

# A Well-Balanced, Conservative Finite-Difference Algorithm for Atmospheric Flows

Debojyoti Ghosh<sup>1</sup> and Emil M. Constantinescu<sup>2</sup>  
*Argonne National Laboratory, Argonne, IL 60439*

The numerical simulation of limited-area atmospheric flows requires the solution of the Euler or the Navier-Stokes equations. Nonhydrostatic weather prediction algorithms often solve the equations in terms of derived quantities such as Exner pressure and potential temperature (and are thus not conservative) and/or as perturbations to the hydrostatically balanced equilibrium state. This paper presents a well-balanced, conservative finite-difference formulation for the Euler equations with a gravitational source term, where the governing equations are solved as conservation laws for mass, momentum, and energy. Preservation of the hydrostatic balance to machine precision on finite grids is essential since atmospheric phenomena are often small perturbations to this balance. The proposed algorithm uses the WENO and CRWENO schemes for spatial discretization that yields high-order accurate solutions for smooth flows and is essentially nonoscillatory across strong gradients; however, the well-balanced formulation may be used with other conservative finite-difference methods. The performance of the algorithm is demonstrated on test problems as well as benchmark atmospheric flow problems, and the results are verified with those in the literature.

---

<sup>1</sup> Postdoctoral Appointee, Mathematics & Computer Science Division, ghosh@mcs.anl.gov, and AIAA member

<sup>2</sup> Computational Mathematician, Mathematics & Computer Science Division, emconsta@mcs.anl.gov.

## Nomenclature

$\mathcal{A}$	= flux Jacobian matrix
$a$	= Runge-Kutta coefficients for stage calculation
$b$	= Runge-Kutta coefficients for step completion
$c$	= optimal weights for WENO scheme
$\mathcal{D}$	= discretization operator
$e$	= energy per unit mass ( $\text{J}/\text{m}^3$ )
$\mathbf{F}, \mathbf{G}$	= flux vectors along $x, y$
$\mathbf{f}$	= one-dimensional flux function (vector)
$f$	= one-dimensional flux function (scalar)
$\mathbf{g}$	= gravitational force (per unit mass) vector ( $\text{m}/\text{s}^2$ )
$g$	= gravitational force per unit mass ( $\text{m}/\text{s}^2$ )
$i, j$	= grid indices
$\mathbf{L}$	= right-hand side operator for semi-discrete ODE
$n$	= index for time integration
$p$	= pressure ( $\text{N}/\text{m}^2$ )
$\mathbf{Q}$	= Runge-Kutta stage values of $\mathbf{q}$
$\mathbf{q}$	= state vector
$R$	= universal gas constant ( $\text{J}/\text{kg K}$ )
$\mathcal{R}$	= interpolation operator
$r$	= order of spatial discretization scheme
$\mathbf{s}$	= source term
$T$	= temperature ( $\text{K}$ )
$t$	= time ( $\text{s}$ )
$\mathbf{u}$	= velocity vector ( $\text{m}/\text{s}$ )
$u, v$	= flow velocity components in $x, y$ ( $\text{m}/\text{s}$ )
$X, X^{-1}$	= matrices with right and left eigenvectors as columns and rows, respectively
$x, y$	= spatial position
$\alpha$	= nonconvex WENO weights
$\beta$	= smoothness indicators for WENO discretization
$\epsilon$	= parameter for WENO discretization

$\kappa$	= dissipation factor due to gravitational field
$\gamma$	= specific heat ratio
$\Lambda$	= diagonal matrix of eigenvalues
$\nu$	= dissipation factor in Rusanov's upwinding
$\phi$	= scalar function
$\varphi$	= pressure variation function for stratified atmosphere
$\pi$	= Exner pressure
$\sigma$	= interpolation coefficients
$\tau$	= paramter for WENO discretization
$\theta$	= potential temperature (K)
$\rho$	= density (kg/m <sup>3</sup> )
$\varrho$	= density variation function for stratified atmosphere
$\omega$	= nonlinear WENO weights

## I. Introduction

Recent decades have seen the development of several numerical algorithms for the accurate simulation of atmospheric flows. Hydrostatic models [1, 2] or simplified models that remove acoustic waves [3–5] are often used for flows with large horizontal scales (such as planetary simulations). Nonhydrostatic effects are significant when simulating mesoscale and limited-area atmospheric phenomena; thus, the solution to the compressible Euler equations is required [6–8]. Various formulations of the Euler equations have been proposed in the literature and used for operational weather prediction software [9, 10]. Algorithms that solve the Euler equations in terms of derived quantities that are relevant to atmospheric flows [11–15] (such as Exner pressure and potential temperature) do not conserve mass, momentum, and energy. Several algorithms [8, 16–20] solve the governing equations in terms of mass and momentum conservation and use the assumption of adiabaticity to simplify the energy conservation equation to the conservation of potential temperature [21]. Recent efforts solve the Euler equations as conservation laws for mass, momentum, and energy [9, 22–24], with no additional assumptions. This form of the governing equations is identical to that used in simulating compressible aerodynamic flows [25, 26]. Thus, a conservative discretization conserves

the mass, momentum, and energy to machine precision, and the true viscous terms may be specified if needed. This approach is followed here.

Several operational algorithms to simulate atmospheric flows [12, 16, 27] are based on the finite-difference method; however, they have been criticized for their low spectral resolution (due to low-order spatial discretization) and the lack of scalability [19]. These drawbacks have been addressed to some extent by high-order finite-volume [8, 17, 20, 22, 24] and finite-element or spectral-element [9, 10, 19] methods. Often, atmospheric phenomena are characterized by strong gradients, and stable solutions are obtained by using a total-variation-diminishing/bounded (TVD/TVB) discretization (such as slope-limited methods [17, 20, 24]), applying a filter [9], or adding artificial diffusion [8, 9]. On the other hand, the weighted essentially nonoscillatory (WENO) schemes [28, 29] use solution-dependent interpolation stencils to yield high-order accurate nonoscillatory solutions and have been applied successfully to several application areas [30] including compressible and incompressible fluid dynamics. The compact-reconstruction WENO (CRWENO) schemes [31] apply the solution-dependent stencil selection to compact finite-difference schemes [32] and thus have higher spectral resolution than the standard WENO schemes. The CRWENO schemes have been applied to turbulent flows [33] and aerodynamic flows [34] where the resolution of small length scales is crucial. Although the CRWENO schemes require the solution to banded systems of equations at every time-integration step or stage, a scalable implementation of the CRWENO scheme [35] demonstrated its performance for massively parallel simulations. The accuracy, spectral resolution, and scalability of the WENO and CRWENO schemes make them well suited for the simulation of atmospheric flows.

The Euler equations, with the addition of gravitational and Coriolis forces as source terms, govern the dynamics of atmospheric flows and constitute a hyperbolic balance law. This paper focuses on mesoscale and limited-area flows, and thus Coriolis forces are not included. Balance laws admit steady states where the flux derivatives are balanced by the source terms. In the context of atmospheric flows, steady states are in hydrostatic balance where the pressure gradient is counteracted by the gravitational body force. Numerical methods must be able to preserve such steady states on a finite grid to machine precision. Atmospheric phenomena are often small perturbations to the hydrostatic balance, and thus errors in balancing the discretized pressure gradient with the

gravitational forces have the potential to overwhelm the flow. One way to ensure the preservation of this balance is by subtracting the hydrostatically balanced quantities from the flow variables and expressing the equations in terms of the perturbations [8, 9, 20]. Alternatively, well-balanced discretization methods for the governing equations can be formulated that preserve the hydrostatic balance on a finite grid. Balanced finite-volume methods have been proposed [36, 37] and applied to the Euler equations with gravitational source terms [17, 22, 24]. A well-balanced, conservative finite-difference formulation for the shallow water equations was introduced [38] and extended to general balance laws [39] as well as finite-volume and discontinuous Galerkin discretizations [40]. This formulation was applied to the Euler equations with gravitational source terms [41]; however, it was derived only for the special case of an isothermal equilibrium, and other balanced equilibria or flow problems relevant to atmospheric flows were not considered.

This paper presents a high-order, well-balanced finite-difference algorithm for atmospheric flows. The governing equations are solved as conservation laws for mass, momentum, and energy. The balanced formulation of Xing and Shu [39, 41] for conservative finite-difference methods is extended to a more general form of the hydrostatic balance that includes as specific cases the isothermal equilibrium [41], as well as other examples of stratified atmosphere encountered in the literature. The fifth-order WENO and CRWENO schemes are used in this study; however, the balanced formulation may be used with other discretization schemes expressed in the conservative finite-difference form. In the absence of gravitational forces, the proposed algorithm reduces to the standard finite-difference discretization of the Euler equations. One of the motivations for this approach is to develop a unified numerical framework for both aerodynamic and atmospheric flows. Explicit, multistage Runge-Kutta methods are used for time integration; in the future, efficient semi-implicit methods [10, 18] will be explored. The ability of the algorithm to maintain the hydrostatically balanced equilibrium to machine precision is demonstrated. The algorithm is verified by solving benchmark atmospheric flow problems, and the results are compared with those obtained with two operational weather prediction solvers: Weather Research and Forecasting (WRF) [16] and Nonhydrostatic Unified Model of the Atmosphere (NUMA) [9].

The outline of the paper is as follows. Section II describes the governing equation. The numerical

method, including the well-balanced formulation is described in Sec. III. The algorithm is verified and results for benchmark flow problems are presented in Sec. IV. The Appendix contains three specific examples of the general well-balanced formulation.

## II. Governing Equations

The dynamics of atmospheric flows are governed by the Navier-Stokes equations [26], with the addition of gravitational and Coriolis forces as source terms. The effects of viscosity are insignificant, and the inviscid Euler equations [25] are solved. Various equation sets have been used in the literature [9, 10]. The algorithm proposed here solves the Euler equations stated as the conservation of mass, momentum, and energy. Mesoscale flows are considered, and Coriolis forces are neglected. The governing equations are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (1)$$

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + p \mathcal{I}_d) = -\rho \mathbf{g}, \quad (2)$$

$$\frac{\partial e}{\partial t} + \nabla \cdot (e + p) \mathbf{u} = -\rho \mathbf{g} \cdot \mathbf{u}, \quad (3)$$

where  $\rho$  is the density,  $\mathbf{u}$  is the velocity vector,  $p$  is the pressure, and  $\mathbf{g}$  is the gravitational force vector (per unit mass).  $\mathcal{I}_d$  denotes the identity matrix of size  $d$ , where  $d$  is the number of space dimensions. The energy is given by

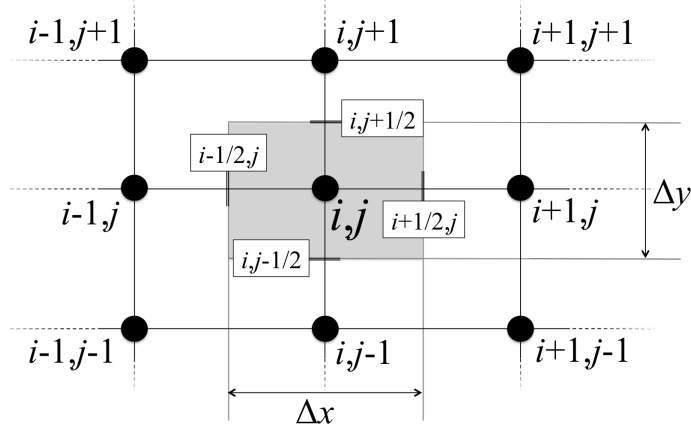
$$e = \frac{p}{\gamma - 1} + \frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u}, \quad (4)$$

where  $\gamma$  is the specific heat ratio. The equation of state relates the pressure, density, and temperature as  $p = \rho R T$ , where  $R$  is the universal gas constant and  $T$  is the temperature. Two additional quantities of interest in atmospheric flows are the Exner pressure  $\pi$  and the potential temperature  $\theta$ , defined as

$$\pi = \left( \frac{p}{p_0} \right)^{\frac{\gamma-1}{\gamma}}, \text{ and } \theta = \frac{T}{\pi}, \quad (5)$$

respectively. The pressure at a reference altitude is denoted by  $p_0$ . Equations (1)–(3) may be nondimensionalized as follows:

$$\mathbf{x}^* = \frac{\mathbf{x}}{L_\infty}, \quad \mathbf{u}^* = \frac{\mathbf{u}}{a_\infty}, \quad t^* = \frac{t}{L_\infty/a_\infty}, \quad \rho^* = \frac{\rho}{\rho_\infty}, \quad p^* = \frac{p}{\rho a_\infty^2}, \quad \mathbf{g}^* = \frac{\mathbf{g}}{a_\infty^2/L_\infty}, \quad (6)$$



**Fig. 1** Illustration of a two-dimensional Cartesian grid with the grid points (cell centers) and cell interfaces on which Eq. (10) is discretized with a conservative finite-difference formulation.

where the subscript  $\infty$  denotes reference quantities, the superscript  $*$  denotes nondimensionalized variables,  $L_\infty$  is a characteristic length scale, and  $a = \sqrt{\gamma p/\rho}$  is the speed of sound. The governing equations, expressed in terms of the nondimensionalized variables, are

$$\frac{\partial \rho^*}{\partial t^*} + \nabla \cdot (\rho^* \mathbf{u}^*) = 0, \quad (7)$$

$$\frac{\partial (\rho^* \mathbf{u}^*)}{\partial t^*} + \nabla \cdot (\rho^* \mathbf{u}^* \otimes \mathbf{u}^* + p^* \mathcal{I}_d) = -\rho^* \mathbf{g}^*, \quad (8)$$

$$\frac{\partial e^*}{\partial t^*} + \nabla \cdot (e^* + p^*) \mathbf{u}^* = -\rho^* \mathbf{g}^* \cdot \mathbf{u}^*, \quad (9)$$

where the  $\nabla$  operator now denotes derivatives with respect to the nondimensionalized vector  $\mathbf{x}^*$ . Equations (7)–(9) are identical in form to (1)–(3). Thus the same equations are used to solve both nondimensional and dimensional problems. Subsequent discussions describing the numerical discretization do not distinguish between the dimensional and nondimensional equations, and the superscript  $*$  is omitted for convenience.

