Advanced Artificial Intelligence

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VISION INNOVATION EXCELLENCE

Uncertainty

- So far, we have learnt about deterministic scenarios
  - I have a specific clause that needs to satisfy
  - No notion of randomness
- In logic, we have to define a variable for each such possibilities
  - In many problems, there are a huge number of such possibilities
  - Example: a person has cough
    - What is the possible cause?
    - There may be a huge number of causes for this
    - It's almost impossible to enumerate all such possibilities
- We often have to make decision based on the possibilities

Sample Space

- The set of possible outcomes
- Rolling a dice
  - Sample space $S = \{1, 2, 3, 4, 5, 6\}$
  - Each element of sample space: sample point

- Sample space can be
  - Finite (above example)
  - Infinite (Amount of rainfall at Jodhpur in July)

Event

- Any subset of the sample space
- Example: In the context of rolling a dice
  - The outcome is a number < 3
  - Event $A = \{1, 2\}$

Random Variables

- In logic, we have symbols

- In probability, we have random variables (rv)
  - A numerical description of the outcomes of a random experiment
  - A function that assigns numerical values (real or Boolean) to each sample point
  - Usually indicated using capital letters (e.g., X)
    - Discrete: takes only a countable number of discrete values
      - Sample space for weather condition: $\{sunny, rainy, cloudy\}$
    - Continuous: takes uncountably infinite number of possible values
      - Sample space for temperature at Jodhpur: $[6.7^o, 46.2^o]$

Probability of an Event

- Consider an event A
- Probability of event A
  - $P(A) = \frac{Number\ of\ elements\ in\ set\ A}{Number\ of\ elements\ in\ the\ sample\ space\ S}$
  - $P(A) = \frac{Number\ of\ favourable\ outcomes}{Total\ number\ of\ outcomes}$
- Example: In the context of rolling a dice
  - Event A: The outcome is an odd number
  - Event $A = \{\ldots\}$

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What is the problem with this definition?

Probability of an Event

- In an experiment with **finite sample space and equally likely outcomes**, Probability of event A
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What if this condition does not hold?

Frequentist Approach of Probability

- Suppose we do an experiment n number of times
- Out of these, event A occurs $n(A)$ number of times

- Relative frequency of A is $f_r(A) = \frac{n(A)}{n}$
- Probability of A is $P(A) =$

Frequentist Approach of Probability

- Suppose we do an experiment n number of times
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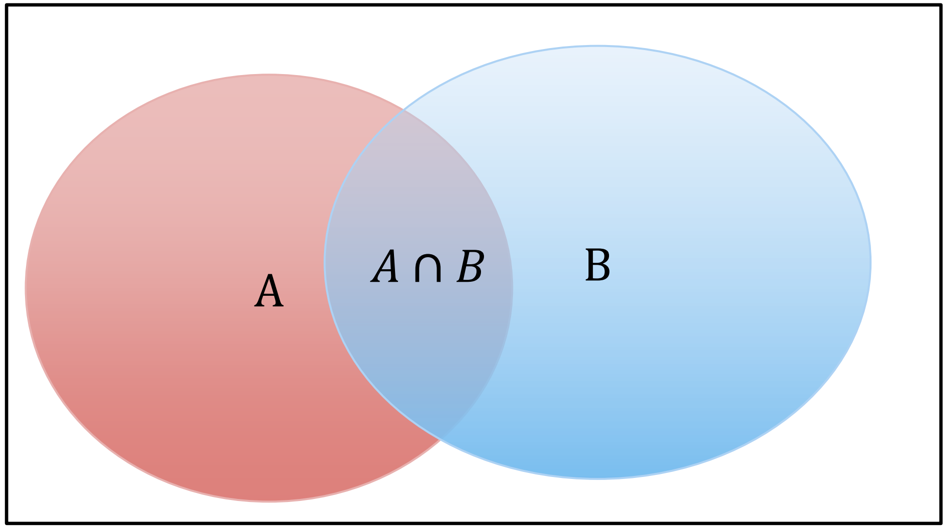
- Relative frequency of A is $f_r(A) = \frac{n(A)}{n}$
- Probability of A is $P(A) = \lim_{n \to \infty} f_r(A) = \lim_{n \to \infty} \frac{n(A)}{n}$

Probability

- A probability measure or probability function $P(\cdot)$ assigns a probability to an event
  - $P(A)$ is the chance that event A occurs

Conditional Probability

$$\boldsymbol{P(A|B) = \frac{P(A \cap B)}{P(B)}}$$

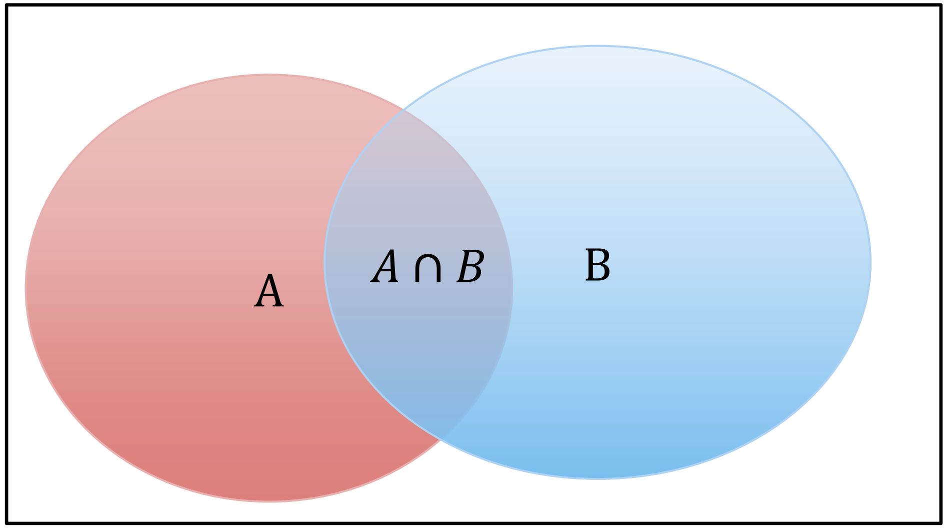


Conditional Probability

Prior Probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)\,P(A)}{P(B)}$$

Indicates my belief about the occurrence of A

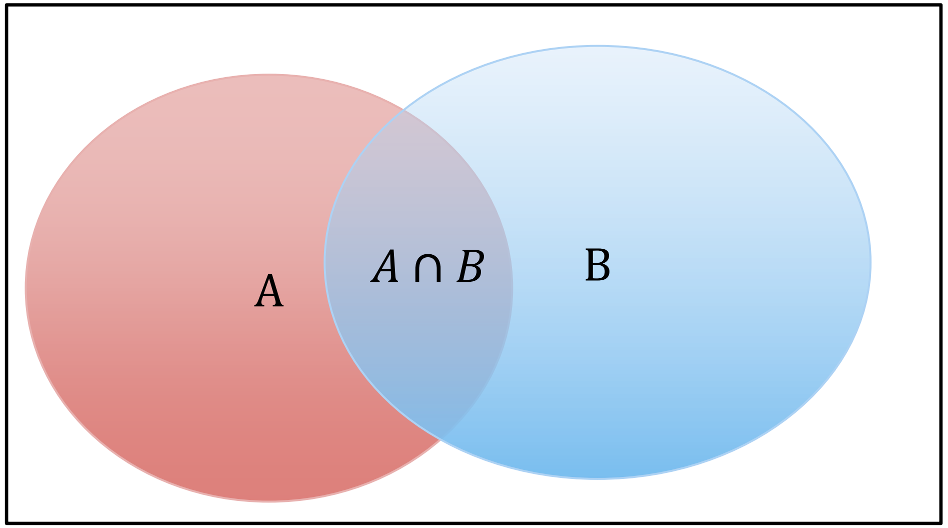


Conditional Probability

$$\boldsymbol{P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)\,P(A)}{P(B)}}$$

Likelihood

Indicates the chance of B to occur given that A has occurred

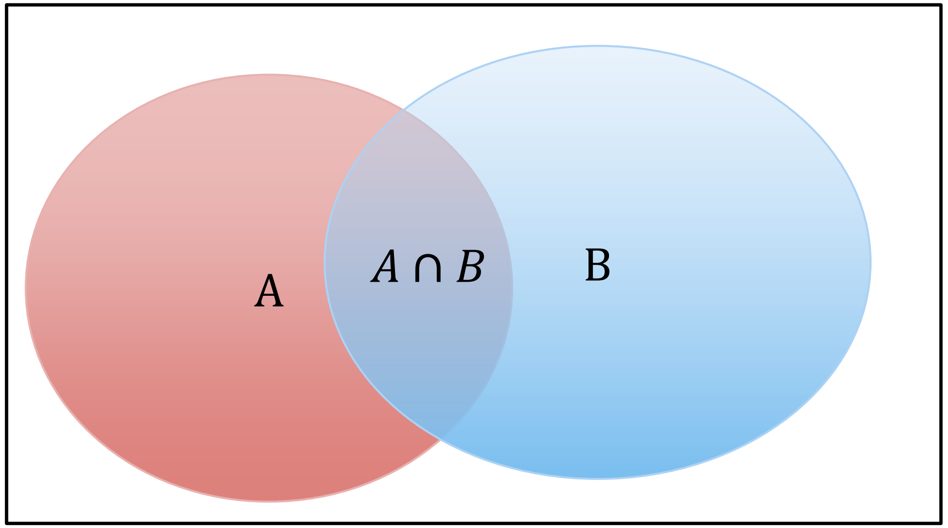


Conditional Probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)\,P(A)}{P(B)}$$

Evidence

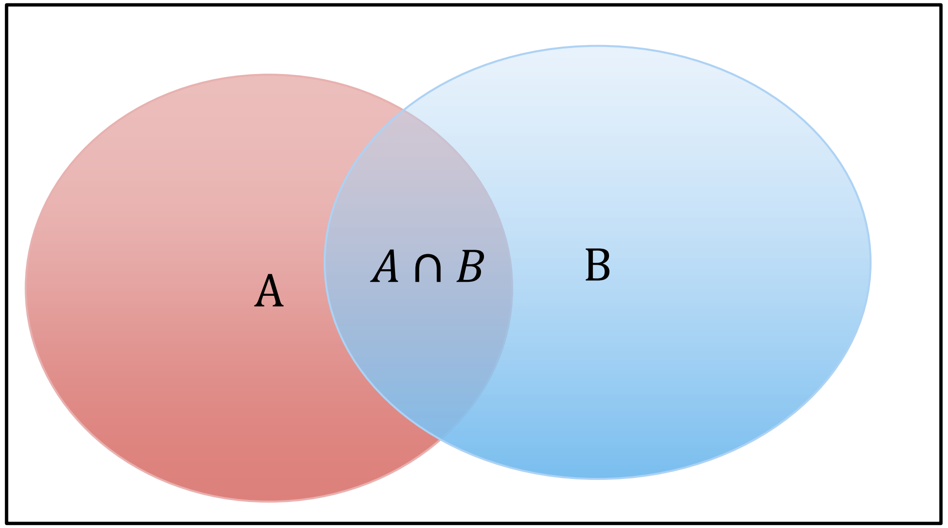
Probability that B occurs



Conditional Probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)\,P(A)}{P(B)}$$

Posterior/
Conditional
probability

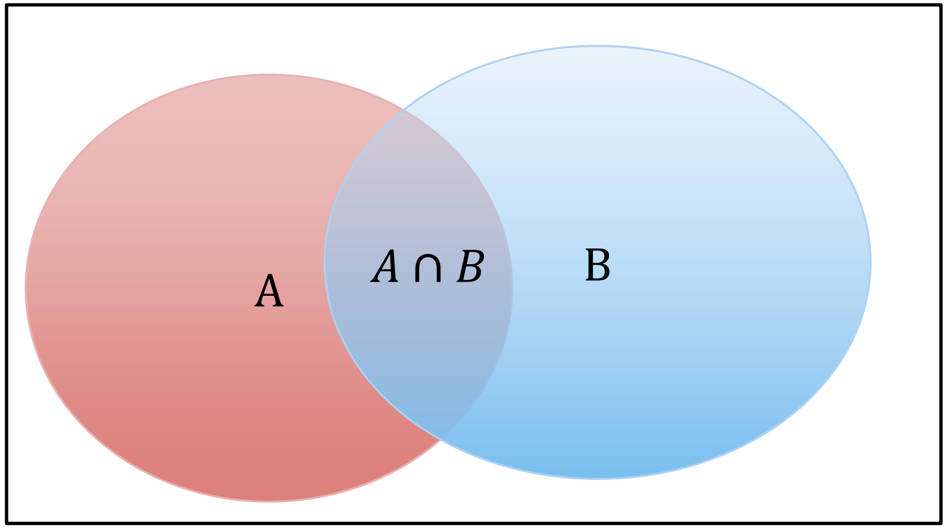


Conditioned on the evidence that we have seen

Conditional Probability

$$\boldsymbol{P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)\,P(A)}{P(B)}}$$

Bayes' Theorem



Conditional Probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)\,P(A)}{P(B)}$$

Bayes' Theorem

$$P(Cause|Effect) = \frac{P(Effect|Cause)\,P(Cause)}{P(Effect)}$$

Conditional Probability

- Consider a system with causes c1, and c2, and effects e1, e2 and e3

- Suppose we want to find the probabilities of different causes given effect e2

$$\boldsymbol{P(Cause|Effect) = \frac{P(Effect|Cause)\,P(Cause)}{P(Effect)}}$$

Independent Events

- Two events A and B are said to be independent if
  - $\boldsymbol{P(A|B) = P(A)}$ (1)

- We already have
  - $\boldsymbol{P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)\,P(A)}{P(B)}}$ (2)

- From (1) and (2), we get
  - $\boldsymbol{P(B|A) = P(B)}$
  - $\boldsymbol{P(A \cap B) = P(A)P(B)}$

Joint Probability Distribution

- Consider two random variables
- X corresponding to weather $\{sunny, rainy, cloudy\}$
  - $P(X) = \{0.6, 0.1, 0.3\}$
- Y corresponding to power cut $\{power\ cut, no\ power\ cut\}$
  - $P(Y) = \{0.15, 0.85\}$
- A joint probability distribution of $X\ and\ Y$
  - Probability distribution on all possible pairs of outputs

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- A joint probability distribution of $X\ and\ Y$
  - Probability distribution on all possible pairs of outputs
- A 3×2 matrix of values

Chain Rule

- If $A_1, A_2, \dots, A_n$ are n events, then
  - $P(A_n \cap A_{n-1} \cap \cdots \cap A_1) = P(A_n | A_{n-1} \cap \cdots \cap A_1)\, P(A_{n-1} \cap \cdots \cap A_1)$ (1)
- Similarly,
  - $P(A_{n-1} \cap A_{n-2} \cap \cdots \cap A_1) = P(A_{n-1} | A_{n-2} \cap \cdots \cap A_1)\, P(A_{n-2} \cap \cdots \cap A_1)$ (2)
- Extending this for the subsequent events and putting in (1), we get,
  - $P(A_n \cap A_{n-1} \cap \cdots \cap A_1)$

    $= P\,(A_n | A_{n-1} \cap \cdots \cap A_1)\, P(A_{n-1} | A_{n-2} \cap \cdots \cap A_1) P(A_{n-2} | A_{n-3} \cap \cdots \cap A_1) \dots P(A_1)$

Chain Rule

- If A_1, A_2, A_3 are 3 events, then we use
  - $P(A_n \cap A_{n-1} \cap \cdots \cap A_1)$

    $= P\ (A_n | A_{n-1} \cap \cdots \cap A_1)\ P(A_{n-1} | A_{n-2} \cap \cdots \cap A_1) P(A_{n-2} | A_{n-3} \cap \cdots \cap A_1) \ldots P(A_1)$

- We get
  - $P(A_4 \cap A_3 \cap A_2 \cap A_1)$

    $= P\ (A_4 | A_3 \cap A_2 \cap A_1)\ P(A_3 | A_2 \cap A_1) P(A_2 | A_1) P(A_1)$

Markov Decision Process

- Many decision making problems happen with uncertainty but for a long time
- Decision theory: episodic (only one decision)
- In MDP, we focus on a long sequence of actions
- Stochastic transitions (Actions have stochastic outcome)
- Many probabilistic problems can be converted into MDP

Markov Decision Process

- A set of states $\mathcal{S}$
- A set of actions $\mathcal{A}$
- A transition function $\mathcal{T}(s, a, s')$: An agent in state s reaches state s' with probability $\mathcal{T}(s, a, s')$ if the agent performs action a
- A cost model $\mathcal{C}(s, a, s')$: If the agent reaches state s' from s by taking action a, $\mathcal{C}(s, a, s')$ is the cost that the agent pays

Markov Process

- Next state depends on the current state and not on the past (first order Markov process)

- $\mathcal{C}(s, a, s'), \mathcal{T}(s, a, s')$: first order Markov process
    - How we arrived at s does not matter to determine cost or the next transition

Markov Decision Process

- A set of goals $\mathcal{G}$ may be given: absorbing / non-absorbing goals
- A start state may be given s_0
- Discount factor γ may be given
- A reward model $\mathcal{R}(s, a, s')$ may be given

What is the Objective of Markov Decision Process?

- In search problems (deterministic), I find a path from a start state to goal state
- In MDP, I am looking for a table that tells me which action to take when the agent is at a particular state
  - Policy
- After the action is taken, the agent knows exactly where (next) it reaches
  - i.e. we assume full observability

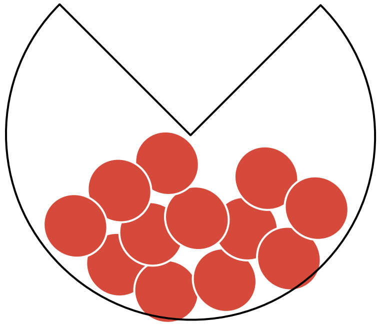
Markov Decision Process

- In search problems, we optimize cost to reach a goals
  - Since it is deterministic

- In MDP, we are going to optimize expected cost / expected rewards to reach the goal
  - Since my transitions are stochastic

Markov Model with an Example

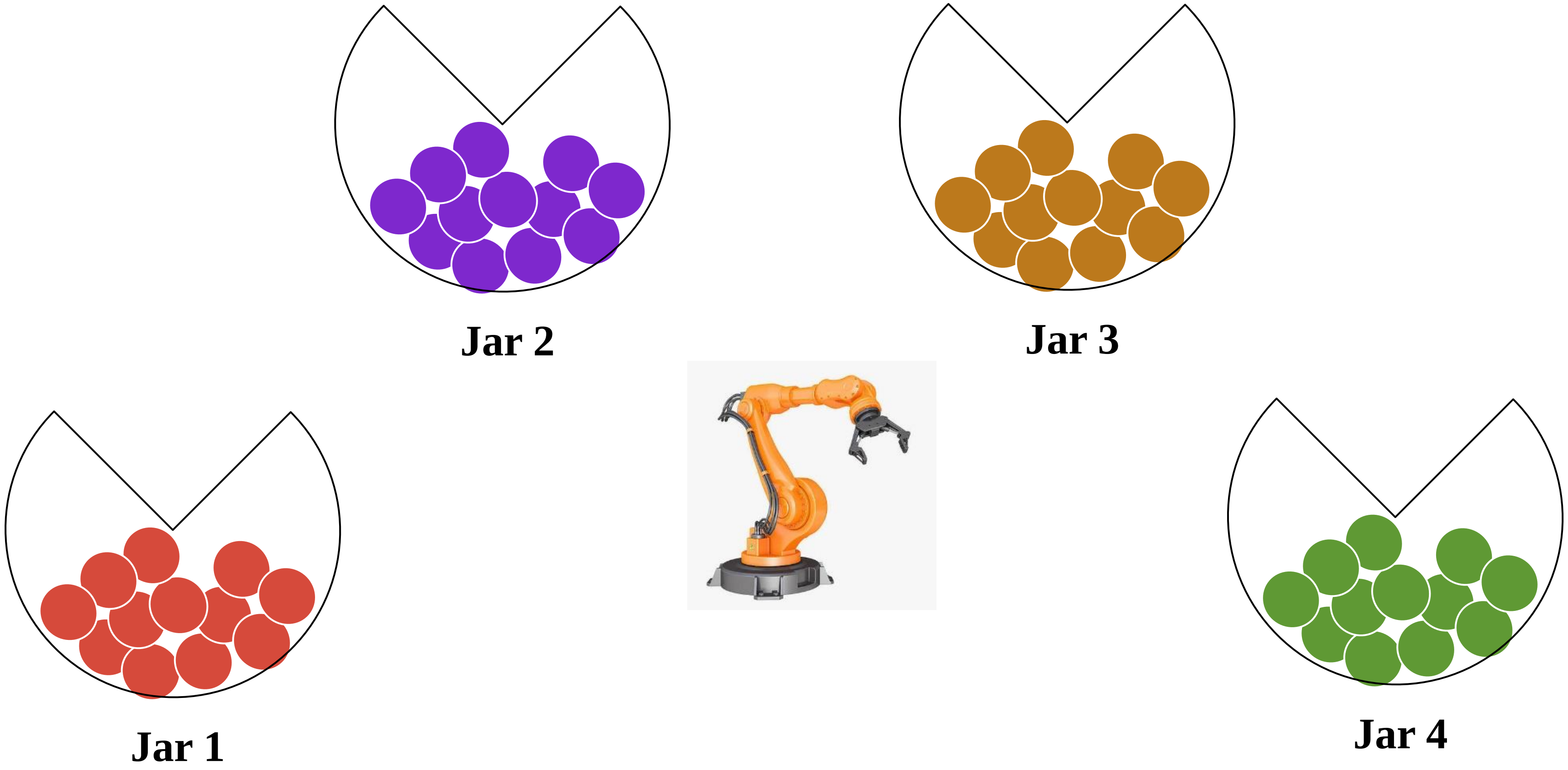
- A robot arm pick up balls from jars



Jar 1

https://www.google.com/imgres?imgurl=https%3A%2F%2F5.imimg.com%2Fdata5%2FEE%2FEM%2FMY-4370182%2Frobot-arm-500x500.jpg&imgrefurl=https%3A%2F%2Fwww.indiamart.com%2Fproddetail%2Findustrial-robot-arm-18072085973.html&tbnid=kEmmq4Dozcd6zM&vet=12ahUKEwjauvvcy4f6AhU2gGMGHZd7A5wQMygCegUIARDvAQ..i&docid=aCDlxZsII6s0RM&w=500&h=500&q=robotic%20arm&ved=2ahUKEwjauvvcy4f6AhU2gGMGHZd7A5wQMygCegUIARDvAQ

Markov Model with an Example



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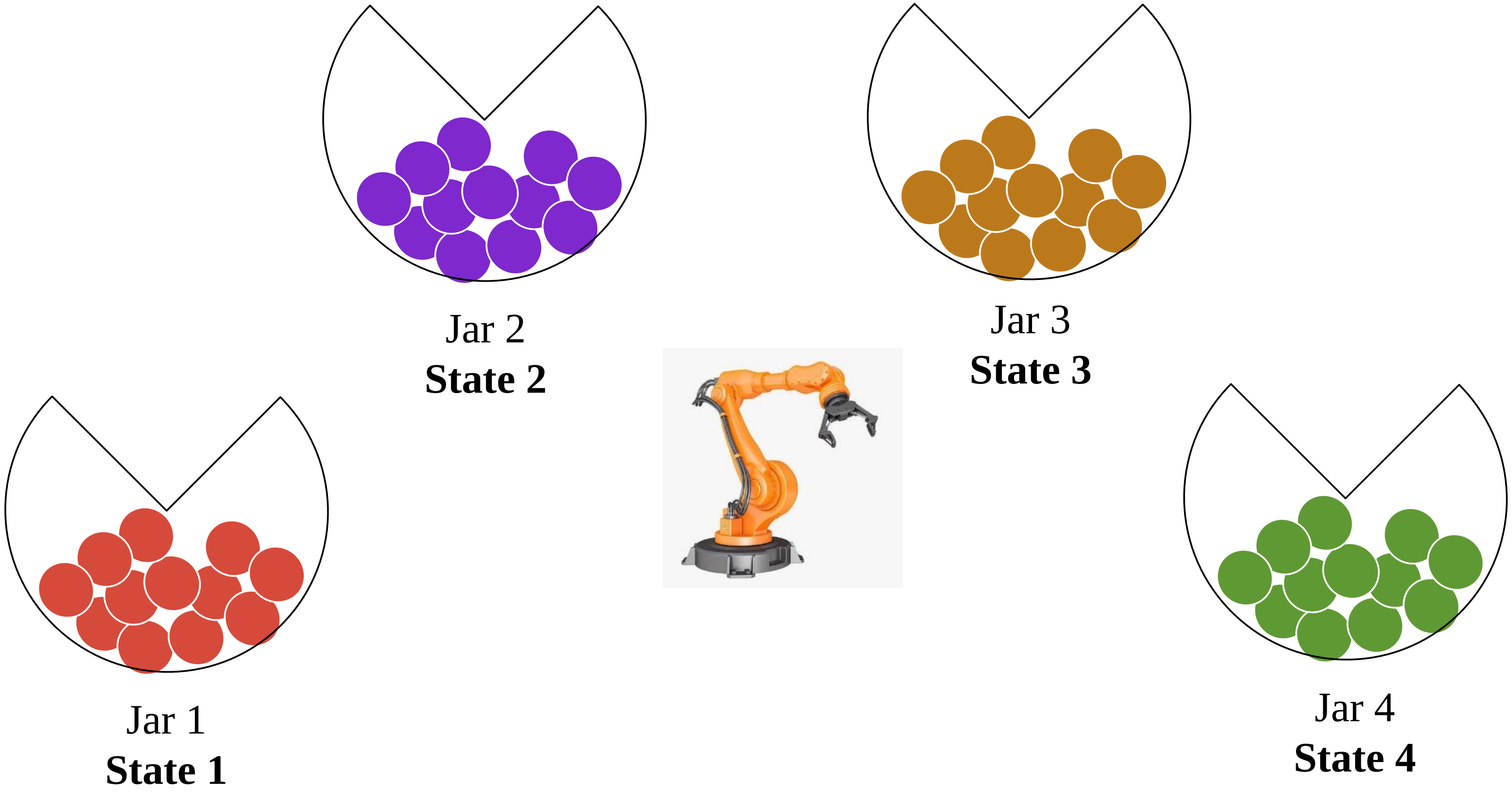

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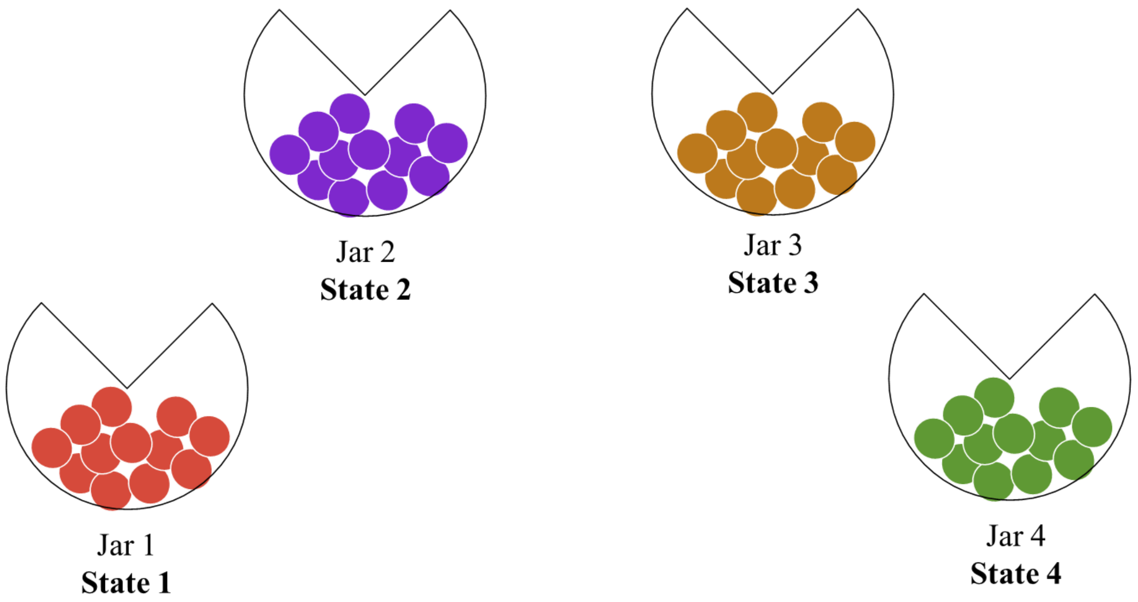
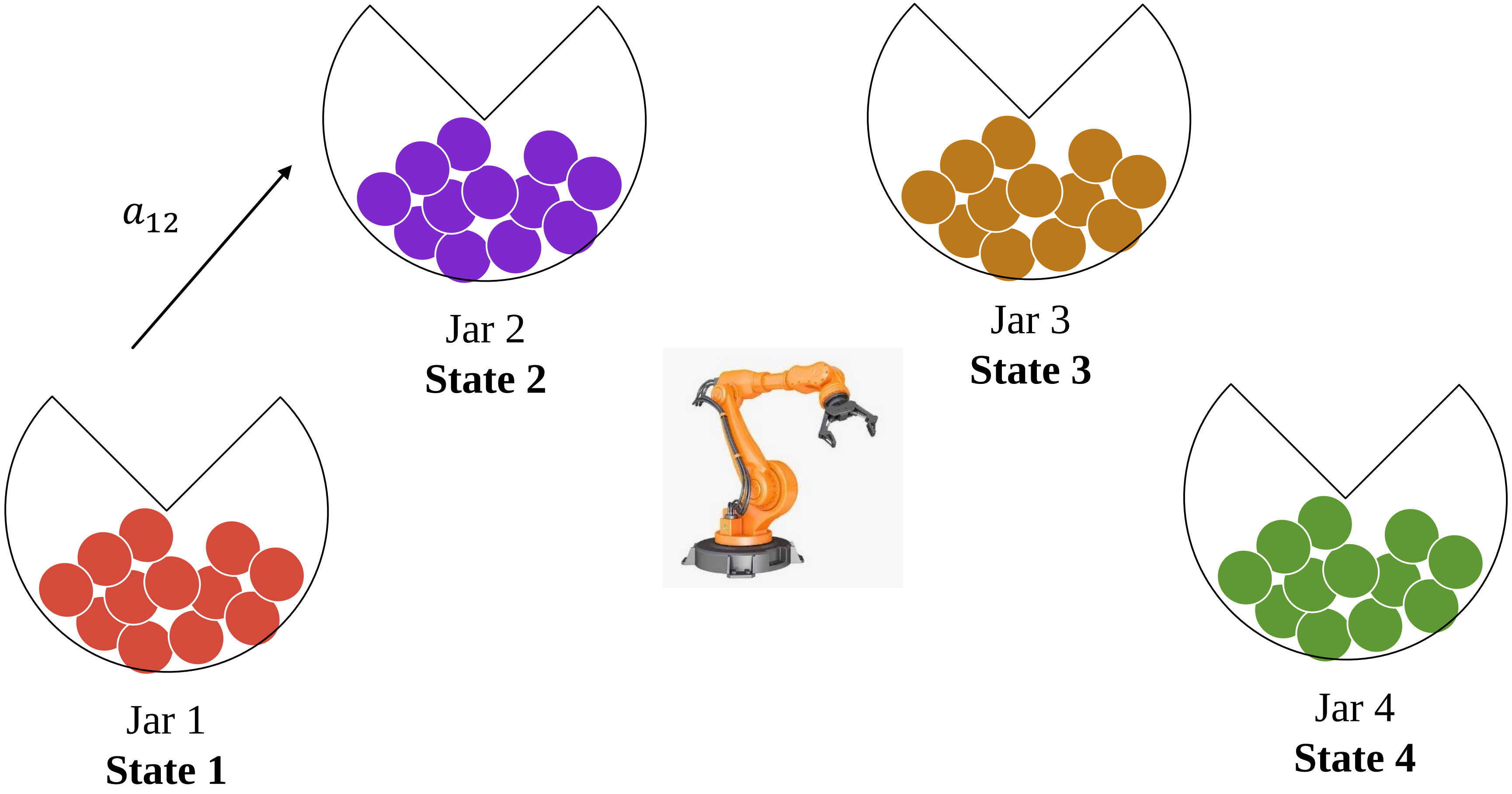


- We have a set of states (jars) and an observation (colour of the ball picked up) at each state
- The next state depends only on the present state

