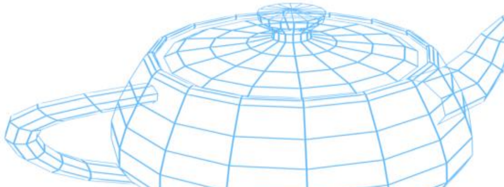
Scuola universitaria professionale della Svizzera italiana
Dipartimento tecnologie innovative

SUPSI

Computer Graphics

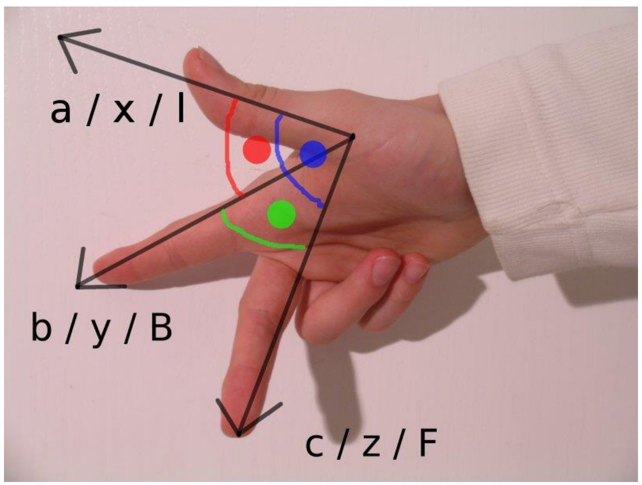
Mathematics for Computer Graphics (1)

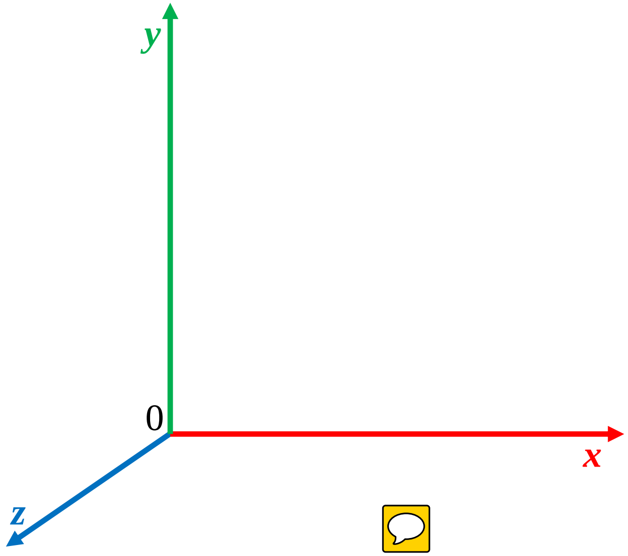
Achille Peternier, lecturer



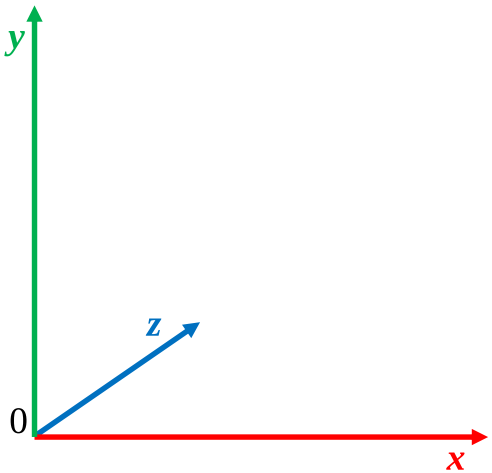
Coordinate systems

- Right-hand rule.