### III. Numerical Method

This paper considers two-dimensional flows with gravity acting along the  $y$ -dimension; however, the numerical methodology and the well-balanced formulation can be trivially extended to three-dimensional flows. Equations (1)–(3) constitute a hyperbolic system of partial differential equations

and are discretized by a conservative finite-difference method. The governing equations can be expressed as a hyperbolic conservation law

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{q})}{\partial x} + \frac{\partial \mathbf{G}(\mathbf{q})}{\partial y} = \mathbf{s}(\mathbf{q}), \quad (10)$$

where the state vector, the flux vectors along  $x$  and  $y$ , and the source terms are

$$\mathbf{q} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ e \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ (e + p)u \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ (e + p)v \end{bmatrix}, \quad \mathbf{s} = \begin{bmatrix} 0 \\ 0 \\ -\rho g \\ -\rho v g \end{bmatrix}. \quad (11)$$

The Cartesian velocity components are  $u$  and  $v$ , and  $g$  is the gravitational force (per unit mass).

Figure 1 shows a part of a two-dimensional Cartesian grid around the grid point  $(i, j)$  whose spatial coordinates are  $(x_i = i\Delta x, y_j = j\Delta y)$ , along with the neighboring grid points and the cell interfaces.

A conservative spatial discretization [42, 43] of Eq. (10) on this grid yields a semi-discrete ordinary differential equation (ODE) in time,

$$\frac{d\mathbf{q}_{ij}}{dt} + \frac{1}{\Delta x} [\hat{\mathbf{f}}_{i+1/2,j} - \hat{\mathbf{f}}_{i-1/2,j}] + \frac{1}{\Delta y} [\hat{\mathbf{g}}_{i,j+1/2} - \hat{\mathbf{g}}_{i,j-1/2}] = \mathbf{s}_{ij}, \quad (12)$$

where  $\mathbf{q}_{ij} = \mathbf{q}(x_i, y_j)$  is the cell-centered solution,  $(x_i = i\Delta x, y_j = j\Delta y)$  are the spatial coordinates of a grid point, and  $i, j$  denote the grid indices. The numerical approximation to the flux function at the cell interfaces  $\hat{\mathbf{f}}_{i+1/2,j} = \hat{\mathbf{f}}(x_{i+1/2,j}, t)$ ,  $\hat{\mathbf{g}}_{i,j+1/2} = \hat{\mathbf{g}}(y_{i,j+1/2}, t)$  satisfy

$$\left. \frac{\partial \mathbf{F}}{\partial x} \right|_{x_i, y_j} = \frac{1}{\Delta x} [\hat{\mathbf{f}}(x_{i+1/2,j}, t) - \hat{\mathbf{f}}(x_{i-1/2,j}, t)] + O(\Delta x^r), \quad (13)$$

$$\left. \frac{\partial \mathbf{G}}{\partial y} \right|_{x_i, y_j} = \frac{1}{\Delta y} [\hat{\mathbf{g}}(y_{i,j+1/2}, t) - \hat{\mathbf{g}}(y_{i,j-1/2}, t)] + O(\Delta y^r), \quad (14)$$

for an  $r$ th-order spatial discretization method, and thus  $\hat{\mathbf{f}}$  and  $\hat{\mathbf{g}}$  are the primitives of  $\mathbf{F}$  and  $\mathbf{G}$ , respectively. The discretized source term  $\mathbf{s}_{ij}$  in Eq. (12) is expressed as its cell-centered value, and this naive treatment does not preserve the hydrostatic equilibrium [41] except in the limit  $\Delta x \rightarrow 0$ . The treatment of the source term for a well-balanced formulation is discussed in subsequent sections.

Equation (12) can be rewritten as

$$\frac{d\tilde{\mathbf{q}}}{dt} = \mathbf{L}(\tilde{\mathbf{q}}), \quad (15)$$



where  $\tilde{\mathbf{q}} = [\mathbf{q}_{ij}; 1 \leq i \leq N_i, 1 \leq j \leq N_j]$  is the entire solution vector on a grid with  $N_i \times N_j$  points and  $\mathbf{L}$  denotes the right-hand side operator comprising the discretized flux and source terms. Equation (15) is integrated with multistage explicit Runge-Kutta schemes, expressed as follows:

$$\mathbf{Q}^{(s)} = \tilde{\mathbf{q}}^n + \Delta t \sum_{t=1}^{s-1} a_{st} \mathbf{L} \left( \mathbf{Q}^{(t)} \right), \quad s = 1, \dots, S \quad (16)$$

$$\tilde{\mathbf{q}}^{n+1} = \tilde{\mathbf{q}}^n + \Delta t \sum_{s=1}^S b_s \mathbf{L} \left( \mathbf{Q}^{(s)} \right), \quad (17)$$

where  $S$  is the number of stages,  $\mathbf{Q}^{(s)}$  is the  $s$ th-stage value,  $a_{st}$  and  $b_s$  are the coefficients of the Butcher table [44], and the superscripts  $n$  and  $n+1$  denote the time levels  $t^n = n\Delta t$  and  $t^{n+1} = (n+1)\Delta t$ , respectively. The strong-stability-preserving third-order Runge-Kutta (SSPRK3) [45] and the classical fourth-order Runge-Kutta (RK4) schemes are used in this study; their Butcher tables are given by

$$\begin{array}{c|ccc} 0 & 0 & & \\ 1 & 1 & 0 & \\ 1/2 & 1/4 & 1/4 & 0 \\ \hline & 1/6 & 1/6 & 2/3 \end{array}, \quad \text{and} \quad \begin{array}{c|cccc} 0 & 0 & & & \\ 1/2 & 1/2 & 0 & & \\ 1/2 & 0 & 1/2 & 0 & \\ 1 & 0 & 0 & 1 & 0 \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}, \quad (18)$$

respectively.

### A. Reconstruction

The reconstruction step computes the numerical flux primitives  $\hat{\mathbf{f}}_{i\pm 1/2,j}$  and  $\hat{\mathbf{g}}_{i,j\pm 1/2}$  in Eq. (12) from the cell-centered values of the flux functions  $\mathbf{F}(\mathbf{q})$  and  $\mathbf{G}(\mathbf{q})$ , respectively. This section describes the approximation of a scalar, one-dimensional function primitive at the cell interface  $\hat{f}_{j+1/2} = \hat{f}(x_{j+1/2})$  from the cell-centered values of the function  $f_j = f(x_j)$ . It is extended to multiple dimensions by carrying out a one-dimensional reconstruction at the interface along the grid line normal to that interface. Vector quantities are reconstructed in a componentwise manner where the reconstruction process described below is applied to each component of the vector independently. The fifth-order WENO scheme [29] and CRWENO [31] schemes are used in this study.

The fifth-order WENO scheme (WENO5) [29] is constructed by considering three third-order accurate interpolation schemes for the numerical flux  $\hat{f}_{j+1/2}$ :

$$\hat{f}_{j+1/2}^1 = \frac{1}{3}f_{j-2} - \frac{7}{6}f_{j-1} + \frac{11}{6}f_j, \quad c_1 = \frac{1}{10}, \quad (19)$$

$$\hat{f}_{j+1/2}^2 = -\frac{1}{6}f_{j-1} + \frac{5}{6}f_j + \frac{1}{3}f_{j+1}, \quad c_2 = \frac{6}{10}, \quad (20)$$

$$\hat{f}_{j+1/2}^3 = \frac{1}{3}f_j + \frac{5}{6}f_{j+1} - \frac{1}{6}f_{j+2}, \quad c_3 = \frac{3}{10}, \quad (21)$$

where  $c_k; k = 1, 2, 3$  are the optimal coefficients. Multiplying Eqs. (19)–(21) by the corresponding  $c_k$  and taking the sum yields a fifth-order interpolation scheme,

$$\hat{f}_{j+1/2} = \frac{1}{30}f_{j-2} - \frac{13}{60}f_{j-1} + \frac{47}{60}f_j + \frac{27}{60}f_{j+1} - \frac{1}{20}f_{j+2}. \quad (22)$$

The optimal coefficients  $c_k$  are replaced by nonlinear weights ( $\omega_k, k = 1, 2, 3$ ), and the WENO5 scheme is the weighted sum of Eqs. (19)–(21) with these nonlinear weights:

$$\hat{f}_{j+1/2} = \frac{\omega_1}{3}f_{j-2} - \frac{1}{6}(7\omega_1 + \omega_2)f_{j-1} + \frac{1}{6}(11\omega_1 + 5\omega_2 + 2\omega_3)f_j + \frac{1}{6}(2\omega_2 + 5\omega_3)f_{j+1} - \frac{\omega_3}{6}f_{j+2}. \quad (23)$$

The weights are evaluated based on the smoothness of the solution [46],

$$\omega_k = \frac{\alpha_k}{\sum_k \alpha_k}; \quad \alpha_k = c_k \left[ 1 + \left( \frac{\tau}{\epsilon + \beta_k} \right)^2 \right], \quad (24)$$

where

$$\tau = (f_{j-2} - 4f_{j-1} + 6f_j - 4f_{j+1} + f_{j+2})^2. \quad (25)$$

The parameter  $\epsilon = 10^{-6}$  prevents division by zero, and the smoothness indicators ( $\beta_k$ ) are given by

$$\beta_1 = \frac{13}{12}(f_{j-2} - 2f_{j-1} + f_j)^2 + \frac{1}{4}(f_{j-2} - 4f_{j-1} + 3f_j)^2, \quad (26)$$

$$\beta_2 = \frac{13}{12}(f_{j-1} - 2f_j + f_{j+1})^2 + \frac{1}{4}(f_{j-1} - f_{j+1})^2, \quad (27)$$

$$\text{and } \beta_3 = \frac{13}{12}(f_j - 2f_{j+1} + f_{j+2})^2 + \frac{1}{4}(3f_j - 4f_{j+1} + f_{j+2})^2. \quad (28)$$

Other definitions for the nonlinear weights exist in the literature [47–49] as well as a comparison of the nonlinear properties of the WENO5 scheme with these weights [33]. When the solution is smooth, the nonlinear weights converge to the optimal coefficients ( $\omega_k \rightarrow c_k$ ), and Eq. (23) reduces to Eq. (22). The scheme is fifth-order accurate for such solutions. Across and near discontinuities,

the weights corresponding to the stencil containing the discontinuity approach zero, and Eq. (23) represents an interpolation scheme with its stencil biased away from the discontinuity. Nonoscillatory solutions are thus obtained.

The fifth-order CRWENO scheme (CRWENO5) [31] is similarly constructed by considering three third-order accurate compact interpolation schemes [32] for the numerical flux  $\hat{f}_{j+1/2}$ :

$$\frac{2}{3}\hat{f}_{j-1/2}^1 + \frac{1}{3}\hat{f}_{j+1/2}^1 = \frac{1}{6}(f_{j-1} + 5f_j); \quad c_1 = \frac{2}{10}, \quad (29)$$

$$\frac{1}{3}\hat{f}_{j-1/2}^2 + \frac{2}{3}\hat{f}_{j+1/2}^2 = \frac{1}{6}(5f_j + f_{j+1}); \quad c_2 = \frac{5}{10}, \quad (30)$$

$$\frac{2}{3}\hat{f}_{j+1/2}^3 + \frac{1}{3}\hat{f}_{j+3/2}^3 = \frac{1}{6}(f_j + 5f_{j+1}); \quad c_3 = \frac{3}{10}, \quad (31)$$

where  $c_k, k = 1, 2, 3$  are the optimal coefficients. A fifth-order compact scheme is obtained by multiplying Eqs. (30)–(31) by their corresponding optimal coefficient  $c_k$  and adding

$$\frac{3}{10}\hat{f}_{j-1/2} + \frac{6}{10}\hat{f}_{j+1/2} + \frac{1}{10}\hat{f}_{j+3/2} = \frac{1}{30}f_{j-1} + \frac{19}{30}f_j + \frac{1}{3}f_{j+1}. \quad (32)$$

The CRWENO5 scheme is constructed by replacing the optimal coefficients  $c_k$  by nonlinear weights  $\omega_k$  and can be expressed as

$$\begin{aligned} \left(\frac{2}{3}\omega_1 + \frac{1}{3}\omega_2\right)\hat{f}_{j-1/2} + \left[\frac{1}{3}\omega_1 + \frac{2}{3}(\omega_2 + \omega_3)\right]\hat{f}_{j+1/2} + \frac{1}{3}\omega_3\hat{f}_{j+3/2} \\ = \frac{\omega_1}{6}f_{j-1} + \frac{5(\omega_1 + \omega_2) + \omega_3}{6}f_j + \frac{\omega_2 + 5\omega_3}{6}f_{j+1}. \end{aligned} \quad (33)$$

The weights  $\omega_k$  are computed by Eq. (24) and Eqs. (26)–(28). The resulting scheme Eq. (33) is fifth-order accurate when the solution is smooth ( $\omega_k \rightarrow c_k$ ) and reduces to Eq. (32). Across and near discontinuities, the weights corresponding to the stencils containing the discontinuity approach zero, and a biased (away from the discontinuity) compact scheme is obtained. Equation (33) results in a tridiagonal system of equations that must be solved at each time-integration step or stage; however, the additional expense is justified by the higher accuracy and spectral resolution of the compact scheme [31, 33, 34]. An efficient and scalable implementation of the CRWENO5 scheme was recently proposed in [35, 50] and is used in this study.