Markov Model with an Example



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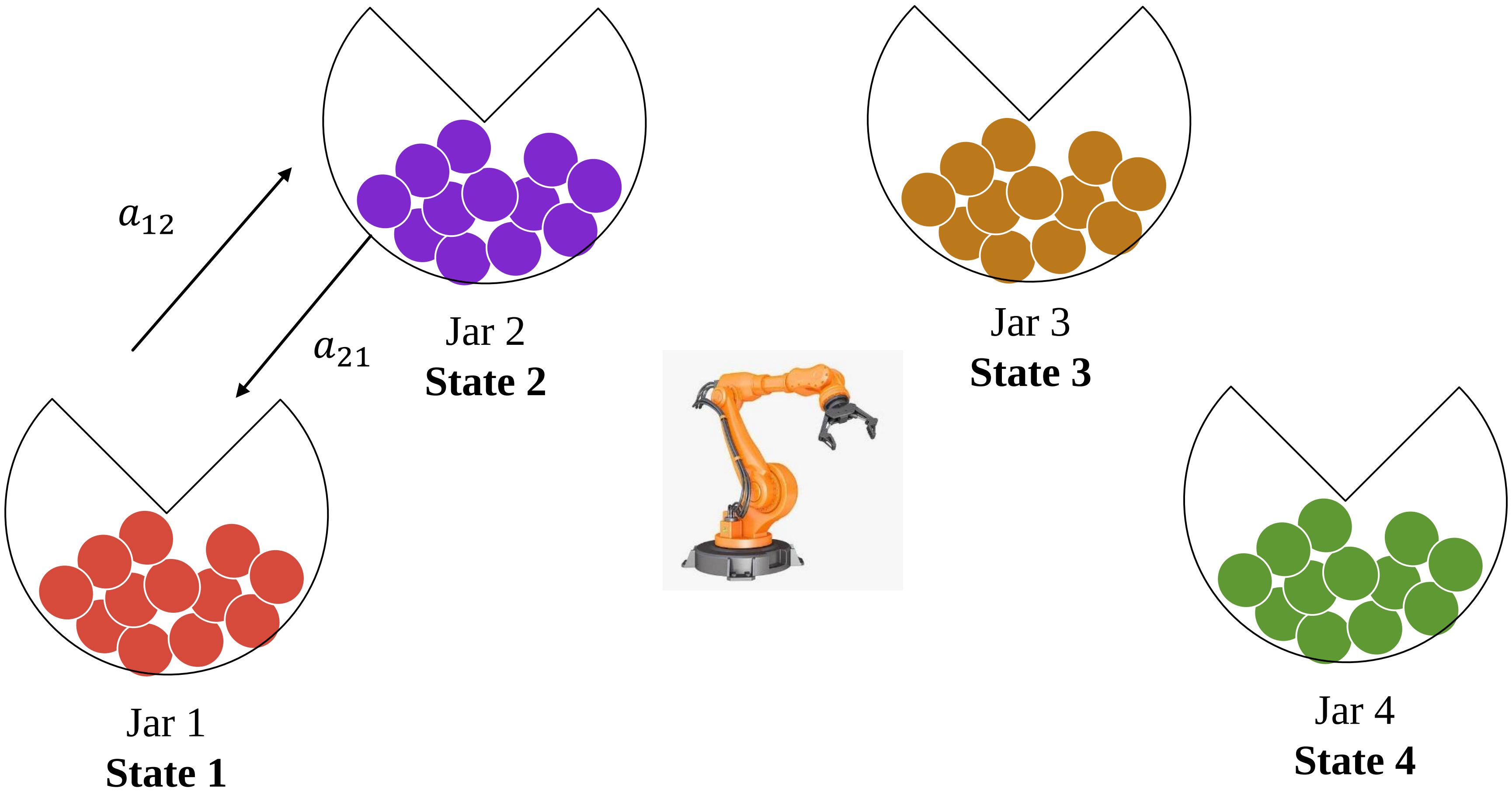

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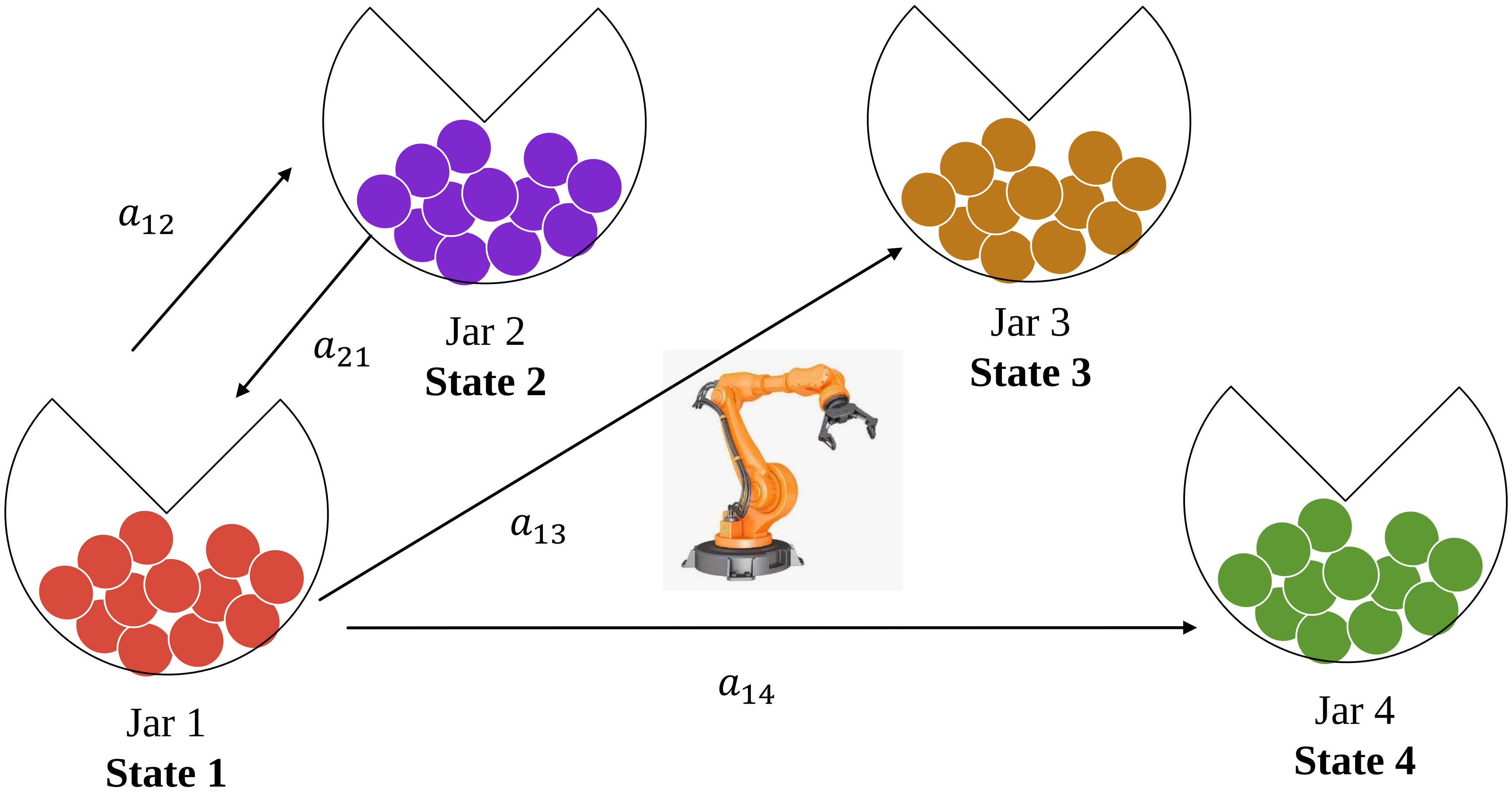
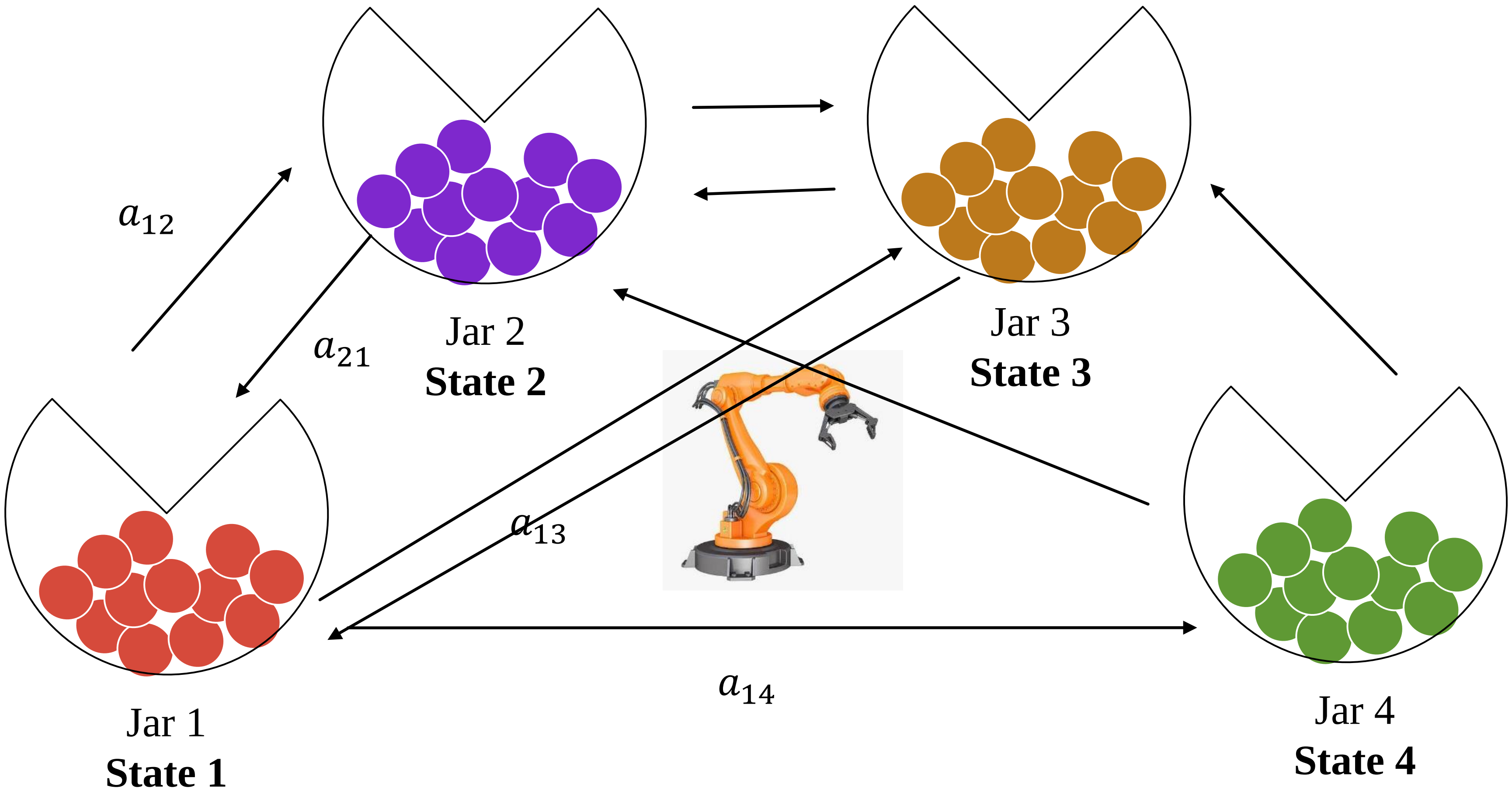
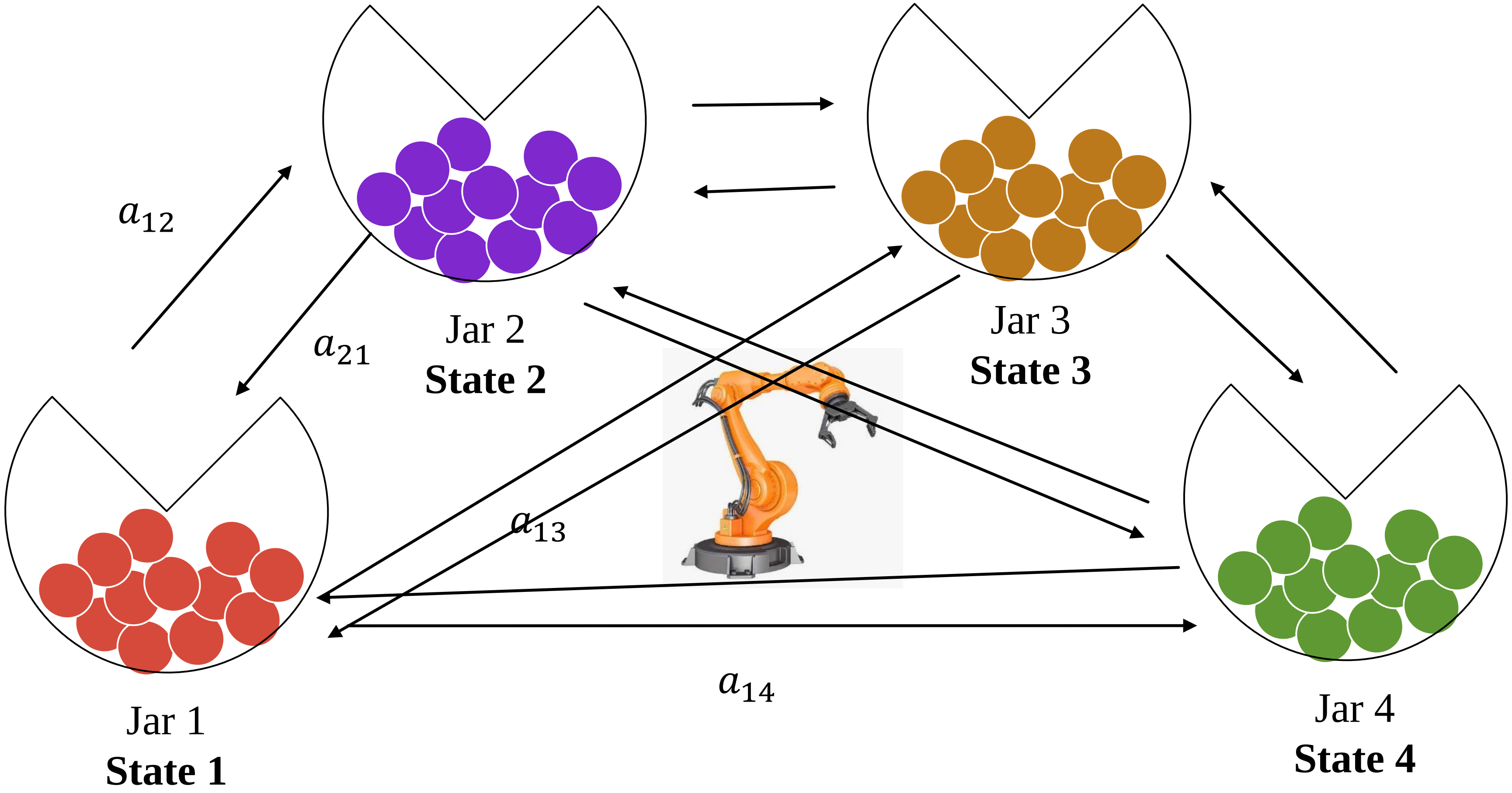


Markov Model with an Example

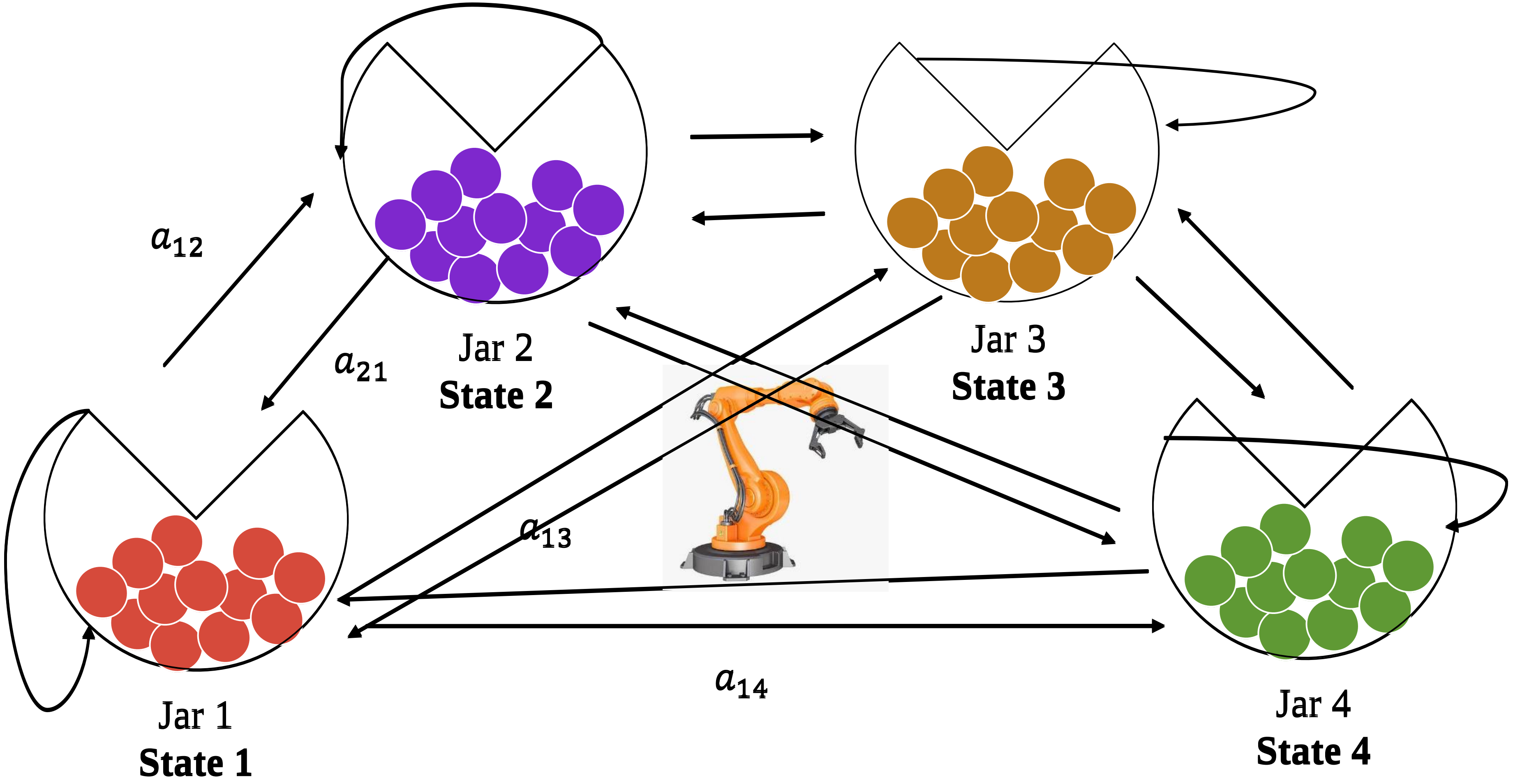


And so on ...

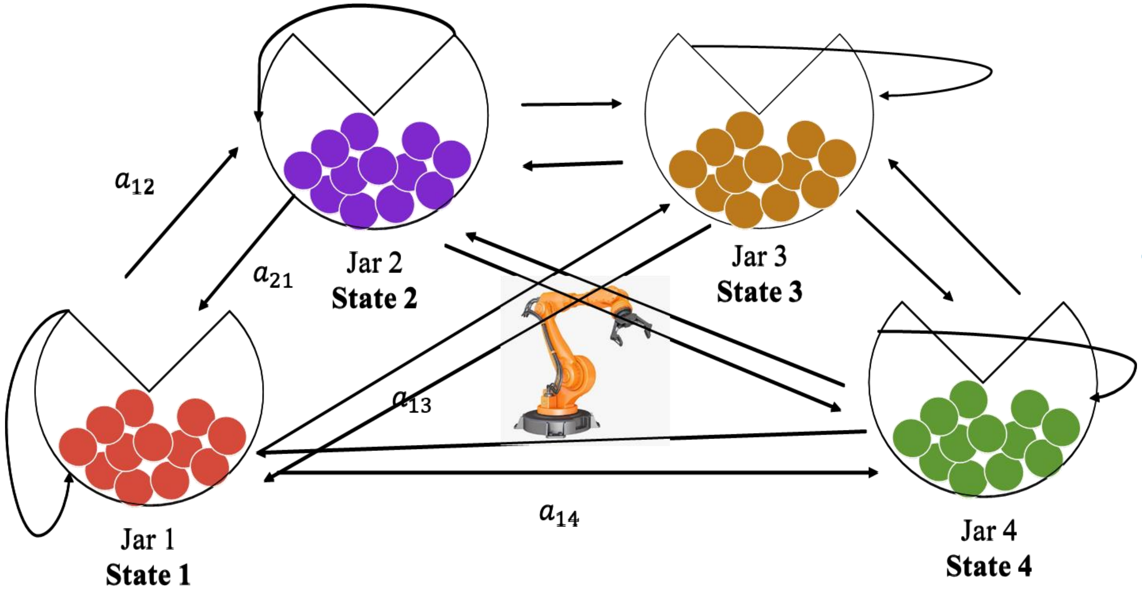
Markov Model with an Example



Markov Model with an Example

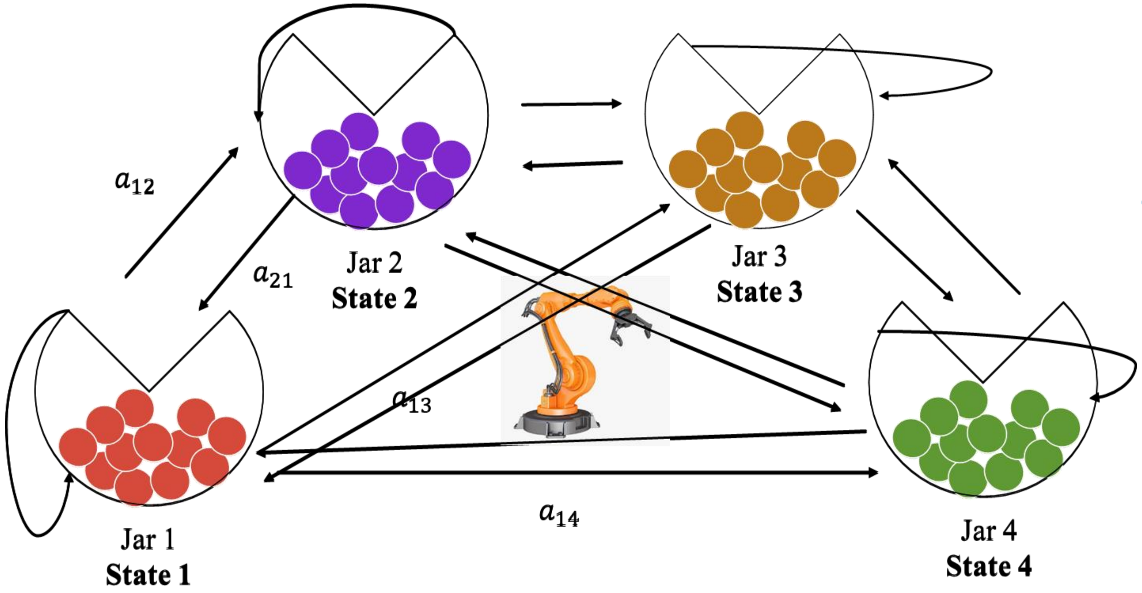


Markov Model with an Example



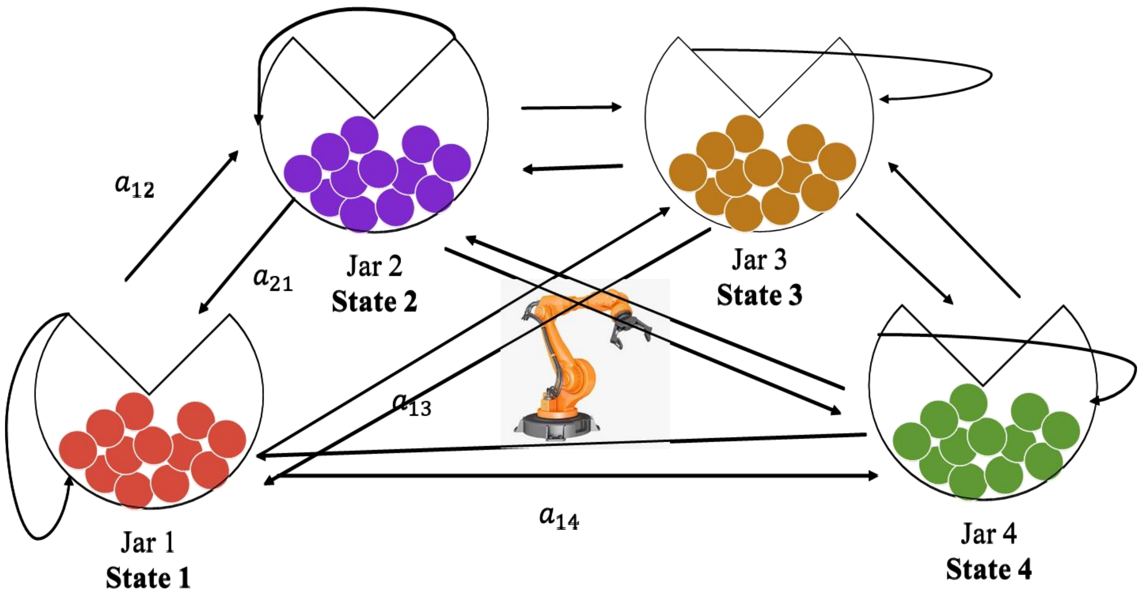
- The arm moves from one state (jar) to another state (jar) with certain probability
- However, when it reaches to a particular state our observation (the colour of the ball) is fixed
- For example, whenever the arm reach state 4, the observation is **green**

Markov Model with an Example



- Suppose, we have an observation sequence

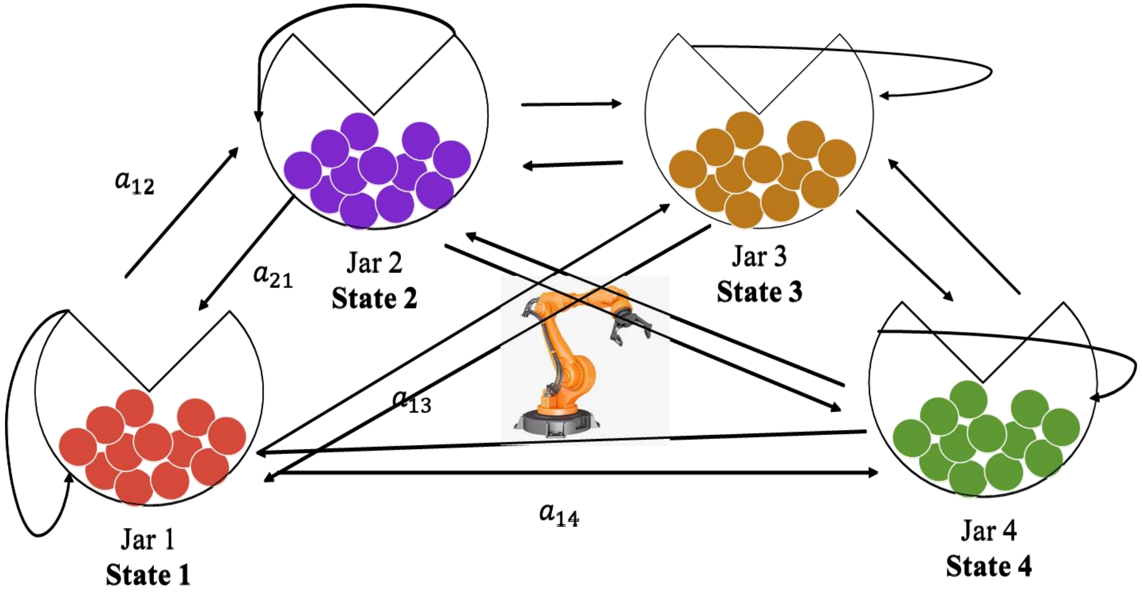
Markov Model with an Example



- Suppose, we have an observation sequence

- **What can we conclude from this?**

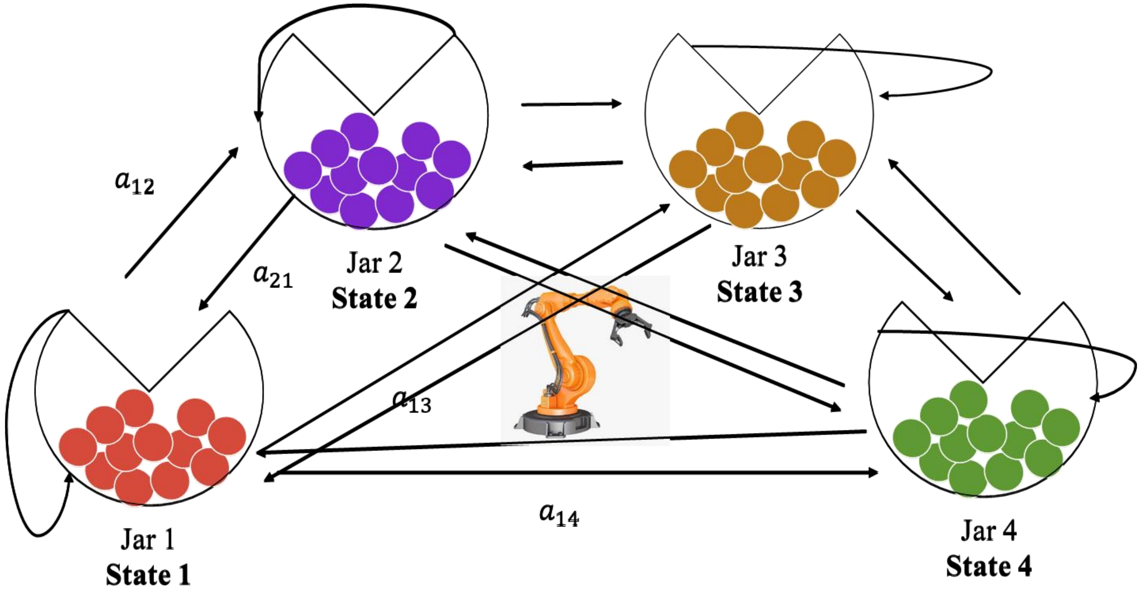
Markov Model with an Example



- Suppose, we have an observation sequence

- **What can we conclude from this?**
- **The arm started in state 2, then moved to state 3, state 3, and finally state 1**

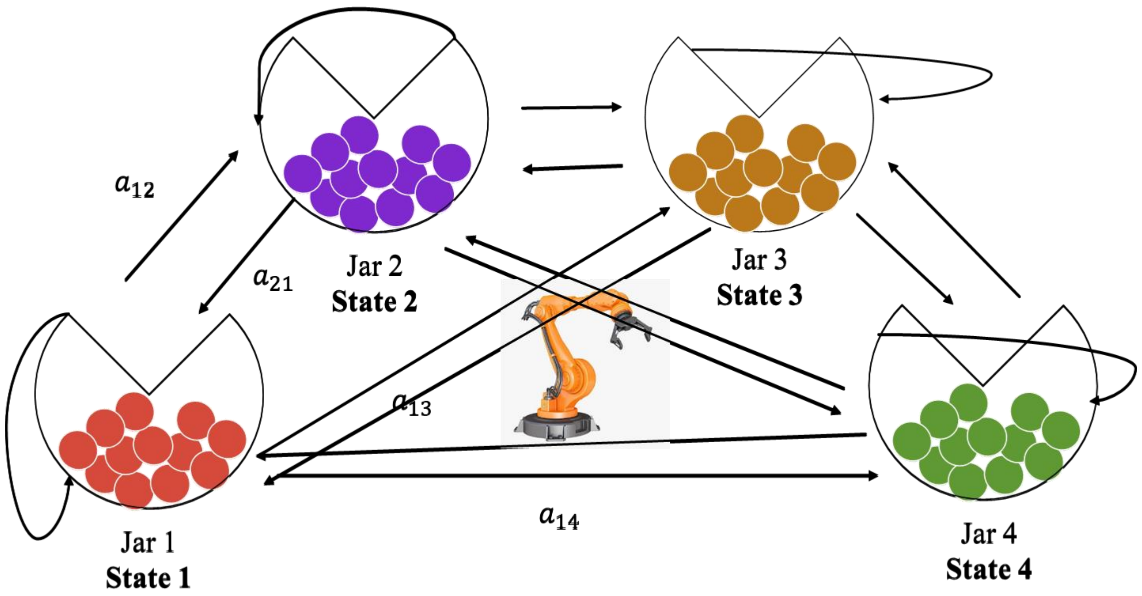
Markov Model with an Example



- Suppose, we have an observation sequence

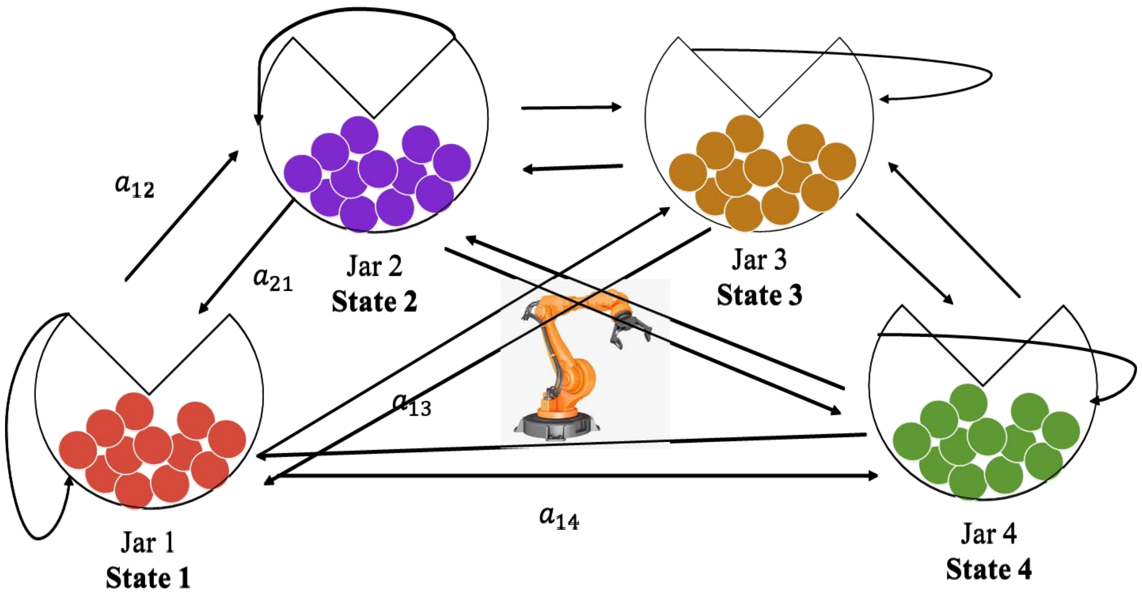
- **Observable Markov Model**

Discrete Markov Process



- State change at regularly discrete times

What Kind of Decisions Can We Make?

