Summation, Subtraction, Scalar Multiplication

Ex $A = 2\hat{i} + 3\hat{j} - \hat{k}$ & $\vec{B} = \hat{i} - \hat{j} + 2\hat{k}$

$\vec{A} + \vec{B} = 3\hat{i} + 2\hat{j} + \hat{k}$ {addition}

$\vec{A} - \vec{B} = \hat{i} + 4\hat{j} - 3\hat{k}$ {Subtraction}

$2\vec{A} = 4\hat{i} + 6\hat{j} - 2\hat{k}$ {Scalar Multiplication}

Unit vector

$$\boxed{\hat{A} = \frac{\vec{A}}{|\vec{A}|}}$$

Ex:- $\vec{A} = 2\hat{i} + 3\hat{j} - \hat{k}$

$$\hat{A} = \frac{2\hat{i} + 3\hat{j} - \hat{k}}{\sqrt{14}}$$

Vector Multiplication

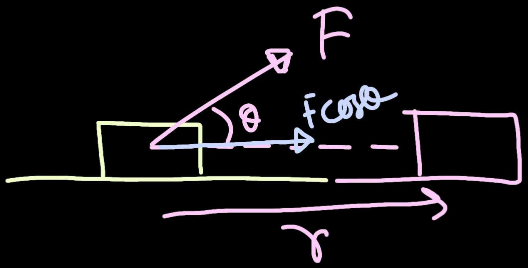
Scalar Multiplication
{Dot product}

Vector Mutiplication
{Cross product}

Dot product

$$\boxed{\vec{A}\cdot\vec{B} = AB\cos\theta = A_xB_x + A_yB_y + A_zB_z}$$

work



$$W = (F\cos\theta)\, r$$

$$= F r \cos\theta$$

$$\boxed{W = \vec{F}\cdot\vec{r}}$$

① If $\vec{A}$ and $\vec{B}$ are perpendicular

then $\vec{A} \cdot \vec{B} = 0$

② If $\vec{A}$ and $\vec{B}$ are parallel

then $\vec{A} \cdot \vec{B} = AB$

Crossproduct

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$|\vec{A} \times \vec{B}| = AB\sin\theta$$

Torque

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Angular velocity

$$\vec{v} = \vec{\omega} \times \vec{r}$$

① If $\vec{A}$ and $\vec{B}$ are are two sides of a parallogram and $\vec{A}$ and $\vec{B}$ are colinear then cross products gives us the area of palallogram

area of parallogram $= |\vec{A} \times \vec{B}|$

② If $\vec{A}$ & $\vec{B}$ are parallel then $\Rightarrow \vec{A} \times \vec{B} = \vec{0}$

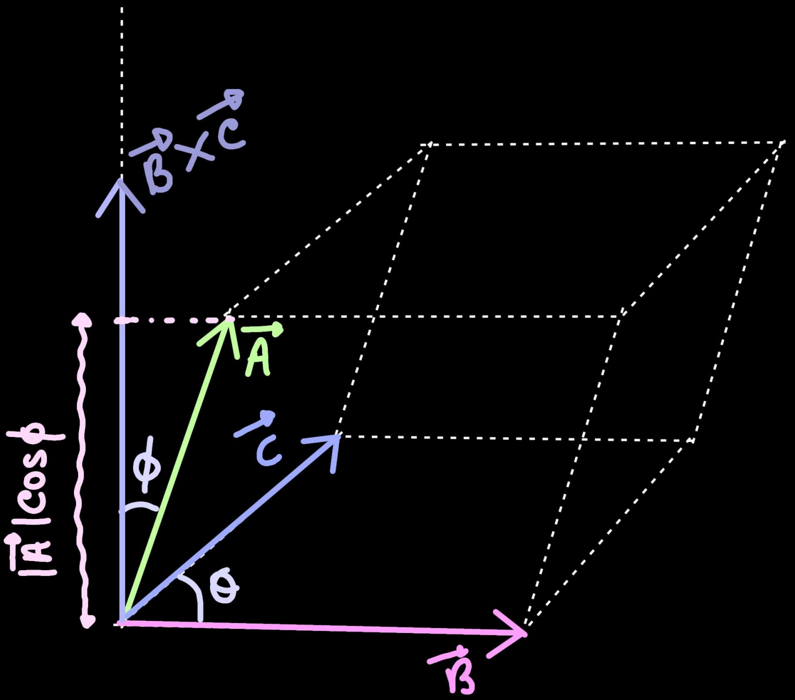
③ If $\vec{A}$ & $\vec{B}$ are perpendicular then $\Rightarrow \vec{A} \times \vec{B} = AB$

④ Area of triangle $= \frac{1}{2} |\vec{A} \times \vec{B}|$

Scalar Triple product

Let $\vec{A}, \vec{B}, \vec{C}$

$$\boxed{\begin{aligned}\text{Scalar Triple product} &= \vec{A}\cdot(\vec{B}\times\vec{C}) \\ &= \vec{B}\cdot(\vec{C}\times\vec{A}) \\ &= \vec{C}\cdot(\vec{A}\times\vec{B})\end{aligned}}$$