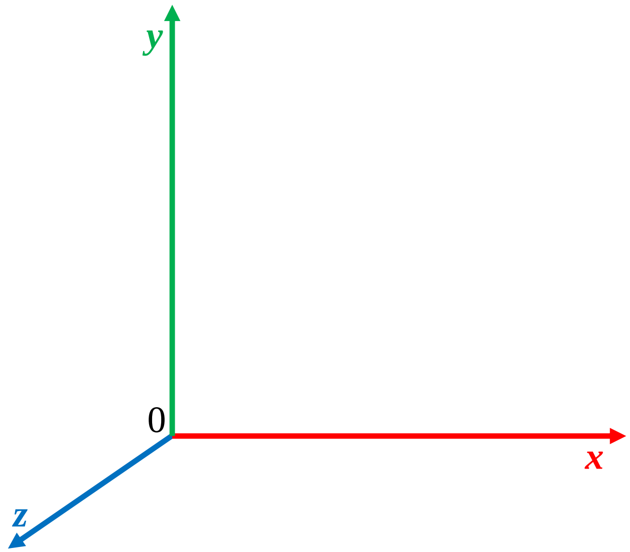


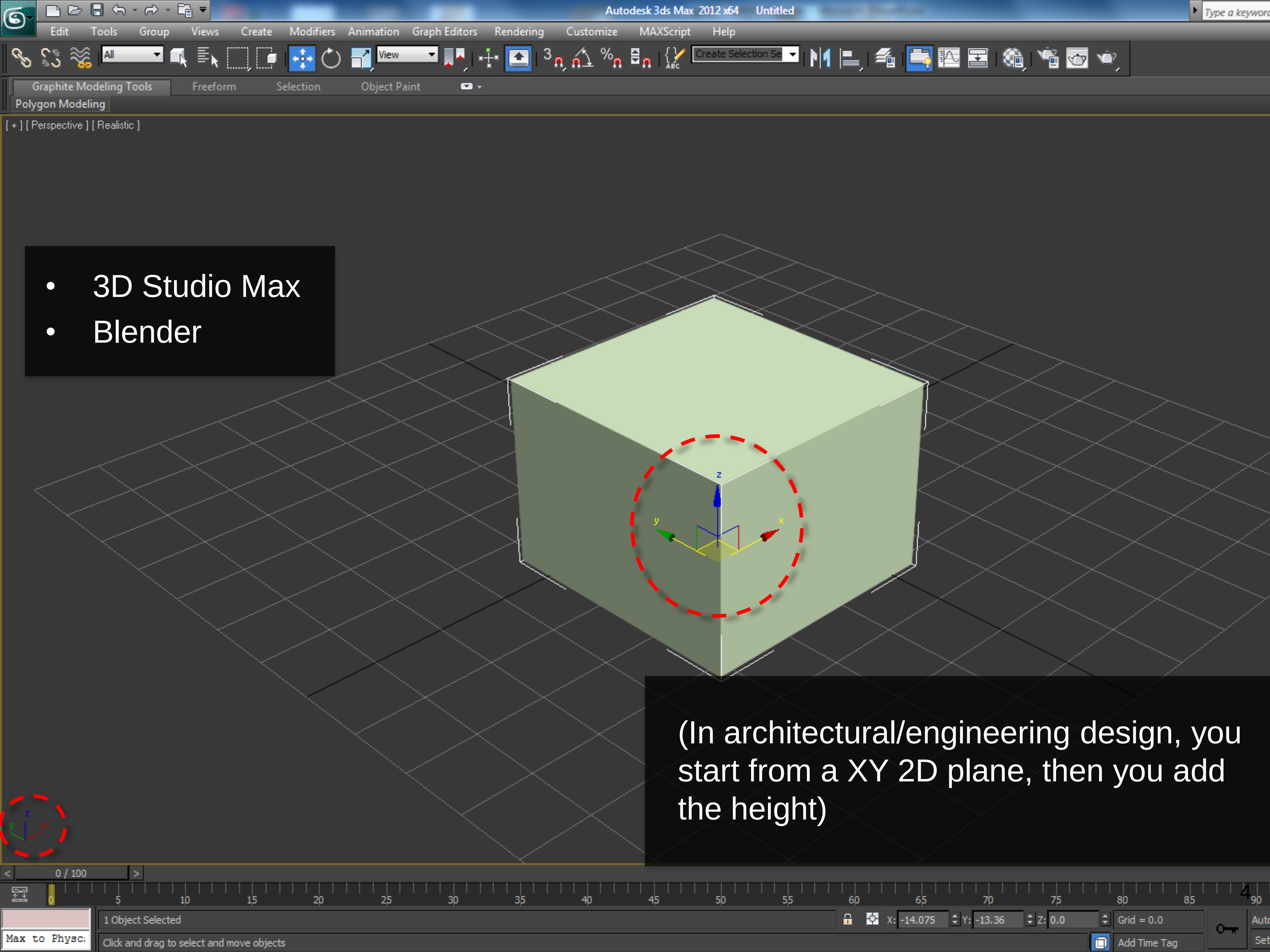


Coordinate systems



(left-hand rule)



