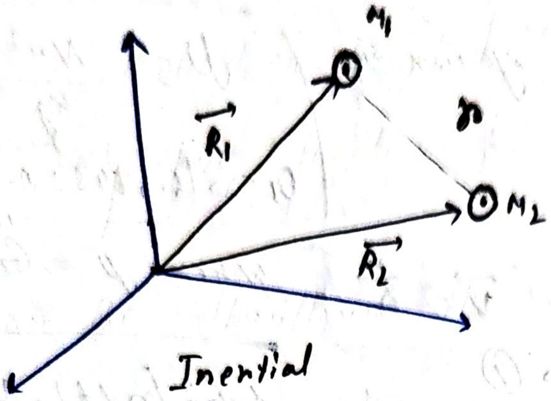
Week-10 Lec-15 Intro to Orbital Mechanics : the two body problem!

$\boxed{\text{Relative motion of two bodies Under gravitational attraction!}}$

① Two body problem:-



Inertial Frame

consider the motion of spacecraft of mass m_2 around a body of mass m_1

assume both the bodies to be spherical in shape.

The eqn's of motion of the two masses,

$$\text{(i)} \quad m_1 \ddot{\vec{R_1}} = G \frac{m_1 m_2}{r^3} (\vec{R_2} - \vec{R_1})$$

$$\text{(ii)} \quad m_2 \ddot{\vec{R_2}} = G \frac{m_1 m_2}{r^3} (\vec{R_1} - \vec{R_2})$$

| Radius vector, $\vec{r}$ | Where $r = |\vec{R_2} - \vec{R_1}|$

||

Radius, r ~~vector~~

position vector of m_2 wrt m_1

$\downarrow$

$\vec{R_2} - \vec{R_1}$

G ~ univ. Grav. Const.

$= 6.67430 \times 10^{-11} \frac{Nm^2}{kg^2}$

$|.|$ ~ euclidean norm.

(i) + (ii) ~ $m_1 \ddot{\vec{R_1}} + m_2 \ddot{\vec{R_2}} = \vec{0}$

$$\boxed{\frac{m_1 \ddot{\vec{R_1}} + m_2 \ddot{\vec{R_2}}}{m_1 + m_2} = \vec{0}}$$ ~ acc. of COM of the two body system is 0.

i.e. COM moves at a const. velocity.

$\frac{\text{(ii)}}{m_2} - \frac{\text{(i)}}{m_1}$ ~ $$\boxed{(\ddot{\vec{R_2}} - \ddot{\vec{R_1}}) = -\frac{G(m_1+m_2)}{r^3} (\vec{R_2} - \vec{R_1})}$$

Where $r = |\vec{R_2} - \vec{R_1}|$

So, $\ddot{\vec{r}} = -\dfrac{G(M_1+M_2)}{r^3}\vec{r}$

$\Rightarrow \boxed{\ddot{\vec{r}} + \dfrac{\mu}{r^3}\vec{r} = 0}$ ~ where $\mu = G(M_1+M_2)$ ↳ Gravitational constant of the two body system

↳ a solution $(r(t), \dot{r}(t))$ with a given IC $(r(0), \dot{r}(0))$ is called an orbit.

It is the Governing differential equⁿ of motion of spacecraft of mass M_2 w.r.t central body of mass M_1 (orbital motion).

$\mu \rightarrow \dfrac{Nm^2}{kg} \equiv km^3 s^{-2}$

$G \sim (6.6743 \pm 0.00015) \times 10^{-11}\ m^3 kg^{-1} s^{-2}$

where $\mu = G(M_1+M_2)$

- $\boxed{\ddot{\vec{r}} + \dfrac{\mu}{r^3}\vec{r} = \vec{0}} \rightarrow (7)$ $\vec{V} = \dot{\vec{r}}$

a solnⁿ $[r(t), V(t)]$ of equⁿ (1) to a given IC $[r(0), V(0)]$ is called an orbit.

- $\vec{r} \times (7)$ taking cross product with $\vec{r}$

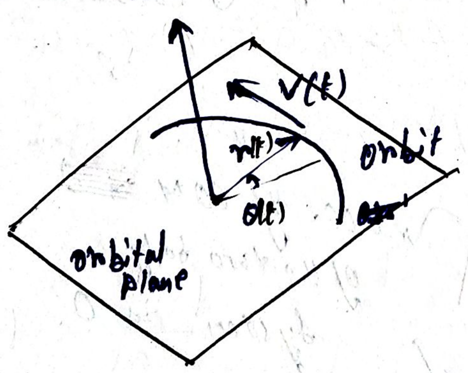
$\Downarrow$

$\vec{r} \times \ddot{\vec{r}} = \vec{0}$

$\Downarrow$

$\boxed{\dfrac{d}{dt}(\vec{r} \times \vec{V}) = \vec{0}} \Rightarrow \boxed{\text{Orbital Angular Momentum},\ \vec{h} = \vec{r} \times \vec{V} = \text{Const.}}$

$\vec{h}(t) = \vec{r}(t) \times \vec{V}(t)$



$\Downarrow$

Orbital motion $[\vec{r}(t), \vec{V}(t)]$ takes place in a const. plane whose normal is in the direction of $\vec{h}$.

the planar orbital motion can be described by the polar coordinates $(\vec{r}, \theta)$

$h = |\vec{h}| = r^2\dot{\theta}$ ~ $m^2 s^{-1}$

is also constant.

$\dfrac{dA}{dt} = \dfrac{1}{2} r^2\dot{\theta} = \dfrac{h}{2} = \text{Const.}$ ~ 2nd law of Kepler.

(St. line joining M_1 & M_2 sweeps out equal area in equal intervals)

By taking scalar product of eqn ① with $\vec{V} = \dot{\vec{r}}$

we have, $\vec{V} \cdot \ddot{\vec{r}} + \vec{V} \cdot \frac{\mu}{r^3} \vec{r} = 0.$

considering scalar product with $\vec{V}$

$$\frac{d}{dt}\left(\frac{V^2}{2} - \frac{\mu}{r} \right) = 0 \sim \boxed{\frac{V^2}{2} - \frac{\mu}{r} = \text{const.}}$$

$\Downarrow$

★ Orbital energy, $\boxed{\epsilon = \frac{V^2}{2} - \frac{\mu}{r} = \text{const.}}$

let us define,

$$\epsilon = \boxed{\frac{V^2}{2} - \frac{\mu}{r} = \frac{-\mu}{2a} \text{ (const.)}}$$

This 'a' is a 'const'

Where a: Semi major axis of the orbit.

(It is a characteristic length of the orbital motion)

$V = |\vec{V}|$ orbital speed

$r = |\vec{r}|$

- Restricted problem: (mass of spacecraft M_2 negligible wrt central body M_1)

Most often the spacecraft's mass, M_2 is negligible wrt. the mass of the central body, M_1

i.e. $M_2 << M_1$

$\Downarrow$

$$\mu = G(M_1 + M_2) \approx GM_1$$

$$\boxed{\text{for earth } GM_1 = 3,98,600.4 \text{ km}^3 s^{-2}}$$

$$\boxed{\text{Unit of } h \text{ is } m^2/s}$$

$\cancel{h^2 = \mu t}$ $\quad p = h^2/\mu.$

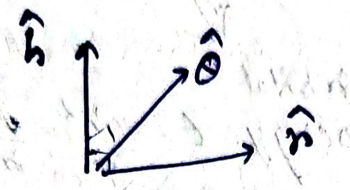
Shape of the orbit : Orbital Shape

Governing eqn of motion ~ $\boxed{\ddot{\vec{r}} + \frac{\mu}{r^3}\vec{r} = \vec{0}}$ → (i)

✓ We want the eqn of orbit! $r(\theta)$

- the motion of the two body will be restricted in a plane.

$$\hat{\theta} = \hat{h} \times \hat{r}$$



$$\hat{r} \times \hat{\theta} = \hat{h} \sim \hat{\theta} = \hat{h} \times \hat{r}$$

$$\hat{h} = \frac{\vec{h}}{|\vec{h}|} \quad \& \quad \hat{r} = \frac{\vec{r}}{|\vec{r}|}$$

