Probability
Think About It!

David Patrick
Art of Problem Solving

Circle for Teachers

Probability is a measure of how likely an event is to occur. If an event has finitely many *equally likely* outcomes, and some subset of the possible outcomes are considered "successful" outcomes, then

$$P(\text{success}) = \frac{\text{number of successful outcomes}}{\text{number of possible outcomes}}.$$

For example, the probability of rolling a ⚁ on a standard 6-sided die is $\frac{1}{6}$. Probability is always between 0 and 1.

Warmups

1. Suppose that we roll two 6-sided dice. What is the probability that the sum of the dice is 8?
2. Suppose that we roll two 6-sided dice. What is the probability that their product is odd?
3. A bag contains 5 red marbles and 5 blue marbles. If we select two of the marbles at random, what is the probability that we select 1 of each color?
4. Two vertices of a regular octagon are chosen at random. What is the probability that they are adjacent?

Harder problems

11. A Senate committee consists of 6 Republicans, 6 Democrats, and 2 Independents. We choose a 3-person subcommittee at random. What is the probability that the subcommittee consists of:
(a) 3 Republicans?
(b) 3 Democrats?
(c) 1 Republican, 1 Democrat, and 1 Independent?
12. I have 120 blocks. Each block is one of 2 different materials, 3 different colors, 4 different sizes, and 5 different shapes. No two blocks have exactly the same of all four properties. I select two blocks at random. What is the probability that the two blocks:
(a) have no properties in common?
(b) have exactly 2 properties in common?
13. Mary and James each sit in a row of 7 chairs, and they choose their seats at random. What is the probability that they don't sit next to each other?

Thought-provoking problems

21. Jenny goes fishing in a lake which has a lot of fish in it. If she fishes for an hour, there is a 90% chance that she catches a fish. Assuming that the probability of catching a fish is uniform the entire hour, what is the probability that Jenny catches a fish in the first 10 minutes?

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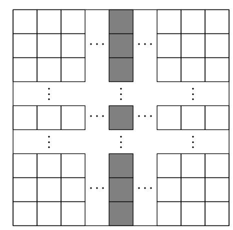
22. Wayne and Mario play a game in which they take turns flipping a fair coin. The first one to flip tails wins. Wayne goes first. What is the probability that Wayne wins?

23. Is a fair pair of dice necessarily a pair of fair dice? That is, can we construct two special dice that, when rolled together, behave the same way as a regular pair of dice?

24. Three friends Xavier, Yul, and Zeb are having a 3-way paintball duel. Xavier is a poor shot, hitting his target with probability $\frac{1}{3}$. Yul is a pretty good shot, hitting his target with probability $\frac{2}{3}$. Zeb is a marksman, always hitting his target. The rules of the duel are as follows: in turn, each person will get one shot at a target of his choice. Once you're hit, you're out of the game. The last person remaining wins. Since he's the worst shot, Xavier gets to go first, followed by Yul, followed by Zeb. What is Xavier's best strategy?

25. A 2011×2011 square consists of $(2011)^2$ unit squares, as shown to the right. The middle square of each row is shaded as indicated. If a rectangle (of any size) from the figure is chosen at random, what is the probability that the rectangle includes a shaded square?



26. I recently took a plane flight on a sold-out plane that had 137 seats. I happened to be the first person to board the plane. Unfortunately, as I was walking down the jetway to get on the plane, I found that I had lost my boarding pass, and I couldn't remember to which seat I had been assigned. Not knowing what else to do, I randomly took any seat. Of course, there's only a $\frac{1}{137}$ probability that I happened to pick my assigned seat. As passengers boarded behind me, naturally they took their assigned seats, unless they happened to find it occupied, in which case they took any available seat at random. When the last passenger boarded, what is the probability that she found her assigned seat unoccupied?

27. You and two other prisoners are in jail. Tomorrow, two of you will be sentenced to life in prison, but the third will be set free; any one of the three of you is equally likely to be the one set free. The jailer is sitting just outside the bars. You ask the jailer if you are the one to be set free.
(a) The jailer says he knows who will be let free, but he won't tell you if you are the one to be freed. He does, however, point at one of the other two and tells you that that person will not be released. What is the probability that you will be set free?
(b) The jailer says that he can't give out that information, but he does have the names of the two to be sentenced to life written on two scraps of paper in his pocket. When the jailer turns away from you, you're able to sneak a hand through the bars and snatch one piece of paper out of his pocket. It doesn't have your name on it. What is the probability that you will be set free?
(c) Are parts (a) and (b) different? Why?