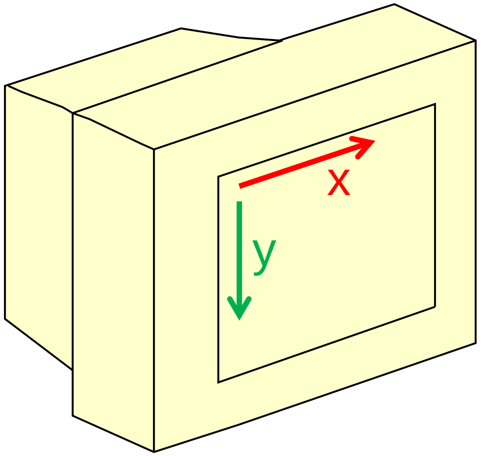


- 3D Studio Max
- Blender

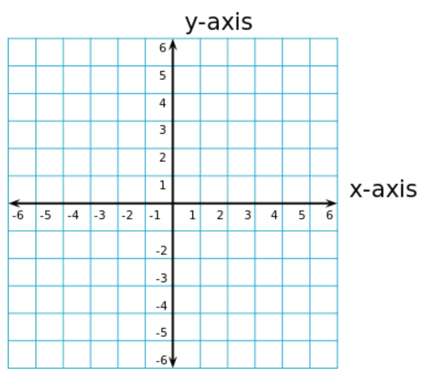
(In architectural/engineering design, you start from a XY 2D plane, then you add the height)

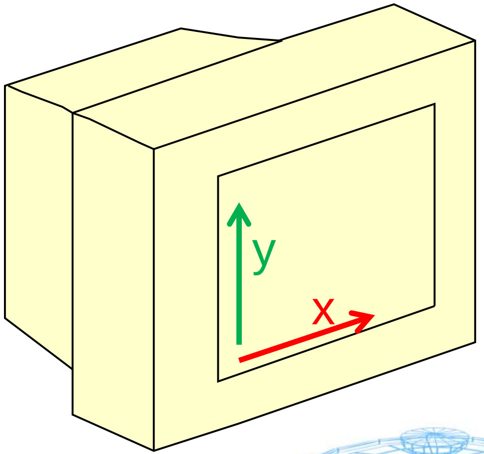
Coordinate systems

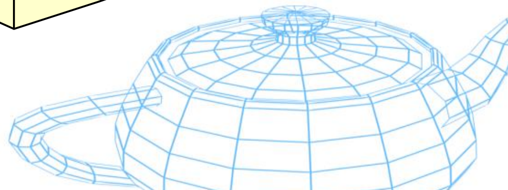
- Screen coordinates:
  - Origin located at the top-left corner:
    - Windows, X11.



  - Origin located at the bottom-left corner:
    - MacOS UI, OpenGL.

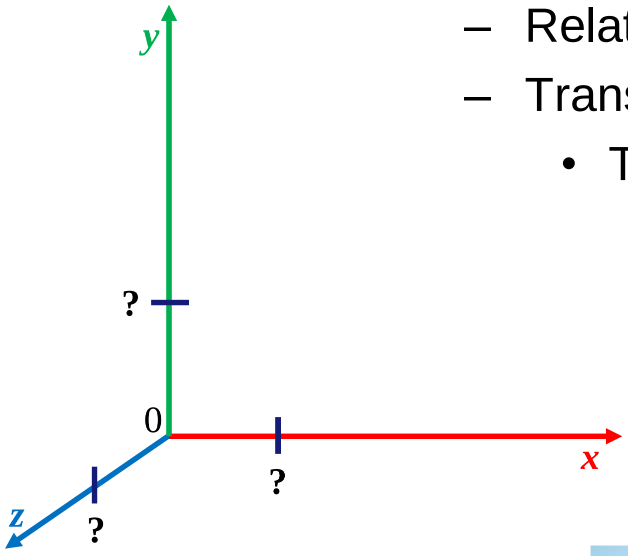






Units

- Generic units without explicit meaning:
  - Relative to each other.
  - Translated into pixels only at the end of the rendering pipeline:
    - To match different screen resolutions and aspect ratios.

