

SEMI-IMPLICIT TIME INTEGRATION OF ATMOSPHERIC FLOWS
WITH CHARACTERISTIC-BASED FLUX PARTITIONING*DEBOJYOTI GHOSH^{†‡} AND EMIL M. CONSTANTINESCU^{†§}

Abstract. This paper presents a characteristic-based flux partitioning for the semi-implicit time integration of atmospheric flows. Nonhydrostatic models require the solution of the compressible Euler equations. The acoustic time-scale is significantly faster than the advective scale, yet it is typically not relevant to atmospheric and weather phenomena. The acoustic and advective components of the hyperbolic flux are separated in the characteristic space. High-order, conservative additive Runge-Kutta methods are applied to the partitioned equations so that the acoustic component is integrated in time implicitly with an unconditionally stable method, while the advective component is integrated explicitly. The time step of the overall algorithm is thus determined by the advective scale. Benchmark flow problems are used to demonstrate the accuracy, stability, and convergence of the proposed algorithm. The computational cost of the partitioned semi-implicit approach is compared with that of explicit time integration.

Key words. atmospheric flows, nonhydrostatic, compressible, Euler equations, implicit-explicit time integration, characteristic-based splitting

AMS subject classifications. 65M-06, 86A-10, 76N-15

1. Introduction. Atmospheric flows are characterized by multiple spatial and temporal scales, and numerical models that simulate them have followed several different approaches. Planetary or global simulations involve large horizontal scales and are often solved by using hydrostatic models [49, 69] or models that remove the acoustic mode [4, 18, 52]. Mesoscale and limited-area modeling require the inclusion of non-hydrostatic effects, and thus solutions to the Euler equations of fluid dynamics are required [17, 42, 70]. Operational weather prediction codes use various formulations of the Euler equations [31, 32]. Models based on expressing the governing equations in terms of the Exner pressure and potential temperature [21, 33, 35, 36, 75] do not conserve mass, momentum, and energy. This is a severe drawback, especially when simulating long durations. Several operational models solve the equations as the conservation of mass, momentum, and potential temperature [2, 30, 40, 66, 70, 76] by assuming adiabatic flows [15]. Some recent efforts [1, 3, 10, 27, 31, 62] proposed solving the Euler equations as the conservation of mass, momentum, and energy [46]. If discretized by a conservative numerical method, this approach yields a truly conservative algorithm, and also allows for the specification of the true viscous terms.

The Euler equations governing compressible flows describe wave motion at two different temporal scales – the acoustic scale characterized by the speed of sound (relative to the fluid) and the advective scale characterized by the flow velocity. Atmospheric flows are often low-Mach flows where the acoustic scale is significantly faster than the advective scale [11]. In addition, the acoustic modes do not affect weather phenomena significantly. Explicit time integration methods are thus inefficient because the largest stable time step is restricted by the physically inconsequential acoustic time scale. Implicit time integration methods can be unconditionally stable; however, they

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[†]Mathematics & Computer Science Division, Argonne National Laboratory, Lemont, IL 60439

[‡]ghosh@mcs.anl.gov

[§]emconsta@mcs.anl.gov

have been rarely applied to atmospheric flows. One of their drawbacks is that they require the solution to either a nonlinear system of equations or a linearized approximation that introduces an error in the overall discretization. A second-order accurate implicit Navier-Stokes solver based on the Crank-Nicholson method was proposed [54] for modeling hurricanes, where the spatial discretization uses finite differences on a staggered grid. A discontinuous Galerkin solver was developed that used the Rosenbrock scheme for time integration [68]. More recently, a finite volume compressible flow solver with second order explicit-first-step, single-diagonal-coefficient, diagonally implicit Runge-Kutta (ESDIRK) time integration was applied to mesoscale flows [76].

An alternative approach to more efficient time integration is an operator-split method, where the flux operator is split into its fast (acoustic) and slow (advective) components and each component is integrated in time separately. A split-explicit time integration method was proposed [44, 64] where the leapfrog method is used for the slow component and a forward-backward method is used with smaller time steps for the fast component. This algorithm was extended by replacing the leapfrog method with Runge-Kutta methods [73] and used in the dynamical core of the Weather Research and Forecasting (WRF) code [43, 65]. A variant of this algorithm is based on the Adams-Bashforth-Moulton method [72]. A split-explicit two-step peer method was proposed [37] that avoids the artificial damping of the fast modes. These methods are a form of decoupled multirate methods [23, 13, 61].

Semi-implicit or implicit-explicit (IMEX) approaches stabilize the fast modes by integrating them implicitly in time; time-step sizes are thus dictated by the slow scales. Semi-implicit methods for the primitive meteorological equations were introduced [45, 11] where the terms involving pressure and gravitational forces are integrated using the trapezoidal scheme and the remaining terms are integrated using the leapfrog method. A split-step semi-implicit method for the Euler equations expressed in terms of the primitive flow variables was proposed [22]; the prognostic variables are perturbations to the hydrostatic mean profile and the acoustic modes are separated by decomposing the velocity into its anelastic, curl-free, and harmonic components. Partially implicit peer methods were applied to atmospheric flows [38]; the governing equations were the linearized Euler equations expressed in terms of the velocity and perturbations to the density and potential temperature. Multistep IMEX methods based on the Adam's method and backward differencing were proposed [20] and applied to the compressible Boussinesq equations [19]. Other notable algorithms include a semi-Lagrangian semi-implicit method [9], all-scale models [67, 8], and a split-step algorithm [71] that does not split the prognostic variables into the hydrostatic mean and perturbations. Drawbacks of these efforts include lack of conservation (due to the form of the governing equations, the operator splitting for semi-implicit time integration, or the choice of the implicit and explicit methods in the semi-implicit time integration) and lack of higher-than-second-order accuracy. An operator splitting was introduced for the governing equations expressed as perturbations to the hydrostatic mean [32]; first-order perturbations are defined as the “fast” terms and treated implicitly, while second- and higher-order perturbations are treated explicitly. Multistep and multistage semi-implicit methods were then applied [32, 30] and the overall algorithm was shown to be consistent, conservative, and high-order accurate in time. In addition to scale separation between the acoustic and advective modes, splitting by dimension is also possible, leading to horizontally explicit, vertically implicit algorithms [62, 70, 30].

This paper presents a characteristic-based partitioning of the hyperbolic flux for

the semi-implicit time integration of limited-area and mesoscale atmospheric flows. Our motivation is the development of a conservative, high-order accurate atmospheric flow solver based on the Euler equations expressed as the conservation of mass, momentum, and energy, with no other assumptions. In addition, the equations are not expressed as perturbations to a hydrostatic mean profile; a well-balanced algorithm [27] is used to ensure numerical accuracy. Thus, the formulation of our governing equations does not involve any assumptions or manipulations specific to atmospheric flows. In contrast to previous approaches, we define the fast and slow components of the hyperbolic flux by partitioning it into its acoustic and advective components in the characteristic space. The ordinary differential equation (ODE) resulting from the spatial discretization of the partitioned flux thus comprises scale-separated terms; eigenvalues of the fast term correspond to the acoustic mode, and the eigenvalues of the flow term correspond to the advective mode. In the context of implicit time integration methods, characteristic-based partitioning has been previously applied to selectively precondition the stiff characteristic modes of a hyperbolic system [55]. We also present a linearization of this partitioning such that the solution to a linear system of equations is required for the semi-implicit time integration; as opposed to implicit time integration, where the solution to a nonlinear system of equations is needed. Moreover, we show that this linearization does not introduce an error in the overall discretization. The partitioned equations are integrated with semi-implicit additive Runge-Kutta (ARK) methods [41, 30] implemented in the time integration module (TS) of the Portable, Extensible Toolkit for Scientific Computing (PETSc) [6, 7]; all the methods considered have an explicit first stage and a single diagonal coefficient. We show that this partitioning of the flux allows time step sizes that depend only on the advective speeds. We also verify that the overall algorithm is conservative and achieves its theoretical orders of convergence.

The characteristic-based semi-implicit time integration is applied to a conservative finite-difference algorithm developed for the Euler equations with gravitational source terms [27]. Several operational weather prediction codes are based on such finite-difference methods [66, 35, 16]. High-order finite-volume [70, 2, 10, 3, 76] and finite-element or spectral-element [31, 32, 40] solvers have been successfully developed and applied to numerical weather prediction. Although atmospheric flows are low-speed flows, they often develop strong gradients, and stabilizing mechanisms are required such as total variation-diminishing/bounded (TVD/TVB) discretizations [2, 3, 76], applying a filter [31], adding artificial diffusion [31, 70], or incorporating a subgrid-scale model [51]. In this paper, we use the fifth-order weighted essentially nonoscillatory (WENO) [50, 39] and the compact-reconstruction WENO (CRWENO) [25, 26, 29] schemes for the spatial discretization; these methods yield solutions that are nonoscillatory across strong gradients and are high-order accurate in smooth regions of the flow. Although the characteristic-based flux partitioning and its use with semi-implicit time integration methods are described for the spatial discretization schemes considered here, it can be easily adapted and applied to other spatial discretizations as well.

The outline of the paper is as follows. Section 2 describes the governing equations, and Section 3 outlines the overall numerical method, including the spatial discretization. The characteristic-based flux partitioning is introduced in Section 4. Section 5 describes the semi-implicit time integration and the implementation of the linearized characteristic-based partitioning with multistage ARK methods. The extension to two dimensional flows is outlined in Section 6. The proposed algorithm is tested for

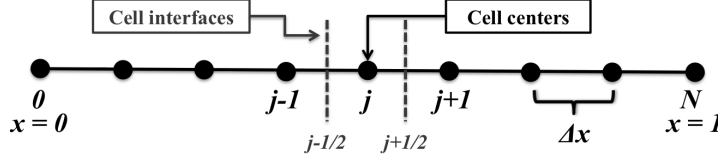


FIG. 3.1. Illustration of a one-dimensional domain and the grid on which (3.1) is discretized.

small problems in Section 7 and applied to atmospheric flow problems in Section 8. Section 9 contains concluding remarks.

2. Governing Equations. The governing equations for nonhydrostatic atmospheric flows are the Euler equations [46], with the addition of gravitational and Coriolis forces as source terms. Various equation sets have been used in the literature [31, 32]. We consider the equations stated as the conservation of mass, momentum, and energy. Mesoscale and limited-area flows are considered, and Coriolis forces are neglected. Thus, the equations are expressed as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (2.1)$$

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + p \mathcal{I}_d) = -\rho \mathbf{g}, \quad (2.2)$$

$$\frac{\partial e}{\partial t} + \nabla \cdot (e + p) \mathbf{u} = -\rho \mathbf{g} \cdot \mathbf{u}, \quad (2.3)$$

where ρ is the density, \mathbf{u} is the velocity vector, p is the pressure, and \mathbf{g} is the gravitational force vector (per unit mass). \mathcal{I}_d denotes the identity matrix of size d , where d is the number of space dimensions. The energy is given by

$$e = \frac{p}{\gamma - 1} + \frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u}, \quad (2.4)$$

where $\gamma = 1.4$ is the specific heat ratio. The equation of state relates the pressure, density, and temperature as $p = \rho R T$, where R is the universal gas constant and T is the temperature. Two additional quantities of interest in atmospheric flows are the Exner pressure π and the potential temperature θ , defined as

$$\pi = \left(\frac{p}{p_0} \right)^{\frac{\gamma-1}{\gamma}}, \text{ and } \theta = \frac{T}{\pi}, \quad (2.5)$$

respectively. The pressure at a reference altitude is denoted by p_0 . We consider one- and two-dimensional flows ($d = 1, 2$) in this paper. The governing equations share the same form as (2.1)–(2.3) when expressed in terms of nondimensional variables [27], and thus these equations can be used for both dimensional and nondimensional problems.

3. Numerical Methodology. The numerical discretization of the governing equations is described in one spatial dimension, and it can be trivially extended to multiple dimensions. Equations (2.1)–(2.3) (with $d = 1$) can be expressed as a system of hyperbolic partial differential equations (PDEs),

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{q})}{\partial x} = \mathbf{s}(\mathbf{q}), \quad (3.1)$$

where

$$\mathbf{q} = \begin{bmatrix} \rho \\ \rho u \\ e \end{bmatrix}, \mathbf{f} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (e + p)u \end{bmatrix}, \text{ and } \mathbf{s} = \begin{bmatrix} 0 \\ -\rho g \\ -\rho u g \end{bmatrix}. \quad (3.2)$$

Equation (3.1) is discretized in space with a conservative finite-difference formulation. Figure 3.1 shows a one-dimensional domain of unit length, discretized by $N + 1$ grid points. The cell centers and interfaces are shown. The resulting semi-discrete ODE in time is given by

$$\frac{d\mathbf{Q}}{dt} = \hat{\mathbf{F}}(\mathbf{Q}) + \hat{\mathbf{S}}(\mathbf{Q}), \quad (3.3)$$

where $\mathbf{Q} = [\mathbf{q}_j; j = 1, \dots, N - 1]$ is the solution vector of the state variable at the cell centers (excluding boundary points), $\hat{\mathbf{S}}$ is the discretized source term, and the discretized hyperbolic flux at a grid point is given by

$$\hat{\mathbf{F}}_j = -\frac{1}{\Delta x} \left(\hat{\mathbf{f}}_{j+1/2} - \hat{\mathbf{f}}_{j-1/2} \right). \quad (3.4)$$

The numerical flux $\hat{\mathbf{f}}$ is an approximation to the primitive of $\mathbf{f}(\mathbf{q})$ at the cell interfaces $x_{j\pm 1/2}$.

