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Compact-Reconstruction Weighted Essentially Non-Oscillatory Schemes for Hyperbolic Conservation Laws

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Doctoral Dissertation Defense
Applied Mathematics & Statistics, and Scientific Computation

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Committee

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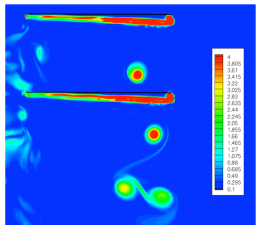
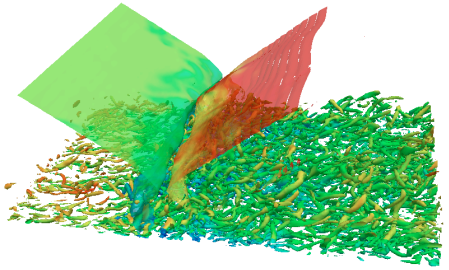
Motivation

Numerical Solution of Compressible Turbulent Flows

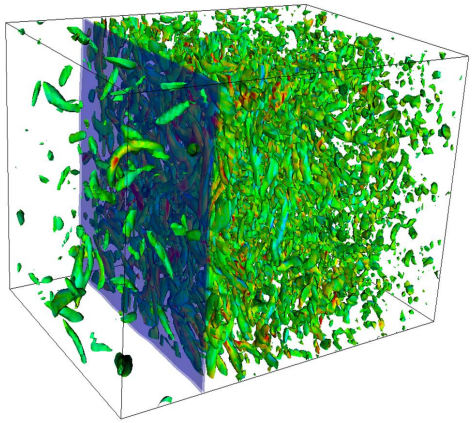
- Aircraft and Rotorcraft wake flows
- Characterized by large range of length scales
- Convection and interaction of eddies
- Compressibility → Shock waves & Shocklets
- Thin shear layers → High gradients in flow





High order accurate Navier-Stokes solver

- High spectral resolution for accurate capturing of smaller length scales
- Non-oscillatory solution across shock waves and shear layers
- Low dissipation errors for preservation of flow structures over large distances



Outline

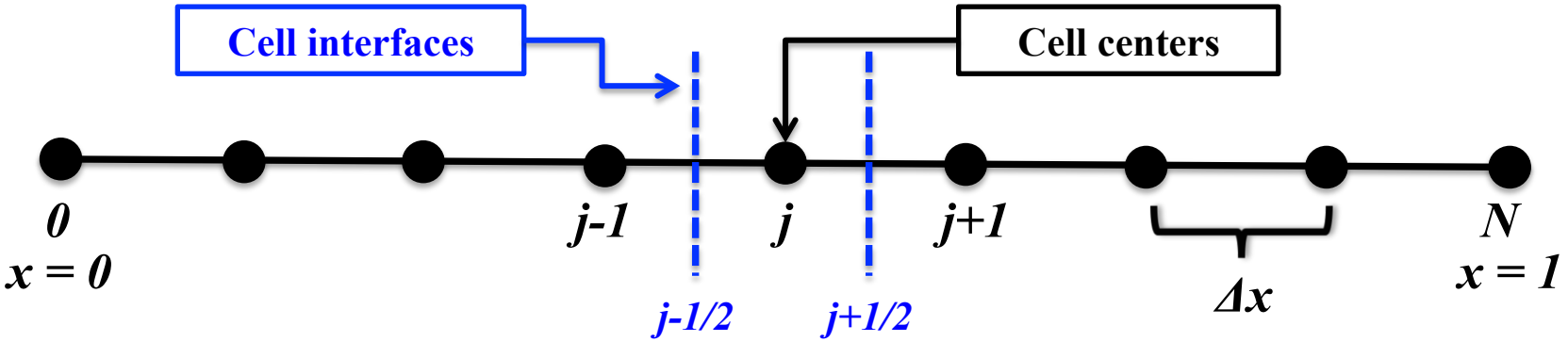
- **Hyperbolic Conservation Laws (Introduction / Background)**
  - Numerical solution (Reconstruction + Time – marching)
  - High-resolution schemes in literature (Previous work)
  - Objectives of this thesis
- **Weighted Non – Linear Compact Schemes**
  - Derivation & analysis of 5th order Compact-Reconstruction WENO (CRWENO) schemes
  - Application to scalar conservation laws (accuracy, order of convergence, resolution)
  - Numerical cost + computational efficiency (compared to non-compact WENO schemes)
  - Alternative formulations for the non-linear weights
- **Application to the Inviscid Euler Equations**
  - Reconstruction of conserved, primitive and characteristic variables
  - Benchmark 1D and 2D inviscid flow problems (accuracy, convergence, resolution)
  - Numerical cost (conserved/primitive vs. characteristic reconstruction)
- **Integration with a finite volume Navier-Stokes solver**
  - Validation/verification for curvilinear, overset grids with relative motion → Application to flows around airfoils, wings, helicopter rotor blades
  - Direct numerical simulation of compressible, turbulent flows
- **Conclusions and Future Work**

Introduction / Background

Hyperbolic Conservation Laws

Scalar hyperbolic partial differential equation

$$u_t + f(u)_x = 0; \quad f'(u) \in \Re$$



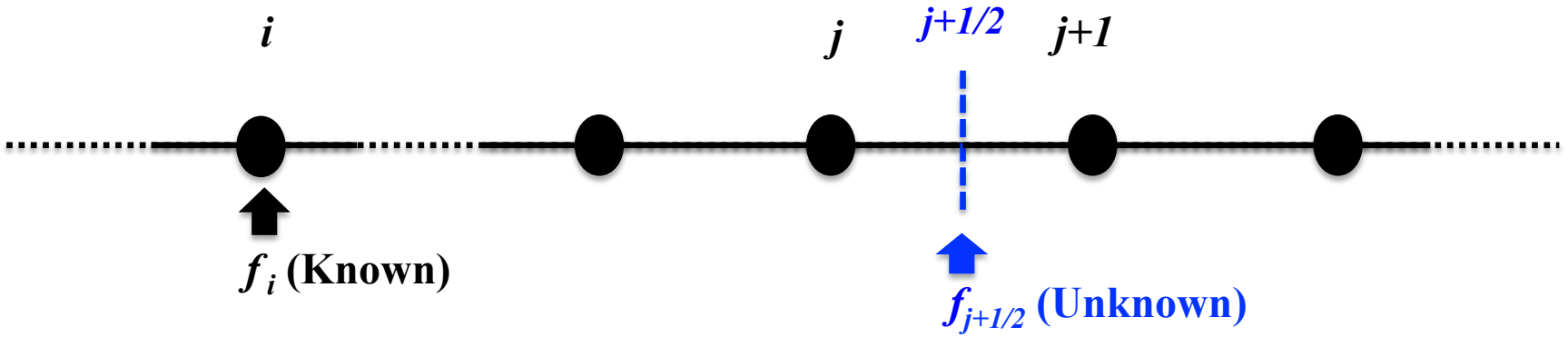
Conservative discretization in space leads to an ordinary differential equation in time (solved by explicit / implicit ODE solvers)

$$\frac{du_j}{dt} + \frac{1}{\Delta x}\left[f(x_{j+1/2}, t) - f(x_{j-1/2}, t) \right] = 0$$

Euler Explicit (1st order)
TVD-RK3 (3rd order)
BDF2 (Implicit 2nd order)

Reconstruction of Interface Flux

Reconstruction – interpolation of *f* at the interfaces from the cell centered / averaged values **(Focus of this thesis)**



$$f_{j+1/2} = \mathrm{f}\left(f_i : i = j-m,\ldots,j+n\right)$$

Interpolation Stencil

Upwinding – biased interpolation stencil to model wave nature of the solution

$$f_{j+1/2} = \begin{cases} f^{L}_{j+1/2} \text{ if } f'(u)\big|_{x=x_{j+1/2}} > 0; \quad (\mathrm{m} \geq \mathrm{n}) & \text{Left – biased} \\ f^{R}_{j+1/2} \text{ if } f'(u)\big|_{x=x_{j+1/2}} < 0; \quad (\mathrm{m} < \mathrm{n}) & \text{Right – biased} \end{cases}$$

WENO Schemes

The Weighted Essentially Non-Oscillatory (WENO) schemes

- Liu, Osher & Chan (*JCP*, 1994) and Jiang & Shu (*JCP*, 1996)
- At each interface, *r* possible *r*-th order interpolation schemes
- Final interface flux = convex combination of the *r*-th order interpolations
- Optimal weights in smooth regions → (*2r-1*)-th order interpolation
- Smoothness – dependent weights for discontinuous solutions
  → Non-oscillatory interpolation

$$f_{j+1/2} = \sum_{k=1}^{r} \omega_k f^k_{j+1/2}$$

Interface flux (*2r-1*)-th order

Interface flux *r*-th order

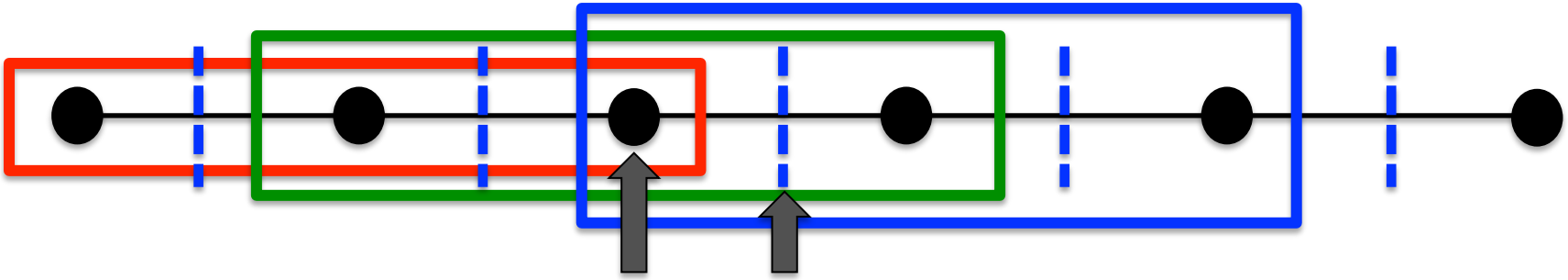
$$\alpha_k = \frac{c_k}{(\beta_k + \varepsilon)^p}; \quad \omega_k = \frac{\alpha_k}{\sum_k \alpha_k}$$

Optimal Weights

WENO Weights

Smoothness Indicators

5th Order WENO scheme



$$f_{j+1/2} = \frac{1}{3} f_{j-2} - \frac{7}{6} f_{j-1} + \frac{11}{6} f_j \quad \rightarrow \quad c_1 = \frac{2}{10} \quad \omega_1$$

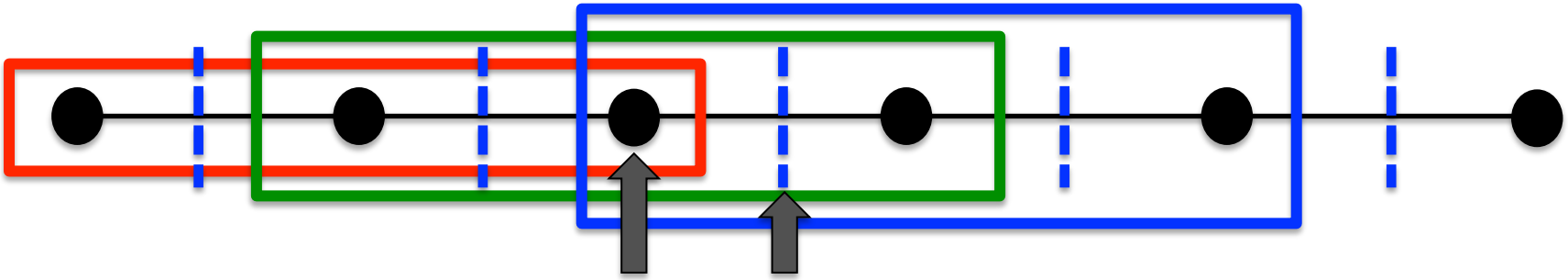
$$f_{j+1/2} = -\frac{1}{6} f_{j-1} + \frac{5}{6} f_j + \frac{1}{3} f_{j+1} \quad \rightarrow \quad c_2 = \frac{5}{10} \quad \omega_2$$

$$f_{j+3/2} = \frac{1}{3} f_j + \frac{5}{6} f_{j+1} - \frac{1}{6} f_{j+2} \quad \rightarrow \quad c_3 = \frac{3}{10} \quad \omega_3$$

$$f_{j+1/2} = \frac{\omega_1}{3} f_{j-2} - \frac{1}{6}(7\omega_1 + \omega_2) f_{j-1} + \frac{1}{6}(11\omega_1 + 5\omega_2 + 2\omega_3) f_j + \frac{1}{6}(2\omega_2 + 5\omega_3) f_{j+1} - \frac{\omega_3}{6} f_{j+2}$$

Non-Linear Weights

Weights are calculated based on smoothness indicators



$$\beta_1 = IS(f_{j-2}, f_{j-1}, f_j)$$
$$\beta_2 = IS(f_{j-1}, f_j, f_{j+1})$$
$$\beta_3 = IS(f_j, f_{j+1}, f_{j+2})$$

Undivided differences

$$\alpha_k = \begin{cases} \dfrac{c_k}{(\beta_k + \varepsilon)^p} \\ c_k \left[1 + \left(\dfrac{\tau}{\varepsilon + \beta_k} \right)^p \right] \end{cases}$$

Implementations of WENO weights:

- Jiang & Shu (1996) → **WENO5-JS**
- Henrick, Aslam & Powers (2005) → **WENO5-M**
- Borges, et. al. (2008) → **WENO5-Z**
- Yamaleev & Carpenter (2009) → **WENO5-YC**

Compact Difference Schemes

Introduced by Lele (*JCP*, 1992) in non-conservative form

$$u_t + f(u)_x = 0 \quad \Rightarrow \quad \frac{du_j}{dt} + \hat{f}_{x,j} = 0$$

(Non – conservative discretization)

$$\beta \hat{f}_{x,j-2} + \alpha \hat{f}_{x,j-1} + \hat{f}_{x,j} + \alpha \hat{f}_{x,j+1} + \beta \hat{f}_{x,j+2} = a \frac{f_{j+1} - f_{j-1}}{2\Delta x} + b \frac{f_{j+2} - f_{j-2}}{4\Delta x} + c \frac{f_{j+3} - f_{j-3}}{3\Delta x}$$

Central differencing schemes

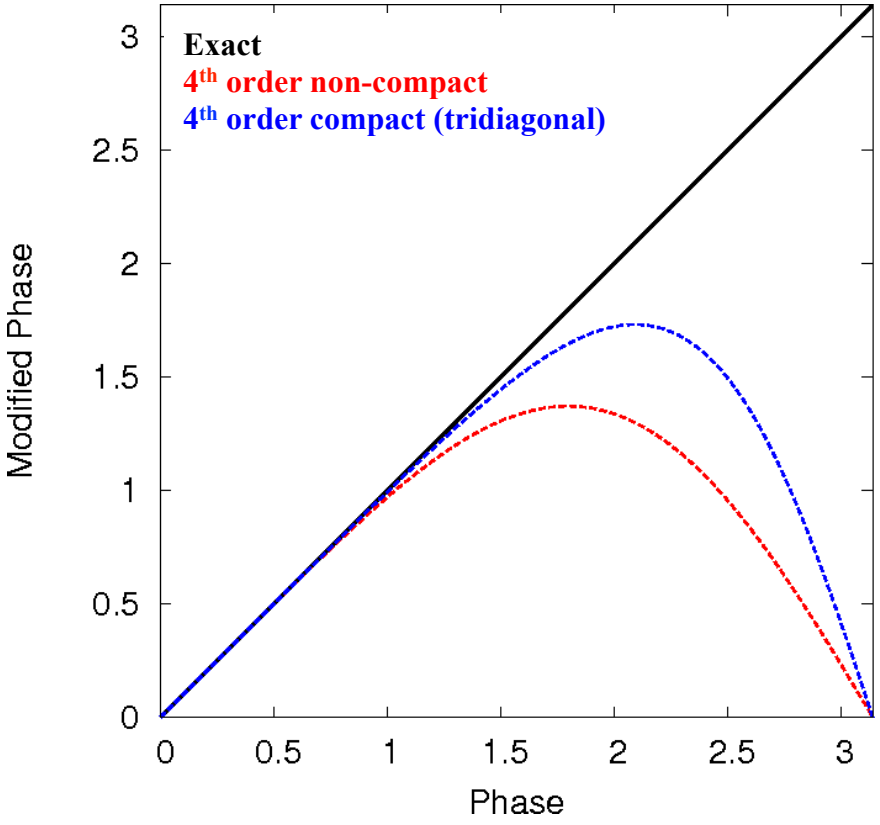
$\alpha = \beta = 0$: Non-compact

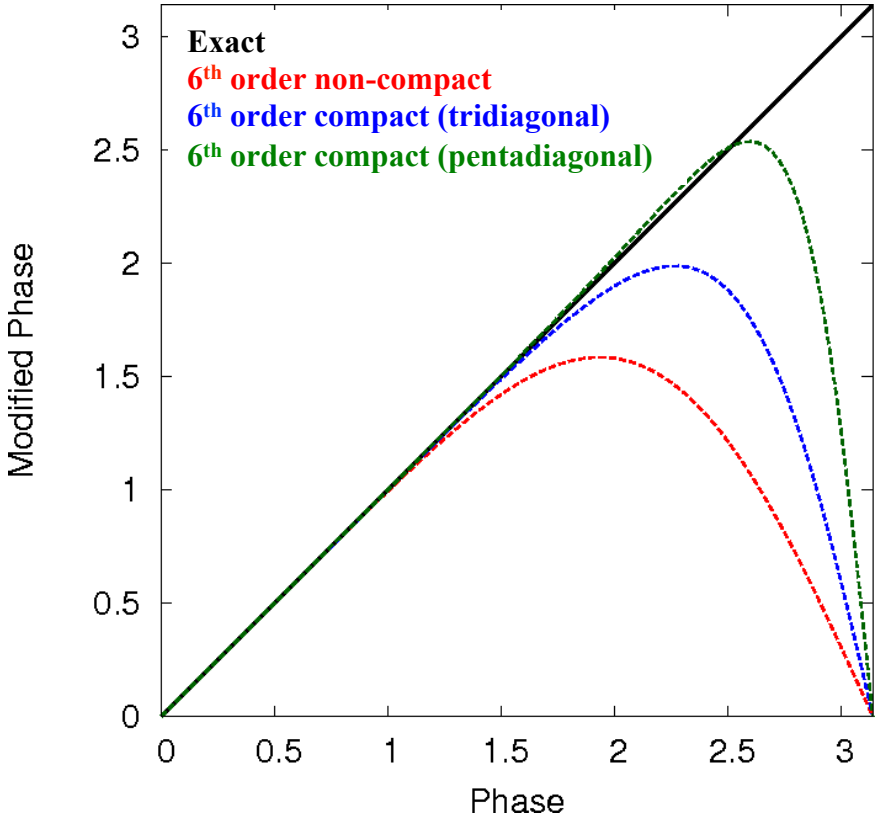
$\beta = 0$: Tridiagonal compact

$\alpha \neq 0, \beta \neq 0$: Penta-diagonal compact

- Coupling between neighboring flux derivative approximations
- Requires the solution of a tridiagonal / penta-diagonal system of equations (sparse LU)
- Constant coefficients ➔ Pre-computed LU decomposition

Spectral Resolution of Compact Schemes





Compact schemes have higher spectral resolution for same order of convergence ➔ Application to problems with large range of length scales

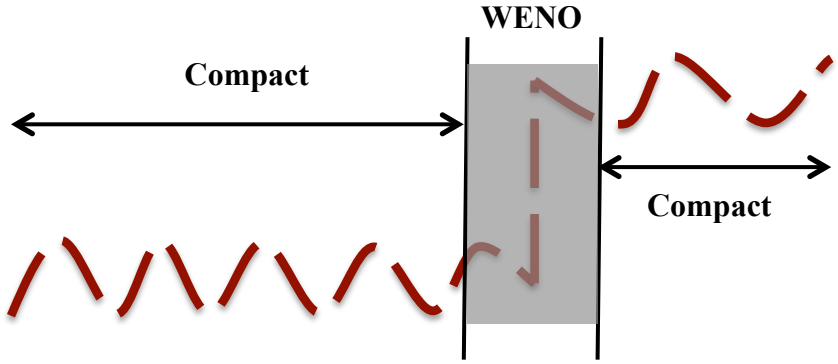
Previous Work

Compressible **Turbulent** **Flows**

Non-Oscillatory (WENO Schemes) + High spectral Resolution (Compact schemes)

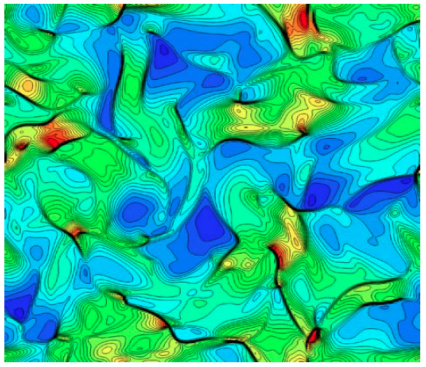
Hybrid Compact-WENO Schemes

- Smoothness indicator marks out regions of near discontinuities
- Compact scheme for smooth regions
- WENO scheme near discontinuities
- Adams & Sherriff (1996), Pirozzoli (2002), Ren et. al. (2003)



Disadvantages:

- Loss of spectral resolution near discontinuities
- Requirement of an arbitrary parameter for switching between schemes
- Presence of shocklets throughout the domain → Use WENO everywhere?



Isotropic Turbulence Decay
Martin &Taylor, 2007,
J. Sci. Comput.