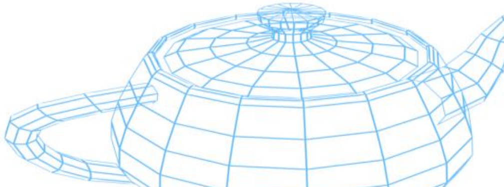


Point

- Defines a location.

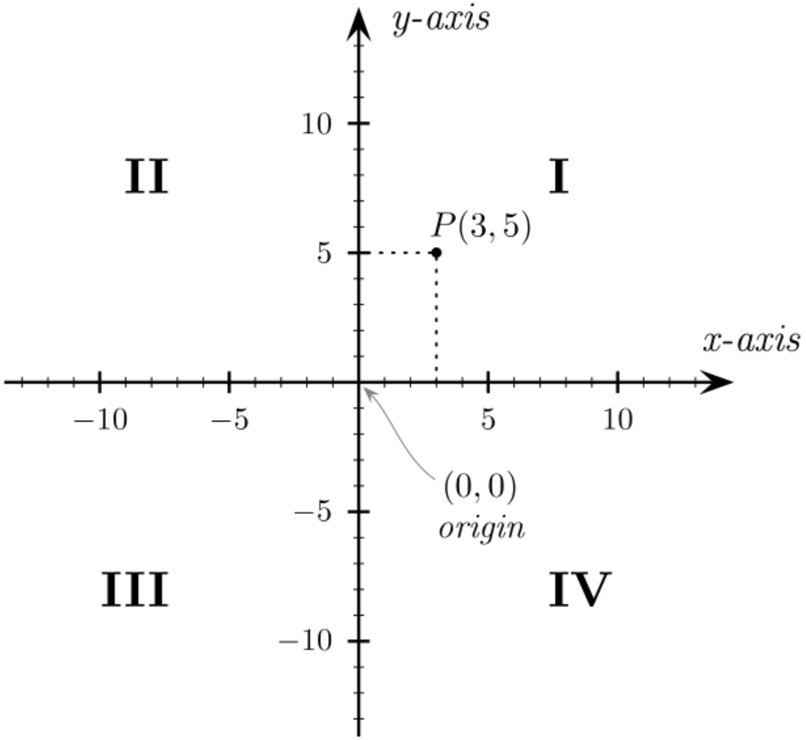
- Abstract entity:
  - no length.
  - no thickness.
  - no direction.

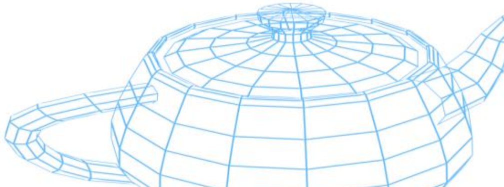
- A point usually specifies one of the vertices of a 3D primitive or the position of a light source.



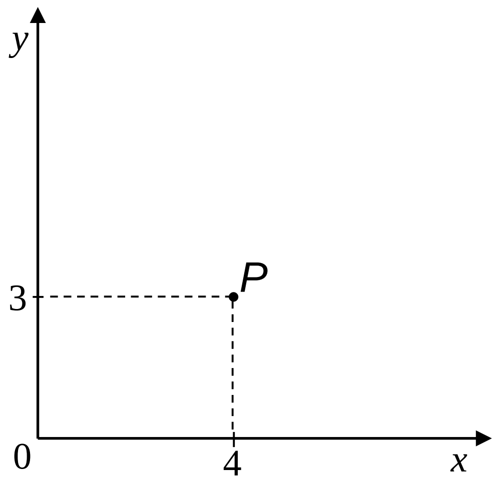
Point

- *P*(3, 5)