The solution of a hyperbolic system is composed of waves propagating at their characteristic speeds along their characteristic directions, and thus the final flux at the interface is an appropriate combination of the left- and right-biased fluxes. The description of the WENO5 and CRWENO5

schemes above considered a left-biased reconstruction of the scalar flux; the corresponding right-biased reconstruction can be obtained by reflecting the expressions around the interface  $j + 1/2$ . The Roe [51, 52] and the Rusanov [52, 53] upwinding schemes are used in this study. The Roe scheme is expressed as

$$\hat{\mathbf{f}}_{j+1/2} = \frac{1}{2} \left[ \hat{\mathbf{f}}_{j+1/2}^L + \hat{\mathbf{f}}_{j+1/2}^R - \left| \mathcal{A} \left( \hat{\mathbf{q}}_{j+1/2}^L, \hat{\mathbf{q}}_{j+1/2}^R \right) \right| \left( \hat{\mathbf{q}}_{j+1/2}^R - \hat{\mathbf{q}}_{j+1/2}^L \right) \right], \quad (34)$$

where

$$\left| \mathcal{A} \left( \hat{\mathbf{q}}_{j+1/2}^L, \hat{\mathbf{q}}_{j+1/2}^R \right) \right| = X_{j+1/2} \left| \Lambda_{j+1/2} \right| X_{j+1/2}^{-1}. \quad (35)$$

The eigenvalues  $\Lambda$  and the eigenvectors  $X, X^{-1}$  are evaluated at the interface from the Roe-averaged flow variables. The superscripts  $L$  and  $R$  indicate the left- and right-biased interpolations, respectively. The Rusanov scheme is given by

$$\hat{\mathbf{f}}_{j+1/2} = \frac{1}{2} \left[ \hat{\mathbf{f}}_{j+1/2}^L + \hat{\mathbf{f}}_{j+1/2}^R - \left( \max_{j,j+1} \nu \right) \left( \hat{\mathbf{q}}_{j+1/2}^R - \hat{\mathbf{q}}_{j+1/2}^L \right) \right]. \quad (36)$$

The dissipation factor is  $\nu = a + |\mathbf{u}|$ , where  $a$  is the speed of sound and  $\mathbf{u}$  is the flow velocity. We note that  $\hat{\mathbf{q}}_{j+1/2}^{L,R}$  in Eqs. (34) and (36) are the left- and right-biased interface values for  $\mathbf{q}$  that are reconstructed in the same manner as  $\hat{\mathbf{f}}_{j+1/2}^{L,R}$ .

## B. Well-Balanced Formulation

A hyperbolic balance law, such as Eq. (10), admits steady-state solutions where the flux derivative is exactly balanced by the source term. For atmospheric flows, the gravitational force on the fluid is balanced by the pressure gradient, resulting in the hydrostatic balance. The numerical algorithm must preserve this balance to machine precision because errors have the potential to overwhelm physically relevant perturbations to the balance. In this section, a well-balanced formulation of the finite-difference discretization of Eq. (10) is presented; this formulation reduces to the balanced discretization previously proposed [41] for the specific case of an isothermal hydrostatic balance. Although the formulation is described for two-dimensional flows with gravity acting along the  $y$ -dimension, it can be easily extended to three dimensions and for domains where the gravity may not be aligned with a specific dimension.

Steady atmospheric flow in hydrostatic balance can be expressed in the following general form:

$$u = \text{constant}, v = 0, \rho = \rho_0 \varrho(y), p = p_0 \varphi(y), \quad (37)$$

where the subscript 0 indicates the flow variables at a reference altitude. The pressure and density at the reference altitude are related by the equation of state  $p_0 = \rho_0 R T_0$ . At equilibrium, Eq. (10) reduces to

$$\frac{dp}{dy} = -\rho g. \quad (38)$$

Therefore, by substituting Eq. (37) in Eq. (38) and considering the equation of state, the functions  $\varrho(y)$  and  $\varphi(y)$  satisfy

$$R T_0 [\varrho(y)]^{-1} \varphi'(y) = -g. \quad (39)$$

The necessity of a well-balanced algorithm can be explained as follows: Let a general, linear finite-difference approximation to the derivative of an arbitrary function  $\phi(y)$  be expressed as follows:

$$\left. \frac{\partial \phi}{\partial y} \right|_{y=y_j} \approx \mathcal{D}[\phi] \equiv \frac{1}{\Delta y} \sum_{k=-m}^n \sigma_k^{\mathcal{D}} \phi_{j+k}, \quad (40)$$

where  $m$  and  $n$  are integers defining the stencil  $[j-m, j-m+1, \dots, j+n-1, j+n]$  of the finite-difference operator  $\mathcal{D}$ , and  $\sigma_k^{\mathcal{D}}$  are the stencil coefficients. With this notation, the discretized form of Eq. (38) at a grid point can be written as

$$\mathcal{D}[p]_j = -(\rho g)_j, \quad (41)$$

where the subscript  $j$  indicates the corresponding terms evaluated at the  $j$ th grid point. If  $\mathcal{D}$  is a consistent finite-difference operator, Eq. (41) is exactly satisfied as  $\Delta x \rightarrow 0$ . On a finite grid with  $\Delta x \neq 0$ , however, the error in satisfying Eq. (41) is nonzero and is related to the spatial discretization error of the finite-difference operator  $\mathcal{D}$ .

A well-balanced algorithm must satisfy Eq. (38) in its discretized form on a finite grid ( $\Delta x \neq 0$ ) as well; thus, the discretized flux derivative must exactly balance the discretized source term. The first step modifies Eq. (10) as

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{q})}{\partial x} + \frac{\partial \mathbf{G}(\mathbf{q})}{\partial y} = \mathbf{s}^*(\mathbf{q}, y), \quad (42)$$

where  $\mathbf{s}^* = \left[0, 0, \rho RT_0 [\varrho(y)]^{-1} \varphi'(y), \rho v RT_0 [\varrho(y)]^{-1} \varphi'(y)\right]^T$ . The relationship between  $\varrho(y)$  and  $\varphi(y)$ , given by Eq. (39), ensures that Eq. (42) is consistent with Eq. (10). The source terms are thus rendered in a form similar to that of the flux term [39, 41]. With this modification, Eq. (38) can be rewritten as

$$\frac{dp}{dy} = \rho RT_0 [\varrho(y)]^{-1} \varphi'(y). \quad (43)$$

It can then be discretized by using the notation in Eq. (40) to yield

$$\mathcal{D}_{\mathbf{G}}[p] - \rho RT_0 \{\varrho(y)\}^{-1} \mathcal{D}_{\mathbf{s}^*}[\varphi(y)] = 0, \quad (44)$$

where  $\mathcal{D}_{\mathbf{G}}$  and  $\mathcal{D}_{\mathbf{s}^*}$  are the finite-difference operators used to approximate the  $y$ -derivatives of the flux function  $\mathbf{G}$  and the source term  $\mathbf{s}^*$ , respectively. Although Eq. (44) holds true for  $\Delta x \rightarrow 0$  if  $\mathcal{D}_{\mathbf{G}}$  and  $\mathcal{D}_{\mathbf{s}^*}$  are both consistent finite-difference operators, it is not exactly satisfied for  $\Delta x \neq 0$  with the error being related to the spatial discretization errors of  $\mathcal{D}_{\mathbf{G}}$  and  $\mathcal{D}_{\mathbf{s}^*}$ . However, Eq. (44) is exactly satisfied for  $\Delta x \neq 0$  if

$$\mathcal{D}_{\mathbf{G}} = \mathcal{D}_{\mathbf{s}^*} = \mathcal{D}. \quad (45)$$

After substituting Eq. (45) and exploiting the linearity of  $\mathcal{D}$ , the left-hand side of Eq. (44) reduces to

$$\mathcal{D} \left[ p - \rho RT_0 \{\varrho(y)\}^{-1} \varphi(y) \right] = \mathcal{D} \left[ p_0 \varphi(y) - \rho_0 \varrho(y) RT_0 \{\varrho(y)\}^{-1} \varphi(y) \right] = 0. \quad (46)$$

Equation (45) implies that a linear finite-difference algorithm to solve Eq. (42) is well balanced (preserves hydrostatically balanced steady states to machine precision) if the flux derivative and the modified source terms are discretized by the same linear operator.

The WENO5 and CRWENO5 schemes are nonlinear finite-difference operators since their coefficients are solution dependent. Additional steps are required to construct a well-balanced algorithm. The discretization of the flux derivative along  $y$  with the WENO5 or CRWENO5 schemes can be

summarized as follows:

$$\hat{\mathbf{G}}_{j+1/2}^{L,R} = \mathcal{R}_{\mathbf{G}}^{L,R}[\mathbf{G}] \equiv \sum_{k=-m}^n \hat{\sigma}_k(\omega) \mathbf{G}_{j+k}, \quad (47)$$

$$\hat{\mathbf{G}}_{j+1/2} = \frac{1}{2} \left[ \hat{\mathbf{G}}_{j+1/2}^L + \hat{\mathbf{G}}_{j+1/2}^R + |\mathcal{A}_{j+1/2}| \left( \hat{\mathbf{q}}_{j+1/2}^L - \hat{\mathbf{q}}_{j+1/2}^R \right) \right] \quad (\text{Roe}) \quad (48)$$

$$\text{or } \hat{\mathbf{G}}_{j+1/2} = \frac{1}{2} \left[ \hat{\mathbf{G}}_{j+1/2}^L + \hat{\mathbf{G}}_{j+1/2}^R + \left( \max_{j,j+1} \nu \right) \left( \hat{\mathbf{q}}_{j+1/2}^L - \hat{\mathbf{q}}_{j+1/2}^R \right) \right] \quad (\text{Rusanov}), \quad (49)$$

$$\left. \frac{\partial \mathbf{G}}{\partial y} \right|_{y=y_j} \approx \frac{1}{\Delta y} \left[ \hat{\mathbf{G}}_{j+1/2} - \hat{\mathbf{G}}_{j-1/2} \right], \quad (50)$$

where  $j$  is the grid index along the  $y$ -coordinate (the index along the  $x$ -coordinate is suppressed for convenience of notation),  $\mathcal{R}_{\mathbf{G}}^{L,R}$  are the reconstruction operators with  $m$  and  $n$  defining the stencil bounds, and the coefficients for the stencil points  $\hat{\sigma}_k$  are functions of the solution-dependent nonlinear weights  $\omega$ . Equations (23) and (33) (representing the WENO5 and CRWENO5 schemes) can be represented through this operator. The subscript  $\mathbf{G}$  denotes that the nonlinear weights ( $\omega$ ) are computed based on  $\mathbf{G}(\mathbf{q})$ . The superscripts  $L$  and  $R$  denote the left- and right-biased operators, respectively.

The derivative of  $\varphi(y)$  in the source term of Eq. (42) is discretized in the same manner as the flux derivative, summarized as follows:

$$\hat{\varphi}_{j+1/2}^{L,R} = \mathcal{R}_{\mathbf{G}}^{L,R}[\varphi] \equiv \sum_{k=-m}^n \hat{\sigma}_k \varphi_{j+k}, \quad (51)$$

$$\hat{\varphi}_{j+1/2} = \frac{1}{2} \left[ \hat{\varphi}_{j+1/2}^L + \hat{\varphi}_{j+1/2}^R \right], \quad (52)$$

$$\left. \frac{\partial \varphi}{\partial y} \right|_{y=y_j} \approx \frac{1}{\Delta y} \left[ \hat{\varphi}_{j+1/2} - \hat{\varphi}_{j-1/2} \right], \quad (53)$$

where the vector  $\varphi$  is simply given by  $\varphi = [0, 0, \varphi(y), \varphi(y)]^T$ . The remaining terms in the source  $\mathbf{s}^*$  are evaluated at the cell center  $j$ . The interpolation operators in Eqs. (47) and (51) are both  $\mathcal{R}_{\mathbf{G}}$ ; the interface values of both  $\mathbf{G}$  and  $\varphi$  are computed with the same interpolation operator. This is achieved in the WENO5 and CRWENO5 schemes by calculating the weights based on the smoothness of the flux function  $\mathbf{G}(\mathbf{q})$  and using these weights to compute the interface values of both  $\mathbf{G}$  and  $\varphi$  at a given time-integration step or stage.

The final step in the construction of a well-balanced method is the suitable modification of the dissipation term in the upwinding step. The Roe and Rusanov schemes, given by Eqs. (48) and (49),

are modified as follows:

$$\hat{\mathbf{G}}_{j+1/2} = \frac{1}{2} \left[ \hat{\mathbf{G}}_{j+1/2}^L + \hat{\mathbf{G}}_{j+1/2}^R + \kappa |\mathcal{A}_{j+1/2}| \left( \hat{\mathbf{q}}_{j+1/2}^{*,L} - \hat{\mathbf{q}}_{j+1/2}^{*,R} \right) \right] \quad (\text{Roe}), \quad (54)$$

$$\hat{\mathbf{G}}_{j+1/2} = \frac{1}{2} \left[ \hat{\mathbf{G}}_{j+1/2}^L + \hat{\mathbf{G}}_{j+1/2}^R + \kappa \left( \max_{j,j+1} \nu \right) \left( \hat{\mathbf{q}}_{j+1/2}^{*,L} - \hat{\mathbf{q}}_{j+1/2}^{*,R} \right) \right] \quad (\text{Rusanov}), \quad (55)$$

where  $\kappa = \max_{j,j+1} \varphi(y)$ ,  $\hat{\mathbf{q}}_{j+1/2}^{*,L}$  and  $\hat{\mathbf{q}}_{j+1/2}^{*,R}$  respectively are the left- and right-biased interpolation (at the interface) of a modified state vector  $\mathbf{q}^* = \left[ \rho \{ \varrho(y) \}^{-1}, \rho u \{ \varrho(y) \}^{-1}, \rho v \{ \varrho(y) \}^{-1}, e^* \right]^T$ . The modified energy  $e^*$  is given by

$$e^* = \frac{p \{ \varphi(y) \}^{-1}}{\gamma - 1} + \frac{1}{2} \rho \{ \varrho(y) \}^{-1} (u^2 + v^2). \quad (56)$$

At steady state,  $\mathbf{q}^*$  is a constant, and the dissipation term in Eqs. (54) and (55) is zero with this modification ( $\hat{\mathbf{q}}_{j+1/2}^{*,L} = \hat{\mathbf{q}}_{j+1/2}^{*,R}$ ).