Equation (3.1) represents a hyperbolic conservation law that admits equilibrium states where the pressure gradient is balanced by the gravitational force. The spatially discretized ODE, (3.3), must preserve this balance on a finite grid to machine precision; failure to do so will result in inaccurate solutions since atmospheric phenomena are often small perturbations around this balanced equilibrium state. A well-balanced formulation for the Euler equations with gravitational source terms for conservative finite-difference methods was introduced [74] and extended to atmospheric flows [27]. A description of the well-balanced approach used in this study to evaluate the source term $\hat{\mathbf{S}}$ [27] is omitted because it is independent of the time integration aspects discussed here; however, it is a necessary component of the overall algorithm.

We briefly summarize the computation of the numerical flux at the cell interfaces $\hat{\mathbf{f}}_{j\pm 1/2}$ in (3.4) from the cell-centered values of the hyperbolic flux $\mathbf{f}_j = \mathbf{f}(\mathbf{q}_j)$ using the fifth-order WENO [50, 39] and the CRWENO [25, 26] schemes. The description below applies to a scalar flux function, and it is extended to the vector flux in (3.3) through a component-wise approach.

3.1. WENO Schemes. The WENO schemes use a solution-dependent interpolation stencil selection [50] to achieve high-order accuracy where the solution is smooth and to avoid oscillations across discontinuities. Low-order interpolation schemes are identified at a given cell interface, and the final interpolation method is their weighted sum. The weights approach optimal values where the solution within the interpolation stencil is smooth, so that the resulting method achieves a higher order of accuracy. If any stencil for the low-order interpolation schemes contains a discontinuity, the corresponding weight approaches zero, and the final method has a stencil biased away from the discontinuity. Thus, numerical oscillations are avoided.

The fifth-order WENO scheme [39] is constructed by three third-order interpola-

tion schemes:

$$\hat{f}_{j+1/2}^1 = \frac{1}{3}f_{j-2} - \frac{7}{6}f_{j-1} + \frac{11}{6}f_j, \quad c_1 = \frac{1}{10}, \quad (3.5)$$

$$\hat{f}_{j+1/2}^2 = -\frac{1}{6}f_{j-1} + \frac{5}{6}f_j + \frac{1}{3}f_{j+1}, \quad c_2 = \frac{6}{10}, \quad (3.6)$$

$$\hat{f}_{j+1/2}^3 = \frac{1}{3}f_j + \frac{5}{6}f_{j+1} - \frac{1}{6}f_{j+2}, \quad c_3 = \frac{3}{10}. \quad (3.7)$$

Multiplying (3.5)–(3.7) with their optimal coefficient c_k , $k = 1, 2, 3$, and then adding them results in the following fifth-order accurate interpolation scheme:

$$\hat{f}_{j+1/2} = \frac{1}{30}f_{j-2} - \frac{13}{60}f_{j-1} + \frac{47}{60}f_j + \frac{27}{60}f_{j+1} - \frac{1}{20}f_{j+2}. \quad (3.8)$$

Solution-dependent weights are computed based on the local solution smoothness as

$$\omega_k = \frac{\alpha_k}{\sum_k \alpha_k}; \quad \alpha_k = \frac{c_k}{(\epsilon + \beta_k)^p}; \quad k = 1, 2, 3, \quad (3.9)$$

where $\epsilon = 10^{-6}$ is a small number to prevent division by zero and β_k are the smoothness indicators for the stencils, given by

$$\beta_1 = \frac{13}{12}(f_{j-2} - 2f_{j-1} + f_j)^2 + \frac{1}{4}(f_{j-2} - 4f_{j-1} + 3f_j)^2, \quad (3.10)$$

$$\beta_2 = \frac{13}{12}(f_{j-1} - 2f_j + f_{j+1})^2 + \frac{1}{4}(f_{j-1} - f_{j+1})^2, \quad (3.11)$$

$$\text{and } \beta_3 = \frac{13}{12}(f_j - 2f_{j+1} + f_{j+2})^2 + \frac{1}{4}(3f_j - 4f_{j+1} + f_{j+2})^2. \quad (3.12)$$

The fifth-order WENO (WENO5) scheme is obtained by multiplying (3.5)–(3.7) by the solution-dependent weights ω_k (instead of the optimal coefficients c_k) and then adding them. It can be expressed as

$$\begin{aligned} \hat{f}_{j+1/2} &= \frac{\omega_1}{3}f_{j-2} - \frac{1}{6}(7\omega_1 + \omega_2)f_{j-1} + \frac{1}{6}(11\omega_1 + 5\omega_2 + 2\omega_3)f_j \\ &\quad + \frac{1}{6}(2\omega_2 + 5\omega_3)f_{j+1} - \frac{\omega_3}{6}f_{j+2}. \end{aligned} \quad (3.13)$$

If the solution is locally smooth, $\omega_k \rightarrow c_k$, $k = 1, 2, 3$, and (3.13) is equivalent to (3.8).

3.2. CRWENO Schemes. Compact finite-difference schemes [47] compute the unknown flux derivative or the flux at the interfaces implicitly. Therefore, they require the solution to a linear system of equations. The benefits are higher spectral resolution and lower absolute errors for the same order of convergence, when compared with standard finite-difference schemes. The CRWENO scheme [25] applies the WENO concept of solution-dependent interpolation stencils to compact finite-difference methods. The fifth-order CRWENO scheme [25, 26] is constructed by considering three third-order compact interpolation schemes:

$$\frac{2}{3}\hat{f}_{j-1/2} + \frac{1}{3}\hat{f}_{j+1/2} = \frac{1}{6}(f_{j-1} + 5f_j); \quad c_1 = \frac{2}{10}, \quad (3.14)$$

$$\frac{1}{3}\hat{f}_{j-1/2} + \frac{2}{3}\hat{f}_{j+1/2} = \frac{1}{6}(5f_j + f_{j+1}); \quad c_2 = \frac{5}{10}, \quad (3.15)$$

$$\frac{2}{3}\hat{f}_{j+1/2} + \frac{1}{3}\hat{f}_{j+3/2} = \frac{1}{6}(f_j + 5f_{j+1}); \quad c_3 = \frac{3}{10}. \quad (3.16)$$

Multiplying (3.14)–(3.16) with their optimal coefficients (c_k , $k = 1, 2, 3$) and adding them results in a fifth-order compact scheme:

$$\frac{3}{10}\hat{f}_{j-1/2} + \frac{6}{10}\hat{f}_{j+1/2} + \frac{1}{10}\hat{f}_{j+3/2} = \frac{1}{30}f_{j-1} + \frac{19}{30}f_j + \frac{1}{3}f_{j+1}. \quad (3.17)$$

Replacing the optimal coefficients c_k with solution-dependent weights ω_k yields the fifth-order CRWENO scheme (CRWENO5):

$$\begin{aligned} \left(\frac{2}{3}\omega_1 + \frac{1}{3}\omega_2\right)\hat{f}_{j-1/2} + \left[\frac{1}{3}\omega_1 + \frac{2}{3}(\omega_2 + \omega_3)\right]\hat{f}_{j+1/2} + \frac{1}{3}\omega_3\hat{f}_{j+3/2} \\ = \frac{\omega_1}{6}f_{j-1} + \frac{5(\omega_1 + \omega_2) + \omega_3}{6}f_j + \frac{\omega_2 + 5\omega_3}{6}f_{j+1}. \end{aligned} \quad (3.18)$$

The weights ω_k are computed by (3.9) and (3.10)–(3.12). If the solution is locally smooth, $\omega_k \rightarrow c_k$, $k = 1, 2, 3$, and (3.18) is equivalent to (3.17). The left-hand side of (3.18) represents a tridiagonal system with solution-dependent coefficients that needs to be solved at each time-integration step or stage. An efficient and scalable implementation of the CRWENO5 scheme [28] is used in this study.

3.3. Upwinding. Solutions to hyperbolic PDEs comprise waves traveling at their characteristic speeds. The numerical flux at the cell interfaces needs to be computed in an upwind manner, and the stencil for the spatial discretization must be biased according to the direction of wave propagation. Equations (3.13) and (3.18) describe the left-biased WENO5 and CRWENO5 schemes, assuming a positive wave speed. The right-biased schemes corresponding to a negative wave speed can be similarly obtained. The Rusanov upwinding scheme [58, 48] is used to obtain the final flux at the interface from the left and right-biased approximations:

$$\hat{\mathbf{f}}_{j+1/2} = \frac{1}{2} \left[\hat{\mathbf{f}}_{j+1/2}^L + \hat{\mathbf{f}}_{j+1/2}^R - \left(\max_{j,j+1} \nu \right) \left(\hat{\mathbf{q}}_{j+1/2}^R - \hat{\mathbf{q}}_{j+1/2}^L \right) \right], \quad (3.19)$$

where the superscripts L and R indicate the left- and right-biased interpolations, respectively. The dissipation factor is $\nu = a + |u|$, where $a = \sqrt{\gamma p / \rho}$ is the speed of sound and $\hat{\mathbf{q}}_{j+1/2}^{L,R}$ are the left- and right-biased interface values for \mathbf{q} that are computed in the same manner as $\hat{\mathbf{f}}_{j+1/2}^{L,R}$.

4. Characteristic-Based Flux Partitioning. The separation of the acoustic and advective components of the hyperbolic flux is described by considering (3.1) and its semi-discrete form (3.3), without the source terms. The one-dimensional Euler equations, although nonlinear, satisfy the following property [46],

$$\mathbf{f}(\mathbf{q}) = \mathcal{A}\mathbf{q}, \quad \mathcal{A}(\mathbf{q}) = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}, \quad (4.1)$$

where \mathcal{A} is the flux Jacobian. Though this property is not essential to the flux partitioning, it is useful as a tool to describe it. Equation (3.3) (without the source term) can be expressed as

$$\frac{d\mathbf{Q}}{dt} = \hat{\mathbf{F}}(\mathbf{Q}) \equiv [\mathcal{D} \otimes \mathcal{A}] \mathbf{Q}, \quad (4.2)$$

where \mathcal{D} represents a finite-difference operator for a scalar function $\phi(x)$ on a grid,

$$-[\phi_{x,j}]^T = [\mathcal{D}][\phi_j]^T + O(\delta x^r), \quad 0 < j < N, \quad \phi_j = \phi(x_j), \quad \phi_{x,j} = \phi_x(x_j) \quad (4.3)$$

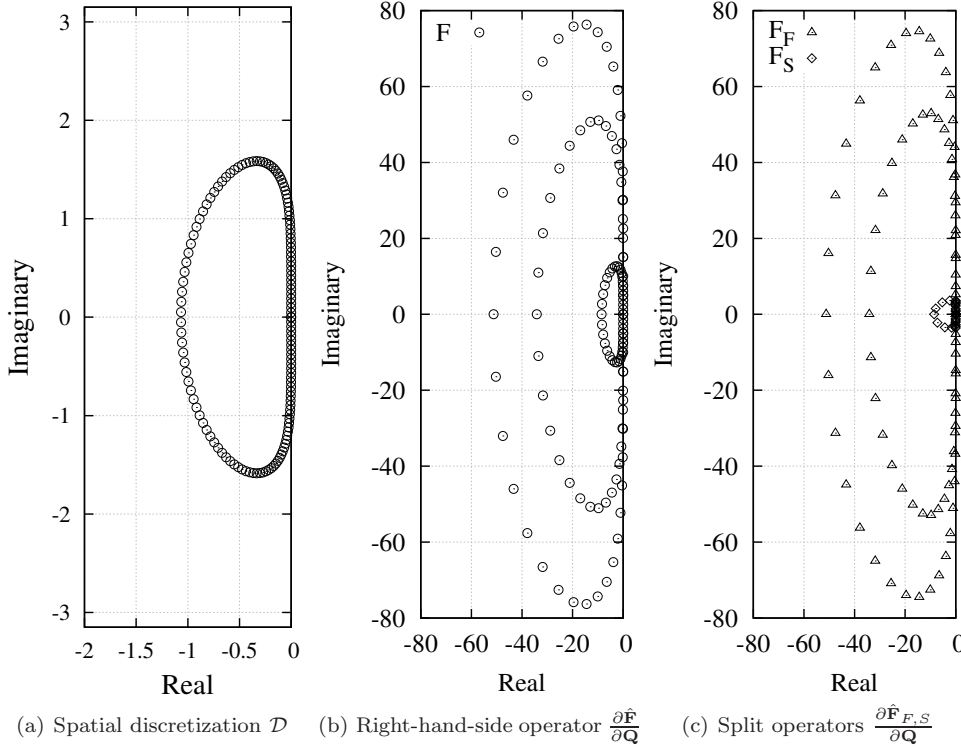


FIG. 4.1. *Eigenvalues of the spatial discretization operator corresponding to WENO5, the Jacobian of the right-hand side of (4.2), and the Jacobians of the fast and slow partitioned terms of (4.13).*

with r being the spatial order of accuracy. The WENO5 and CRWENO5 schemes, described in the previous section, can be expressed in this form [24]. Therefore, the eigenvalues of the right-hand side (RHS) operator of (4.2) are the products of the eigenvalues of the discretization operator \mathcal{D} and the eigenvalues of the flux Jacobian that are the characteristic wave speeds of the Euler equations,

$$\lambda \left(\frac{\partial \hat{\mathbf{F}}}{\partial \mathbf{Q}} \right) = \lambda(\mathcal{D}) * \lambda(\mathcal{A}). \quad (4.4)$$

The flux Jacobian has three real eigenvalues [46],

$$\lambda(\mathcal{A}) = \{u, u + a, u - a\}, \quad (4.5)$$

where u is the flow velocity and a is the local speed of sound. Figure 4.1(a) shows the eigenvalues of the finite-difference operator \mathcal{D} representing the WENO5 scheme, computed by using a linear spectral analysis [26]. Figure 4.1(b) shows the eigenvalues of the Jacobian of $\hat{\mathbf{F}}$ evaluated on a periodic domain of unit length, discretized by a grid with 40 points and the WENO5 scheme, with the following initial flow:

$$\rho = \rho_\infty + \hat{\rho} \sin(2\pi x), \quad u = u_\infty, \quad p = p_\infty, \quad (4.6)$$

where $\rho_\infty = 1$, $u_\infty = 0.2$, $p_\infty = 1/\gamma$, and $\hat{\rho} = 0.1$. The mean speed of sound is $a_\infty = \sqrt{\gamma p_\infty / \rho_\infty} = 1$, and therefore the mean Mach number is $M_\infty = u_\infty / a_\infty = 0.2$.