Angular velocity $\vec{\omega} = \dot{\theta}\hat{h}$

put this in (i)

$$\begin{cases} \vec{r} = r\hat{r} \\ \dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \\ \ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} \end{cases}$$

$$\left(\ddot{r} - r\dot{\theta}^2 + \frac{\mu}{r^2}\right)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} = 0$$

$$\boxed{\ddot{r} - \frac{h^2}{r^3} + \frac{\mu}{r^2} = 0} \sim (*)$$

Where $r^2\dot{\theta} = h$ = const

∴ Now we will introduce another variable, $\boxed{f = \frac{1}{r}}$

(flight path curvature)

$$(\cdot)' = \frac{d(\cdot)}{d\theta}$$

$$(\cdot)'' = \frac{d^2(\cdot)}{d\theta^2}$$

(i) ←
$$\begin{cases} r = \frac{1}{f} \\ r' = \frac{-f'}{f^2} \\ r'' = -\frac{f''}{f^2} + \frac{2f'^2}{f^3} \end{cases}$$

Consider this $\rightsquigarrow$ $\dot{r} = r'\dot{\theta}$

$$\left.\begin{aligned} \dot{r} &= r'\dot{\theta} \\ \ddot{r} &= r''\dot{\theta}^2 + r'\ddot{\theta} \\ \ddot{\theta} &= -\frac{2h^2}{r^5} r' \\ \dot{\theta} &= \frac{h}{r^2} \end{aligned}\right\} \rightarrow (ii)$$

put (i) & (ii) in (*) to get,

$$\boxed{f'' + f = \mu/h^2}$$

$\rightsquigarrow$ linear and order ODE in f

$$f(\theta) = c_1 \cos\theta + c_2 \sin\theta + \frac{\mu}{h^2}$$

We will measure θ from the point of min. radius $r = r_p$ along the direction of motion! $\rightsquigarrow$ this angle θ is called "True Anomaly."

$\Downarrow$

$\theta = 0$ at $r = r_p$

$h = |r_p \times V_p|$

$h = r_p V_p$

f gives a unique position along an orbit!

$\Downarrow$

~~$r'(0) =$~~ $r'(\theta = 0) = 0$ [as r is min. at $\theta = 0$]

$r''(\theta = 0) > 0$

$f'(\theta = 0) = 0$ [as f will be max. at $\theta = 0$]

$f''(\theta = 0) < 0$

So,

$$\boxed{f(\theta) = c_1 \cos\theta + \frac{\mu}{h^2}}$$

~~we will~~ $r_p = \dfrac{1}{f(0)}$ using this we can obtain

$$c_1 = \frac{1}{r_p} - \frac{\mu}{h^2}$$

$$\Rightarrow \quad r(\theta) = \frac{1}{f(\theta)} = \frac{h^2/\mu}{1 + \left(\frac{h^2}{\mu r_p} - 1\right)\cos\theta}$$

$$\Rightarrow \quad \boxed{r(\theta) = \frac{h^2/\mu}{1 + \left(h^2/\mu r_p - 1\right)\cos\theta}}$$

orbital parameter, p defined as $p = h^2/\mu > 0$.

orbital eccentricity, $e = \left(\frac{p}{r_p} - 1\right)$

So, $\boxed{r(\theta) = \frac{p}{1 + e\cos\theta}}$ where $p = h^2/\mu > 0$ & $e = p/r_p - 1$

Orbit eqn

Conditions:

for $p > 0$ & $0 \le e < 1$ the orbit $r(\theta)$ is a closed curve → elliptic orbit → $a > 0, \varepsilon < 0$

for $p > 0$ & $e = 1$ the orbit $r(\theta)$ is an Open curve → parabolic orbit → $a = \infty, \varepsilon = 0$

for $p > 0$ & $e > 1$ " " $r(\theta)$ is a closed curve → hyperbolic orbit → $a < 0, \varepsilon > 0$

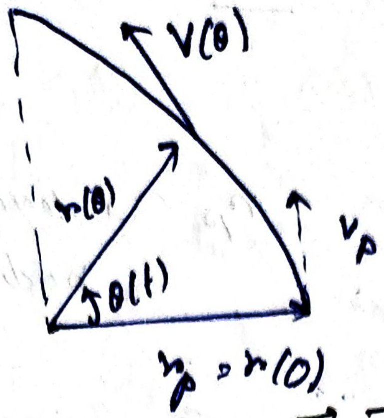
conic Section Orbits

- $p = 0$ ($\equiv h = 0$) → (rectilinear) Straight line trajectory.

one can define

$a > 0$	$a = \infty$	$a < 0$
rectilinear ellipse	rectilinear parabola	rectilinear hyperbola

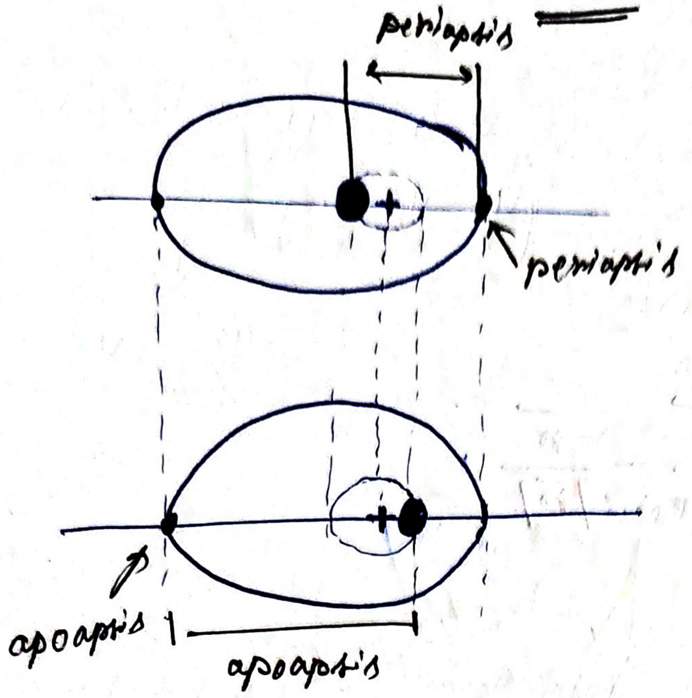
→ it does NOT satisfy "orbit eqn"



- eccentricity vector, $\vec{e} = \dfrac{\vec{V} \times \vec{h}}{\mu} - \dfrac{\vec{r}}{|\vec{r}|}$

$\boxed{\vec{e} \text{ points to the periapse.}}$

Week-11 Lec-16 Conic Section Orbits



r_p ~ sometimes also called radius of periapsis

The extreme point on the major axis of the orbit are called apses or apsis

- Earth orbit of altitude 250 KM → parking orbit.
- Geosynchronous Equatorial Orbit (GEO) → T_s = 23hr 56min 4.09 s

which gives r = 42164.17 KM

3 fundamental formula

$r(\theta) = p/1+e\cos\theta$

$p = h^2/\mu$

$\mathcal{E} = v^2/2 - \mu/r = \frac{-\mu}{2a}$

orbital shape constants (e, p)

- for the case of parabola, $\mathcal{E} = 0$ & $e = 1$
- Flight path angle, ϕ : $\tan\phi = \frac{e\sin\theta}{1+e\cos\theta}$

$h = rv\cos\phi$

- perigee : periapsis for an orbit around Earth.
- perihelion : periapsis for an orbit around Sun.