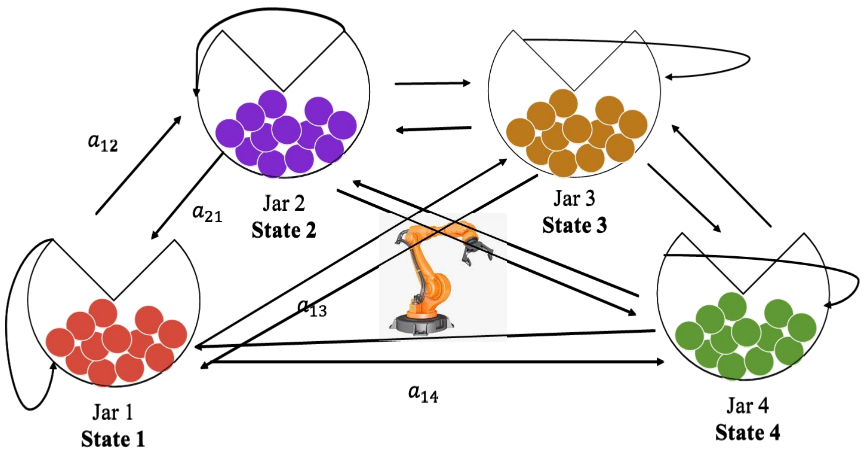


- Let's say we have an observation sequence

$$O = \{B, G, Y, Y, R, G\}$$

- What is $P(O|Model)$?

What Kind of Decisions Can We Make?



Sequence of states is
$\boldsymbol{S_2, S_4, S_3, S_3, S_1, S_4}$

$$P(O|Model) = P(B, G, Y, Y, R, G|Model)$$

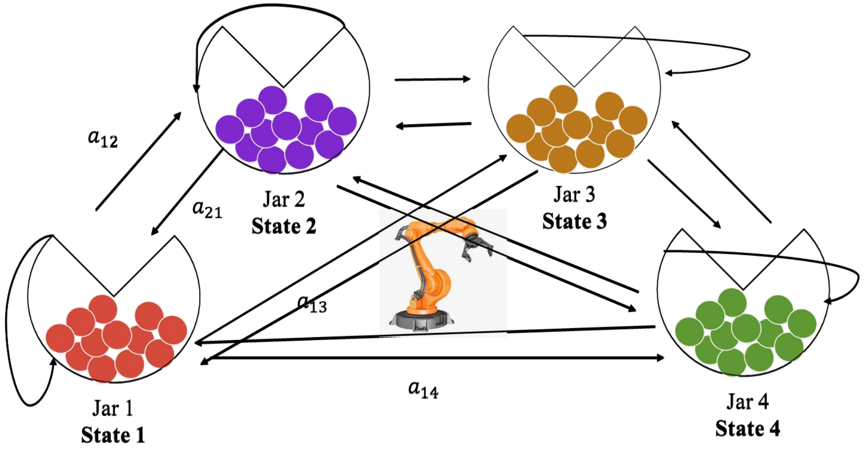
$$= P(S_2, S_4, S_3, S_3, S_1, S_4|Model)$$

$$= P(S_2|Model)P(S_4|S_2, Model)P(S_3|S_4, Model)P(S_3|S_3, Model)P(S_1|S_3, Model)P(S_4|S_1, Model)$$

$$= \pi_2 a_{24} a_{43} a_{33} a_{31} a_{14}$$

What Kind of Decisions Can We Make?

Let the model be λ



Sequence of states is
$\boldsymbol{S_2, S_4, S_3, S_3, S_1, S_4}$

$$P(O|\lambda) = P(B, G, Y, Y, R, G|\lambda)$$

$$= P(S_2, S_4, S_3, S_3, S_1, S_4|\lambda)$$

$$= P(S_2|\lambda)P(S_4|S_2, \lambda)P(S_3|S_4, \lambda)P(S_3|S_3, \lambda)P(S_1|S_3, \lambda)P(S_4|S_1, \lambda)$$

$$= \pi_2 a_{24} a_{43} a_{33} a_{31} a_{14}$$

Can We Prove?

Sequence of states is
$\boldsymbol{S_2, S_4, S_3, S_3, S_1, S_4}$

$$P(S_2, S_4, S_3, S_3, S_1, S_4|\lambda) = P(S_2|\lambda)P(S_4|S_2, \lambda)P(S_3|S_4, \lambda)P(S_3|S_3, \lambda)P(S_1|S_3, \lambda)P(S_4|S_1, \lambda)$$

Can We Prove?

Sequence of states is
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$$\boldsymbol{P(S_2, S_4, S_3, S_3, S_1, S_4|\lambda) = P(S_2|\lambda)P(S_4|S_2,\lambda)P(S_3|S_4,\lambda)P(S_3|S_3,\lambda)P(S_1|S_3,\lambda)P(S_4|S_1,\lambda)}$$

$$P(S_2, S_4, S_3, S_3, S_1, S_4, \lambda) = P(S_2, S_4, S_3, S_3, S_1, S_4|\lambda)P(\lambda)$$

$$P(S_2, S_4, S_3, S_3, S_1, S_4|\lambda) = \frac{P(S_2, S_4, S_3, S_3, S_1, S_4, \lambda)}{P(\lambda)}$$

Sequence of states is
$\boldsymbol{S_2, S_4, S_3, S_3, S_1, S_4}$

Can We Prove?

$$\boldsymbol{P(S_2, S_4, S_3, S_3, S_1, S_4|\lambda) = P(S_2|\lambda)P(S_4|S_2, \lambda)P(S_3|S_4, \lambda)P(S_3|S_3, \lambda)P(S_1|S_3, \lambda)P(S_4|S_1, \lambda)}$$

$$P(S_2, S_4, S_3, S_3, S_1, S_4, \lambda) = P(S_4|S_2, S_4, S_3, S_3, S_1, \lambda)P(S_2, S_4, S_3, S_3, S_1, \lambda)$$

$$= P(S_4|S_2, S_4, S_3, S_3, S_1, \lambda)P(S_1|S_2, S_4, S_3, S_3, \lambda)P(S_2, S_4, S_3, S_3, \lambda)$$

$$= P(S_4|S_2, S_4, S_3, S_3, S_1, \lambda)P(S_1|S_2, S_4, S_3, S_3, \lambda)P(S_3|S_2, S_4, S_3, \lambda)P(S_2, S_4, S_3, \lambda)$$

$$= P(S_4|S_2, S_4, S_3, S_3, S_1, \lambda)P(S_1|S_2, S_4, S_3, S_3, \lambda)P(S_3|S_2, S_4, S_3, \lambda)P(S_3|S_2, S_4, \lambda)P(S_2, S_4, \lambda)$$

$$= P(S_4|S_2, S_4, S_3, S_3, S_1, \lambda)P(S_1|S_2, S_4, S_3, S_3, \lambda)P(S_3|S_2, S_4, S_3, \lambda)P(S_3|S_2, S_4, \lambda)P(S_4|S_2, \lambda)P(S_2, \lambda)$$

$$= P(S_4|S_2, S_4, S_3, S_3, S_1, \lambda)P(S_1|S_2, S_4, S_3, S_3, \lambda)P(S_3|S_2, S_4, S_3, \lambda)P(S_3|S_2, S_4, \lambda)P(S_4|S_2, \lambda)P(S_2|\lambda)P(\lambda)$$

Sequence of states is
$\boldsymbol{S_2, S_4, S_3, S_3, S_1, S_4}$

Can We Prove?

$$\boldsymbol{P(S_2, S_4, S_3, S_3, S_1, S_4|\lambda) = P(S_2|\lambda)P(S_4|S_2, \lambda)P(S_3|S_4, \lambda)P(S_3|S_3, \lambda)P(S_1|S_3, \lambda)P(S_4|S_1, \lambda)}$$

$$P(S_2, S_4, S_3, S_3, S_1, S_4, \lambda)$$
$$= P(S_4|S_2, S_4, S_3, S_3, S_1, \lambda)P(S_1|S_2, S_4, S_3, S_3, \lambda)P(S_3|S_2, S_4, S_3, \lambda)P(S_3|S_2, S_4, \lambda)P(S_4|S_2, \lambda)P(S_2|\lambda)P(\lambda)$$

Sequence of states is
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Can We Prove?

$$\boldsymbol{P(S_2, S_4, S_3, S_3, S_1, S_4|\lambda) = P(S_2|\lambda)P(S_4|S_2, \lambda)P(S_3|S_4, \lambda)P(S_3|S_3, \lambda)P(S_1|S_3, \lambda)P(S_4|S_1, \lambda}$$

$$P(S_2, S_4, S_3, S_3, S_1, S_4, \lambda)$$
$$= P(S_4|S_2, S_4, S_3, S_3, S_1, \lambda)P(S_1|S_2, S_4, S_3, S_3, \lambda)P(S_3|S_2, S_4, S_3, \lambda)P(S_3|S_2, S_4, \lambda)P(S_4|S_2, \lambda)P(S_2|\lambda)P(\lambda)$$

$$= P(S_4|S_1, \lambda)P(S_1|S_3, \lambda)P(S_3|S_3, \lambda)P(S_3|S_4, \lambda)P(S_4|S_2, \lambda)P(S_2|\lambda)P(\lambda)$$

Sequence of states is $S_2, S_4, S_3, S_3, S_1, S_4$

Can We Prove?

$$\boldsymbol{P(S_2, S_4, S_3, S_3, S_1, S_4|\lambda) = P(S_2|\lambda)P(S_4|S_2,\lambda)P(S_3|S_4,\lambda)P(S_3|S_3,\lambda)P(S_1|S_3,\lambda)P(S_4|S_1,\lambda)}$$

$$P(S_2, S_4, S_3, S_3, S_1, S_4, \lambda)$$
$$= P(S_4|S_1,\lambda)P(S_1|S_3,\lambda)P(S_3|S_3,\lambda)P(S_3|S_4,\lambda)P(S_4|S_2,\lambda)P(S_2|\lambda)P(\lambda)$$

$$\boldsymbol{P(S_2, S_4, S_3, S_3, S_1, S_4|\lambda) = \frac{P(S_2, S_4, S_3, S_3, S_1, S_4, \lambda)}{P(\lambda)}}$$

Can We Prove?

Sequence of states is $\boldsymbol{S_2, S_4, S_3, S_3, S_1, S_4}$

$$\boldsymbol{P(S_2, S_4, S_3, S_3, S_1, S_4|\lambda) = P(S_2|\lambda)P(S_4|S_2,\lambda)P(S_3|S_4,\lambda)P(S_3|S_3,\lambda)P(S_1|S_3,\lambda)P(S_4|S_1,\lambda}$$

$$P(S_2, S_4, S_3, S_3, S_1, S_4, \lambda)$$
$$= P(S_4|S_1,\lambda)P(S_1|S_3,\lambda)P(S_3|S_3,\lambda)P(S_3|S_4,\lambda)P(S_4|S_2,\lambda)P(S_2|\lambda)P(\lambda)$$

$$\boldsymbol{P(S_2, S_4, S_3, S_3, S_1, S_4|\lambda) = \frac{P(S_4|S_1,\lambda)P(S_1|S_3,\lambda)P(S_3|S_3,\lambda)P(S_3|S_4,\lambda)P(S_4|S_2,\lambda)P(S_2|\lambda)P(\lambda)}{P(\lambda)}}$$

Can We Prove?

Sequence of states is
$\boldsymbol{S_2, S_4, S_3, S_3, S_1, S_4}$

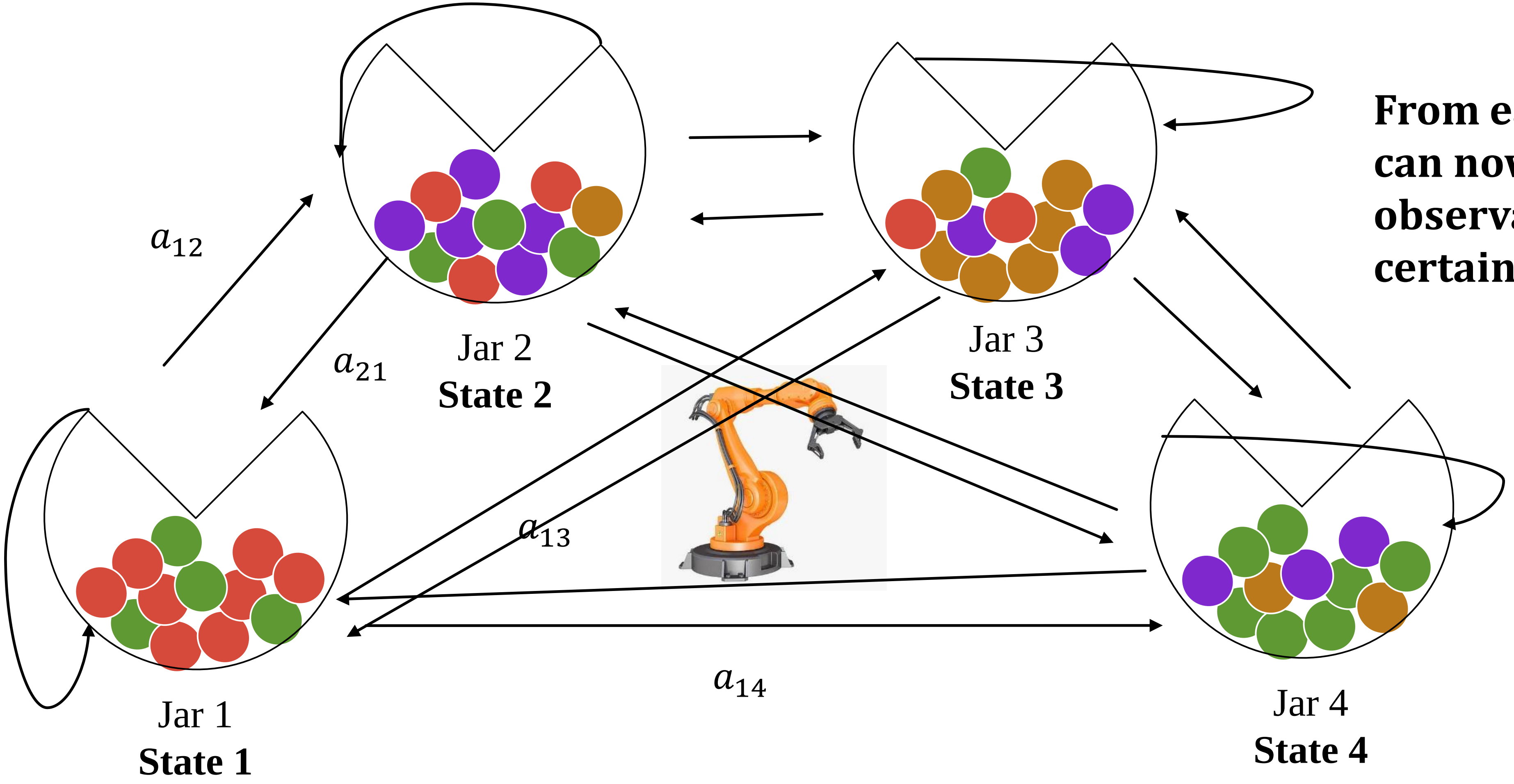
$$\boldsymbol{P(S_2, S_4, S_3, S_3, S_1, S_4|\lambda) = P(S_2|\lambda)P(S_4|S_2, \lambda)P(S_3|S_4, \lambda)P(S_3|S_3, \lambda)P(S_1|S_3, \lambda)P(S_4|S_1, \lambda}$$

$$P(S_2, S_4, S_3, S_3, S_1, S_4, \lambda)$$
$$= P(S_4|S_1, \lambda)P(S_1|S_3, \lambda)P(S_3|S_3, \lambda)P(S_3|S_4, \lambda)P(S_4|S_2, \lambda)P(S_2|\lambda)P(\lambda)$$

$$\boldsymbol{P(S_2, S_4, S_3, S_3, S_1, S_4|\lambda) = P(S_4|S_1, \lambda)P(S_1|S_3, \lambda)P(S_3|S_3, \lambda)P(S_3|S_4, \lambda)P(S_4|S_2, \lambda)P(S_2|\lambda)}$$

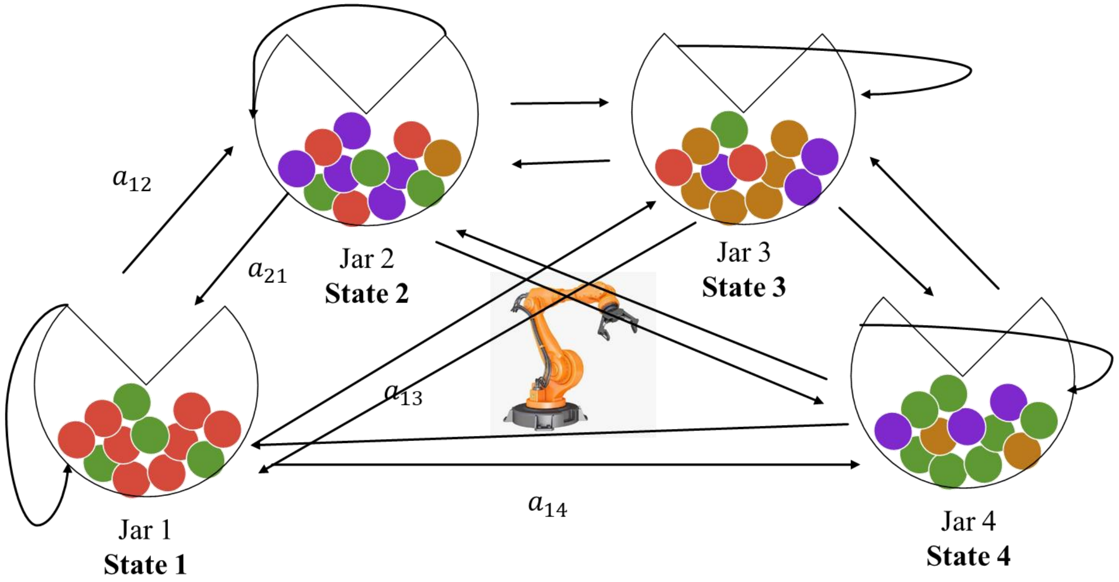
$$\boldsymbol{= P(S_2|\lambda)P(S_4|S_2, \lambda)P(S_3|S_4, \lambda)P(S_3|S_3, \lambda)P(S_1|S_3, \lambda)P(S_4|S_1, \lambda)}$$

Now, Consider This Situation



From each state, we can now have multiple observations with certain probabilities

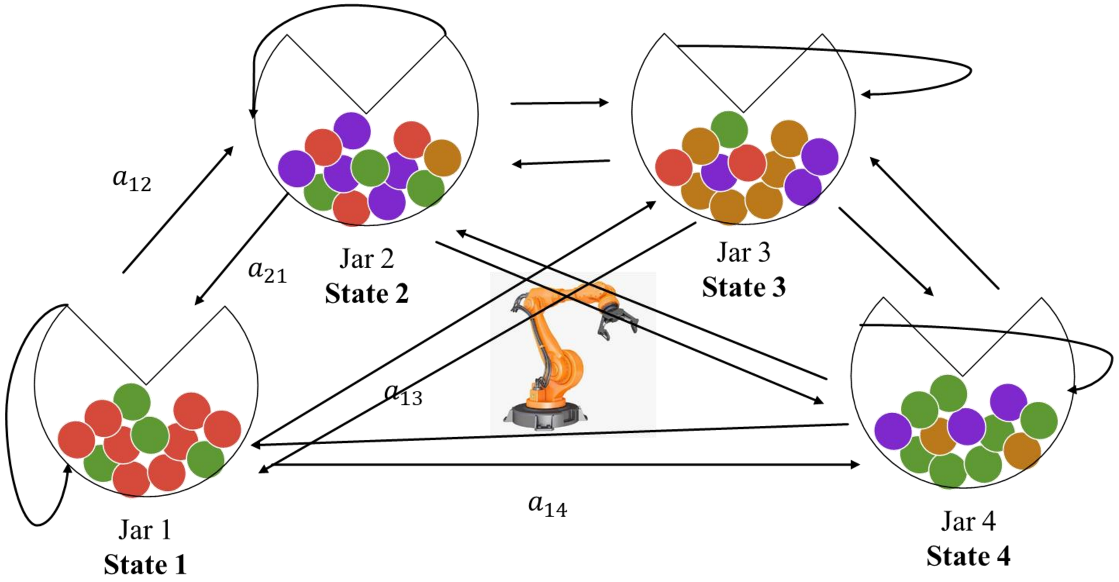
The New Situation



- Suppose, we have an observation sequence

- **What can we conclude from this?**

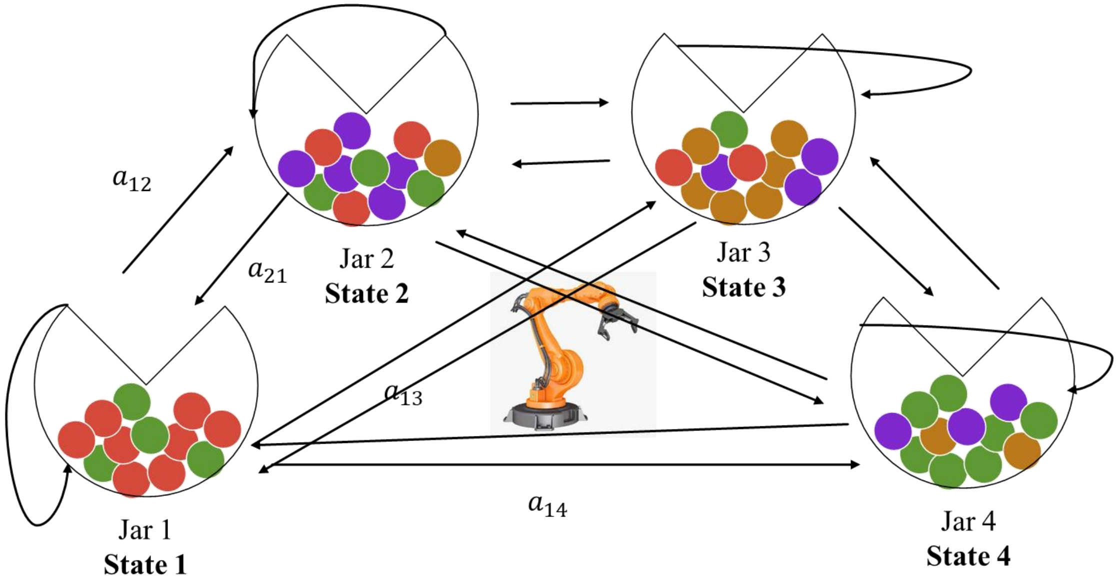
The New Situation



- Suppose, we have an observation sequence

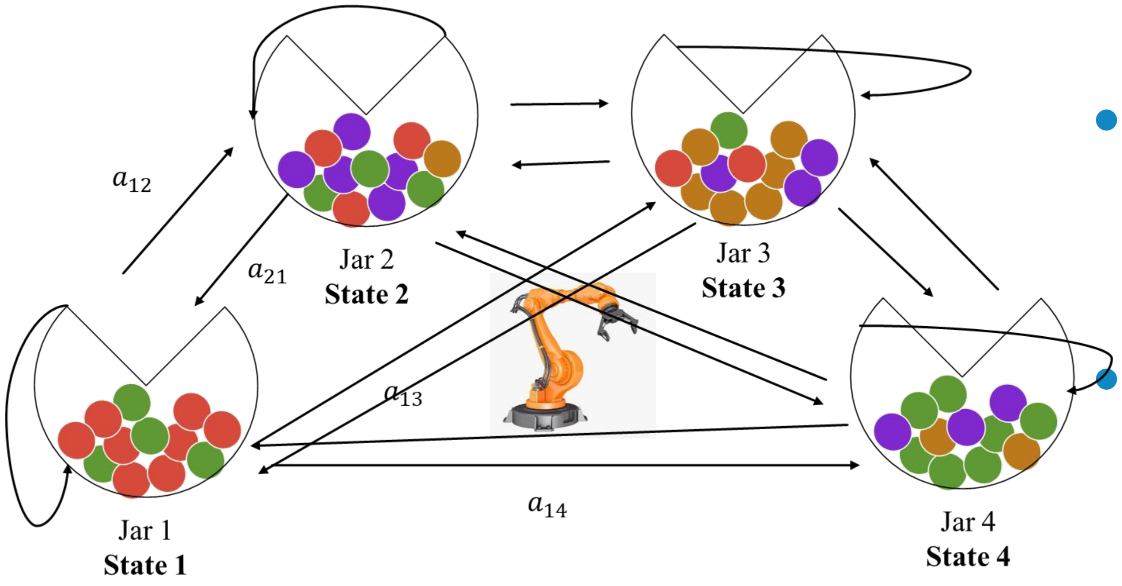
- **What can we conclude from this?**
- **We can't conclude about the movement of the arm**
- **we can only make probabilistic decisions about the movement through the states (jars)**

The Hidden Markov Model



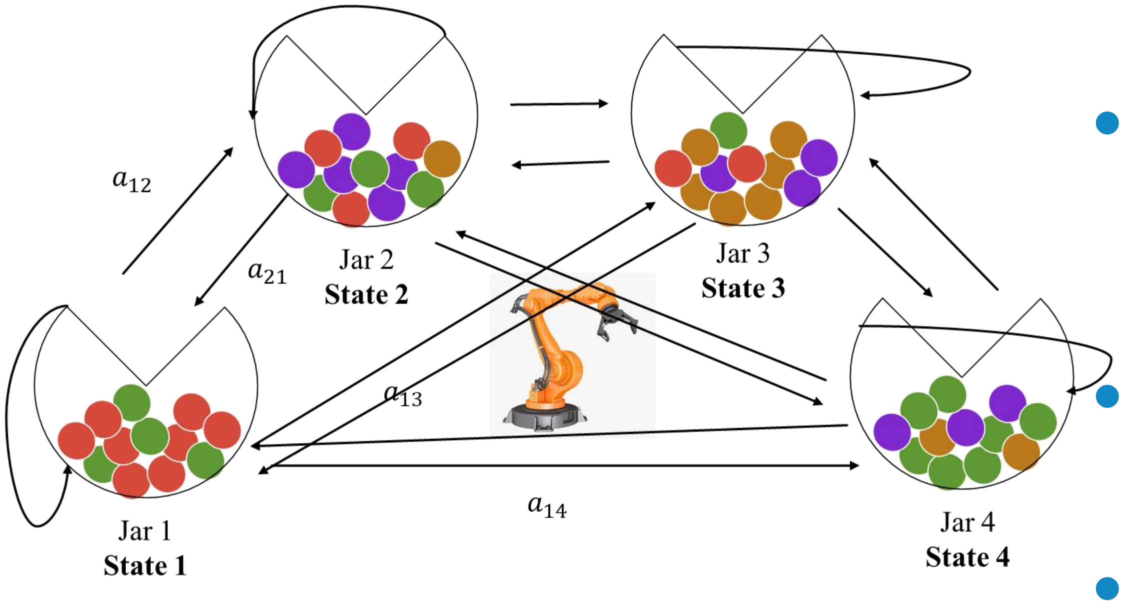
- The states (jars) traversed during the movement are hidden (not known to us)
- We have a set of observations

The Hidden Markov Model



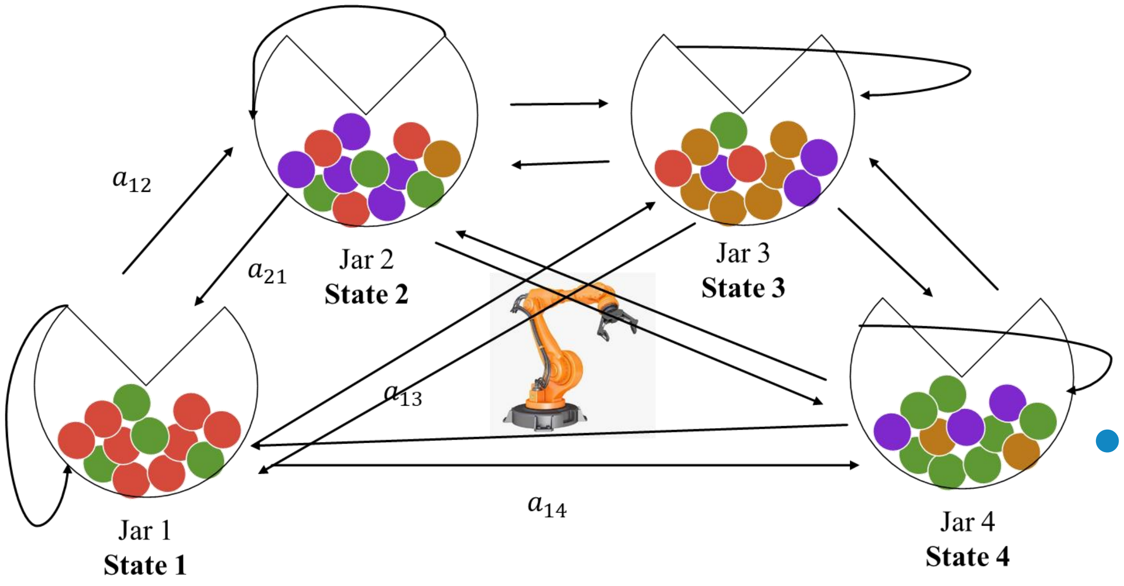
- A set of states S
  - N states
- A set of observations O
  - At each state, we may have M different observations
  - $v_1, v_2, \ldots, v_M$
- State transition matrix A
  - State transition probabilities a_{ij}
  - $a_{ij} = P(q_t = S_j | q_{t-1} = S_i) \quad 1 \leq i, j \leq N$
  - $\sum_{j=1}^{N} a_{ij} = 1$

The Hidden Markov Model



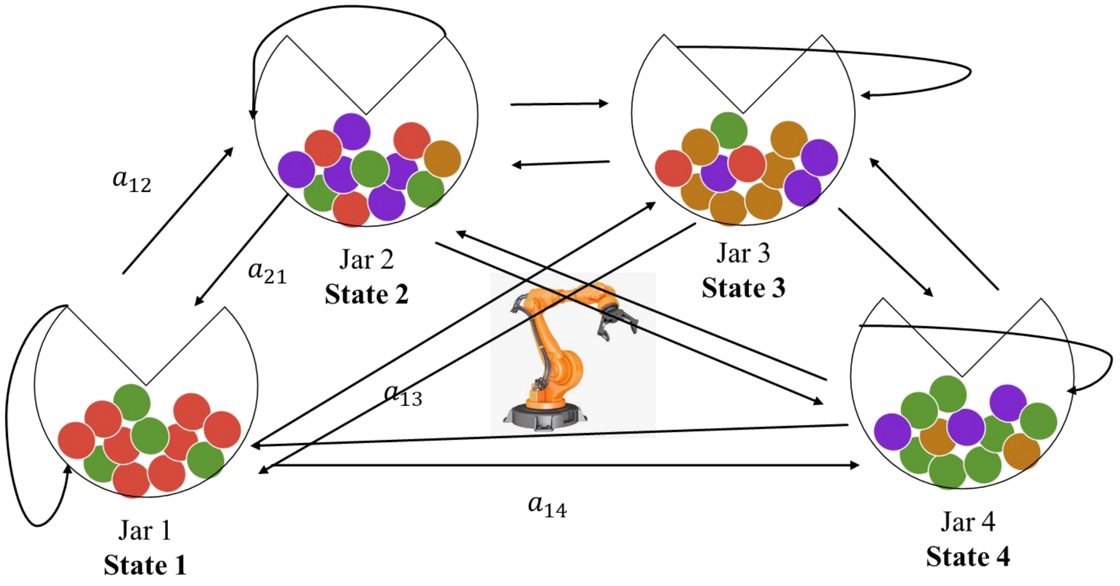
- A set of states S
  - N states
- A set of observations O
  - At each state, we may have M different observations
  - $v_1, v_2, \dots, v_M$
- State transition matrix A
  - State transition probabilities a_{ij}
- States are ergodic
  - From one state, we can move to any other state

The Hidden Markov Model



- Observation probabilities
  - $B = \{b_j(k)\}$
  - At each state, we may have M different observations/ symbols
  - $v_1, v_2, \dots, v_M$
- $b_j(k) = P(v_k \; at \; t | q_t = S_j)$
  - $1 \leq j \leq N$
  - $1 \leq k \leq M$

