \underset{\sim}{u}}\right|_{\underset{\sim}{u}=\underset{\sim}{u}^*} = 0$$

$$\left.\frac{\partial^2 L}{\partial \underset{\sim}{u}^2}\right|_{\underset{\sim}{u}=\underset{\sim}{u}^*} \geq 0 \rightarrow ①$$

$$L_U = \frac{\partial L(u)}{\partial \underset{\sim}{u}} = \left(\frac{\partial L}{\partial u_1}, \frac{\partial L}{\partial u_2}, \dots, \frac{\partial L}{\partial u_m}\right) \in \mathbb{R}^{1\times m}$$

$$L_{UU} = \frac{\partial^2 L(u)}{\partial \underset{\sim}{u}^2} = \begin{pmatrix} \frac{\partial^2 L}{\partial u_1^2} & \frac{\partial^2 L}{\partial u_2 \partial u_1} & \cdots & \frac{\partial^2 L}{\partial u_m \partial u_1} \\ \frac{\partial^2 L}{\partial u_1 \partial u_2} & \frac{\partial^2 L}{\partial u_2^2} & \cdots & \frac{\partial^2 L}{\partial u_m \partial u_2} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 L}{\partial u_1 \partial u_m} & \frac{\partial^2 L}{\partial u_2 \partial u_m} & \cdots & \frac{\partial^2 L}{\partial u_m^2} \end{pmatrix} \in \mathbb{R}^{m\times m}$$

if is symmetric hence all its EVs are real.

$$\underset{\sim}{u} = (u_1, u_2, u_3, \dots, u_m)^T \in \mathbb{R}^{m\times 1}$$

$$= \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

(in eqn ①)

here L_{UU} is semi +ve definite ⇔ all EVs of L_{UU} are ≥ 0

• Stationary points are points where $\frac{\partial L}{\partial \underset{\sim}{y}} = \underset{\sim}{0}$

① Sufficient condition for a pt. to be minima provided that there is No constraint on $\underset{\sim}{y}$

if at $\underset{\sim}{y} = \underset{\sim}{y}^*$, $\left.\frac{\partial L}{\partial \underset{\sim}{y}}\right|_{\underset{\sim}{y} = \underset{\sim}{y}^*} = \underset{\sim}{0}$

$\left.\frac{\partial^2 L}{\partial \underset{\sim}{y}^2}\right|_{\underset{\sim}{y} = \underset{\sim}{y}^*} > 0$ → (i)

⇓ provided $\underset{\sim}{y} \in \mathbb{R}^n$ has no constraint

$\underset{\sim}{y} = \underset{\sim}{y}^*$ is a point of minima

in eqn (i) L_{uu} is +ve definite matrix

⇕

all EVs of $L_{uu} > 0$

★ if any of EVs of $L_{uu} = 0$ then $\underset{\sim}{y} = \underset{\sim}{y}^*$ will be a Singular point!

[In such a case we will need additional constraints to determine whether a singular point is minima or not]

• if at $\underset{\sim}{y} = \underset{\sim}{y}^*$; $\left.\frac{\partial L}{\partial \underset{\sim}{y}}\right|_{\underset{\sim}{y} = \underset{\sim}{y}^*} = 0$ & $\left.\frac{\partial^2 L}{\partial \underset{\sim}{y}^2}\right|_{u = u^*}$

[provided no constraint is placed on $\underset{\sim}{y}$]

> 0 : $u = u^*$ is pt. of ~~Maxima~~ minima

< 0 : $u = u^*$ is pt. of Maxima

$= 0 \Rightarrow$ we can't decide!

[$u = u^*$ is singular pt.]

① if at $\underset{\sim}{y} = \underset{\sim}{y}^*$; $\left.\frac{\partial L}{\partial \underset{\sim}{y}}\right|_{\underset{\sim}{y} = \underset{\sim}{y}^*} = 0$ & $\left.\frac{\partial^2 L}{\partial \underset{\sim}{y}^2}\right|_{\underset{\sim}{y} = \underset{\sim}{y}^*} = \underline{\underline{M}}$

($\underset{\sim}{y}^*$ stationary pt.)

if all EigenValues of $\underline{\underline{M}}$ > 0 [$\underset{\sim}{y} = \underset{\sim}{y}^*$ is a pt. of minima]

< 0 [$\underset{\sim}{y} = \underset{\sim}{y}^*$ is a pt. of maxima]

if some EVs of M is/are 0 ~~else~~ → $\underset{\sim}{y} = \underset{\sim}{y}^*$ is a Singular point.

else $\underset{\sim}{y} = \underset{\sim}{y}^*$ is a Saddle pt.