Volume of parallelepiped

= (Base) (Perpendicular Hight)

$= (BC\sin\theta)(A\cos\phi)$

$= |\vec{B}\times\vec{C}|\ A\cos\phi$

$= \vec{A}\cdot(\vec{B}\times\vec{C})$

Area of Base $= |\vec{B}\times\vec{C}| = BC\sin\theta$

* If $\vec{A}$, $\vec{B}$ & $\vec{C}$ are coplaner

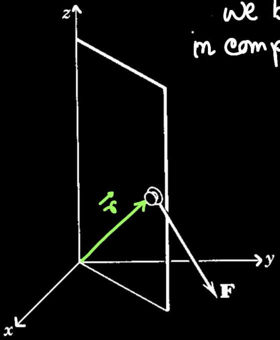
then, $\vec{A} \cdot (\vec{B} \times \vec{C}) = 0$

* Application of Vector Triple product :-

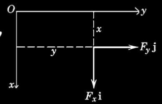
we shall now show that torque of $\vec{F}$ about the line L through o is $\boxed{\vec{n}\cdot(\vec{r}\times\vec{F})}$

where n is unit vector along L

To simplyfy the calculation chose the positive z axis the direction of $\hat{n}$, then $\hat{n}=\hat{K}$



we break force $\vec{F}$ in components,



The torque about z axis Produce by F_x & F_y is $xF_y - yF_x$ by the elementry defination of torque

we want to show that this is same as $\vec{n}\cdot(\vec{r}\times\vec{F})$ or here $\hat{K}\cdot(\vec{r}\times\vec{F})$

$$K\cdot(\vec{r}\times\vec{F}) = \begin{vmatrix} 0 & 0 & 1 \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} = xF_y - yF_x$$

Example of Scalar Triple Product

Example 1. If $\mathbf{F} = \mathbf{i} + 3\mathbf{j} - \mathbf{k}$ acts at the point $(1,1,1)$, find the torque of $\mathbf{F}$ about the line $\mathbf{r} = 3\mathbf{i} + 2\mathbf{k} + (2\mathbf{i} - 2\mathbf{j} + \mathbf{k})t$.

Solution

We first find the vector torque about a point on the line, say the point $(3,0,2)$. By (3.10) and Figure 3.5, this is $\mathbf{r} \times \mathbf{F}$ where $\mathbf{r}$ is the vector *from* the point about which we want the torque, *to* the point at which $\mathbf{F}$ acts, that is, from $(3,0,2)$ to $(1,1,1)$; then $\mathbf{r} = (1,1,1) - (3,0,2) = (-2,1,-1)$. The vector torque is

$$\mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & -1 \\ 1 & 3 & -1 \end{vmatrix} = 2\mathbf{i} - 3\mathbf{j} - 7\mathbf{k}.$$

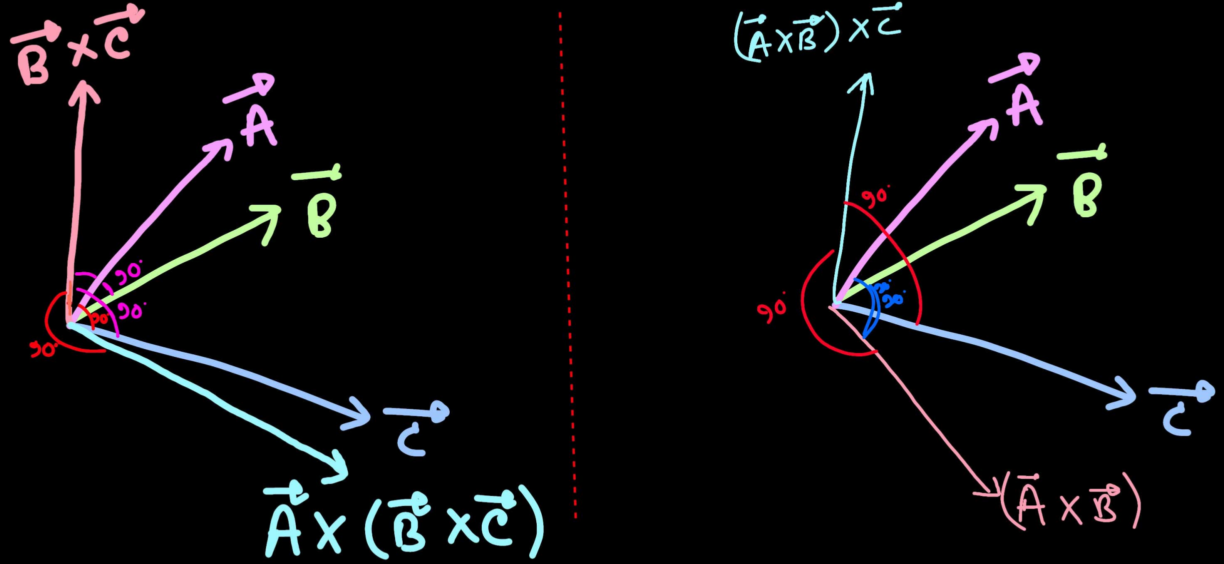
The torque about the line is $\mathbf{n} \cdot (\mathbf{r} \times \mathbf{F})$ where $\mathbf{n}$ is a unit vector along the line, namely $\mathbf{n} = \frac{1}{3}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$. Then the torque about the line is

$$\mathbf{n} \cdot (\mathbf{r} \times \mathbf{F}) = \frac{1}{3}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \cdot (2\mathbf{i} - 3\mathbf{j} - 7\mathbf{k}) = 1.$$