This flow represents the advection of a smooth (sinusoidal) density wave at a constant speed u_∞ . The Jacobian of $\hat{\mathbf{F}}$ is computed by using finite differences. The eigenvalues in Figure 4.1(b) form three distinct sets that correspond to the eigenvalues of \mathcal{D} multiplied by each of the characteristic wave speeds of the Euler equations. The smallest ring represents the advective mode (u) where the eigenvalues of \mathcal{D} are scaled by $u/\Delta x$. The two larger rings represent the acoustic modes ($u \pm a$) where the eigenvalues of \mathcal{D} are scaled by $(u \pm a)/\Delta x$. The separation in magnitude of the acoustic and advective eigenvalues is a function of the Mach number $M = u/a$; lower Mach numbers result in a larger separation.

The acoustic and the advective modes in the RHS operator $\hat{\mathbf{F}}$ of (4.2) are partitioned in the characteristic space as follows:

$$\begin{aligned}\hat{\mathbf{F}}(\mathbf{Q}) &= [\mathcal{D} \otimes \mathcal{A}] \mathbf{Q} = [\mathcal{D} \otimes (\mathcal{A}_F + \mathcal{A}_S)] \mathbf{Q} \\ &= [\mathcal{D} \otimes \mathcal{A}_F] \mathbf{Q} + [\mathcal{D} \otimes \mathcal{A}_S] \mathbf{Q} = \hat{\mathbf{F}}_F + \hat{\mathbf{F}}_S,\end{aligned}\quad (4.7)$$

where the subscripts F and S denote “fast” and “slow” time scales, respectively. The partitioning of the flux Jacobian is defined as

$$\mathcal{A} = \mathcal{A}_S + \mathcal{A}_F; \quad \mathcal{A}_{F,S} = \mathcal{X} \Lambda_{F,S} \mathcal{X}^{-1}, \quad (4.8)$$

where

$$\Lambda_F + \Lambda_S = \Lambda = \text{diag}[u, u - a, u + a] = \mathcal{X}^{-1} \mathcal{A} \mathcal{X}. \quad (4.9)$$

Λ is the diagonal matrix with the eigenvalues of \mathcal{A} as its elements, \mathcal{X} is the matrix with the right eigenvectors as its columns, and \mathcal{X}^{-1} is the matrix with the left eigenvectors as its rows. $\Lambda_{F,S}$ represent the fast (acoustic) and slow (advective) characteristic modes, defined as

$$\Lambda_F = \text{diag}[0, u - a, u + a], \quad \Lambda_S = \text{diag}[u, 0, 0]. \quad (4.10)$$

Thus, the fast flux term $\hat{\mathbf{F}}_F$ represents only the acoustic modes, while the slow flux term $\hat{\mathbf{F}}_S$ represents the advective mode. Figure 4.1(c) shows the eigenvalues of the Jacobians of the partitioned flux terms for the same flow as in Figure 4.1(b). The partitioning results in a clear separation of the advective and acoustic eigenvalues; the eigenvalues of the slow term are significantly smaller in magnitude than those of the fast term. We note that the eigenvalues of the partitioned terms $\hat{\mathbf{F}}_{F,S}$ do not correspond exactly to the eigenvalues of $\hat{\mathbf{F}}$ because of the nonlinearity of the Euler equations,

$$\frac{\partial \hat{\mathbf{F}}_{F,S}}{\partial \mathbf{Q}} \neq \mathcal{D} \otimes \mathcal{A}_{F,S} \Rightarrow \Lambda \left[\frac{\partial \hat{\mathbf{F}}_F}{\partial \mathbf{Q}} \right] \cup \Lambda \left[\frac{\partial \hat{\mathbf{F}}_S}{\partial \mathbf{Q}} \right] \neq \Lambda \left[\frac{\partial \hat{\mathbf{F}}}{\partial \mathbf{Q}} \right]. \quad (4.11)$$

Atmospheric flows are low-speed flows where the advective mode is significantly slower than the acoustic modes ($u \ll a$). The separation of the two time scales is useful in the context of semi-implicit time integration, discussed in the next section.

With the partitioning defined as (4.8), the governing equations (3.1) can be written as

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}_S(\mathbf{q})}{\partial x} + \frac{\partial \mathbf{f}_F(\mathbf{q})}{\partial x} = \mathbf{s}(\mathbf{q}), \quad (4.12)$$

where $\mathbf{f}_{F,S} = \mathcal{A}_{F,S}\mathbf{q}$ are the partitioned hyperbolic flux terms and the corresponding semi-discrete ODE can be expressed as

$$\frac{d\mathbf{Q}}{dt} = \left\{ \hat{\mathbf{F}}_F(\mathbf{Q}) + \hat{\mathbf{F}}_S(\mathbf{Q}) \right\} + \hat{\mathbf{S}}(\mathbf{Q}). \quad (4.13)$$

We note that (4.7) holds true if and only if both the slow and fast flux terms $\mathbf{f}_{F,S}$ are discretized by the same finite-difference operator \mathcal{D} . In the context of the nonlinear WENO5 and CRWENO5 schemes, this implies that the same solution-dependent coefficients for the interpolation operators (3.13) or (3.18) need to be used for discretizing \mathbf{f}_S and \mathbf{f}_F . In our implementation, the WENO coefficients (3.9) are computed based on $\mathbf{f}(\mathbf{q})$. The exact forms for $\mathcal{A}_{F,S}(\mathbf{q})$ and $\mathbf{f}_{F,S}(\mathbf{q})$ are provided in the Appendix A for completeness.

5. Time Integration. Equation (4.13) is integrated in time by using semi-implicit additive Runge-Kutta (ARK) methods [5, 41, 53] implemented in the time integration module (TS) of PETSc [6, 7]. These methods apply two different integrators for the slow and the fast terms; the fast terms are integrated in time implicitly, and thus the largest stable time step size of the algorithm is determined by the eigenvalues of the slow term. ARK methods can be represented with the following Butcher tableaux [12]:

$$\left(\begin{array}{c|c} c_i & a_{ij} \\ \hline & b_j \end{array}, \begin{array}{c|c} \tilde{c}_i & \tilde{a}_{ij} \\ \hline & \tilde{b}_j \end{array}; i, j = 1, \dots, s \right), c_i = \sum_{j=1}^s a_{ij}, \tilde{c}_i = \sum_{j=1}^s \tilde{a}_{ij}, \quad (5.1)$$

where a_{ij}, b_j, c_i define the explicit integrator for the slow term, $\tilde{a}_{ij}, \tilde{b}_j, \tilde{c}_i$ define the implicit integrator for the fast term, and s is the number of stages. The coefficients satisfy $a_{ij} = 0, j \geq i$ and $\tilde{a}_{ij} = 0, j > i$. The ARK methods applied to (4.13) and using coefficients (5.1) results in the following:

Stage computations $i = 1, \dots, s$:

$$\mathbf{Q}^{(i)} = \mathbf{Q}^n + \Delta t \sum_{j=1}^{i-1} a_{ij} \hat{\mathbf{F}}_S(\mathbf{Q}^{(j)}) + \Delta t \sum_{j=1}^i \tilde{a}_{ij} \left\{ \hat{\mathbf{F}}_F(\mathbf{Q}^{(j)}) + \hat{\mathbf{S}}(\mathbf{Q}^{(j)}) \right\}, \quad (5.2a)$$

Step completion :

$$\mathbf{Q}^{n+1} = \mathbf{Q}^n + \Delta t \sum_{i=1}^s b_i \hat{\mathbf{F}}_S(\mathbf{Q}^{(i)}) + \Delta t \sum_{i=1}^s \tilde{b}_i \left\{ \hat{\mathbf{F}}_F(\mathbf{Q}^{(i)}) + \hat{\mathbf{S}}(\mathbf{Q}^{(i)}) \right\}, \quad (5.2b)$$

where \mathbf{Q}^n is the solution at the current time step and \mathbf{Q}^{n+1} is the solution at the next time step. The gravitational source term is treated implicitly in time.

Three high-order ARK methods are considered in this study. A second-order ARK method [30] with the following Butcher tableaux is referred to as ARK 2c:

$$\left(\begin{array}{c|ccc} 0 & 0 & & \\ \hline 2 - \sqrt{2} & 2 - \sqrt{2} & 0 & \\ 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ \hline & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 1 - \frac{1}{\sqrt{2}} \end{array}, \begin{array}{c|ccc} 0 & & & \\ \hline 2 - \sqrt{2} & 1 - \frac{1}{\sqrt{2}} & 1 - \frac{1}{\sqrt{2}} & \\ 1 & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 1 - \frac{1}{\sqrt{2}} \\ \hline & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 1 - \frac{1}{\sqrt{2}} \end{array} \right). \quad (5.3)$$

This method has an L-stable implicit part, and the explicit part has a large stability region along the real axis. In addition, we consider the third-order (four stages) and

TABLE 5.1

List of time integration methods and their orders and number of stages.

Name	Type	Order	Stages (s)	Comments/Reference
ARK 2c	Semi-implicit	2	3	[30]
ARK 3	Semi-implicit	3	4	[41]
ARK 4	Semi-implicit	4	6	[41]
RK 2a	Explicit	2	2	Explicit midpoint method
RK 3	Explicit	3	3	Kutta's third-order method
RK 4	Explicit	4	4	Classical fourth-order method

fourth-order (six stages) methods constructed in [41] and refer to them as ARK 3 and ARK 4, respectively. The implicit parts of the ARK methods used here are ESDIRK (explicit first-stage, single-diagonal coefficient) and L-stable. The performance of the ARK methods is compared with that of the explicit RK methods: second-order, two-stage RK 2a, third-order, three-stage RK 3, and the classical fourth-order four-stage RK 4. Table 5.1 summarizes the time integration methods used in this paper.

5.1. Linearization. The stage calculations (5.2a) require the solution of a nonlinear system of equations. We modify the partitioning of the RHS so that only a linear system needs to be solved instead. The fast term is linearized and the implicit integrator is applied on this linear part. The slow term, redefined as total RHS with the linearized fast term subtracted from it, is treated explicitly. The linearized fast term removes the stiffness from the original RHS and reduces the computational cost of solving the implicit part. We note that the linearization does not introduce an error in the overall discretized equations.

Ignoring the source term for now, we rewrite (5.2a) as the following nonlinear system of equations for an implicit ARK stage:

$$\begin{aligned} \mathbf{Q}^{(i)} - \sigma \hat{\mathbf{F}}_F(\mathbf{Q}^{(i)}) &= \mathbf{Q}^n + \Delta t \sum_{j=1}^{i-1} \left\{ a_{ij} \hat{\mathbf{F}}_S(\mathbf{Q}^{(j)}) + \tilde{a}_{ij} \hat{\mathbf{F}}_F(\mathbf{Q}^{(j)}) \right\}, \\ \Rightarrow [\mathcal{I} - \sigma \mathcal{D} \otimes \mathcal{A}_F] \mathbf{Q}^{(i)} &= \mathbf{Q}^n + \Delta t \sum_{j=1}^{i-1} \left\{ a_{ij} \hat{\mathbf{F}}_S(\mathbf{Q}^{(j)}) + \tilde{a}_{ij} \hat{\mathbf{F}}_F(\mathbf{Q}^{(j)}) \right\}, \end{aligned} \quad (5.4)$$

where $\sigma = \Delta t \tilde{a}_{ii}$. The nonlinearity of (5.4) arises from two sources: the fast Jacobian $\mathcal{A}_F = \mathcal{A}_F(\mathbf{Q})$ and the WENO5/CRWENO5 finite-difference operator $\mathcal{D} = \mathcal{D}(\omega)$, where $\omega = \omega(\mathbf{f}(\mathbf{q}))$ are the solution-dependent weights given by (3.9).

The fast Jacobian is evaluated at the beginning of the step and kept fixed for all the stages. The partitioning of the flux at stage i is modified as follows:

$$\mathbf{f}_F(\mathbf{Q}^{(i)}) = \mathcal{A}_F(\mathbf{Q}^n) \mathbf{Q}^{(i)}, \quad \mathbf{f}_S(\mathbf{Q}^{(i)}) = \mathbf{f}(\mathbf{Q}^{(i)}) - \mathbf{f}_F(\mathbf{Q}^{(i)}). \quad (5.5)$$

The corresponding expressions for spatially discretized partitioned flux terms are

$$\begin{aligned} \hat{\mathbf{F}}_F(\mathbf{Q}^{(i)}) &= [\mathcal{D} \otimes \mathcal{A}_F(\mathbf{Q}^n)] \mathbf{Q}^{(i)}, \\ \hat{\mathbf{F}}_S(\mathbf{Q}^{(i)}) &= \hat{\mathbf{F}}(\mathbf{Q}^{(i)}) - \hat{\mathbf{F}}_F(\mathbf{Q}^{(i)}) = \left[\mathcal{D} \otimes \left\{ \mathcal{A}(\mathbf{Q}^{(i)}) - \mathcal{A}_F(\mathbf{Q}^n) \right\} \right] \mathbf{Q}^{(i)}. \end{aligned} \quad (5.6)$$

Equation (5.6) satisfies $\hat{\mathbf{F}}_F(\mathbf{Q}^{(i)}) + \hat{\mathbf{F}}_S(\mathbf{Q}^{(i)}) = \hat{\mathbf{F}}(\mathbf{Q}^{(i)})$ exactly. Therefore, the linearized partitioning is consistent with the unpartitioned RHS, and does not introduce an error in the overall algorithm.