$\underset{\sim}{y} = \underset{\sim}{y}^*$ is a Saddle point if some EVs of $\frac{\partial^2 L}{\partial \underset{\sim}{y}^2}$ are +ve & some EVs are -ve !

• if a pt. $\underset{\sim}{y} = \underset{\sim}{y}^*$ does NOT satisfy the "sufficient condition" check manually ! (primarily by plotting graph)

Lec-07

- In the previous lecture we had ~~discussed~~ studied how an objective function is optimized wrt static variables, when there are NO constraint placed upon their values!

- In this lecture we will extend the treatment to multivariable optimization when "static equality constraints" are imposed on the variables of optimization!

Objective function : $\boxed{L(\underset{\sim}{x}, \underset{\sim}{u}) \in \mathbb{R}}$

State variable $\underset{\sim}{x} \in \mathbb{R}^n$

Control variable $\underset{\sim}{u} \in \mathbb{R}^m$

Static equality Contraint : $\boxed{f(\underset{\sim}{x}, \underset{\sim}{u}) = \underset{\sim}{0}_{n\times 1}}$ → (i)

here $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a smooth mapping function i.e. it posses continuous pds wrt $\underset{\sim}{x}$ & $\underset{\sim}{u}$

This constraint should give a unique $\underset{\sim}{x}$ for ~~...~~ a given $\underset{\sim}{u}$

also assume that $L(\underset{\sim}{x}, \underset{\sim}{u})$ posses continuous pds wrt $\underset{\sim}{x}$ & $\underset{\sim}{u}$

An arbitary, small variation in control variable, $\underset{\sim}{u}$ causes a small corresponding change in state variable, $\underset{\sim}{x}$ in order to satisfy eqn (i)

$$df = \underset{\sim}{0}_{n\times 1}$$

$$\Rightarrow f_{x_{n\times n}} d\underset{\sim}{x}_{n\times 1} + f_{u_{n\times m}} d\underset{\sim}{u}_{m\times 1} = \underset{\sim}{0}_{n\times 1} \rightarrow \text{(ii)}$$

$$f_x = \left(\frac{\partial f}{\partial \underset{\sim}{x}}\right)_{n\times n}$$

$$f_u = \left(\frac{\partial f}{\partial \underset{\sim}{u}}\right)_{n\times m}$$

$$\boxed{d\underset{\sim}{x} = -f_x^{-1} f_u d\underset{\sim}{u}}$$

provided f_x is NOT singular.

Stationary Condition:

Suppose $\underset{\sim}{u} = \underset{\sim}{u}^*$ is the point which ~~which~~ minimizes $L(\underset{\sim}{x}, \underset{\sim}{u})$ subject to constraint (i)

$\Downarrow$ then

$\underset{\sim}{u}^*$ stationary pt.

$$\boxed{\begin{aligned} & dL(\underset{\sim}{x}, \underset{\sim}{u}) \Big|_{\underset{\sim}{u}=\underset{\sim}{u}^*} = 0 \\ \Rightarrow\ & \left(\underset{\sim}{L_x} \cdot d\underset{\sim}{x} + \underset{\sim}{L_u} \cdot d\underset{\sim}{u}\right)\Big|_{\underset{\sim}{u}=\underset{\sim}{u}^*} = 0 \quad \text{(v)} \\ & \underset{\sim}{L_x} = \left(\frac{\partial L}{\partial \underset{\sim}{x}}\right)_{1\times n} \\ & \underset{\sim}{L_u} = \left(\frac{\partial L}{\partial \underset{\sim}{u}}\right)_{1\times m} \end{aligned}}$$

• Combining both (i) & (v) we have,

$$\left(\underset{\sim}{L_u} - L_x \underline{\underline{f_x}}^{-1} \cdot \underline{\underline{f_u}}\right) \cdot d\underset{\sim}{u} = 0$$

$$\Downarrow$$

$$\boxed{\left(\underset{\sim}{L_u}_{1\times m} - \underset{\sim}{L_x}_{1\times n} \cdot \underline{\underline{f_x}}^{-1}_{n\times n} \cdot \underline{\underline{f_u}}_{n\times m}\right)\Big|_{\underset{\sim}{u}=\underset{\sim}{u}^*} = \underset{\sim}{0}_{1\times m}}$$

→ You will have m equns.