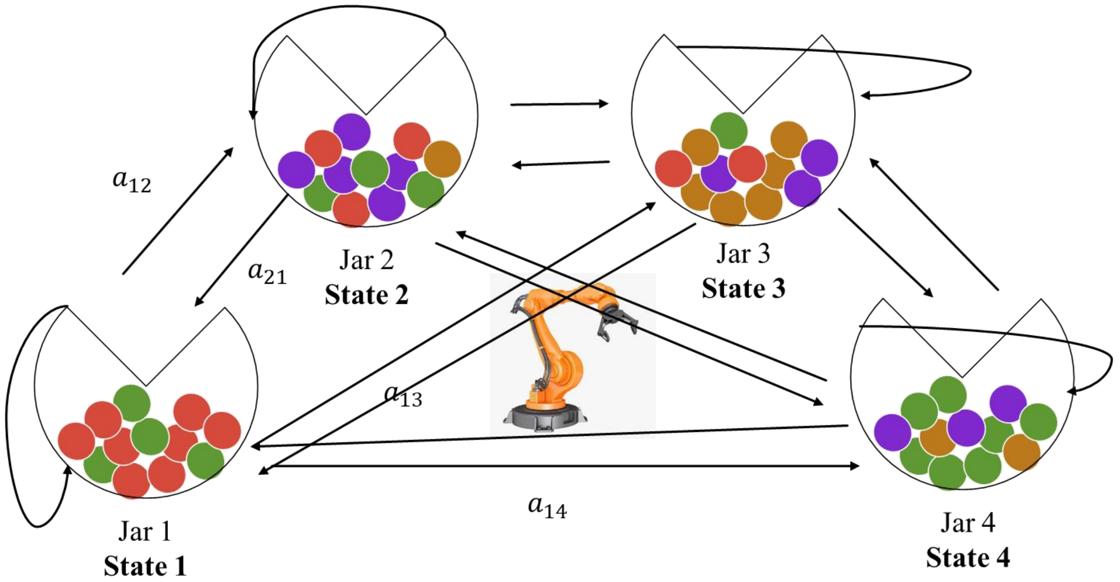
The Hidden Markov Model



- Initial state probabilities $\pi = \{\pi_i\}$

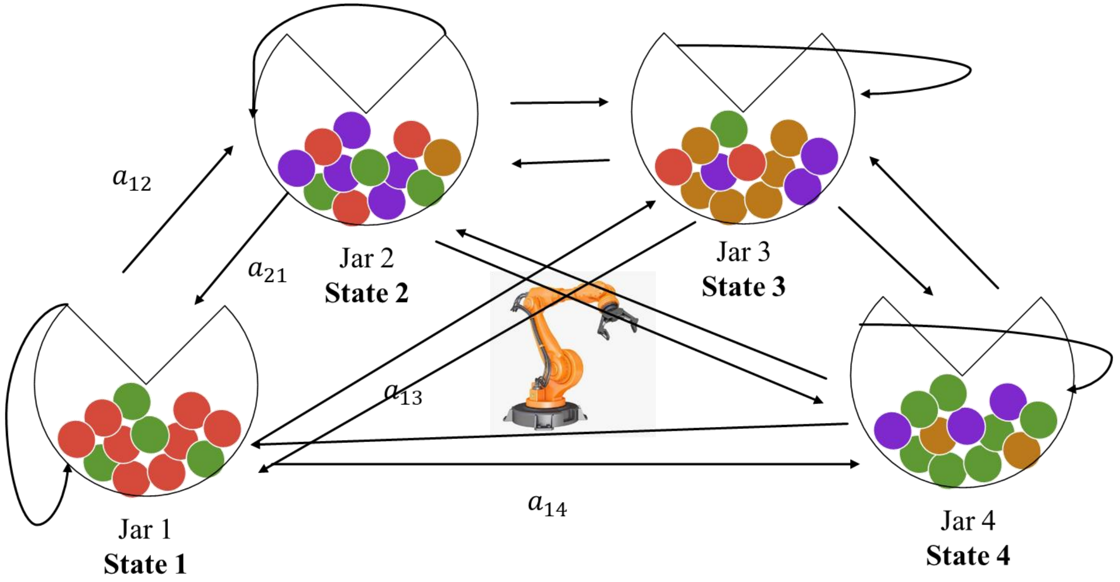
- $\pi_i = P(q_1 = S_i) \qquad 1 \le i \le N$

The Hidden Markov Model



- $\text{HMM} = \{N, M, A, B, \pi\}$
- $O = \{O_1, O_2, \ldots, O_T\}$

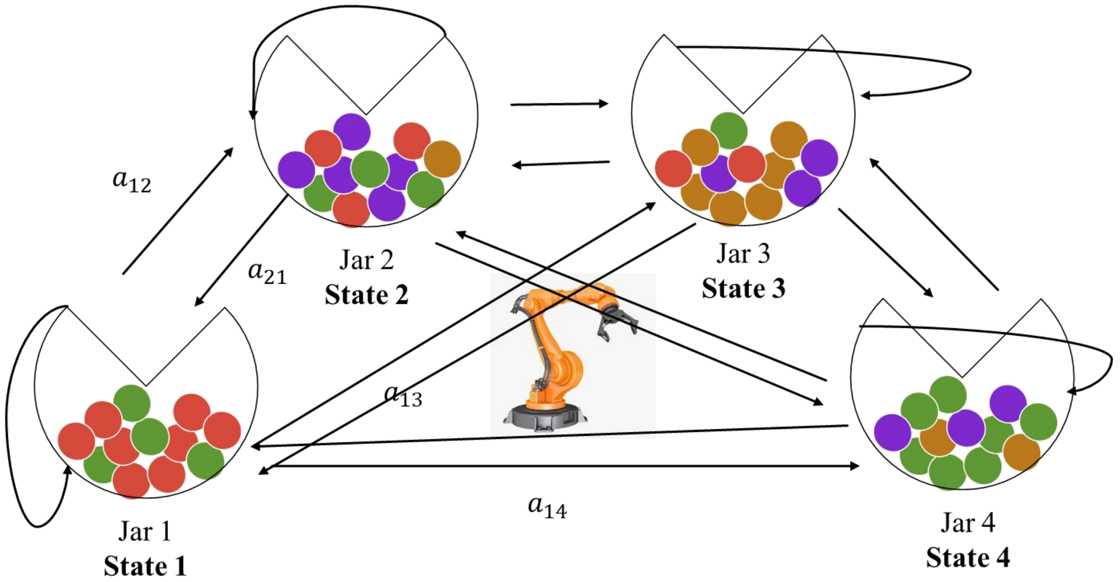
The Hidden Markov Model



- HMM $= \{N, M, A, B, \pi\}$
- $O = \{O_1, O_2, \ldots, O_T\}$

- To generate observations
  - Choose $q_1 = S_i$ according to π_i
  - Set $t = 1$
  - Choose $O_t = v_k$ according to B i.e. $b_i(k)$
  - Transit to a new state $q_{t+1} = S_j$ according to A i.e. a_{ij}
  - Set $t = t + 1$
  - Repeat until $t > T$

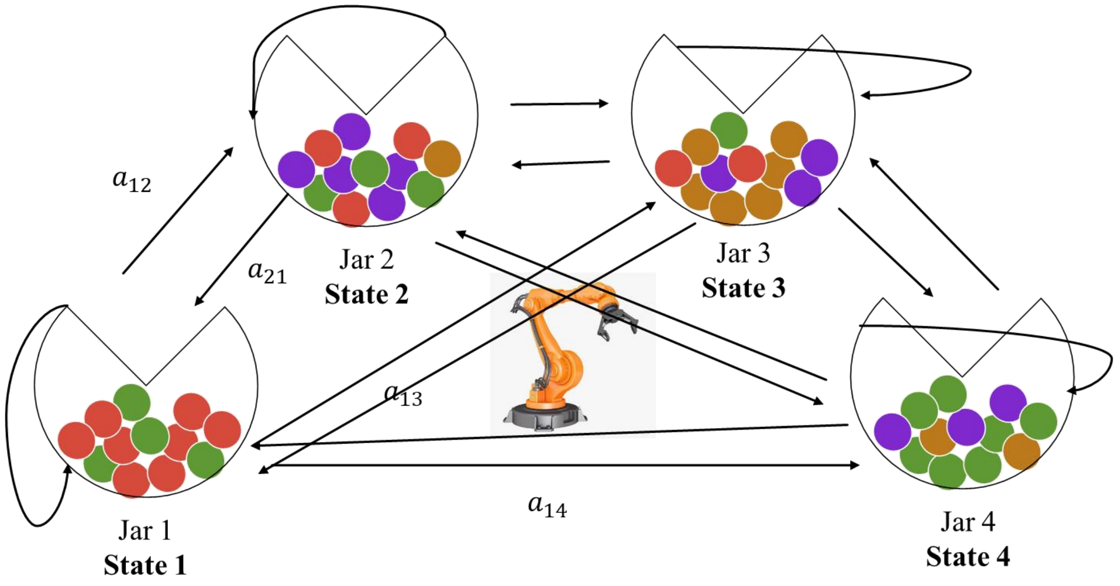
The Hidden Markov Model



- $\text{HMM} = \{N, M, A, B, \pi\}$
- $O = \{O_1, O_2, \dots, O_T\}$

- Observable Markov Model is a special case of HMM with only one non zero observation probability at each state
  - In OMM, observation distribution has only one non zero probability

What Kind of Questions Can We Answer?



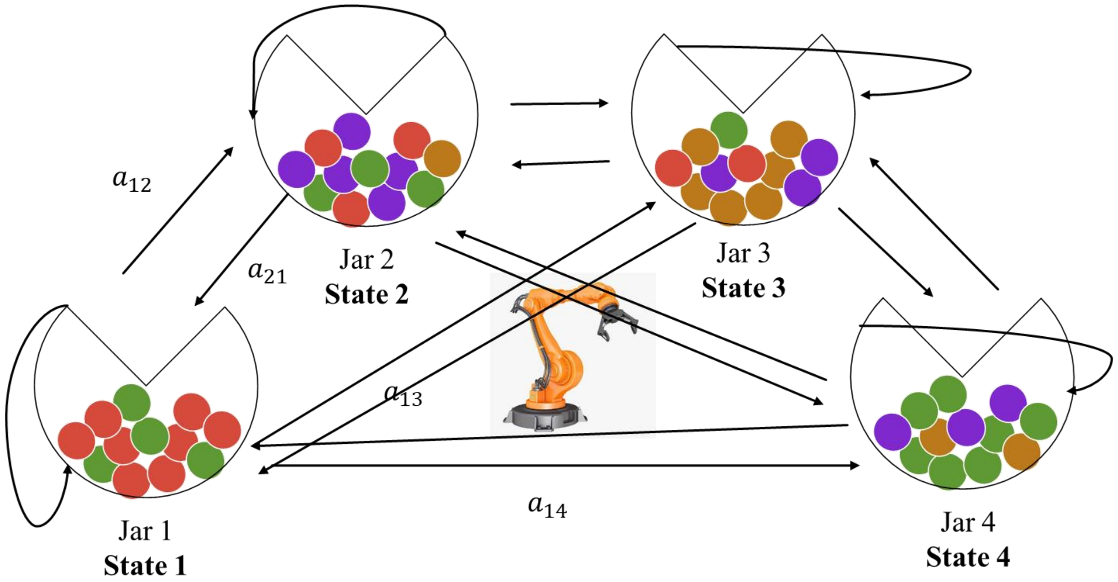
- Suppose, we have a model λ
- What is the probability that this model generates an observation sequence

$$O = \{O_1, O_2, \dots, O_T\}$$

for example, given our model, what is the probability of observing the following sequence

Given the model means given the information $\boldsymbol{\lambda = \{A, B, \pi\}}$

What Kind of Questions Can We Answer?



Given the model means given the information $\lambda = (A, B, \pi)$

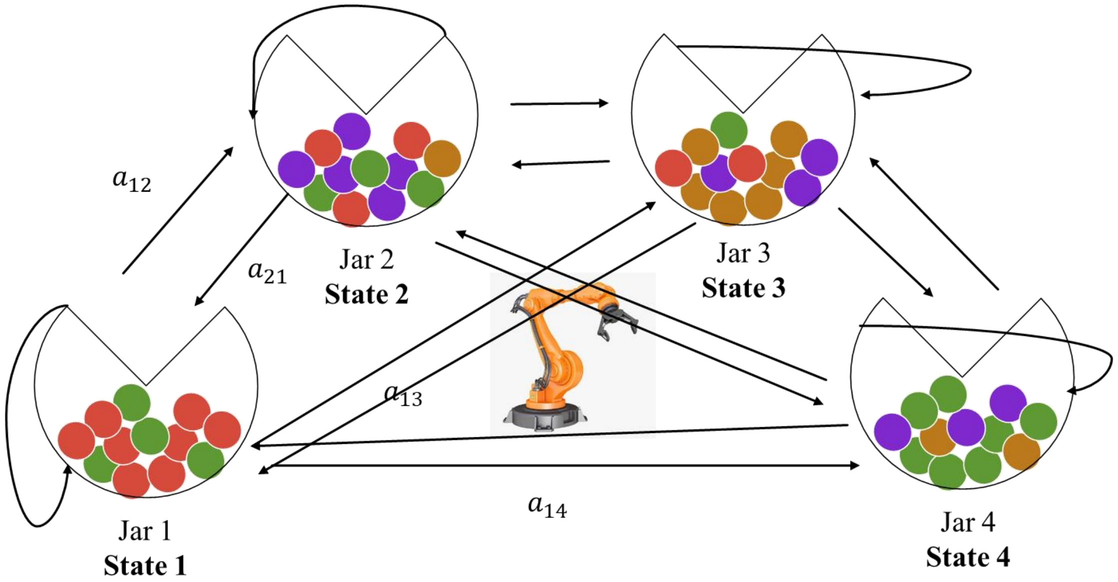
- Suppose, we have a model λ
- What is the probability that this model generates an observation sequence

$$O = \{O_1, O_2, \ldots, O_T\}$$

for example, given our model, what is the probability of observing the following sequence

What is $P(O|\lambda)$?

What Kind of Questions Can We Answer?



Given the information $\boldsymbol{\lambda = (A, B, \pi)}$

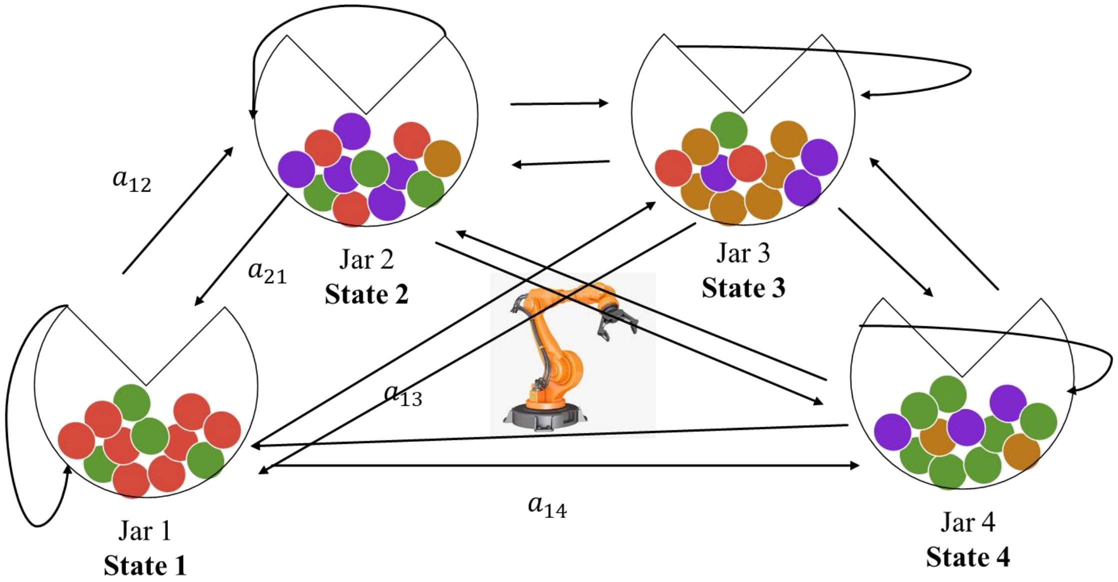
- Suppose, we have an observation sequence

  $O = \{O_1, O_2, \dots, O_T\}$

- Given the model, what sequence of states best explains the above observation?

What is $\mathbf{Q = \{q_1, q_2, \dots, q_T\}}$?

What Kind of Questions Can We Answer?



Given $\boldsymbol{O} = \{\boldsymbol{O_1}, \boldsymbol{O_2}, \ldots, \boldsymbol{O_T}\}$

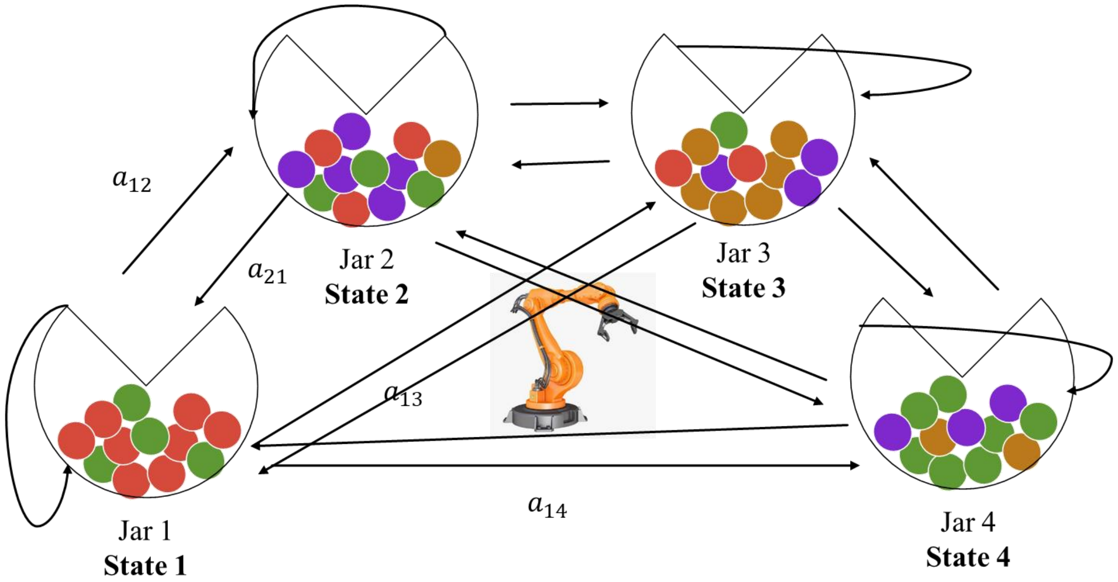
What is $\boldsymbol{\lambda} = \{\mathbf{A}, \mathbf{B}, \boldsymbol{\pi}\}$?

- Given an observation sequence

  $O = \{O_1, O_2, \ldots, O_T\}$

- How to learn the model parameters that will maximize the chance of generating the above sequence?

Question 1



Given the model means given the information $\boldsymbol{\lambda = (A, B, \pi)}$

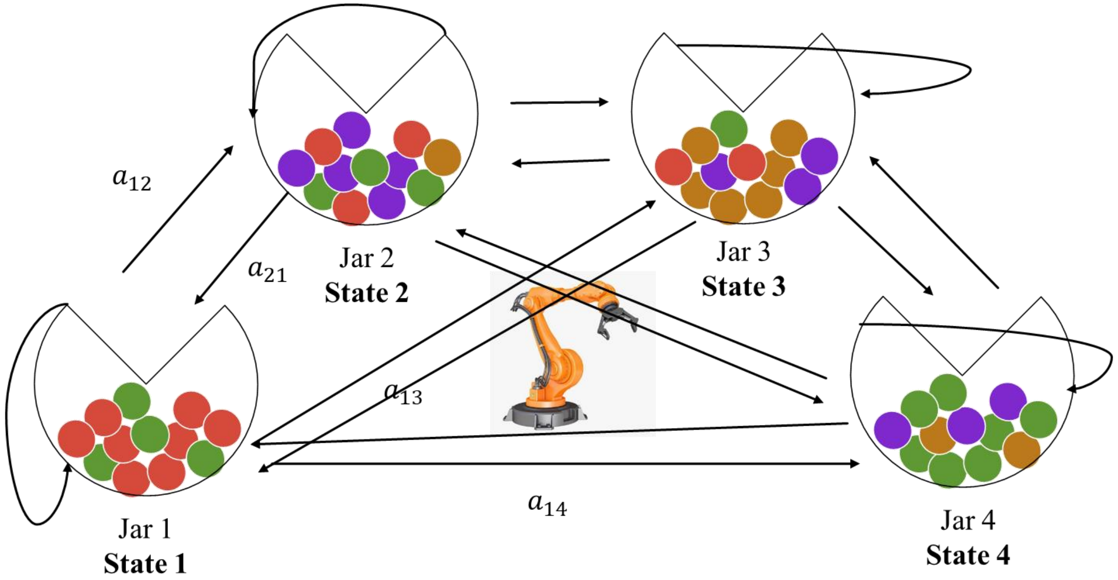
- Suppose, we have a model λ
- What is the probability that this model generates an observation sequence

$$O = \{O_1, O_2, \ldots, O_T\}$$

for example, given our model, what is the probability of observing the following sequence

What is $\boldsymbol{P(O|\lambda)}$?

Question 1

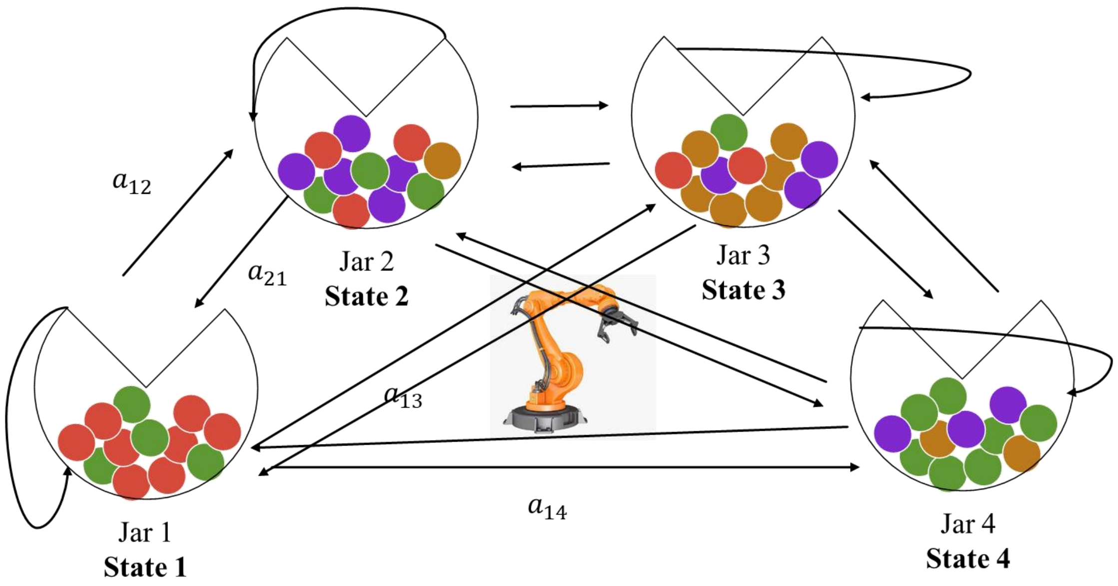


Given the model means given the information $\boldsymbol{\lambda = (A, B, \pi)}$

- Suppose, we have three models (three different sets of jars with different number of balls of these four colours, different state transition probabilities)
  - λ_1
  - λ_2
  - λ_3

Which model is more likely to generate the above observation?

Question 1



Given the model means given the information $\boldsymbol{\lambda = (A, B, \pi)}$

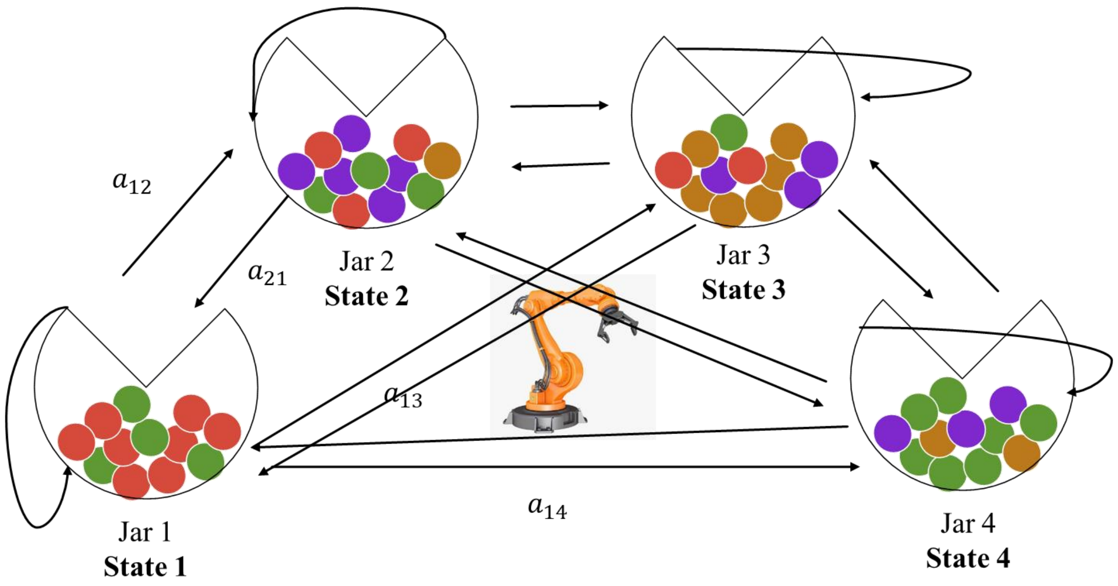
We may calculate $\boldsymbol{P(O|\lambda_1), P(O|\lambda_2), P(O|\lambda_3)}$

Find out which of these probabilities is maximum and decide which model is most likely to generate the observed sequence

- Suppose, we have three models (three different sets of jars with different number of balls of these four colours, different state transition probabilities)
  - λ_1
  - λ_2
  - λ_3

Which model is more likely to generate the above observation?

Question 1



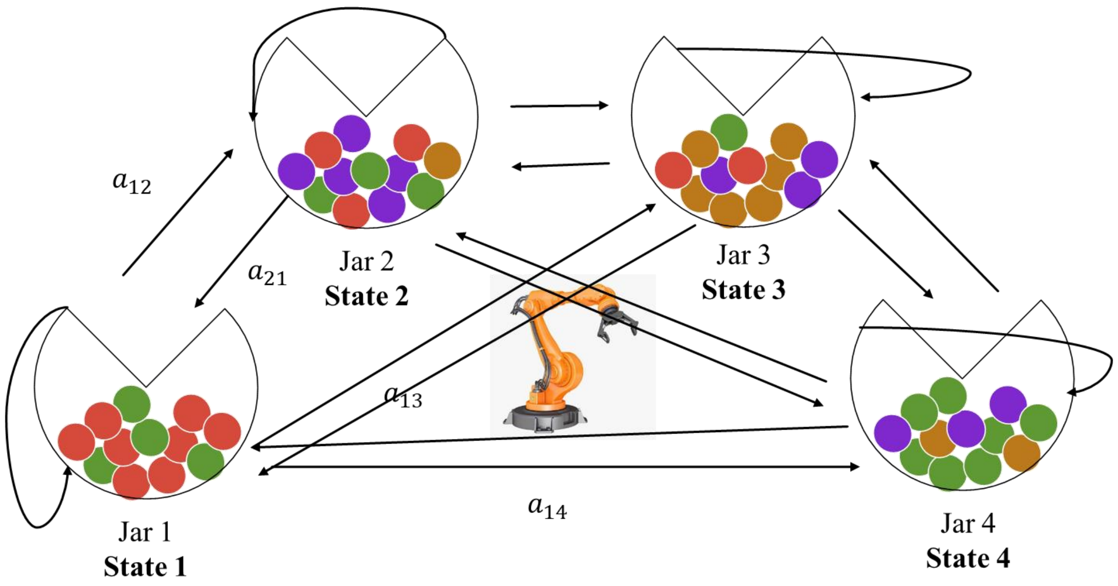
- Let's consider all possible state sequence

$$Q = q_1, q_2, \dots, q_T$$

- $O = \{O_1, O_2, \dots, O_T\}, \lambda = (A, B, \pi)$
- Probability of above observation given the state sequence

$$P(O|Q,\lambda) = P(O_1|q_1,\lambda)P(O_2|q_2,\lambda) \dots P(O_T|q_T,\lambda)$$

Question 1



- Let's consider a possible state sequence

$$Q_i = q_{1i}, q_{2i}, \dots, q_{Ti}$$

- $O = \{O_1, O_2, \dots, O_T\}, \lambda = (A, B, \pi)$
- Probability of above observation given the state sequence

$$P(O|Q_i, \lambda) = P(O_1|q_{1i}, \lambda)P(O_2|q_{2i}, \lambda) \dots P(O_T|q_{Ti}, \lambda) = \prod_{t=1}^{T} P(O_t|q_{ti}, \lambda)$$

Question 1

- Let's consider a possible state sequence

$$Q_i = q_{1i}, q_{2i}, \dots, q_{Ti}$$

- $O = \{O_1, O_2, \dots, O_T\}, \lambda = (A, B, \pi)$
- Probability of above observation given the state sequence

$$P(O|Q_i, \lambda) = P(O_1|q_{1i}, \lambda)P(O_2|q_{2i}, \lambda) \dots P(O_T|q_{Ti}, \lambda) = \prod_{t=1}^{T} P(O_t|q_{ti}, \lambda)$$

- $P(O_1|q_{1i}, \lambda) = b_{q_{1i}}(O_1), \quad P(O_2|q_{2i}, \lambda) = b_{q_{2i}}(O_2), \quad P(O_T|q_{Ti}, \lambda) = b_{q_{Ti}}(O_T)$

Question 1

- Let's consider a possible state sequence

$$Q_i = q_{1i}, q_{2i}, \ldots, q_{Ti}$$

- $O = \{O_1, O_2, \ldots, O_T\}, \lambda = (A, B, \pi)$
- Probability of above observation given the state sequence

$$P(O|Q_i, \lambda) = P(O_1|q_{1i}, \lambda)P(O_2|q_{2i}, \lambda) \ldots P(O_T|q_{Ti}, \lambda) = \prod_{t=1}^{T} P(O_t|q_{ti}, \lambda)$$

- $P(O_1|q_{1i}, \lambda) = b_{q_{1i}}(O_1), \quad P(O_2|q_{2i}, \lambda) = b_{q_{2i}}(O_2), \quad P(O_T|q_{Ti}, \lambda) = b_{q_{Ti}}(O_T)$

- $P(O|Q_i, \lambda) = b_{q_{1i}}(O_1) b_{q_{2i}}(O_2) \ldots b_{q_{Ti}}(O_T)$

Question 1

- Let's consider a possible state sequence

$$Q_i = q_{1i}, q_{2i}, \dots, q_{Ti}$$

- $O = \{O_1, O_2, \dots, O_T\}, \lambda = (A, B, \pi)$
- Probability of above observation given the state sequence

$$P(O|Q_i, \lambda) = P(O_1|q_{1i}, \lambda)P(O_2|q_{2i}, \lambda) \dots P(O_T|q_{Ti}, \lambda) = \prod_{t=1}^{T} P(O_t|q_{ti}, \lambda)$$

- $P(O|Q_i, \lambda) = b_{q_{1i}}(O_1) b_{q_{2i}}(O_2) \dots b_{q_{Ti}}(O_T)$

- Probability of observing the above state transition given the model λ

$$P(Q_i|\lambda) = \pi_{q_{1i}} a_{q_{1i}q_{2i}} a_{q_{2i}q_{3i}} \dots a_{q_{(T-1)i}q_{Ti}}$$

Question 1

- Probability of seeing the observation and the specific state transitions is
$$P(O|Q_i,\lambda)P(Q_i|\lambda) = P(O,Q_i|\lambda)$$

- Probability of seeing the observation considering specific state transitions Q_1 is
$$P(O,Q_1|\lambda) = P(O|Q_1\ \lambda)P(Q_1|\lambda)$$
- Probability of seeing the observation considering specific state transitions Q_2 is
$$P(O,Q_2|\lambda) = P(O|Q_2\ \lambda)P(Q_2|\lambda)$$

…

…

Question 1

- Probability of seeing the observation and the specific state transitions is

$$P(O|Q_i, \lambda)P(Q_i|\lambda) = P(O, Q_i|\lambda)$$

- Probability of seeing the observation considering specific state transitions Q_1 is

$$P(O, Q_1|\lambda) = P(O|Q_1, \lambda)P(Q_1|\lambda)$$

- Probability of seeing the observation considering specific state transitions Q_2 is

$$P(O, Q_2|\lambda) = P(O|Q_2, \lambda)P(Q_2|\lambda)$$

…

…

- Probability of seeing the observation regardless of any specific state transitions is

$$P(O\ |\lambda) = \sum_{\forall i} P(O|Q_i, \lambda)P(Q_i|\lambda)$$

Law of total probability

Question 1

- $P(O|Q_i, \lambda) = b_{q_{1i}}(O_1) b_{q_{2i}}(O_2) \dots b_{q_{Ti}}(O_T)$
- Probability of observing the above state transition given the model λ

$$P(Q_i|\lambda) = \pi_{q_{1i}} a_{q_{1i}q_{2i}} a_{q_{2i}q_{3i}} \dots a_{q_{(T-1)i}q_{Ti}}$$

- Probability of seeing the observation regardless of any specific state transitions is

$$P(O\,|\lambda) = \sum_{\forall i} P(O|Q_i, \lambda) P(Q_i|\lambda)$$

$$= \sum_{\forall i} \pi_{q_{1i}} b_{q_{1i}}(O_1) a_{q_{1i}q_{2i}} b_{q_{2i}}(O_2) a_{q_{2i}q_{3i}} \dots a_{q_{(T-1)i}q_{Ti}} b_{q_{Ti}}(O_T)$$

Calculation

- Probability of seeing the observation regardless of any specific state transitions is

$$= \sum_{\forall i} \pi_{q_{1i}} b_{q_{1i}}(O_1) a_{q_{1i}q_{2i}} b_{q_{2i}}(O_2) a_{q_{2i}q_{3i}} \dots a_{q_{(T-1)i}q_{Ti}} b_{q_{Ti}}(O_T)$$

- How many possible state sequence: N^T
- How many multiplications per state sequence: $(2T-1)$
- Total number of operations: $(2T-1)N^T + (N^T - 1)$

Calculation

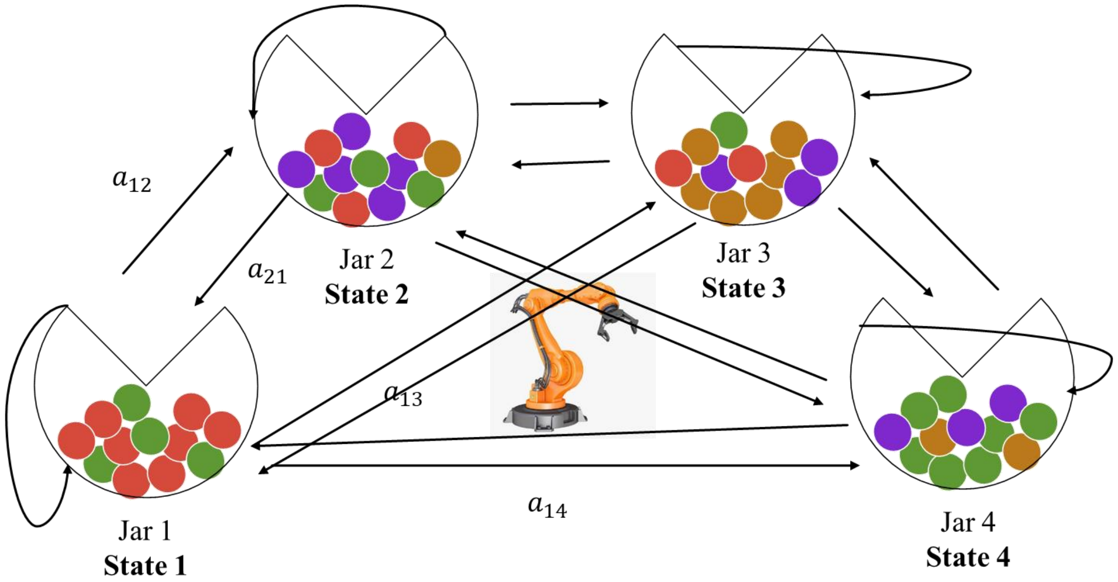
- Probability of seeing the observation regardless of any specific state transitions is

$$= \sum_{\forall i} \pi_{q_{1i}} b_{q_{1i}}(O_1) a_{q_{1i}q_{2i}} b_{q_{2i}}(O_2) a_{q_{2i}q_{3i}} \dots a_{q_{(T-1)i}q_{Ti}} b_{q_{Ti}}(O_T)$$

- How many possible state sequence: N^T
- How many multiplications per state sequence: $(2T - 1)$
- Total number of operations: $(2T - 1)N^T + (N^T - 1)$

- If total number of observations $T = 100$, number of states $N = 10$
  - Total number of operations $\sim 10^{100}$

Question 1



Given the model means given the information $\boldsymbol{\lambda = (A, B, \pi)}$

- Suppose, we have a model λ
- What is the probability that this model generates an observation sequence

$$O = \{O_1, O_2, \dots, O_T\}$$

for example, given our model, what is the probability of observing the following sequence

What is $\boldsymbol{P(O|\lambda)}$?