Discretization of the flux derivatives in Eq. (42) by Eqs. (47), (54), or (55), and (50) and evaluation of the source term as Eqs. (51), (52), and (53) result in the following discretized form of Eq. (43) at grid point  $j$ :

$$p_0 \left[ \frac{\hat{\varphi}_{j+1/2} - \hat{\varphi}_{j-1/2}}{\Delta y} \right] = \rho_j R T_0 \{ \varrho(y_j) \}^{-1} \left[ \frac{\hat{\varphi}_{j+1/2} - \hat{\varphi}_{j-1/2}}{\Delta y} \right], \quad (57)$$

where the  $\hat{\varphi}_{j+1/2}$  denotes the interface approximation of the scalar function  $\varphi(y)$ . It is evaluated on the left-hand side through Eq. (47), and Eq. (54) or (55), and is evaluated on the right-hand side through Eq. (51) and Eq. (52). Equation (57) is exactly satisfied if the operator  $\mathcal{R}_{\mathbf{G}}$  is linear. Although  $\mathcal{R}_{\mathbf{G}}$  represents nonlinear finite-difference operators Eqs. (23) and (33), the nonlinearity of these schemes arises from the solution-dependent weights  $\omega_k$ . Within a time-integration step or stage, these weights are computed and fixed, and the operator  $\mathcal{R}_{\mathbf{G}}$  is essentially linear. Therefore, Eq. (57) is exactly satisfied.

*Summary* The steps to construct a well-balanced conservative finite-difference algorithm are summarized as follows:

1. The governing equations are modified as Eq. (42).
2. The flux derivatives are computed through Eqs. (47), (54) or (55), and (50).
3. The derivatives in the modified source term are computed through Eqs. (51), (52), and (53) (the remaining terms are evaluated at the cell centers).



The resulting algorithm preserves a hydrostatically balanced steady state to machine precision. Three examples of the steady state, Eq. (37), that occur in atmospheric flow problems are presented in the Appendix, as well as the resulting expressions for the modified source term in Eq. (42) and modified solution in Eqs. (54) and (55). If gravitational forces are not present ( $g = 0 \Rightarrow \varrho(y) = \varphi(y) = 1$ ), the discretization of the hyperbolic flux reduces to the standard conservative finite-difference discretization.

#### IV. Verification and Results

This section demonstrates the performance of the numerical algorithm and verifies the computed results with those in the literature. The ability of the algorithm to preserve the hydrostatic balance to machine precision and to accurately capture small perturbations is demonstrated. Comparisons are made with a naive discretization of the source term to show the necessity for the well-balanced formulation. Further, benchmark atmospheric flow problems are solved, and the results obtained agree with those obtained with solvers based on other forms of the governing equations and using different discretization techniques. Both dimensional and nondimensional problems are considered; the descriptions of the former have the relevant units specified. One-dimensional problems in  $y$  are solved by specifying an arbitrary domain size and periodic boundary conditions along  $x$  (the number of grid points in  $x$  is taken as the minimum number of ghost points required to implement the boundary treatment).

##### A. Well-Balanced Tests

A one-dimensional problem is initially considered to demonstrate the need for a well-balanced discretization. The initial flow is static and in hydrostatic balance. An isothermal equilibrium is considered, and a sinusoidal gravitational field potential [41, 54]

$$\phi = -\frac{1}{2\pi} \sin(2\pi y) \quad (58)$$

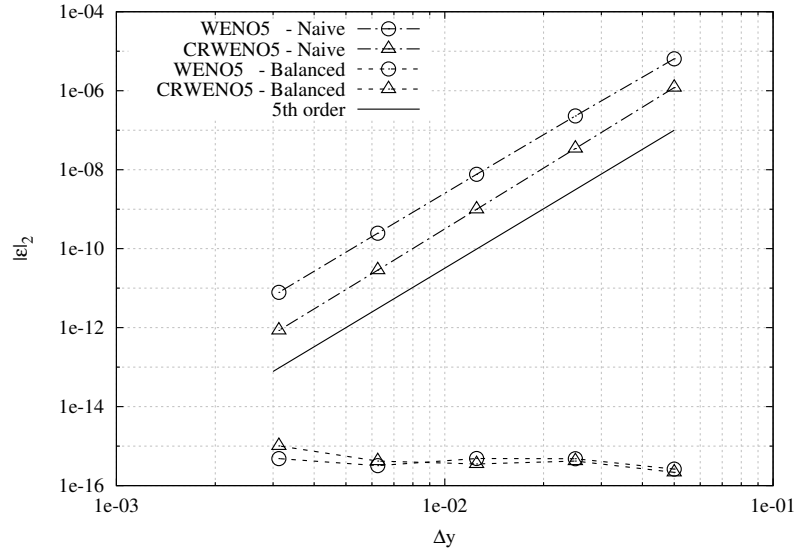
is specified on a periodic domain  $0 \leq y < 1$ . The gravitational force in Eq. (10) is  $g(y) = \phi_y$  (where the subscript denotes the derivative), and the steady state is given by

$$\rho = \exp(-\phi), \quad p = \exp(-\phi). \quad (59)$$

The specific heat ratio is  $\gamma = 1.4$ , and the solutions are obtained at a final time of 1.0. This problem is solved with the balanced discretization, as well as with a naive discretization that can be described by Eq. (12), where the source term is discretized as  $\mathbf{s}_{ij} = [0, 0, -(\rho g)_{ij}, -(\rho v g)_{ij}]^T$  and the flux terms are discretized by the standard WENO5 and CRWENO5 schemes. The Roe scheme is used for upwinding (Eq. (34) for the naive algorithm and Eq. (54) for the balanced algorithm). Since the problem is steady, the error in the final solution is defined as

$$\|\epsilon\|_{(\cdot)} = \frac{\|\mathbf{q}(\mathbf{x}, t_f) - \mathbf{q}(\mathbf{x}, 0)\|_{(\cdot)}}{\|\mathbf{q}(\mathbf{x}, 0)\|_{(\cdot)}}, \quad (60)$$

where  $t_f$  denotes the final time. Figure 2 shows the  $L_2$  norm of the error as a function of the grid resolution  $\Delta y$  for the solutions obtained with the balanced and naive implementations. The RK4 method is used for time integration with a time-step size of  $10^{-3}$  for the cases. The naive implementation results in a non-zero error for all the grid sizes, which shows that it is unable to preserve the hydrostatically balanced equilibrium. In addition, the error in preserving the equilibrium shows fifth-order convergence as the grid is refined for both the WENO5 and CRWENO5 schemes. The errors for the CRWENO5 schemes are an order of magnitude lower than those of the WENO5 scheme at all grid resolutions. These observations are consistent with the numerical properties of



**Fig. 2** Relative error ( $L_2$  norm) with respect to the initial solution for a steady-state problem with naive and balanced implementations of the algorithm.

these schemes [31]. Therefore, the error in preserving the hydrostatic balance is indeed the spatial discretization error of the algorithm, as discussed regarding Eq. (41). The balanced implementation preserves the steady state to machine precision for both the WENO5 and CRWENO5 schemes.

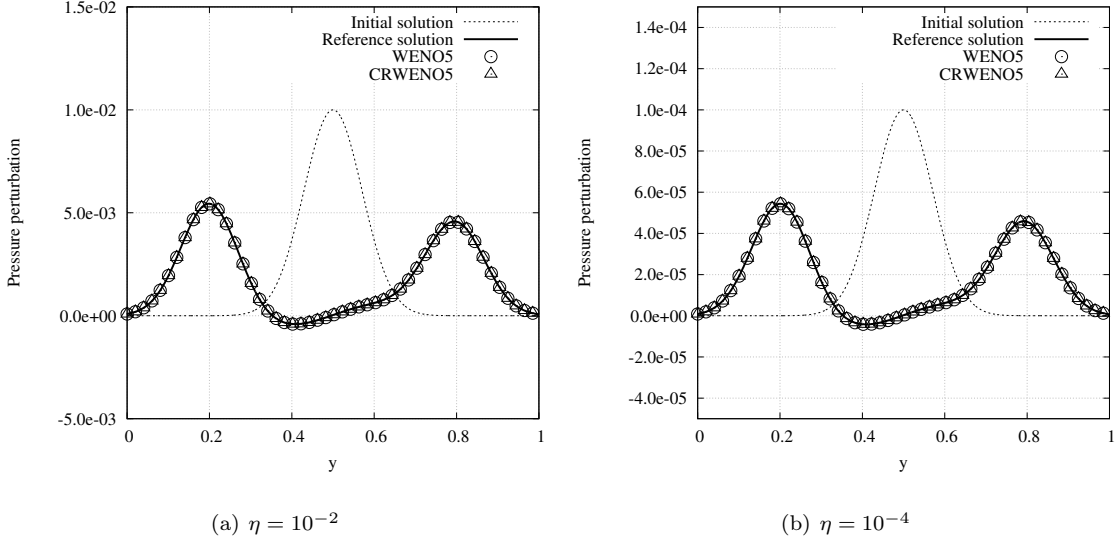
The ability of the proposed formulation to preserve two examples of hydrostatic balance encountered in atmospheric flows is tested. These correspond to the second and third cases in the Appendix; the specific flow conditions are described below. The initial solution is specified as the balanced steady-state flow. The specific heat ratio is  $\gamma = 1.4$  in all the examples.

*Case 1* The first case corresponds to a stratified atmosphere with constant potential temperature  $\theta$  (Example 2 in the Appendix). The initial solution is specified by Eq. (68) with  $u = v = 0 \text{ m/s}^{-1}$ ,  $R = 287.058 \text{ J/kg K}$ ,  $T = 300 \text{ K}$ ,  $p_0 = 10^5 \text{ N/m}^2$ , and  $g = 9.8 \text{ m/s}^2$ . The domain is  $1000 \times 1000 \text{ m}^2$  discretized by  $51 \times 51$  points. No-slip wall conditions are specified on all boundaries. This case represents the hydrostatic equilibrium for the two-dimensional rising thermal bubble problem [9]. A time-step size of  $0.02 \text{ s}$  is taken, and the solution is evolved until a final time of  $1000 \text{ s}$  with the RK4 method.

*Case 2* The second case corresponds to a stratified atmosphere with a specified Brunt-Väisälä frequency  $\mathcal{N}$  (Example 3 in the Appendix). The initial solution is specified by Eq. (75) with  $\mathcal{N} = 0.01 / \text{s}$ ,  $u = 20 \text{ m/s}$ ,  $v = 0 \text{ m/s}$ ,  $R = 287.058 \text{ J/kg K}$ ,  $T = 300 \text{ K}$ ,  $p_0 = 10^5 \text{ N/m}^2$ , and  $g = 9.8 \text{ m/s}^2$ . The domain is  $300,000 \times 10,000 \text{ m}^2$  discretized by  $1200 \times 50$  points. Periodic boundaries are specified along  $x$ , and no-slip wall boundaries are specified along  $y$ . This case

**Table 2** Relative error with respect to the initial solution for the three equilibria with the WENO5 and CRWENO5 schemes.

Case	Algorithm	$\ \epsilon\ _1$	$\ \epsilon\ _2$	$\ \epsilon\ _\infty$	$\ \epsilon\ _1$	$\ \epsilon\ _2$	$\ \epsilon\ _\infty$
		WENO5			CRWENO5		
Case 1	Balanced	6.02E-15	7.11E-15	1.31E-14	1.50E-14	1.53E-14	2.09E-14
	Naive	3.75E-02	4.39E-02	7.34E-02	2.86E-02	3.37E-02	6.17E-02
Case 2	Balanced	3.63E-15	4.35E-15	8.15E-15	1.58E-14	1.83E-14	6.11E-14
	Naive	1.62E-02	1.60E-02	1.72E-02	1.89E-02	1.86E-02	1.99E-02



**Fig. 3** Pressure perturbation at a final time of 0.25 on a grid with 200 points with the WENO5 and CRWENO5 schemes (every fourth grid point is plotted). The reference solution is obtained with the CRWENO5 scheme on a grid with 2,000 points.

represents the hydrostatic equilibrium for the inertia-gravity wave problem [9]. A time-step size of 0.25 s is taken, and the solution is evolved until a final time of 3000 s with the SSPRK3 method.

The two cases are solved with the balanced formulation, as well as a naive discretization of the source term. The Roe scheme is used for upwinding in all the cases. Table 2 shows the  $L_1$ ,  $L_2$ , and  $L_\infty$  norms of the error defined by Eq. (60). When the balanced formulation is used, the errors are zero to machine precision for both the CRWENO5 and WENO5 schemes for all the cases. However, the naive discretization of the source term results in a significant error in preserving the steady state. Thus, these tests demonstrate that the balanced algorithm is necessary to accurately preserve the hydrostatic balance.

A one-dimensional problem [41] is used to test the accurate simulation of small perturbations to the hydrostatic balance. The initial solution represents a pressure perturbation to an isothermal hydrostatic equilibrium with a constant gravitational field of unit magnitude,

$$\rho(y, 0) = \exp(-y), \quad p(y, 0) = \exp(-y) + \eta \exp\left[-100(y - 0.5)^2\right], \quad u(y, 0) = 0, \quad (61)$$

on a unit domain  $y \in [0, 1]$  with extrapolative boundary conditions. The specific heat ratio is  $\gamma = 1.4$ . Solutions are obtained at a final time of 0.25 with the RK4 method, and a time-step size of

0.0025 (corresponding to a CFL of  $\sim 0.6$ ). Figure 3 shows the initial and final pressure perturbations  $(p(y, t) - \exp(-y))$ , obtained with the WENO5 and CRWENO5 schemes on a grid with 200 points. The reference solutions are obtained with the CRWENO5 scheme on a grid with 2,000 points. Two values of the perturbation strength ( $\eta$ ) are considered:  $10^{-2}$  and  $10^{-4}$ . The computed solutions agree well with the reference solutions, as well as with results in the literature [41]. These results demonstrate that the algorithm is able to capture accurately both strong and weak perturbations to the balanced steady state.