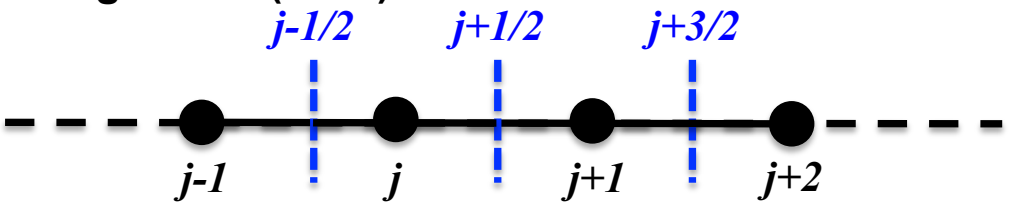
Previous Work

Compressible Turbulent Flows

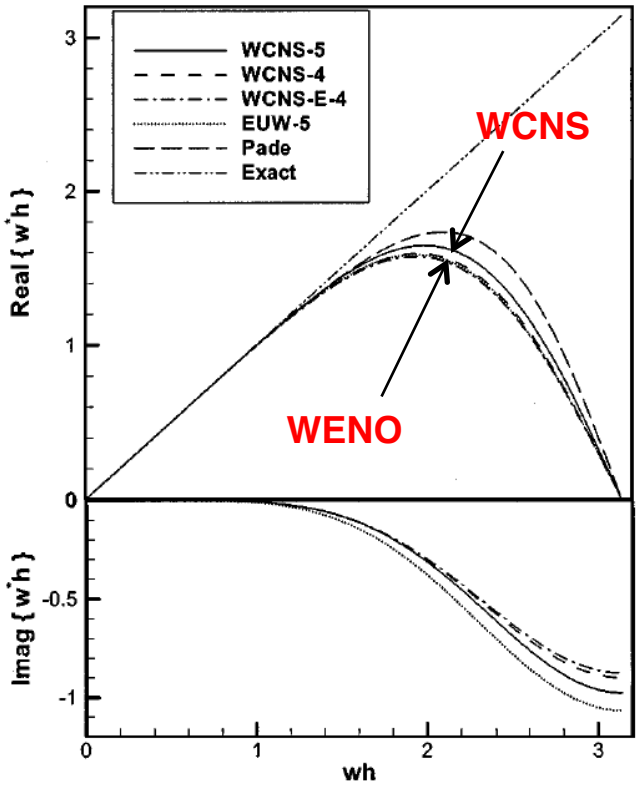
Non-Oscillatory (WENO Schemes) + High spectral Resolution (Compact schemes)

Weighted Compact Non-Linear Schemes (WCNS):

- Non-conservative finite difference discretization for a staggered mesh arrangement
- Step 1: $f_{j+1/2}$ computed from f_j using the WENO scheme (Non-oscillatory)
- Step 2: f_x computed from $f_{j+1/2}$ using high order central compact difference scheme
- Deng & Zhang (2000), Wang & Huang (2002), Zhang, Jiang & Shu (2008)



Disadvantage: Marginal increase in spectral resolution (due to reconstruction of interface flux with WENO scheme)



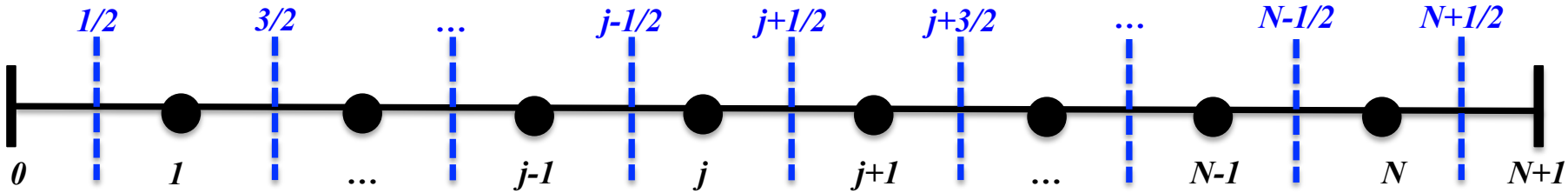
Deng & Zhang, 2000, J. Comput. Phys.

Objectives

- **Derivation and implementation of CRWENO schemes**
  - Formulation: Identification of lower-order *compact* interpolation schemes & application of WENO weights
  - Numerical analysis of underlying linear schemes
  - Formulation and analysis of boundary closure
  - Application to scalar conservation laws: Study accuracy, convergence, non-oscillatory behavior, and computational efficiency
- **Extension to system of equation: inviscid Euler equations**
  - Application to vector quantities: conserved, primitive & characteristic variables
  - Numerical properties: assessed on benchmark inviscid flow problems
- **Integration with a finite-volume Navier-Stokes solver**
  - Application to steady and unsteady flows around airfoils, wings and rotors: Comparison of the resolution for near-blade and wake flow features
  - Direct numerical simulation of canonical compressible, turbulent flows

Weighted Non-Linear Compact Schemes

CRWENO Schemes



- **General form of a conservative compact scheme:**

$$\mathrm{A}\left(\hat{f}_{j+1/2-m},\ldots,\hat{f}_{j+1/2},\ldots,\hat{f}_{j+1/2+m}\right)=\mathrm{B}\left(f_{j-n},\ldots,f_{j},\ldots,f_{j+n}\right) \quad \Rightarrow \quad [A]\hat{\mathbf{f}}=[B]\mathbf{f}$$

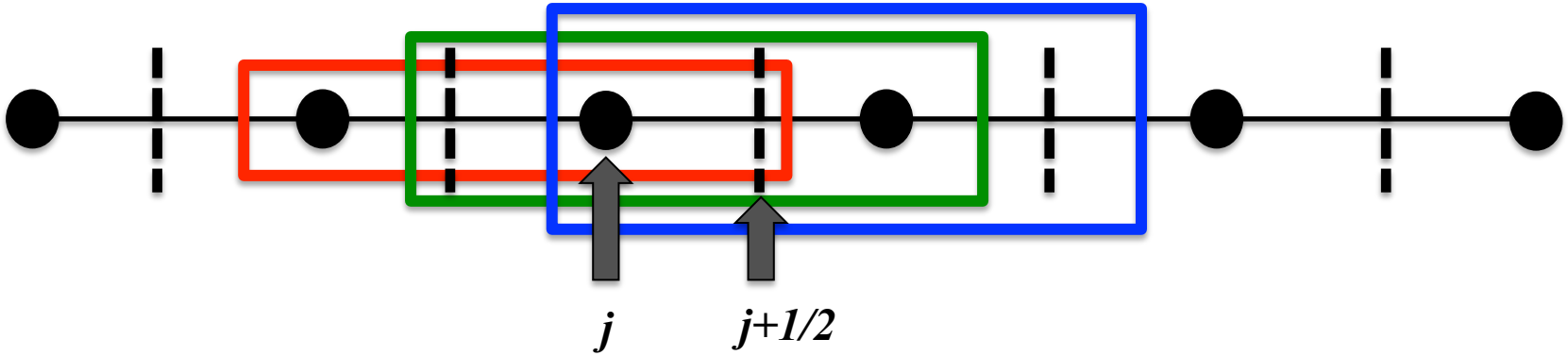
- **At each interface, r possible (r)-th order compact interpolations, combined using optimal weights c_k to yield $(2r-1)$-th order compact interpolation scheme:**

$$\sum_{k=1}^{r} c_k\,\mathrm{A}_k^r\left(\hat{f}_{j+1/2-m},\ldots,\hat{f}_{j+1/2+m}\right)=\sum_{k=1}^{r} c_k\,\mathrm{B}_k^r\left(f_{j-n},\ldots,f_{j+n}\right)\Rightarrow \mathrm{A}^{2r-1}\left(\hat{f}_{j+1/2-m},\ldots,\hat{f}_{j+1/2+m}\right)=\mathrm{B}^{2r-1}\left(f_{j-n},\ldots,f_{j+n}\right)$$

- **Apply WENO algorithm on the optimal weights c_k – scale them according to local smoothness**

$$\sum_{k=1}^{r} \omega_k\,\mathrm{A}_k^r\left(\hat{f}_{j+1/2-m},\ldots,\hat{f}_{j+1/2+m}\right)=\sum_{k=1}^{r} \omega_k\,\mathrm{B}_k^r\left(f_{j-n},\ldots,f_{j+n}\right) \qquad \alpha_k=\frac{c_k}{\left(\beta_k+\varepsilon\right)^p};\ \ \omega_k=\alpha_k\Big/\sum_k \alpha_k$$

5th Order CRWENO scheme (CRWENO5)



$$\frac{2}{3}f_{j-1/2}+\frac{1}{3}f_{j+1/2}=\frac{1}{6}f_{j-1}+\frac{5}{6}f_j \quad \rightarrow \quad c_1=\frac{2}{10} \quad \omega_1$$

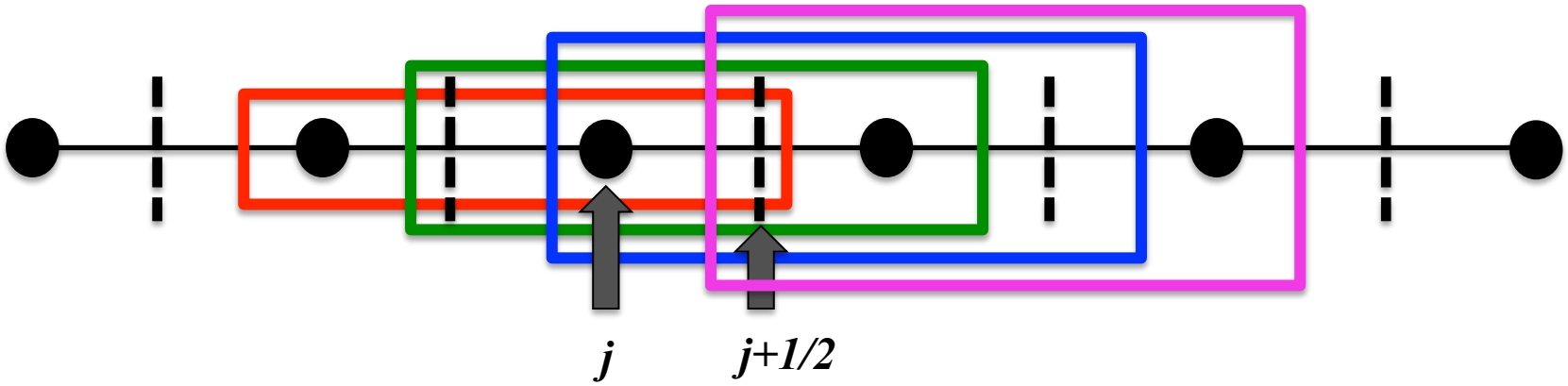
$$\frac{1}{3}f_{j-1/2}+\frac{2}{3}f_{j+1/2}=\frac{5}{6}f_j+\frac{1}{6}f_{j+1} \quad \rightarrow \quad c_2=\frac{5}{10} \quad \omega_2$$

$$\frac{2}{3}f_{j+1/2}+\frac{1}{3}f_{j+3/2}=\frac{1}{6}f_j+\frac{5}{6}f_{j+1} \quad \rightarrow \quad c_3=\frac{3}{10} \quad \omega_3$$

$$\left(\frac{2}{3}\omega_1+\frac{1}{3}\omega_2\right)f_{j-1/2}+\left(\frac{1}{3}\omega_1+\frac{2}{3}\omega_2+\omega_3\right)f_{j+1/2}=\frac{1}{3}\omega_1 f_{j-1}+\left(\frac{2}{3}\omega_1+\frac{1}{3}\omega_2+\frac{2}{3}\omega_3\right)f_j+\left(\frac{1}{3}\omega_2+\frac{1}{3}\omega_3\right)f_{j+1}$$

$$\frac{1}{30}f_{j-1}+\frac{19}{30}f_j+\frac{10}{30}f_{j+1} \qquad \frac{5(\omega_1+\omega_2)}{6}f_j+\frac{\omega_2+5\omega_3}{6}f_{j+1}$$

Low Dissipation 5th order CRWENO scheme (CRWENO5-LD)



$$\frac{2}{3}f_{j-1/2}+\frac{1}{3}f_{j+1/2}=\frac{1}{6}f_{j-1}+\frac{5}{6}f_j \quad \rightarrow \quad c_1=\frac{3}{20} \quad \omega_1$$

$$\frac{1}{3}f_{j-1/2}+\frac{2}{3}f_{j+1/2}=\frac{5}{6}f_j+\frac{1}{6}f_{j+1} \quad \rightarrow \quad c_2=\frac{9}{20} \quad \omega_2$$

$$\frac{2}{3}f_{j+1/2}+\frac{1}{3}f_{j+3/2}=\frac{1}{6}f_j+\frac{5}{6}f_{j+1} \quad \rightarrow \quad c_3=\frac{7}{20} \quad \omega_3$$

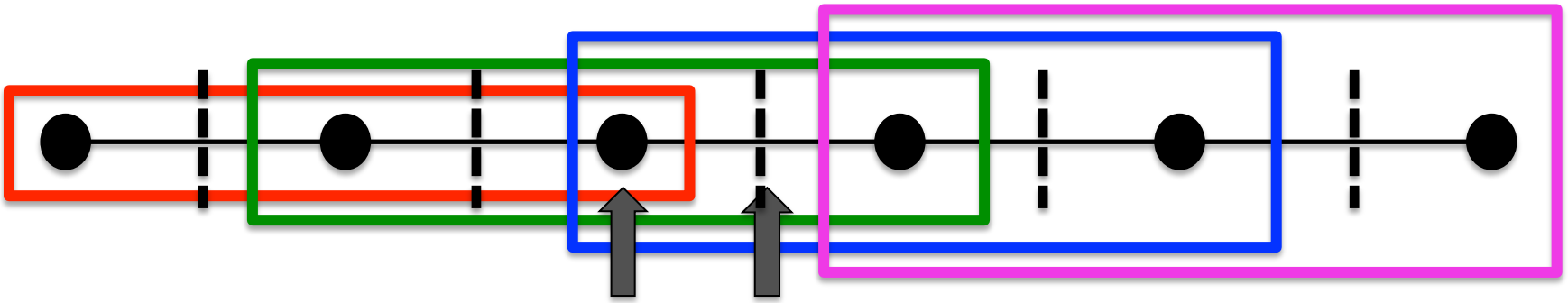
$$\frac{1}{3}f_{j+1/2}+\frac{2}{3}f_{j+3/2}=\frac{5}{6}f_{j+1}+\frac{1}{6}f_{j+2} \quad \rightarrow \quad c_4=\frac{1}{20} \quad \omega_4$$

$$\left(\frac{2}{3}\omega_1\frac{5}{20}\right)f_{j-1/2}+\left(\frac{1}{3}\omega_1\frac{12}{20}+\frac{2}{3}\omega_2\frac{3}{20}\right)f_{j+1/2}+\left(\frac{2}{3}\omega_2\frac{3}{20}+\frac{1}{3}\omega_3\frac{7}{20}\right)f_{j+3/2}=\frac{3}{120}f_{j-1}+\frac{2}{3}\omega_1\frac{67}{120}f_j+\frac{49}{120}f_{j+1}+\frac{1}{6}f_{j+2}$$

$$\frac{\omega_1}{6}f_{j-1}+\frac{2\omega_1+\omega_2}{6}f_j+\frac{\omega_1+2\omega_2+2\omega_3+\omega_4}{6}f_{j+1}+\frac{\omega_2+5(\omega_3+\omega_4)}{6}f_{j+1}+\frac{\omega_4}{6}f_{j+2}$$

Smoothness Indicators

Weights are calculated based on smoothness indicators of corresponding explicit stencils (same as WENO5 scheme)



$$\beta_1 = IS(f_{j-2}, f_{j-1}, f_j)$$

$$\beta_2 = IS(f_{j-1}, f_j, f_{j+1})$$

$$\beta_3 = IS(f_j, f_{j+1}, f_{j+2})$$

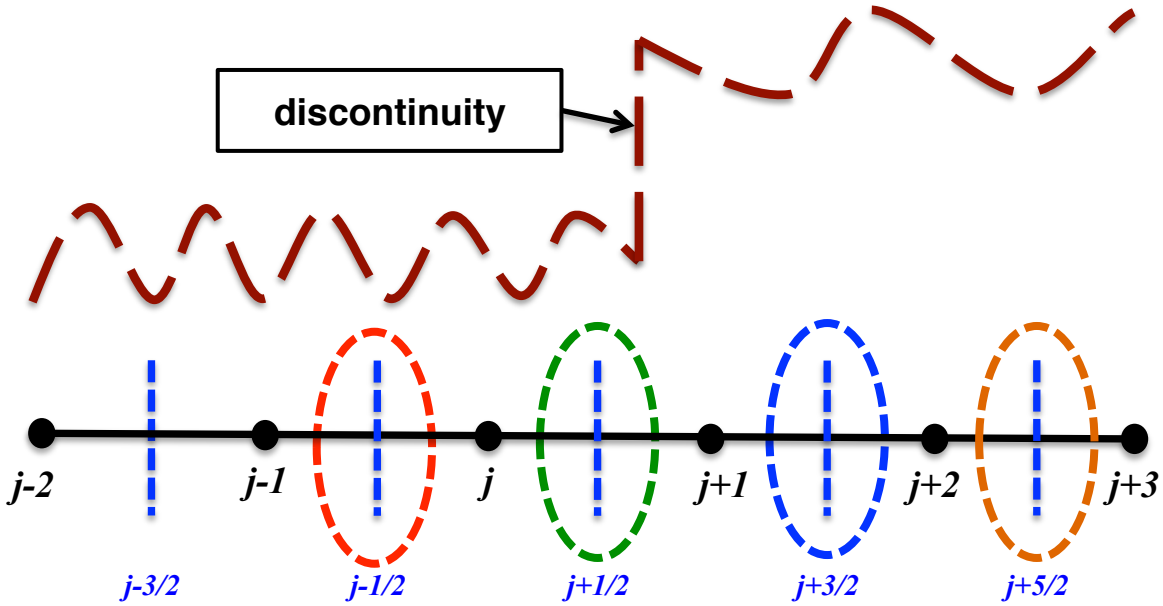
$$\beta_4 = IS(f_{j+1}, f_{j+2}, f_{j+3})$$

$$\alpha_k = \frac{c_k}{(\beta_k + \varepsilon)^p}; \quad \omega_k = \frac{\alpha_k}{\sum_k \alpha_k}; \quad k = 1,2,3$$

$$\beta_4 = \max(\beta_3, \beta_4)$$ (Avoid downwind interpolation)

CRWENO5-LD

Behavior across Discontinuities



Decoupling of domain across a discontinuity

$$\frac{2}{3}f_{j-1/2}+\frac{1}{3}f_{j+1/2}=\frac{1}{6}f_{j-1}+\frac{5}{6}f_j \quad \Rightarrow \quad \omega_1 \times \times 0\ 0$$

$$\frac{1}{3}f_{j-1/2}+\frac{2}{3}f_{j+1/2}=\frac{5}{6}f_j+\frac{1}{6}f_{j+1} \quad \Rightarrow \quad \omega_2 \times 0\ 0 \times$$

$$\frac{2}{3}f_{j+1/2}+\frac{1}{3}f_{j+3/2}=\frac{1}{6}f_j+\frac{5}{6}f_{j+1} \quad \Rightarrow \quad \omega_3\ 0\ 0 \times \times$$

$$\begin{bmatrix} b & c & & & & \\ a & b & c & & & \\ & a & b & c & & \\ & & a & b & \mathbf{0} & \\ & & & \mathbf{0} & b & c \\ & & & & a & b & c \\ & & & & & a & b & c \\ & & & & & & a & b \end{bmatrix} \begin{bmatrix} f_{1/2} \\ \\ \\ f_{j+1/2} \\ f_{j+3/2} \\ \\ \\ f_{N+1/2} \end{bmatrix} = \text{RHS}$$

Numerical Analysis

Underlying optimal (linear) schemes:

$$f_{j+1/2} = \frac{1}{30} f_{j-2} - \frac{13}{60} f_{j-1} + \frac{47}{60} f_j + \frac{27}{60} f_{j+1} - \frac{1}{20} f_{j+2}$$ **WENO5**

$$\frac{3}{10} f_{j-1/2} + \frac{6}{10} f_{j+1/2} + \frac{1}{10} f_{j+3/2} = \frac{1}{30} f_{j-1} + \frac{19}{30} f_j + \frac{10}{30} f_{j+1}$$ **CRWENO5**

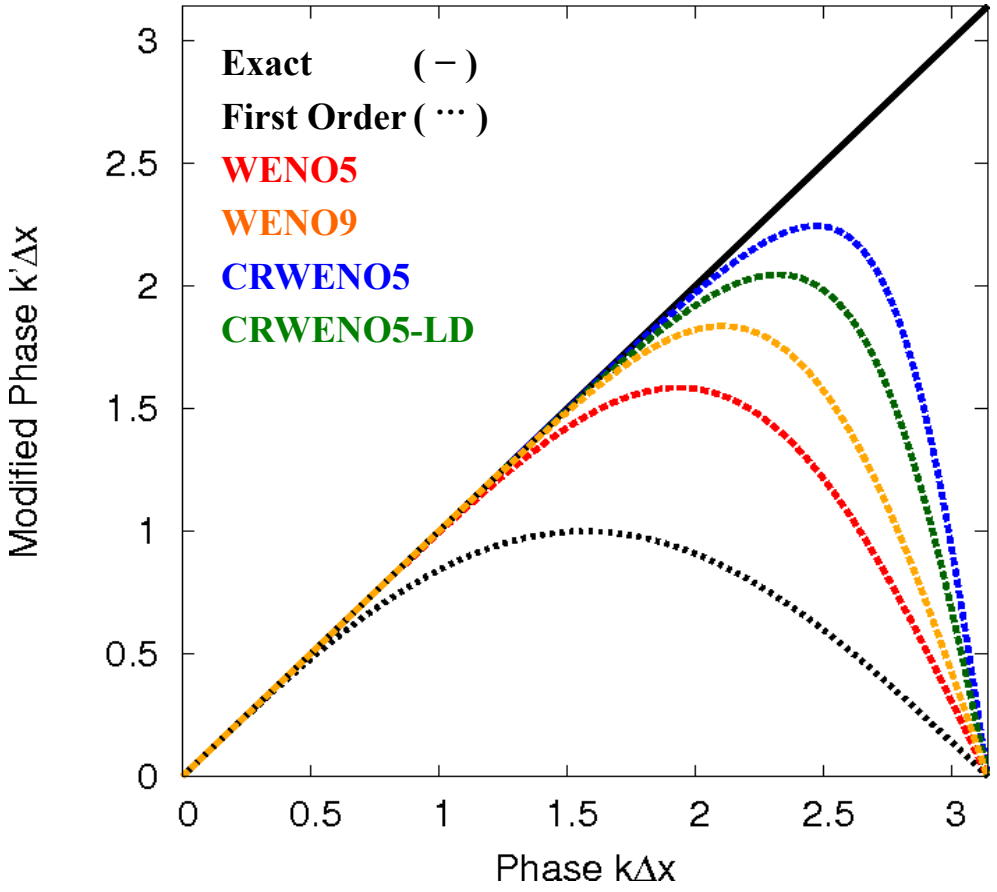
$$\frac{5}{20} f_{j-1/2} + \frac{12}{20} f_{j+1/2} + \frac{3}{20} f_{j+3/2} = \frac{3}{120} f_{j-1} + \frac{67}{120} f_j + \frac{49}{120} f_{j+1} + \frac{1}{120} f_{j+2}$$ **CRWENO5-LD**