Calculation

- Probability of seeing the observation regardless of any specific state transitions is

$$= \sum_{\forall i} \pi_{q_{1i}} b_{q_{1i}}(O_1) a_{q_{1i}q_{2i}} b_{q_{2i}}(O_2) a_{q_{2i}q_{3i}} \dots a_{q_{(T-1)i}q_{Ti}} b_{q_{Ti}}(O_T)$$

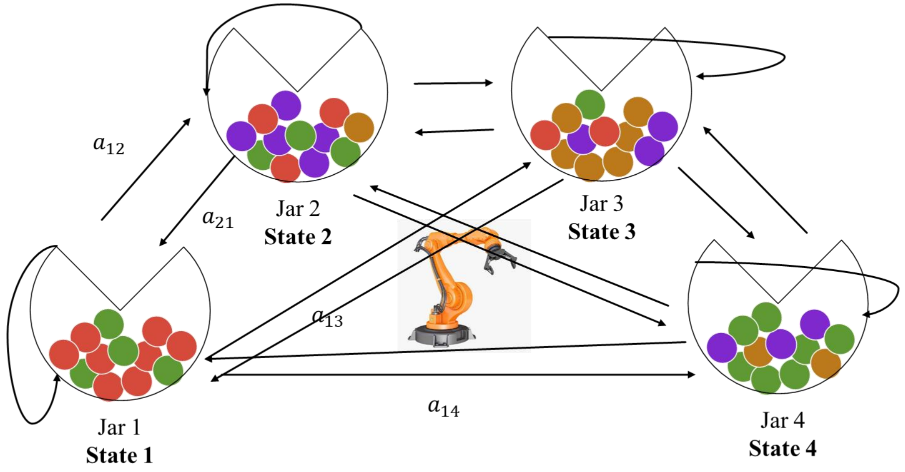
- How many possible state sequence: N^T
- How many multiplications per state sequence: $(2T - 1)$
- Total number of operations: $(2T - 1)N^T + (N^T - 1)$

- If total number of observations $T = 100$, number of states $N = 10$
  - Total number of operations $\sim 10^{100}$

Solution: Forward backward Algorithm

- Probability of seeing the observation regardless of any specific state transitions is

$$= \sum_{\forall i} \pi_{q_{1i}} b_{q_{1i}}(O_1) a_{q_{1i}q_{2i}} b_{q_{2i}}(O_2) a_{q_{2i}q_{3i}} \dots a_{q_{(T-1)i}q_{Ti}} b_{q_{Ti}}(O_T)$$



- Helper function (to reduce the number of repeating calculations)

$$\alpha_t(i) = P(O_1, O_2, \dots O_t, q_t = S_i | \lambda)$$

$\alpha_t(i)$ is the probability of seeing observations $O_1, O_2, \dots O_t$ and reaching at state S_i at t given our model

Solution: Forward backward Algorithm

- Helper function (to reduce the number of repeating calculations)

$$\alpha_t(i) = P(O_1, O_2, \ldots O_t, q_t = S_i|\lambda)$$

$\alpha_t(i)$ is the probability of seeing observations $O_1, O_2, \ldots O_t$ and reaching at state S_i at t given our model

- Inductive solution
  - Base case: $\alpha_1(i) = P(O_1, q_1 = S_i|\lambda) = \pi_i b_i(O_1) \qquad 1 \leq i \leq N$

(probability of starting at state i and seeing O_1)

Solution: Forward backward Algorithm

- Helper function (to reduce the number of repeating calculations)

$$\alpha_t(i) = P(O_1, O_2, \dots O_t, q_t = S_i | \lambda)$$

- Inductive solution
  - Base case: $\alpha_1(i) = P(O_1, q_1 = S_i|\lambda) = \pi_i b_i(O_1)$ $\quad 1 \leq i \leq N$
  - Inductive step

$$\alpha_{t+1}(j) = \left[\sum_{i=1}^{N} \alpha_t(i)\, a_{ij}\right] b_j(O_{t+1}) \qquad 1 \leq i \leq N$$

$$1 \leq t \leq T - 1$$

Solution: Forward backward Algorithm

- Helper function (to reduce the number of repeating calculations)

$$\alpha_t(i) = P(O_1, O_2, \dots O_t, q_t = S_i|\lambda)$$

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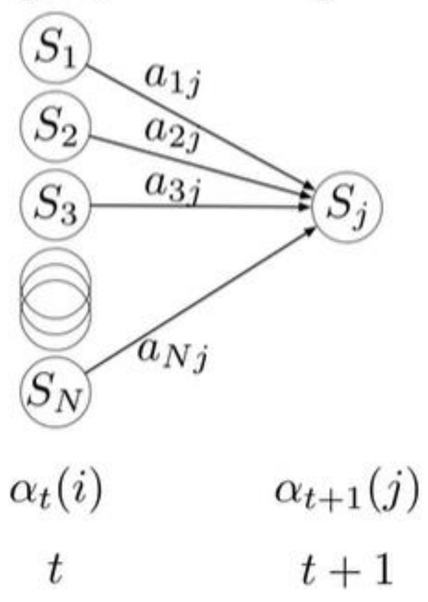

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$$1 \leq t \leq T-1$$

  - Find $\alpha_T(i)$ using above equations

Solution: Forward backward Algorithm

- Helper function (to reduce the number of repeating calculations)

$$\alpha_t(i) = P(O_1, O_2, \dots O_t, q_t = S_i|\lambda)$$

- Inductive solution
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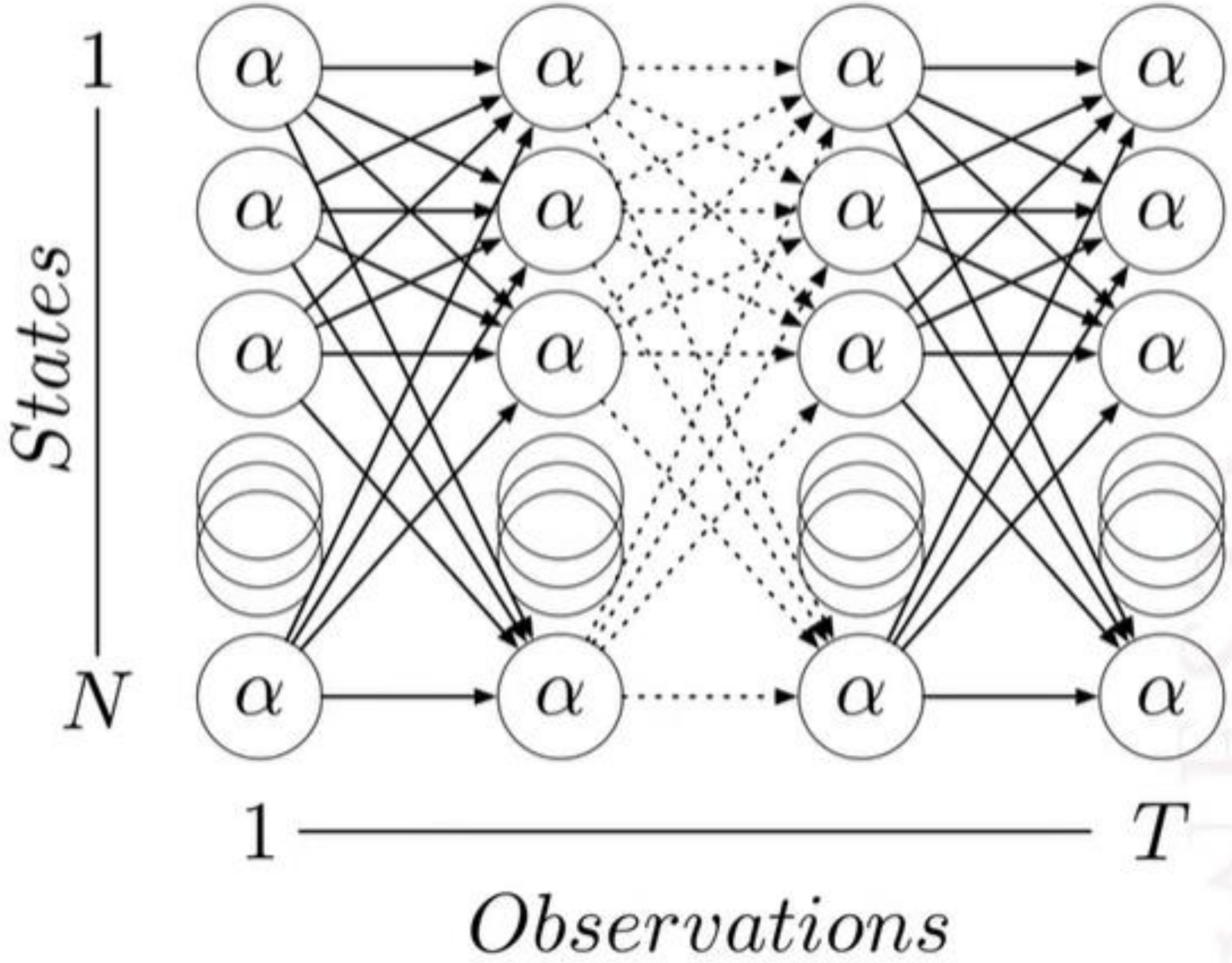
  - Find $\alpha_T(i)$ using above equations
  - Final step $P(O|\lambda) = \sum_{i=1}^{N} \alpha_T(i)$

Solution: Forward backward Algorithm

Final step $P(O|\lambda) = \sum_{i=1}^{N} \alpha_T(i)$



Solution: Forward backward Algorithm

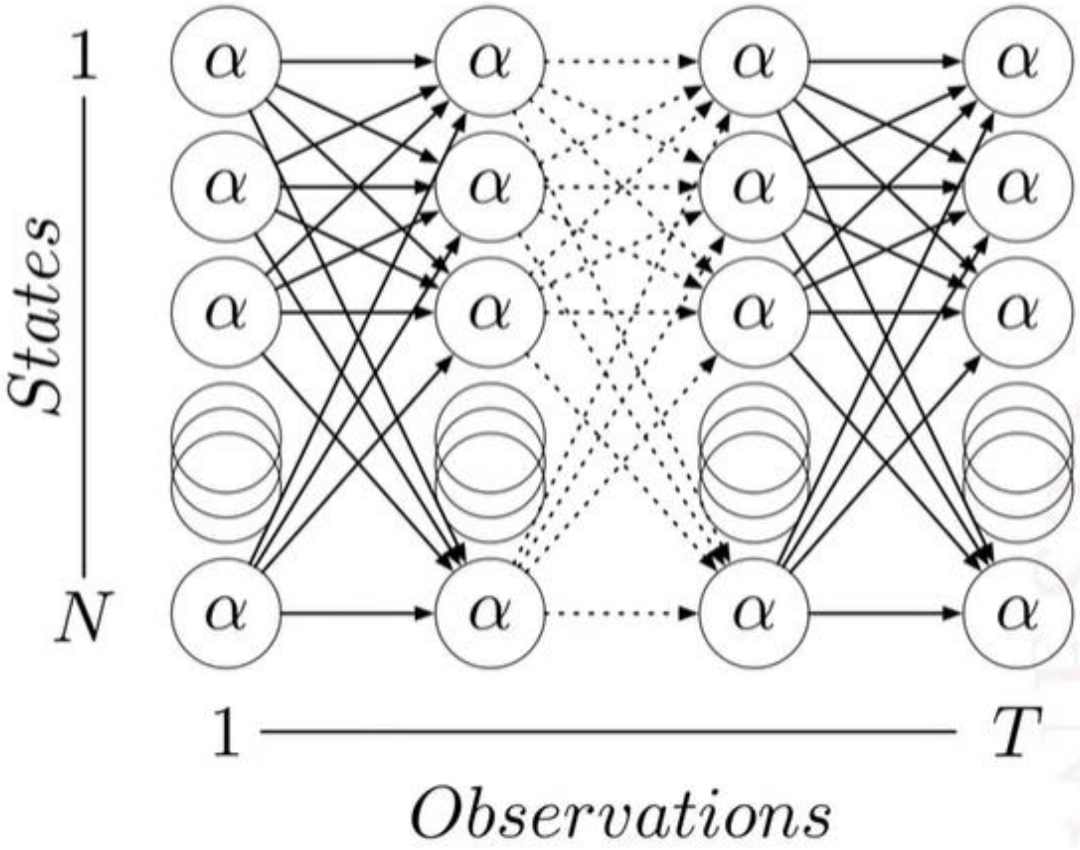
Final step $P(O|\lambda) = \sum_{i=1}^{N} \alpha_T(i)$

Number of computations $\sim N^2 T$

For $N = 10, T = 100$

Number of computations $\sim 10^2 \times 100 = 10^4$

Earlier, it was $\sim 10^{100}$



Forward backward Algorithm

- Helper function 1

$$\alpha_t(i) = P(O_1, O_2, \dots O_t, q_t = S_i | \lambda)$$

Forward backward Algorithm

- Helper function 1 (forward)

$$\alpha_t(i) = P(O_1, O_2, \ldots O_t, q_t = S_i|\lambda)$$

- Helper function 2 (backward)

$$\beta_t(i) = P(O_{t+1}, O_{t+2}, \ldots O_T|q_t = S_i, \lambda)$$

probability of seeing observations $O_{t+1}, O_{t+2}, \ldots O_T$ in future given that we are starting at state S_i at t and given our model

Solution: Forward backward Algorithm

- Helper function (to reduce the number of repeating calculations)

$$\beta_t(i) = P(O_{t+1}, O_{t+2}, \dots O_T | q_t = S_i, \lambda)$$

- Inductive solution
  - Base case: $\beta_T(i) = 1 \qquad\qquad 1 \leq i \leq N$
  - Inductive step

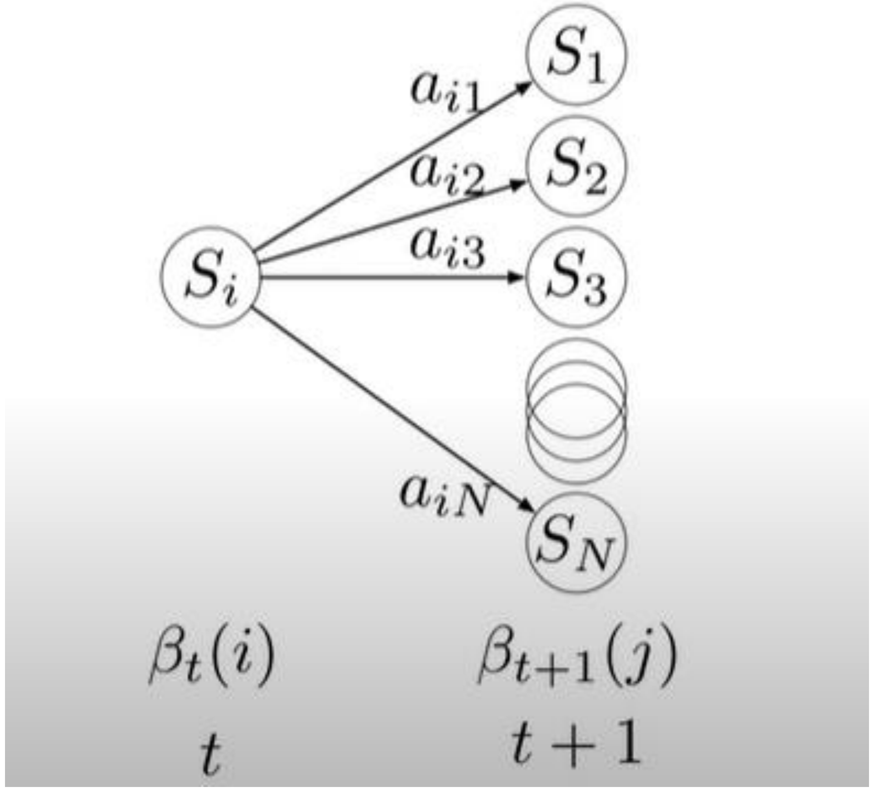
$$\beta_t(i) = \sum_{j=1}^{N} a_{ij}\, b_j(O_{t+1})\beta_{t+1}(j) \qquad 1 \leq i \leq N$$

$$t = T-1, T-2, \dots, 1$$

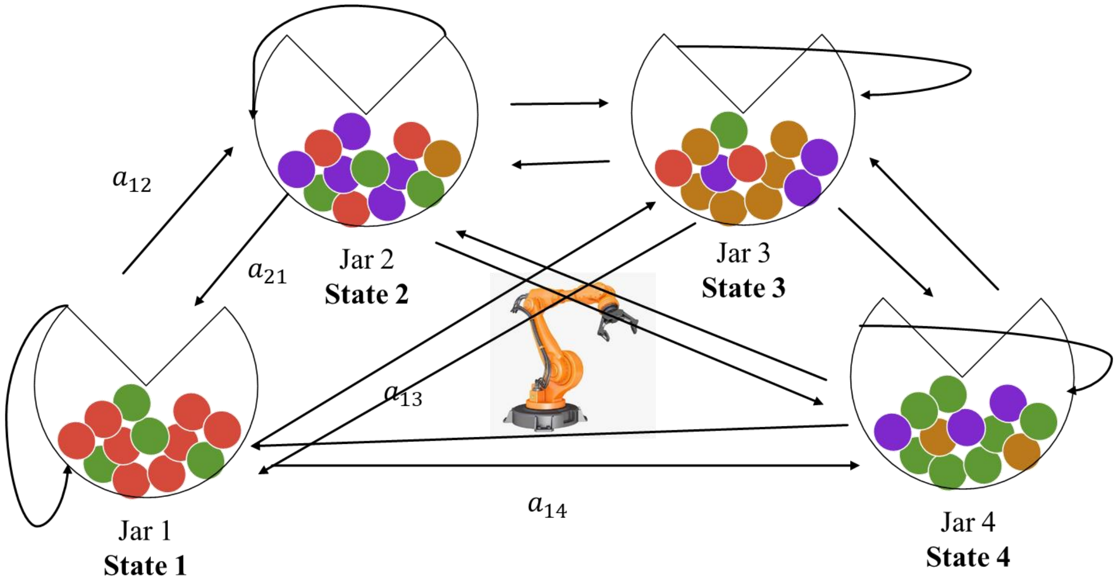
Solution: Forward backward Algorithm

- Helper function (to reduce the number of repeating calculations)

$$\beta_t(i) = P(O_{t+1}, O_{t+2}, \ldots O_T | q_t = S_i, \lambda)$$



Question 2



Given the information $\boldsymbol{\lambda = (A, B, \pi)}$

- Suppose, we have an observation sequence

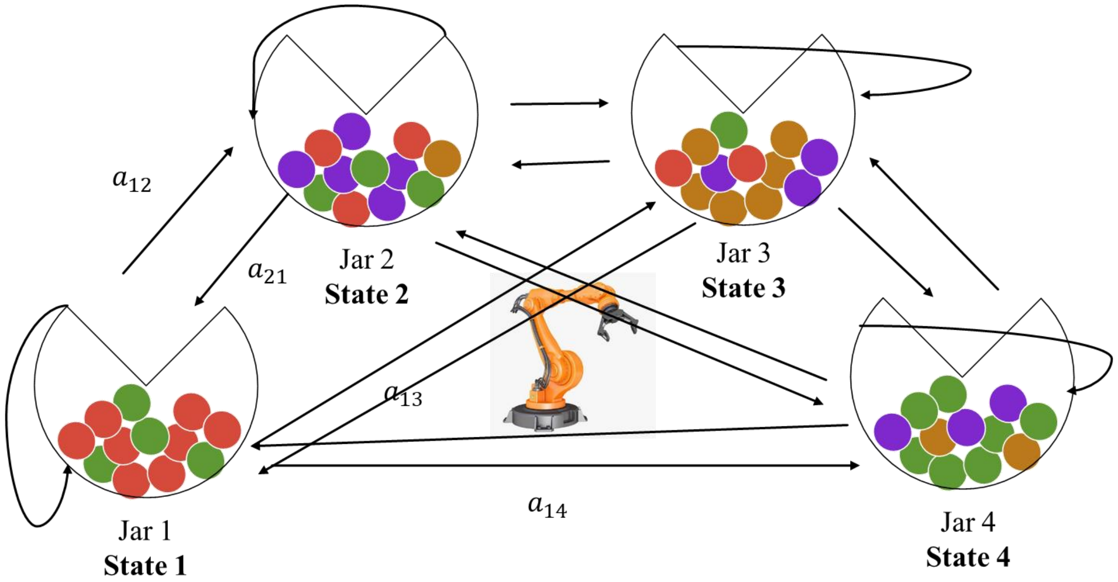
  $$O = \{O_1, O_2, \ldots, O_T\}$$

- Given the model, what sequence of states best explains the above observation?

What is $\mathbf{Q = \{q_1, q_2, \ldots, q_T\}}$?

Question 2



Given the information $\boldsymbol{\lambda = (A, B, \pi)}$

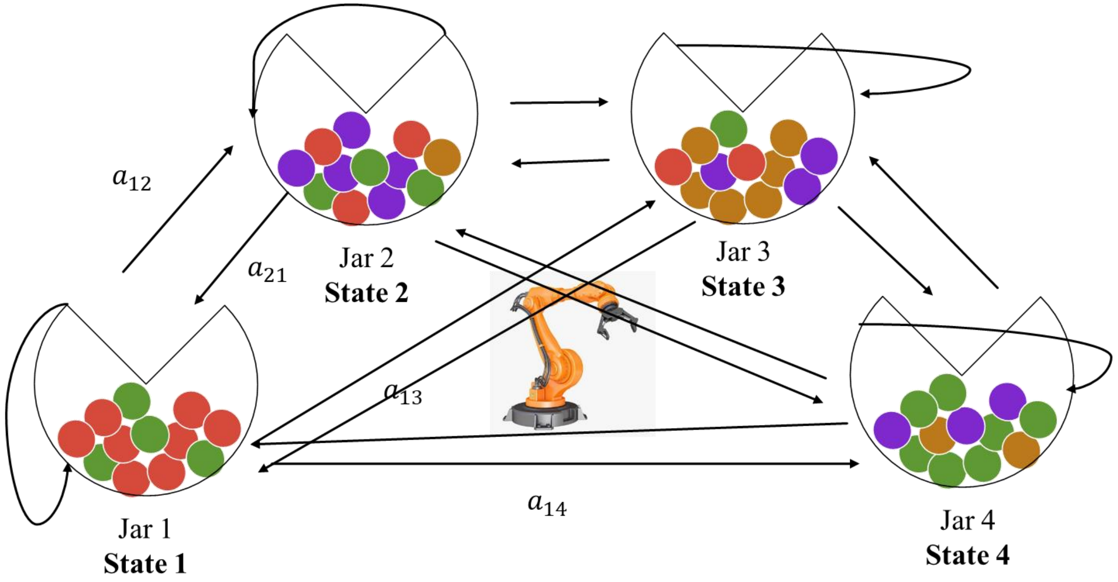
- Suppose, we have an observation sequence

  $$O = \{O_1, O_2, \dots, O_T\}$$

- Given the model, what sequence of states **best** explains the above observation?
- What is the meaning of best?

What is $\mathbf{Q = \{q_1, q_2, \dots, q_T\}}$**?**

Question 2



Given the information $\boldsymbol{\lambda = (A, B, \pi)}$

What is $\mathbf{Q = \{q_1, q_2, \dots, q_T\}}$?

- Suppose, we have an observation sequence

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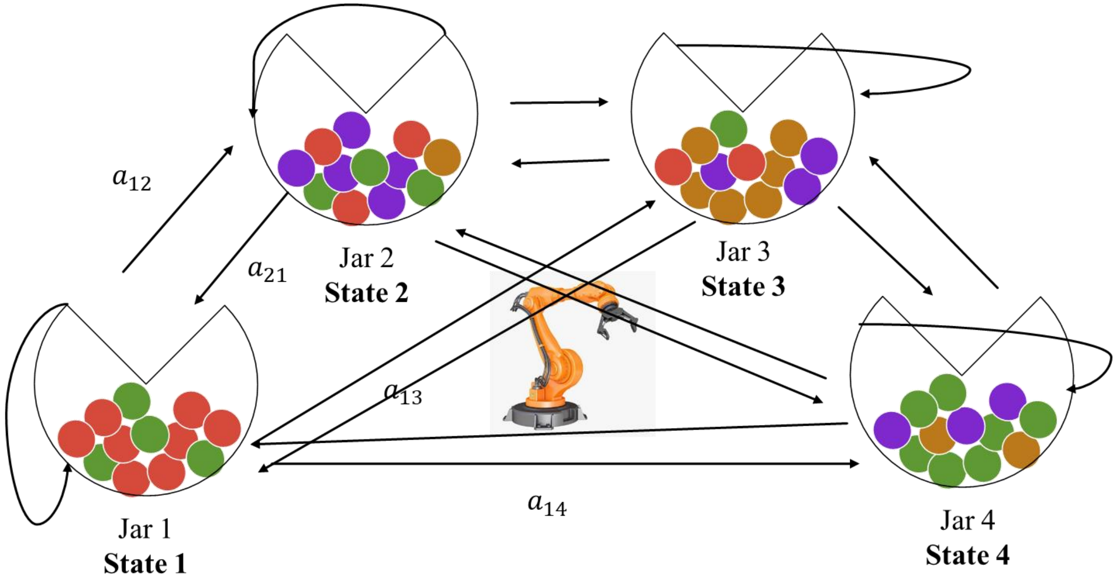
- Given the model, what sequence of states **best** explains the above observation?
- What is the meaning of best?
- What is the most likely state at any given time t given the observation

Question 2



Given the information $\boldsymbol{\lambda = (A, B, \pi)}$

What is $\mathbf{Q = \{q_1, q_2, \dots, q_T\}}$?

- Suppose, we have an observation sequence

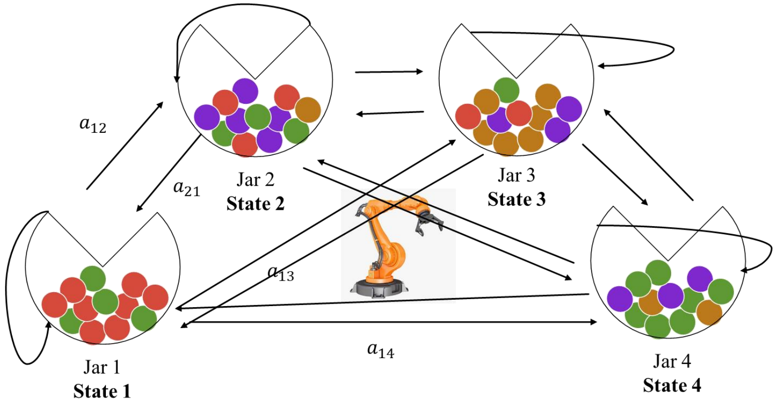
  $O = \{O_1, O_2, \dots, O_T\}$

- What is the most likely state at any given time t given the observation

- What is the probability of state i at time t given the observation

  $$\gamma_t(i) = P(q_t = S_i | O, \lambda)$$

Question 2



$$\boldsymbol{\alpha_t(i) = P(O_1, O_2, \ldots O_t, q_t = S_i|\lambda)}$$

probability of seeing observations $O_1, O_2, \ldots O_t$ and reaching at state S_i at t given our model

$$\boldsymbol{\beta_t(i) = P(O_{t+1}, O_{t+2}, \ldots O_T|q_t = S_i, \lambda)}$$

probability of seeing observations $O_{t+1}, O_{t+2}, \ldots O_T$ in future given that we are starting at state S_i at t and given our model

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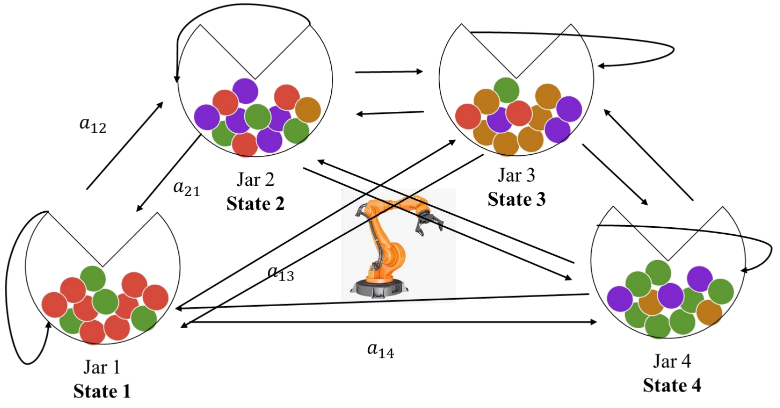
- What is the probability of state i at time t given the observation

  $$\gamma_t(i) = P(q_t = S_i|O, \lambda)$$

- $\gamma_t(i) = \dfrac{\alpha_t(i)\beta_t(i)}{\sum_{j=1}^{N} \alpha_t(j)\beta_t(j)}$

Question 2



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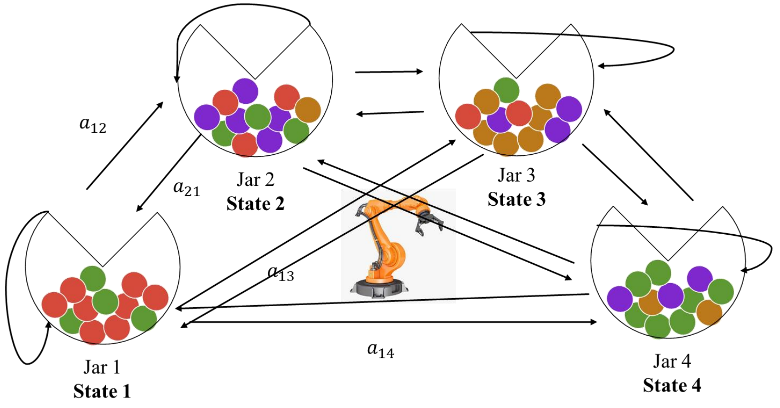
- $\gamma_t(i) = \frac{\alpha_t(i)\beta_t(i)}{\sum_{j=1}^{N} \alpha_t(j)\beta_t(j)} = \frac{\alpha_t(i)\beta_t(i)}{P(O|\lambda)}$

Question 2



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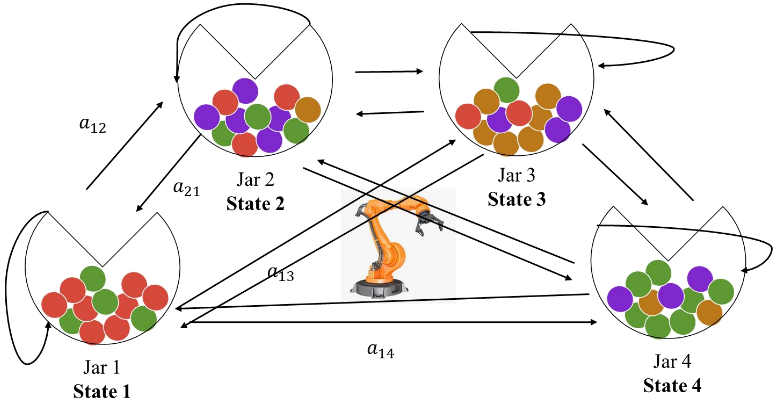
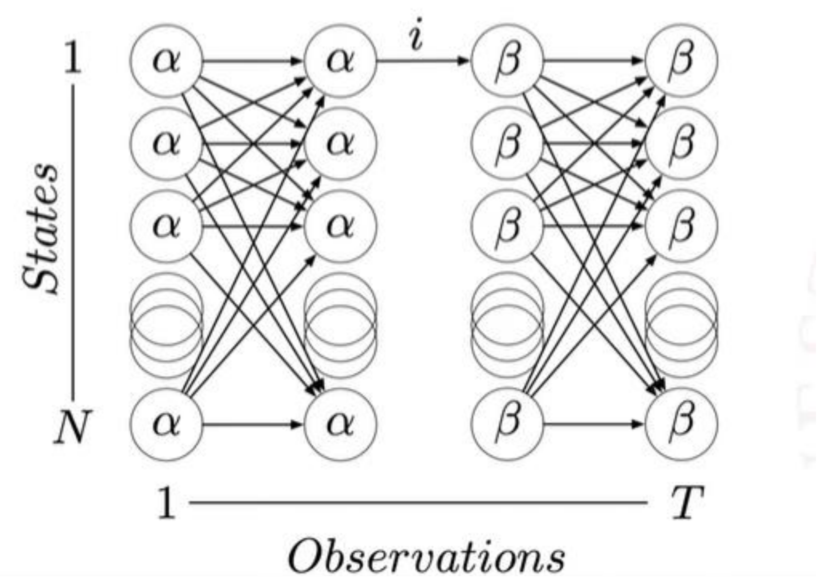
- $\sum_{i=1}^{N} \gamma_t(i) = 1$