Point

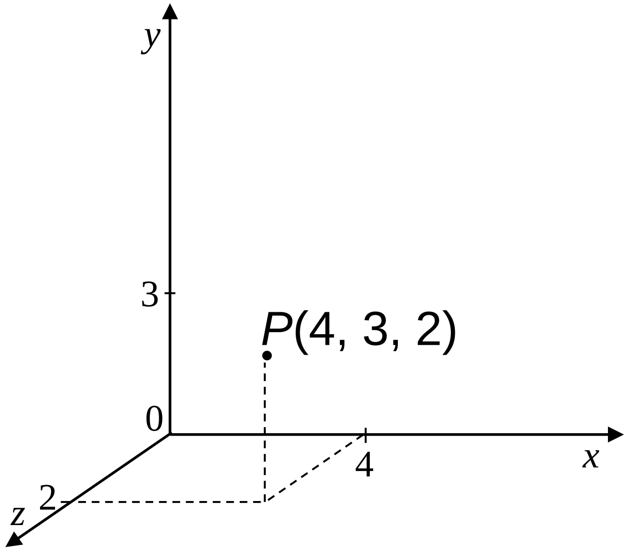


P(4, 3)

(4, 3)

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$[4 \quad 3]^{\mathrm{T}}$$



(4, 3, 2)

$$\begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

$$[4 \quad 3 \quad 2]^{\mathrm{T}}$$

Point

- Standard notation: (x, y, z)
- Row matrix notation: $[x \quad y \quad z]$ or $(x \quad y \quad z)$
- Column matrix notation: $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ or $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ OpenGL
- Transposed notation: $[x \quad y \quad z]^{\mathrm{T}}$

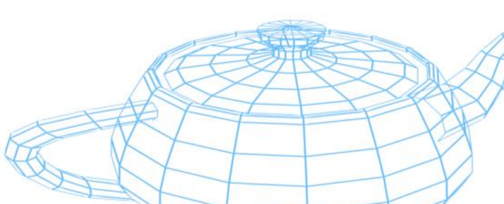
Point

$$(4, 3) \neq \begin{bmatrix} 4 \\ 3 \end{bmatrix} \qquad (4, 3, 2) \neq \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

$$(4, 3) \neq [4 \;\; 3]^T \qquad (4, 3, 2) \neq [4 \;\; 3 \;\; 2]^T$$

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix} = [4 \;\; 3]^T \qquad \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}^T = [4 \;\; 3 \;\; 2]$$

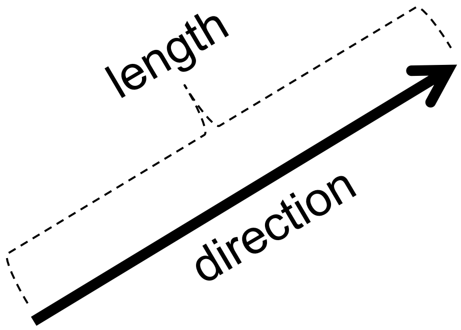
$$\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \qquad \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$$



Vector

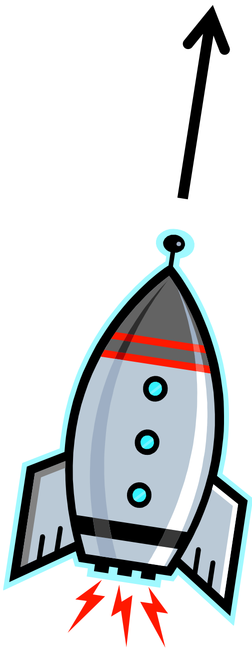
- Specifies a displacement:
  - Direction.
  - Length (magnitude).

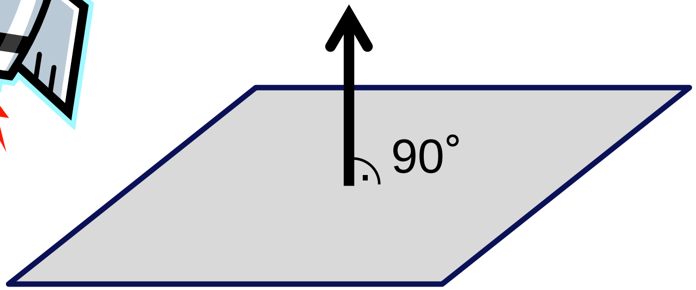
- <u>No</u> position.