mean radius of earth

$r_{earth} = 6378.14$ KM

$\mu_{earth} = GM_1 = 398600.4 \ km^3/s^2$

$G = 6.6743 \times 10^{-11} \ m^3 kg^{-1} s^{-2}$

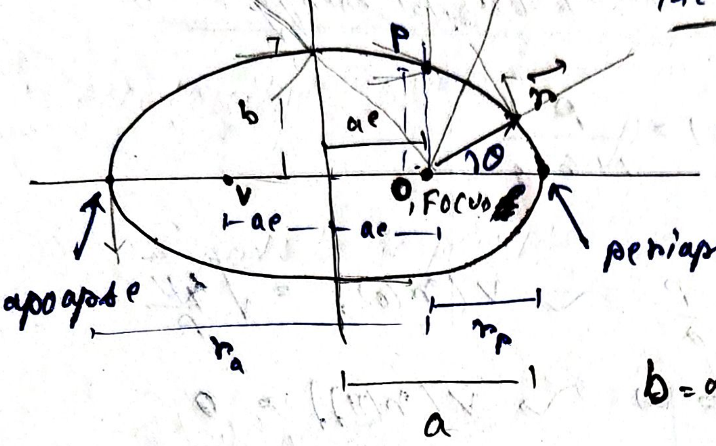
Elliptic orbit:

(orbital energy, $\epsilon < 0$)

Orbital eccentricity, $e \in [0, 1)$

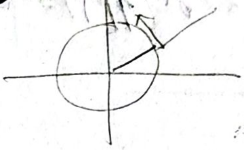
Semi major axis, a
Semi minor axis, b $\rightsquigarrow$ $b = a\sqrt{1-e^2}$

the COM of the two body system is located at one of the focus of the elliptical orbit.

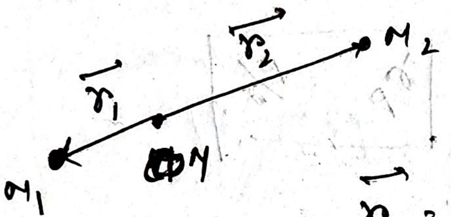


Reduced mass, $\mu = \frac{m_1 m_2}{m_1 + m_2}$

$b = a\sqrt{1-e^2}$



We can describe the motion of the two body system using the ~~orbi~~ elliptical orbit motion of an imaginary body of mass, $\mu = \frac{m_1 m_2}{m_1 + m_2}$ w.r.t. focus, F (COM of the two body system.



$$\vec{r_1} = -\frac{\mu}{m_1}\vec{r}$$

$$\vec{r_2} = \frac{\mu}{m_2}\vec{r}$$

where $\vec{r} = \vec{r_2} - \vec{r_1}$

$$\boxed{r(\theta) = \frac{p}{1 + e\cos\theta}}$$

$$\boxed{\begin{aligned} r_p &= a(1-e) \\ r_a &= a(1+e) \end{aligned}} \rightsquigarrow a = \frac{r_p + r_a}{2}$$

$$\boxed{\begin{aligned} r(\theta = 0) &= r_p = \frac{p}{1+e} \\ r(\theta = \pi) &= r_a = \frac{p}{1-e} \end{aligned}}$$

$\hookrightarrow \frac{p}{a} = 1 - e^2$

$\Rightarrow \boxed{e^2 = 1 - p/a}$

- for an elliptical orbit $p = a(1-e^2)$

Total area of ellipse, $\boxed{A = \pi a b}$

- orbital period, $\boxed{T = \frac{2\pi}{\sqrt{\mu}} a^{3/2}}$

$\hookrightarrow$ which verifies Kepler's 3rd law of planetary motion.

① Parabolic orbit:

orbital eccentricity, $e = 1$
semi major axis, $a = \infty$

Orbital energy, $\epsilon = 0$

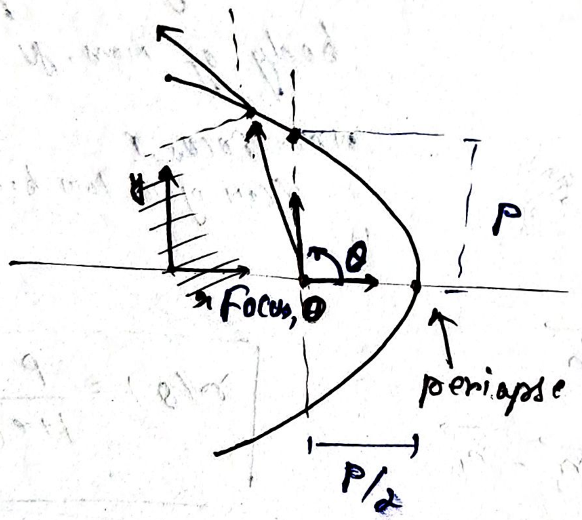
$\left(\epsilon = \frac{V^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}\right)$ it is an open orbit that can be used to escape the gravity of mass, m, with minimum orbital energy $\epsilon = 0$.

$V = \sqrt{\frac{2\mu}{r}}$

$r(\theta) = \frac{p}{1+\cos\theta}$

$r_p = r(0) = \frac{p}{2} \rightsquigarrow V(r(0)) = \sqrt{\frac{4\mu}{p}}$

$r_a = r(\pi) = \infty \rightsquigarrow V(r(\pi)) = 0$



$\boxed{r_p = p/2}$

- The vacant focus, V is at ∞

Hyperbolic Orbit:

$\boxed{\begin{array}{l} \text{orbital eccentricity, } e > 1 \\ \text{Semi major axis, } \underline{a < 0} \end{array}}$

~~Semi minor axis, $b = a\sqrt{e^2-1}$~~

orbital energy, $\epsilon > 0$

it is an open orbit that can be used to escape the gravity of mass m_1 with orbital energy

$\epsilon > 0$

$$\left(\epsilon = \frac{v^2}{2} - \frac{\mu}{r} = \frac{-\mu}{2a} \right)$$

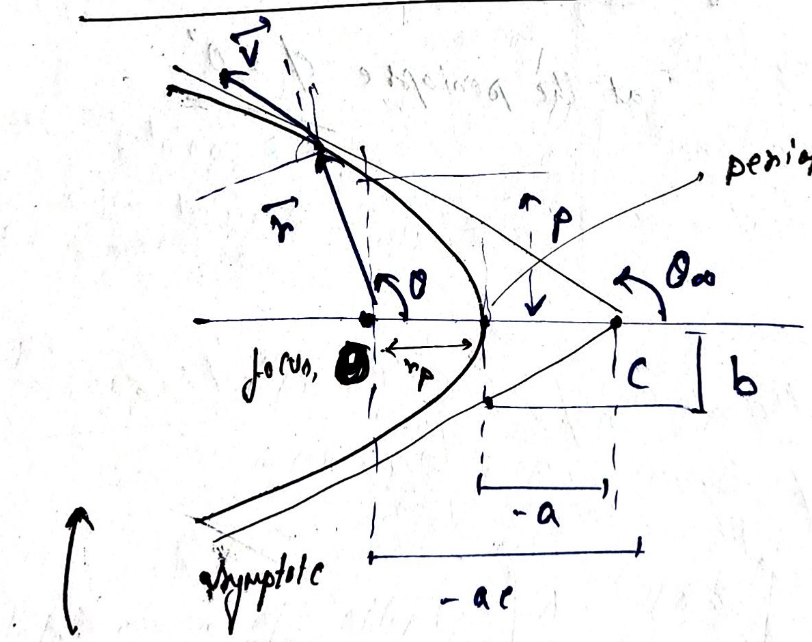
$$r(\theta) = \frac{p}{1 + e\cos\theta}$$

$r \longrightarrow \infty \Rightarrow \boxed{\theta_\infty = \cos^{-1}\left(-\frac{1}{e}\right)}$ angle of asymptotes

$\Rightarrow \boxed{V_\infty = \sqrt{\frac{-\mu}{a}}}$ hyperbolic excess velocity.

$\underline{a < 0}$

Only concave ~~branch~~ (left) branch of the hyperbola will be there!