Vector Triple Product

written as $\vec{A} \times (\vec{B} \times \vec{C})$

① $\vec{A} \times (\vec{B} \times \vec{C})$ & $(\vec{A} \times \vec{B}) \times \vec{C}$ are different



② $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A}.\vec{C})\vec{B} - (\vec{A}.\vec{B})\vec{C}$

$(\vec{A} \times \vec{B}) \times \vec{C} = (\vec{A}.\vec{C})\vec{B} - (\vec{B}.\vec{C})\vec{A}$

③ If $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{m}$

then $\vec{m}$ is perpendicular to $\vec{A}$ and $(\vec{B} \times \vec{C})$

& $\vec{B}, \vec{C}, \vec{m}$ are in same plane

Applications of the Triple Vector Product

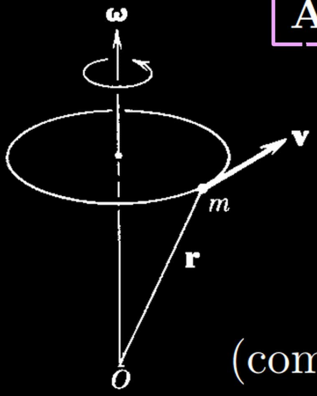
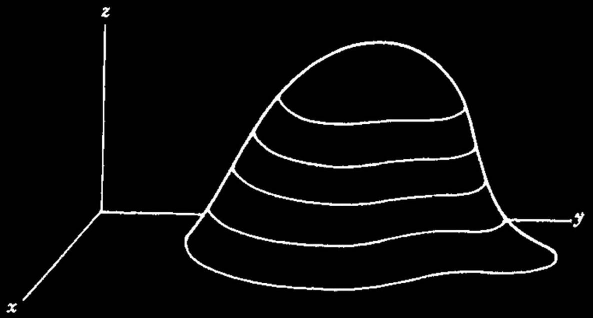


Figure 3.8

In Figure 3.8 (compare Figure 2.6), suppose the particle m is at rest on a rotating rigid body (for example, the earth). Then the *angular momentum* $\mathbf{L}$ of m about point O is defined by the equation $\mathbf{L} = \mathbf{r} \times (m\mathbf{v}) = m\mathbf{r} \times \mathbf{v}$. In the discussion of Figure 2.6, we showed that $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$. Thus, $\mathbf{L} = m\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})$.

Scalar Field and vector field

Many physical quantities have different values at different points in space. For example, the temperature in a room is different at different points: high near a register, low near an open window, and so on. The electric field around a point charge is large near the charge and decreases as we go away from the charge. Similarly, the gravitational force acting on a satellite depends on its distance from the earth. The velocity of flow of water in a stream is large in rapids and in narrow channels and small over flat areas and where the stream is wide. In all these examples there is a particular region of space which is of interest for the problem at hand; at every point of this region some physical quantity has a value. The term *field* is used to mean both the region and the value of the physical quantity in the region (for example, electric field, gravitational field). If the physical quantity is a scalar (for example, temperature), we speak of a *scalar field*. If the quantity is a vector (for example, electric field, force, or velocity), we speak of a *vector field*. Note again a point which we discussed in "endpoint problems" in Chapter 4, Section 10: Physical problems are often restricted to certain regions of space, and our mathematics must take account of this.



A simple example of a scalar field is the gravitational potential energy near the earth; its value is $V = mgz$ at every point of height z above some arbitrary reference level [which we take as the (x, y) plane]. Suppose that on a hill (Figure 5.1) we mark a series of curves each corresponding to some value of z (curves of constant elevation, often called *contour lines* or *level lines*). Any curve or surface on which a potential is constant is called an *equipotential*. Thus these level lines are equipotentials of the gravitational field since along any one curve the value of the gravitational potential energy mgz is constant. The horizontal planes which intersect the hill in these curves are equipotential surfaces (or level surfaces) of the gravitational field. (See Problems, Section 6 for more examples.)

As another example, let us ask for the equipotential surfaces in the field of an electric point charge q. The potential is $V = 9 \cdot 10^9 q/r$ (in SI units) at a point which is a distance r from the charge. The potential V is constant if r is constant; that is, the equipotentials of this electric field are spheres with centers at the charge. Similarly we could imagine drawing a set of surfaces (probably very irregular) in a room so that at every point of a single surface the temperature would be constant. These surfaces would be like equipotentials; they are called *isothermals* when the constant quantity is the temperature.

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