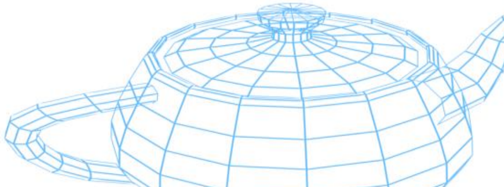


Vector

- Typical usage:
  - Light direction and intensity.
  - Object displacement.
  - Surface orientation (normal).
  - Physical properties (e.g., gravity, acceleration).
  - Wind direction.
  - …







Notation

- Conventions:
  - bold lowercase letter for column and row matrices, using the first letters of the alphabet for known elements:

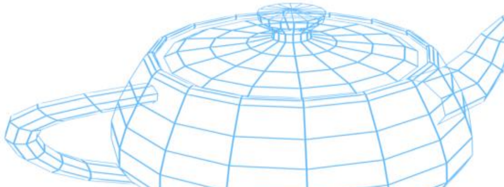
$$\mathbf{a} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} \qquad \mathbf{b} = [1 \;\; 2 \;\; 3] \qquad \mathbf{c} = [0 \;\; -1 \;\; 2]^T$$

  - …and letters from the end of the alphabet when elements are variables:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \mathbf{r} = [r_0 \;\; r_1 \;\; r_2] \qquad \mathbf{t} = [t_1 \;\; t_2 \;\; t_3]^T$$

  - Bar or arrow instead of bold:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \vec{b} = [1 \;\; 2 \;\; 3]$$



Zero matrix

- Defined by **0** (bold zero) or $\vec{0}$:

$$\mathbf{0} = [0 \;\; 0 \;\; \dots \;\; 0]$$

- …such as:

$$\mathbf{a} + \mathbf{0} = \mathbf{a} \qquad\qquad \mathbf{b} - \mathbf{0} = \mathbf{b}$$

Vector operations

- Addition:

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} = [a_1 + b_1 \quad a_2 + b_2 \quad \dots \quad a_n + b_n]$$

- Subtraction:

$$\mathbf{a} - \mathbf{b} = [a_1 - b_1 \quad a_2 - b_2 \quad \dots \quad a_n - b_n] \qquad \mathbf{a} - \mathbf{b} \neq \mathbf{b} - \mathbf{a} \qquad -\mathbf{a} = [-a_1 \quad -a_2 \quad \dots \quad -a_n]$$

Vector operations

- Vector length/magnitude:

$$|\mathbf{a}| = \sqrt{a_1{}^2 + a_2{}^2 + \cdots + a_n{}^2}$$

- Vector normalization:

$$\hat{\mathbf{a}} = \left[\frac{a_1}{|\mathbf{a}|} \quad \frac{a_2}{|\mathbf{a}|} \quad \dots \quad \frac{a_n}{|\mathbf{a}|}\right]$$

Vector operations

- Dot/scalar product:

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} = \mathrm{a}_1\mathrm{b}_1 + \mathrm{a}_2\mathrm{b}_2 + \ \ldots + \mathrm{a}_n\mathrm{b}_n$$

$$\mathbf{a} \cdot \mathbf{0} = 0 \qquad\qquad \mathbf{0} \cdot \mathbf{0} = 0$$

  – If **a** and **b** are normalized, then $\cos^{-1}(\mathbf{a} \cdot \mathbf{b})$ is the angle between the two vectors.

- Cross/vector product (for 3D vectors):

$$\mathbf{a} \times \mathbf{b} = [\mathrm{a}_2\mathrm{b}_3 - \mathrm{a}_3\mathrm{b}_2 \quad \mathrm{a}_3\mathrm{b}_1 - \mathrm{a}_1\mathrm{b}_3 \quad \mathrm{a}_1\mathrm{b}_2 - \mathrm{a}_2\mathrm{b}_1]$$

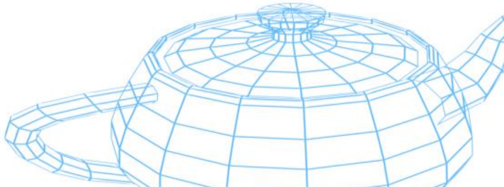
$$\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a} \qquad\qquad \mathbf{a} \times \mathbf{a} = \mathbf{0}$$

Matrix

$$\begin{bmatrix} 1 & 3.14 & 0 \\ -3.14 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Points and vectors:
  → column/row matrices.