(only the concave branch of the hyperbola constitutes the hyperbolic orbit, because gravity is an attractive force)

$$\boxed{\begin{array}{l} r_p = a(e-1) \\ b = a\sqrt{e^2-1} \end{array}}$$

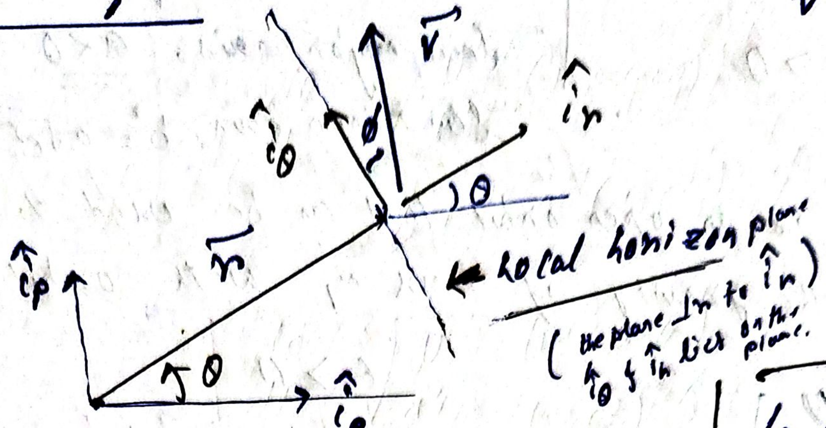
r_p is +ve

b is -ve

$$\boxed{p = a(1-e^2)}$$

Flight Path Angle:

$$\vec{V} = \dot{r}\,\hat{i}_r + r\dot{\theta}\,\hat{i}_\theta$$



$\phi \to$ Flight path angle

$$\tan\phi = \frac{\dot{r}}{r\dot{\theta}} = \frac{e\sin\theta}{1+e\cos\theta}$$

$$\vec{h} = rV\cos\phi\,\hat{i}_h$$

$$\hat{i}_h = \hat{i}_r \times \hat{i}_\theta$$

1st & 2nd quadrant

$\phi > 0$, $\theta \in [0, \pi]$

$\phi < 0$, $\theta \in [\pi, 2\pi]$

3rd & 4th quadrant

★ v imp.

$$\vec{V} = V(\sin\phi\,\hat{i}_r + \cos\phi\,\hat{i}_\theta)$$

$p = \frac{h^2}{\mu}$ $\quad h = r_p V_p = |\vec{r}_p \times \vec{V}_p|$

at the periapse $\phi = 0°$.

Lec-17 Week-12

we will discuss the time evolution of true anomaly, $\theta(t)$

using which we will be able to find $[\underset{\sim}{r}(t), \underset{\sim}{v}(t)]$ for a given $[\underset{\sim}{r}(0), \underset{\sim}{v}(0)]$

~~So far we have~~

we can find $\underset{\sim}{r}(t)$, $\underset{\sim}{v}(t)$ & $\phi(t)$ once the true anomaly $\theta(t)$ is known!

- A two body ~~trajectory~~ is completely described ~~by six scalar constants~~ which are called orbital elements

We need the knowledge of five scalar constants or integrals for the two body orbits

$\underset{\sim}{h}$, $\mathcal{E}$, e

angular momentum — orbital energy — orbital eccentricity

We need a sixth const. to decide $\theta(t)$ which will be t_0

t_0 — time of periapse

- So far we have derived orbit eqn for the shape of conic section orbits which gives variation of radius $\underset{\sim}{r}$, speed $\underset{\sim}{v}$, flight path angle ϕ as a function of θ ← true anomaly

Now we wish to find $\theta(t)$ & hence $\underset{\sim}{r}(t)$, $\underset{\sim}{v}(t)$ & $\phi(t)$

⇒ a gen approach

$$h = r^2\dot{\theta} \Rightarrow \frac{d\theta}{dt} = \frac{h}{r^2}$$

$$\& \quad r = \frac{p}{1+e\cos\theta}$$

$$p = h^2/\mu$$

$$\boxed{\frac{d\theta}{dt} = \sqrt{\frac{\mu}{p^3}}\,(1+e\cos\theta)^2}$$

for $\theta(t)$

Analytical soln exists only at
$e = 0$ → circular orbit
$e = 1$ → ~~ellip~~ parabolic orbit

① Elliptic Orbit: $\rightsquigarrow$ $0 \le e < 1$

$$\frac{d\theta}{dt} = \sqrt{\frac{\mu}{p^3}}\,(1+e\cos\theta)^2$$

$\Downarrow$ $p^2 = a^2(1-e^2)$ [valid for elliptic orbit]

$$\boxed{\frac{d\theta}{dt} = \sqrt{\frac{\mu}{a^3}}\,\frac{(1+e\cos\theta)^2}{(1-e^2)^{3/2}}} \longrightarrow (*)$$

$\rightarrow$ at periapse $\theta = 0$, $\dot{\theta}$ is max

at apoapse $\theta = \pi$, $\dot{\theta}$ is min

$\boxed{\theta(t) \leftarrow \text{True anomaly!}}$

We define $n = \sqrt{\frac{\mu}{a^3}}$

Mean motion

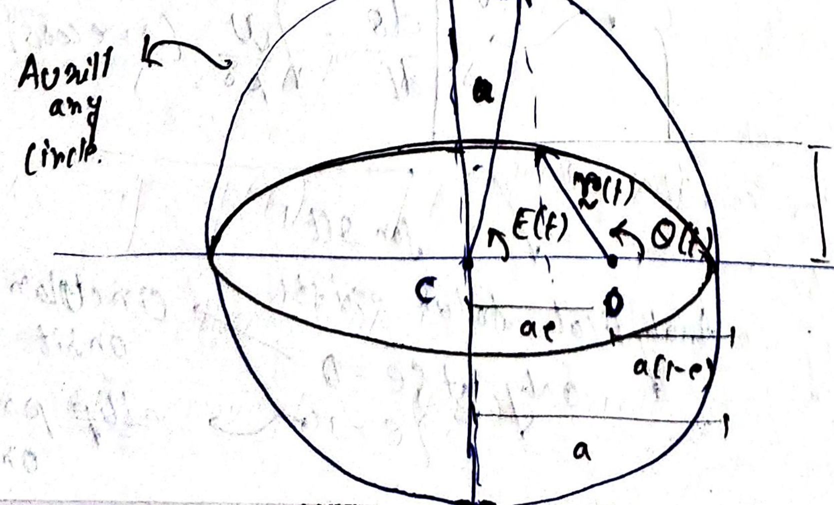
Which is the avg. angular frequency $\dot{\theta}$ over a complete orbit, as total time period is

$$T = \frac{2\pi}{\sqrt{\mu}}\,a^{3/2} = \frac{2\pi}{\sqrt{\mu/a^3}} = \frac{2\pi}{n}$$

$\boxed{\text{for } e=0 \text{ (circular orbit)}, \quad \dot{\theta} = n \text{ (const.)}}$

point P rotates with constant angular velocity n

- Now we will introduce the notion of eccentric anomaly, E as an alternative measurement of angular position $\theta(t)$ $\leftarrow$ true anomaly.



$$\boxed{\begin{aligned} r &= \frac{p}{1+e\cos\theta} \\ &\parallel \\ r &= a(1-e\cos E) \end{aligned}}$$

relation between E & θ

① $\boxed{\cos E = \frac{e+\cos\theta}{1+e\cos\theta}}$ or $\boxed{\cos\theta = \frac{\cos E - e}{1-e\cos E}}$

also for elliptic orbit $p = a(1-e^2)$, $0 \le e < 1$

vvimp $\boxed{\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}}$ ✓ → is used to determine E from θ (& vice versa) without any quadrant ambiguity, as angles $E/2$ & $\theta/2$ are always in the same quadrant !!

when $\theta = 0$, $E = 0$

$\theta = \pi$, $E = \pi$

$\theta = \pm\frac{\pi}{2}$, $E = \cos^{-1}(e)$

• differentiating ① & substituting

$$\sin E = \frac{\sin\theta\sqrt{1-e^2}}{1+e\cos\theta}$$

& using ⊛ we obtain the following :

$$(1-e\cos E)\,dE = \frac{\mu}{a^3}\,dt$$

integrating from $t=t_0$, $E=0$ results in

$$\boxed{E - e\sin E = n(t-t_0)}$$ Kepler's eqn

a transcendental eqn ⇑ if ~~required~~ has to be solved numerically ~~based~~

valid only for elliptic orbit.

t_0 : time of periapse

$M = n(t-t_0)$ where $n = \sqrt{\frac{\mu}{a^3}}$

Mean anomaly

Sidereal ~~month~~ time ~ time taken by ~~moon~~ earth to go around the Sun & return to the same starting position.