Use these m equns + (i) to get $\underset{\sim}{u}^*_{m\times 1}$

Q. problem : Minimize $l(\underset{\sim}{x}, \underset{\sim}{u})$ subject to the constraint

$$f(\underset{\sim}{x}, \underset{\sim}{u}) = \underset{\sim}{0}_{n\times 1}$$

Algo :

Step I :

objective funcn : $l(\underset{\sim}{x}, \underset{\sim}{u}) \in \mathbb{R}$

equality constraint: $f(\underset{\sim}{x}, \underset{\sim}{u}) = \underset{\sim}{0}_{n\times 1}$, $\underset{\sim}{x} \in \mathbb{R}^{n\times 1}$, $\underset{\sim}{u} \in \mathbb{R}^{m\times 1}$

Step II : check whether $f(\underset{\sim}{x}, \underset{\sim}{u})$ posses continuous pds wrt $\underset{\sim}{x}$ & $\underset{\sim}{u}$ $\boxed{\underset{=}{f_x}_{n\times n}, \underset{=}{f_u}_{n\times m}}$

Check " $l(\underset{\sim}{x}, \underset{\sim}{u})$ " pds wrt $\underset{\sim}{x}$ & $\underset{\sim}{u}$ $\boxed{\underset{\sim}{l_x}_{1\times n}, \underset{\sim}{l_u}_{1\times m}}$

If both of the conditions are satisfied then proceed!!

Step III : If minimization is wrt $\underset{\sim}{u}$

then

$$d\underset{\sim}{f} = \underset{\sim}{0}_{n\times 1}$$

$$\Rightarrow \underset{=}{f_x}_{n\times n} \cdot d\underset{\sim}{x}_{n\times 1} + \underset{=}{f_u}_{n\times m} \cdot d\underset{\sim}{u}_{m\times 1} = \underset{\sim}{0}_{n\times 1}$$

$$\Rightarrow \boxed{d\underset{\sim}{x}_{n\times 1} = -\left(\underset{=}{f_x}^{-1}_{n\times n} \cdot \underset{=}{f_u}_{n\times m}\right) \cdot d\underset{\sim}{u}_{m\times 1}}$$

$\underset{=}{f_x}$ should be Non Singular.

If minimization is wrt $\underset{\sim}{x}$

$$d\underset{\sim}{u}_{m\times 1} = -\left(\underset{=}{f_u}^{-1}_{n\times m} \cdot \underset{=}{f_x}_{n\times n}\right) \cdot d\underset{\sim}{x}_{m\times 1}$$

[NOTE: the way by which the problem has been set uped the minimization has to be always wrt $\underset{\sim}{u}$]

Step IV: at $\underset{\sim}{u} = \underset{\sim}{u}^*$

$\cancel{dL(x, u)}$ $dL = 0$

$\Rightarrow \underset{\sim}{L_x}_{1\times n} \cdot d\underset{\sim}{x}_{n\times 1} + \underset{\sim}{L_u}_{1\times m} \cdot d\underset{\sim}{u}_{m\times 1} = 0$

$\Rightarrow \left(\underset{\sim}{L_u}_{1\times m} - \underset{\sim}{L_x}_{1\times n} \cdot \underset{=}{f_x}^{-1}_{n\times n} \cdot \underset{=}{f_u}_{n\times m} \right) \cdot d\underset{\sim}{u}_{m\times 1} = 0$

$$\boxed{\underset{\sim}{L_u}_{1\times m} - \underset{\sim}{L_x}_{1\times n} \cdot \underset{=}{f_x}^{-1}_{n\times n} \cdot \underset{=}{f_u}_{n\times m} = \underset{\sim}{0}_{1\times m}} \longrightarrow (*)$$

Step V: use eqn $(*)$ & $f(\underset{\sim}{x}, \underset{\sim}{u}) = \underset{\sim}{0}_{n\times 1}$

to get $\underset{\sim}{u}^*$ & the corresponding $\underset{\sim}{x}^*$!

