3 Same Side Probability

Suppose we have two coins. One is fair and the other is biased where the probability of it coming up heads is $3/4$. Let's say we select a coin at random and flip it two times. What is the probability that both flips result in the same side?

For the biased coin, probability that both flips result in the same side =

$$\left(\frac{3}{4} \times \frac{3}{4}\right) + \left(\frac{1}{4} \times \frac{1}{4}\right)$$

$$= \frac{9}{16} + \frac{1}{16} = \frac{10}{16}.$$

For the fair coin, probability that both flips result in the same side =

$$\left(\frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2}\right)$$

$$= \frac{1}{2}.$$

Final probability given a random selection of coins $= \frac{1}{2} \times \frac{10}{16} + \frac{1}{2} \times \frac{1}{2}$

$$= 0.5625.$$

4 Estimated Rounds

Let's say there are six people trying to divide up into two equally separate teams. Because they want to pick random teams, on each roun each person shows their hand in either a face-up or face-down position. If there are three of each position, then they'll split into teams. What's the expected number of rounds that everyone will hav pick a hand side before they split into teams?

Since "they want to pick random teams" and there is no additional information given, we can assume that there is a 50/50 chance th each person puts a face down or face up. Thus the face of every individual person follows a Bernoulli distribution with probability success $p = 0.5$. We are looking for the total number of face up be exactly 3, as that would imply that the rest of the group h their faces down. Let F denote the number of faces that are up. F is the sum of Bernoulli random variables and as such follows a binomial distribution, meaning the probability of $F = 3$ is

$$P(F=3) = \binom{6}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^3 = 0.3125$$

Let R be the number of rounds before having 3 faces up. Clearly, follows a geometric distibution because it denotes the number of tria before one success. For geometric random variable G with success probability p, $E[G] = \frac{1}{p}$. Thus, the expected number of rounds until teams form is:

$$E[R] = \frac{1}{0.3125} = 3.2$$

5) Ad Raters

Let's say we use people to rate ads.

There are two types of raters. Random and independent from our point of view:

80% of raters are careful and they rate an ad as good (60% chance) bad (40% chance)

20% of raters are lazy and they rate every ad as good (100% chance).

Suppose we have 100 raters each rating one ad independently. What's the expected number of good ads?

Let G be the event that the ad is rated good and C be the event a rater being careful.

$$P(G) = P(C \cap G) + P(C^c \cap G) = P(G|C)\,P(G) + P(G|C^c)\,P(G)$$

$$= 0.8 \times 0.6 + 0.2 \times 1 = 0.68$$

The total number of good ads follows a binomial distribution because it looks for the results of many individual trials. In this case our parameters are numbers of trials $n = 100$ and success probability $p = 0.68$. The expected value of a binomial random variable is np, thus the expected number of ads rated good will be 68.

2. Now suppose we have 1 rater rating 100 ads. What's the expected number of good ads?

Let G_t be the total number of ads rated good.

$$E[G_t] = P(C)\,E[G_t \mid C] + P(C^c)\,E[G_t \mid C^c]$$

$$= 0.8 \times 100 \times 0.6 + 0.2 \times 100 \times 1$$

$$= 68.$$

This is the same answer as question 1, showing that from a mathematical perspective, it doesn't matter if we have one random rater rating 100 ads or have 100 random raters rating one ad.

3. Suppose we have one ad, rated as bad. What's the probability that the rater was lazy?

$$P(C^c \mid G^c) = \frac{P(G^c \cap C^c)\,P(C^c)}{P(G^c \cap C^c)\,P(C^c) + P(G^c \cap C)\,P(C)}$$

$$= \frac{0 \times 0.2}{0 \times 0.2 + 0.8 \times 0.4} = 0.$$

6) First to Six

Amy and Brad take turns in rolling a fair six-sided die. Whoever rolls a "6" first wins the game. Amy starts by rolling first. What's the probability that Amy wins?

Probability that Amy wins = P(Amy rolls 6) + P(Amy doesn't roll 6) × P(Brad doesn't roll 6) × P(Amy rolls 6) + ...

$$= \frac{1}{6} + \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} + \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} + \cdots$$

$$= \frac{1}{6} + \left(\frac{5}{6}\right)^2 \times \frac{1}{6} + \left(\frac{5}{6}\right)^4 \times \frac{1}{6} + \cdots$$

$$= \frac{\frac{1}{6}}{1 - \left(\frac{5}{6}\right)^2} = \frac{\frac{1}{6}}{1 - \frac{25}{36}} = \frac{\frac{1}{6}}{\frac{11}{36}}$$

$$= \frac{1}{6} \times \frac{36}{11}$$

$$= \frac{6}{11}$$

7 Jars and Coins

A jar holds 1000 coins. Out of all the coins, 999 are fair and one is double-sided with two heads. Picking a coin at random, you toss the coin ten times. Given that you see 10 heads, what is the probability that the coin is double headed and the probability that next toss of the coin is also a head?