### B. Sod's Shock Tube with Gravitational Forcing

Sod's shock tube test [55] is a benchmark one-dimensional Riemann problem. A modified test case with gravitational forcing [41] is solved. The initial solution is given by

$$(\rho, v, p) = \begin{cases} (1, 0, 1) & y < 0.5 \\ (0.125, 0, 0.1) & y \geq 0.5 \end{cases}, \quad (62)$$

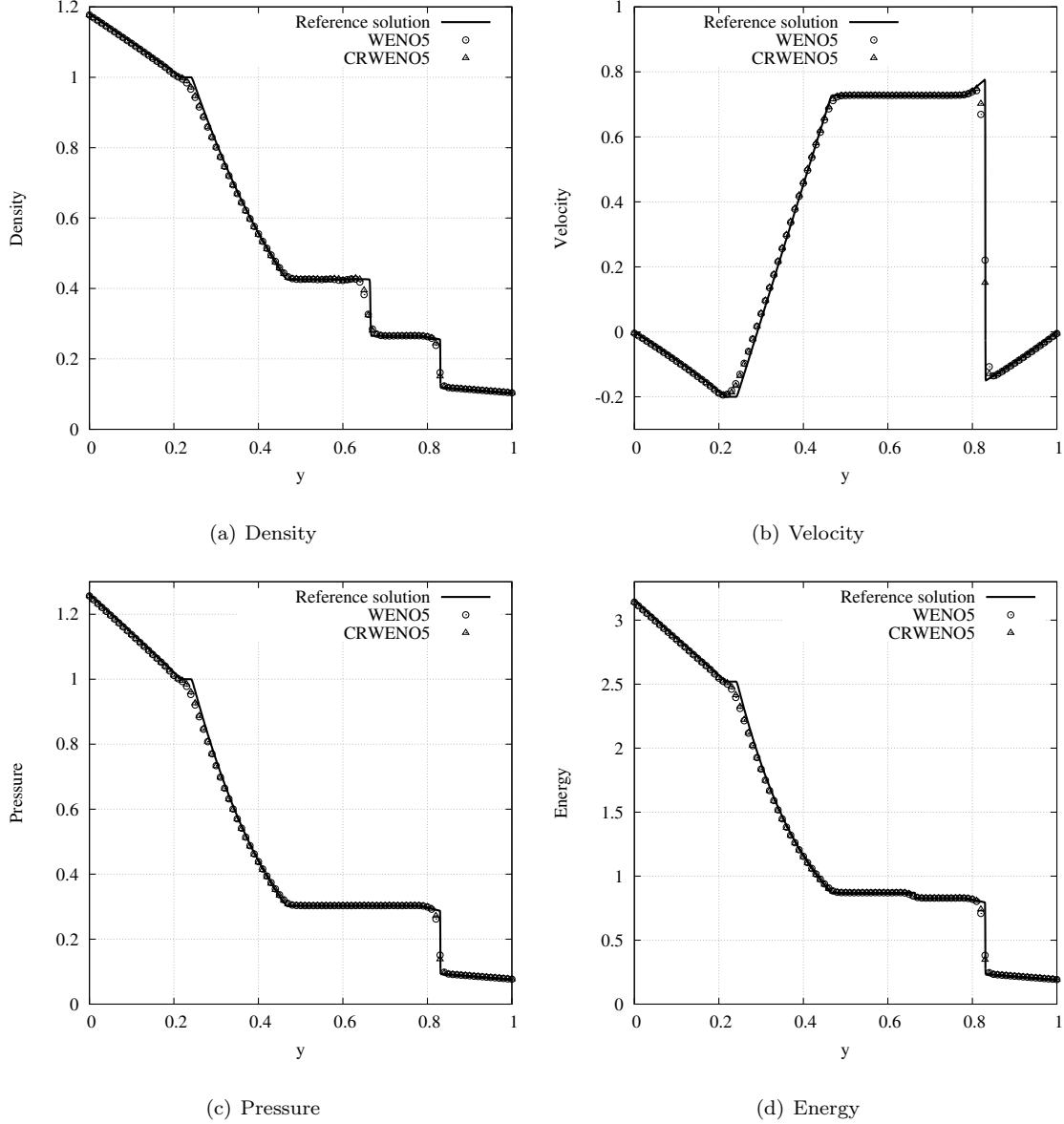
on a unit domain  $y \in [0, 1]$  discretized by a grid with 101 points. The specific heat ratio is  $\gamma = 1.4$ . Reflective boundary conditions are applied at both ends of the domain. The flow is subjected to a gravitational field  $g = 1$ . Solutions are obtained at a final time of 0.2 with the SSPRK3 method and a time-step size of 0.002 (corresponding to a CFL of  $\sim 0.4$ ). The Roe upwinding scheme is used. Figure 4 shows the solutions obtained with the WENO5 and CRWENO5 schemes; the reference solution is obtained with the CRWENO5 scheme on a grid with 2,001 points. The computed solutions agree well with the reference solution and results in the literature [41].

### C. Inertia-Gravity Waves

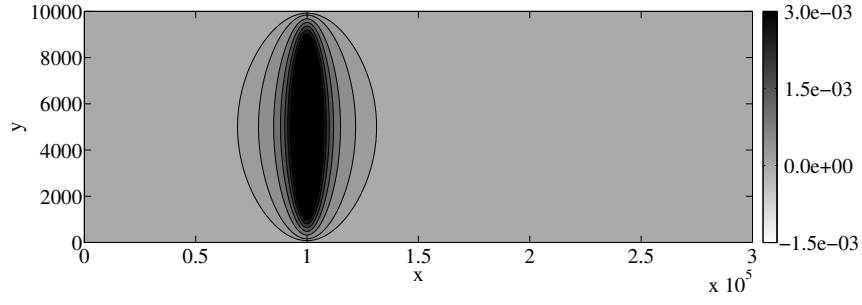
The inertia-gravity wave [9, 56] is a two-dimensional benchmark atmospheric flow problem that involves the evolution of a potential temperature perturbation. The domain consists of a channel with dimensions  $300,000 \text{ m} \times 10,000 \text{ m}$ . Periodic boundary conditions are applied on the left ( $x = 0 \text{ m}$ ) and right ( $x = 300,000 \text{ m}$ ) boundaries, while inviscid wall boundary conditions are applied on the bottom ( $y = 0 \text{ m}$ ) and top ( $y = 10,000 \text{ m}$ ) boundaries. The initial flow is a perturbation added to a stratified atmosphere in hydrostatic balance (Example 3 in the Appendix). The Brunt-Väisälä frequency is specified as  $\mathcal{N} = 0.01 \text{ /s}$ , the gravitational force per unit mass is

9.8 m/s<sup>2</sup>, and the horizontal flow velocity is  $u = 20$  m/s throughout the domain. The reference pressure and temperature at  $y = 0$  m are  $10^5$  N/m<sup>2</sup> and 300 K, respectively. The perturbation is added to the potential temperature, specified as

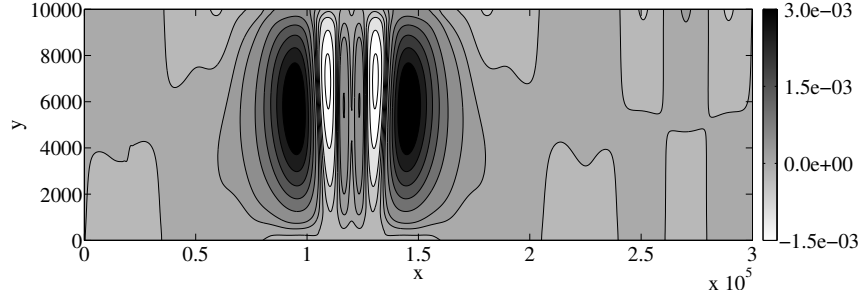
$$\Delta\theta(x, y, t = 0) = \theta_c \sin\left(\frac{\pi c y}{h_c}\right) \left[1 + \left(\frac{x - x_c}{a_c}\right)^2\right]^{-1}, \quad (63)$$



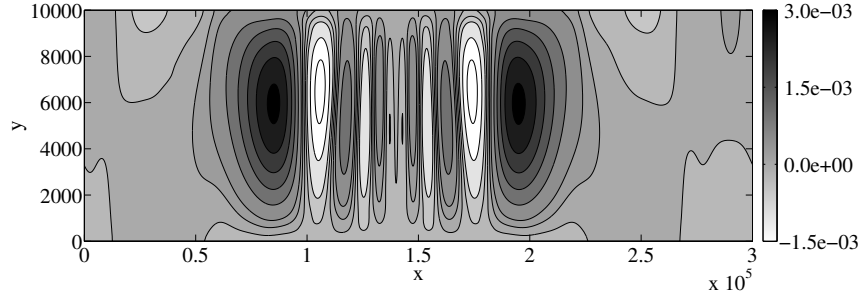
**Fig. 4** Solution to the modified Sod's shock tube problem (with gravity) obtained with the WENO5 and CRWENO5 scheme on a grid with 101 points. The reference solution is obtained with the CRWENO5 scheme on a grid with 2,001 points.



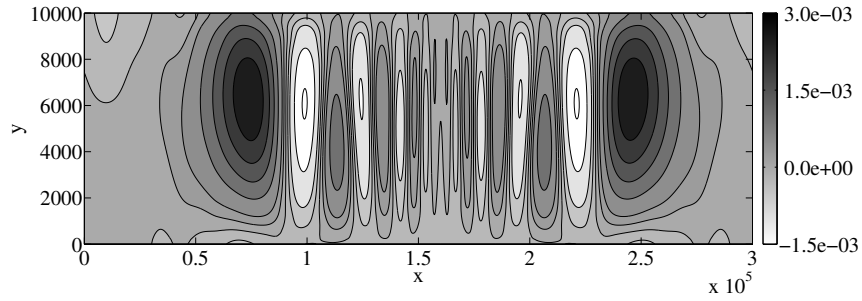
(a)  $t = 0$  s



(b)  $t = 1000$  s

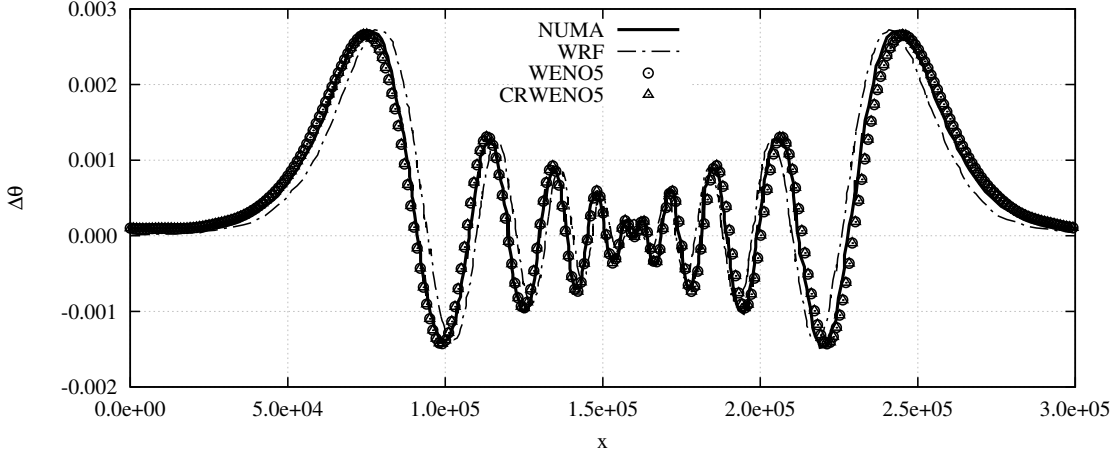


(c)  $t = 2000$  s



(d)  $t = 3000$  s

**Fig. 5 Inertia-gravity waves: Potential temperature perturbation contours for the solution obtained with the CRWENO5 scheme on a grid with  $1200 \times 50$  points.**



**Fig. 6 Inertia-gravity waves: Cross-sectional potential temperature perturbation at  $y = 5000$  m for the solution obtained at 3000 s with the WENO5 and CRWENO5 scheme on a grid with  $1200 \times 50$  points ( $250 \text{ m} \times 200 \text{ m}$  resolution. The reference solutions are computed by using two operational numerical weather prediction codes: NUMA [9] and WRF [16].**

where  $\theta_c = 0.01 \text{ K}$  is the perturbation strength,  $h_c = 10,000 \text{ m}$  is the height of the domain,  $a_c = 5,000 \text{ m}$  is the perturbation half-width,  $x_c = 100,000 \text{ m}$  is the horizontal location of the perturbation, and  $\pi_c \approx 3.141592654$  is the Archimedes (trigonometric) constant. The evolution of the perturbation is simulated until a final time of 3000 s.

Solutions are obtained with the WENO5 and CRWENO5 schemes on a grid with  $1200 \times 50$  points that results in a resolution of 250 m in  $x$  and 200 m in  $y$ . The SSPRK3 method is used for time integration with a time-step size of 0.25 s (corresponding to a CFL of  $\sim 0.4$ ). The Rusanov scheme is used for upwinding. Figure 5 shows the potential temperature perturbation ( $\Delta\theta$ ) for the initial, intermediate, and final solutions. The results agree well with those in the literature [17, 20, 24, 56]. The initial perturbation is centered at  $x = 100,000 \text{ m}$  while the flow features in the final solution are centered at  $x = 160,000 \text{ m}$ ; this translation is expected because of the mean horizontal velocity of 20 m/s.

Figure 6 shows the cross-sectional potential temperature perturbation at an altitude of  $y = 5,000 \text{ m}$ . The solutions obtained with the WENO5 and CRWENO5 schemes are compared those of two reference solutions. NUMA [9] refers to a spectral-element solver, and the solution is obtained



with 10th-order polynomials and 250 m effective grid resolution. WRF [16] uses a finite-difference discretization, and the solution is obtained with a fifth-order upwind scheme in  $x$  and a third-order upwind scheme in  $y$  [17]. Good agreement is observed with NUMA, while there is a slight difference in the perturbation propagation speed with WRF.