Synodic time : ~~Going~~ time taken to Go from a new moon day to the next new moon day

① **Kepler's eqn** (Valid for elliptic orbit only , $0 \leq e < 1$)

$$E - e\sin E = n(t - t_0)$$

t_0 → time of periapse

n → mean motion which is the avg. angular frequency, $\dot{\theta}$ over a complete orbit, as total time period is given by $T = \frac{2\pi}{\sqrt{\mu}} a^{3/2}$

$= \frac{2\pi}{n}$

where $n = \sqrt{\frac{\mu}{a^3}}$

E → eccentric anomaly

e → orbital eccentricity $e \in [0, 1)$

Mean anomaly,

$M = n(t - t_0)$

i.e.

$$\boxed{E - e\sin E = M}$$ → Kepler's eqn

• **Solution to Kepler's eqn**

$$f(E) \equiv E - e\sin E - M$$

we are searching for an E such that $f(E) = 0$

$f(E) = -f(-E)$

& $f(E) = 0$ has a unique solution E (one & only one) in each of the intv. $k\pi \leq E \leq (k+1)\pi$

where $k = 0, 1, 2, \ldots$

} properties of $f(E)$

① Newton's Method:

Newton gave a numerical solution procedure for finding the roots of Algebraic & transcendental equn $f(x)=0$, employing a first order Taylor series approximation of function $f(x)$ as follows:

$$f(x+\Delta x) \approx f(x) + f'(x)\,\Delta x$$ ← is step size

$f'(x)$ ↓ $\frac{df(x)}{dx}$ ⤳ it is the 1st order derivative of $f(x)$

- Over here we want to solve for E

$$f(E) \equiv E - e\sin E - M$$

- M will be given

Step 1. $E \equiv E^{guess} = M$

Step 2. find ΔE, such that $f(E+\Delta E)=0$

⇓

$$\Delta E = -\frac{f(E)}{f'(E)} = -\frac{(E - e\sin E - M)}{1 - e\cos E}$$

$$\Rightarrow \boxed{\Delta E = \frac{-E + e\sin E + M}{1 - e\cos E}}$$

Step 3. $E \leftarrow E^{guess} + \Delta E$

4. if $f(E) \equiv f(E^{guess} + \Delta E) \leq \delta$ ⤳ some fixed tolerance (which is already decided by you)

Then Stop!

else move to step 2 with $E = E^{guess} + \Delta E$ ~~& find~~

- Newton's method usually convgs. in a few iterations to a very small tolerance $|f(E)| < \delta$

- Large no. of iterations may be required when e is close to 1

Orbital eccentricity

- **NOTE:** if e is very close to 1 → we may require large no. of itn.

To reduce no. of itn. we can start with E_{given} = eccentric anomaly calculated from $e = 1$ assuming θ to be true anomaly in parabola.

(2) Parabolic orbit : $\rightarrow e = 1$. $\theta(t) \rightsquigarrow$ true anomaly

for $e=1$, $\frac{d\theta}{dt} = \sqrt{\frac{\mu}{p^3}} (1 + e\cos\theta)^2$

$\Downarrow$

$$\frac{d\theta}{dt} = \sqrt{\frac{\mu}{p^3}} (1+\cos\theta)^2 \equiv \frac{1}{4} \sec^4\frac{\theta}{2} \frac{d\theta}{dt} = \sqrt{\mu/p^3}$$

which on integration from $t = t_0$, $\theta = 0$ yields

Barker's equⁿ $\Rightarrow$

$$\boxed{\tan^3\frac{\theta}{2} + 3\tan\frac{\theta}{2} = 6\sqrt{\frac{\mu}{p^3}} (t - t_0)}$$

$\hookrightarrow$ it has unique closed form solution which is derived by substituting

$$\tan\frac{\theta}{2} = \alpha - \frac{1}{\alpha}$$

$\Downarrow$ & solving for α^3

its only real root is $\alpha = \left(C + \sqrt{C^2+1}\right)^{1/3}$

where $C = 3\sqrt{\frac{\mu}{p^3}} (t - t_0)$

$\Downarrow$

equⁿ mean anomaly

therefore true anomaly $\theta(t)$

$$\boxed{\tan\left(\frac{\theta(t)}{2}\right) = \left(C + \sqrt{1+C^2}\right)^{1/3} - \left(C + \sqrt{C^2+1}\right)^{-1/3}}$$

(3) Hyperbolic Orbit : $\rightsquigarrow$ $e>1$

Let us define hyperbolic ~~anomaly~~ anomaly, H as follows:

$$r\cos\theta = a(\cosh H - e)$$
$$r\sin\theta = b\sinh H$$

$$\underline{r = a(1 - e\cosh H)}$$

$$d\theta = \frac{\sin\theta}{\sinh H}\, dH = \frac{b}{r}\, dH$$

$$dr = -ae\sinh H\, dH$$

+

angular momentum $h = r^2\dot{\theta} = \text{Const}$

$$(e\cosh H - 1)\, dH = \sqrt{\frac{-\mu}{a^3}}\, dt$$

$\underline{a<0}$ for hyperbolic orbits

integrating from $H=0,\ t=t_0$ to $H(t),\ t$

$$\boxed{e\sinh H - H = n(t - t_0)}$$

where $n = \sqrt{\frac{-\mu}{a^3}}$

hyperbolic mean motion

equiv of Kepler's law for Hyperbolic Orbit

has to be solved numerically using Newton's method in manner similar to Kepler equi

However it will take longer time to conv.

Lec-18 Orbital Transfer

- The problem of guiding the spacecraft from an Initial position $\underset{\sim}{r_1}$ to final position $\underset{\sim}{r_2}$ is the problem of determining a trajectory that passes through two given position $\underset{\sim}{r_1}$ & $\underset{\sim}{r_2}$ by application of thrust I/ps.

The orbital transfer problem is a two point, BVP whose solution is the transfer orbit $[\underset{\sim}{r}(t), \underset{\sim}{v}(t)]$ taking spacecraft from $\underset{\sim}{r_1}$ to $\underset{\sim}{r_2}$

- two type of Orbital Transfer:

Impulsive Orbital Transfer:

Large magnitude of thrust applied for short duration resulting in an (instantaneous) change in the velocity, $\underset{\sim}{V}$ at each instant the thrust is applied.

Low thrust Orbital Transfer:

Small but continuous thrust acting on the spacecraft slowly changing its trajectory ~~with~~ w.r.t. time.

Rocket Equation :

$$(M + dM)\underset{\sim}{V} = M(\underset{\sim}{V} + d\underset{\sim}{V}) + dM(\underset{\sim}{V} + \underset{\sim}{V_e})$$

$$\Rightarrow M\,d\underset{\sim}{V} = -dM\,\underset{\sim}{V_e} \longrightarrow ⊛$$

here $\underset{\sim}{V_e}$ ~ exhaust velocity

M ~ Total mass of the rocket

$\underset{\sim}{V}$ ~ Spacecraft's velocity

for optimum exhaust direction,

$$\underset{\sim}{V_e} = -V_e \frac{\underset{\sim}{V}}{V} \longrightarrow ⊛$$

from ⊛ & ⊛

$$M\frac{dV}{dt} = -V_e\frac{dM}{dt}$$

the rate of change of Spacecraft's Mass, $\dot{M} = \frac{dM}{dt}$

$$\boxed{V(t) - V(0) = V_e \ln\left(\frac{M_0}{M(t)}\right)}$$

assuming V_e to be const.