The nonlinear finite-difference operator $\mathcal{D}(\omega)$ is linearized by computing and fixing the solution-dependent weights (3.9) at the beginning of each stage. The computation of $\mathbf{F}(\mathbf{Q}^{(i)})$ and $\mathbf{F}_F(\mathbf{Q}^{(i)})$ during the iterative solution of (5.4) does not recalculate the weights ω based on the smoothness of the current guess for $\mathbf{Q}^{(i)}$. We define the finite-difference operator at stage i as

$$\bar{\mathcal{D}} = \mathcal{D}(\bar{\omega}), \text{ where } \bar{\omega} = \begin{cases} \omega(\mathbf{f}(\mathbf{Q}^{(i-1)})) & i > 1 \\ \omega(\mathbf{f}(\mathbf{Q}^n)) & i = 1 \end{cases}. \quad (5.7)$$

Thus, during the stage computation, the interpolation coefficients in (3.13) or (3.18) are constant, and the resulting operators are linear.

Inspection of (3.1) shows that the source term is linear if the gravitational forces do not depend on the solution (this is true for our application). As previously mentioned, a well-balanced formulation [27] is used to evaluate it on the discretized domain; this formulation preserves its linearity. Denoting $\mathcal{S} = \partial \mathbf{S} / \partial \mathbf{Q}$ as the Jacobian of the source term, (5.6) and (5.7) are applied to (5.2a) to obtain the following linear system of equations for the implicit ARK stages:

$$\begin{aligned} & [\mathcal{I} - \sigma \{\bar{\mathcal{D}} \otimes \mathcal{A}_F(\mathbf{Q}^n) + \mathcal{S}\}] \mathbf{Q}^{(i)} \\ &= \mathbf{Q}^n + \Delta t \sum_{j=1}^{i-1} \left\{ a_{ij} \hat{\mathbf{F}}_S(\mathbf{Q}^{(j)}) + \tilde{a}_{ij} [\bar{\mathcal{D}} \otimes \mathcal{A}_F(\mathbf{Q}^n) + \mathcal{S}] \mathbf{Q}^{(j)} \right\}, \end{aligned} \quad (5.8)$$

where $\hat{\mathbf{F}}_S$ is defined by (5.6). Equation (5.8) is solved iteratively by using the generalized residual method (GMRES) [59, 60] implemented in the Krylov solver module (KSP) of PETSc, and a Jacobian-free approach is adopted where the Jacobian

$$\mathcal{J} \equiv [\mathcal{I} - \sigma \bar{\mathcal{D}} \otimes \mathcal{A}_F(\mathbf{Q}^n) + \mathcal{S}] \quad (5.9)$$

is specified as its action on a vector. The stopping criterion for iterative solver is specified as

$$\|\mathbf{r}_{k+1} - \mathbf{r}_k\|_2 \leq \max(\tau_r \|\mathbf{r}_0\|_2, \tau_a), \quad (5.10)$$

where τ_a and τ_r are the absolute and relative tolerances, respectively, \mathbf{r} is the residual given by

$$\begin{aligned} \mathbf{r}_k &= [\mathcal{I} - \sigma \{\bar{\mathcal{D}} \otimes \mathcal{A}_F(\mathbf{Q}^n) + \mathcal{S}\}] \mathbf{Q}_k^{(i)} \\ &- \left[\mathbf{Q}^n + \Delta t \sum_{j=1}^{i-1} \left\{ a_{ij} \hat{\mathbf{F}}_S(\mathbf{Q}^{(j)}) + \tilde{a}_{ij} [\bar{\mathcal{D}} \otimes \mathcal{A}_F(\mathbf{Q}^n) + \mathcal{S}] \mathbf{Q}^{(j)} \right\} \right], \end{aligned} \quad (5.11)$$

and the subscript k denotes the k th guess for the stage solution $\mathbf{Q}^{(i)}$.

5.2. Modified Upwinding. The interpolated flux at a grid interface is computed by using (3.19), which can be written for the total and the fast flux terms as follows:

$$\hat{\mathbf{f}}_{j+1/2} = \frac{1}{2} \left[\hat{\mathbf{f}}_{j+1/2}^L + \hat{\mathbf{f}}_{j+1/2}^R - \delta_{j+1/2} \left(\hat{\mathbf{q}}_{j+1/2}^R - \hat{\mathbf{q}}_{j+1/2}^L \right) \right], \quad (5.12)$$

$$\hat{\mathbf{f}}_{F,j+1/2} = \frac{1}{2} \left[\hat{\mathbf{f}}_{F,j+1/2}^L + \hat{\mathbf{f}}_{F,j+1/2}^R - \delta_{j+1/2}^F \left(\hat{\mathbf{q}}_{j+1/2}^R - \hat{\mathbf{q}}_{j+1/2}^L \right) \right], \quad (5.13)$$

where δ is the diffusion coefficient for the upwinding scheme. Subtracting (5.13) from (5.12) results in the expression for the slow flux given by

$$\begin{aligned} \hat{\mathbf{f}}_{S,j+1/2} = & \frac{1}{2} \left[\left(\hat{\mathbf{f}}_{j+1/2}^L + \hat{\mathbf{f}}_{j+1/2}^R \right) - \left(\hat{\mathbf{f}}_{F,j+1/2}^L + \hat{\mathbf{f}}_{F,j+1/2}^R \right) \right] \\ & - \frac{1}{2} \left[\left(\delta_{j+1/2} - \delta_{j+1/2}^F \right) \left(\hat{\mathbf{q}}_{j+1/2}^R - \hat{\mathbf{q}}_{j+1/2}^L \right) \right]. \end{aligned} \quad (5.14)$$

By using the same diffusion coefficient for both total and the fast flux

$$\delta_{j+1/2} = \delta_{j+1/2}^F = \max_{j,j+1} \nu, \quad (5.15)$$

we obtain a central discretization of the slow flux term (5.14) with no diffusion. The eigenvalues of the discretized slow flux are thus purely imaginary. This imposes tight restrictions on the classes of methods that can be used in practice because of the stability properties of the time integration methods. Constructing semi-implicit time integrators with large imaginary stability of the explicit part is challenging. Moreover, the imaginary stability of the explicit component is negatively impacted by the presence of eigenvalues with small real parts in the implicit component [14], as is the case in our applications. Semi-implicit methods with explicit imaginary stability that are less dependent of the implicit operator have been constructed [41, 20, 30]; however, relaxing this restriction allows for more efficient methods. As an example, we are forced to fix the coefficients $a_{3,1} = a_{3,2} = \frac{1}{2}$ in (5.3); a detailed discussion is presented in [30].

We modify the Rusanov upwinding such that it applies the diffusion specifically to the characteristic fields that the flux term represents. The diffusion coefficients are matrices given by

$$\left[\tilde{\delta} \right]_{j+1/2} = \mathcal{X} \begin{bmatrix} \bar{\mu} & & \\ & \bar{\nu} & \\ & & \bar{\nu} \end{bmatrix} \mathcal{X}^{-1}, \quad \left[\tilde{\delta}^F \right]_{j+1/2} = \mathcal{X} \begin{bmatrix} 0 & & \\ & \bar{\nu} & \\ & & \bar{\nu} \end{bmatrix} \mathcal{X}^{-1}, \quad (5.16)$$

where

$$\bar{\nu} = \max_{j,j+1} (|u| + a), \quad \bar{\mu} = \max_{j,j+1} |u|. \quad (5.17)$$

Thus, the equations to compute the flux at the grid interfaces from their left- and right-biased interpolated values are

$$\hat{\mathbf{f}}_{j+1/2} = \frac{1}{2} \left\{ \hat{\mathbf{f}}_{j+1/2}^L + \hat{\mathbf{f}}_{j+1/2}^R - \left[\tilde{\delta} \right]_{j+1/2} \left(\hat{\mathbf{q}}_{j+1/2}^R - \hat{\mathbf{q}}_{j+1/2}^L \right) \right\}, \quad (5.18)$$

$$\hat{\mathbf{f}}_{F,j+1/2} = \frac{1}{2} \left\{ \hat{\mathbf{f}}_{F,j+1/2}^L + \hat{\mathbf{f}}_{F,j+1/2}^R - \left[\tilde{\delta}^F \right]_{j+1/2} \left(\hat{\mathbf{q}}_{j+1/2}^R - \hat{\mathbf{q}}_{j+1/2}^L \right) \right\}. \quad (5.19)$$

The modified Rusanov scheme used here resembles the Roe upwinding scheme [56].

5.3. Preconditioning. The block Jacobi preconditioner [59], implemented in the preconditioning module (PC) of PETSc, is used in the current work. Although the Jacobian of the implicitly treated operator is specified in a matrix-free way (5.9), an approximation to the Jacobian is provided as a sparse matrix. The approximate Jacobian for the preconditioner is defined as

$$\mathcal{J}_p \equiv [\mathcal{I} - \sigma \bar{\mathcal{D}}_{1st} \otimes \mathcal{A}_F(\mathbf{Q}^n) + \mathcal{S}] \approx \mathcal{J}, \quad (5.20)$$

where \mathcal{D}_{1st} represents a first-order upwind discretization operator. This results in a block tridiagonal matrix for the one-dimensional system and will result in block penta- and septa-diagonal systems for two- and three-dimensional flows, respectively. Development of more advanced preconditioning techniques for the algorithm presented here is beyond the scope of this paper and will be studied in the future.

6. Extension to Two-Dimensional Flows. The two-dimension Euler equations with gravitational source terms can be expressed as the following hyperbolic conservation law:

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{q})}{\partial x} + \frac{\partial \mathbf{h}(\mathbf{q})}{\partial y} = \mathbf{s}(\mathbf{q}), \quad (6.1)$$

where

$$\mathbf{q} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ e \end{bmatrix}, \mathbf{f} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ (e + p)u \end{bmatrix}, \mathbf{h} = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ (e + p)v \end{bmatrix}, \mathbf{s} = \begin{bmatrix} 0 \\ -\rho \mathbf{g} \cdot \hat{\mathbf{i}} \\ -\rho \mathbf{g} \cdot \hat{\mathbf{j}} \\ -(\rho u \mathbf{g} \cdot \hat{\mathbf{i}} + \rho v \mathbf{g} \cdot \hat{\mathbf{j}}) \end{bmatrix}.$$

The Cartesian unit vectors along x and y are denoted by $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, respectively, and u, v are the velocity components along x, y . The spatial discretization described in the previous sections is extended to the two-dimension equations through a dimension-by-dimension approach, where the derivatives along one dimension are computed independently of the other dimension. This paper considers only problems solved on Cartesian grids. The eigenvalues of the two-dimensional system are given by

$$\Lambda \left[\frac{\partial (\mathbf{f}, \mathbf{h})}{\partial \mathbf{q}} \right] = \{(u, v), (u, v), (u, v) - a, (u, v) + a\}, \quad (6.2)$$

and they are split into their advective and acoustic components as

$$\Lambda_S = \{(u, v), (u, v), 0, 0\}, \Lambda_F = \{0, 0, (u, v) - a, (u, v) + a\}. \quad (6.3)$$

The slow and fast Jacobians are obtained by using the similarity transformations given by (4.8), and the partitioned flux and its spatially discretized counterpart are then computed. The left and right eigenvectors for the two-dimensional Euler equations are provided in [34, 57], and we use these in this paper. The resulting semi-discrete ODE can be expressed as

$$\frac{d\mathbf{Q}}{dt} = \left\{ \hat{\mathbf{F}}_F(\mathbf{Q}) + \hat{\mathbf{H}}_F(\mathbf{Q}) \right\} + \left\{ \hat{\mathbf{F}}_S(\mathbf{Q}) + \hat{\mathbf{H}}_S(\mathbf{Q}) \right\} + \hat{\mathbf{S}}(\mathbf{Q}), \quad (6.4)$$

where $\hat{\mathbf{H}}_{F,S}$ denotes the spatially discretized partitioned fluxes along the y -direction. Equation (6.4) is integrated in time by using an ARK method given by (5.1), where the fast flux terms and the source term $\left\{ \hat{\mathbf{F}}_F + \hat{\mathbf{H}}_F \right\} + \hat{\mathbf{S}}$ are treated implicitly and the slow flux terms $\left\{ \hat{\mathbf{F}}_S + \hat{\mathbf{H}}_S \right\}$ are treated explicitly.

7. Numerical Tests. The performance of the semi-implicit time integrators with characteristic-based flux partitioning is tested in this section with two simple flow problems. The tests verify that the integration of the acoustic modes in time using an implicit method results in a largest stable time step that is determined by the advective scale. In addition, the accuracy and convergence of the time integration methods are demonstrated. The two problems solved in this section are formulated in terms of nondimensional flow variables.

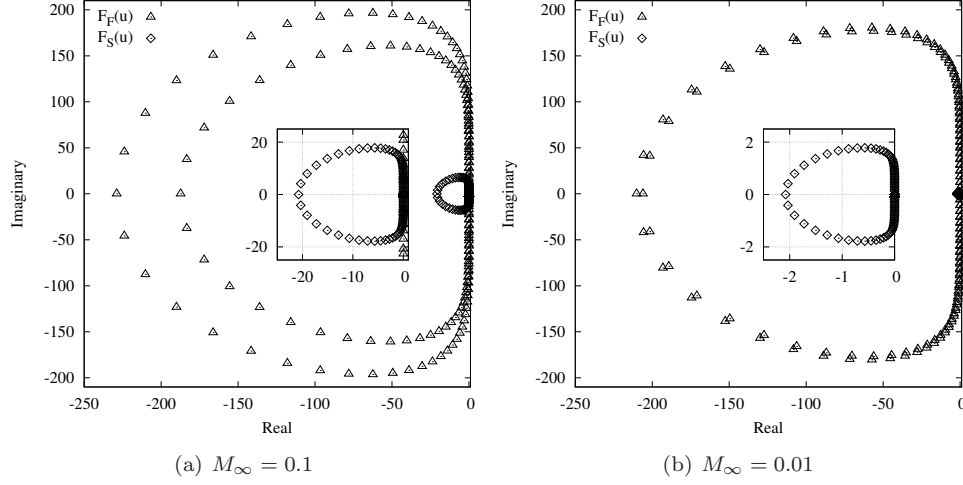


FIG. 7.1. Eigenvalues of Jacobians of the partitioned flux terms $\hat{\mathbf{F}}_{F,S}$ for the one-dimensional density wave advection at two Mach numbers. The CRWENO5 scheme is used, and the problem is discretized on a grid with 80 points. The insets are magnified plots of the eigenvalues of the slow flux term $\hat{\mathbf{F}}_S$.