~~Let F denote the event that the coin is fair and D denote the event that the coin is double headed.~~

$$P(\text{fair}) = \cancel{P(F)} = \frac{999}{1000} = 0.999, \quad P(\text{double-headed}) = \cancel{P(D)} = \frac{1}{1000} = 0.001$$

Let H denote the event of getting a head.

$$P(D \mid 10H) = \frac{P(10H \mid D) \cdot P(D)}{P(10H)}.$$

$$P(10H \mid D) = 1.$$

$$P(10H) = P(10H \mid D)P(D) + P(10H \mid F)P(F)$$

$$= 1 \times 0.001 + (0.5)^{10} \times 0.999$$

$$= \cancel{0.001 + 0.000} \; 0.0019755859\,3.$$

$$P(D \mid 10H) = \frac{1 \times 0.001}{0.00197558593} = 0.506.$$

$\cancel{P(H) = P(H|D)P(D) + P(H|F)P(F)}$

$P(\text{next toss of coin is a head}) = \cancel{P(H|D)P(D) + P(H|F)P(F)}$

$$= P(\text{Heads} \mid \text{Double-headed coin})\, P(\text{Double-headed coin}) + P(\text{Heads} \mid \text{Fair Coin})\, P(\text{Fair Coin})$$

$$= 1 \times 0.506 + ((1 - 0.506) \times 0.5$$

$$= 0.753.$$

Here we are considering that the next toss of the coin is dependent on the previou 10 tosses.

500 Cards

magine a deck of 500 cards numbered from 1 to 500. If all the ards are shuffled randomly and you are asked to pick three cards, one t a time, what's the probability of each subsequent card being larger than e previous drawn card?

Imagine this as a sample space problem ignoring all other distracting details. If you have to draw three different numbered cards without replacement, and they are all unique, then we are assuming that there will be effectively a lowest card, a middle card and a high card.

Let's make it easy and assume that we drew the numbers 1, 2 and 3. In our scenario, if we drew (1, 2, 3), then that would be the winning scenario. But what's the full range of outcomes we could draw? Let map out all the possibilities: (3, 2, 1), (3, 1, 2), (2, 1, 3), (2, 3, 1), (1, 3, 2) (1, 2, 3). So six possibilites in the total sample space. And only one of them is the partition that we want. Given this, the answer is $\frac{1}{6}$. The trick is not to be distracted by the size of the population. The order within the population does not matter if you are looking at the order within the sample.

(9) Second Ace

Let's say you have to draw two cards from a shuffled deck, one at a time. What is the probability that the second card is not an ace?

$$\text{Probability} = \underbrace{\frac{4}{52}}_{\text{first card ace}} \times \underbrace{\frac{48}{51}}_{\text{Second card not ace}} + \underbrace{\frac{48}{52}}_{\text{First card not ace}} \times \underbrace{\frac{47}{51}}_{\text{Second card not ace}} = \frac{48 \times 51}{52 \times 51} = \frac{48}{52} \approx 0.$$

10 Four Person Elevator

There are four people on the ground floor of a building that has five levels not including the ground floor. They all get into the same elevator. If each person is equally likely to get on any floor and they ave independently of each other, what is the probability that no two assengers will get off at the same floor ?

Probability that no two ~~passengers~~ persons can get off on the same floor

= Probability that each person has to get off on their own floor

$$= \frac{\text{Number of ways of assigning five floors to four different people without repetition}}{\text{Number of ways of assigning five floors to four different people}}$$

$$= \frac{5 \times 4 \times 3 \times 2}{5 \times 5 \times 5 \times 5} = 0.192$$

1 Dice Worth Rolling

et's play a game. You are given two fair six-sided dice and asked to ll them. If the sum of the values on the dice equals seven, then u win \$21. However, you must pay \$10 for each roll. Is the ame worth playing?

Let D_1 and D_2 be the result of the first and second dice respectivel
(

There are six ways that the sum $(D_1 + D_2) = 7$.

$$D_1 = 1,\ D_2 = 6$$
$$D_1 = 2,\ D_2 = 5$$
$$D_1 = 3,\ D_2 = 4$$
$$D_1 = 4,\ D_2 = 3$$
$$D_1 = 5,\ D_2 = 2$$
$$D_1 = 6,\ D_2 = 1$$

$$P(D_1 + D_2 = 7) = \frac{6}{36} = \frac{1}{6}.$$

We can use the expected value of $(D_1 + D_2)$ to get an estimate of
much we expect to gain from this game, on average. Let's call th
profit you make from this game P. The sample space of P is $\{11, -$
because we either make \$21 ~~dollars~~ by rolling a seven or lose \$10 b
(
not. Thus the possible results from the game are

$$P(P = p) = \begin{cases} 11, & \text{with probability } \frac{1}{6} \\ -10, & \text{with probability } \frac{5}{6} \end{cases}$$

Expected value of P is $E[P] = 11 \cdot \frac{1}{6} - 10 \cdot \frac{5}{6} \approx -6.5$

Since $E[P] < 0$, we on average lose money from playing this gam

Thus, this game is not worth playing.