Rocket eqn

- Specific impulse, $I_{sp} = \frac{V_e}{g_0}$ where $g_0 = 9.81\ ms^{-2}$

So, $V(t) - V(0) = I_{sp}\, g_0 \ln\left(\frac{M_0}{M(t)}\right)$

is a measure of a rocket's efficiency.

For an ideal rocket, the specific impulse is solely the property of the propellant

• Rocket can be broadly divided into two categories

ion propulsion (or plasma) rockets

mass of charged particle is accelerated through a magnetic field.

[Ion propulsion rocket is used to carry out low thrust maneuver where the small thrust is applied continuously over a long time]

NASA's Deep Space One probe!

chemical rockets:

propellant = fuel + oxidizer burned together in a combustion chamber & results in a mixture of gases exhausted in nozzle to produce thrust.

propellant	I_{sp} (s)
Cold gas	50
Hydrazine	230
Solid Aluminium perchlorate	268
Liquid Oxygen/Kerosene	304
Liquid NO_4 / MMH (mono methyl hydrazine)	313
Liquid oxygen / Liquid Hydrogen	460
Liquid Lithium/Fluorine	542

[chem. rockets apply large thrust in a relatively small duration, negligible in comparison with orbital time scale, $\frac{2\pi\sqrt{p^3}}{\sqrt{\mu}}$]

reasonable to assume that velocity change occurs instantaneously at the pt. of thrust application.

$$\Delta V = V_f - V_i = I_{sp} g_0 \ln\left(1 + \frac{\Delta M}{M_f}\right)$$

where $\Delta M = M_i - M_f$

$$\Delta M = M_f \left[e^{\Delta V / g_0 I_{sp}} - 1\right]$$

$$\Rightarrow \Delta M = M_i \left[1 - e^{-\Delta V / g_0 I_{sp}}\right]$$

• As the cost of space mission rises nearly exponentially with the required propellant mass, it is necessary to design a mission in which the sum of the magnitude of all the velocity changes is the minimum → optimal maneuver.

⊙ Impulsive Orbital Transfer:

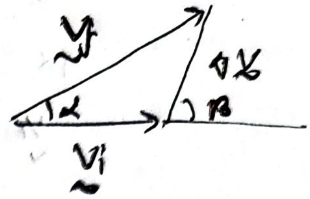
~~when the impulse is applied~~ $\cancel{\underset{\sim}{r_f} = \underset{\sim}{r_i}}$

$$\underset{\sim}{v_f} = \underset{\sim}{v_i} + \Delta \underset{\sim}{v}$$

$\uparrow$ final velocity vector (just after the impulse)

$\uparrow$ Initial velocity vector (just before the impulse)

$\uparrow$ velocity impulse. (applied thrust act as velocity impulse)



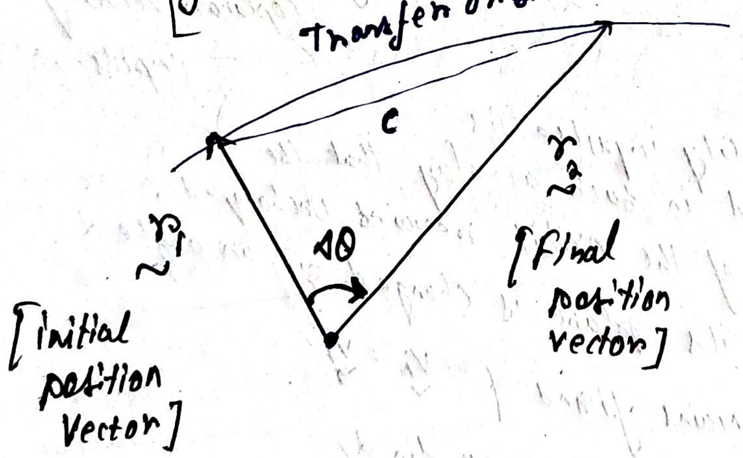
$\boxed{\underset{\sim}{r_f} = \underset{\sim}{r_i}}$

$\underset{\sim}{r_f}$: position vector, just after the impulse.

$\underset{\sim}{r_i}$: " " , just before " "

- Since approximately No time elapses during the velocity impulse, the radius is assumed to be const. While the impulse is applied i.e. $\underset{\sim}{r_f} = \underset{\sim}{r_i}$

⊙ Between any two velocity impulses, the motion around a spherical central body follows a two body orbit defined by [final radius vector $\underset{\sim}{r_f}$, final velocity vector $\underset{\sim}{v_f}$] $(\underset{\sim}{r_f}, \underset{\sim}{v_f})$ produced after the application of the last impulse.



$$\boxed{\dot{\underset{\sim}{r}} = \underset{\sim}{v}, \quad \dot{\underset{\sim}{v}} + \mu/r^3 \, \underset{\sim}{r} = \underset{\sim}{0}}$$

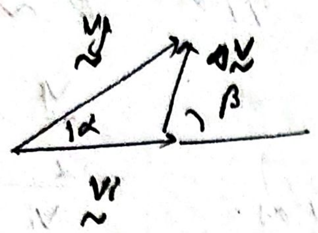
$\Delta\theta = \theta_1 - \theta_2$

True anomaly angles of the respective orbits.

- Lambert's problem: refers to the transfer of Spacecraft from radius $\underset{\sim}{r_1}(t_1)$ to $\underset{\sim}{r_2}(t_2)$ where both t_1 & t_2 are known in advance.

here

$(t_2 - t_1) = f$ (semimajor axis, a of transfer orbit , $\underset{\sim}{r_1} + \underset{\sim}{r_2}$, $\underset{\uparrow \text{chord}}{c}$, joining the two position)



$\underset{\sim}{V_f} = \underset{\sim}{V_i} + \Delta\underset{\sim}{V}$ → When $\alpha = 0$

$\underset{\sim}{V_f} = \underset{\sim}{V_i} - \Delta\underset{\sim}{V}$ → When $\alpha = \pi$

⇓

Orbital speed either ↑se or ↓se at a given point, which results in an instantaneous change in the shape of the orbit given by the orbital elements (a, e)

- Coplanar Maneuver: is when both the initial & final orbit is in the same plane.

⇓

$\Delta V = \sqrt{V_i^2 + V_f^2 - 2V_iV_f\cos\alpha}$

in such a case, the velocity impulse is applied to the plane of the orbit, & makes an angle α w.r.t the initial velocity $\underset{\sim}{V_i}$

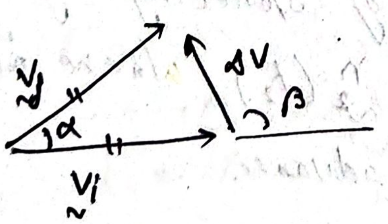
For a gen. α, there is a change in both the → orbital speed & → Flight path angle (after the application of coplanar velocity impulse)

- Plane change Maneuver: the velocity impulse is applied in such a way that the shape of the orbit remains unchanged, but its plane is changed by an angle α.

⇓

i.e. (a, e) remains fixed & $\underset{\sim}{V_i} = \underset{\sim}{V_f}$

so, $\Delta V = 2V_i \sin \alpha/2$

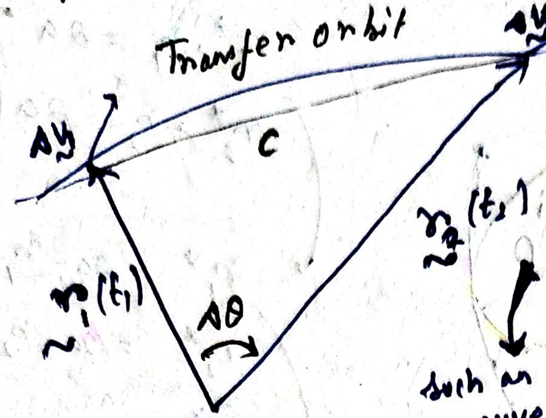


$$\frac{V_f}{\sin\beta} = \frac{\Delta V}{\sin\alpha}$$

$$\Rightarrow \frac{V_f}{\Delta V}\sin\alpha = \sin\beta, \quad \beta > \pi/2$$

Week-13 Lec-19 Optimal impulsive transfer

• The problem of guiding the spacecraft from an initial position to a desired final position is the problem of determining trajectory that passes through two given position



Consider the problem of $\underset{\sim}{r}_1$ & $\underset{\sim}{r}_2$, determining a two body orbit that passes through the two given points $\underset{\sim}{r}_1$ & $\underset{\sim}{r}_2$ by application of velocity impulses at either end.

by the application of multiple velocity impulses.

such an impulsive maneuver is called Multi-Impulse Orbital Transfer.