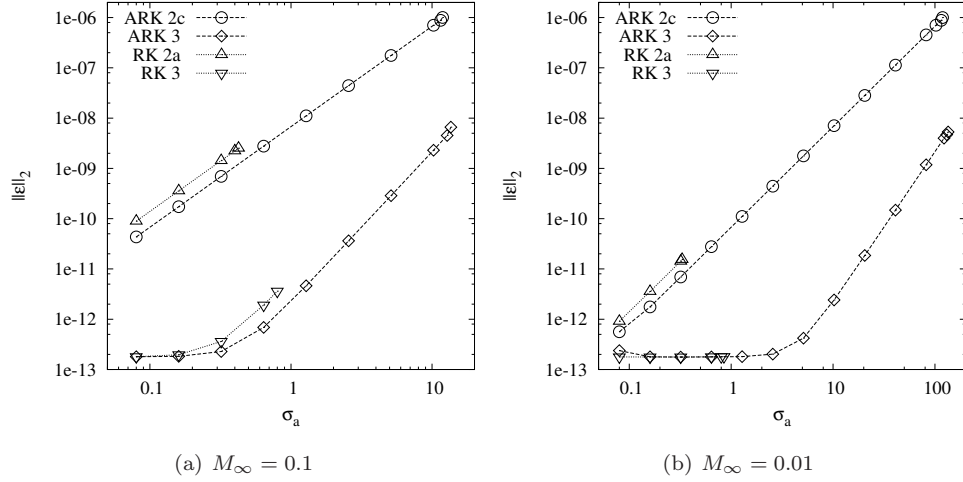


FIG. 7.2. Density wave advection: L_2 norm of the error as a function of the acoustic CFL number for the ARK and explicit RK methods.

7.1. Density Wave Advection. This one-dimensional test problem involves the advection of a sinusoidal density wave over a periodic domain. The exact solution is given by

$$\rho(x, t) = \rho_\infty + \hat{\rho} \sin[2\pi(x - u_\infty t)], \quad u(x, t) = u_\infty, \quad p(x, t) = p_\infty. \quad (7.1)$$

With this solution, the Euler equations are equivalent to the linear advection equation. The mean density and pressure are taken as $\rho_\infty = 1$ and $p_\infty = 1/\gamma$, resulting in the mean speed of sound as $a_\infty = 1$. We consider two values for the mean Mach number (given by $M_\infty = u_\infty/a_\infty$): 0.1 and 0.01. The domain is $x \in [0, 1]$, and periodic

boundary conditions are applied at the boundaries.

Figure 7.1 shows the eigenvalues of the partitioned flux Jacobians for the CR-WENO5 scheme on a grid with 80 points. The Jacobians are evaluated from the discretized operators $\hat{\mathbf{F}}_{F,S}$ through finite differences. The eigenvalues for the case with mean Mach number 0.1 is shown in Figure 7.1(a), with the magnified subplot showing the eigenvalues of the slow flux term $\hat{\mathbf{F}}_S$. As shown earlier, the flux partitioning results in a separation of the advective and acoustic modes. The eigenvalues of the slow flux correspond to the advective mode, and they are smaller in magnitude than those of the fast flux by an approximate factor of 10 (the inverse of the Mach number). Figure 7.1(b) shows the eigenvalues for the case with a mean Mach number of 0.01. At this smaller Mach number, the separation between the advective and acoustic scales is larger. The magnitudes of the eigenvalues of the slow flux are smaller than those of the fast flux by an approximate factor of $M_\infty^{-1} = 100$. The two acoustic modes are characterized by the wave speeds $u \pm a$; and thus, as Mach number decreases, they converge to a .

Figure 7.2 shows the error as a function of the acoustic Courant-Friedrichs-Lewy (CFL) for the second- and third-order ARK methods, ARK 2c and ARK 3, as well as the two explicit RK methods of the same orders, RK 2a and RK 3. The solutions are obtained with the tolerances for the iterative solver specified as $\tau_r = \tau_a = 10^{-10}$. The final times for both cases correspond to one cycle over the periodic domain (10 for $M_\infty = 0.1$ and 100 for $M_\infty = 0.01$). The time step sizes are increased from a small value until it reaches a value for which the solution blows up, thus indicating the largest stable time step size of that time integrator. The error and the acoustic CFL are defined as

$$\epsilon = \mathbf{Q}(x, t) - \mathbf{Q}_{\text{exact}}(x, t), \quad \sigma_a = a_\infty \frac{\Delta t}{\Delta x}, \quad (7.2)$$

where $\mathbf{Q}_{\text{exact}}$ is given by (7.1). The ARK methods converge at their theoretical orders for all the cases. The case with $M_\infty = 0.1$ is shown in Figure 7.2(a), and the largest stable time steps for the ARK methods are observed to be larger than those of the explicit RK methods by a factor of approximately $M_\infty^{-1} = 10$. The explicit RK methods are restricted in their time step size by the acoustic mode, while the implicit treatment of the acoustic modes in the ARK method allows time step sizes restricted by the advective mode. Figure 7.2(b) shows the case with $M_\infty = 0.01$. The advective eigenvalues are smaller in magnitude for this lower Mach number, and therefore larger time step sizes are possible. The largest time step sizes for the ARK methods are again approximately $M_\infty^{-1} = 100$ times larger than those of the explicit RK methods. This demonstrates that the stability limits for the ARK methods are determined by the advective time scale because of the characteristic-based flux partitioning.

7.2. Isentropic Vortex Convection. The convection of an isentropic vortex [63] is used to test the flux partitioning in two dimensions. The flow involves the inviscid convection of a vortex over a periodic domain and tests the ability of the numerical method to preserve the shape and strength of the vortex. We modify the original test case by reducing the Mach number. The domain is specified as $(x, y) \in [0, 10]^2$, and the mean (freestream) flow is

$$\rho_\infty = 1, \quad u_\infty = 0.1, \quad v_\infty = 0, \quad p_\infty = 1. \quad (7.3)$$

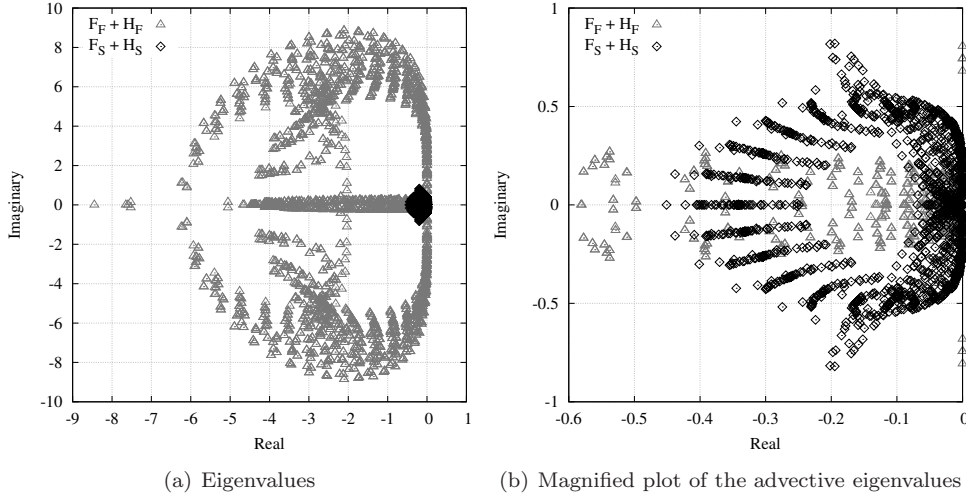


FIG. 7.3. *Eigenvalues of Jacobians of the partitioned flux terms $(\hat{\mathbf{F}}_F + \hat{\mathbf{H}}_F)$ and $(\hat{\mathbf{F}}_S + \hat{\mathbf{H}}_S)$ in (6.4) for the isentropic vortex convection case. The WENO5 scheme is used and the problem is discretized on a grid with 32^2 points.*

A vortex is introduced in the flow, whose density and pressure are specified as

$$\rho = \left[1 - \frac{(\gamma - 1)b^2}{8\gamma\pi^2} e^{1-r^2} \right]^{\frac{1}{\gamma-1}}, \quad p = \rho^\gamma, \quad (7.4)$$

and thus, $\rho, p \rightarrow \rho_\infty, p_\infty$ as $r \rightarrow \infty$. The velocity field is

$$u = u_\infty - \frac{b}{2\pi} e^{\frac{1}{2}(1-r^2)} (y - y_c), \quad v = v_\infty + \frac{b}{2\pi} e^{\frac{1}{2}(1-r^2)} (x - x_c), \quad (7.5)$$

where $b = 0.5$ is the vortex strength and $r = [(x - x_c)^2 + (y - y_c)^2]^{1/2}$ is the distance from the vortex center $(x_c, y_c) = (5, 5)$. Periodic boundary conditions are applied at all boundaries. As the solution is evolved in time, the vortex convects over the periodic domain with a time period of $T_p = 100$.

Figure 7.3 shows the eigenvalues of the partitioned Jacobians for the WENO5 scheme on a grid with 32^2 points. The freestream Mach number for this example is $M_\infty \approx 0.08$, and thus we see a significant separation in the magnitudes of the advective and acoustic eigenvalues. Figure 7.3(b) is a magnified plot of the advective eigenvalues. This demonstrates that the extension of the characteristic-based partitioning to two dimensions, as described in Section 6, works as expected.

Figures 7.4(a)–7.4(d) show the density contours of the flow for solutions obtained on a grid with 64^2 points and the WENO5 scheme. The final time is 200, corresponding to 2 cycles over the periodic domain. The ARK 4 time integration method is used, and solutions for time step sizes corresponding to acoustic CFL numbers from 0.8 to 7.6 are shown. The solution obtained with the explicit RK 4 method at an acoustic CFL number of 0.4 is also shown in Figure 7.4(e). The horizontal cross-sectional density profile through $y = 5$ for these solutions is shown in Figure 7.4(f). The solutions obtained with the ARK 4 schemes at all the CFL numbers considered agree well with that obtained with the explicit RK 4 scheme.

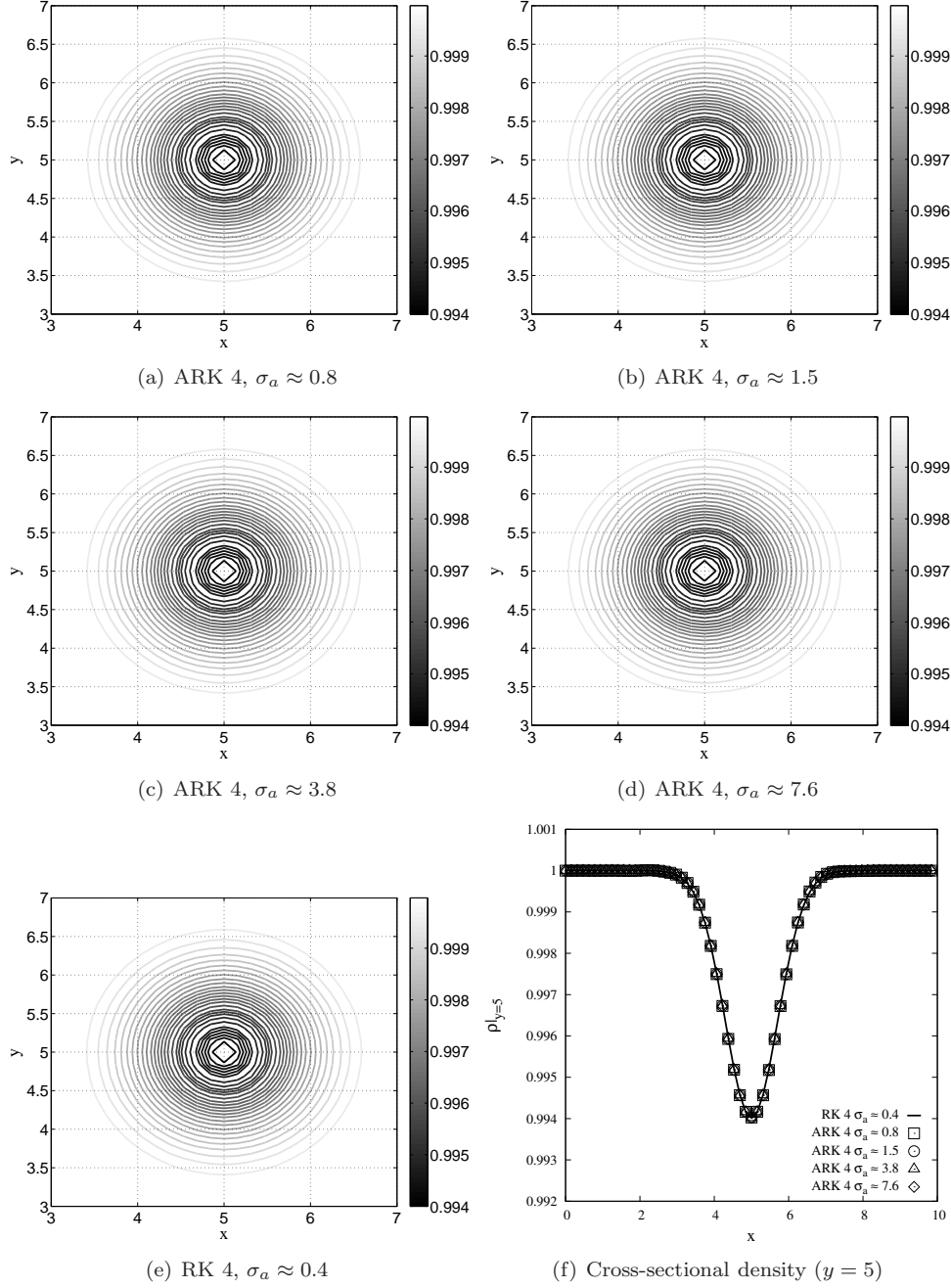


FIG. 7.4. Density contours of the isentropic vortex convection case after 2 cycles over the periodic domain obtained with the WENO5 scheme on a grid with 64^2 points, and the cross-sectional density profile at $y = 5$; σ_a is the acoustic CFL number.