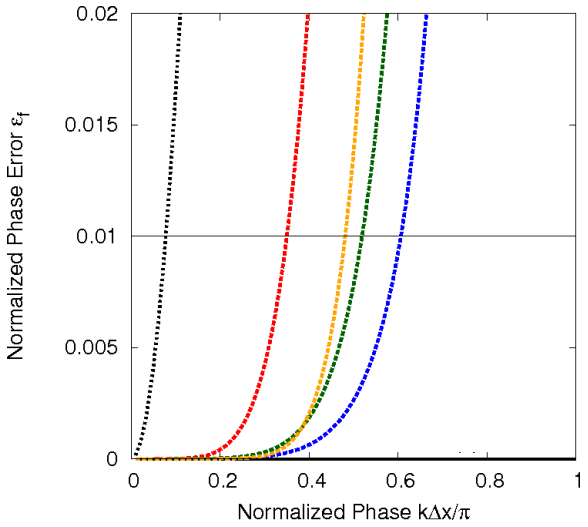
Taylor Series Analysis: Dissipation and dispersion errors

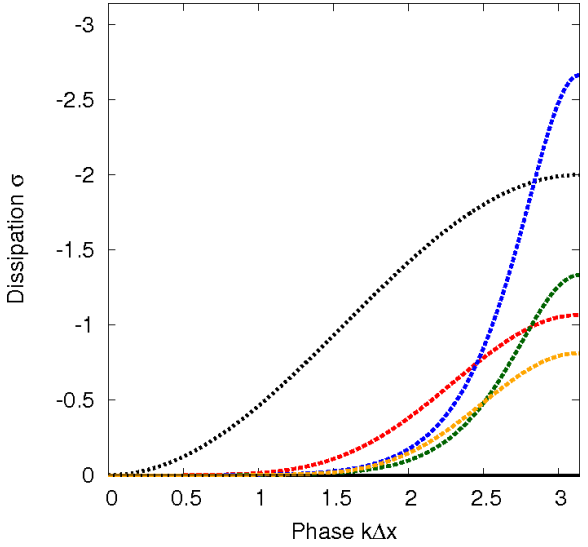
	WENO5	CRWENO5	CRWENO5-LD
Dissipation	$\frac{1}{60}\frac{\partial^6 f}{\partial x^6}\Big\|_j \Delta x^5$	$\frac{1}{600}\frac{\partial^6 f}{\partial x^6}\Big\|_j \Delta x^5$	$\frac{1}{1200}\frac{\partial^6 f}{\partial x^6}\Big\|_j \Delta x^5$
Dispersion	$\frac{1}{140}\frac{\partial^7 f}{\partial x^7}\Big\|_j \Delta x^6$	$\frac{1}{2100}\frac{\partial^7 f}{\partial x^7}\Big\|_j \Delta x^6$	$\frac{1}{2100}\frac{\partial^7 f}{\partial x^7}\Big\|_j \Delta x^6$

Numerical Analysis

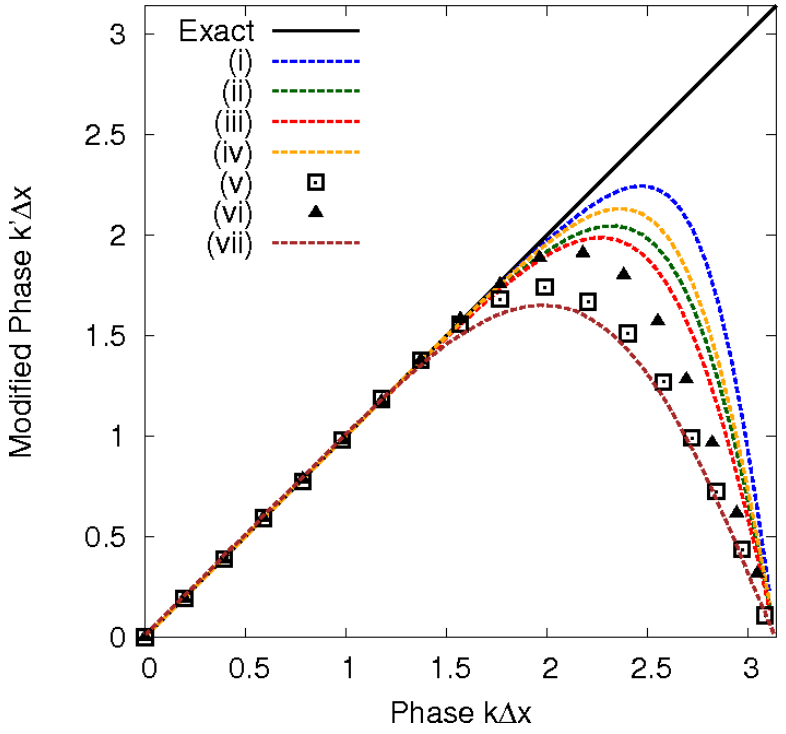
Fourier Analysis: spectral properties







Comparison of Spectral Resolutions



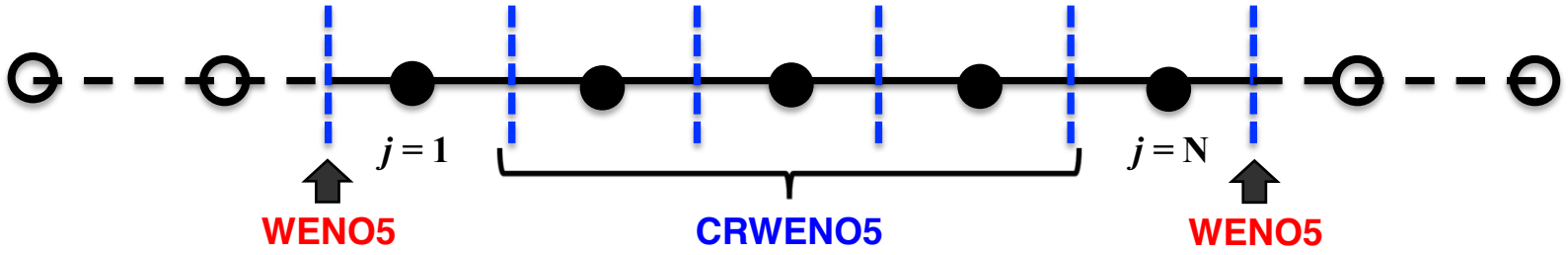
i. CRWENO5
ii. CRWENO5-LD
iii. 6th order central compact (Lele, 1992)
iv. 8th order central compact (Lele, 1992)
v. WENO-SYMBO (*r=3*) (Martin, et. al., 2006)
vi. WENO-SYMBO (*r=4*) (Martin, et. al., 2006)
vii. WCNS5 (Deng & Zhang, 2000)

Comparison of spectral resolution and bandwidth resolving efficiency – CRWENO5 scheme with high-resolution schemes in literature

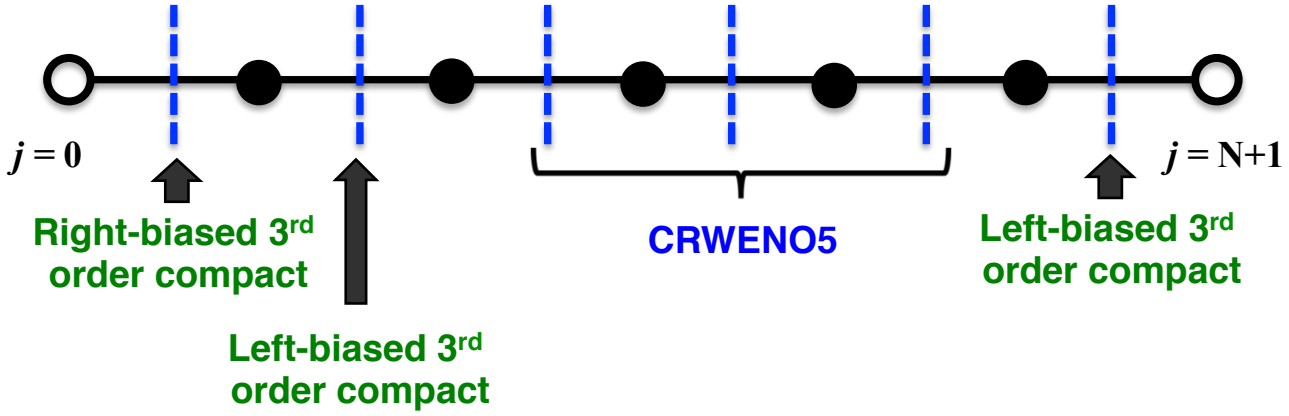
Bandwidth Resolving Efficiency	
WENO5 (Jiang & Shu, 1996)	0.35
WENO7 (Balsara & Shu, 2000)	0.42
WENO9 (Balsara & Shu, 2000)	0.48
CRWENO5	0.61
CRWENO5-LD	0.52
6th-order central compact (Lele, 1992)	0.50
8th-order central compact (Lele, 1992)	0.58
WENO-SYMBO (r = 3) (Martin, et. al., 2006)	0.49
WENO-SYMBO (r = 4) (Martin, et. al., 2006)	0.56

Boundary Closure

Cell interface aligned with physical boundary – Boundary conditions implemented on **"ghost cells"**



Cell center coincident on physical boundary – **Reduced-order / interior-biased** interpolation scheme at interfaces near the boundary



Scalar Conservation Laws

Schemes validated for the linear advection equation and the inviscid Burgers equation

Smooth problems

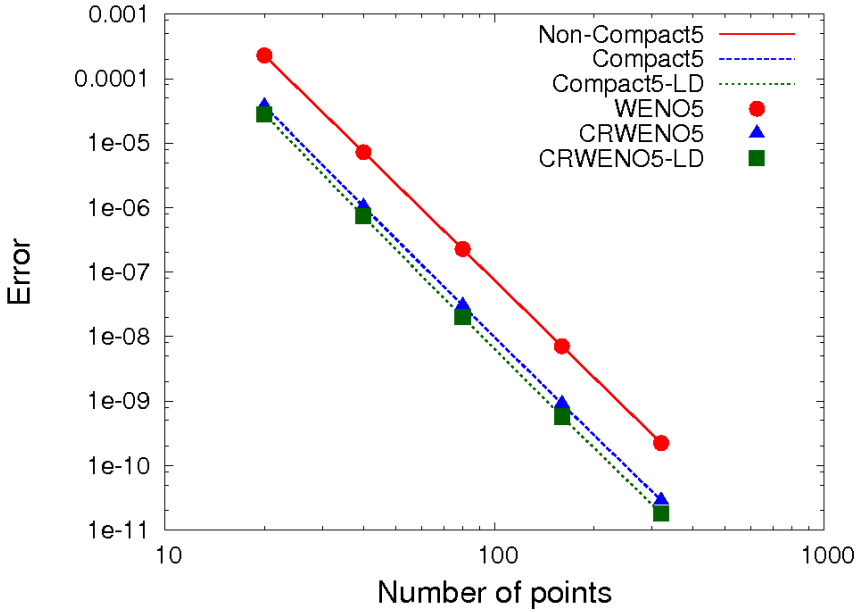
- **5th order convergence verified for the new schemes**
- **Errors for CRWENO5 order of magnitude lower than WENO5, errors for CRWENO5-LD half those of CRWENO5**

Linear advection equation

$u_0(x) = \sin(2\pi x);\ 0 \le x \le 1$

(Periodic domain)

Solution obtained after 1 cycle with TVD-RK3 time-stepping

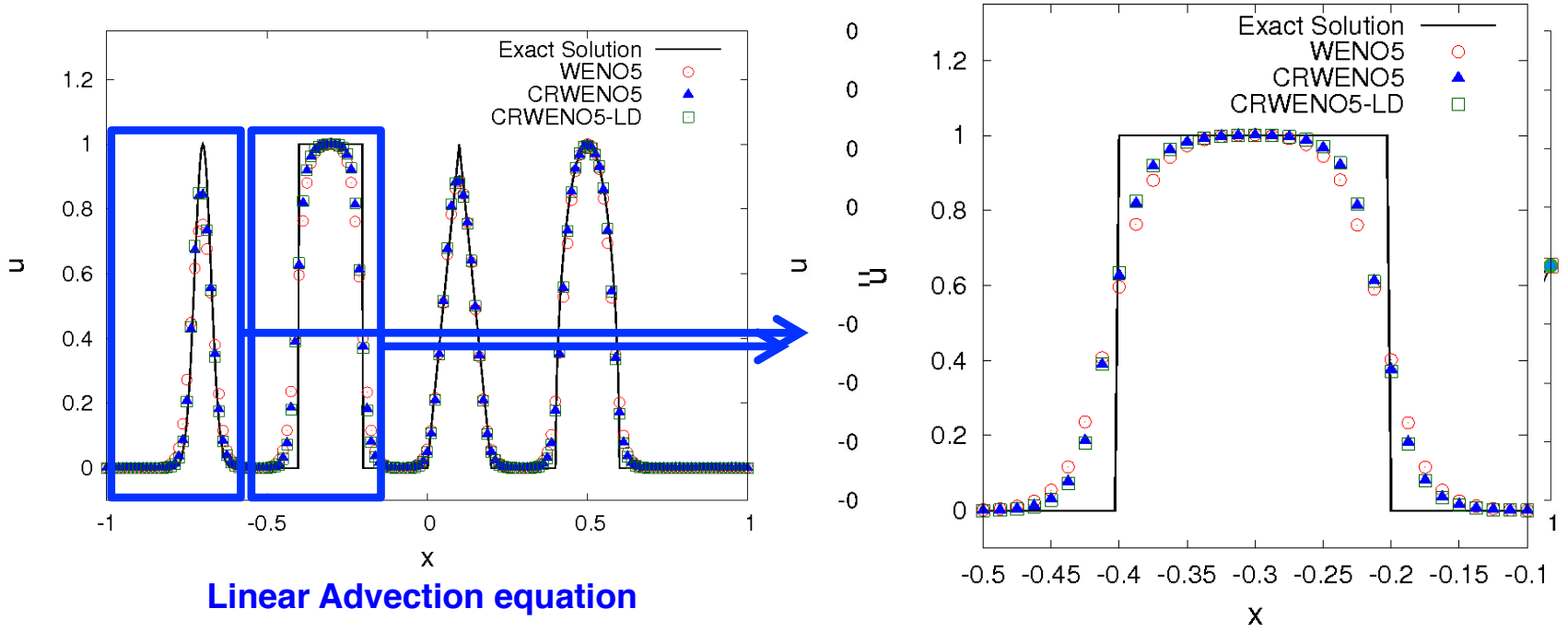


Scalar Conservation Laws

Schemes validated for the linear advection equation and the inviscid Burgers equation

Discontinuous problems

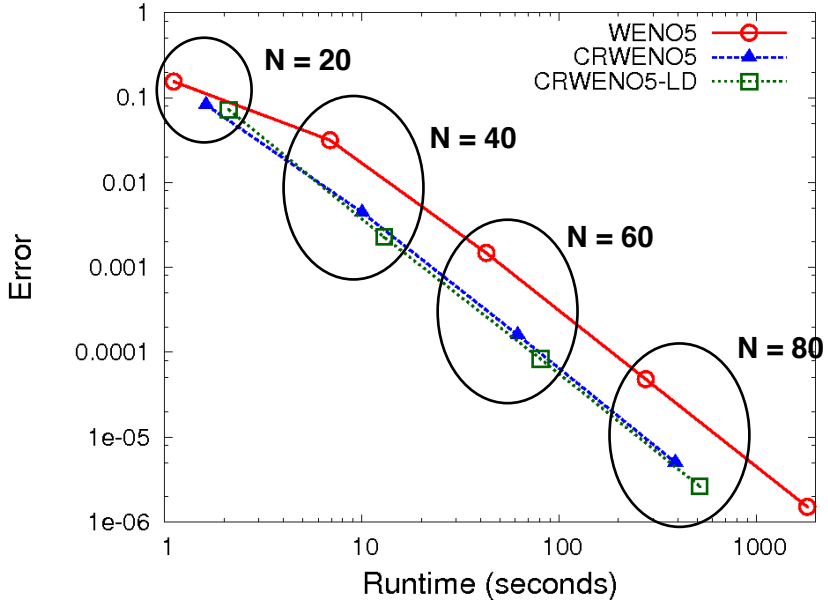
- **Non-oscillatory behavior validated across discontinuities**
- **CRWENO schemes show better resolution of discontinuous data (lower smearing and clipping)**



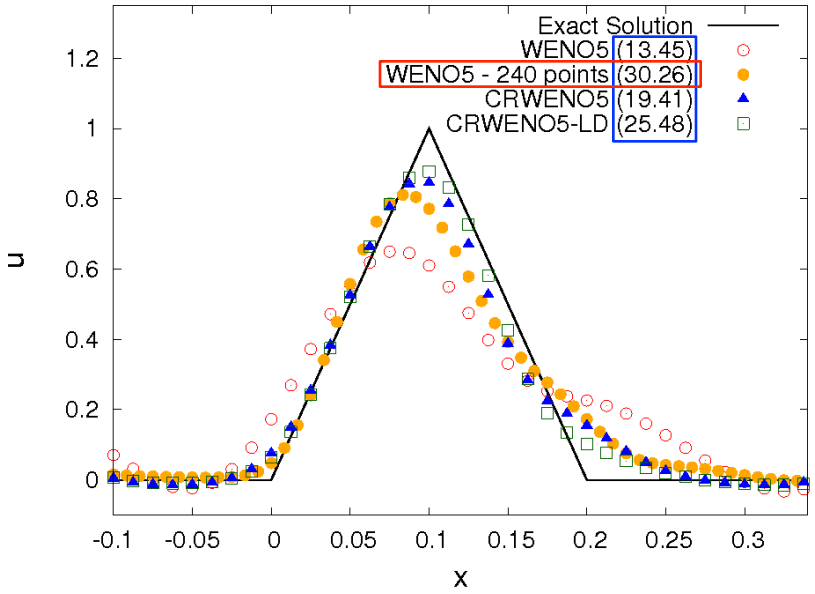
Linear Advection equation

Computational Efficiency

- CRWENO schemes require solution to tridiagonal system at each iteration (solution-dependent weights) ➔ Higher numerical cost for same number of points
- Higher accuracy (error is 1/10th that of WENO) ➔ Comparable solution obtained on coarser mesh ➔ Computationally more efficient



Error vs. runtime for smooth problem



Periodic advection of a triangular wave after 100 cycles (120 point mesh)

Implementation of Non-Linear Weights

- **Definition of the WENO weights by Jiang & Shu (1996):**

$$\alpha_k = \frac{c_k}{\left(\beta_k + \varepsilon\right)^p} \qquad \begin{matrix} p = 2 \\ \varepsilon = 10^{-6} \end{matrix}$$

CRWENO5-JS

  - ε – **sensitivity: Lower values of ε (10^{-40}) results in sub-optimal convergence for smooth problems with critical points (vanishing derivatives)**
  - **Excessive dissipation: Original weights defined by Jiang & Shu were found to be too dissipative across discontinuities**

- **Alternative formulations proposed in literature:**
  - **Mapped WENO (Henrick, Aslam & Powers, 2005): Defined a mapping function that improved convergence to WENO weights**

$$g_k\left(\omega\right) = \frac{\omega\left(c_k + c_k^2 - 3c_k\omega + \omega^2\right)}{c_k^2 + \omega\left(1 - 2c_k\right)}$$

CRWENO5-M

  - **WENO-Z (Borges, et. al., 2008)**
  - **ESWENO (Yamaleev & Carpenter, 2009)**

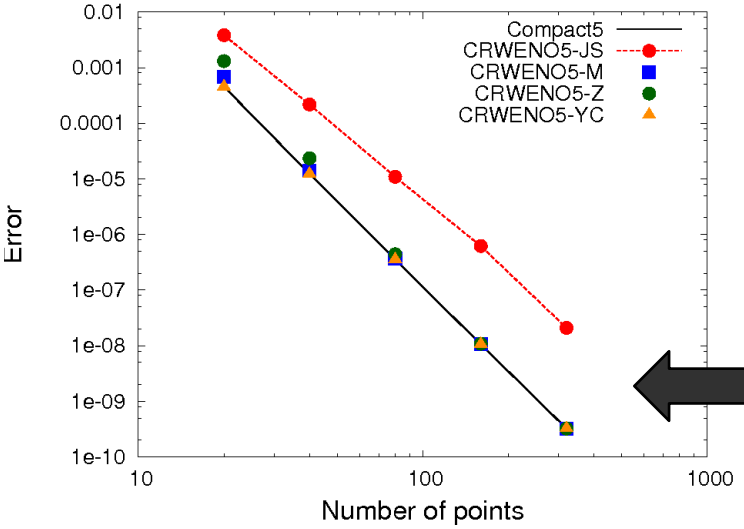
$$\alpha_k = c_k\left[1 + \left(\frac{\tau}{\varepsilon + \beta_k}\right)^p\right]$$