⑤ Lagrange Multipliers : $\underset{\sim}{\lambda}_{n\times 1}$

The problem of minimizing $L(\underset{\sim}{x}, \underset{\sim}{u})$ wrt $\underset{\sim}{u}$, subject to $f(\underset{\sim}{x}, \underset{\sim}{u}) = \underset{\sim}{0}$

is well posed only if $\left.\begin{array}{l} df = 0_{n\times 1} \\ \& \; dL = 0 \end{array}\right\}$ are consistent linear equation in the arbitary small variations $d\underset{\sim}{x} = 0$ & $d\underset{\sim}{u} = 0$

$$\left[\begin{array}{l} \underset{=}{f_x}_{n\times n} \cdot d\underset{\sim}{x}_{n\times 1} + \underset{=}{f_u}_{n\times m} \cdot d\underset{\sim}{u}_{m\times 1} = \underset{\sim}{0}_{n\times 1} \\ \\ \underset{\sim}{L_x}_{1\times n} \cdot d\underset{\sim}{x}_{n\times 1} + \underset{\sim}{L_u}_{1\times m} \cdot d\underset{\sim}{u}_{m\times 1} = 0 \end{array}\right.$$

for this to be consistent, we must be able to find

$$\underset{\sim}{\lambda}_{n\times 1} = (\lambda_1, \lambda_2, \dots, \lambda_n)^T \in R^n$$

such that

$$\left.\begin{array}{ll} \text{(i)} & \underset{\sim}{L_x}_{1\times n} + (\underset{\sim}{\lambda})^T_{1\times n} \cdot \underset{=}{f_x}_{n\times n} = \underset{\sim}{0}_{1\times n} \\ & \& \\ \text{(ii)} & \underset{\sim}{L_u}_{1\times m} + (\underset{\sim}{\lambda}^T)_{1\times n} \cdot \underset{=}{f_u}_{n\times m} = \underset{\sim}{0}_{1\times m} \end{array}\right.$$

if the matrix $\underset{=}{f_x}_{n\times n}^{-1}$ exists only then $\underset{\sim}{\lambda}$ exists!!

then from ① we have $\boxed{\underset{\sim}{\lambda}^T_{1\times n} = - \underset{\sim}{L_x}_{1\times n} \cdot \underset{=}{f_x}^{-1}_{n\times n}}$

$$\boxed{\underset{\sim}{\lambda}^T_{1\times n} = -\left(\frac{\partial L}{\partial f}\right)_{d\underset{\sim}{u} = \underset{\sim}{0}_{m}}}$$

putting this back in (ii) we have,

$$\left(\underset{\sim}{L_u}_{1\times m} - \underset{\sim}{L_x}_{1\times n} \cdot \underset{=}{f_x}^{-1}_{n\times n} \cdot \underset{=}{f_u}_{n\times m} \right) = \underset{\sim}{0}_{1\times m}.$$

using this & $f(\underset{\sim}{x}, \underset{\sim}{u}) = \underset{\sim}{0}$

we can get hold of $\underset{\sim}{x}^*$ & $\underset{\sim}{u}^*$!!

① *Hamiltonian Formulation :*

$$H(\underset{\sim}{x}, \underset{\sim}{u}, \underset{\sim}{\lambda}) \equiv \underbrace{L(\underset{\sim}{x}, \underset{\sim}{u})}_{\text{Langrangian}} + (\lambda^T)_{1\times n} f(\underset{\sim}{x}, \underset{\sim}{u})_{n\times 1}$$

The problem of minimising the Langrangian, $L(\underset{\sim}{x}, \underset{\sim}{u})$ wrt $\underset{\sim}{u}$ subject to $f(\underset{\sim}{x}, \underset{\sim}{u}) = \underset{\sim}{0}_{n\times 1}$ is $\equiv$

minimising $H(\underset{\sim}{x}, \underset{\sim}{u}, \underset{\sim}{\lambda})$ wrt $\underset{\sim}{u}$