• If the transfer time $(t_2 - t_1)$ is specified then there is unique two body orbit given by (a, e, t_0) which passes between the two radii & is calculated using Lambert's problem.

• If transfer time $(t_2 - t_1)$ is NOT specified then there are infinitely many solutions to the transfer problem, each given by (a, e, t_0) lying in the plane formed by $\underset{\sim}{r}_1$ & $\underset{\sim}{r}_2$

To close this problem optimization is employed, resulting in impulsive orbit transfer!! — minimizing the sum of the |velocity impulses|, which leads to a unique transfer orbit called optimal transfer orbit.

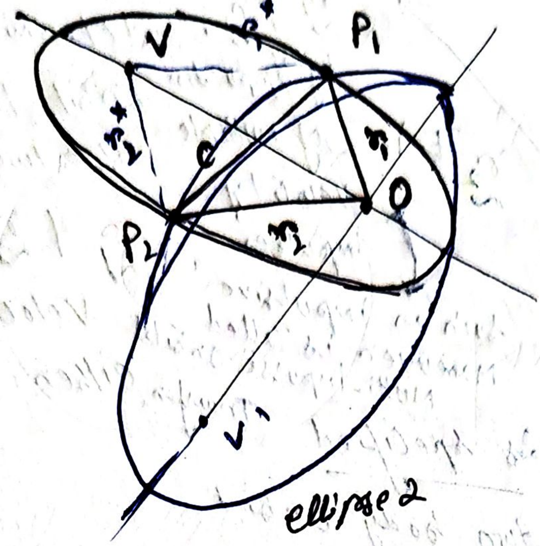
Minimum Energy Transfer:

If no constraint ~~is placed~~ on the velocity vector (v, ϕ) are specified at the end of the transfer, then the minimum energy required for transfer is determined by minimizing the orbital energy of the transfer orbit.

$$\hookrightarrow \epsilon = \frac{v^2}{2} - \frac{\mu}{r} = \frac{-\mu}{2a}$$

i.e. we need to minimize a (such that ϵ is as $-$ve as possible) provided the transfer ellipse ~~orbit~~ passes through $\underset{\sim}{r}_1$ & $\underset{\sim}{r}_2$

Let the min. value ~~of~~ of a satisfying above is $\underline{a = a_m}$.



$r_1^* + r_2^* + r_1 + r_2 = 4a$

$r_1^* + r_1 = 2a$

$r_2^* + r_2 = 2a$

$r_1^* = 2a - r_1$

$r_2^* = 2a - r_2$

virtual radius. virtual radius.

V', V : Vacant focus

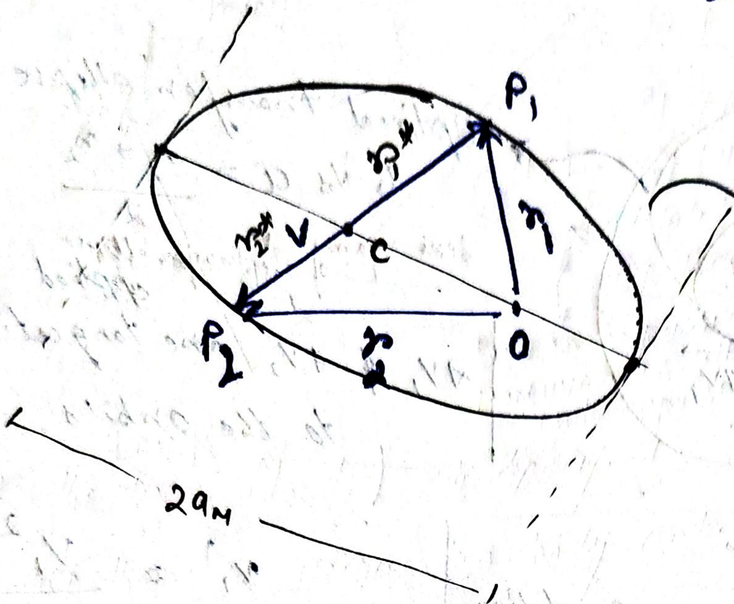
Position on ellipse
can be described from
the vacant focus V, V'
using the virtual radius, $r^* = 2a - r$

- For a given transfer Δ, $(r_1, r_2, \Delta\theta)$ of transfer semi-major axis, a there are two possible transfer ellipse $\rightarrow$ ellipse #1, ellipse #2 $\}$ of different eccentricity, e.

corresponding to
vacant focus V & V'

for $a = a_M$ $\rightsquigarrow$ when Minimum Energy Transfer takes places

$$\mathcal{E}_M = -\frac{\mu}{2a_M}$$



METO (Minimum Energy Transfer Orbit)

for $a = a_M$, V (& V') lies on the chord joining P_1 & P_2

i.e. $r_1^* + r_2^* = c$

$$r_1 + r_2 + r_1^* + r_2^* = 4a_M$$

$$\Rightarrow r_1 + r_2 = 4a_M - c$$

$$\Downarrow$$

$$\boxed{2a_M = \frac{r_1 + r_2 + c}{2}}$$

$$2a_M = s$$

$\Rightarrow$ ~~METOO~~ Major axis of METO = Semiperimeter of the transfer Δ.

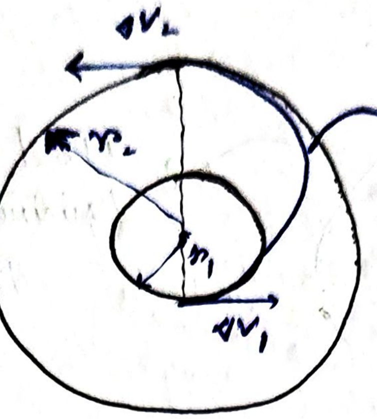
So, Minimum Energy Transfer along the ellipse between $\underset{\sim}{r_1}$ & $\underset{\sim}{r_2}$ is equivalent to a rectilinear transfer between $\underset{\sim}{r_1^*}$ & $\underset{\sim}{r_2^*}$ along the chord, for which we have $c = r_1^* + r_2^*$

$\Downarrow$ provided there is NO constraint on (V, ϕ) at both ends P_1 & P_2

Optimal Transfer between two Circular orbits : of radius r_1 & r_2

↳ called Hohmann transfer

$$\frac{V^2}{2} - \frac{\mu}{r} = \frac{-\mu}{2a}$$



Optimal transfer ellipse
↳ its $a = \frac{r_1 + r_2}{2}$
semi major axis of the transfer ellipse

ΔV_1 & ΔV_2 applied are tangential to the orbits.

" Hohmann transfer ellipse satisfies the neccessary condition for optimality

$$v_1' = \frac{V_1^2}{\sqrt{\frac{V_1^2+V_2^2}{2}}}$$

$$V_2' = \frac{v_2^2}{\sqrt{\frac{V_1^2+V_2^2}{2}}}$$

- The Hohmann transfer ~~orbit~~ time

$$= \frac{1}{2}\left(\frac{2\pi}{\sqrt{\mu}} a^{3/2}\right) = t_H$$

where $a = (r_1+r_2)/2$

$$\Delta V_1 = \sqrt{\frac{2\mu}{r_1} - \frac{\mu}{a}} - \sqrt{\frac{\mu}{r_1}}$$

$$\Delta V_2 = \sqrt{\frac{\mu}{r_2}} - \sqrt{\frac{2\mu}{r_2} - \frac{\mu}{a}}$$

- Hohmann transfer is an economical transfer only when $\frac{r_2}{r_1}$ lies in the range $(1, 15.6)$

So if $\frac{r_2}{r_1} > 15.5817$ Hohmann transfer is not the most economical transfer!