#### D. Rising Thermal Bubble

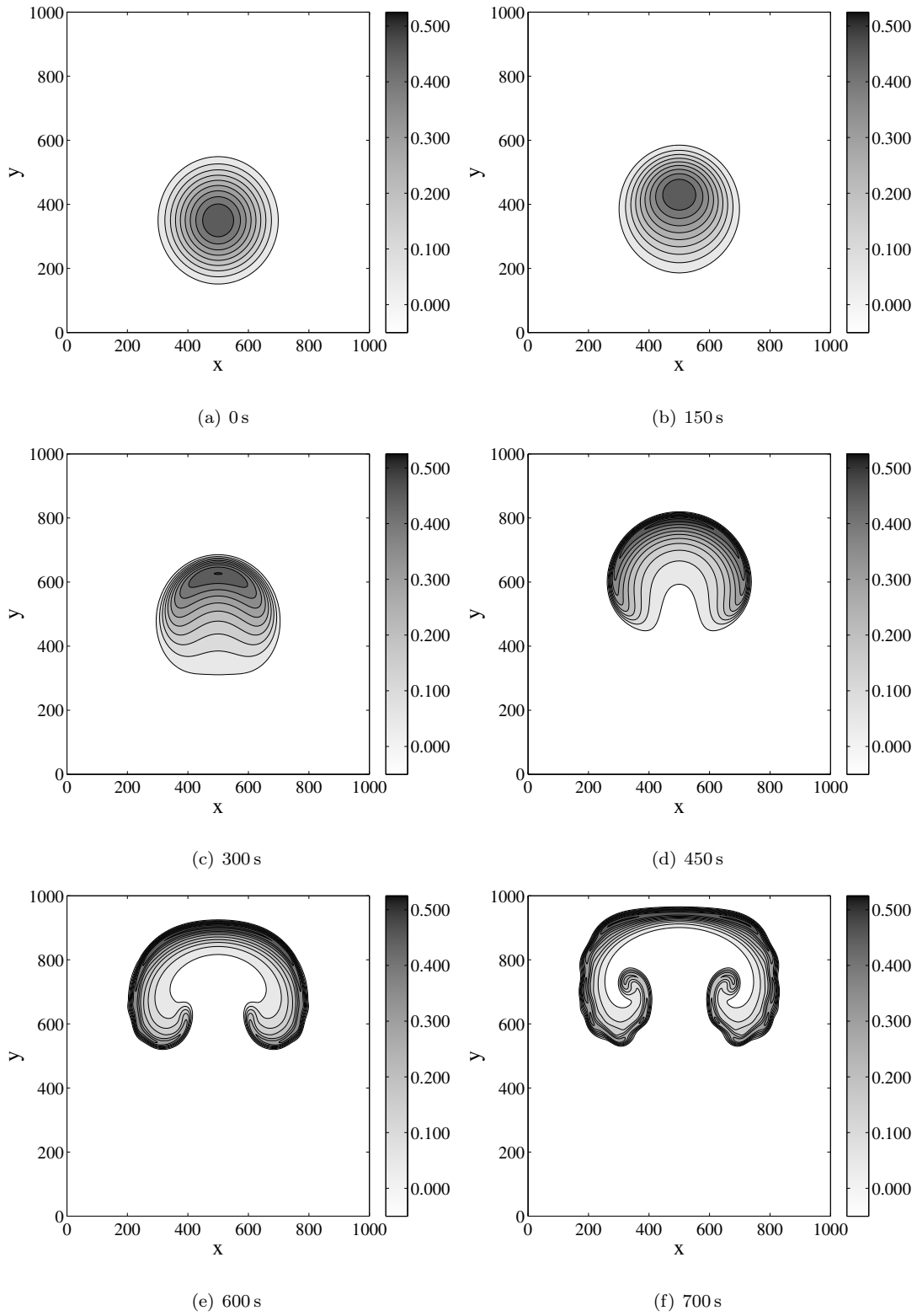
The two-dimensional rising thermal bubble [9] is another benchmark atmospheric flow that simulates the dynamics of a warm bubble. The domain is specified as  $(x, y) \in [0, 1000] \times [0, 1000] \text{ m}^2$  with inviscid wall boundary conditions on all sides. The initial solution is a stratified atmosphere in hydrostatic balance corresponding to Example 2 in the Appendix. The constant potential temperature (and thus the reference temperature at  $y = 0 \text{ m}$ ) is 300 K, and the reference pressure is  $10^5 \text{ N/m}^2$ . The ambient flow is at rest and experiences a constant gravitation force per unit mass of  $9.8 \text{ m/s}^2$ . The warm bubble is added as a potential temperature perturbation, specified as

$$\Delta\theta(x, y, t = 0) = \begin{cases} 0 & r > r_c \\ \frac{\theta_c}{2} \left[ 1 + \cos\left(\frac{\pi_c r}{r_c}\right) \right] & r \leq r_c \end{cases}, \quad r = \sqrt{(x - x_c)^2 + (z - z_c)^2}, \quad (64)$$

where  $\theta_c = 0.5 \text{ K}$  is the perturbation strength,  $(x_c, y_c) = (500, 350) \text{ m}$  is the initial location at which the bubble is centered,  $r_c = 250 \text{ m}$  is the radius of the bubble, and  $\pi_c$  is the trigonometric constant. The flow is simulated to a final of 700 s.

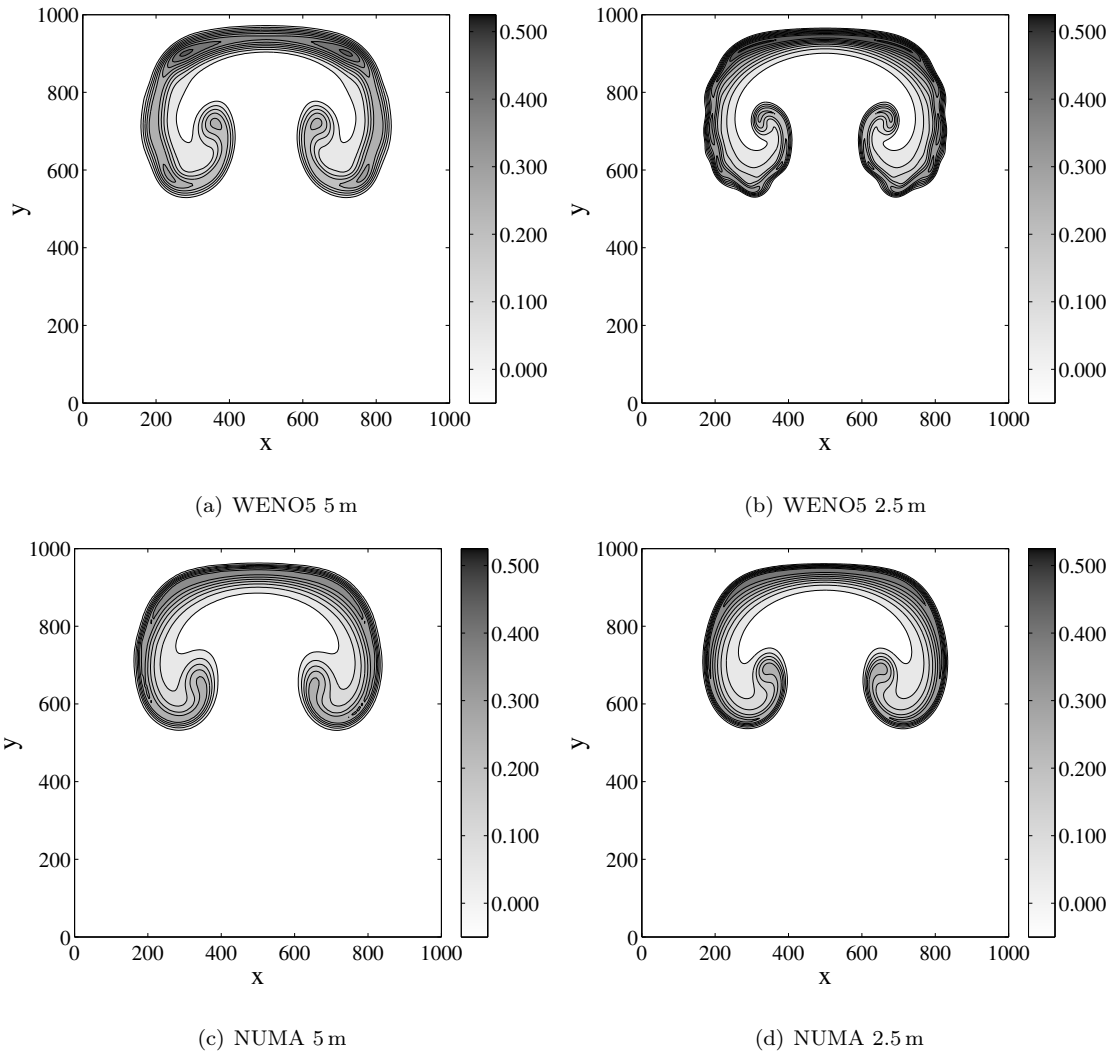
Figure 7 shows the solution obtained on a grid with  $401^2$  points, corresponding to a resolution of 2.5 m. The Rusanov scheme is used for upwinding. The RK4 method is used for time integration with a time-step size of 0.005 s (corresponding to a maximum CFL of  $\sim 0.7$ ). The potential temperature perturbation ( $\Delta\theta$ ) is shown at 0 s (initial bubble), 250 s, 500 s, and 700 s. The warm bubble rises as a result of buoyancy. The temperature differential within the bubble causes velocity gradients that shear and deform the bubble to a mushroom-like cloud. At the top boundary, the deformed bubble interacts with the inviscid wall to form a thin layer of warm air, while the trailing edges roll up due to the local temperature difference. The solution agrees with results in the literature [8, 9].

The solutions obtained by the proposed algorithm are compared with those obtained with the two-dimensional version of NUMA [9] (Nonhydrostatic Unified Model of the Atmosphere) with the



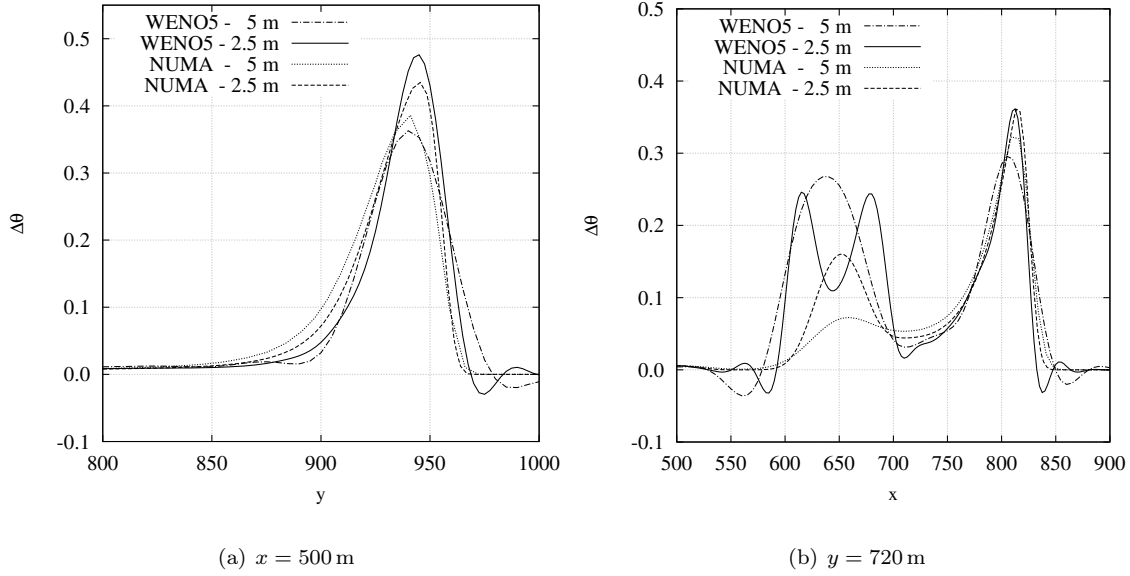
**Fig. 7** Rising thermal bubble: Potential temperature perturbation ( $\Delta\theta$ ) contours for the solution obtained with the WENO5 scheme on a grid with  $401 \times 401$  points (2.5 m resolution).

same order of accuracy and grid resolution. NUMA solves the governing equations in terms of mass,



**Fig. 8 Rising thermal bubble: Comparison of potential temperature perturbation ( $\Delta\theta$ ) contours at 700s for the solution obtained with the WENO5 scheme and NUMA [9] for two grid resolutions.**

momentum, and potential temperature discretized by the continuous and discontinuous Galerkin methods with spectral elements. Figures 8(a) and 8(b) show the final solution at 700s obtained with the WENO5 scheme on grids with  $201^2$  and  $401^2$  points (5 m and 2.5 m grid resolutions), respectively. Figures 8(c) and 8(d) show the solutions obtained with NUMA using the continuous Galerkin discretization. The domain is discretized with  $40^2$  and  $80^2$  elements with fifth-order polynomials, respectively, resulting in effective grid resolutions of 5 m and 2.5 m. These solutions are obtained at a CFL of  $\sim 0.7$  with the RK4 time integration method. Good agreement is observed for the overall flow at both grid resolutions. Figure 9 shows the cross-sectional potential temperature perturbation for



**Fig. 9 Rising thermal bubble: Cross-sectional potential temperature perturbation ( $\Delta\theta$ ) at  $x = 500$  m and  $y = 720$  m for the solution obtained at 700 s with the WENO5 scheme and NUMA [9] on grids with resolutions of 5 m and 2.5 m.**

the solutions shown in Fig. 8 along  $y$  ( $x = 500$  m) and  $x$  ( $y = 720$  m). While the solutions obtained with NUMA exhibit smooth flow features, the WENO5 solutions predict stronger gradients that result in qualitatively different flow features at small length scales. As an example, WENO5 predicts a stronger roll-up of the trailing edges at 2.5 m grid resolution at  $600 \text{ m} \leq x \leq 700 \text{ m}$ ,  $y \sim 720 \text{ m}$ , as observed in Fig. 9(b). This difference is due to the treatment of subgrid-length scales. The NUMA solver stabilizes the solution through a residual-based dynamic subgrid-scale model [57, 58], whereas the WENO5 solution to the inviscid Euler equations relies on the linear and nonlinear numerical diffusion to stabilize the unresolved scales. Incorporation of a subgrid-scale model in the current solver will be explored in the future.