12 Biased five out of six

let's say we are given a biased coin that comes up heads 30% of
he time when tossed. What is the probability of the coin landing
heads exactly 5 times out of 6 tosses?

$$P(5 \text{ heads}) = \binom{6}{5} (0.3)^5 (1-0.3)^{6-5} = 0.010206.$$

Three Zebras

hree zebras are chilling in the desert. Suddenly a lion attacks. Each
bra is sitting on a corner of an equilateral triangle. Each zebra
ndomly picks a direction and only runs along the outline of the triangle to
her edge of the triangle. What is the probability that none of the zebras
lide?

e only possibility in which they fail to collide is if they choose to
n in a clockwise direction or a counter-clockwise direction.

bability that zebras will run in a clockwise direction = Probability that
zebras will run in an anticlockwise direction $= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$.

Total probability $= \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$.

(14) Raining in Seattle

You are about to get on a plane to Seattle. You want to know if you should bring an umbrella. You call 3 random friends of yours who live there and ask each independently if it's raining. Each of your fri has a $\frac{2}{3}$ chance of telling you the truth and a $\frac{1}{3}$ chance of mes with you by lying. All 3 friends tell you that "Yes" it is rainin What is the probability that it's actually raining in Seattle?

$P(\text{Not Raining}) = P(\text{3 Friends Lying}) = P(\text{Friend 1 Lying})$ AND $P(\text{Friend}$

AND $(\text{Friend 3 Lyin}$

$$= \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}$$

$$P(\text{Raining}) = 1 - P(\text{Not Raining}) = 1 - \frac{1}{27} = \frac{26}{27}.$$

15 Lazy Raters

Netflix has hired people to rate movies. Out of all the raters, 80%
f the raters carefully rate movies and rate 60% of the movies as
ood and 40% as bad. The other 20% are lazy raters and rate 100%
f the movies as good. Assuming all raters rate the same amount
f movies, what is the probability that a movie is rated good?

et G be the event a movie is rated good. P(G) is the probability
the event.

rior probabilities are:

Given the rater is careful: 60% are rated good

Given the rater is lazy: 100% are rated good.

$P(G \mid \text{careful rater}) = 0.6$

$P(G \mid \text{lazy rater}) = 1.$

nt probabilities are:

$P(\text{careful rater}) = 0.8$

$P(\text{lazy rater}) = 0.2.$

$$P(G) = P(G \mid \text{careful rater}) \times P(\text{careful rater}) + P(G \mid \text{lazy rater}) \times P(\text{lazy rater})$$

$$= 0.6 \times 0.8 + 1 \times 0.2 = 0.68.$$

16 Secret Wins

There are 100 students that are playing a coin-tossing game. The students are given a coin to toss. If a student tosses the coin and it turns up heads, they win. If it comes up tails, they must flip agai

If the coin comes up heads the second time, the students will lie and say they have won when they didn't. If it comes up tails, then t will say they have lost. If 30 students at the end say they won how many students actually did win the game?

Let W denote the number of students that actually won. The question lets us know that 70 students didn't say that they won. This means that $W \leq 30$. Consider the probability of someone winning given they said they won. Assuming the coin is fair, the probability of a student winning is 0.5. But the probability of a student saying they won 0.75, since it includes the 0.25 (0.5×0.5) probability that a stud flips heads on the second try and lies about winning. Thus:

$$P(\text{Won} \mid \text{Said won}) = \frac{P(\text{Won} \cap \text{Said won})}{P(\text{Said won})}$$

$$= \frac{P(\text{Won})}{P(\text{Said won})} = \frac{0.5}{0.5 + 0.25} = \frac{2}{3}.$$

Since everyone that actually won said that they won, we can model the students who actually won as a binomial distribution with population size 30 and success probability $\frac{2}{3}$. Since if $X \sim B(n,p)$, then $E[X] = np$, our expected number of students who actually won is

$0 \times \frac{2}{3} = 20$ students.

) Marble Bucket

e have two buckets full of marbles. There are 30 red marbles and black marbles in Bucket 1 and 20 red and 20 black marbles in cket 2. Your friend secretly pulls a marble from one of the two ckets and shows you that the marble is red.

What is the probability that it was pulled from Bucket 1?