CRWENO5-Z

CRWENO5-YC

- **These improved convergence and resolution of the WENO schemes**

Implementation of Non-Linear Weights



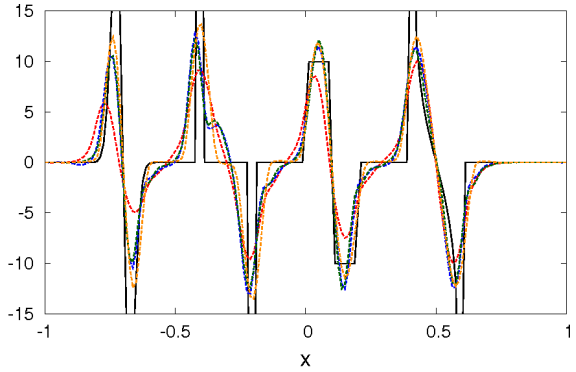
- Improved accuracy and convergence for smooth problems
- Sharper resolution for extrema and discontinuities
- Yamaleev & Carpenter weights show non-oscillatory behavior for higher derivatives (for this case)

Solution after 1 cycle

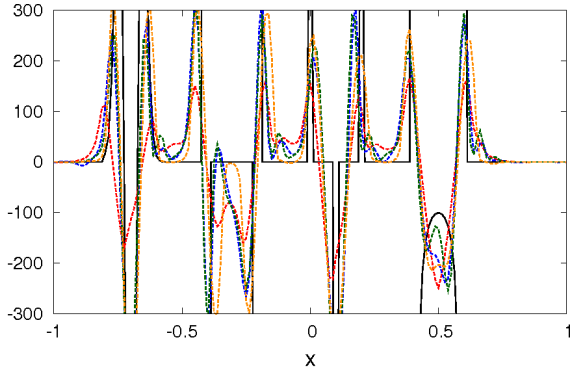
$$u_0(x) = \sin\left(\pi x - \frac{\sin(\pi x)}{\pi}\right)$$

$$-1 \le x \le 1 \text{ (Periodic)}$$

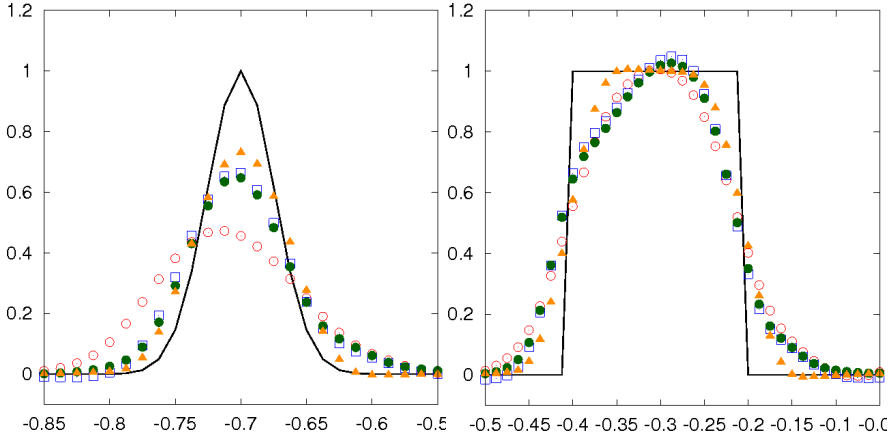
First derivative



Second derivative



Solutions of exponential and square waves after 50 cycles



CRWENO5-JS
CRWENO5-M
CRWENO5-Z
CRWENO5-YC

Application to Inviscid Euler Equations

Euler Equations

- **Compressible Euler equations in 1D given by**

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho u\\ e\end{bmatrix}+\frac{\partial}{\partial x}\begin{bmatrix}\rho u\\ \rho u^2+p\\ (e+p)u\end{bmatrix}=0$$

$\rho =$ Density

$u =$ Velocity

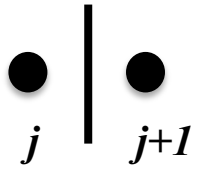
$p =$ Pressure

$$e=\frac{p}{\gamma-1}+\frac{1}{2}\rho u^2 \text{ (Internal energy)}$$

- **In the form of a general hyperbolic PDE** $u_t+f(u)_x=0$
- **Extension of scalar interpolation schemes to a system of equations**
  - **Component-wise reconstruction of conserved variables (ρ, ρu, e)**
  - **Reconstruction of primitive (flow) variables (ρ, u, p)**
  - **Reconstruction of characteristic variables**
- **Extension to conserved/primitive variables trivial**
  - **Solution of *D+2* tridiagonal systems of equations (D is the number of dimensions)**

CRWENO for Euler Equations

Characteristic based reconstruction respects the physics of the problem – 1D scalar wave propagation along each characteristic



$$a\alpha^k_{j-1/2} + b\alpha^k_{j+1/2} + c\alpha^k_{j+3/2} = \tilde{a}\alpha^k_{j-1} + \tilde{b}\alpha^k_{j} + \tilde{c}\alpha^k_{j+1} \quad \left(\alpha^k_i = \mathbf{l}^k_{j+1/2} \bullet \mathbf{f}_i\right)$$

$\Downarrow$

U^{avg} (Roe averaged)

$\hookrightarrow \lambda^k_{j+1/2}, \mathbf{l}^k_{j+1/2}, \mathbf{r}^k_{j+1/2}$

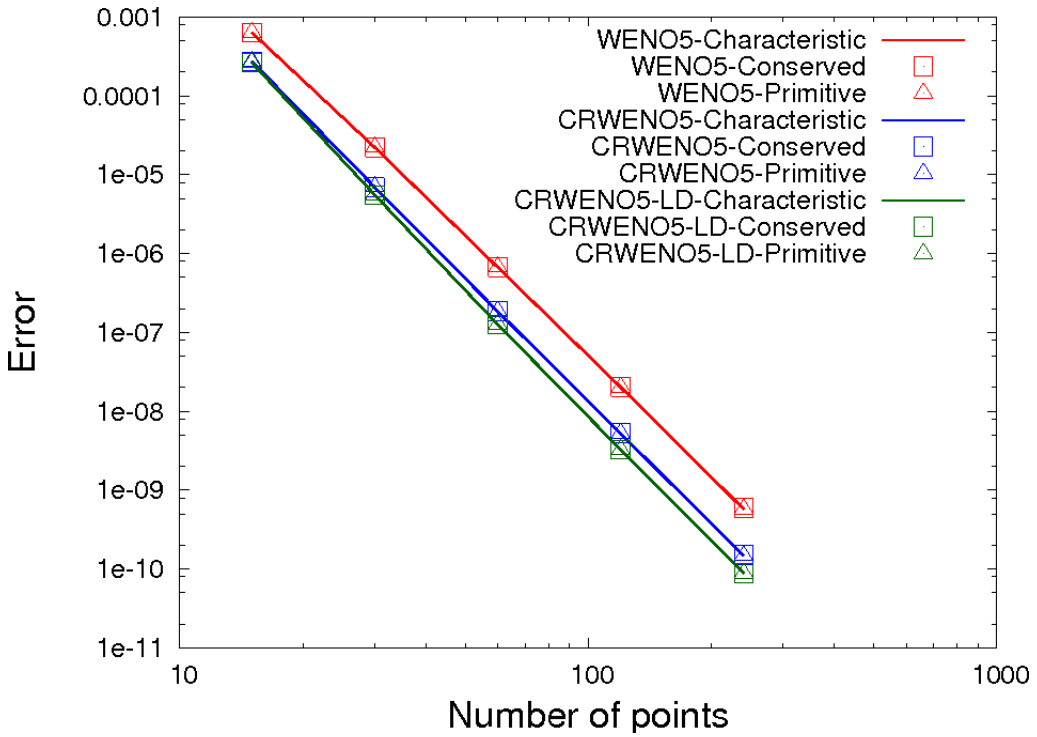
Eigenvalues, left and right eigenvectors

$$\begin{aligned} & a(l^{k1}_{j+1/2} f^1_{j-1/2} + l^{k2}_{j+1/2} f^2_{j-1/2} + l^{k3}_{j+1/2} f^3_{j-1/2}) \\ + & \; b(l^{k1}_{j+1/2} f^1_{j+1/2} + l^{k2}_{j+1/2} f^2_{j+1/2} + l^{k3}_{j+1/2} f^3_{j+1/2}) \quad = \tilde{a}\alpha^k_{j-1} + \tilde{b}\alpha^k_{j} + \tilde{c}\alpha^k_{j+1} \\ + & \; c(l^{k1}_{j+1/2} f^1_{j+3/2} + l^{k2}_{j+1/2} f^2_{j+3/2} + l^{k3}_{j+1/2} f^3_{j+3/2}) \end{aligned}$$

- **Results in a block tri-diagonal linear system along each dimension (as compared to tri-diagonal system for conserved/primitive reconstruction)**
- **For multi-dimensions, solution of linear system required along each grid line**
- **Upwinding –**
  - **Left and right biased fluxes computed**
  - **Roe-Fixed formulation used**

$$\alpha^{k,L}_{j+1/2}, \alpha^{k,R}_{j+1/2} \Rightarrow \alpha^k_{j+1/2}$$

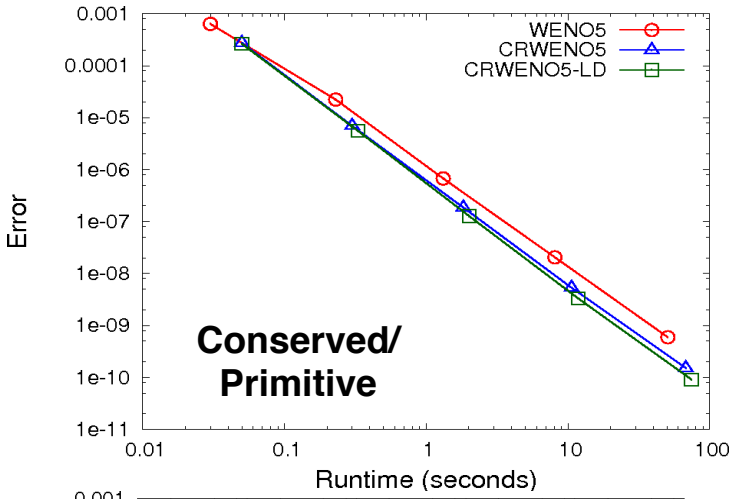
Entropy Wave Advection

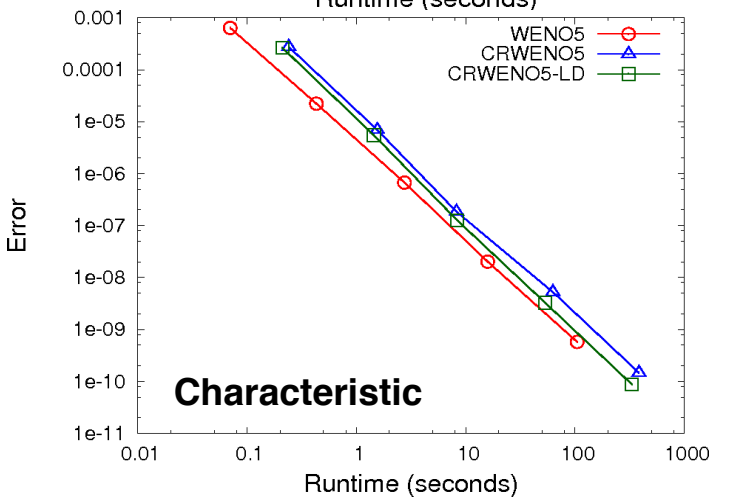


Reconstruction of characteristic, primitive & conserved variables

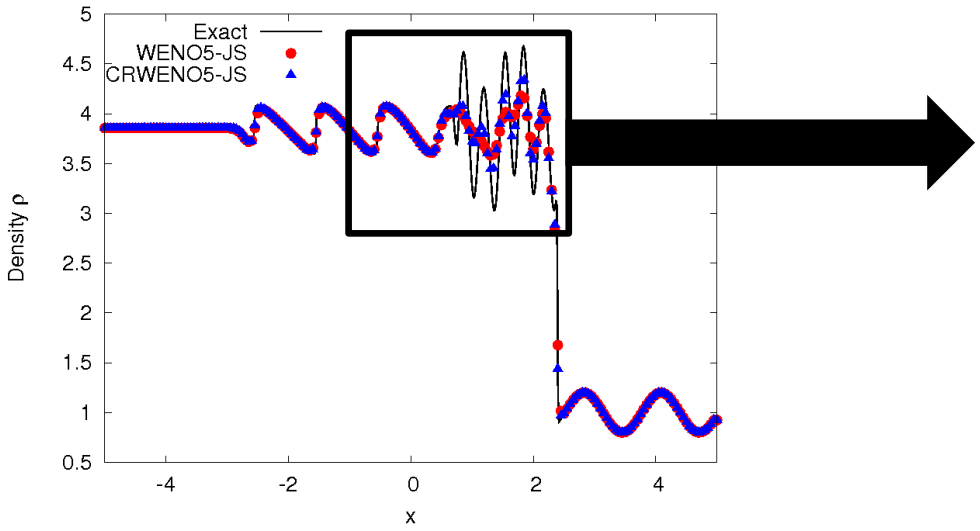
- **Accuracy – CRWENO5 errors 1/10th that of WENO5**
- **Smooth problem – Reconstruction of conserved, primitive and characteristic yield identical solutions**
- **Computational expense - CRWENO5 is more efficient for conserved/primitive reconstruction, but not for characteristic reconstruction**

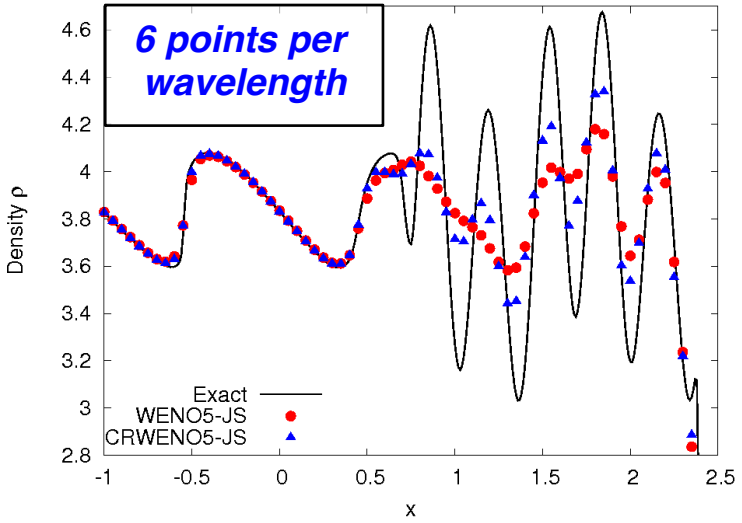
Numerical Cost



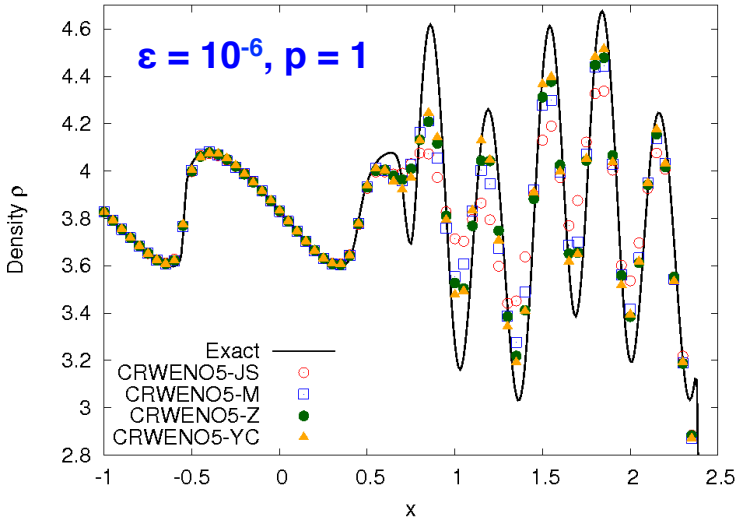


Shock – Entropy Interaction





- Interaction of a shock wave with a density wave resulting in high-frequency waves and discontinuities
- CRWENO scheme shows better resolution of high-resolution waves than WENO5
- Further improvement by using the alternative formulations for the WENO weights

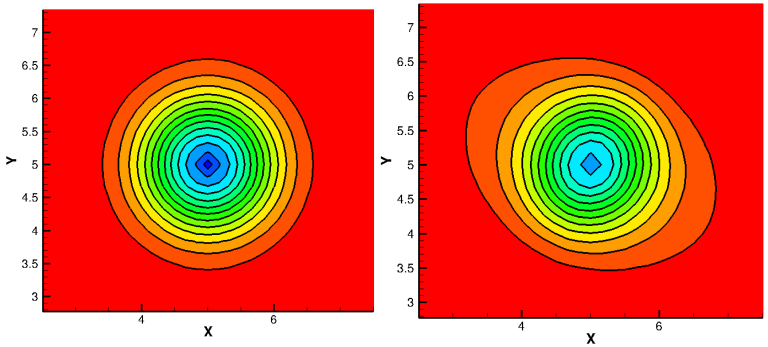


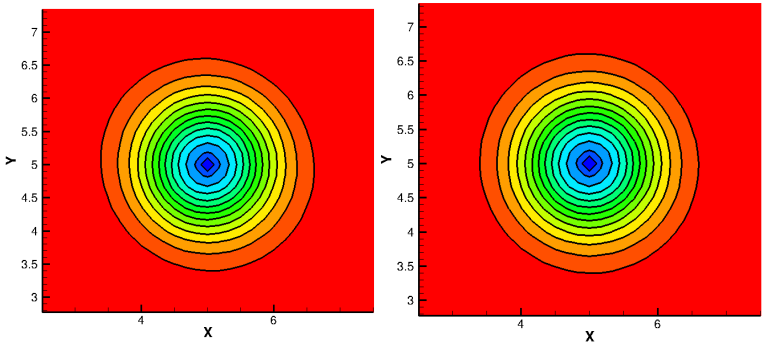
Isentropic Vortex Convection

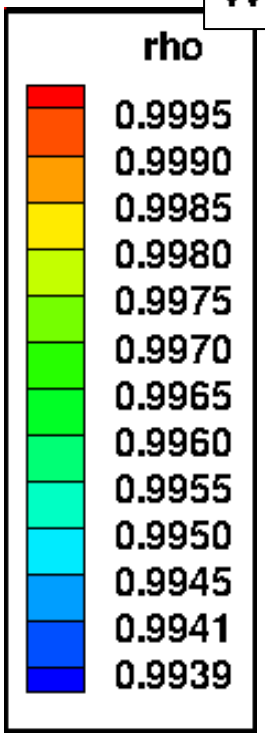
Solution after travelling 1000 core radii

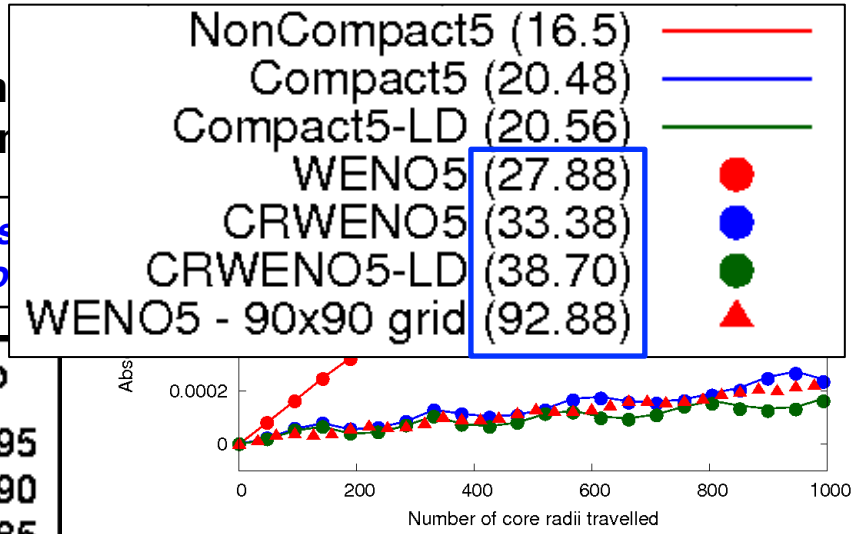
Compact schemes show better shape an
strength preservation for long terr
convection