## V. Conclusions

A well-balanced conservative finite-difference algorithm is proposed in this paper for the numerical simulation of atmospheric flows. The governing equations (inviscid Euler equations) are solved as conservation laws for mass, momentum, and energy, with no additional assumptions; thus, they are of the same form as used in the computational aerodynamics community. The discretization

of the hyperbolic flux and the treatment of the source term are modified so that the overall algorithm preserves hydrostatically balanced equilibria to machine precision; this formulation is an extension of the previous work by Xing and Shu (2013). In the absence of gravitational forces, the algorithm reduces to the standard finite-difference discretization of the Euler equations. The fifth-order weighted essentially nonoscillatory and the compact-reconstruction weighted essentially nonoscillatory schemes are used in this paper for spatial discretization; however, the well-balanced formulation can be used with any conservative finite-difference methods. The ability of the proposed algorithm to preserve a hydrostatically balanced equilibrium is demonstrated on examples of stratified atmosphere. We show that the balanced formulation preserves such steady states to machine precision, while a naive treatment of the source term does not. Benchmark flow problems are solved to verify the algorithm. The solutions are compared with those obtained with operational, state-of-the-art atmospheric flow solvers (WRF and NUMA), and good agreement is observed. Future work will focus on incorporating semi-implicit and implicit time-integration methods, as well as a subgrid-scale model similar to the models used in established atmospheric flow solvers.

## Appendix

Three examples of the well-balanced finite-difference formulation are presented. These examples are representative of the hydrostatic equilibrium encountered in benchmark atmospheric flow problems.

*Example 1* The first example is the isothermal steady state. The resulting formulation is identical to the previously proposed well-balanced WENO scheme [41]. The steady state can be derived by assuming the temperature  $T = T_0$  to be a constant and applying the hydrostatic balance Eq. (38). It is given by

$$u = \text{constant}, \quad v = 0, \quad \rho = \rho_0 \exp\left(-\frac{gy}{RT}\right), \quad p = p_0 \exp\left(-\frac{gy}{RT}\right), \quad (65)$$

where the reference density  $\rho_0$  and pressure  $p_0$  are related by the equation of state. Thus, the functions  $\varrho(y)$  and  $\varphi(y)$  in Eq. (37) are

$$\varrho(y) = \exp\left(-\frac{gy}{RT}\right), \quad \varphi(y) = \exp\left(-\frac{gy}{RT}\right). \quad (66)$$

The modified source term in Eq. (42), and the modified solution in Eqs. (54) and (55) are

$$\mathbf{s}^* = \begin{bmatrix} 0 \\ 0 \\ \rho RT_0 \exp\left(\frac{gy}{RT}\right) \left\{ \exp\left(-\frac{gy}{RT}\right) \right\}_y \\ \rho v RT_0 \exp\left(\frac{gy}{RT}\right) \left\{ \exp\left(-\frac{gy}{RT}\right) \right\}_y \end{bmatrix}, \text{ and } \mathbf{q}^* = \begin{bmatrix} \rho \exp\left(\frac{gy}{RT}\right) \\ \rho u \exp\left(\frac{gy}{RT}\right) \\ \rho v \exp\left(\frac{gy}{RT}\right) \\ e \exp\left(\frac{gy}{RT}\right) \end{bmatrix}. \quad (67)$$

*Example 2* The second example is a hydrostatic balance that is frequently encountered in atmospheric flows of practical relevance [9, 10, 17]. The steady state is derived by specifying a stratified atmosphere with constant potential temperature  $\theta = T_0$ . The hydrostatic balance is thus expressed as

$$\frac{\gamma R}{\gamma - 1} \theta \frac{d\pi}{dy} = -g \Rightarrow \rho = \rho_0 \left[ 1 - \frac{(\gamma - 1)gy}{\gamma R\theta} \right]^{1/(\gamma-1)}, \quad p = p_0 \left[ 1 - \frac{(\gamma - 1)gy}{\gamma R\theta} \right]^{\gamma/(\gamma-1)}, \quad u = \text{constant}, \quad v = 0. \quad (68)$$

where  $\pi$  is the Exner pressure (see section II). Thus, the functions  $\varrho(y)$  and  $\varphi(y)$  in Eq. (37) are

$$\varrho(y) = \left[ 1 - \frac{(\gamma - 1)gy}{\gamma R\theta} \right]^{1/(\gamma-1)}, \quad \varphi(y) = \left[ 1 - \frac{(\gamma - 1)gy}{\gamma R\theta} \right]^{\gamma/(\gamma-1)}, \quad (69)$$

and the modified source term in Eq. (42), and the modified solution in Eqs. (54) and (55) are

$$\mathbf{s}^* = \begin{bmatrix} 0 \\ 0 \\ \rho RT_0 \left\{ 1 - \frac{(\gamma-1)gy}{\gamma R\theta} \right\}^{-1/(\gamma-1)} \left\{ \left( 1 - \frac{(\gamma-1)gy}{\gamma R\theta} \right)^{\gamma/(\gamma-1)} \right\}_y \\ \rho v RT_0 \left\{ 1 - \frac{(\gamma-1)gy}{\gamma R\theta} \right\}^{-1/(\gamma-1)} \left\{ \left( 1 - \frac{(\gamma-1)gy}{\gamma R\theta} \right)^{\gamma/(\gamma-1)} \right\}_y \end{bmatrix}, \quad (70)$$

and

$$\mathbf{q}^* = \begin{bmatrix} \rho \left\{ 1 - \frac{(\gamma-1)gy}{\gamma R\theta} \right\}^{-1/(\gamma-1)} \\ \rho u \left\{ 1 - \frac{(\gamma-1)gy}{\gamma R\theta} \right\}^{-1/(\gamma-1)} \\ \rho v \left\{ 1 - \frac{(\gamma-1)gy}{\gamma R\theta} \right\}^{-1/(\gamma-1)} \\ e^* \end{bmatrix}, \quad (71)$$

where

$$e^* = \frac{p}{\gamma - 1} \left[ 1 - \frac{(\gamma - 1)gy}{\gamma R\theta} \right]^{-\gamma/(\gamma-1)} + \frac{1}{2} \rho \left[ 1 - \frac{(\gamma - 1)gy}{\gamma R\theta} \right]^{-1/(\gamma-1)} (u^2 + v^2). \quad (72)$$

*Example 3* The third example is a stratified atmosphere with a specified Brunt-Väisälä frequency ( $\mathcal{N}$ ) [9, 56]

$$\mathcal{N}^2 = g \frac{d}{dy} (\log \theta) \Rightarrow \theta = T_0 \exp \left( \frac{\mathcal{N}^2}{g} y \right). \quad (73)$$

Assuming hydrostatic balance, the Exner pressure is given by

$$\pi = 1 + \frac{(\gamma - 1)g^2}{\gamma RT_0 \mathcal{N}^2} \left[ \exp \left( -\frac{\mathcal{N}^2}{g} y \right) - 1 \right], \quad (74)$$

and the steady state flow variables are

$$p = p_0 \left[ 1 + \frac{(\gamma - 1)g^2}{\gamma RT_0 \mathcal{N}^2} \left\{ \exp \left( -\frac{\mathcal{N}^2}{g} y \right) - 1 \right\} \right]^{\gamma/(\gamma-1)}, \quad (75)$$

$$\rho = \rho_0 \exp \left( -\frac{\mathcal{N}^2}{g} y \right) \left[ 1 + \frac{(\gamma - 1)g^2}{\gamma RT_0 \mathcal{N}^2} \left\{ \exp \left( -\frac{\mathcal{N}^2}{g} y \right) - 1 \right\} \right]^{1/(\gamma-1)}, \quad (76)$$

$$u = \text{constant}, \quad v = 0. \quad (77)$$

Thus, the functions  $\varrho(y)$  and  $\varphi(y)$  in Eq. (37) are

$$\varrho(y) = \exp \left( -\frac{\mathcal{N}^2}{g} y \right) \left[ 1 + \frac{(\gamma - 1)g^2}{\gamma RT_0 \mathcal{N}^2} \left\{ \exp \left( -\frac{\mathcal{N}^2}{g} y \right) - 1 \right\} \right]^{1/(\gamma-1)}, \quad (78)$$

$$\varphi(y) = \left[ 1 + \frac{(\gamma - 1)g^2}{\gamma RT_0 \mathcal{N}^2} \left\{ \exp \left( -\frac{\mathcal{N}^2}{g} y \right) - 1 \right\} \right]^{\gamma/(\gamma-1)}. \quad (79)$$

The modified source term in Eq. (42), and the modified solution in Eqs. (54) and (55) are

$$\mathbf{s}^* = \begin{bmatrix} 0 \\ 0 \\ \rho RT_0 \exp \left( \frac{\mathcal{N}^2}{g} y \right) \left[ 1 + \frac{(\gamma-1)g^2}{\gamma RT_0 \mathcal{N}^2} \left\{ \exp \left( -\frac{\mathcal{N}^2}{g} y \right) - 1 \right\} \right]^{-1/(\gamma-1)} \left\{ \left( 1 + \frac{(\gamma-1)g^2}{\gamma RT_0 \mathcal{N}^2} \left\{ \exp \left( -\frac{\mathcal{N}^2}{g} y \right) - 1 \right\} \right)^{\gamma/(\gamma-1)} \right\}_y \\ \rho v RT_0 \exp \left( \frac{\mathcal{N}^2}{g} y \right) \left[ 1 + \frac{(\gamma-1)g^2}{\gamma RT_0 \mathcal{N}^2} \left\{ \exp \left( -\frac{\mathcal{N}^2}{g} y \right) - 1 \right\} \right]^{-1/(\gamma-1)} \left\{ \left( 1 + \frac{(\gamma-1)g^2}{\gamma RT_0 \mathcal{N}^2} \left\{ \exp \left( -\frac{\mathcal{N}^2}{g} y \right) - 1 \right\} \right)^{\gamma/(\gamma-1)} \right\}_y \\ e^* \end{bmatrix} \quad (80)$$

and

$$\mathbf{q}^* = \begin{bmatrix} \rho \exp \left( \frac{\mathcal{N}^2}{g} y \right) \left[ 1 + \frac{(\gamma-1)g^2}{\gamma RT_0 \mathcal{N}^2} \left\{ \exp \left( -\frac{\mathcal{N}^2}{g} y \right) - 1 \right\} \right]^{-1/(\gamma-1)} \\ \rho u \exp \left( \frac{\mathcal{N}^2}{g} y \right) \left[ 1 + \frac{(\gamma-1)g^2}{\gamma RT_0 \mathcal{N}^2} \left\{ \exp \left( -\frac{\mathcal{N}^2}{g} y \right) - 1 \right\} \right]^{-1/(\gamma-1)} \\ \rho v \exp \left( \frac{\mathcal{N}^2}{g} y \right) \left[ 1 + \frac{(\gamma-1)g^2}{\gamma RT_0 \mathcal{N}^2} \left\{ \exp \left( -\frac{\mathcal{N}^2}{g} y \right) - 1 \right\} \right]^{-1/(\gamma-1)} \\ e^* \end{bmatrix}, \quad (81)$$

where

$$e^* = \frac{p}{\gamma - 1} \left[ 1 + \frac{(\gamma - 1)g^2}{\gamma RT_0 \mathcal{N}^2} \left\{ \exp \left( -\frac{\mathcal{N}^2}{g} y \right) - 1 \right\} \right]^{-\gamma/(\gamma-1)} \quad (82)$$

$$+ \frac{1}{2} \rho \exp \left( \frac{\mathcal{N}^2}{g} y \right) \left[ 1 + \frac{(\gamma - 1)g^2}{\gamma RT_0 \mathcal{N}^2} \left\{ \exp \left( -\frac{\mathcal{N}^2}{g} y \right) - 1 \right\} \right]^{-1/(\gamma-1)} (u^2 + v^2). \quad (83)$$

## Acknowledgments

This material is based upon work supported by the U.S. Department of Energy, Office of Science, Advanced Scientific Computing Research, under contract DE-AC02-06CH11357. We thank Dr. Francis X. Giraldo (Naval Postgraduate School) for giving us access to the Nonhydrostatic Unified Model of the Atmosphere (NUMA) code and helping us with generating the benchmark solutions, as well as his valuable feedback on our work.

## References

- [1] Taylor, M. A., Edwards, J., and Cyr, A. S., “Petascale atmospheric models for the Community Climate System Model: New developments and evaluation of scalable dynamical cores,” *Journal of Physics: Conference Series*, Vol. 125, No. 1, 2008, p. 012023, doi:10.1088/1742-6596/125/1/012023.
- [2] Lin, S.-J., “A “vertically Lagrangian” finite-volume dynamical core for global models,” *Monthly Weather Review*, Vol. 132, No. 10, 2004, pp. 2293–2307, doi:10.1175/1520-0493(2004)132<2293:AVLFDC>2.0.CO;2.
- [3] Ogura, Y. and Phillips, N., “Scale analysis of deep and shallow convection in the atmosphere,” *Journal of the Atmospheric Sciences*, Vol. 19, No. 2, 1962, pp. 173–79, doi:10.1175/1520-0469(1962)019<0173:SAODAS>2.0.CO;2.
- [4] Durran, D., “Improving the anelastic approximation,” *Journal of the Atmospheric Sciences*, Vol. 46, No. 11, 1989, pp. 1453–1461, doi:10.1175/1520-0469(1989)046<1453:ITAA>2.0.CO;2.
- [5] Arakawa, A. and Konor, C., “Unification of the anelastic and quasi-hydrostatic systems of equations,” *Monthly Weather Review*, Vol. 137, No. 2, 2009, pp. 710–726, doi:10.1175/2008MWR2520.1.
- [6] Davies, T., Staniforth, A., Wood, N., and Thuburn, J., “Validity of anelastic and other equation sets as inferred from normal-mode analysis,” *Quarterly Journal of the Royal Meteorological Society*, Vol. 129, No. 593, 2003, pp. 2761–2775, doi:10.1256/qj.02.1951.
- [7] Klein, R., Achatz, U., Bresch, D., Knio, O., and Smolarkiewicz, P., “Regime of validity of soundproof atmospheric flow models,” *Journal of the Atmospheric Sciences*, Vol. 67, No. 10, 2010, pp. 3226–3237, doi:10.1175/2010JAS3490.1.