~~Probability~~ $\cancel{= \frac{1}{2} \times \frac{30}{30+10} = \frac{1}{2} \times \frac{3}{4}}$

Bucket 1 = 10 Black + 30 Red

ucket 2 = 20 Black + 20 Red.

$$P(\text{Bucket}=1 \mid \text{Marble}=\text{Red}) = \frac{P(\text{Marble}=\text{Red} \mid \text{Bucket}=1)\, P(\text{Bucket}=1)}{P(\text{Marble}=\text{Red})}$$

$$= \frac{\frac{3}{4} \times \frac{1}{2}}{\frac{5}{8}} = \frac{\frac{3}{8}}{\frac{5}{8}} = \frac{3}{5}.$$

2. Let's say your friend puts the marble back in and now picks two marbles. She draws one marble, puts it back in the same bucket, and then draws a second. They both happen to be red. What is the probability that they both came from Bucket 1?

$$P(\text{Bucket} = 2 \mid \text{Marble} = \text{Red}) = 1 - P(\text{Bucket} = 1 \mid \text{Marble} = \text{Red})$$
$$= 1 - \frac{3}{5} = \frac{2}{5}.$$

~~P(1st Red Marble = Bucket 1 AND 2nd Red Marble = Bucket 1)~~

$$P(\text{Both red marbles from Bucket } 1) = \frac{3}{5} \times \frac{3}{5} = \frac{9}{25}.$$

Coin Flip Probability

(18) Let's say you are playing a coin flip game. You start with \$ If you flip heads, you win one dollar, but if you get tails, you lose one dollar. You keep playing until you run out of money or you win \$100. What is the probability that you win \$100?

This is a variation on an old applied probability problem called The Gambler's Ruin. We invoke a probabilistic structure called a random w

The gambler can walk from \$30 to \$100 by taking numerous paths. Maybe he first walks from \$30 to \$29, or from \$30 to \$31. At each step in the walk, we can have two possible states we can to next. Having k dollars, we can only walk to (k+1) dollars or (k-1 dollars in one step.

Our problem is a special case where moving to either of the possible next two states has equal probability of 0.5. Additionally, for each step, the previous outcome has no influence on the next. That is, each flip of the coin is an independent event.

Because of these facts, we expect the gambler to have an equal number of wins and losses, thus remaining at \$30 in the long run. If we name our process X, then $E[X] = 30$.

Now, we know our walk has two possible end points and that we have 100% chance of reaching one or the other. The two end points are \$100 and \$0. Let's define the variable r as the probability of reaching \$100. Then our probability of reaching \$0 is $(1-r)$.

If Y is any discrete random variable or process Y is the expected value of the sum of the products of the values of each event with the probability of their occurrence. That is;

$$E[Y] = \sum_{y \in Y} y * p(y).$$

In our case, we have $E[X] = 30$. Our two possible events are 100 and 0 with probability of occurrence r and $(1-r)$ respectively.

$$E[X] = 100r + 0(1-r)$$

$$\Rightarrow 30 = 100r \Rightarrow r = 0.3.$$ So probability of winning is 30%.

(19) Fake Algorithm Reviews

Let's say we are trying to determine fake reviews on our products. Based on past data, 98% reviews are legitimate and 2% are fake. If a review is fake, there is a 95% chance that the machine learning algorithm identifies it as fake. If a review is legitimate, there is a 90% chance that the machine learning algorithm identifies as legitimate. What is the percentage chance that the review is actually fake when the algorithm detects it as fake?

Let L be the event that a review is actually legitimate and I the event that the review is identified as legitimate.

$P(I|L) = 0.9$

$P(L) = 0.98$ $\quad P(I^c|L^c) = 0.95$

Baye's Rule:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

$$P(L^c|I^c) = \frac{P(I^c|L^c)P(L^c)}{P(I^c|L^c)P(L^c) + P(I^c|L)P(L)} = \frac{P(I^c|L^c)P(L^c)}{P(I^c|L^c)P(L^c) + [1-P(I|L)]P(L)}$$

$$= \frac{0.95 \times 0.02}{0.95 \times 0.02 + (1-0.9)\times 0.98} =$$

(20) Probability of picking a biased coin

Suppose you have a bag of 100 coin of which 1 is biased with both sides as heads. You pick a coin from the bag and toss it three times. The result of all three tosses is heads. What is the probability that the selected coin is biased?

Let A be the event that the selected coin is biased and B be the event that the result of three tosses is heads.

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

$P(B|A) = 1.$

$P(A) = \frac{1}{100}.$

$P(A^c) = 1 - \frac{1}{100} = \frac{99}{100}$

$P(B|A^c) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}.$

$$P(A|B) = \frac{1 \times \frac{1}{100}}{1 \times \frac{1}{100} + \frac{1}{8} \times \frac{99}{100}} = \frac{\frac{1}{100}}{\frac{1}{100}\left(1 + \frac{99}{8}\right)}$$

$$= \frac{8}{107}$$