↓

In such case 3 impulse transfer
where all 3 impulses are applied tangentially is the most economical.

↙

Such a transfer is called OBE transfer
(Olden Bielliptic Transfer)

↙ a generalization of Hohmann transfer for three impulse case

- Further if waiting time for rendezvous by Hohmann transfer is large, the same rendezvous can be achieved by OBE transfer in smaller time.

The net velocity change, V_H in Hohmann transfer is

$$a = \frac{r_i + r_f}{2}$$

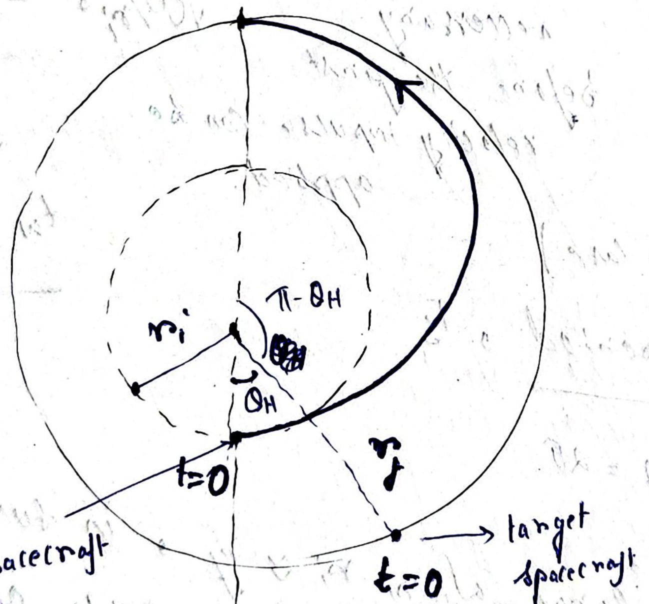
$$V_H = \Delta V_1 + \Delta V_2$$

- Time required for a Hohmann transfer, $t_H = \pi\sqrt{\frac{a^3}{\mu}} =$

⊙ "Optimal Rendezvous" in a Circular Orbit:

is defined to be the meeting of two / more spacecraft such that their positions & velocities match at a given time.

→ The optimal rendezvous between two spacecraft, which are in circular orbit of different radii around the same central body, takes place by Hohmann transfer.



Target spacecraft } orbit of in radius r_f

Other spacecraft } is in orbit of radius r_i

this spacecraft has to be maneuvered by applying two tangential velocity impulses, such that the rendezvous between spacecraft takes place at radius r_f at time $t = t_f$ on t_H

- at $t=0$
  first impulse is applied
- at $t = t_H$
  second impulse is applied

$$t_H = \frac{\pi - \theta_H}{\sqrt{\mu/r_f^3}} = \frac{\pi (r_i + r_f)^{3/2}}{\sqrt{8\mu}} \Rightarrow \theta_H = \pi\left[1 - \left(\frac{1 + r_i/r_f}{2}\right)^{3/2}\right]$$

time for Hohmann transfer

• $0 \le r_i/r_f \le 1 \Rightarrow 0 \le \theta_H \le 0.61645\pi$

• if the two orbits are **NOT** coplanar then the rendezvous pt. must be located at the intersection of the two planes, & a plane change maneuvre must be carried out at the final time, $t = t_f$.

• The Rendezvous problem requires the correct timing of the velocity impulses to be ~~provided~~ applied to the Maneuvering Spacecraft.

If the initial phase angle between the two spacecraft ~~θ_H~~ is different from the correct phase angle, θ_H, then there is a waiting time,

$\Delta\theta = \theta - \theta_H$ (angle by which TS leads MS)

$$t_w = \frac{\Delta\theta}{\sqrt{\mu/r_i^3} - \sqrt{\mu/r_f^3}}$$

necessary before the first velocity impulse can be applied.

TS : Target Spacecraft
MS : Maneuvering "

$$t_w = T_s * \frac{\Delta\theta}{2\pi}$$

When $\Delta\theta = 2\pi$ (extreme case)

$t_w \longrightarrow$ Synodic period, T_s

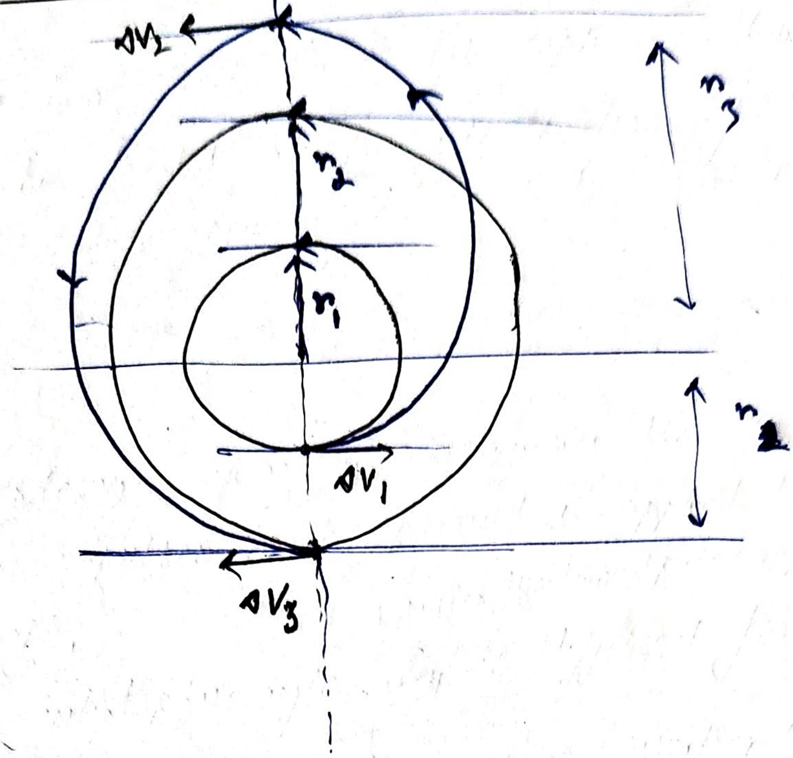
$T_s = t_w$ @ $\Delta\theta = 2\pi$

NOTE: T_s will be very large when $r_i \approx r_f$, in such a case Hohmann transfer is NO longer a viable option for Rendezvous, We need to switch to other (non-optimal) transfer trajectories, which have a reasonable waiting time.

First impulse is applied at $t = t_w$

& second impulse " " " $t = t_w + t_H \longrightarrow$ time during Hohmann transfer

· Outer Bielliptic Transfer: (OBE)



$r_3 > r_2 > r_1$

The sole design parameter for OBE is r_3.

Net velocity Impulse of the OBE transfer

$$\Delta V_B = \Delta V_1 + \Delta V_2 + \Delta V_3$$

$$\Delta V_1 = \sqrt{\frac{2\mu}{r_1} - \frac{\mu}{(r_1+r_3)/2}} - \sqrt{\frac{\mu}{r_1}}$$

$$\Delta V_3 = \sqrt{\frac{2\mu}{r_2} - \frac{\mu}{(r_2+r_3)/2}} - \sqrt{\frac{\mu}{r_2}}$$

$$\Delta V_2 = \sqrt{\frac{2\mu}{r_3} - \frac{\mu}{(r_2+r_3)/2}} - \sqrt{\frac{2\mu}{r_3} - \frac{\mu}{(r_1+r_3)/2}}$$

✓ for $r_2/r_1 > 15.5817$

OBE is more efficient than Hohman transfer.

✓ if tw for Hohmann transfer is very large, in that case using the OBE transfer can reduce the total time of transfer for Rendezvous at final radius r_2