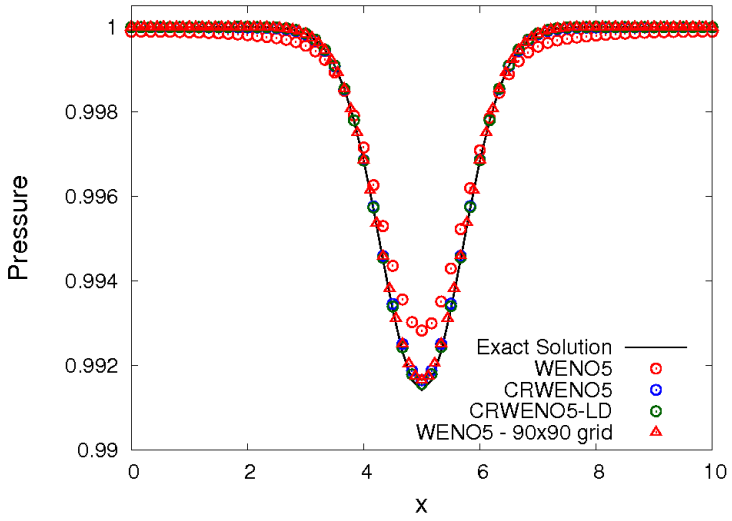
12 points
across co









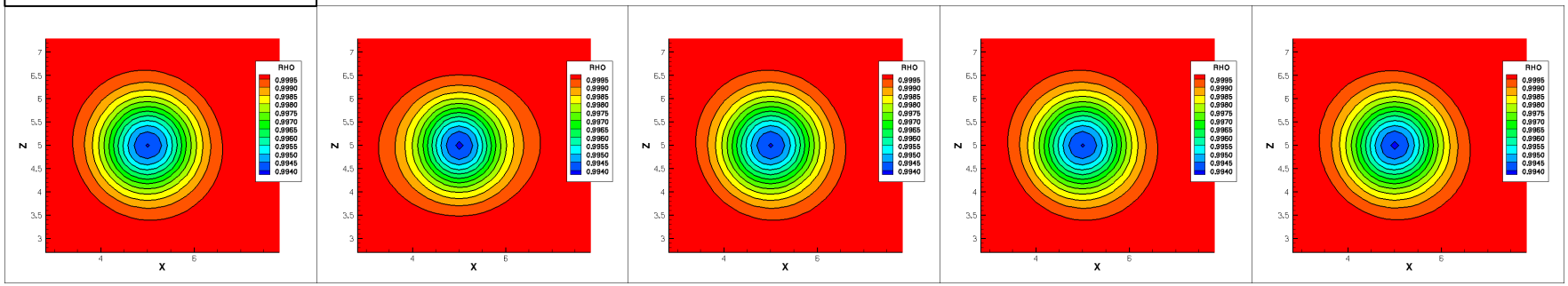


Isentropic Vortex Convection

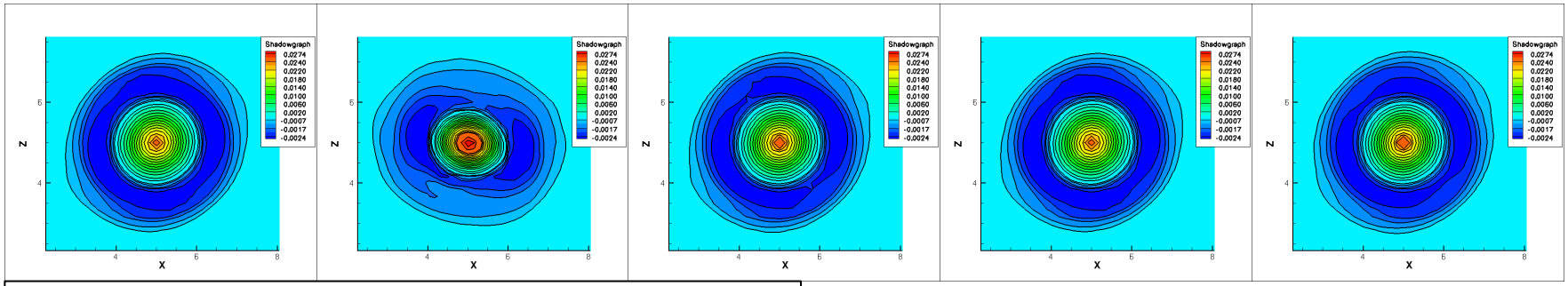
Long term inviscid convection (1000 core radii) – Preservation of strength and shape

Density Contours

$\varepsilon = 10^{-6}$, p = 2



No Limiting (5th order compact)	CRWENO5-JS	CRWENO5-Mapped	CRWENO5-YC	CRWENO5-Borges



Numerical Shadowgraph (Laplacian of density)

- Smooth, non-oscillatory density field for all schemes (almost identical)
- Oscillations in 2nd derivative for CRWENO5-JS & CRWENO5-Mapped, while CRWENO5-YC and CRWENO5-Borges are smooth

Double Mach Reflection

Double Mach Reflection of a Mach 10 shock on a 480 X 120 grid

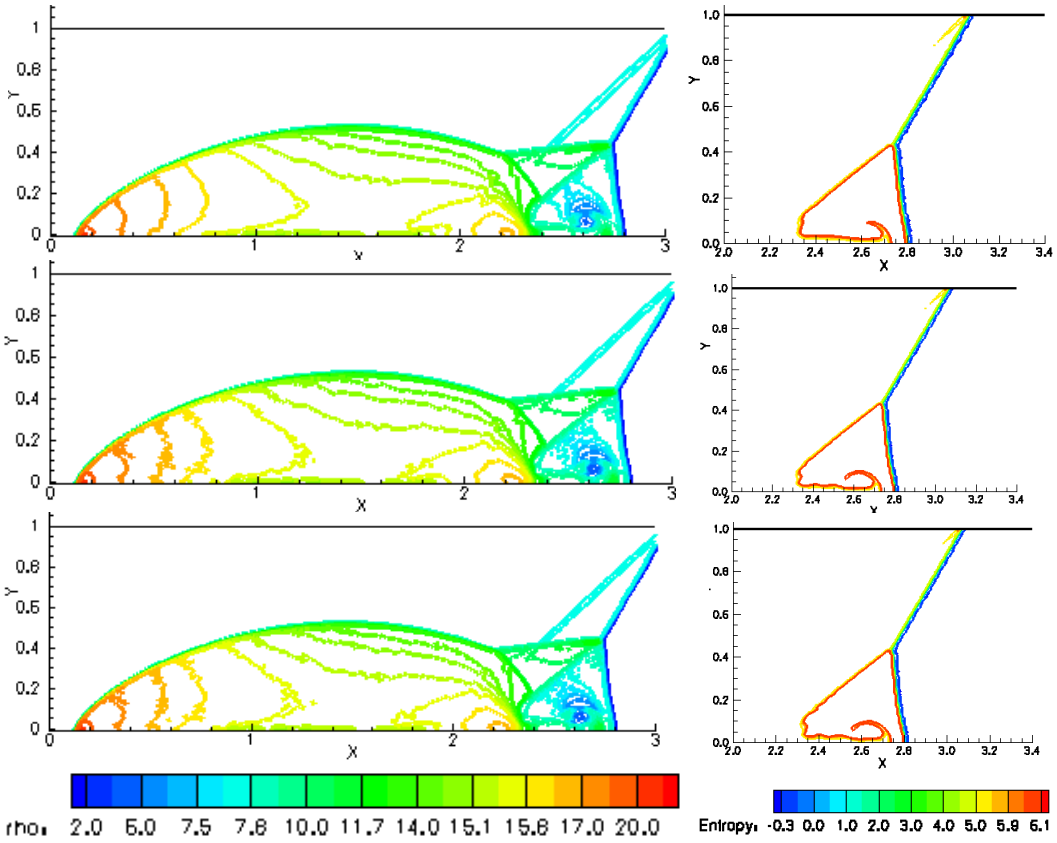
- **CRWENO schemes validation for 2D problem with strong discontinuities**
- **Better capturing of the contact discontinuity roll-up**

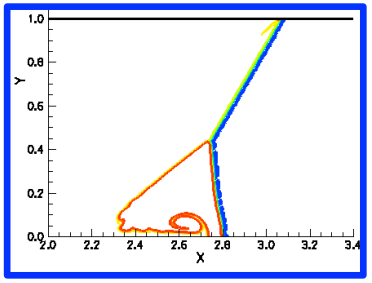
Density **Entropy**

WENO5

CRWENO5

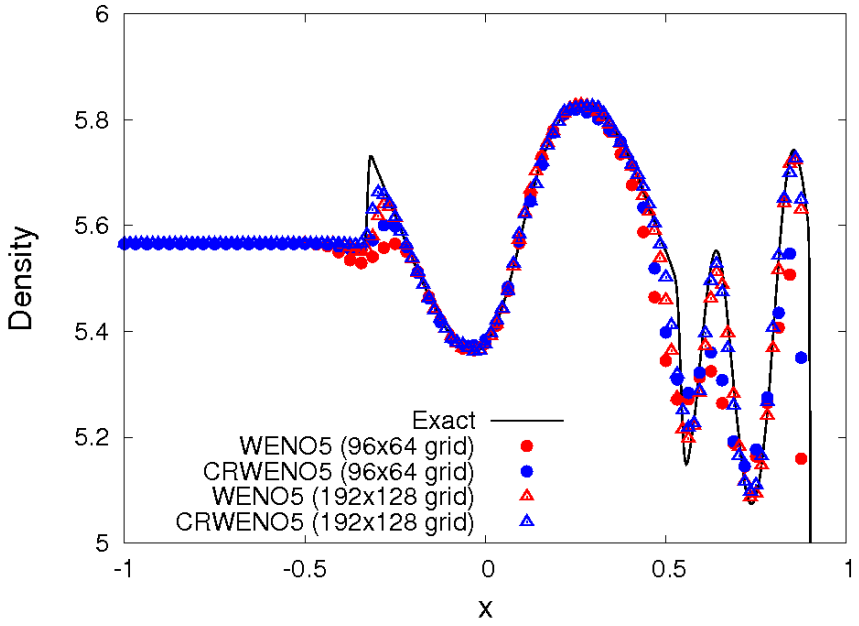
CRWENO5-LD



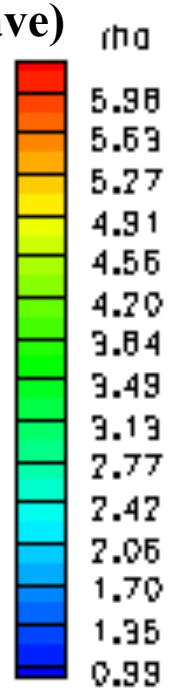


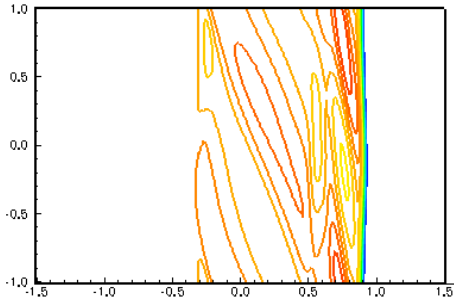
720 X 180 grid

Shock – Vorticity Interaction

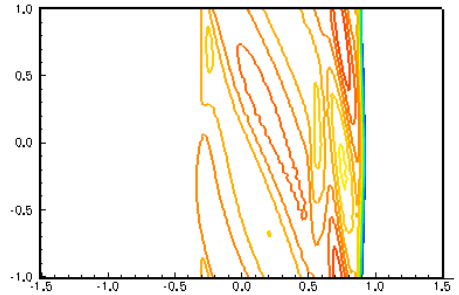


$\theta = \pi/6$ (Angle of vorticity wave)

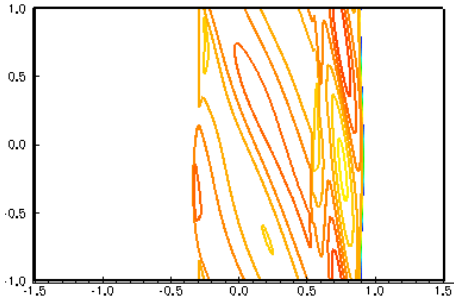




WENO5 (192x128 grid)



CRWENO5 (192x128 grid)



WENO5 (960x640 grid) ("Exact")

Interaction of a shock with a vorticity wave:

- Accurate capturing of acoustic, vorticity and entropy waves
- Solutions obtained on 96x64 and 192x128 grids
- CRWENO5 shows reduced clipping of the waves at both grid resolutions

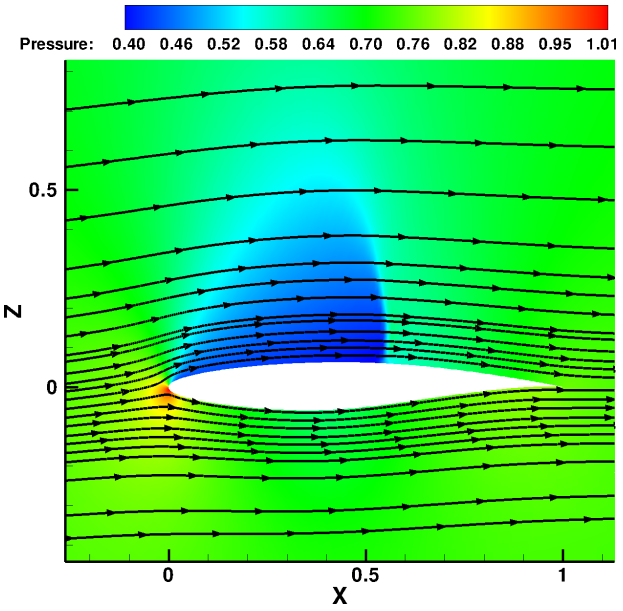
Integration with a Finite-Volume Navier-Stokes Solver

Baseline Solver

Integration of the CRWENO scheme with a compressible Navier Stokes solver for overset structured meshes

- **Time Marching:** 2nd order Backward Differencing (BDF2) and 3rd order Total Variation Diminishing Runge Kutta (TVDRK3)
- **Dual time-stepping** for time-accurate computations
- **Implicit Inversion:** Diagonalized ADI and LU-SGS
- **Spatial reconstruction:**
  - *5th order CRWENO scheme (compact)*
  - 3rd order MUSCL and 5th order WENO schemes (non-compact)
- **Upwinding:** Roe's flux differencing
- **Turbulence Modeling:** Spallart-Almaras one-equation model
- **Implicit hole-cutting** for overset meshes
- **Viscous Terms** discretized by 2nd order central differences

Steady Flow around RAE2822 Airfoil



Turbulent, transonic flow around RAE2822

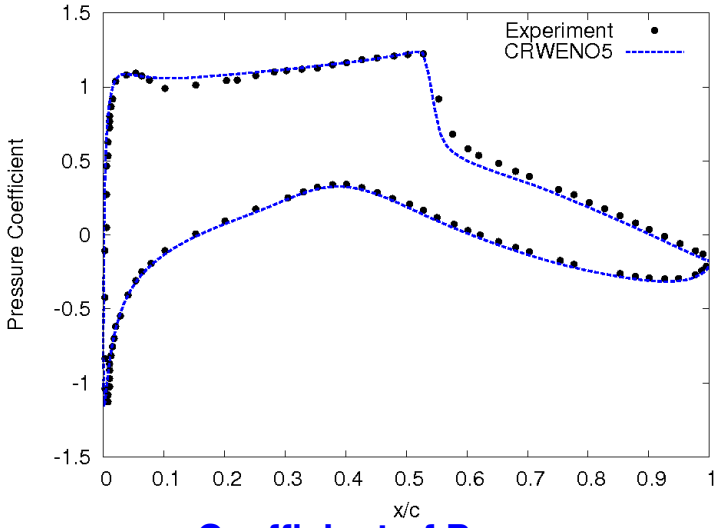
Flow Conditions:

Reynolds number 6.5 million, angle of attack 2.51°, freestream Mach number 0.731

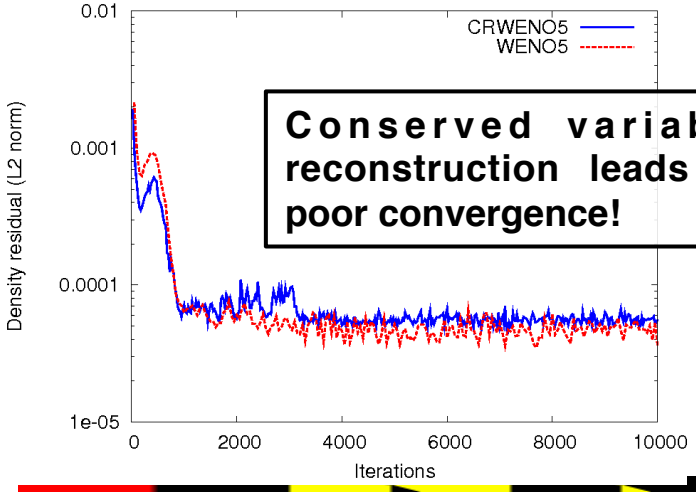
Numerical Solution:

Grid: 521x171 C-type
Outer boundary: 50c
CRWENO5 in space
BDF2 in time

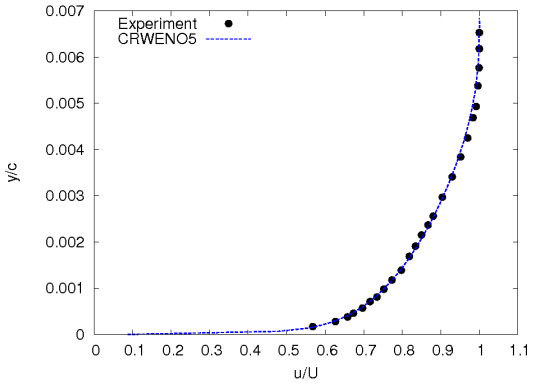
Validation w/ expt. results

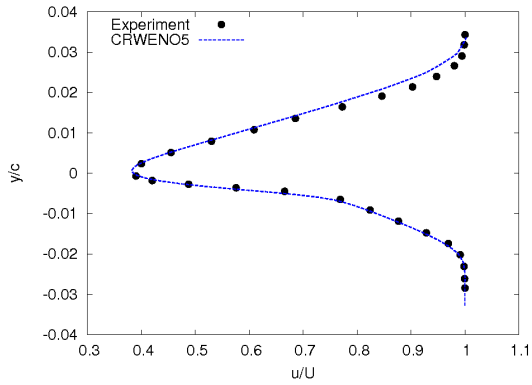


Coefficient of Pressure



Conserved variable reconstruction leads to poor convergence!



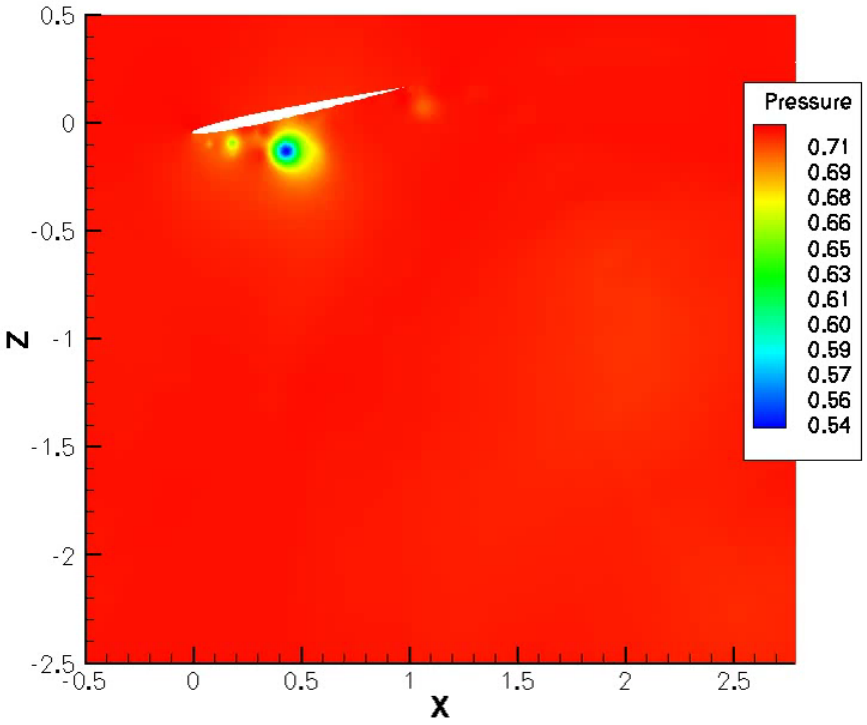


Boundary Layer and Wake velocity profiles

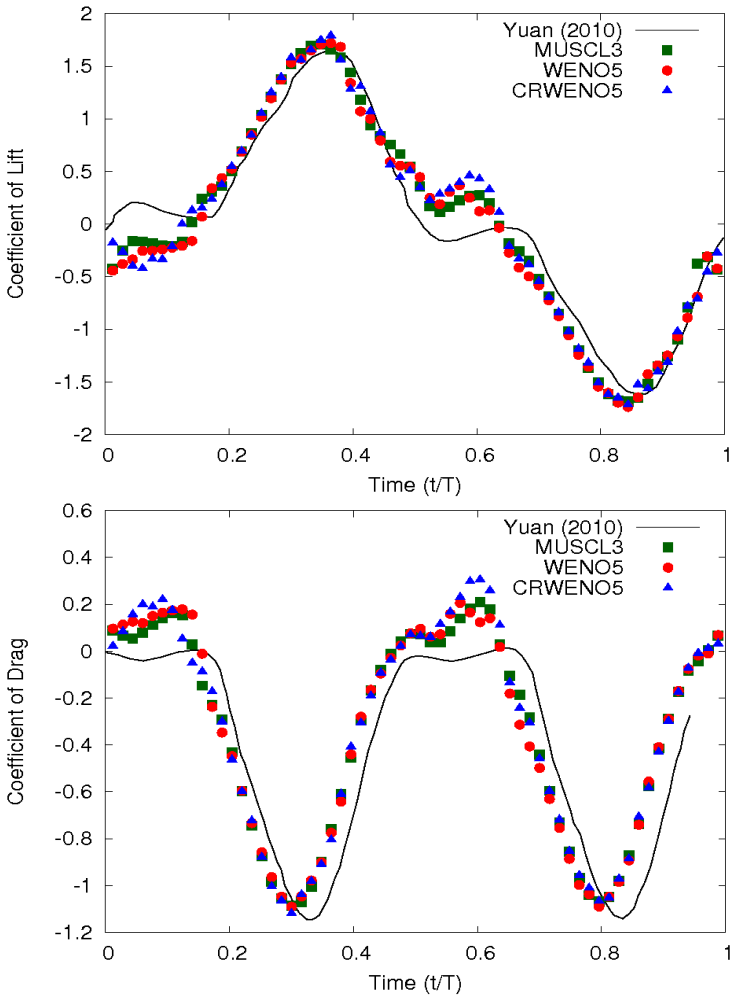
Pitching – Plunging NACA0005

Flow Conditions

Reynolds No. = 15000, Freestream Mach = 0.1
Pitch amplitude = 40^o, Plunge amplitude = 1.0
Reduced frequency = 0.795