- [8] Ullrich, P. and Jablonowski, C., “Operator-split Runge-Kutta-Rosenbrock methods for nonhydrostatic atmospheric models,” *Monthly Weather Review*, Vol. 140, No. 4, 2012, pp. 1257–1284, doi:10.1175/MWR-D-10-05073.1.
- [9] Giraldo, F. X. and Restelli, M., “A study of spectral element and discontinuous Galerkin methods for the Navier-Stokes equations in nonhydrostatic mesoscale atmospheric modeling: Equation sets and test cases,” *Journal of Computational Physics*, Vol. 227, No. 8, 2008, pp. 3849–3877, doi:10.1016/j.jcp.2007.12.009.
- [10] Giraldo, F. X., Restelli, M., and Läuter, M., “Semi-implicit formulations of the Navier-Stokes equations: Application to nonhydrostatic atmospheric modeling,” *SIAM Journal on Scientific Computing*, Vol. 32, No. 6, 2010, pp. 3394–3425, doi:10.1137/090775889.
- [11] Grell, G., Dudhia, J., Stauffer, D., et al., “A description of the fifth-generation Penn State/NCAR mesoscale model (MM5),” Tech. rep., 1994.
- [12] Hodur, R., “The Naval Research Laboratory’s coupled ocean/atmosphere mesoscale prediction system (COAMPS),” *Monthly Weather Review*, Vol. 125, No. 7, 1997, pp. 1414–1430, doi:10.1175/1520-0493(1997)125<1414:TNRLSC>2.0.CO;2.
- [13] Xue, M., Droegemeier, K. K., and Wong, V., “The Advanced Regional Prediction System (ARPS) – A multi-scale nonhydrostatic atmospheric simulation and prediction model. Part I: Model dynamics and verification,” *Meteorology and Atmospheric Physics*, Vol. 75, No. 3-4, 2000, pp. 161–193, doi:10.1007/s007030070003.
- [14] Janjic, Z., “A nonhydrostatic model based on a new approach,” *Meteorology and Atmospheric Physics*, Vol. 82, No. 1-4, 2003, pp. 271–285, doi:10.1007/s00703-001-0587-6.
- [15] Gassmann, A., “An improved two-time-level split-explicit integration scheme for non-hydrostatic compressible models,” *Meteorology and Atmospheric Physics*, Vol. 88, No. 1-2, 2005, pp. 23–38, doi:10.1007/s00703-003-0053-8.
- [16] Skamarock, W. C., Klemp, J. B., Dudhia, J., Gill, D. O., Barker, D. M., Wang, W., and Powers, J. G., “A description of the advanced research WRF version 2,” Tech. rep., DTIC Document, 2005.
- [17] Ahmad, N. and Lindeman, J., “Euler solutions using flux-based wave decomposition,” *International Journal for Numerical Methods in Fluids*, Vol. 54, No. 1, 2007, pp. 47–72, doi:10.1002/fld.1392.
- [18] Giraldo, F. X., Kelly, J. F., and Constantinescu, E., “Implicit-explicit formulations of a three-

- dimensional nonhydrostatic unified model of the atmosphere (NUMA),” *SIAM Journal on Scientific Computing*, Vol. 35, No. 5,  
doi:10.1137/120876034.
- [19] Kelly, J. F. and Giraldo, F. X., “Continuous and discontinuous Galerkin methods for a scalable three-dimensional nonhydrostatic atmospheric model: Limited-area mode,” *Journal of Computational Physics*, Vol. 231, No. 24, 2012, pp. 7988–8008,  
doi:10.1016/j.jcp.2012.04.042.
- [20] Yang, C. and Cai, X., “A scalable fully implicit compressible Euler solver for mesoscale nonhydrostatic simulation of atmospheric flows,” *SIAM Journal on Scientific Computing*, Vol. 36, No. 5, 2014, pp. S23–S47,  
doi:10.1137/130919167.
- [21] Das, P., “A non-Archimedean approach to the equations of convection dynamics,” *Journal of the Atmospheric Sciences*, Vol. 36, No. 11, 1979, pp. 2183–2190,  
doi:10.1175/1520-0469(1979)036<2183:ANAATT>2.0.CO;2.
- [22] Botta, N., Klein, R., Langenberg, S., and Lützenkirchen, S., “Well balanced finite volume methods for nearly hydrostatic flows,” *Journal of Computational Physics*, Vol. 196, No. 2, 2004, pp. 539–565,  
doi:10.1016/j.jcp.2003.11.008.
- [23] Ahmad, N., Bacon, D., Sarma, A., Koraćin, D., Vellore, R., Boybeyi, Z., and Lindeman, J., “Simulations of non-hydrostatic atmosphere using conservation laws package,” in “45th AIAA Aerospace Sciences Meeting and Exhibit, Reno, NV,” American Institute of Aeronautics and Astronautics, 2007,  
doi:10.2514/6.2007-84.
- [24] Ahmad, N. and Proctor, F., “The high-resolution wave-propagation method applied to meso- and micro-scale flows,” in “50th AIAA Aerospace Sciences Meeting and Exhibit, Nashville, TN,” American Institute of Aeronautics and Astronautics, 2012,  
doi:10.2514/6.2012-430.
- [25] Laney, C. B., *Computational Gasdynamics*, Cambridge University Press, 1998.
- [26] Hirsch, C., *Numerical Computation of Internal and External Flows: The Fundamentals of Computational Fluid Dynamics: The Fundamentals of Computational Fluid Dynamics*, Vol. 1 & 2, Elsevier Science, 2007.
- [27] Davies, T., Cullen, M. J. P., Malcolm, A. J., Mawson, M. H., Staniforth, A., White, A. A., and Wood, N., “A new dynamical core for the Met Office’s global and regional modelling of the atmosphere,” *Quarterly Journal of the Royal Meteorological Society*, Vol. 131, No. 608, 2005, pp. 1759–1782,

doi:10.1256/qj.04.101.

- [28] Liu, X.-D., Osher, S., and Chan, T., “Weighted essentially non-oscillatory schemes,” *Journal of Computational Physics*, Vol. 115, No. 1, 1994, pp. 200–212,  
doi:10.1006/jcph.1994.1187.
- [29] Jiang, G.-S. and Shu, C.-W., “Efficient Implementation of Weighted ENO Schemes,” *Journal of Computational Physics*, Vol. 126, No. 1, 1996, pp. 202–228,  
doi:10.1006/jcph.1996.0130.
- [30] Shu, C.-W., “High order weighted essentially nonoscillatory schemes for convection dominated problems,” *SIAM Review*, Vol. 51, No. 1, 2009, pp. 82–126,  
doi:10.1137/070679065.
- [31] Ghosh, D. and Baeder, J. D., “Compact Reconstruction Schemes with Weighted ENO Limiting for Hyperbolic Conservation Laws,” *SIAM Journal on Scientific Computing*, Vol. 34, No. 3, 2012, pp. A1678–A1706,  
doi:10.1137/110857659.
- [32] Lele, S. K., “Compact finite difference schemes with spectral-like resolution,” *Journal of Computational Physics*, Vol. 103, No. 1, 1992, pp. 16–42,  
doi:10.1016/0021-9991(92)90324-R.
- [33] Ghosh, D. and Baeder, J. D., “Weighted non-linear compact schemes for the direct numerical simulation of compressible, turbulent flows,” *Journal of Scientific Computing*, Vol. 61, No. 1, 2014, pp. 61–89,  
doi:10.1007/s10915-014-9818-0.
- [34] Ghosh, D., Medida, S., and Baeder, J. D., “Application of compact-reconstruction weighted essentially nonoscillatory schemes to compressible aerodynamic flows,” *AIAA Journal*, Vol. 52, No. 9, 2014, pp. 1858–1870,  
doi:10.2514/1.J052654.
- [35] Ghosh, D., Constantinescu, E. M., and Brown, J., “Efficient implementation of nonlinear compact schemes on massively parallel platforms,” *SIAM Journal on Scientific Computing*, Vol. 37, No. 3, 2015, pp. C354–C383,  
doi:10.1137/140989261.
- [36] LeVeque, R. J., “Balancing source terms and flux gradients in high-resolution Godunov methods: The quasi-steady wave-propagation algorithm,” *Journal of Computational Physics*, Vol. 146, No. 1, 1998, pp. 346–365,  
doi:10.1006/jcph.1998.6058.

- [37] Bale, D., LeVeque, R., Mitran, S., and Rossmanith, J., “A Wave Propagation Method for Conservation Laws and Balance Laws with Spatially Varying Flux Functions,” *SIAM Journal on Scientific Computing*, Vol. 24, No. 3, 2003, pp. 955–978, doi:10.1137/S106482750139738X.
- [38] Xing, Y. and Shu, C.-W., “High order finite difference {WENO} schemes with the exact conservation property for the shallow water equations,” *Journal of Computational Physics*, Vol. 208, No. 1, 2005, pp. 206 – 227, doi:10.1016/j.jcp.2005.02.006.
- [39] Xing, Y. and Shu, C.-W., “High-Order Well-Balanced Finite Difference WENO Schemes for a Class of Hyperbolic Systems with Source Terms,” *Journal of Scientific Computing*, Vol. 27, No. 1-3, 2006, pp. 477–494, doi:10.1007/s10915-005-9027-y.
- [40] Xing, Y. and Shu, C.-W., “High order well-balanced finite volume WENO schemes and discontinuous Galerkin methods for a class of hyperbolic systems with source terms,” *Journal of Computational Physics*, Vol. 214, No. 2, 2006, pp. 567–598, doi:10.1016/j.jcp.2005.10.005.
- [41] Xing, Y. and Shu, C.-W., “High Order Well-Balanced WENO Scheme for the Gas Dynamics Equations Under Gravitational Fields,” *Journal of Scientific Computing*, Vol. 54, No. 2-3, 2013, pp. 645–662, doi:10.1007/s10915-012-9585-8.
- [42] Shu, C.-W. and Osher, S., “Efficient implementation of essentially non-oscillatory shock-capturing schemes,” *Journal of Computational Physics*, Vol. 77, No. 2, 1988, pp. 439–471, doi:10.1016/0021-9991(88)90177-5.
- [43] Shu, C.-W. and Osher, S., “Efficient implementation of essentially non-oscillatory shock-capturing schemes, II,” *Journal of Computational Physics*, Vol. 83, No. 1, 1989, pp. 32–78, doi:10.1016/0021-9991(89)90222-2.
- [44] Butcher, J., *Numerical Methods for Ordinary Differential Equations*, Wiley, 2003.
- [45] Gottlieb, S., Ketcheson, D. I., and Shu, C.-W., “High Order Strong Stability Preserving Time Discretizations,” *Journal of Scientific Computing*, Vol. 38, No. 3, 2009, pp. 251–289, doi:10.1007/s10915-008-9239-z.
- [46] Yamaleev, N. K. and Carpenter, M. H., “A systematic methodology for constructing high-order energy stable WENO schemes,” *Journal of Computational Physics*, Vol. 228, No. 11, 2009, pp. 4248–4272, doi:10.1016/j.jcp.2009.03.002.

- [47] Henrick, A. K., Aslam, T. D., and Powers, J. M., “Mapped weighted essentially non-oscillatory schemes: Achieving optimal order near critical points,” *Journal of Computational Physics*, Vol. 207, No. 2, 2005, pp. 542–567, doi:10.1016/j.jcp.2005.01.023.
- [48] Borges, R., Carmona, M., Costa, B., and Don, W. S., “An improved weighted essentially non-oscillatory scheme for hyperbolic conservation laws,” *Journal of Computational Physics*, Vol. 227, No. 6, 2008, pp. 3191–3211, doi:10.1016/j.jcp.2007.11.038.
- [49] Castro, M., Costa, B., and Don, W. S., “High order weighted essentially non-oscillatory WENO-Z schemes for hyperbolic conservation laws,” *Journal of Computational Physics*, Vol. 230, No. 5, 2011, pp. 1766–1792, doi:10.1016/j.jcp.2010.11.028.
- [50] Ghosh, D., Constantinescu, E. M., and Brown, J., “Scalable nonlinear compact schemes,” Tech. Rep. ANL/MCS-TM-340, Argonne National Laboratory, Argonne, IL, 2014.
- [51] Roe, P. L., “Approximate Riemann solvers, parameter vectors, and difference schemes,” *Journal of Computational Physics*, Vol. 43, No. 2, 1981, pp. 357–372, doi:10.1016/0021-9991(81)90128-5.
- [52] LeVeque, R. J., *Finite Volume Methods for Hyperbolic Problems*, Cambridge Texts in Applied Mathematics, Cambridge University Press, 2002.
- [53] Rusanov, V. V., “The calculation of the interaction of non-stationary shock waves and obstacles,” *USSR Computational Mathematics and Mathematical Physics*, Vol. 1, No. 2, 1962, pp. 304–320, doi:10.1016/0041-5553(62)90062-9.
- [54] Slyz, A. and Prendergast, K., “Time-independent gravitational fields in the BGK scheme for hydrodynamics,” *Astron. Astrophys. Suppl. Ser.*, Vol. 139, No. 1, 1999, pp. 199–217, doi:10.1051/aas:1999389.
- [55] Sod, G. A., “A survey of several finite difference methods for systems of nonlinear hyperbolic conservation laws,” *Journal of Computational Physics*, Vol. 27, No. 1, 1978, pp. 1 – 31, doi:10.1016/0021-9991(78)90023-2.
- [56] Skamarock, W. C. and Klemp, J. B., “Efficiency and accuracy of the Klemp-Wilhelmson time-splitting technique,” *Monthly Weather Review*, Vol. 122, 1994, pp. 2623–2630, doi:10.1175/1520-0493(1994)122<2623:EAAOTK>2.0.CO;2.
- [57] Nazarov, M. and Hoffman, J., “Residual-based artificial viscosity for simulation of turbulent compress-

ible flow using adaptive finite element methods,” *International Journal for Numerical Methods in Fluids*, Vol. 71, No. 3, 2013, pp. 339–357, doi:10.1002/flid.3663.

- [58] Marras, S., Nazarov, M., and Giraldo, F. X., “A stabilized spectral element method based on a dynamic SGS model for LES,” *Submitted*.

Government License The submitted manuscript has been created by UChicago Argonne, LLC, Operator of Argonne National Laboratory (“Argonne”). Argonne, a U.S. Department of Energy Office of Science laboratory, is operated under Contract No. DE-AC02-06CH11357. The U.S. Government retains for itself, and others acting on its behalf, a paid-up nonexclusive, irrevocable worldwide license in said article to reproduce, prepare derivative works, distribute copies to the public, and perform publicly and display publicly, by or on behalf of the Government.