Angewandte Computer- und Biowissenschaften

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Convolution Theorem and Key Examples of Fourier Series

Math. Seminar

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The Convolution Theorem

- It is one of the most powerful Fourier transform formulas.

- Has applications in Probability & Statistics (i.e. *The sum of two random variables is a convolution of their individual distributions*), Acoustics, Signal and Image Processing, Engineering, Computer Vision and in many other streams.

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The Convolution Theorem

Definition:

- If f and g are functions on R, their convolution is the function f * g

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy, \qquad \forall\ x \in \mathbb{R}.$$

- With a change of variables we have evidently

$$\int_{-\infty}^{\infty} f(x-y)g(y)\,dy = \int_{-\infty}^{\infty} f(y)g(x-y)\,dy.$$

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The Convolution Theorem

Convolution obeys the same algebraic laws as ordinary multiplication, such as-

(i) The associative law: for a , b constants,

$$f*(ag+bh) = a(f*g)+b(f*h).$$

(ii) The commutative law : $f * g = g * f.$

(iii) The distributive law : $f * (g * h) = (f * g) * h.$

Theorem: If f and g are differentiable and the convolutions f * g, f' * g and f * g' are well defined. Then f * g is differentiable and

$$(f * g)'(x) = (f' * g)(x) = (f * g')(x).$$

The Convolution Theorem

The Convolution Theorem

If f(x) and g(x) are integrable functions with Fourier transforms $\hat{f}$ and $\hat{g}$ respectively, then the Fourier transform of the convolution is given by the product of the Fourier transforms $\hat{f}$ and $\hat{g}$. i.e, if f, g $\in L^1$

$$\mathcal{F}[f * g] = (f * g)\hat{} = \hat{f}\hat{g}.$$

The Convolution Theorem

Proof: By the definition

$$(f * g)\hat{}(\xi) = \int_{-\infty}^{\infty} (f * g)(x) e^{-i\xi x} dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-y) g(y) e^{-i\xi x} dy dx.$$

We can use Fubinis theorem to change the order of integration. Substituting also x – y = z, it follows that

$$\begin{aligned}(f * g)\hat{}(\xi) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-y) g(y) e^{-i\xi x} dx dy \\ &= \int_{-\infty}^{\infty} g(y) \Big\{ \int_{-\infty}^{\infty} f(z) e^{-i\xi(y+z)}\, dz \Big\} dy \\ &= \Big(\int_{-\infty}^{\infty} g(y)) e^{-i\xi y} dy \Big) \Big(\int_{-\infty}^{\infty} f(z) e^{-i\xi z} dz \Big) = \hat{f}(\xi) \hat{g}(\xi)\end{aligned}$$

thus we have,

$$(f * g)\hat{}(\xi) = \hat{f}(\xi) \hat{g}(\xi) \qquad \square$$

Examples of Fourier Transform

Example : Determine the Fourier transform for the function f(x) = $e^{-|x|}$

Solution : Using the definition of the Fourier transformation it follows that

$$\mathcal{F}\left[e^{-|x|}\right](\xi) = \int_{-\infty}^{\infty} e^{-|x|} e^{-i\xi x} dx = \int_{-\infty}^{0} e^{(1-i\xi)x} dx + \int_{0}^{\infty} e^{-(1+i\xi)x} dx$$

$$= \left[\frac{e^{(1-i\xi)x}}{1-i\xi}\right]_{-\infty}^{0} + \left[\frac{e^{-(1+i\xi)x}}{-(1+i\xi)}\right]_{0}^{\infty} = \frac{1}{1-i\xi} + \frac{1}{1+i\xi} = \frac{2}{1+\xi^2}.$$

Let us try to plot the graph now,

Examples of Fourier Transform

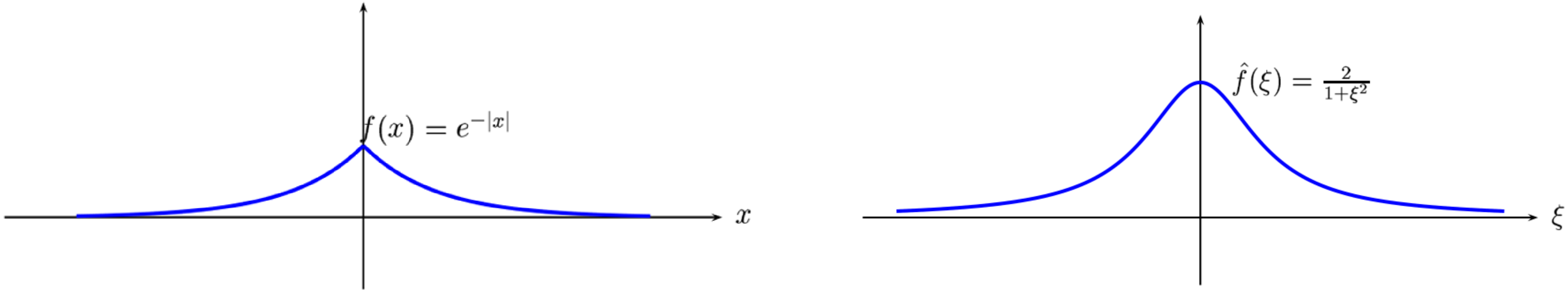

Figure 4.1: The function $f(t) = e^{-|x|}$ and its Fourier transform $\hat{f}(\xi) = \frac{2}{1+\xi^2}$.

Examples of Fourier Transform

Lemma: *Let* $f(x) = sign(x) \cdot e^{-a|x|}$, *then* $\hat{f}(\xi) = \frac{-2i\xi}{a^2+\xi^2}$.

Proof.

$$\mathcal{F}\left[sign(x) \cdot e^{-a|x|}\right] = \int_{-\infty}^{\infty} sign(x) \cdot e^{-a|x|} e^{-i\xi x} dx$$

$$= \int_{-\infty}^{0} -e^{(a-i\xi)x} dx + \int_{0}^{\infty} e^{-(a+i\xi)x} dx$$

$$= \left[\frac{e^{(a-i\xi)x}}{a-i\xi}\right]_{-\infty}^{0} + \left[\frac{e^{-(a+i\xi)x}}{-(a+i\xi)}\right]_{0}^{\infty}$$

$$= \frac{-1}{a-i\xi} + \frac{1}{a+i\xi} = \frac{-2i\xi}{a^2+\xi^2}.$$

$\square$

Examples of Fourier Transform

Example: Find the Fourier transform for the function $f(x) = e^{-x^2}$.

Solution: I will try to solve it by writing it down.

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Thank You For Your Time!

Please feel free to ask, if you have doubts.

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