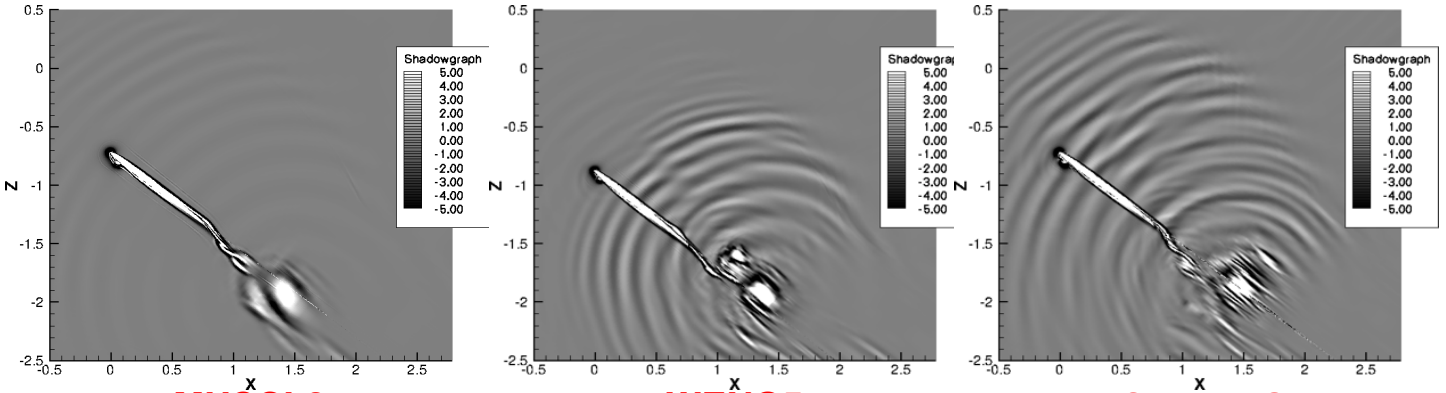
Grid dimensions: 391x161
Time stepping: BDF2 w/ 15 sub-iterations



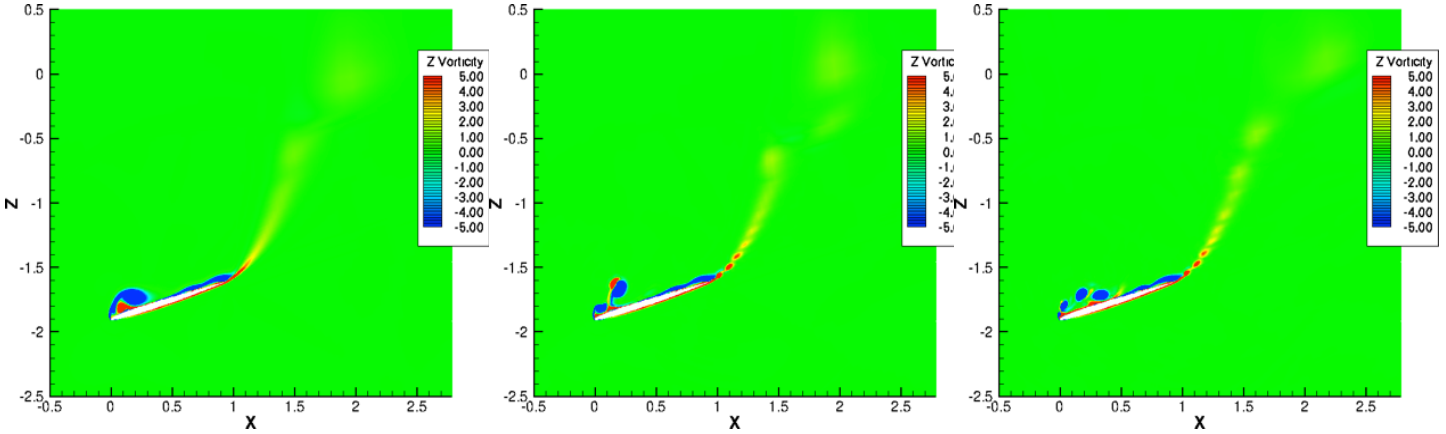
Lift and drag over one cycle

Pitching – Plunging NACA0005

Numerical Shadowgraph at t/T = 0.75



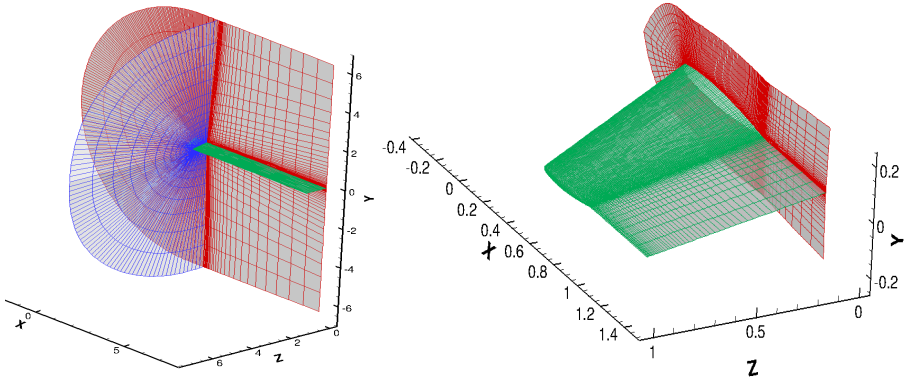
MUSCL3 WENO5 CRWENO5



Out-of-plane Vorticity at t/T = 0.40

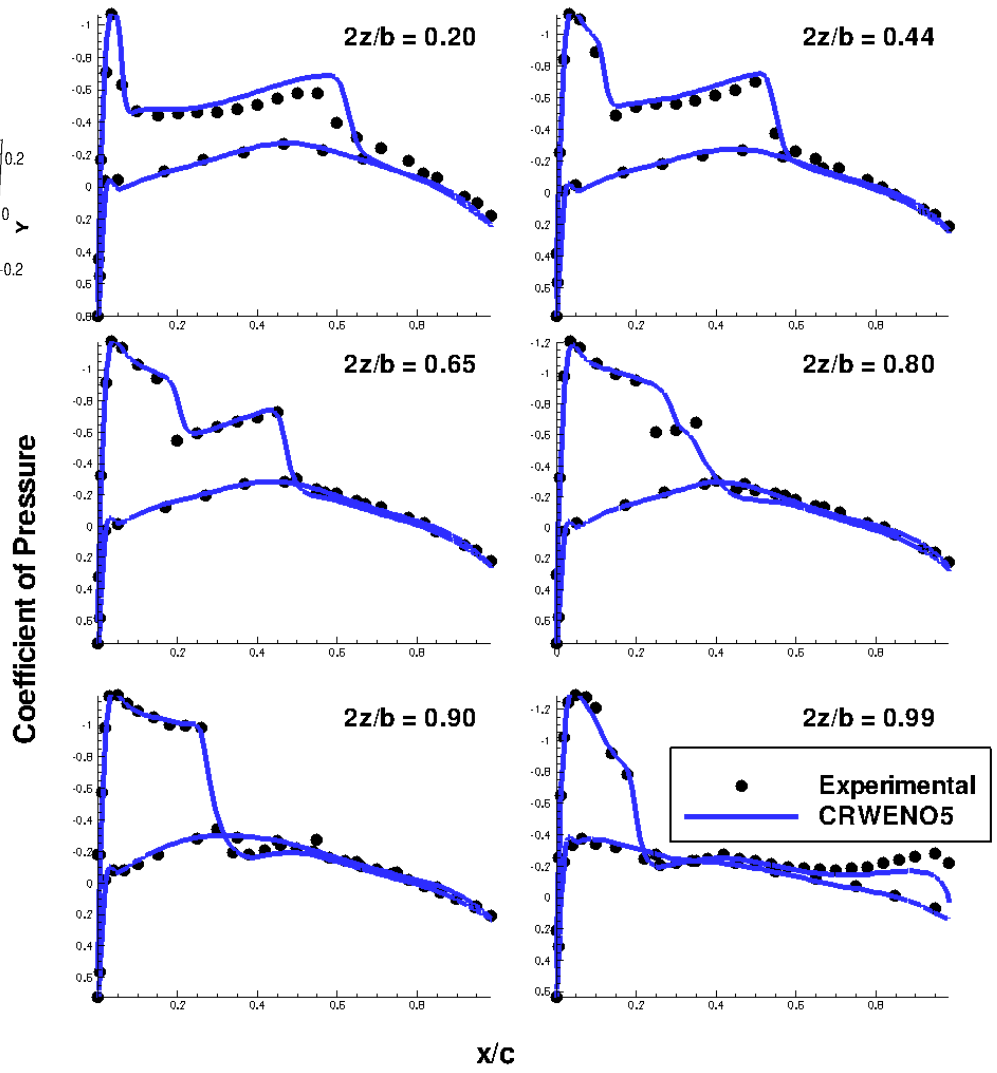
CRWENO5 shows improvement in capturing acoustic waves and vortical structures

ONERA-M6 Wing

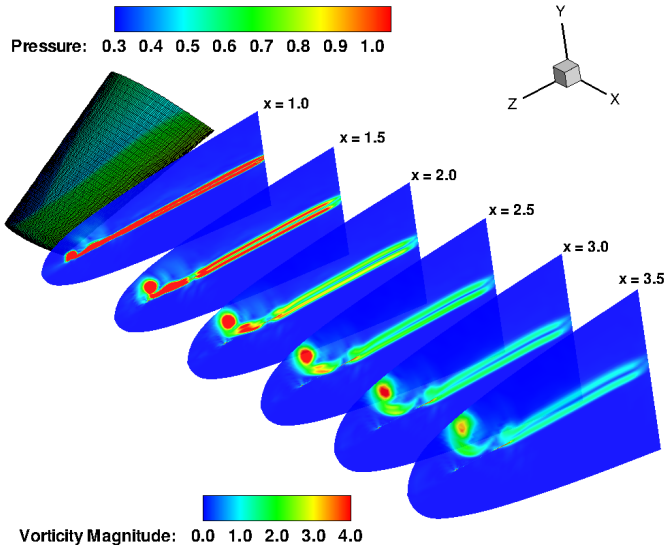


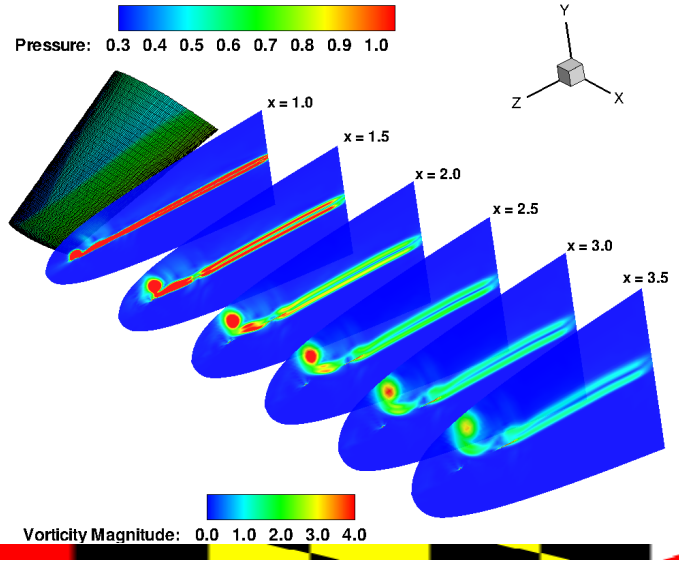
ONERA-M6 grid

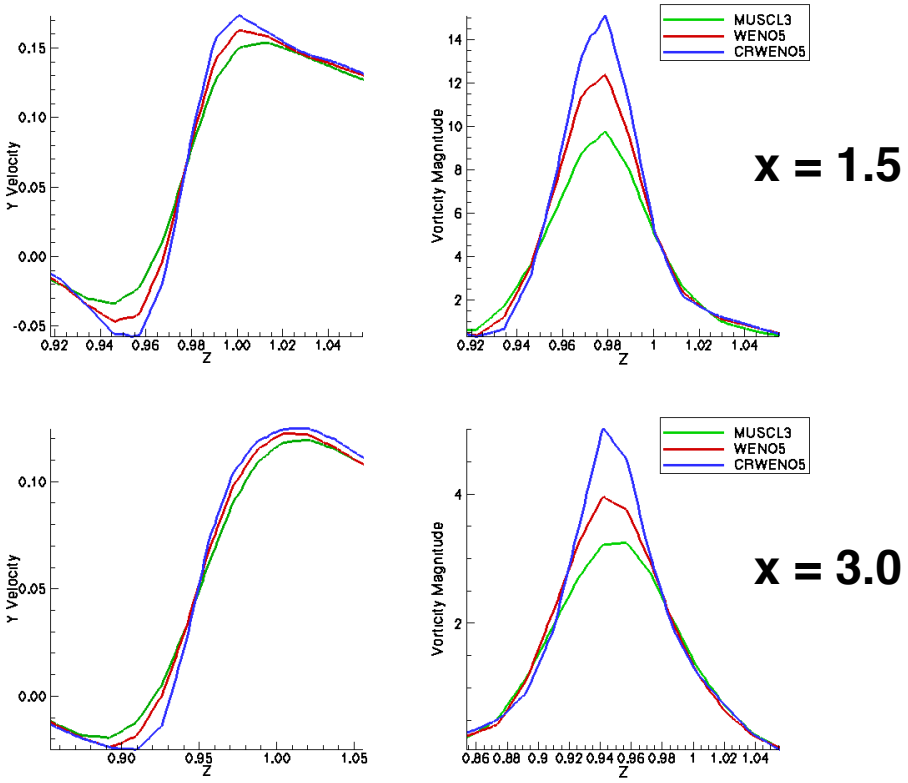
- Steady, transonic flow around the ONERA-M6 wing
- C-O mesh with 289x65x49 points
- Freestream Re = 21.67 million
- Freestream Mach number = 0.84
- Angle of attack = 3.06^{O}
- Surface pressure coefficient validated with experimental data



ONERA-M6 Wing







CRWENO5 shows an improvement in the tip vortex resolution and preservation in the wake region

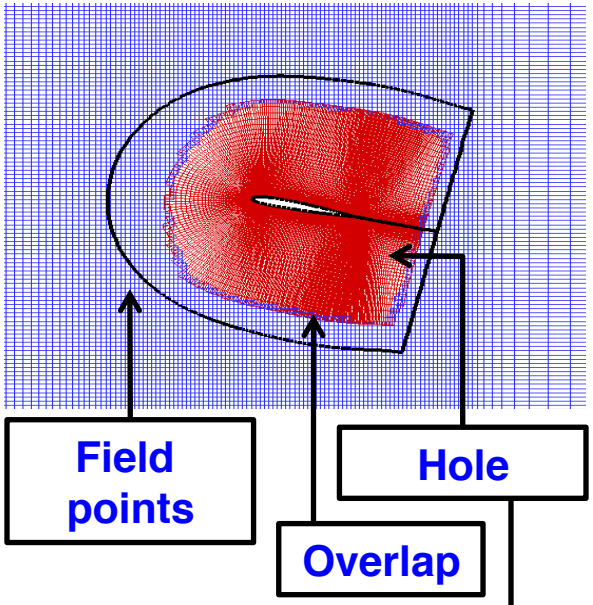
Overset Grids

Solution algorithm on overset meshes

- Identification of field, overlap and hole regions
- Field points → Governing equations are solved
- Overlap region → Solution exchanged with other meshes
- Hole region → Blanked out, contains non-physical values
- Implicit Hole-Cutting (Lee & Baeder, 2008)
- Tri-linear interpolation of solution between donor and receiver points

Application of compact schemes

- Coupled solution for the interface fluxes
- Solution in hole region coupled with solution at field points
- System of equations contain non-physical values from the hole region

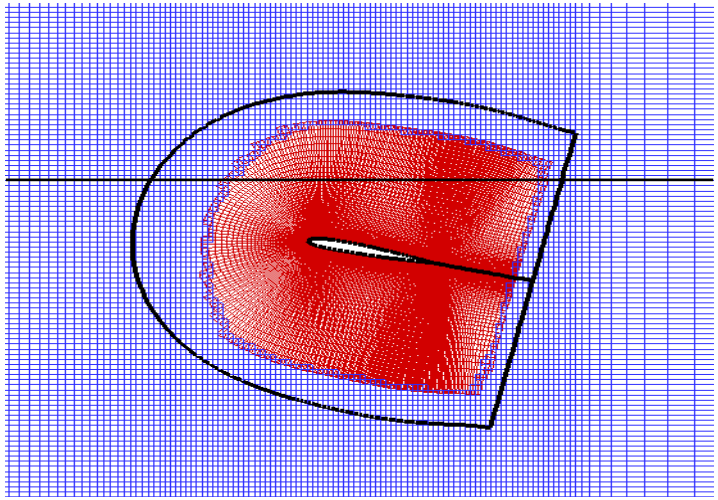


$$\begin{bmatrix} b & c & & & & & \\ a & b & c & & & & \\ & a & b & c & & & \\ & & a & b & c & & \\ & & & a & b & c & \\ & & & & a & b & c \\ & & & & & a & b \end{bmatrix} \begin{bmatrix} f_{1/2} \\ \\ \\ f_{j-1/2} \\ f_{j+1/2} \\ \\ f_{N+1/2} \end{bmatrix} = \text{RHS}$$

CRWENO on Overset Grids

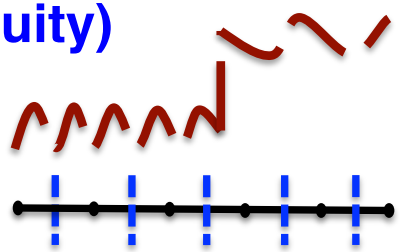
Behavior across discontinuity ⟷ Behavior across hole cut

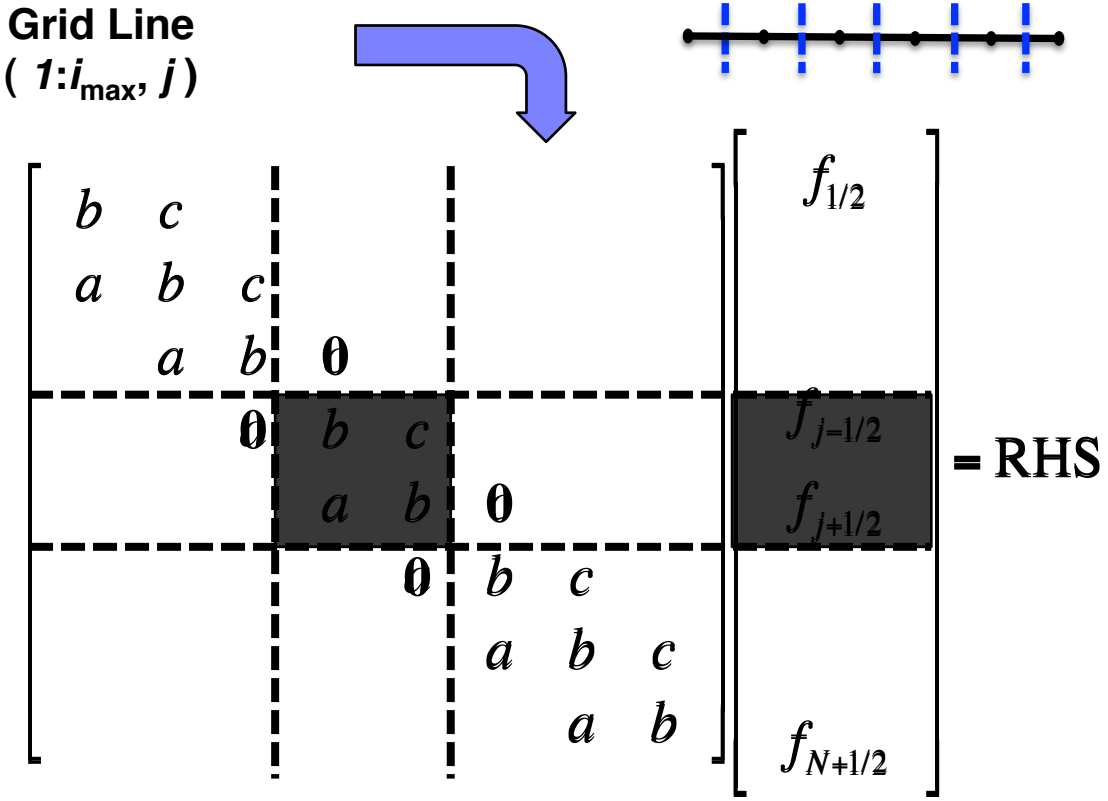
(Non-physical values appear as a discontinuity)



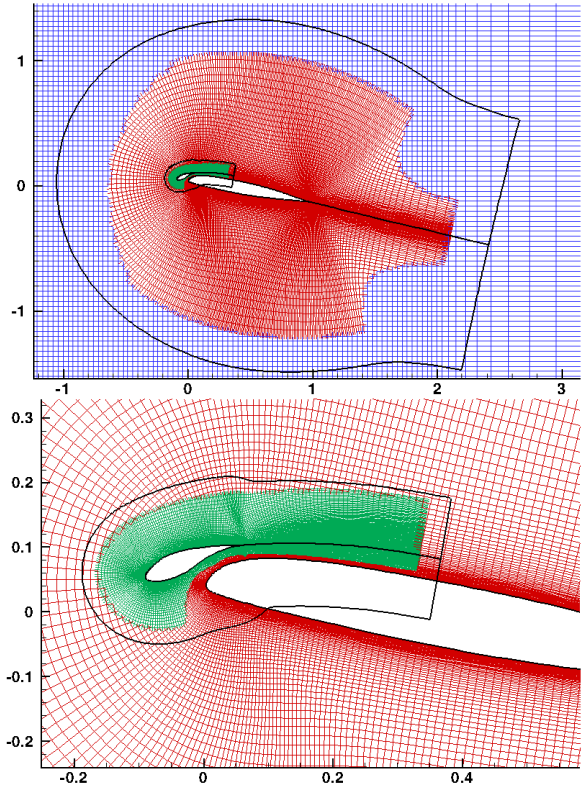
Adaptive stenciling of the CRWENO scheme

Decoupling of solution between field and hole points





SC2110 Airfoil w/ Slat in Wind Tunnel



Wind Tunnel Mesh – Clustered Cartesian, 151x101 points

Airfoil Mesh – C-type, 365x138 points

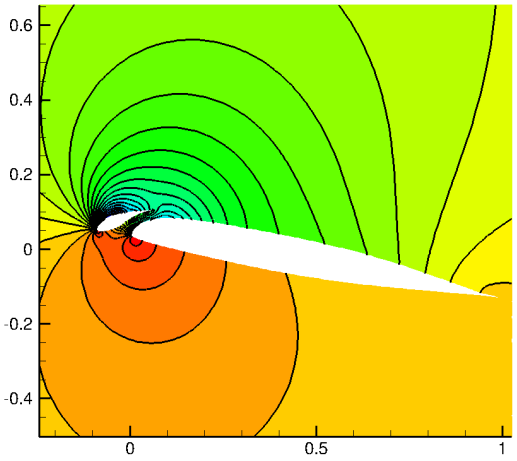
Slat Mesh – C-type, 317x97 points

Flow Conditions:

Reynolds number = 4.15e6

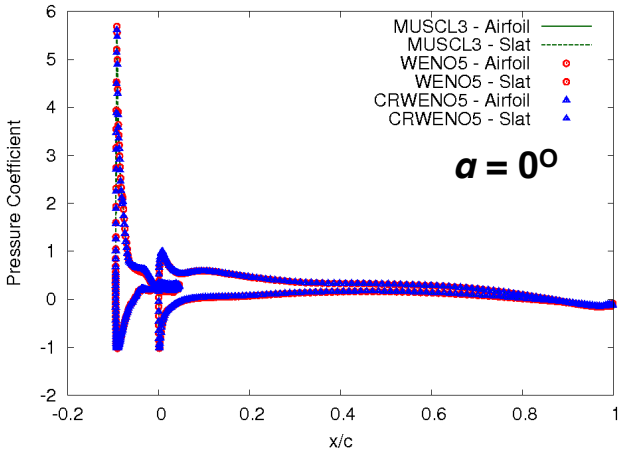
Freestream Mach = 0.283

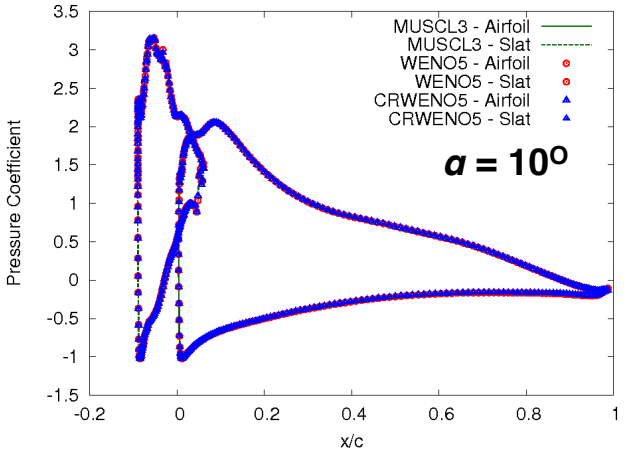
Angle of Attack = 0^O, 10^O



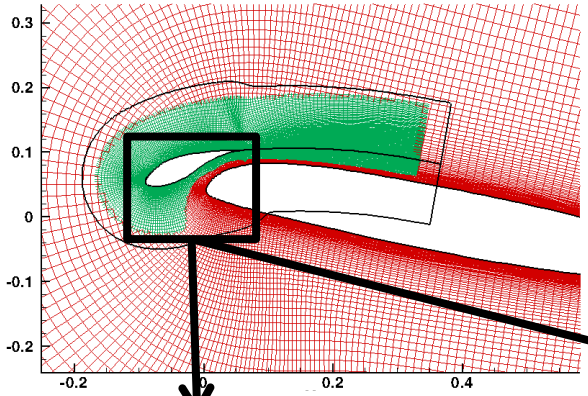
Pressure contours (CRWENO5 w/ BDF2)

Verification of CRWENO5 scheme with non-compact MUSCL3 and WENO5 schemes

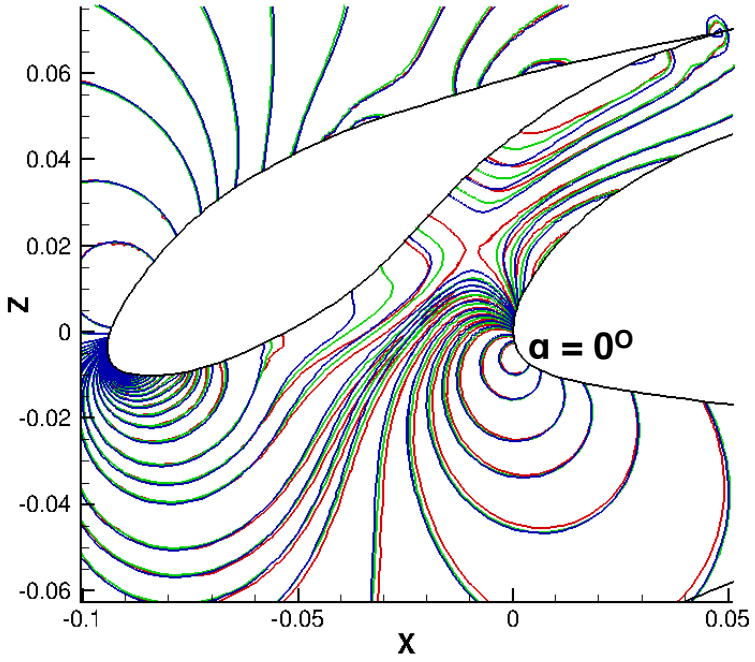


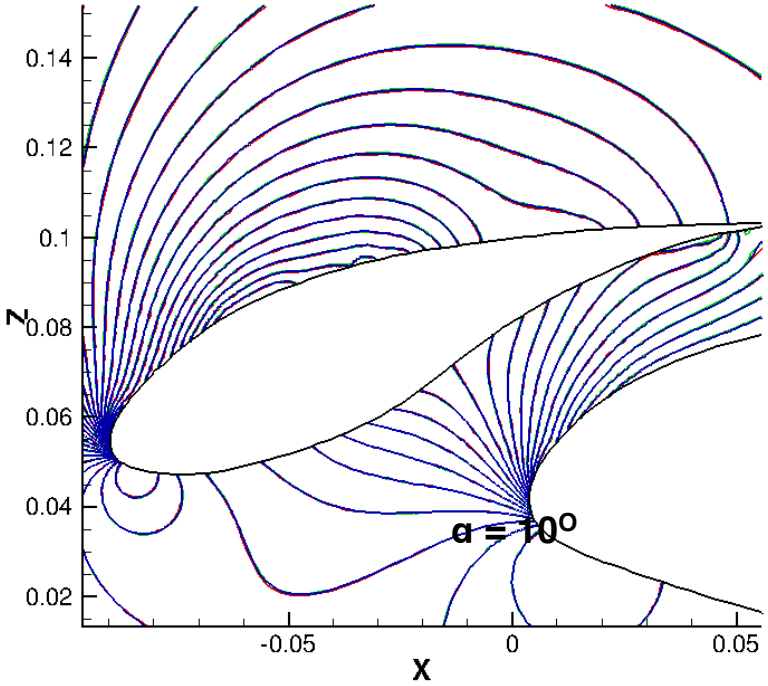


SC2110 Airfoil w/ Slat in Wind Tunnel



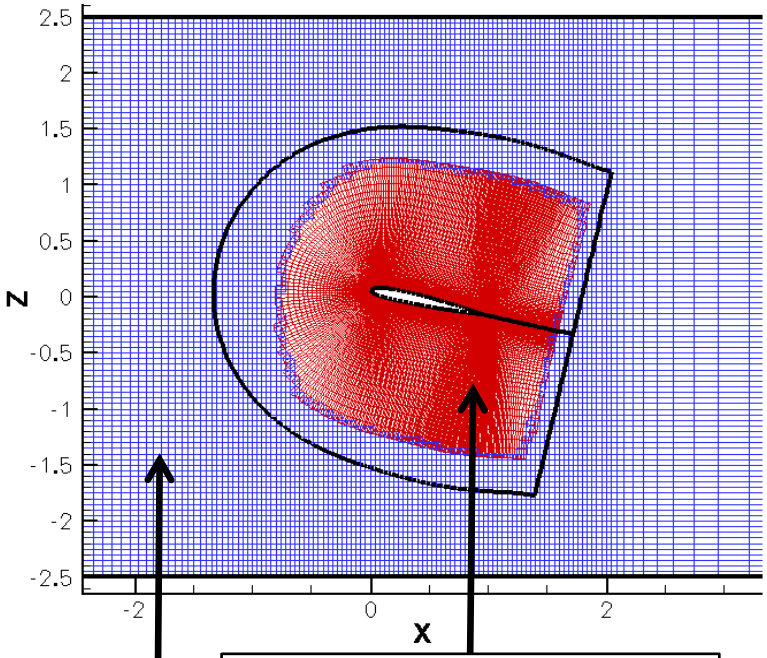
- High gradients in flow between slat and airfoil
- Overlap and solution transfer between slat and airfoil meshes
- Pressure contours from CRWENO5 compared with those from non-compact schemes

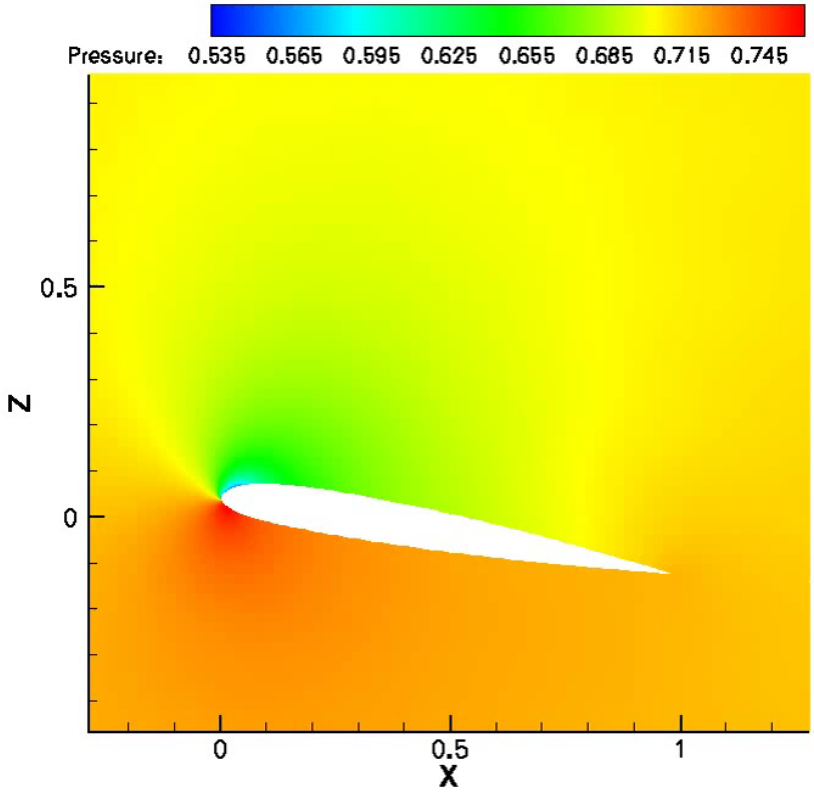




MUSCL
WENO5
CRWENO

SC1095 Dynamic Stall





Airfoil Mesh – C-type, 365x138 points

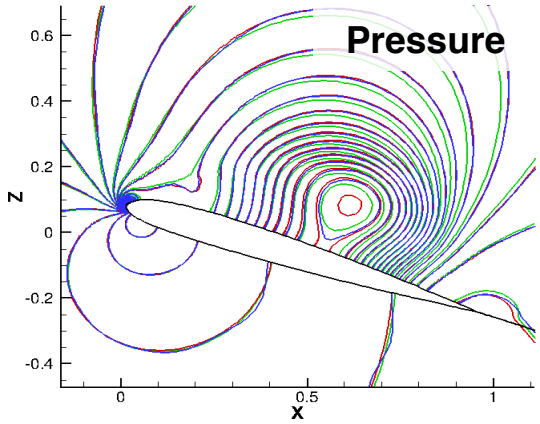
Wind Tunnel Mesh – Clustered Cartesian, 151x101 points

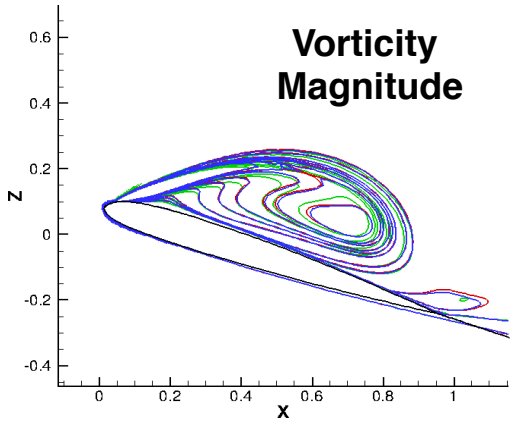
Flow Conditions:
Reynolds number: 3.92 million, Freestream Mach number 0.302
Mean angle of angle: 9.78º, Pitch Amplitude: 9.9º, Reduced Frequency: 0.099, Tunnel height: 5c

Numerical Solution: Time stepping: BDF2 w/ 15 sub-iterations

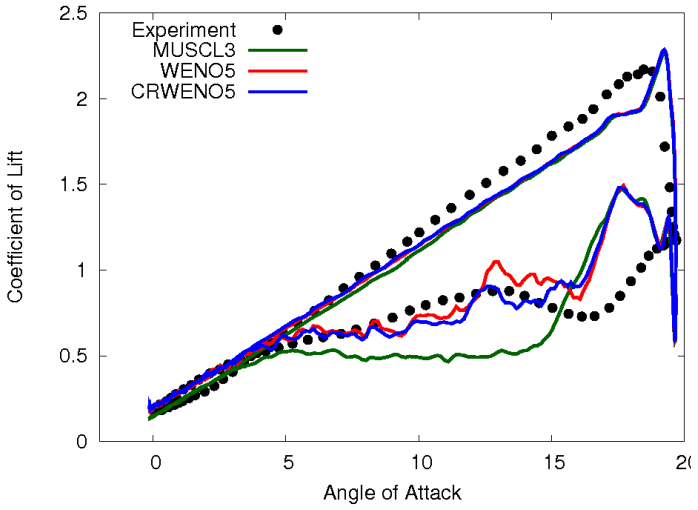
SC1095 Dynamic Stall

Validation for Overset Meshes w/ Grid Motion



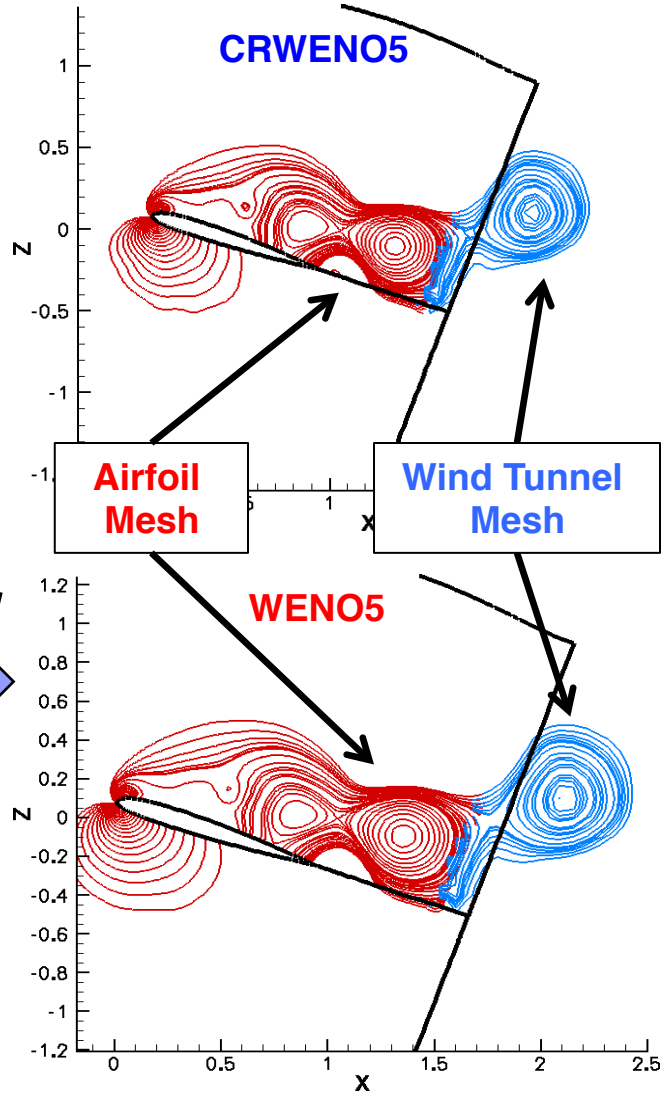


MUSCL3 WENO5 CRWENO5

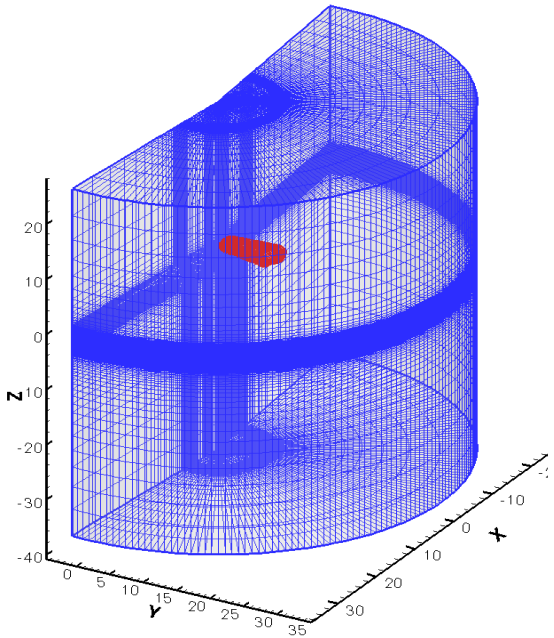


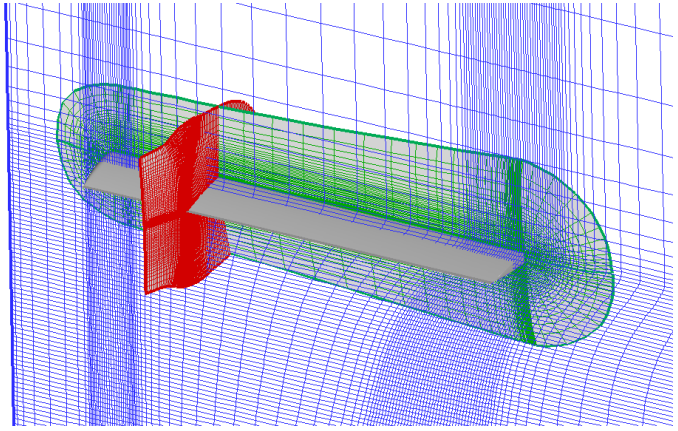
Shed vortices from upper surface

- Contour lines are continuous between airfoil and wind tunnel meshes
- Shed vortices pass smoothly between the two domains



Harrington 2-Bladed Rotor





Blade Mesh – C-O type, 267x78x56 points

Flow Conditions:

- M_{tip}: 0.352
- Re_{tip}: 3.5 million

Validation of thrust & power coefficients and figure of merit

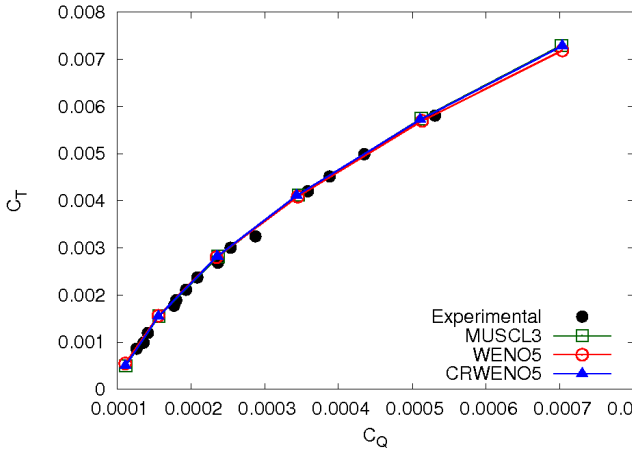
Cylindrical Back-ground Mesh

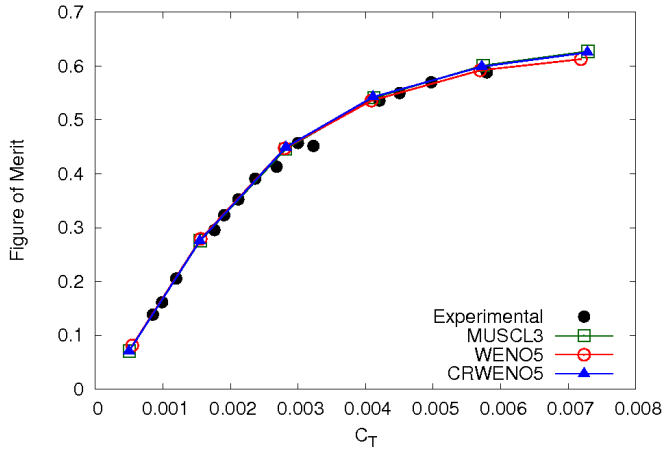
127 x 116 x 118 points

Rotor Geometry:

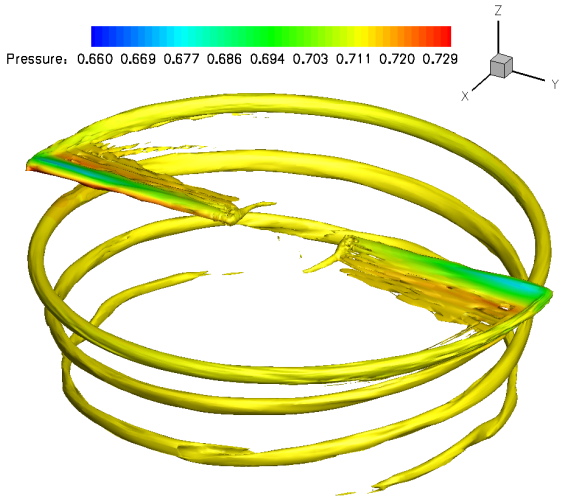
- **Aspect Ratio – 8.33**
- **Airfoil section – NACA**

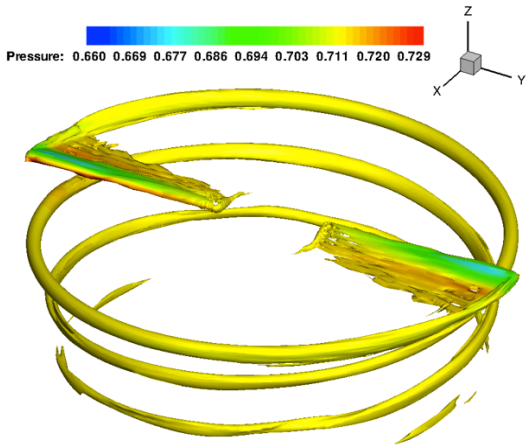
(t/c: 27.5% @ 0.2R, 15% @ R)