Question 2





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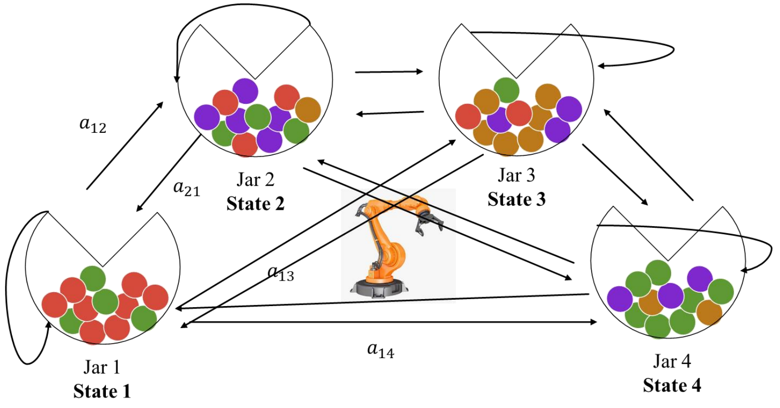
Question 2



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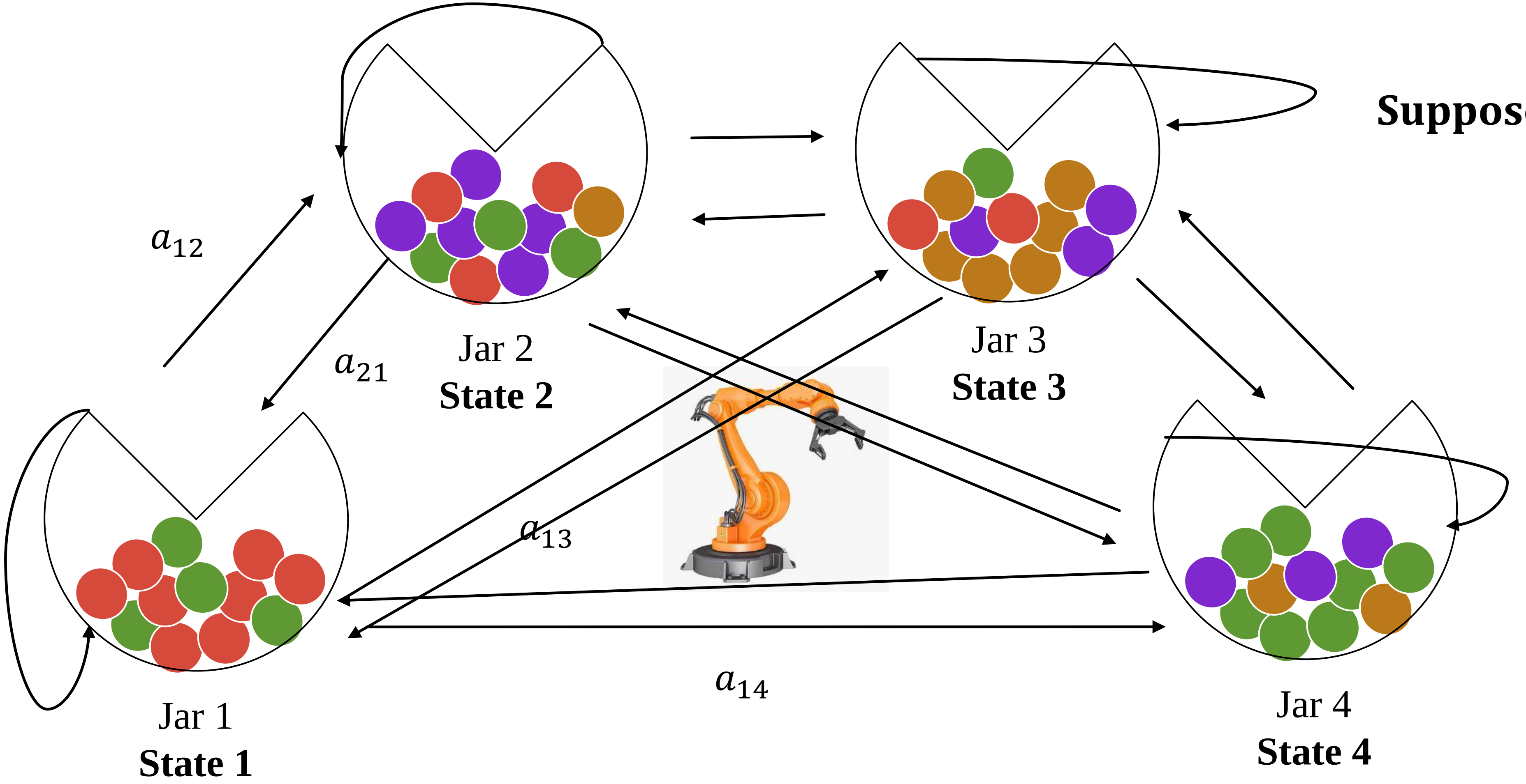
- Most likely state at time t

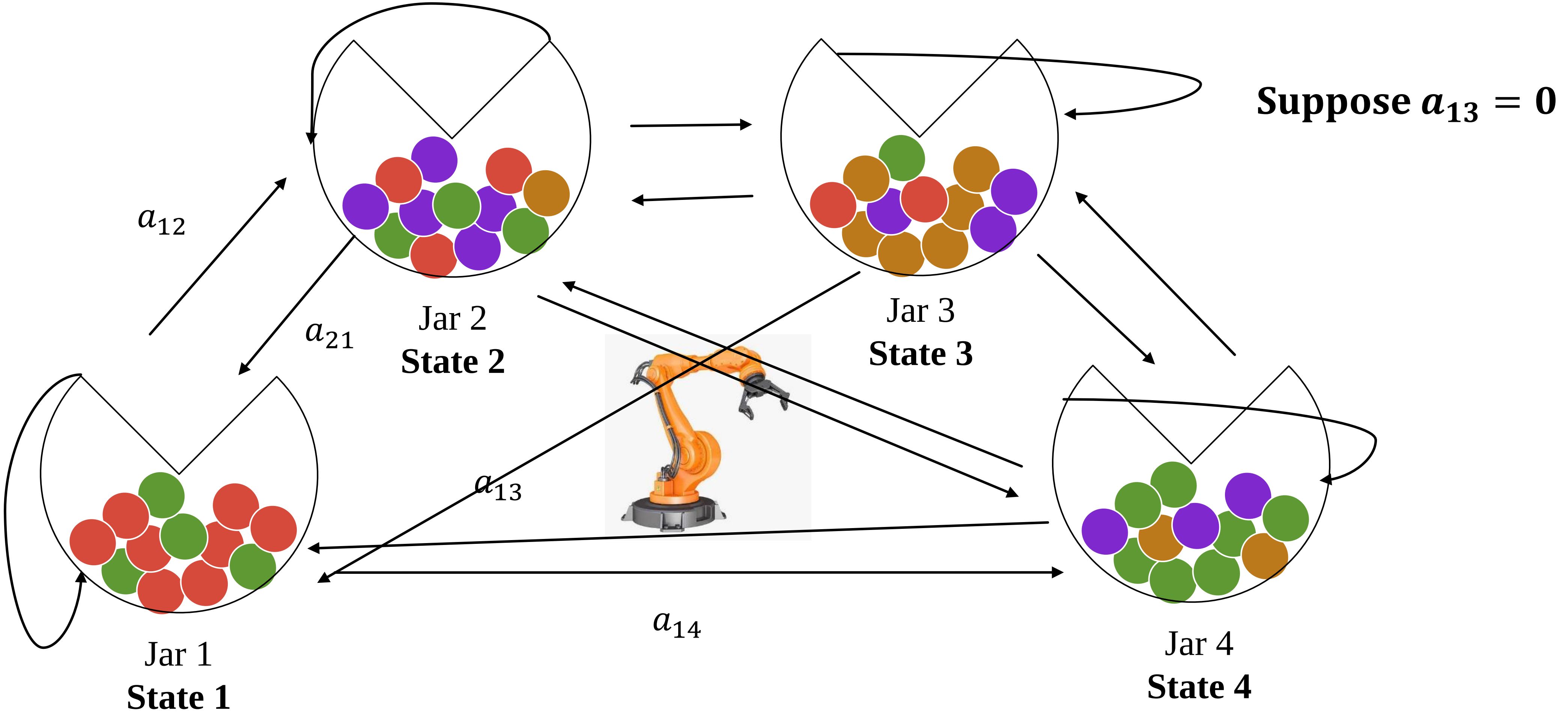
$$q_t = \underbrace{argmax}_{1 \le i \le N} [\gamma_t(i)], \qquad 1 \le t \le T$$

Problem with This Approach



Suppose $a_{13} = 0$

Problem with This Approach



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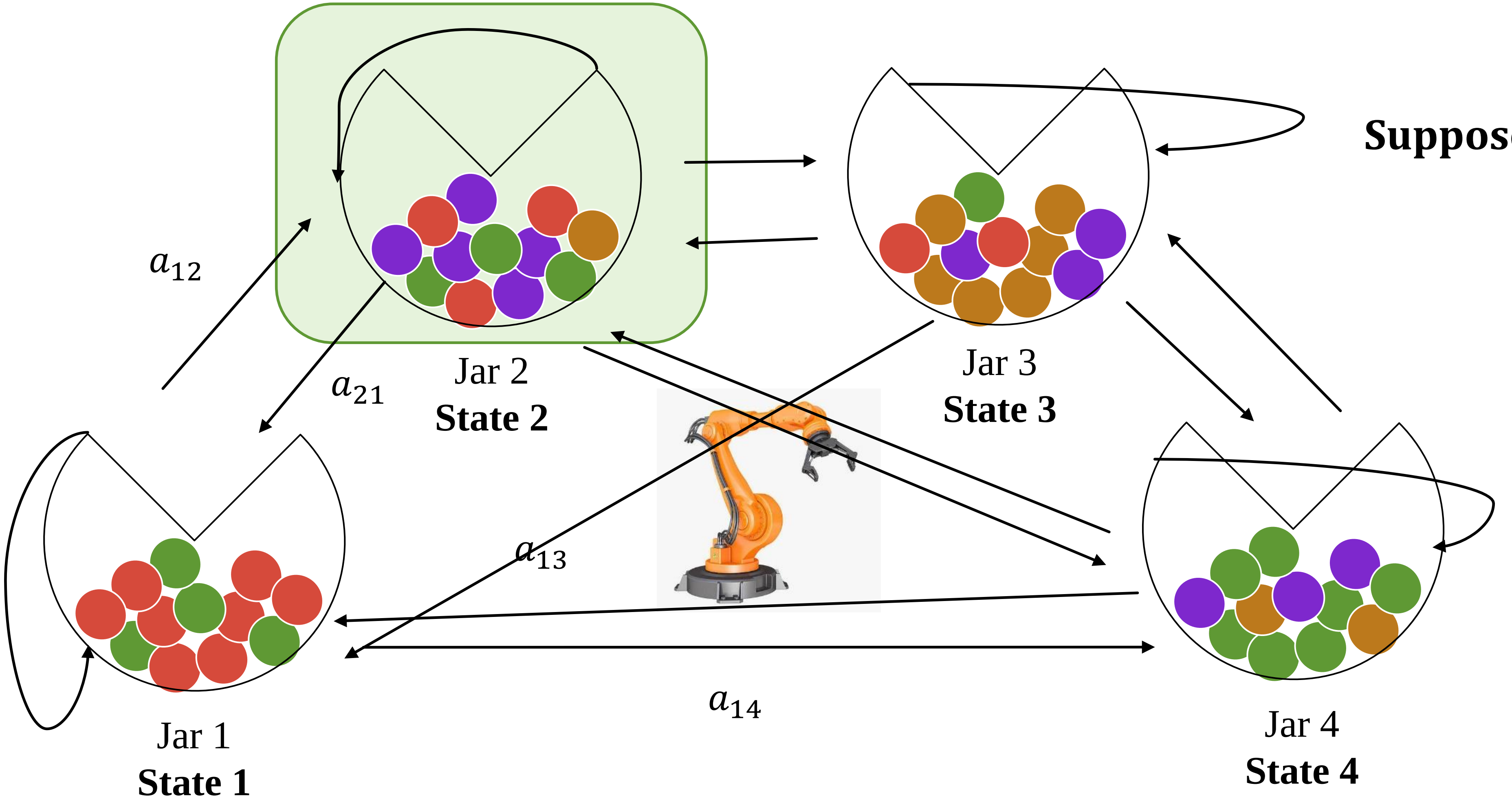
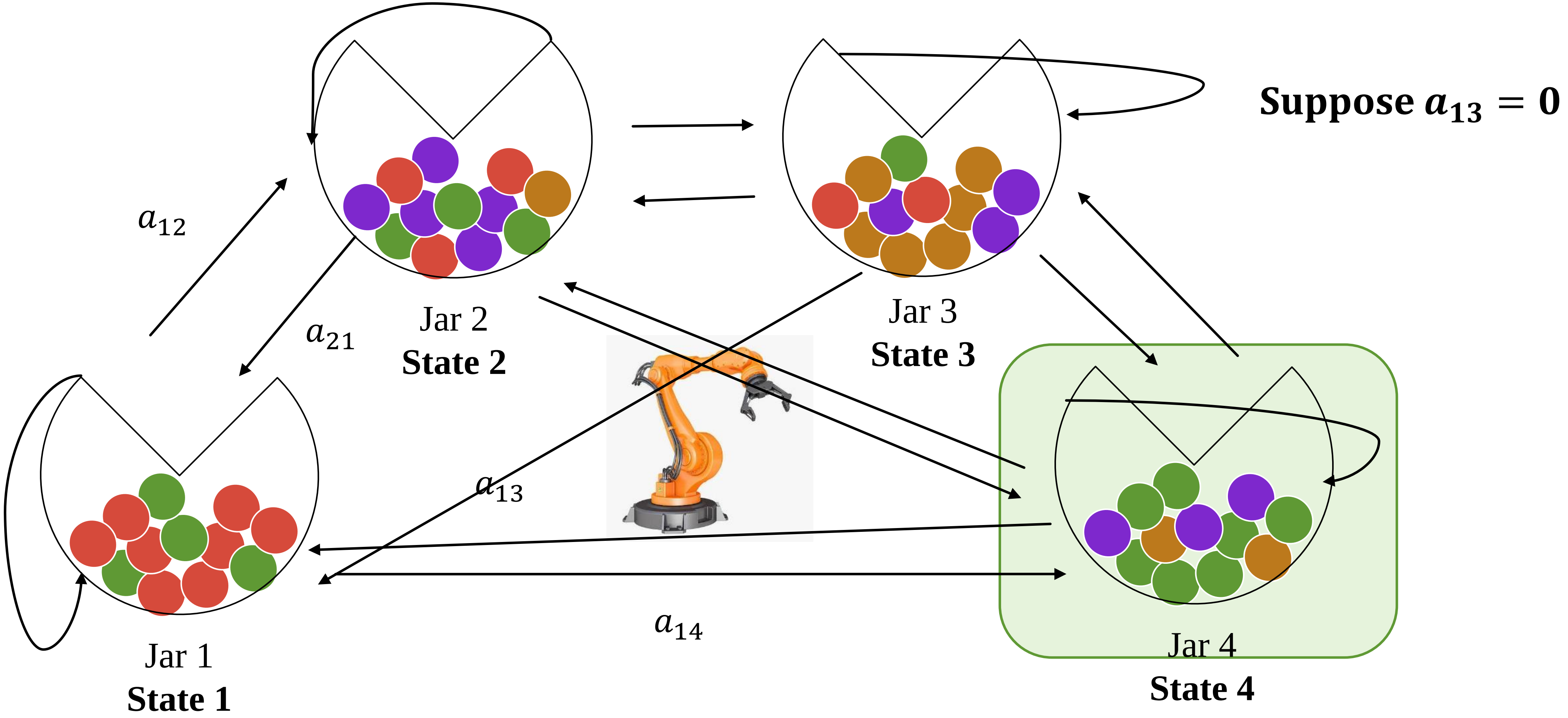


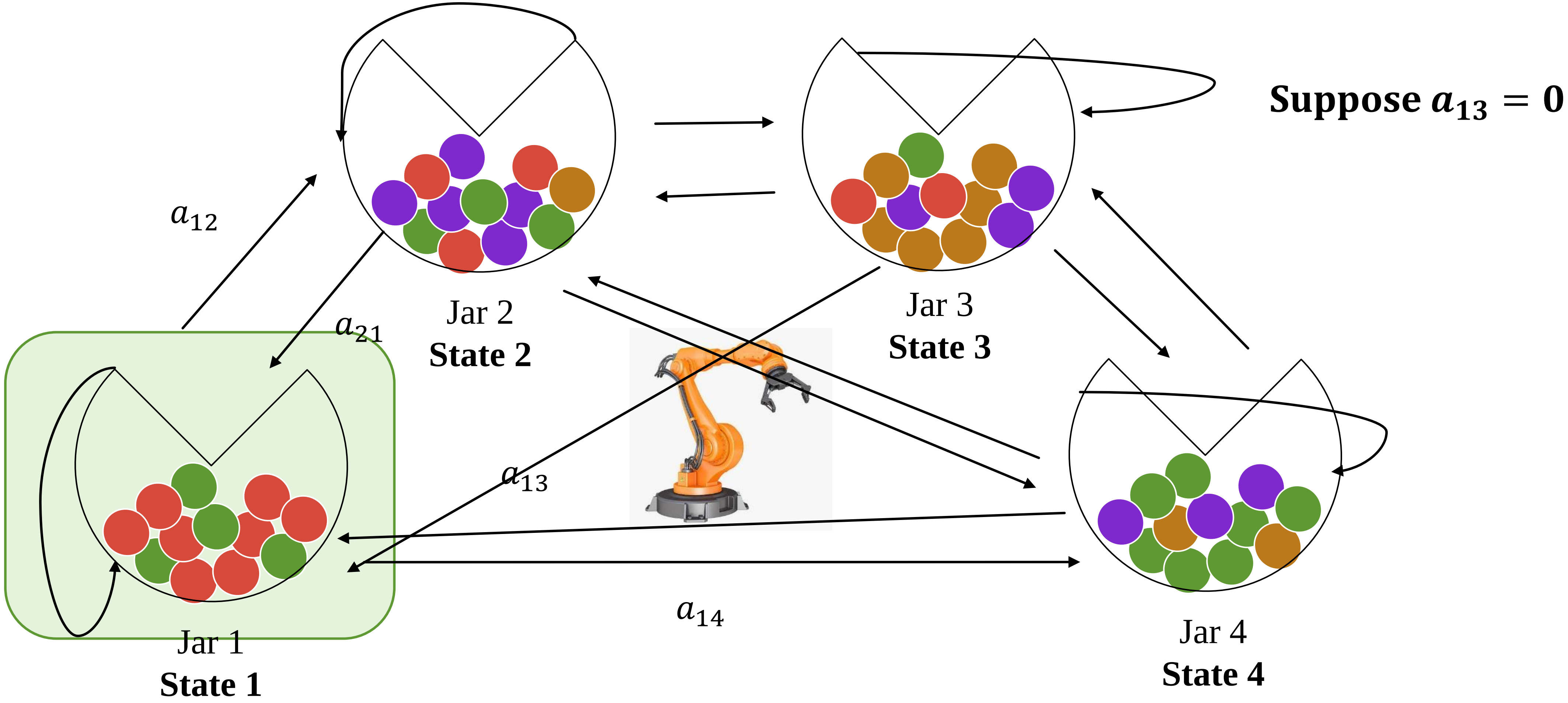
S2,

Problem with This Approach



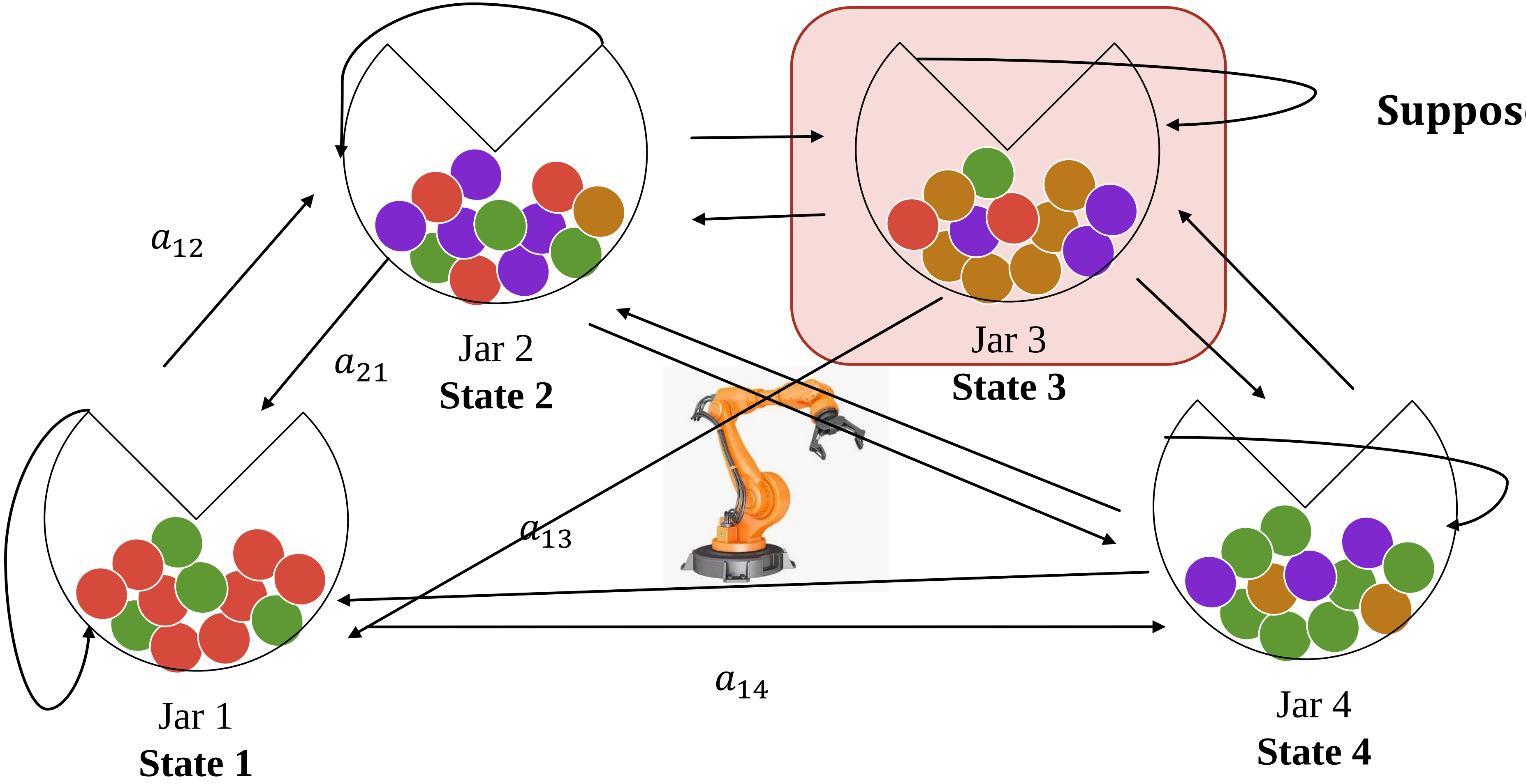
S2, S4

Problem with This Approach



S2, S4, S1

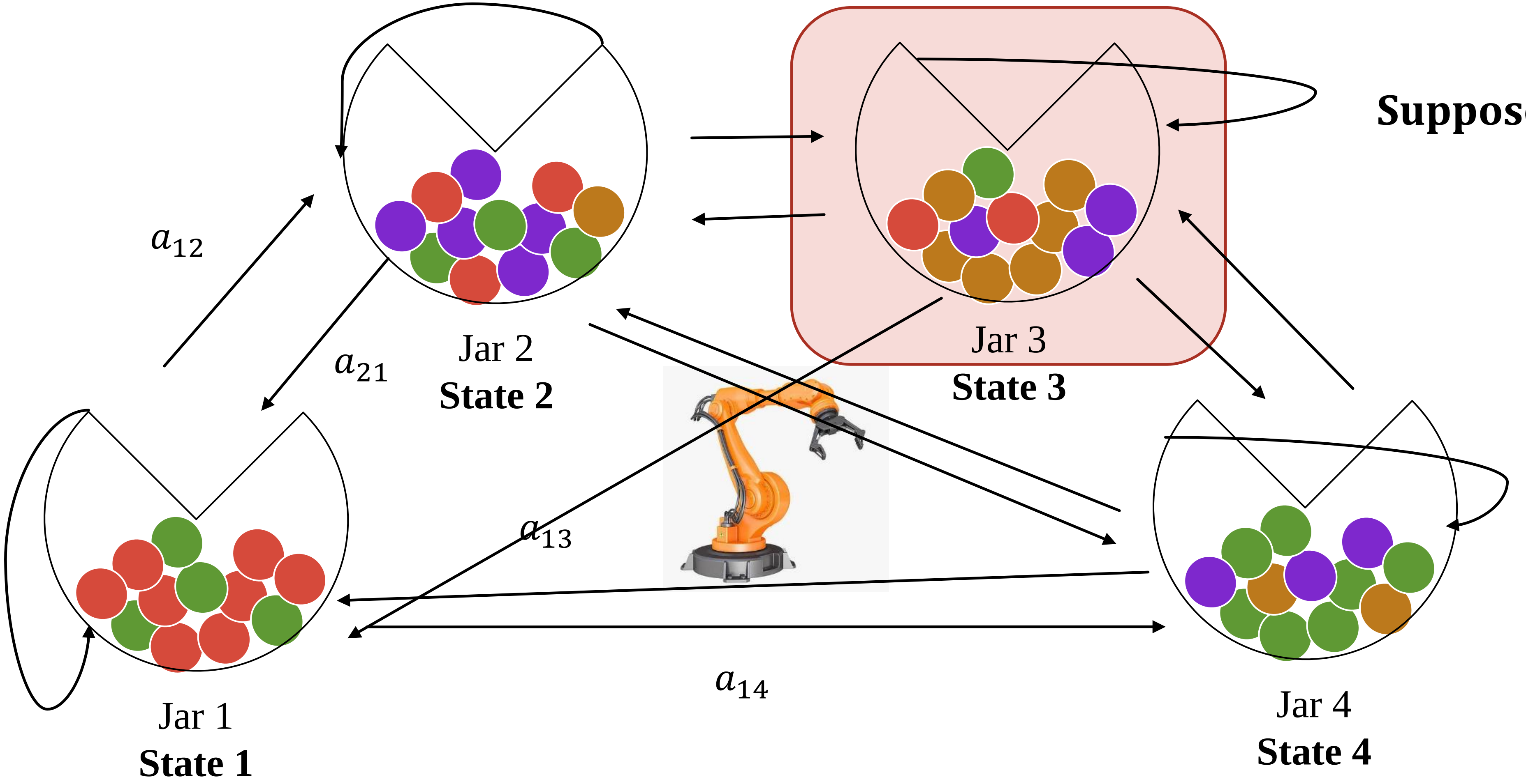
Problem with This Approach



Suppose $a_{13} = 0$

S2, S4, S1, S3

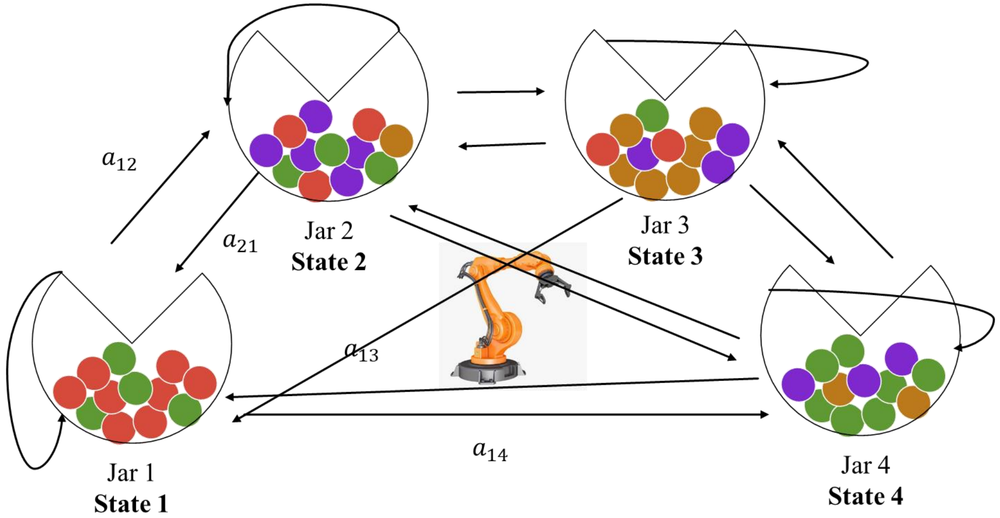
Problem with This Approach



Suppose $a_{13} = 0$

S2, S4, S1, ~~S3~~

Question 2



Choose a path sequence that maximizes

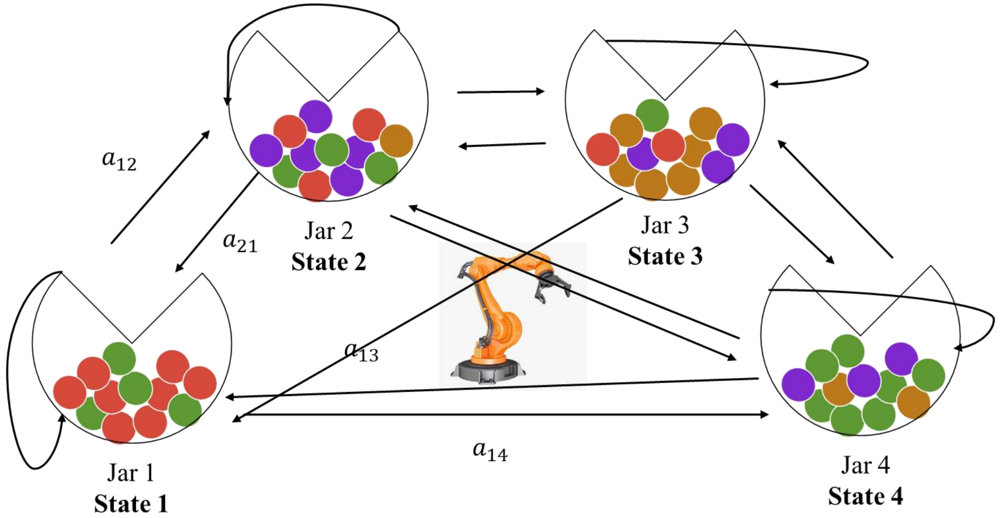
$$P(Q|O,\lambda)$$

- Suppose, we have an observation sequence

$$O = \{O_1, O_2, \ldots, O_T\}$$

- Given the model, what sequence of states **best** explains the above observation?
- What is the meaning of best?