The error as function of the acoustic CFL is shown in Figure 7.5(a). The solutions are obtained on a grid with 32^2 points with the WENO5 scheme after one cycle over the periodic domain. The tolerances for the GMRES solver are specified as

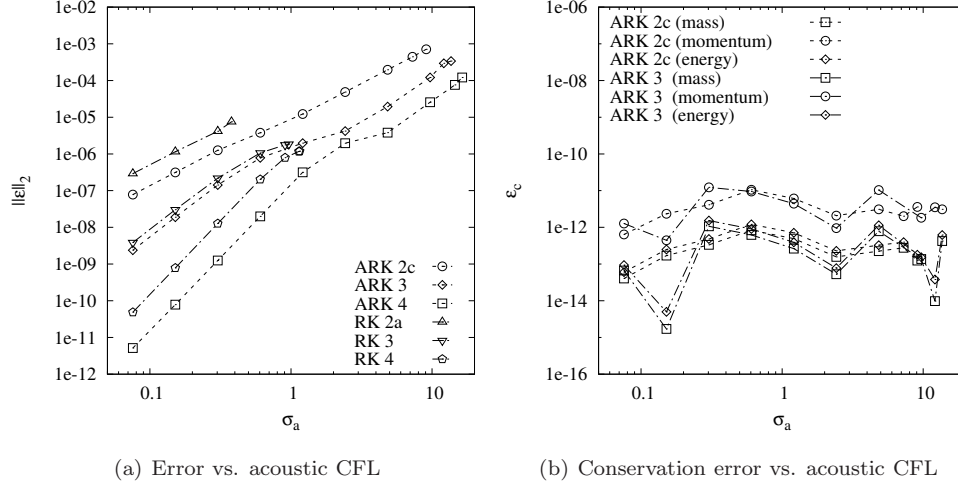


FIG. 7.5. Solution error (ϵ) and conservation error (ϵ_c) as a function of the acoustic CFL σ_a for the isentropic vortex convection. The solutions are obtained on a 32^2 grid with the WENO5 scheme at a final time of 100 (one cycle over the domain).

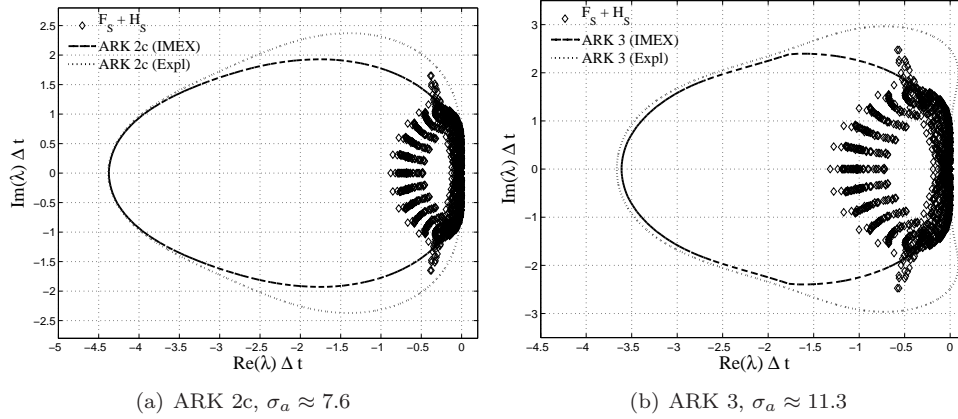


FIG. 7.6. Eigenvalues of the slow partitioned term ($\hat{\mathbf{F}}_S + \hat{\mathbf{H}}_S$) multiplied by the time step Δt , and the stability regions of the explicit components of the ARK time integration methods. “IMEX” denotes the stability region of the explicit method when the implicit method handles the eigenvalues of ($\hat{\mathbf{F}}_F + \hat{\mathbf{H}}_F$), and “Expl” denotes its stability region when it is used by itself as an explicit time integrator.

$\tau_a = \tau_r = 10^{-10}$. We start the tests with an initially small time step and increase it until it reaches the stability limit of the time integrator being used. The error and the acoustic CFL are defined as

$$\epsilon = \mathbf{Q}(x, y, t) - \mathbf{Q}_{\text{ref}}(x, y, t), \quad \sigma_a = a_\infty \frac{\Delta t}{\min(\Delta x, \Delta y)}, \quad (7.6)$$

where $\mathbf{Q}_{\text{ref}}(x, y, t)$ is the reference solution obtained with the explicit RK 4 time integration method with a very small time step of 0.0005. The ARK methods converge at their theoretical orders for acoustic CFL numbers less than 1; at higher CFL

numbers, the acoustic mode is not resolved and thus convergence is only second order. However, the absolute errors for a higher-order ARK method (say, ARK 4) are smaller than those for a lower-order ARK method (say, ARK 2c). The largest stable time step for the ARK methods are larger than those of the explicit RK methods by a factor of approximately M_∞^{-1} , thus demonstrating that the time step size is determined by the advective scale. Figure 7.6 shows the eigenvalues of the slow operator $(\hat{\mathbf{F}}_S + \hat{\mathbf{H}}_S)$ scaled by the time step Δt and the stability regions of the explicit components of the ARK 2c and ARK 3 methods. The time step Δt corresponds to acoustic CFL numbers of ~ 7.6 for ARK 2c and ~ 11.3 for ARK 3. These are close to the observed largest stable CFL numbers for these methods in Figure 7.5(a). At these time step sizes, the advective eigenvalues have started spilling out of the respective stability regions. Comparison of the stability regions of the explicit method by itself (denoted by “Expl”) and when it is a part of an ARK method with the implicit part handling the eigenvalues of $(\hat{\mathbf{F}}_F + \hat{\mathbf{H}}_F)$ (denoted by “IMEX”) shows significant reduction in the imaginary stability [14].

Figure 7.5(b) shows the conservation errors ϵ_c for mass (ρ), momentum ($\rho\mathbf{u}$), and energy (e) as a function of the acoustic CFL, for the ARK 2c and ARK 3 methods. The conservation error is defined as

$$\epsilon_c = \frac{1}{\bar{Q}^k(0)} [\bar{Q}^k(t) - \bar{Q}^k(0)], \quad \bar{Q}^k(t) = \int_V \|\mathbf{Q}^k(x, y, t)\|_2 dV, \quad (7.7)$$

where \bar{Q} is the volume integral over the domain, V denotes the two-dimensional domain, and the superscript k denotes the component ($k = 1$ for mass, $k = 2, 3$ for momentum, and $k = 4$ for energy). The conservation errors are of the order of round-off errors with the specified GMRES tolerances, for both the methods and at all the CFL numbers considered, thus demonstrating that the partitioned semi-implicit algorithm is conservative.

8. Application to Atmospheric Flows. In this section, the algorithm is applied to atmospheric flows, which are governed by the two-dimensional Euler equations with a gravitational source term. Two benchmark flow problems are solved—the inertia-gravity wave and the rising thermal bubble. The flow solver used in this study has been previously verified for atmospheric flows with explicit Runge-Kutta schemes [27]; therefore, the focus of this section is to demonstrate the accuracy, stability, and numerical cost of the ARK methods. We note that the problems solved in this section are in terms of dimensional quantities, unlike the previous section where all quantities were nondimensional.

8.1. Inertia-Gravity Waves. The inertia-gravity wave [64, 31] involves the evolution of a potential temperature perturbation. The domain is a channel with dimensions $300,000 \text{ m} \times 10,000 \text{ m}$. The initial flow consists of a perturbation introduced into a hydrostatically balanced (stratified) atmosphere. The mean flow is the stratified atmosphere with a specified Brunt-Väisälä frequency (\mathcal{N}). The potential temperature and Exner pressure are given by

$$\theta = T_0 \exp\left(\frac{\mathcal{N}^2}{g}y\right), \quad \pi = 1 + \frac{(\gamma - 1)g^2}{\gamma R T_0 \mathcal{N}^2} \left[\exp\left(-\frac{\mathcal{N}^2}{g}y\right) - 1 \right], \quad (8.1)$$

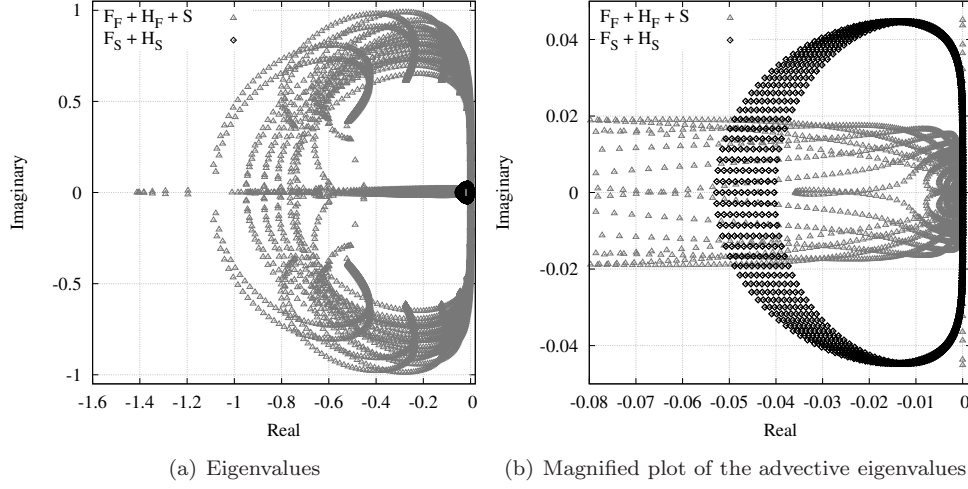


FIG. 8.1. *Eigenvalues of Jacobians of the partitioned terms $(\hat{\mathbf{F}}_F + \hat{\mathbf{H}}_F) + \hat{\mathbf{S}}$ and $(\hat{\mathbf{F}}_S + \hat{\mathbf{H}}_S)$ in (6.4) for the inertia-gravity wave problem. The CRWENO5 scheme is used, and the problem is discretized on a grid with 300×10 points.*

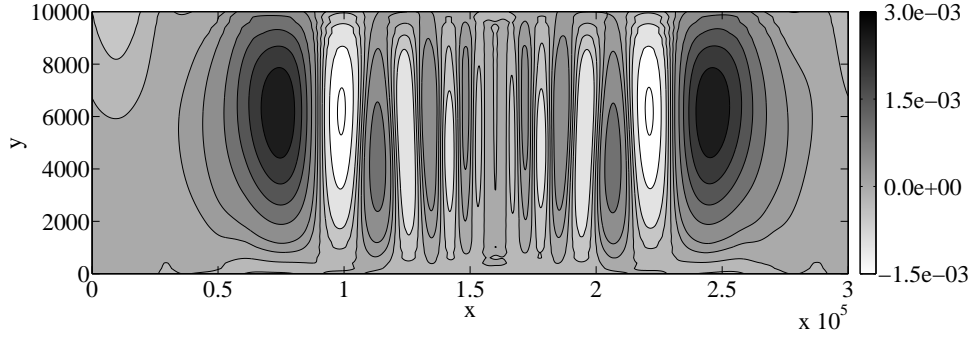


FIG. 8.2. *Inertia-gravity waves: Potential temperature perturbation $\Delta\theta$ at $t = 3000$ s, obtained with the CRWENO5 scheme and the ARK 4 time integrator on a grid with 1200×50 points. The time step is $\Delta t = 12$ s, corresponding to an acoustic CFL number of $\sigma_a \approx 20.8$.*

and the density and pressure are

$$p = p_0 \left[1 + \frac{(\gamma - 1)g^2}{\gamma R T_0 \mathcal{N}^2} \left\{ \exp\left(-\frac{\mathcal{N}^2}{g} y\right) - 1 \right\} \right]^{\gamma/(\gamma-1)}, \quad (8.2)$$

$$\rho = \rho_0 \exp\left(-\frac{\mathcal{N}^2}{g} y\right) \left[1 + \frac{(\gamma - 1)g^2}{\gamma R T_0 \mathcal{N}^2} \left\{ \exp\left(-\frac{\mathcal{N}^2}{g} y\right) - 1 \right\} \right]^{1/(\gamma-1)}. \quad (8.3)$$

The initial velocity components are $u = 20$ m/s and $v = 0$ m/s. Periodic boundary conditions are applied on the left ($x = 0$ m) and right ($x = 300,000$ m) boundaries, while inviscid wall boundary conditions are applied on the bottom ($y = 0$ m) and top ($y = 10,000$ m) boundaries. The Brunt-Väisälä frequency is $\mathcal{N} = 0.01$ /s, and the gravitational force per unit mass is 9.8 m/s² along the y -direction. The reference pressure (p_0) and temperature (T_0) at $y = 0$ m are 10^5 N/m² and 300 K, respectively,