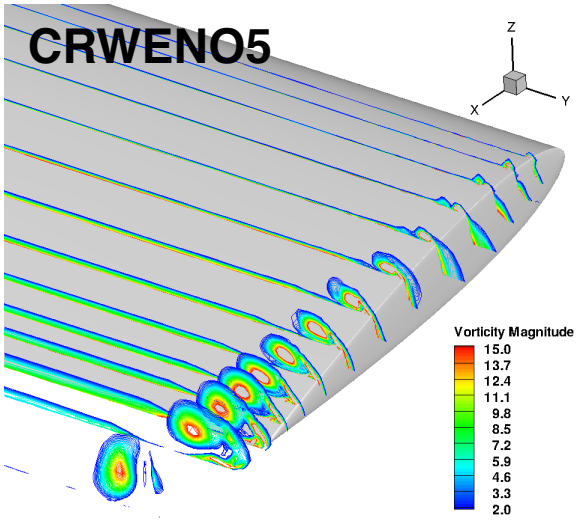


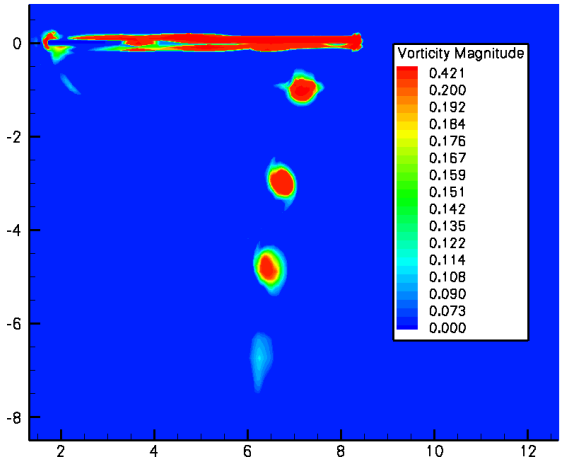


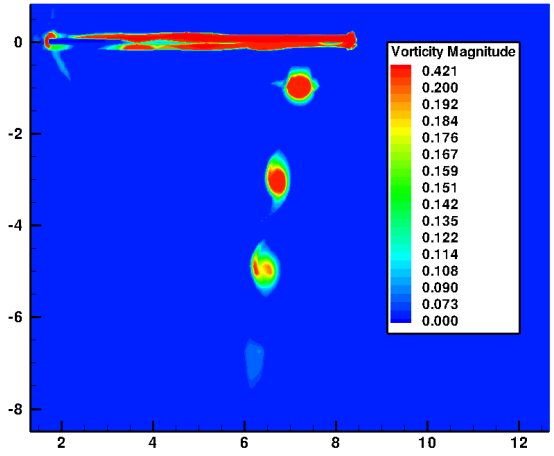
Harrington 2-Bladed Rotor (Near-Blade and Wake Flowfield)

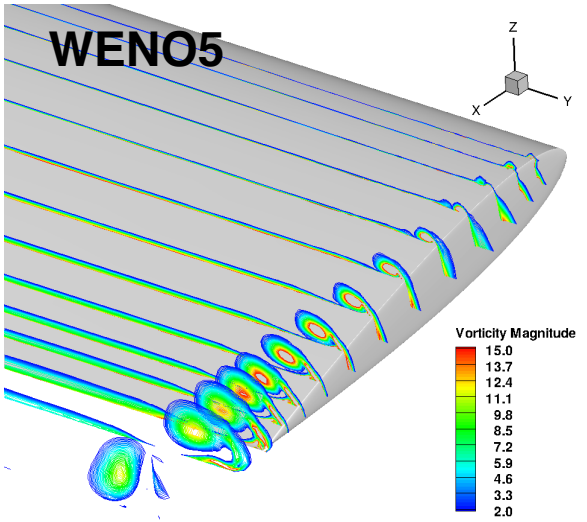








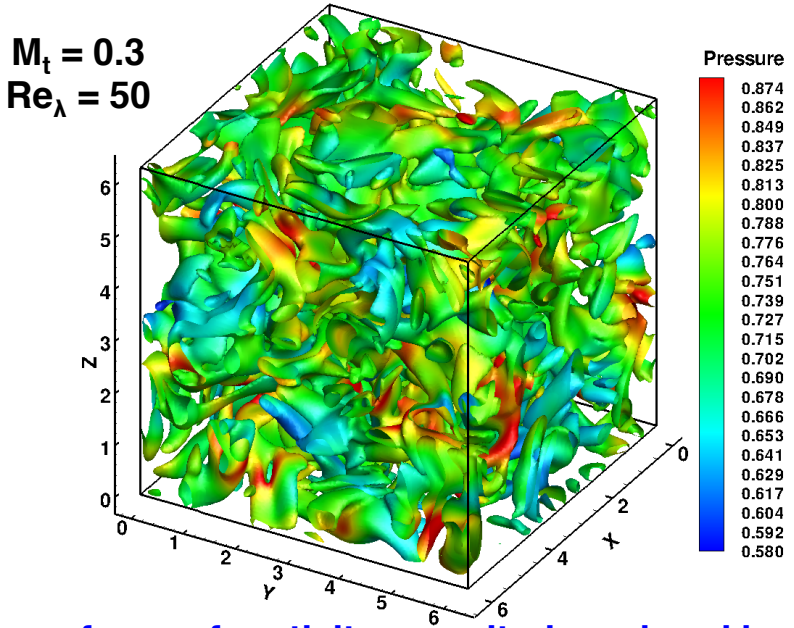




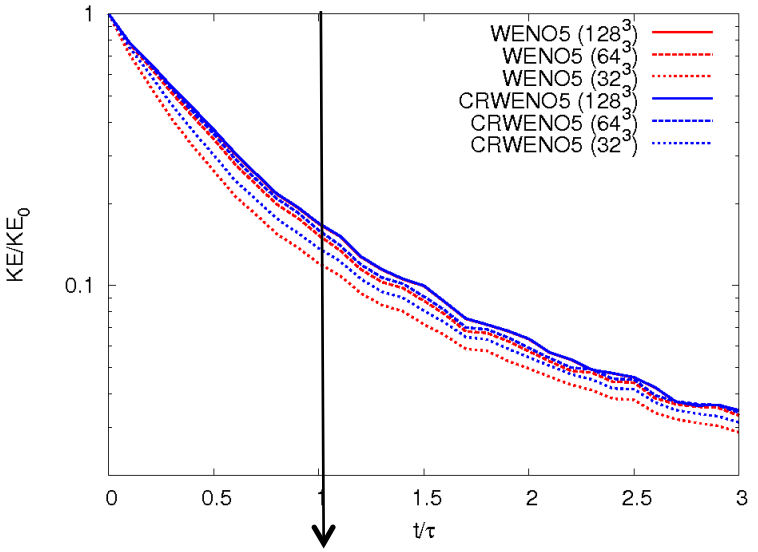
CRWENO5

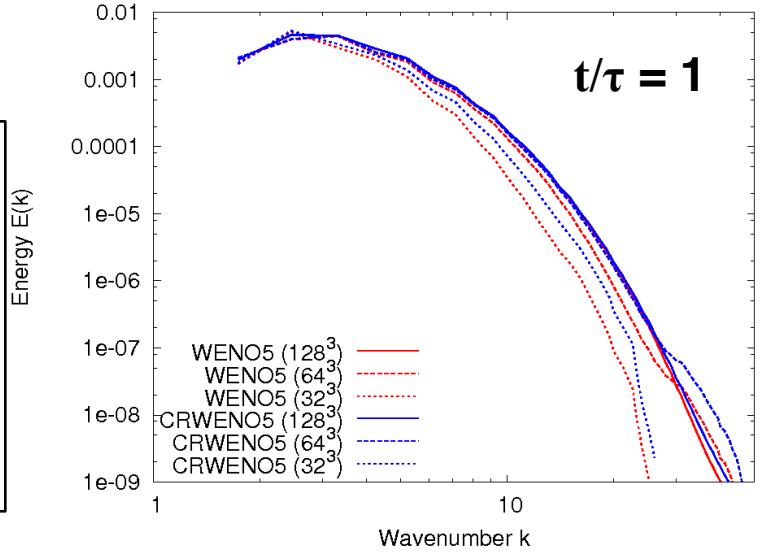
WENO5

Decay of Isotropic Turbulence



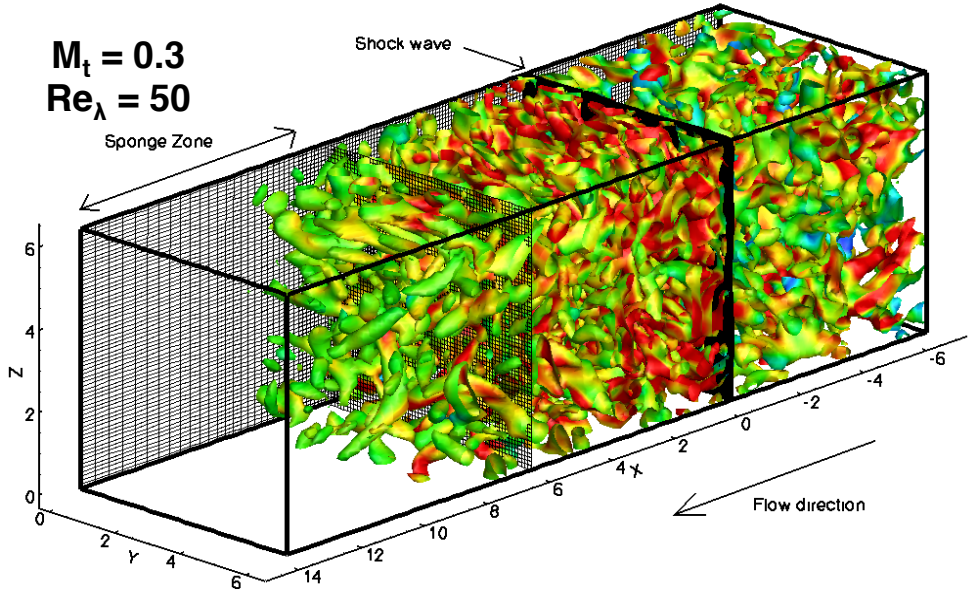
Iso-surfaces of vorticity magnitude, colored by pressure



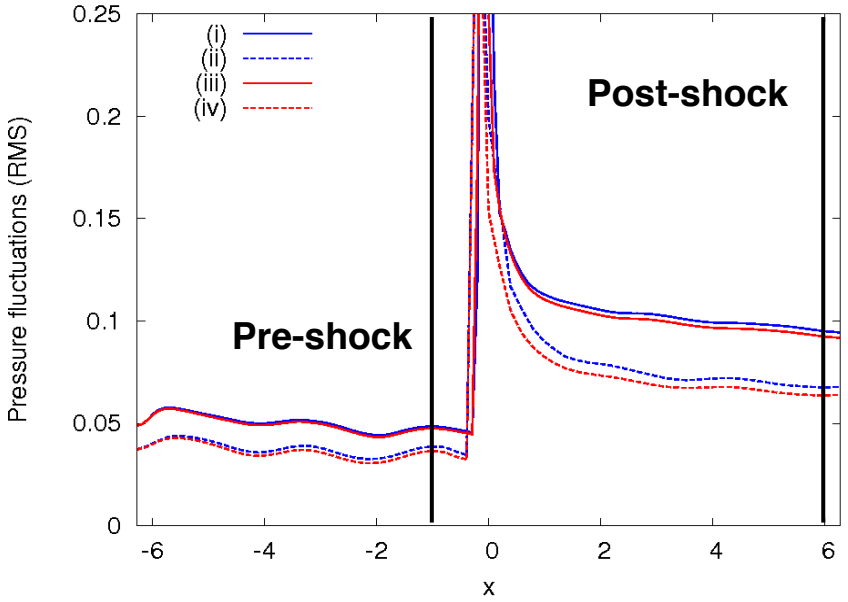


- Flow involves energy transfer to smaller length scales
- Grid-converged solutions obtained on 128^3 grid (WENO5 & CRWENO5 agree)
- CRWENO5 shows better resolution of intermediate and higher wavenumbers

Shock – Turbulence Interaction



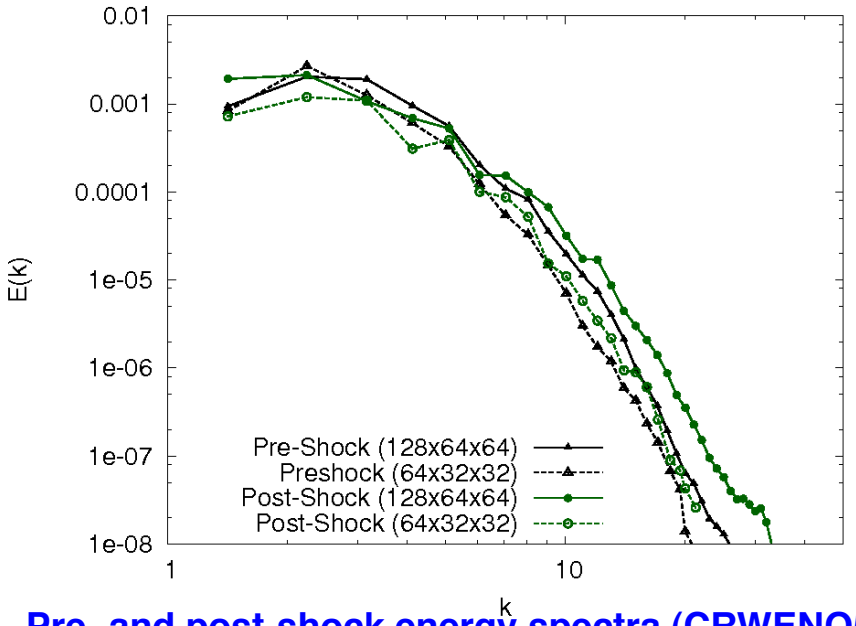
Iso-surfaces of 2nd invariant of velocity tensor, colored by vorticity magnitude



Stream-wise pressure fluctuations (RMS)

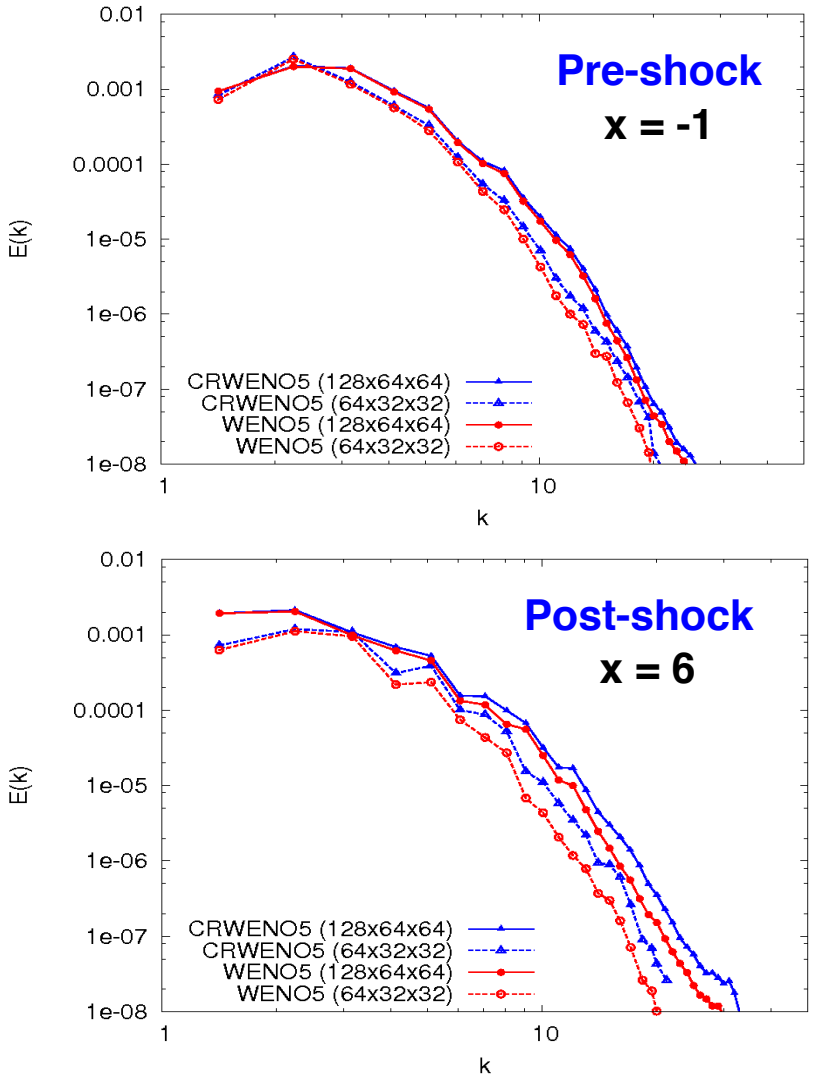
- Inflow: Fluctuations from isotropic turbulence decay added to mean flow at Mach 2
- Interaction with a shock wave magnifies the turbulent fluctuations
- Problem solved on two grids: 64x32x32 and 128x64x64 points (uniform)
- CRWENO5 → Lower dissipation → Predicts higher levels of fluctuations on both grids

Shock – Turbulence Interaction



Pre- and post-shock energy spectra (CRWENO5)

- Interaction with a shock wave amplifies intermediate and higher wavenumbers
- CRWENO5 shows improved resolution of the smaller length scales (for both grids)



Conclusions and Future Work

Conclusions

- **Derivation and implementation of 5th order CRWENO schemes**
  - Higher accuracy: Lower absolute errors than 5th order WENO scheme
  - Reduced clipping and smearing for discontinuities and extrema
  - Higher computational efficiency: Lower runtime than the WENO scheme for solutions with same accuracy and resolution
- **Extension to system of equations: inviscid Euler equations**
  - Implemented for reconstruction of conserved, primitive & characteristic variables
  - Higher accuracy and reduced smearing & clipping of discontinuities
  - Improved resolution of small-length scale waves (Shock-entropy wave interaction)
  - Improved preservation of vortex strength and shape (Isentropic vortex convection)
  - Higher computational efficiency for conserved/primitive reconstruction, NOT for characteristic based reconstruction
- **Integration with a finite-volume Navier-Stokes solver**
  - Applied to steady/unsteady flow past airfoils / wing / rotor
  - Validated for curvilinear grids + overset meshes with relative motion
  - Improved resolution of near-blade and wake flow features (vortical structures)
  - DNS of compressible turbulent flows: Lower dissipation of smaller length scales

Key Contributions

Compressible **Turbulent** **Flows**

Non-Oscillatory (WENO Schemes) + High spectral Resolution (Compact schemes)

Introduction of the Compact-Reconstruction WENO scheme

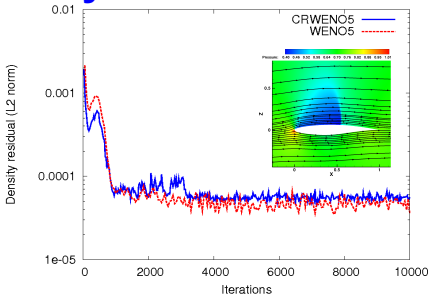
- Applies the WENO algorithm (solution-dependent interpolation) to compact schemes
- High spectral resolution ➔ Improved resolution of small length scales
- Non-oscillatory interpolation across discontinuities and steep gradients
- Higher accuracy ➔ Preservation of flow features for long-term convection
- Applicable to overset meshes with no special treatment

Comparison with compact-WENO hybrid schemes

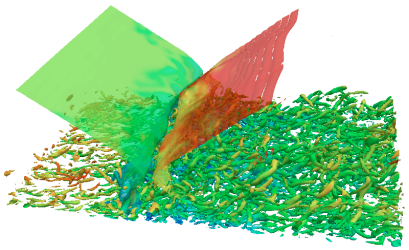
- No need for an additional switching parameter
- Does not revert to non-compact scheme (poor spectral resolution) near discontinuities

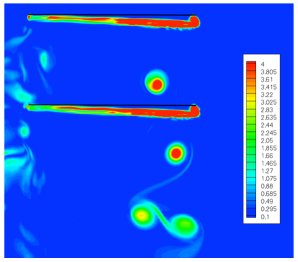
Future Work

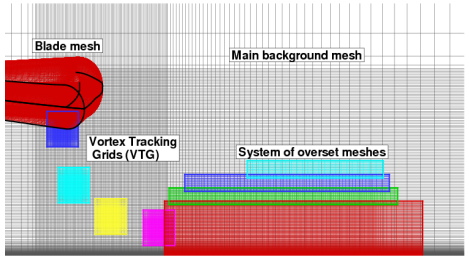
- **Improvements / extensions of the numerical scheme**
  - Implementation of non-linear weights (convergence for stationary shock and steady flow around airfoils)
  - Derivation for non-uniform grids
  - Implementation of a 9th order CRWENO scheme
  - Fine & medium grain parallelization issues
  - Use of monotonicity-preserving (MP) limits (Suresh & Huynh, 1997)

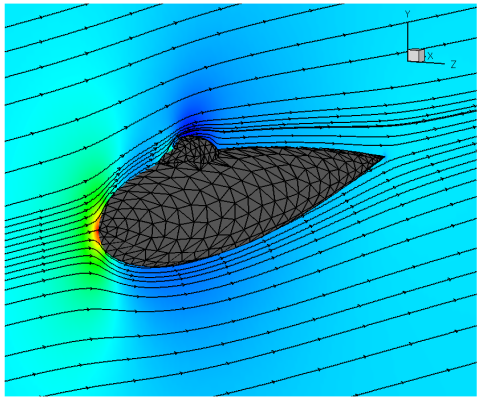


- **Applications**
  - High – resolution solutions to aircraft / rotorcraft wake flow and interaction with ground plane for rotorcraft operating IGE
  - Implementation for domains with immersed boundaries
  - DNS of shock – boundary layer interactions









Acknowledgments

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http://www.mast-cta.org/