Question 2



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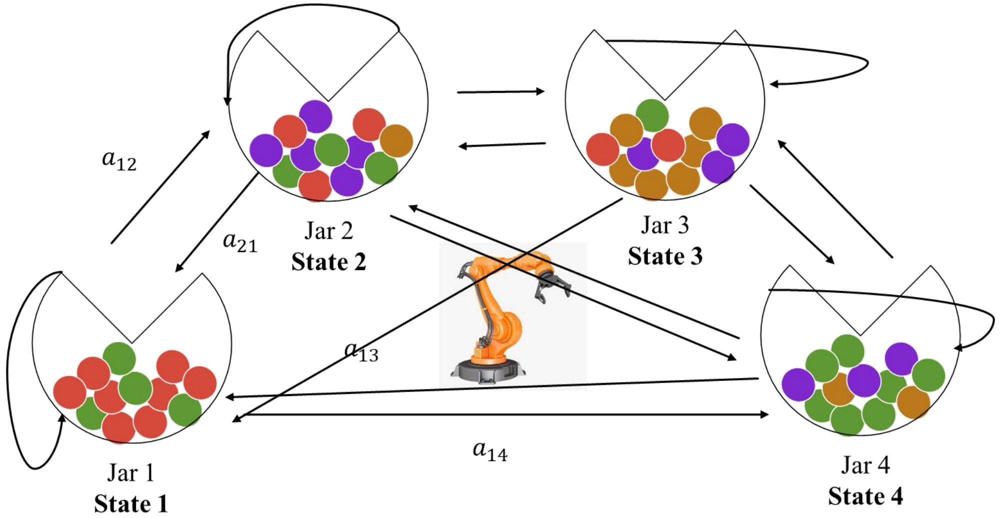
This is equivalent to saying

$$P(Q, O|\lambda)$$

Question 2



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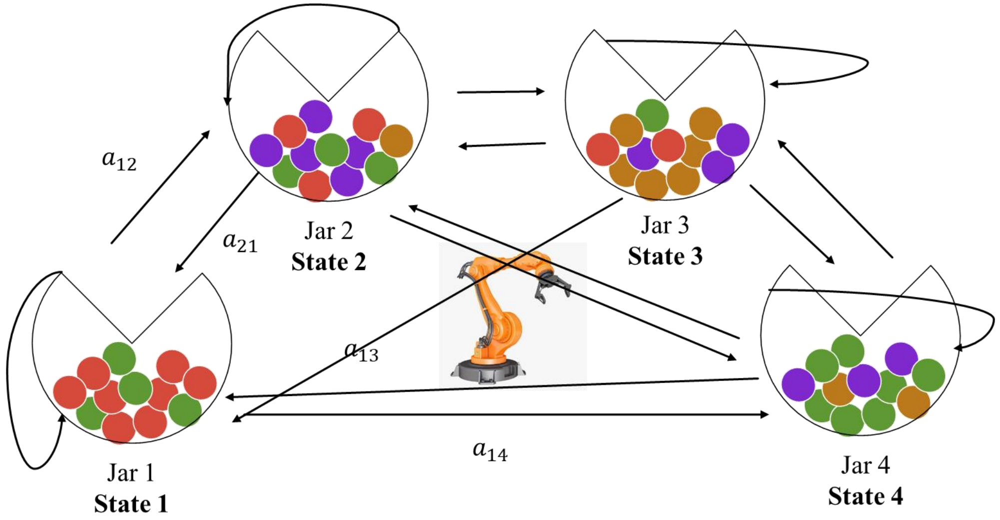
$$\boldsymbol{P(Q,O|\lambda) = P(\{q_1, q_2, \ldots, q_T\}, \{O_1, O_2, \ldots, O_T\}|\lambda)}$$

Solution: Viterbi Algorithm



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Solution: Viterbi Algorithm

$$\boldsymbol{P(Q, O|\lambda) = P(\{q_1, q_2, \ldots, q_T\}, \{O_1, O_2, \ldots, O_T\}|\lambda)}$$

- To solve this, we need a variable

$$\delta_t(j) = \underbrace{max}_{q_1, q_2, \ldots, q_{t-1}} P(\{q_1, q_2, \ldots, q_t = j\}, \{O_1, O_2, \ldots, O_t\}|\lambda)$$

The highest probability of observing the sequence up to time t through a valid path that ends at state S_j

Solution: Viterbi Algorithm

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The highest probability of observing the sequence up to time t through a valid path that ends at state S_j

- Induction step
  - $\delta_{t+1}(j) = [\underbrace{max}_{i} \{\delta_t(i)a_{ij}\}]b_j(O_{t+1})$

Solution: Viterbi Algorithm

- To solve this, we need a variable

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The highest probability of observing the sequence up to time t through a valid path that ends at state S_j

- Induction step
  - $\delta_{t+1}(j) = [\underbrace{max}_{i} \{\delta_t(i)a_{ij}\}]b_j(O_{t+1})$

- We have to keep track of where we came from (to do that, we will use another variable ψ)

Viterbi Algorithm

$$\delta_t(j) = \underbrace{max}_{q_1,q_2,\ldots,q_{t-1}} P(\{q_1, q_2, \ldots, q_t = j\}, \{O_1, O_2, \ldots, O_t\}|\lambda)$$

- Initialization
  - $\delta_1(i)$ =Highest probability of starting at state i and observing O_1

Viterbi Algorithm

$$\delta_t(j) = \underbrace{max}_{q_1,q_2,\dots,q_{t-1}} P(\{q_1, q_2, \dots, q_t = j\}, \{O_1, O_2, \dots, O_t\}|\lambda)$$

- Initialization
  - $\delta_1(i)$ =Highest probability of starting at state i and observing O_1
  - $\delta_1(i) = \pi_i b_i(O_1)$

Viterbi Algorithm

$$\delta_t(j) = \underbrace{max}_{q_1,q_2,\dots,q_{t-1}} P(\{q_1, q_2, \dots, q_t = j\}, \{O_1, O_2, \dots, O_t\}|\lambda)$$

- Initialization
  - $\delta_1(i)$ =Highest probability of starting at state i and observing O_1
  - $\delta_1(i) = \pi_i b_i(O_1)$
  - $\psi_1(i)$ indicates what is the path that we came through
  - $\psi_1(i) = 0$

Viterbi Algorithm

- Initialization
  - $\delta_1(i) = \pi_i b_i(O_1)$
  - $\psi_1(i) = 0$

- Inductive step
  - $\delta_t(j) = [\underbrace{max}_{1 \le i \le N} \{\delta_{t-1}(i) a_{ij}\}] b_j(O_t)$
  - $\psi_t(j) = \underbrace{argmax}_{1 \le i \le N} \{\delta_{t-1}(i) a_{ij}\}$

$2 \leq t \leq T$
$1 \leq j \leq N$

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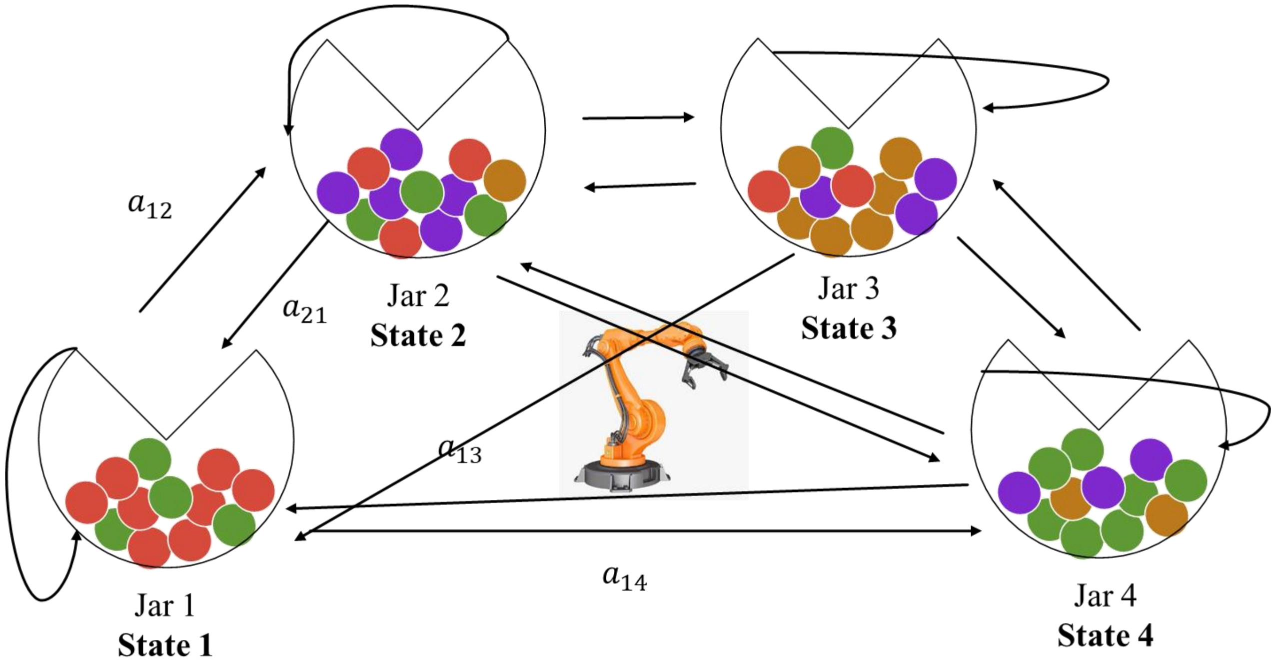

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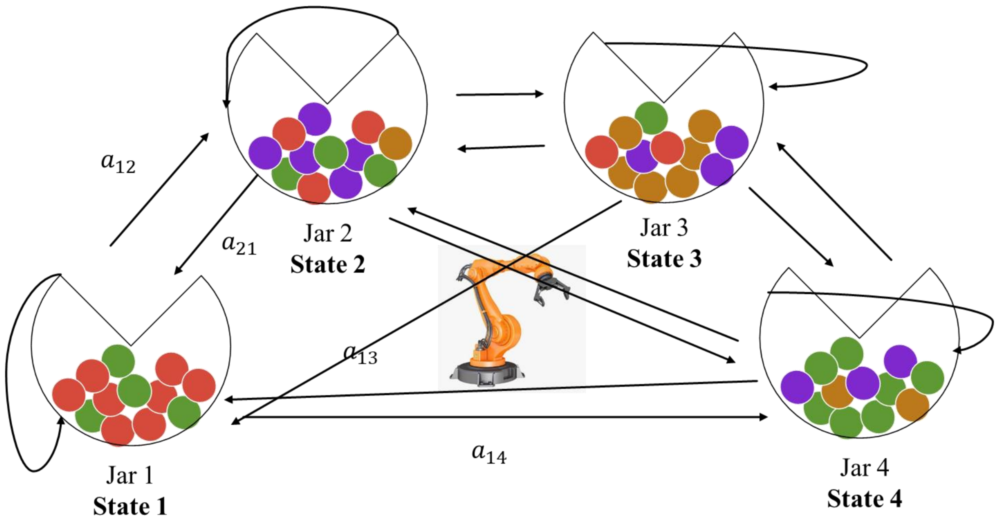
Question 3

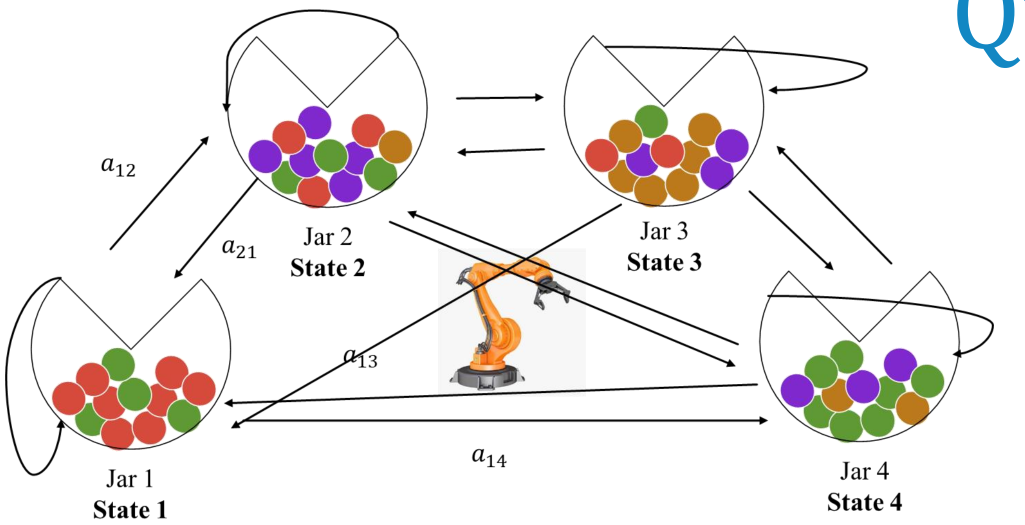


- Suppose, we have an observation sequence

  $$O = \{O_1, O_2, \ldots, O_T\}$$

- How can we find the model parameters $\lambda = \{A, B, \pi\}$ that would generate the above observation?

Question 3



- We have

$$\alpha_t(i) = P(O_1, O_2, \dots O_t, q_t = S_i | \lambda)$$

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- **We can find locally optimal parameters (a good solution), but can't guarantee globally optimal parameters (the best result)**

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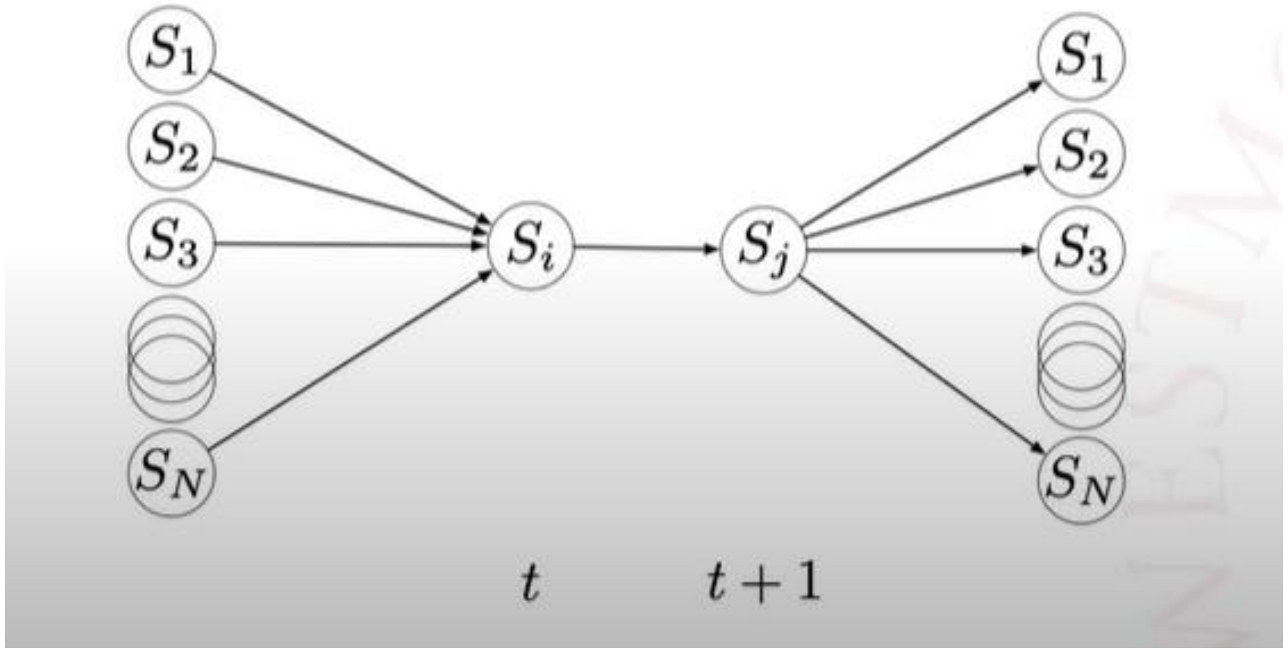

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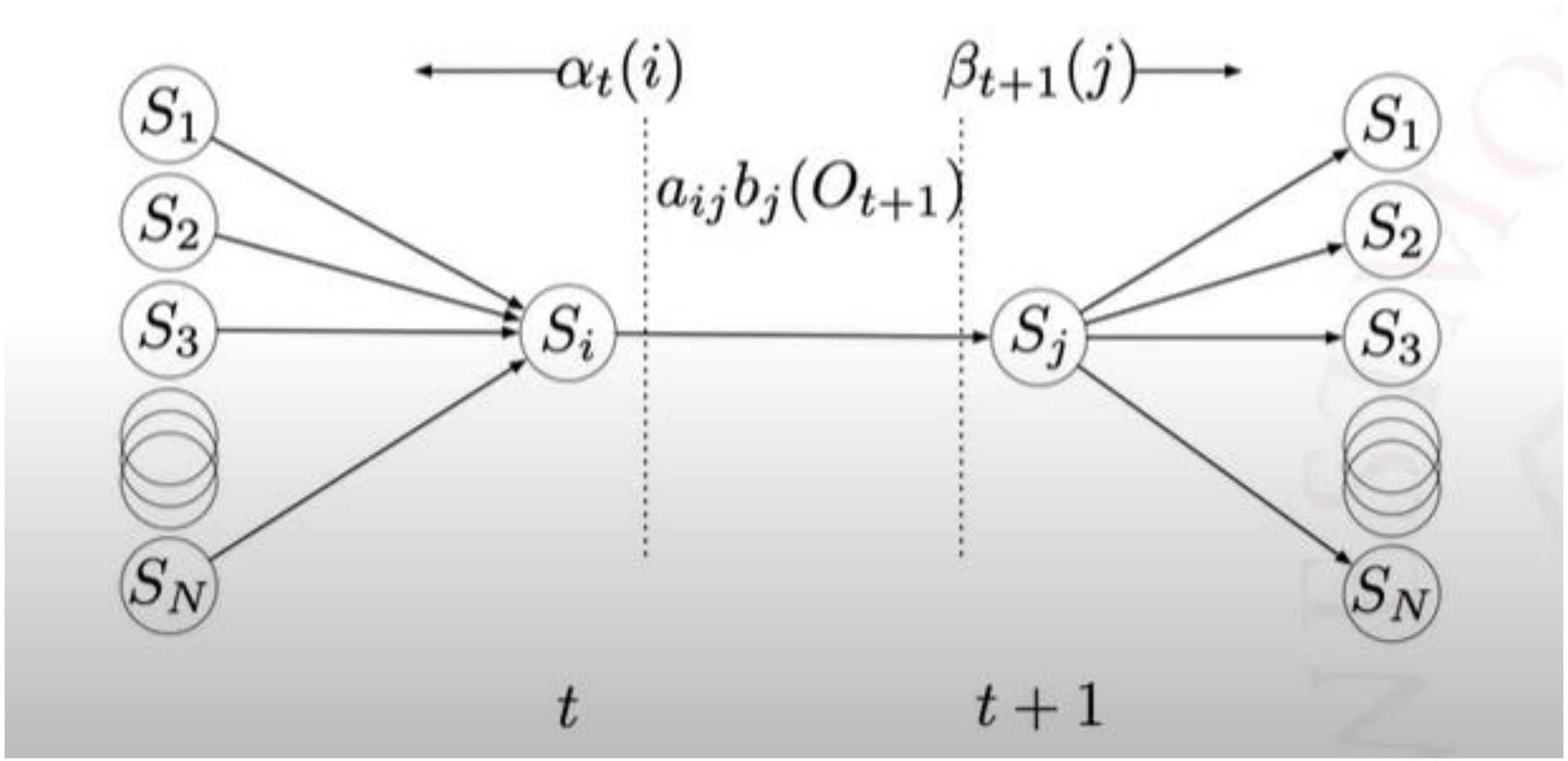

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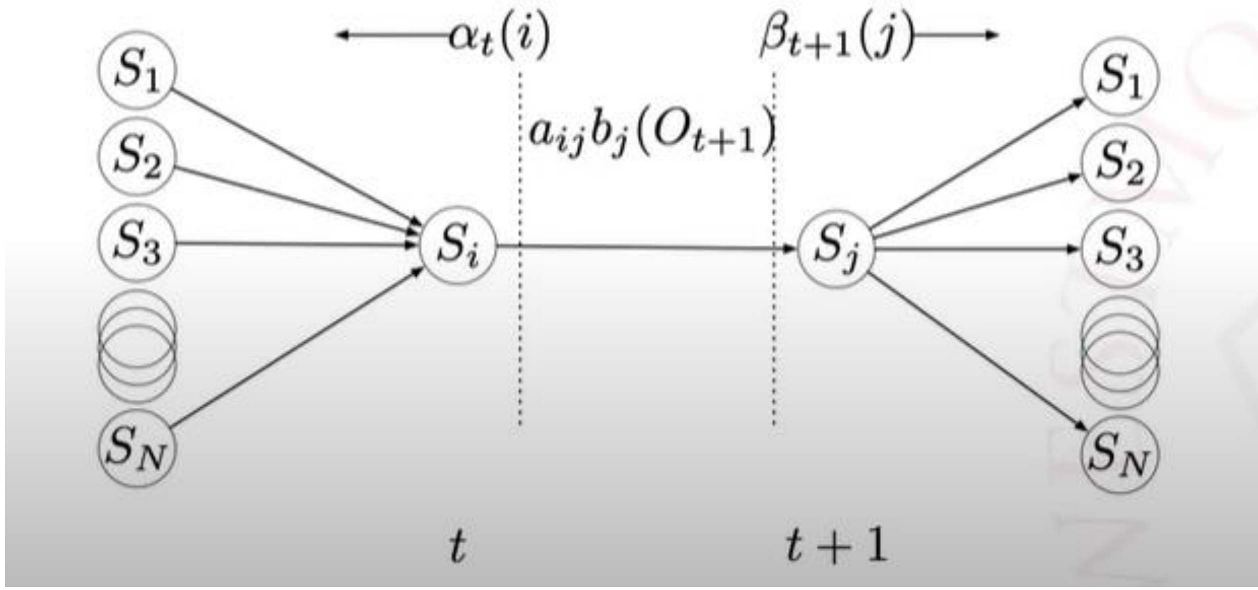

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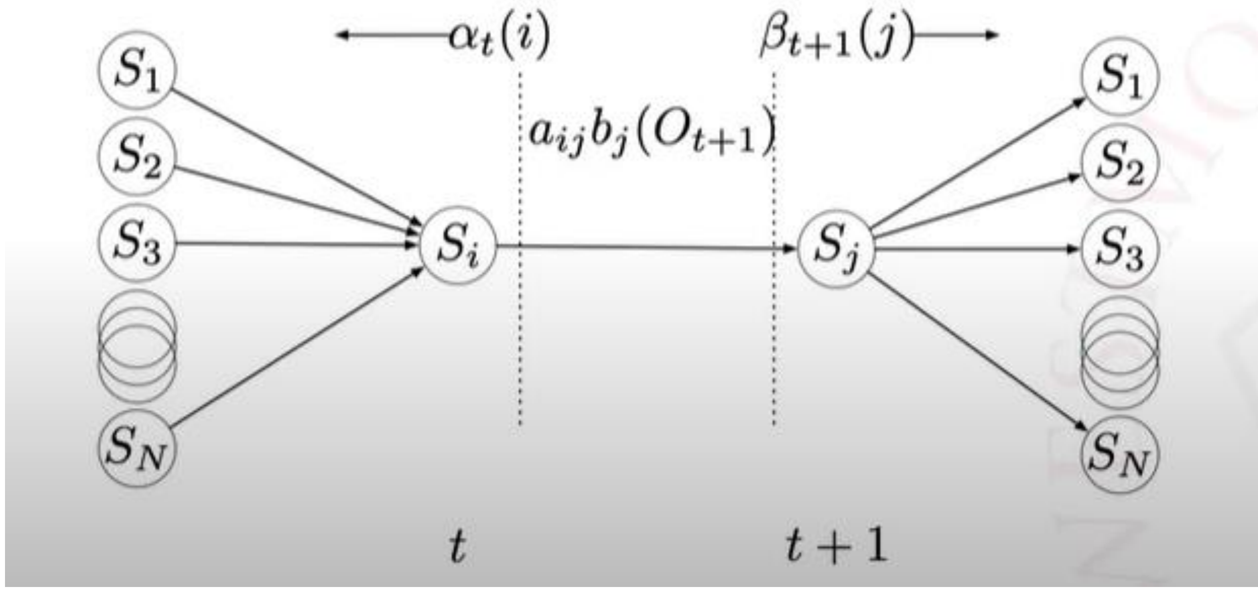

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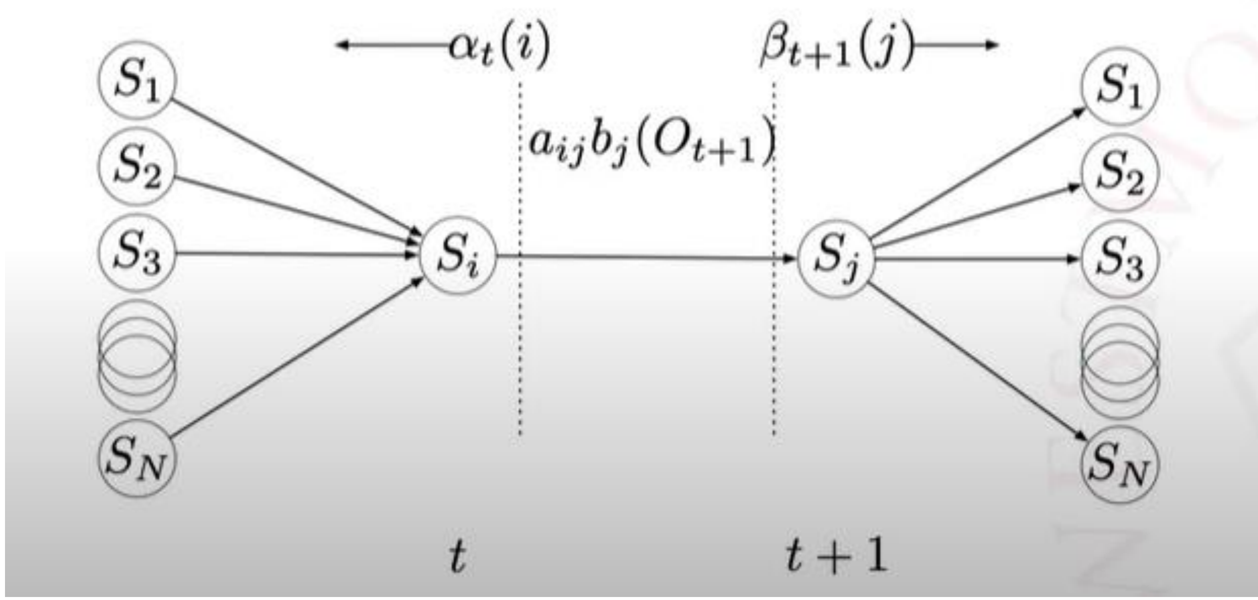

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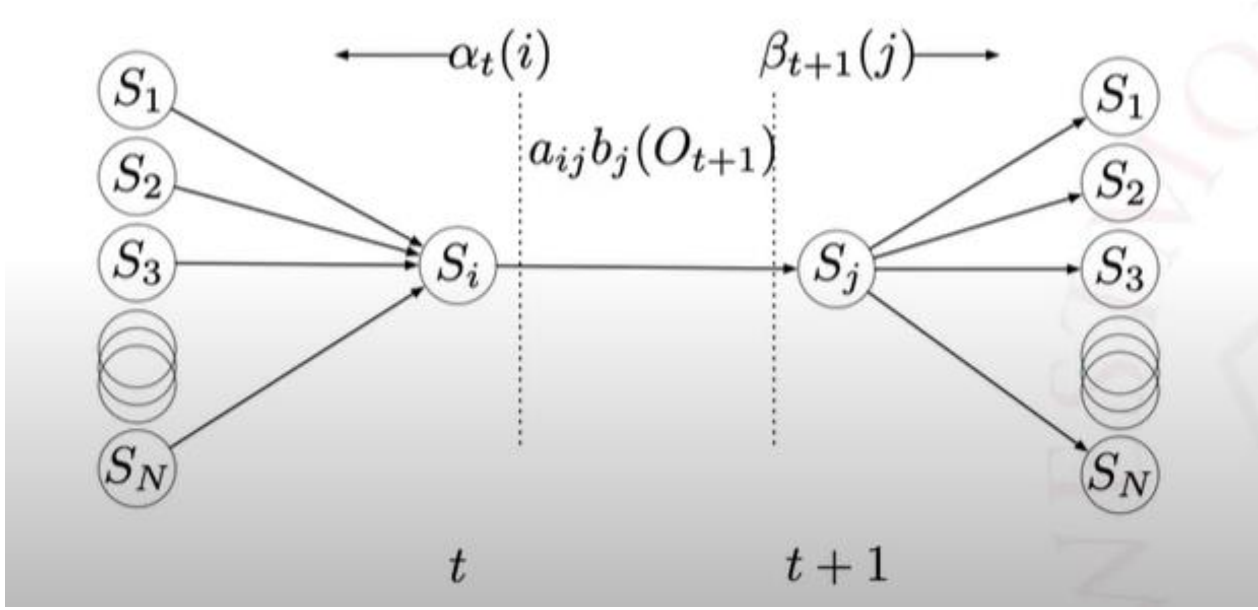

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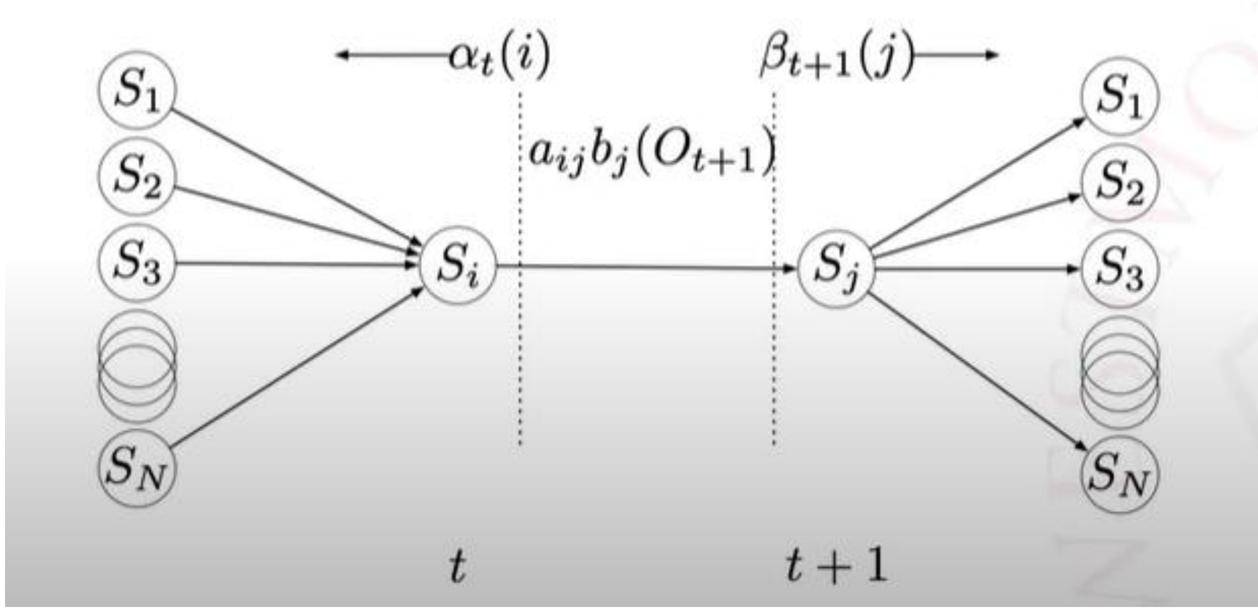

Baum-Welch Algorithm

Randomly initialize A, B, π

Baum-Welch Algorithm

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Use Sequence of observations O

Baum-Welch Algorithm

Randomly initialize A, B, π

Use Sequence of observations O

Calculate $\alpha_t, \beta_t, \gamma_t, \xi_t$

Baum-Welch Algorithm

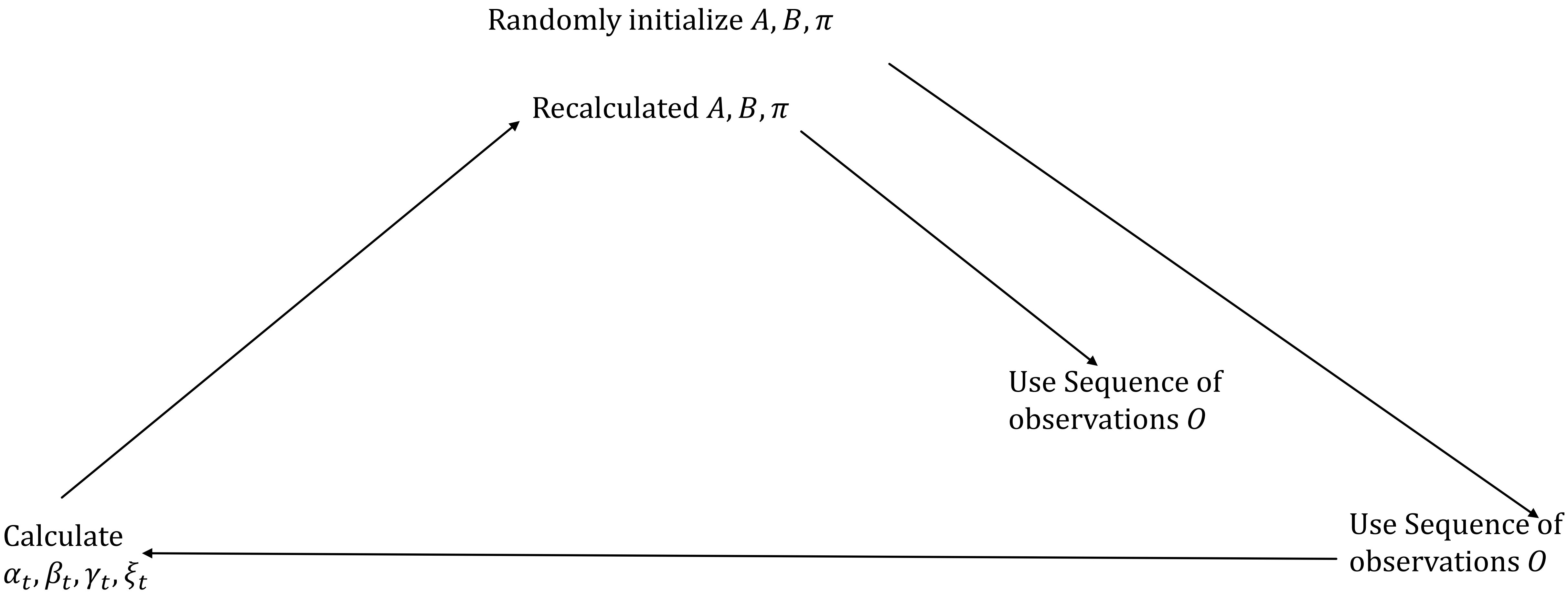
Randomly initialize A, B, π

Recalculated A, B, π

Calculate $\alpha_t, \beta_t, \gamma_t, \xi_t$

Use Sequence of observations O

Baum-Welch Algorithm



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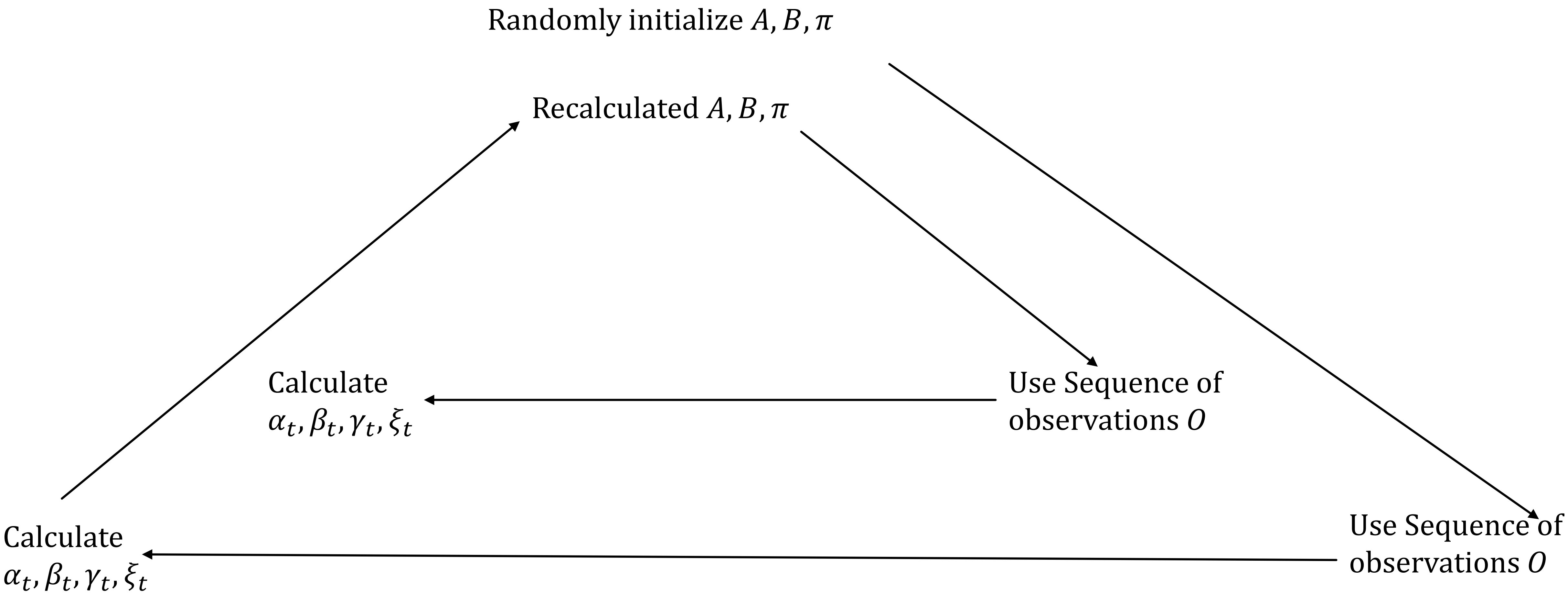