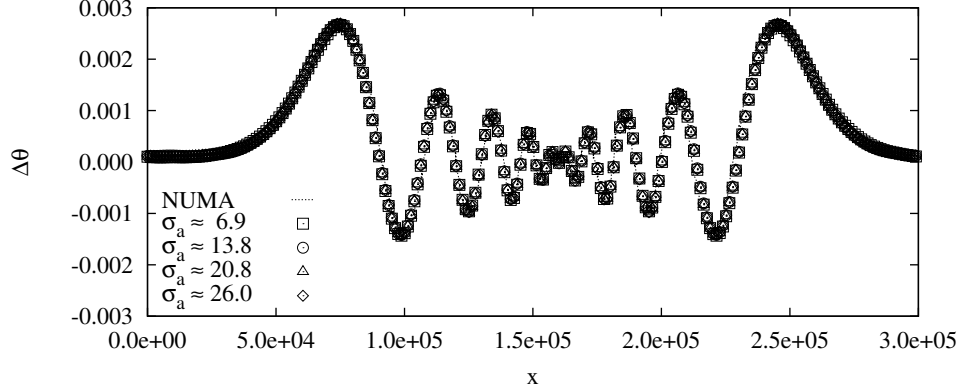


FIG. 8.3. *Inertia-gravity waves: Cross-sectional potential temperature perturbation $\Delta\theta$ at $y = 5000$ m and $t = 3000$ s, obtained with the CRWENO5 scheme and the ARK 4 time integrator on a grid with 1200×50 points. “NUMA” refers to the reference solution obtained with a spectral element solver [31].*

and the reference density is computed from the equation of state $p_0 = \rho_0 R T_0$. The universal gas constant R is 287.058 J/kg K. The perturbation is added to the potential temperature, specified as

$$\Delta\theta(x, y, t = 0) = \theta_c \sin\left(\frac{\pi_c y}{h_c}\right) \left[1 + \left(\frac{x - x_c}{a_c}\right)^2\right]^{-1}, \quad (8.4)$$

where $\theta_c = 0.01$ K is the perturbation strength, $h_c = 10,000$ m is the height of the domain, $a_c = 5,000$ m is the perturbation half-width, $x_c = 100,000$ m is the horizontal location of the perturbation, and $\pi_c \approx 3.141592654$ is the Archimedes (trigonometric) constant. The evolution of the perturbation is simulated until a final time of 3000 s. The reference speed of sound is $a_0 = \sqrt{\gamma R T_0} = 347.22$ m/s, and the reference Mach number for this flow is approximately 0.06 . Figure 8.1 shows the eigenvalues of the slow and the fast operators for the problem discretized on a 300×10 -point grid with the CRWENO5 scheme; the fast operator includes the gravitational source term.

Figure 8.2 shows the potential temperature perturbation $\Delta\theta = (\theta - \theta_0)$ contours for the solution obtained with the CRWENO5 scheme on a grid with 1200×50 points. The ARK 4 method is used for time integration with a time step of $\Delta t = 12$ s, corresponding to an acoustic CFL number of $\sigma_a \approx 20.8$. A good agreement is observed with results in the literature [2, 64, 3, 76]. Figure 8.3 shows the cross-sectional potential temperature perturbation at an altitude of $y = 5,000$ m. The solutions obtained with the ARK 4 methods at various CFL numbers are compared with a reference solution. The reference solution “NUMA” refers to the solution obtained with a spectral-element solver [31], with 10th-order polynomials, 3rd-order explicit RK time integration, and 250 m effective grid resolution. The solutions obtained with the partitioned semi-implicit approach agree well with the reference solution.

Figure 8.4(a) shows the L_2 norm of the solution error as a function of the acoustic CFL for solutions obtained with the CRWENO5 scheme on a 600×20 grid. The error and the acoustic CFL are as defined in (7.6). The reference speed of sound a_0 is used to compute the acoustic CFL, and the reference solution is obtained with the explicit RK 4 time integrator and a very small time step 0.005 . The tolerances for

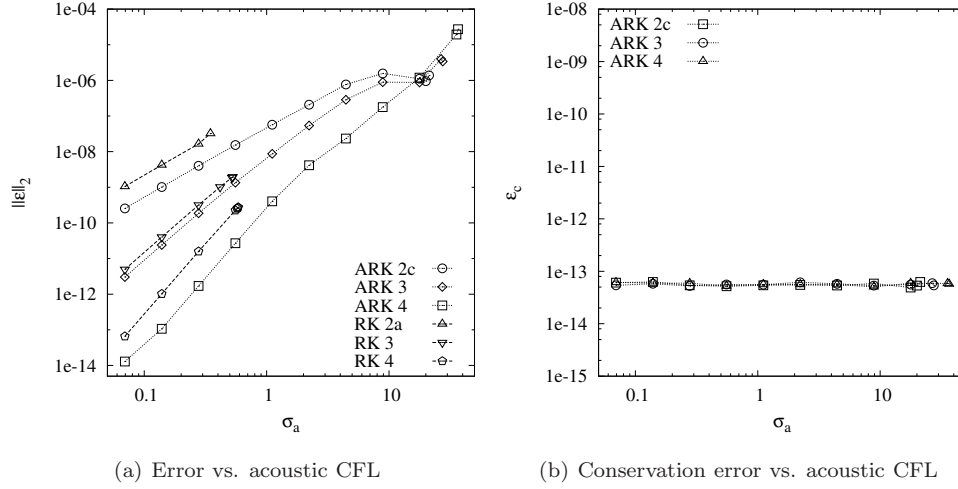


FIG. 8.4. Solution error (ϵ) and conservation error (ϵ_c) as a function of the acoustic CFL σ_a for the inertia-gravity waves. The solutions are obtained on a 600×20 grid with the CRWENO5 scheme at a final time of 500 s.

the GMRES solver are specified as $\tau_a = \tau_r = 10^{10}$. The errors for the partitioned ARK methods are shown, as well as the explicit RK 2a, RK 3, and RK 4 methods. The ARK methods converge at their theoretical orders of convergence, and the largest stable time steps are observed to be approximately $M_\infty^{-1} \approx 15$ times larger than those of the explicit RK methods. The mass conservation errors, defined in (7.7) with $k = 1$, are shown for the ARK methods in Figure 8.4(b). The partitioned algorithm is conservative to round-off error with the specified GMRES tolerances at all the CFL numbers considered.

8.2. Rising Thermal Bubble. The two-dimensional rising thermal bubble [31] simulates the dynamics of a warm bubble. A square domain of size $1000 \text{ m} \times 1000 \text{ m}$ is specified with inviscid wall boundary conditions on all sides. The initial solution is a warm bubble introduced in a hydrostatically balanced atmosphere. The mean flow is the stratified atmosphere with a constant potential temperature $\theta = T_0 = 300 \text{ K}$; and the density, pressure, and velocity are given by

$$\rho = \rho_0 \left[1 - \frac{(\gamma - 1)gy}{\gamma R\theta} \right]^{1/(\gamma-1)}, \quad p = p_0 \left[1 - \frac{(\gamma - 1)gy}{\gamma R\theta} \right]^{\gamma/(\gamma-1)}, \quad u = v = 0. \quad (8.5)$$

The reference pressure is 10^5 N/m^2 , and the reference density is computed from the equation of state $p_0 = \rho_0 R T_0$. The universal gas constant R is 287.058 J/kg K . A constant gravitation force per unit mass of 9.8 m/s^2 is specified along the y -direction. The warm bubble is added as a potential temperature perturbation,

$$\Delta\theta(x, y, t = 0) = \begin{cases} 0 & r > r_c \\ \frac{\theta_c}{2} \left[1 + \cos\left(\frac{\pi_c r}{r_c}\right) \right] & r \leq r_c \end{cases}, \quad r = \sqrt{(x - x_c)^2 + (y - y_c)^2}, \quad (8.6)$$

where $\theta_c = 0.5 \text{ K}$ is the perturbation strength, $(x_c, y_c) = (500, 350) \text{ m}$ is the initial location at which the bubble is centered, $r_c = 250 \text{ m}$ is the radius of the bubble, and π_c is the trigonometric constant. The flow is simulated to a final time of 400 s.

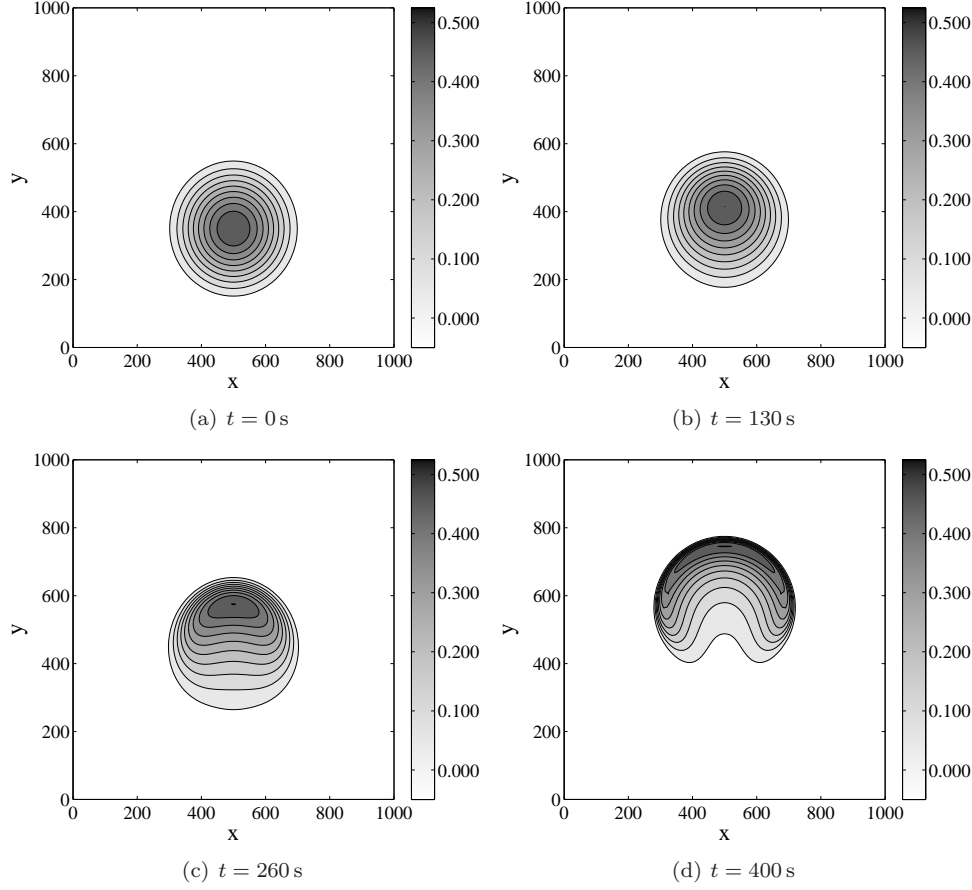


FIG. 8.5. *Rising thermal bubble: Potential temperature perturbation $\Delta\theta$ for the solution obtained with the WENO5 scheme and the ARK 4 time integrator on a grid with 201^2 points. The time step is $\Delta t = 2$ s, corresponding to an acoustic CFL number of $\sigma_a \approx 139$.*

Figure 8.5 shows the evolution of the bubble from $t = 0$ s to $t = 400$ s as it rises due to buoyancy and deforms as a result of the temperature and velocity gradients. The potential temperature perturbation $\Delta\theta = \theta - \theta_0$ is shown. The solution is obtained with the WENO5 scheme and the ARK 4 time integrator on a grid with 201^2 points. The time step size is 2 s, which results in an acoustic CFL number of approximately 139. The acoustic CFL is given by (7.6), and the reference speed of sound a_0 is used. The flow is initially at rest, and thus the advective eigenvalues are all zero. As the bubble rises, it induces a velocity field; at $t = 400$ s, the maximum velocity magnitude in the domain is approximately 2.1 m/s, corresponding to a maximum local Mach number of approximately 0.006. Thus, the disparity between the advective and acoustic scales is very large, and the semi-implicit approach allows time steps that are much larger than those allowed by an explicit time integrator. Figure 8.6 shows the cross-sectional profiles of $\Delta\theta$ for the solutions obtained with the ARK 4 method at various CFL numbers and that obtained with the explicit RK 4 method. The WENO5 scheme is used for the spatial discretization. The solutions obtained with the partitioned ARK method agree well with that obtained with the explicit time

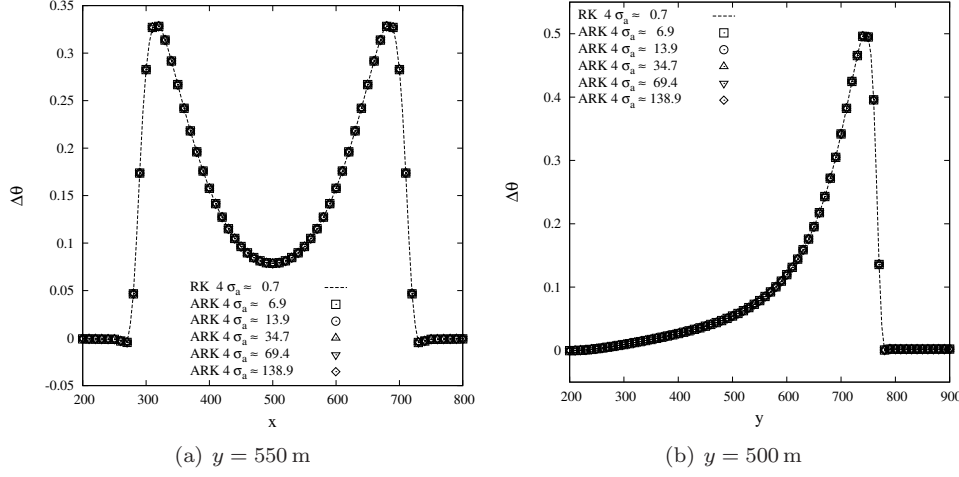


FIG. 8.6. *Rising thermal bubble: Cross-sectional potential temperature perturbation at $y = 550$ m and $x = 500$ m for solutions obtained with the WENO5 scheme on a grid with 201^2 points. The solutions obtained with the ARK 4 method are compared with that obtained with the explicit RK 4 method.*