- So, $dH = H_{x\,1\times n}\, d\underset{\sim}{x}_{n\times 1} + H_{u\,1\times m}\, d\underset{\sim}{u}_{m\times 1}$

- at stationary pt. $\underset{\sim}{u} = \underset{\sim}{u}^*$, $dH = 0$

now, $\underset{\sim}{H}_{x\,1\times n} \equiv \underset{\sim}{L}_{x\,1\times n} + (\underset{\sim}{\lambda}^T)_{1\times n} f_{x\,n\times n}$

$\underset{\sim}{H}_{u\,1\times m} = \underset{\sim}{L}_{u\,1\times m} + (\underset{\sim}{\lambda}^T)_{1\times n} f_{u\,n\times m}$

- to find $\underset{\sim}{\lambda}$, impose $\underset{\sim}{H}_x \equiv 0$ @ stationary pt. $\Rightarrow$ $\boxed{\underset{\sim}{\lambda}^T_{1\times n} = -\underset{\sim}{L}_{x\,1\times n}\, f_{x\,n\times n}^{-1}}$

- $\underset{\sim}{H}_{u\,1\times m} \equiv \left(\underset{\sim}{L}_{u\,1\times m} - \underset{\sim}{L}_{x\,1\times n}\, f_{x\,n\times n}^{-1}\, f_{u\,n\times m}\right)$

- as $\underset{\sim}{H}_{u\,1\times m} = \underset{\sim}{0}_{1\times m}$ @ stationary pt.

So, $\boxed{\left(\underset{\sim}{L}_{u\,1\times m} - \underset{\sim}{L}_{x\,1\times n}\, f_{x\,n\times n}^{-1}\, f_{u\,n\times m}\right) = \underset{\sim}{0}_{1\times m}}$

+

$f(\underset{\sim}{x}, \underset{\sim}{u}) = 0_{n\times 1}$

Gives us $\underset{\sim}{x}^*$ & $\underset{\sim}{u}^*$.

$$\frac{\partial}{\partial \underset{\sim}{z}}\left(\underset{\sim}{z}^T \cdot \underset{=}{A} \cdot \underset{\sim}{z}\right) = \underset{\sim}{z}^T \cdot \left(\underset{=}{A}^T + \underset{=}{A}\right)$$ [in numerator layout]

$$\left[\begin{array}{l} \frac{\partial}{\partial \underset{\sim}{z}}\left(\underset{=}{A} \cdot \underset{\sim}{z}\right) = \underset{=}{A} \\ \frac{\partial}{\partial \underset{\sim}{z}}\left(\underset{\sim}{z}^T \cdot \underset{=}{A}\right) = \underset{=}{A}^T \end{array}\right\}$$

$$\frac{\partial (v \underset{\sim}{z})}{\partial \underset{\sim}{z}} = \underset{\sim}{z} \frac{\partial v}{\partial \underset{\sim}{z}}$$

$$\frac{\partial (\underset{=}{A} \cdot \underset{\sim}{y})}{\partial \underset{\sim}{z}} = \underset{=}{A} \cdot \left(\frac{\partial \underset{\sim}{y}}{\partial \underset{\sim}{z}}\right)$$

$$\int_{-\infty}^{\infty} \delta(x-a)\, g(x)\, dx = g(a)$$

$$\int_{a-\epsilon_1}^{a+\epsilon_2} \delta(x-a)\, g(x)\, dx = g(a)$$

?

- $\mathcal{L}^{-1}\left\{ \dfrac{as+b}{s^2+2\xi\omega s+\omega^2} \right\}$; $0 \leq \xi \leq 1$

$$= e^{-\xi\omega t}\left[a\cos(\omega\sqrt{1-\xi^2}\, t) + \frac{(b-a\xi\omega)}{\omega\sqrt{1-\xi^2}} \sin(\omega\sqrt{1-\xi^2}\, t) \right]$$

where $0 \leq \xi \leq 1$

- A problem is well posed if :
  - A solution exists
  - The solution is unique
  - The solution depends continuously on the data (boundary &/or ICs)

Well posed problems have a good chance to be solved numerically with a stable algo!

problems which do NOT satisfy these criteria are illposed problems!!