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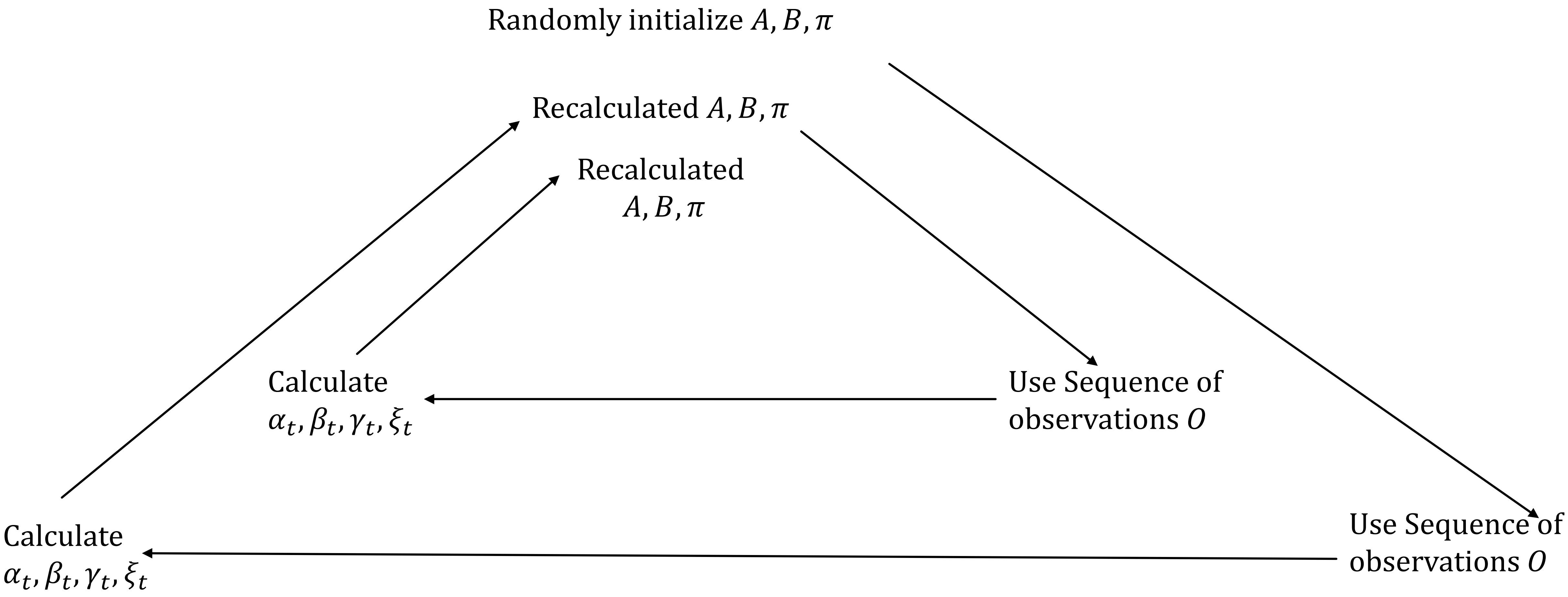

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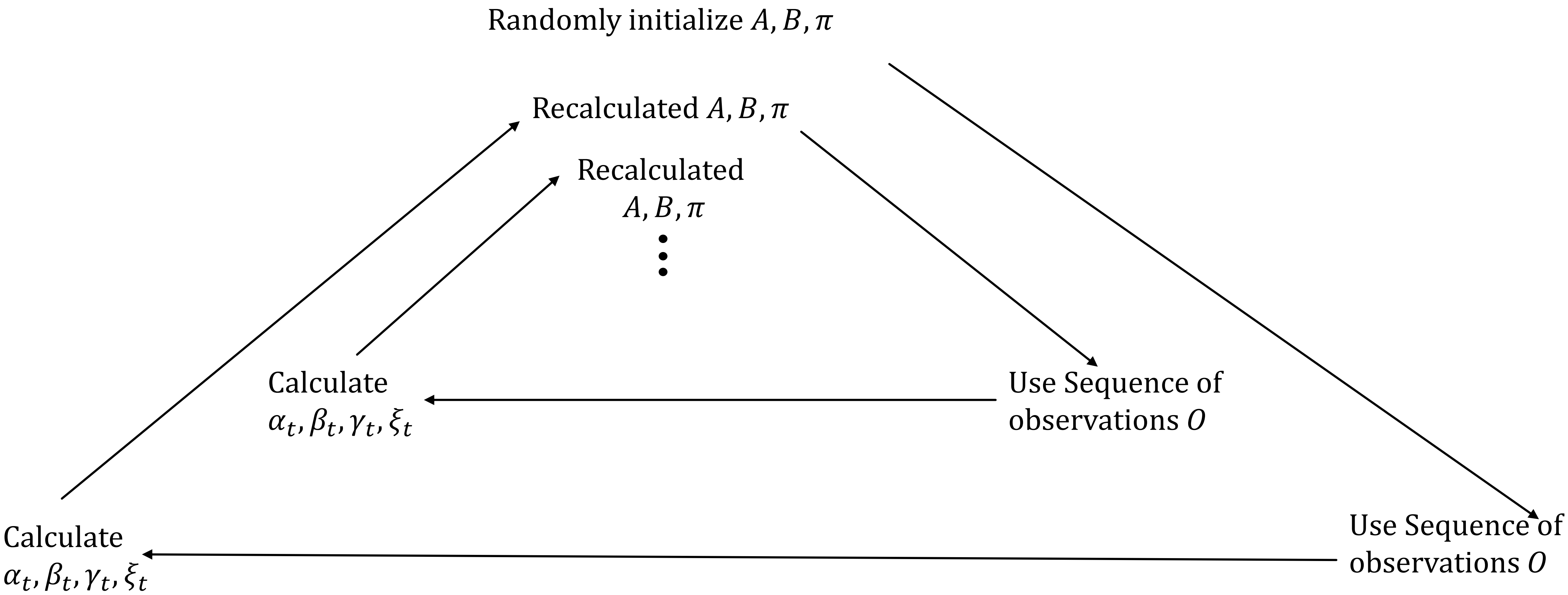

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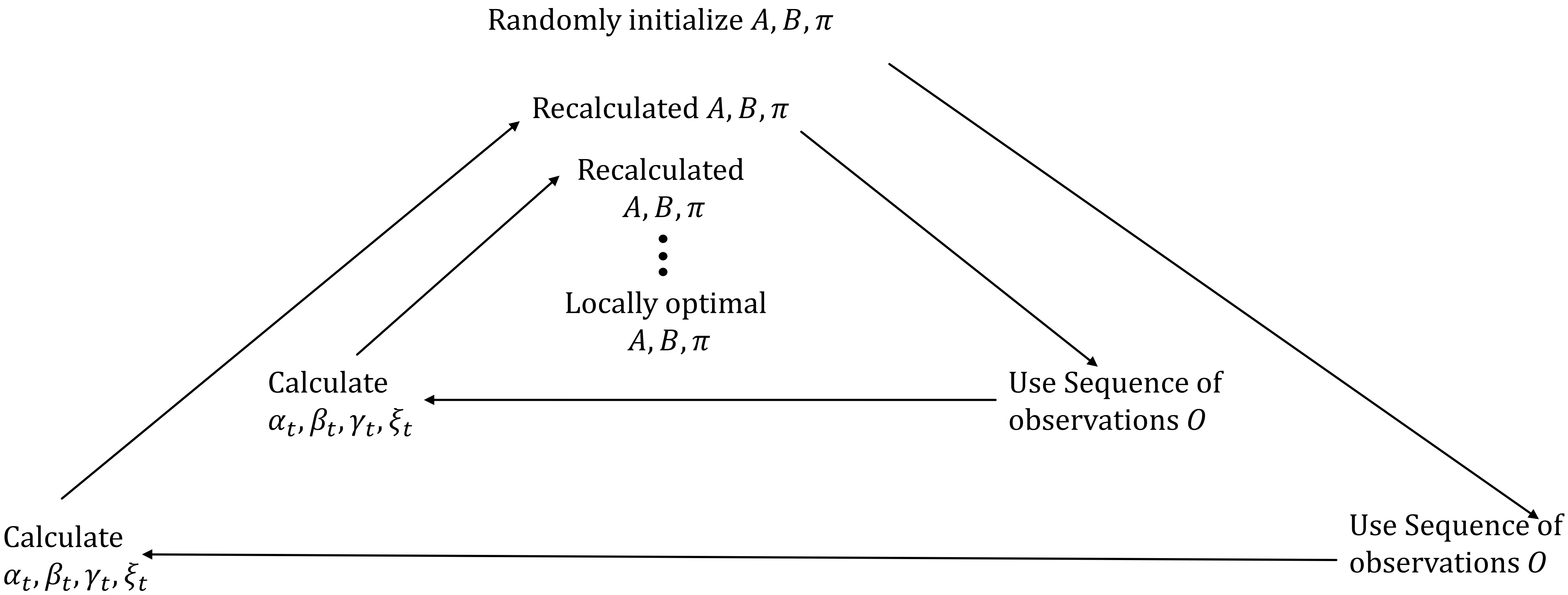

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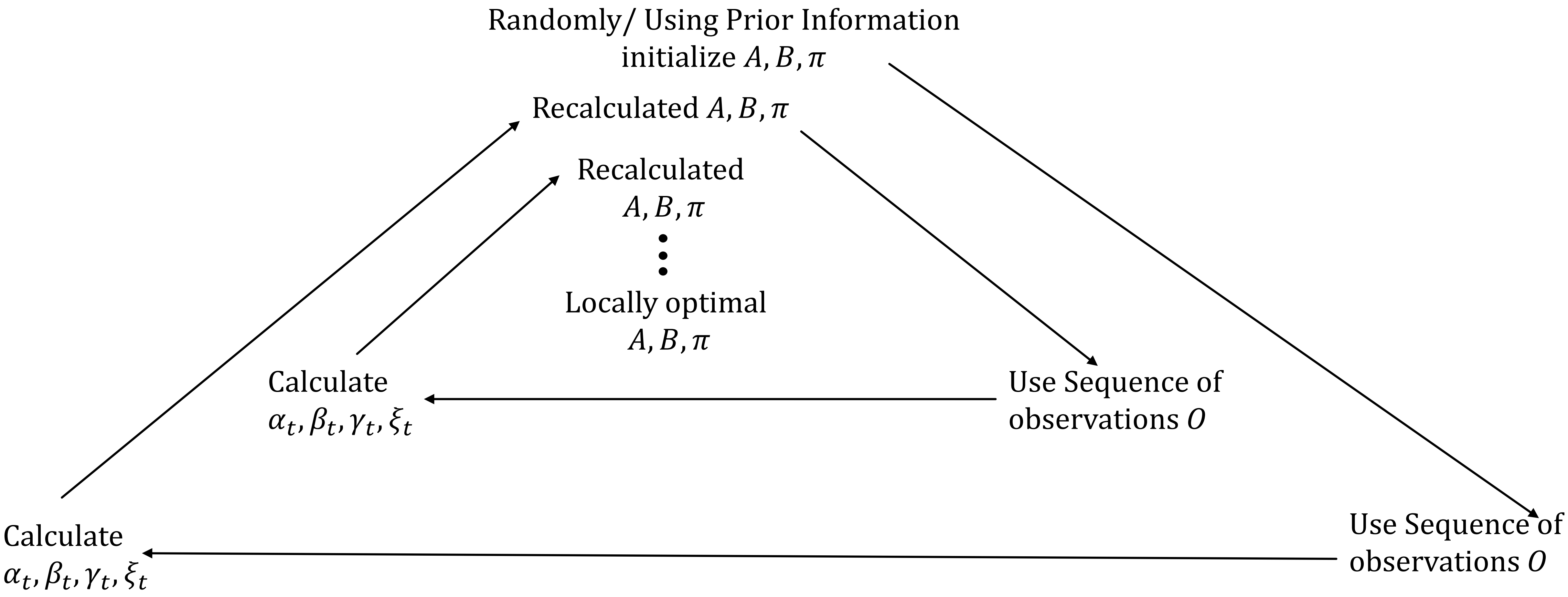
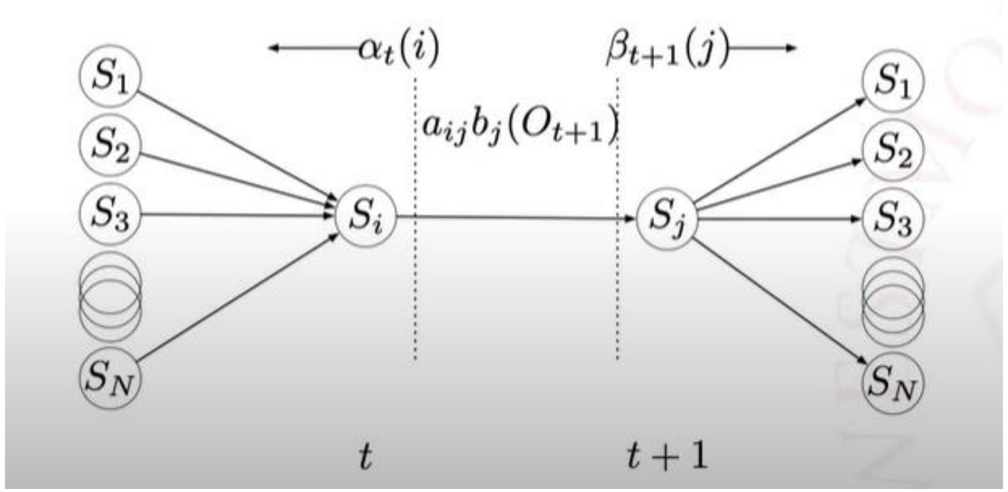


Finding Different Quantities

$$\boldsymbol{\xi_t(i,j) = P(q_t = S_i, q_{t+1} = S_j | O, \lambda)}$$

- What is the probability of state i at time t given the observation

$$\gamma_t(i) = P(q_t = S_i | O, \lambda)$$



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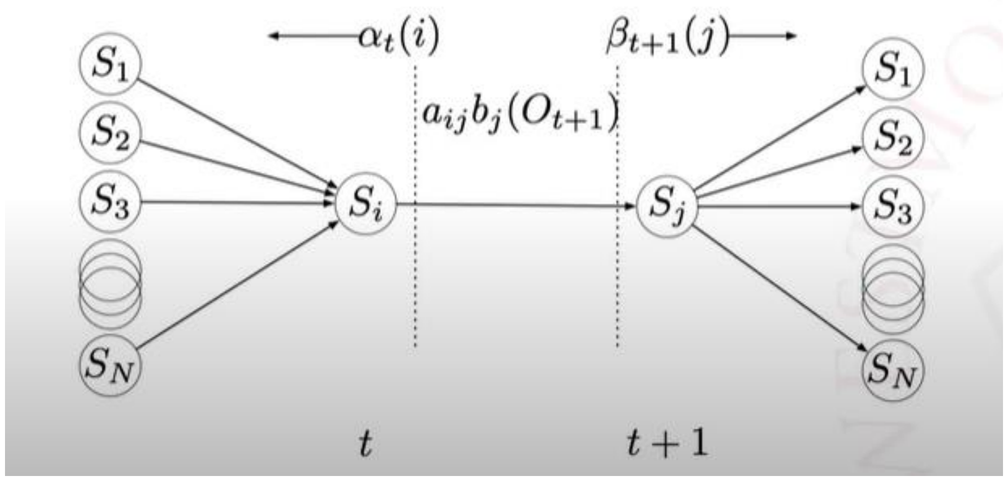
- What is the probability of state i at time t given the observation

$$\gamma_t(i) = P(q_t = S_i | O, \lambda)$$

$$\boldsymbol{\gamma_t(i) = \sum_{j=1}^{N} \xi_t(i,j)}$$



Finding Different Quantities

- Expected number of times S_i is ever visited and we moved out of S_i during our observations

$$\sum_{t=1}^{T-1} \gamma_t(i)$$

Finding Different Quantities

- Expected number of transitions from S_i to S_j during our observations

$$\sum_{t=1}^{T-1} \xi_t(i,j)$$

Finding Different Quantities

- Expected frequency of starting at state S_i

$$\boldsymbol{\gamma_1(i)}$$

Finding Different Quantities

- Expected frequency of starting at state S_i

$$\overline{\boldsymbol{\pi}_{\boldsymbol{i}}} = \boldsymbol{\gamma}_{\mathbf{1}}(\boldsymbol{i})$$

Finding Different Quantities

- Transition probability from S_i to S_j

$$\overline{a_{ij}} = \frac{expected\ number\ of\ transitions\ from\ S_i\ to\ S_j}{expected\ number\ of\ transitions\ from\ S_i}$$

$$= \frac{\sum_{t=1}^{T-1} \xi_t(i,j)}{\sum_{t=1}^{T-1} \gamma_t(i)}$$

Finding Different Quantities

- Observation probability at state S_i

$$\bar{b}_i(k) = \frac{expected\ number\ of\ times\ of\ being\ in\ state\ i\ and\ observing\ v_k}{expected\ number\ of\ times\ of\ being\ in\ state\ i}$$

$$= \frac{\sum^{T}_{\substack{t=1 \\ s.t.\ O_t = v_k}} \gamma_t(i)}{\sum_{t=1}^{T} \gamma_t(i)}$$

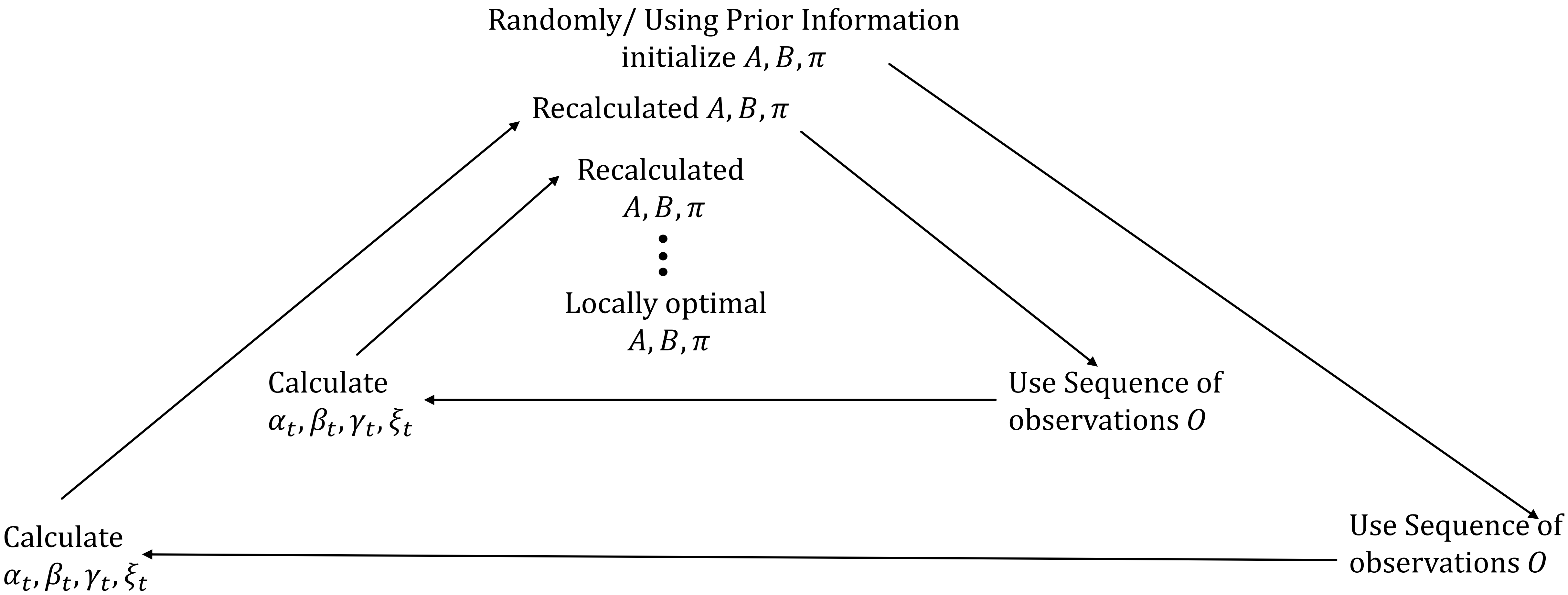
Finding Different Quantities

- So, we got new values $\overline{\boldsymbol{\pi_i}}$, $\overline{a_{ij}}$, $\overline{b}_i(k)$

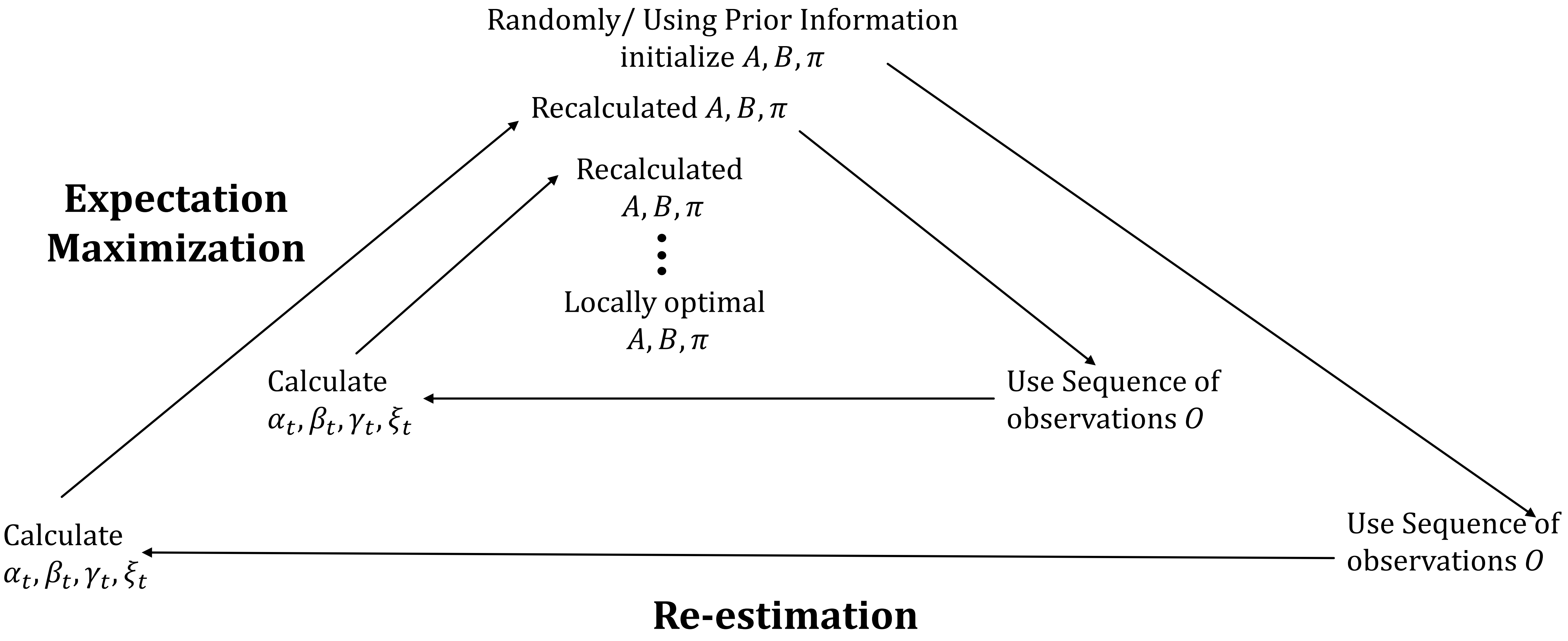
Finding Different Quantities

- So, we got new values $\overline{\boldsymbol{\pi_i}}$, $\overline{a_{ij}}$, $\overline{b}_i(k)$
- **That means we got a new model $\bar{\lambda}$**

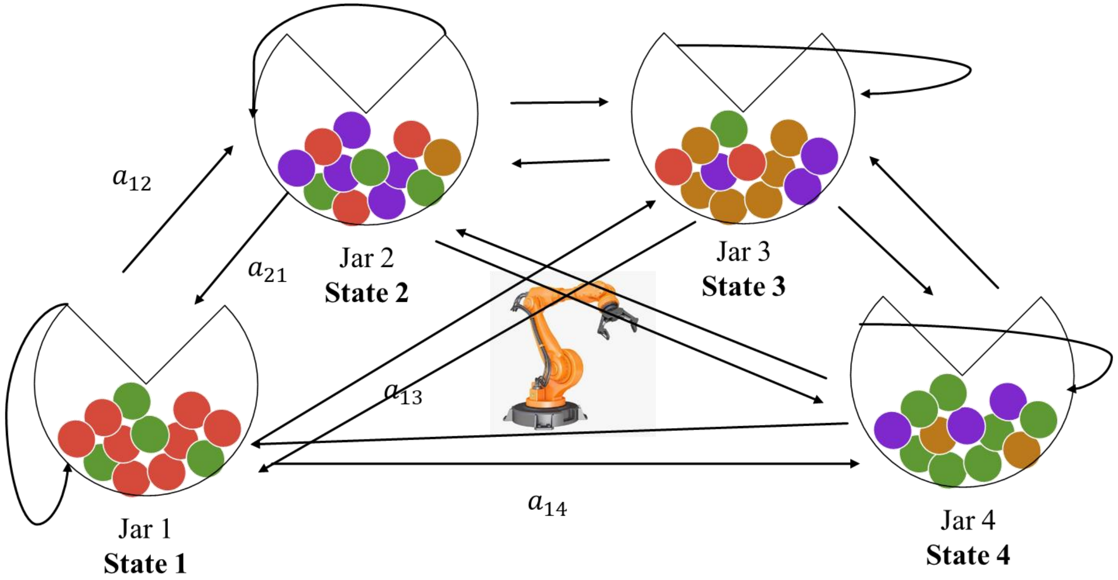
Baum-Welch Algorithm



Baum-Welch Algorithm



What Kind of Questions Can We Answer?



Given the model means given the information $\boldsymbol{\lambda = (A, B, \pi)}$

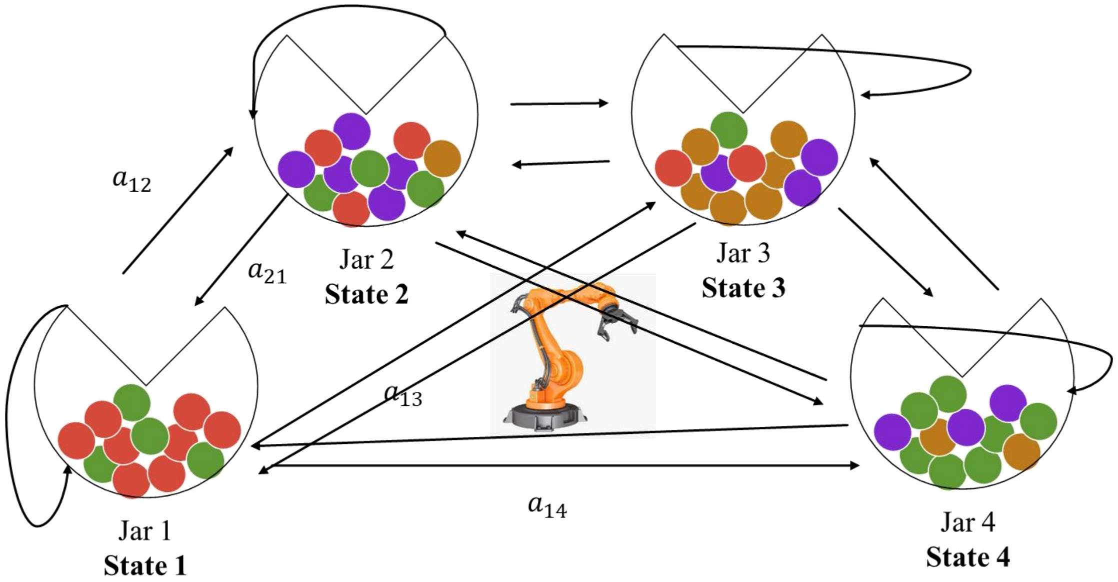
- Suppose, we have a model λ
- What is the probability that this model generates an observation sequence

$$O = \{O_1, O_2, \dots, O_T\}$$

for example, given our model, what is the probability of observing the following sequence

What is $\boldsymbol{P(O|\lambda)}$?

What Kind of Questions Can We Answer?



Given the information $\boldsymbol{\lambda = (A, B, \pi)}$

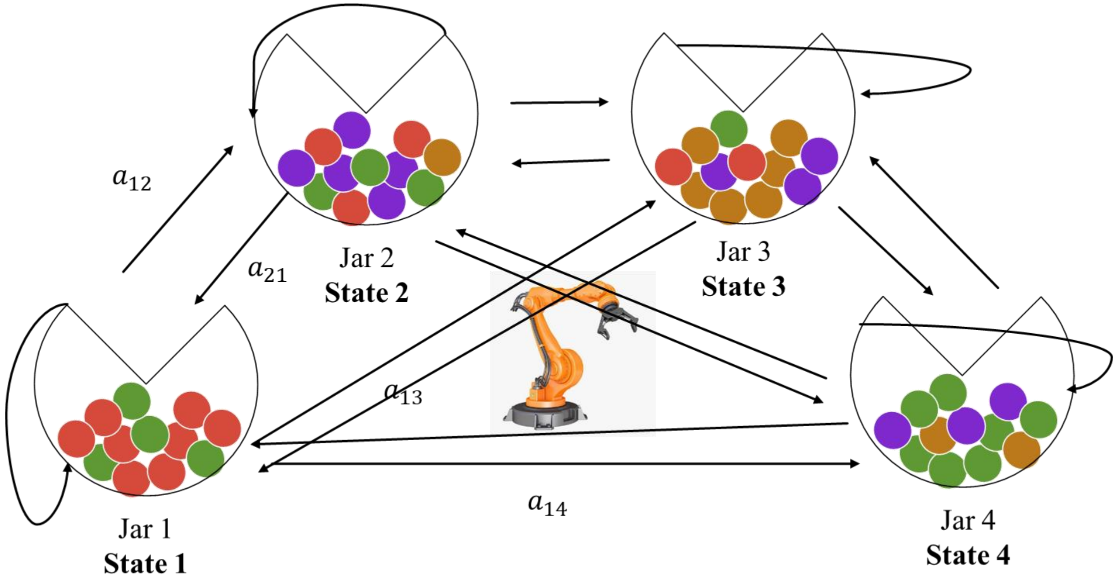
- Suppose, we have an observation sequence

$$O = \{O_1, O_2, \dots, O_T\}$$

- Given the model, what sequence of states best explains the above observation?

What is $\mathbf{Q = \{q_1, q_2, \dots, q_T\}}$?

What Kind of Questions Can We Answer?



Given $\boldsymbol{O} = \{\boldsymbol{O_1}, \boldsymbol{O_2}, \dots, \boldsymbol{O_T}\}$

What is $\boldsymbol{\lambda} = \{\mathbf{A}, \mathbf{B}, \boldsymbol{\pi}\}$?

- Given an observation sequence

  $O = \{O_1, O_2, \dots, O_T\}$

- How to learn the model parameters that will maximize the chance of generating the above sequence?