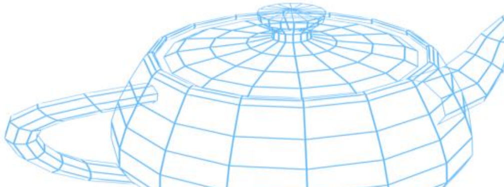
- Operations on points and vectors:
  → rectangular matrices.



Matrix

columns

$$\text{rows}\quad \begin{bmatrix} 1 & 3.14 & 0 \\ -3.14 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad\qquad \begin{bmatrix} 1 & 3.14 & 0 \\ -3.14 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Matrix

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

- Referred to as *nrOfRows* x *nrOfColumns* matrices:
  - e.g., 3x3, 4x4, 2x2, 2x4, 3x2, 4x3, …
- Named using a bold uppercase letter (**A**, **M**, **R**, …):
  - For clarity also $\mathbf{A}_{3x3}$, $\mathbf{M}_{4x4}$, $\mathbf{R}_{4x4}$

Matrix

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

- A matrix with the same number of rows and columns is called **square matrix** (e.g., 2x2, 3x3, 4x4, …).
- The main diagonal is defined by the elements e_{ij} with i=j

Matrix operations

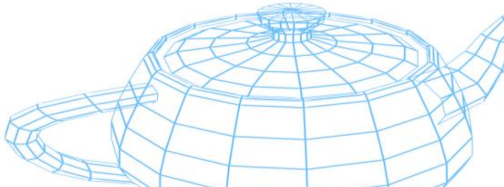
- Addition:

$$\mathbf{A}_{mxn} + \mathbf{B}_{mxn} = \mathbf{B}_{mxn} + \mathbf{A}_{mxn}$$

  – defined only if **A** and **B** have the same number of rows and columns:

e.g.:

$$\begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0.5 & 1 \\ -0.5 & 1.5 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 1.5 & 1 \\ 3.5 & 4.5 \\ 0.5 & 1.5 \end{bmatrix}$$



Matrix operations

- Matrix by vector (column matrix) multiplication (post-multiplication):

$$\mathbf{A}_{mxn}\mathbf{b}_{nx1} = \mathbf{r}_{mx1}$$

  – defined only if $\mathbf{A}_{mxn}$ and $\mathbf{b}_{nx1}$, i.e., if the number of columns in matrix **A** is equal to the number of rows in vector **b**

Matrix operations

- Vector (row matrix) by matrix multiplication (pre-multiplication):

$$\mathbf{b}_{1xm}\mathbf{A}_{mxn} = \mathbf{r}_{1xn}$$

Matrix operations

- Row matrix by matrix:
  - vector to the left:

$$\mathbf{b}_{1xm}\mathbf{A}_{mxn} = \mathbf{r}_{1xn}$$

- Column matrix by matrix:
  - vector to the right:

$$\mathbf{A}_{nxm}\mathbf{b}_{mx1} = \mathbf{r}_{nx1}$$

Matrix operations

$$\mathbf{A}_{3x3}\mathbf{b}_{3x1} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_{11}b_1 + a_{12}b_2 + a_{13}b_3 \\ a_{21}b_1 + a_{22}b_2 + a_{23}b_3 \\ a_{31}b_1 + a_{32}b_2 + a_{33}b_3 \end{bmatrix}$$

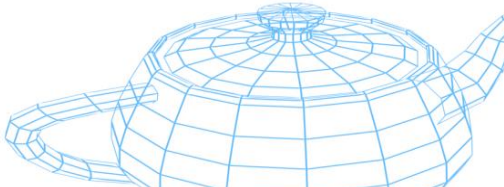
$$\mathbf{b}_{1x3}\mathbf{A}_{3x3} = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} =$$

$$\begin{bmatrix} b_1a_{11} + b_2a_{21} + b_3a_{31} & b_1a_{12} + b_2a_{22} + b_3a_{32} & b_1a_{13} + b_2a_{23} + b_3a_{33} \end{bmatrix}$$