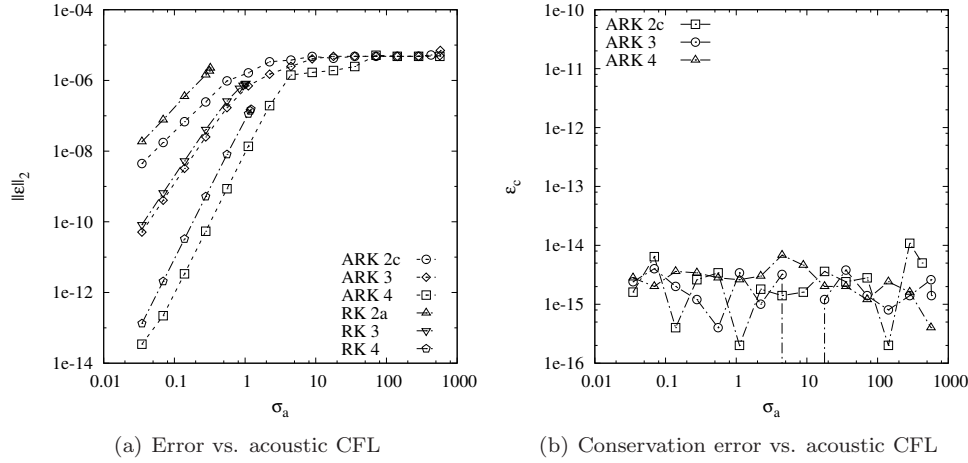


FIG. 8.7. *Solution error (ϵ) and conservation error (ϵ_c) as a function of the acoustic CFL σ_a for the rising thermal bubble. The solutions are obtained on a 51^2 grid with the WENO5 scheme at a final time of 400 s.*

integrator.

The L_2 norm of the solution error is shown in Figure 8.7(a) as a function of the acoustic CFL. The error, defined in (7.6), is computed with respect to a reference solution that is obtained on the same grid with the same spatial discretization and with the explicit RK 4 time integrator with a very small time step size of 10^{-4} . The figure shows the errors for the ARK 2c, ARK 3, and ARK 4 methods, as well as the explicit RK 2a, RK 3, and RK 4 methods. The tolerances specified for the GMRES solver are $\tau_a = \tau_r = 10^{-10}$. In the region where the acoustic waves are resolved and the explicit methods are stable, all the methods converge at their theoretical orders of accuracy. At higher CFL numbers, the acoustic mode is not resolved, and thus the

TABLE 8.1

Inertia-gravity waves: Computational cost as a function of time step size and acoustic CFL number of the ARK and RK methods for solutions on a grid with 1200×50 points discretized in space with the CRWENO5 scheme. The final time is 3000 s. Boldfaced rows indicate the performance at the largest stable time step for the ARK methods.

Method	Δt	n_T	σ_a	n_{FC}	Wall time (s)
RK 2a	0.15	20,000	0.26	40,000	12,353
RK 4	0.30	10,000	0.45	40,000	12,072
ARK 2c	1.0	3,000	1.74	95,968	39,174
	2.0	1,500	3.47	57,121	23,180
	4.0	750	6.94	34,530	14,086
	8.0	375	13.89	21,164	8,797
ARK 4	1.0	3,000	1.74	215,969	93,300
	2.0	1,500	3.47	134,745	54,430
	4.0	750	6.94	80,479	33,296
	8.0	375	13.89	47,258	19,875
	12.0	250	20.83	34,900	14,398
	15.0	200	26.04	29,556	12,608

errors for the ARK methods (relative to the reference solution, in which the acoustic mode is resolved) converge toward a similar value. This behavior has been previously analyzed and discussed for the semi-implicit time integration of the perturbation form of the governing equations [30]. Figure 8.7(b) shows the mass conservation error, as defined in (7.7 with $k = 1$, as a function of the acoustic CFL number for the ARK methods. The partitioned semi-implicit algorithm is conservative to round-off errors for all the methods and at all the CFL numbers considered.

8.3. Numerical Cost. The main objective of using semi-implicit time integration is to obtain well-resolved solutions at a lower computational cost than with explicit time integrators. These methods allow time step sizes that step over the fast acoustic scales; however, they require the solution of a system of equations. Thus, their performance depends on the cost and accuracy of the linear solver. In this section, we compare the computational cost of the ARK methods with the explicit RK methods in terms of the minimum wall time and the number of function calls required to obtain a stable and resolved solution. In the following discussion, the number of function calls (n_{FC}) refers to the total number of calls to the functions that compute the partitioned flux components $\hat{\mathbf{F}}_F$ or $\hat{\mathbf{F}}_S$. Since a matrix-free implementation of the Jacobian is used, n_{FC} is the sum of the total number of time iterations (n_T) times the number of stages s (of the time integration method), and the total number of GMRES iterations. It is thus an estimate of the total computational cost; however, it does not include the cost of assembling and inverting the preconditioning matrix. The algorithm is implemented in serial; its performance and scalability on parallel platforms are being currently investigated. The reported simulations are run on a 2200 MHz AMD Opteron processor.

Table 8.1 shows the wall times (in seconds) and the number of function calls for the inertia-gravity wave problem, solved on a grid with 1200×50 points with the CRWENO5 scheme. The tolerances for the GMRES solver are $\tau_a = \tau_r = 10^{-6}$. The ARK 2c and ARK 4 methods are considered and compared with the explicit RK 2a and RK 4. The time steps for the explicit RK methods are chosen close to their

TABLE 8.2

Rising thermal bubble: Computational cost as a function of time step size and acoustic CFL number of the fourth-order ARK and RK methods for solutions on a grid with 201^2 points discretized in space with the WENO5 scheme. The final time is 400 s. Boldfaced rows indicate the performance at the largest stable time step for the ARK method.

Method	Δt	n_T	σ_a	n_{FC}	Wall time (s)
RK 4	0.01	40,000	0.69	160,000	30,154
ARK 4	0.02	20,000	1.39	1,041,121	239,517
	0.05	8,000	3.47	571,456	121,949
	0.10	4,000	6.94	360,016	73,111
	0.20	2,000	13.89	219,821	44,163
	0.50	800	34.72	111,824	22,104
	1.00	400	69.44	71,452	13,868
	2.00	200	138.89	45,969	8,569

stability limits; thus, the reported wall times are the fastest time to solution for the explicit methods. The final row for each ARK method reports the cost with the largest stable time step and thus represents their fastest time to solution. The cost of the ARK methods decreases as the time step size increases (both the number of function calls and the wall times). ARK 2c is the fastest method among those considered. The acoustic scale is approximately 17 times faster than the advective scale for this problem. While the ARK 2c is faster than the explicit methods by 25%, the ARK 4 is generally slower at all the CFL numbers except at the largest CFL, where its cost is comparable.

Table 8.2 shows the cost of the ARK 4 method for the rising thermal bubble, solved on a grid with 201^2 points with the WENO5 scheme. The tolerances for the GMRES solver are $\tau_a = \tau_r = 10^{-6}$. The cost of the explicit RK 4 method is used as a reference. The separation between the acoustic and advective scales is very large; the flow is initially at rest, with the Mach number at the final time being ~ 0.006 . The semi-implicit method is thus able to take much larger time steps. The cost of the ARK method decreases as the time step size increases, and for CFL numbers greater than ~ 30 , the ARK 4 is faster than the RK 4. At the largest stable time step, the ARK 4 is faster than the RK 4 method by a factor of approximately 3.5.

The results reported here are obtained with basic preconditioning of the linear system, as described in Section 5.3. The primary focus of this paper is to introduce a flux partitioning for semi-implicit time integration based on the governing equations expressed as (2.1)–(2.3). Improving the efficiency of the time integrator by developing suitable preconditioning techniques for the GMRES solver is currently being investigated.

9. Conclusion. This paper presents a characteristic-based partitioning of the hyperbolic flux in the compressible Euler equations for semi-implicit time integration. The acoustic and the advective modes are separated; the former is integrated in time implicitly because of its stiffness, while the latter is integrated explicitly. The stiff term is linearized, and thus the semi-implicit algorithm only requires the solution to a linear system. The nonstiff term, defined as the total nonlinear flux with the linearized stiff term subtracted from it, is treated explicitly. High-order additive Runge-Kutta methods are applied to the partitioned equations, and the WENO and CRWENO schemes are used for the spatial discretization. This approach is tested for simple

inviscid flow problems at low Mach numbers. The results show that the largest stable time step is determined by the advective scale. The algorithm is then applied to atmospheric flows where the acoustic modes are much faster than the advective mode but are not physically relevant. The accuracy and convergence of the algorithm are demonstrated for benchmark problems, and it is shown that the partitioned semi-implicit approach is conservative. Moreover, the computational cost is assessed and compared with that of explicit time integrators. The extension of this algorithm to parallel platforms and the development of more effective preconditioning techniques are areas of current research.

Appendix A. Partitioned Jacobian and Flux Terms.

The flux Jacobian \mathcal{A} for the one-dimensional Euler equations is

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{\gamma-3}{2}u^2 & (3-\gamma)u & \gamma-1 \\ -\frac{\gamma u e}{\rho} + (\gamma-1)u^3 & \frac{\gamma e}{\rho} - \frac{3}{2}(\gamma-1)u^2 & \gamma u \end{bmatrix}. \quad (\text{A.1})$$

The matrices $\mathcal{X}, \mathcal{X}^{-1}$ with the right and left eigenvectors as columns and rows, respectively, are

$$\mathcal{X} = \begin{bmatrix} 1 & \frac{\rho}{2a} & -\frac{\rho}{2a} \\ u & \frac{\rho}{2a}(u+a) & -\frac{\rho}{2a}(u-a) \\ \frac{u^2}{2} & \frac{\rho}{2a}\left(\frac{u^2}{2} + \frac{a^2}{\gamma-1} + au\right) & -\frac{\rho}{2a}\left(\frac{u^2}{2} + \frac{a^2}{\gamma-1} - au\right) \end{bmatrix}, \quad (\text{A.2})$$

$$\mathcal{X}^{-1} = \frac{\gamma-1}{\rho a} \begin{bmatrix} \frac{\rho}{a}\left(-\frac{u^2}{2} + \frac{a^2}{\gamma-1}\right) & \frac{\rho}{a}u & -\frac{\rho}{a} \\ \frac{u^2}{2} - \frac{au}{\gamma-1} & -u + \frac{a}{\gamma-1} & 1 \\ -\frac{u^2}{2} - \frac{au}{\gamma-1} & u + \frac{a}{\gamma-1} & -1 \end{bmatrix}. \quad (\text{A.3})$$

The advective and acoustic modes are partitioned as

$$\Lambda_S = \begin{bmatrix} u & & \\ & 0 & \\ & & 0 \end{bmatrix}, \Lambda_S = \begin{bmatrix} 0 & & \\ & u+a & \\ & & u-a \end{bmatrix}. \quad (\text{A.4})$$

The slow partitioned Jacobian \mathcal{A}_S is computed by using (4.8),

$$\mathcal{A}_S = \mathcal{X} \Lambda_S \mathcal{X}^{-1} = \begin{bmatrix} -\frac{1}{2}\left(\frac{\gamma-1}{\gamma}\right)\frac{\rho u^3}{p} + u & \left(\frac{\gamma-1}{\gamma}\right)\frac{\rho u^2}{p} & -\left(\frac{\gamma-1}{\gamma}\right)\frac{\rho u}{p} \\ -\frac{1}{2}\left(\frac{\gamma-1}{\gamma}\right)\frac{\rho u^4}{p} + u^2 & \left(\frac{\gamma-1}{\gamma}\right)\frac{\rho u^3}{p} & -\left(\frac{\gamma-1}{\gamma}\right)\frac{\rho u^2}{p} \\ -\frac{1}{4}\left(\frac{\gamma-1}{\gamma}\right)\frac{\rho u^5}{p} + \frac{u^3}{2} & \frac{1}{2}\left(\frac{\gamma-1}{\gamma}\right)\frac{\rho u^4}{p} & -\frac{1}{2}\left(\frac{\gamma-1}{\gamma}\right)\frac{\rho u^3}{p} \end{bmatrix}, \quad (\text{A.5})$$

and the fast partitioned Jacobian is obtained from (4.8) as

$$\mathcal{A}_F = \mathcal{A} - \mathcal{A}_S. \quad (\text{A.6})$$

The slow and fast flux $\mathbf{f}_{S,F}(\mathbf{q})$ are

$$\mathbf{f}_S(\mathbf{q}) = \mathcal{A}_S \mathbf{q} = \begin{bmatrix} \left(\frac{\gamma-1}{\gamma}\right)\rho u \\ \left(\frac{\gamma-1}{\gamma}\right)\rho u^2 \\ \frac{1}{2}\left(\frac{\gamma-1}{\gamma}\right)\rho u^3 \end{bmatrix}, \mathbf{f}_F(\mathbf{q}) = \mathcal{A}_F \mathbf{q} = \begin{bmatrix} \left(\frac{1}{\gamma}\right)\rho u \\ \left(\frac{1}{\gamma}\right)\rho u^2 + p \\ (e+p)u - \frac{1}{2}\left(\frac{\gamma-1}{\gamma}\right)\rho u^3 \end{bmatrix}. \quad (\text{A.7})$$

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