Matrix operations

- Matrix by matrix multiplication:
  - same rules as before.
  - row and column vectors are a special case of matrix.
  - warning (in general):

$$\mathbf{AB} \neq \mathbf{BA}$$



Matrix operations

$$\mathbf{A}_{3x3}\mathbf{B}_{3x3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{bmatrix}$$

Matrix operations

- Matrix transpose:
  - turns columns into rows and rows into columns.
  - noted as $\mathbf{A}^{\mathrm{T}}$

$$\begin{bmatrix} 1 & 2 & 0.5 \\ -1 & 3 & -2 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 0.5 & -2 \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ e & f & g \\ h & i & j \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} a & e & h \\ b & f & i \\ c & g & j \end{bmatrix}$$

$(\mathbf{A}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{A}$

Identity matrix

- All zeros but ones on the main diagonal:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Properties:

$$\mathbf{AI} = \mathbf{IA} = \mathbf{A} \qquad \mathbf{I}^{\mathrm{T}} = \mathbf{I}$$

Matrix operations

- Matrix inverse:
  - the **inverse** of a matrix **A** is noted $\mathbf{A}^{-1}$ and is such that:

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

  - not all the matrices have an inverse matrix, e.g.:

$$\mathbf{0}\mathbf{0}^{-1} \neq \mathbf{I}$$

  - a matrix without inverse matrix is called **singular**.

Matrix operations

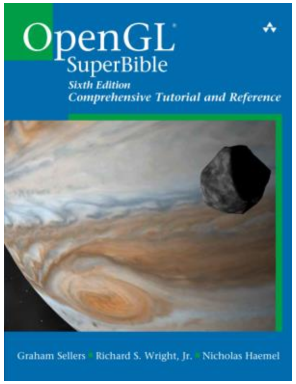
- Matrix inverse:
  - complex operation on large matrices.
  - first compute the determinant:
    - if the determinant is equal to 0, the matrix is singular.

- E.g., Cayley-Hamilton method:

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\left[\frac{1}{6}\left((\mathrm{tr}\mathbf{A})^3 - 3\mathrm{tr}\mathbf{A}\,\mathrm{tr}\mathbf{A}^2 + 2\mathrm{tr}\mathbf{A}^3\right)\mathbf{I} - \frac{1}{2}\mathbf{A}\left((\mathrm{tr}\mathbf{A})^2 - \mathrm{tr}\mathbf{A}^2\right) + \mathbf{A}^2\mathrm{tr}\mathbf{A} - \mathbf{A}^3\right]$$

```
Matrix inverse() // The OpenGL-way for transposed matrices
{
   Matrix t;
   const float fDetInverse = 1.0f / ((_11 * (_22 * _33 - _23 * _32)) -
                                  (_12 * (_21 * _33 - _23 * _31)) +
                                  (_13 * (_21 * _32 - _22 * _31)));
   t._11 =  fDetInverse * (_22 * _33 - _23 * _32);
   t._12 = -fDetInverse * (_12 * _33 - _13 * _32);
   t._13 =  fDetInverse * (_12 * _23 - _13 * _22);
   t._14 =  0.0f;
   t._21 =  -fDetInverse * (_21 * _33 - _23 * _31);
   t._22 =   fDetInverse * (_11 * _33 - _13 * _31);
   t._23 =  -fDetInverse * (_11 * _23 - _13 * _21);
   t._24 =  0.0f;
   t._31 =  fDetInverse * (_21 * _32 - _22 * _31);
   t._32 = -fDetInverse * (_11 * _32 - _12 * _31);
   t._33 =  fDetInverse * (_11 * _22 - _12 * _21);
   t._34 =  0.0f;
   t._41 = -(_41 * t._11 + _42 * t._21 + _43 * t._31);
   t._42 = -(_41 * t._12 + _42 * t._22 + _43 * t._32);
   t._43 = -(_41 * t._13 + _42 * t._23 + _43 * t._33);
   t._44 =  1.0f;
   return t;
}
```

Bibliography



Sellers, Wright, Haemel,
OpenGL SuperBible 6th edition,
Addison-Wesley

Chapter 4: Math for 3D Graphics

Tutorials

Central Connecticut University, tutorial on vector math for 3D Computer Graphics (including examples and exercises):

http://chortle.ccsu.edu/VectorLessons/index.html