Discussion Topic

Simpson's Paradox—how combining multiple events can lead to apparent contradictions.

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Answers / Instructor's Notes

1. $\frac{5}{36}$, it usually helps to visualize this to have two dice of different colors.
2. $\frac{1}{4}$, you can count odd outcomes but it's easier to realize that each die has to be odd, so it's just $\frac{1}{2} \cdot \frac{1}{2}$.
3. $\frac{5}{9}$, you can count outcomes but easier to see that the second marble has to be a different color than the first; also this makes "2 red" and "2 blue" both easily $\frac{2}{9}$ by symmetry.
4. $\frac{2}{7}$ (similar reasoning to #3 above)

11. (a) is $\frac{6 \cdot 5 \cdot 4}{14 \cdot 13 \cdot 12} = \frac{5}{91}$. (b) is the same as (a) by symmetry. (c) is $\frac{3! \cdot 6 \cdot 6 \cdot 2}{14 \cdot 13 \cdot 12} = \frac{18}{91}$.
12. (a) The second block must be different in all components, so there are $1 \cdot 2 \cdot 3 \cdot 4 = 12$ choices, thus the probability is $\frac{12}{119}$. (b) Casework on which two components are different, so there are 6 cases, total is

$$(1)(2) + (1)(3) + (1)(4) + (2)(3) + (2)(4) + (3)(4) = 2 + 3 + 4 + 6 + 8 + 12 = 35.$$

Probability is $\frac{35}{119}$.

13. They *do* sit next to each other in $2 \cdot 6!$ ways (6! ways with Mary on the left, 6! ways with James on the left). So probability of them sitting next to each other is $\frac{2 \cdot 6!}{7!} = \frac{2}{7}$. Probability of them not next to each other is thus $1 - \frac{2}{7} = \frac{5}{7}$.

21. Answer is *not* (90%)/6 = 15% — very important to be sure one knows why this is incorrect. Doing an absurd counterexample usually helps (e.g. if the probability of a fish in 10 minutes is 20%, does that make the probability of a fish in a hour to be 120%??). Correct method is to let p be the probability we want, and noting that $(1-p)^6 = 0.1$. Solving gives $1 - p = \sqrt[6]{0.1} \approx 68.1\%$, so $p \approx 31.9\%$.
22. Wayne wins right away with probability $\frac{1}{2}$, and Mario wins on his first turn with probability $\frac{1}{4}$ (first two flips must be HT). Otherwise the game starts over again. On any cycle, Wayne has double the chance of winning as Mario. So Wayne's probability of winning the game is $\frac{2}{3}$ (and Mario's is $\frac{1}{3}$). Can discuss why the probability of the game never ending is 0.
23. Can play with this experimentally, and see that it is possible: one die is 1,3,4,5,6,8 and the other is 1,2,2,3,3,4. More systematic method is to use a *generating function*: refactor $(x + x^2 + x^3 + x^4 + x^5 + x^6)^2$ into two factors, each with coefficients that sum to 6. Some algebra can be used to get

$$(x + x^2 + x^3 + x^4 + x^5 + x^6)^2 = (x + x^3 + x^4 + x^5 + x^6 + x^8)(x + 2x^2 + 2x^3 + x^4),$$

and a little more algebra (and some reasoning) shows that this is the only possibility.

24. Best strategy is for Xavier to shoot into the air (that is, to deliberately miss). Then, one of Yul or Zeb will eliminate the other, and Xavier will get a free shot to win. His probability of winning this way is at least $\frac{1}{3}$ (it's actually $\frac{25}{63}$).

25. The height is irrelevant. For the width, the correct computation is

$$\frac{(1006)^2}{\binom{2012}{2}} = \frac{1006 \cdot 1006}{1006 \cdot 2011} = \frac{1006}{2011}.$$

Even easier is that the wherever we pick the first column to be, the second column has to be on the opposite side of the shaded area.

26. Experimenting with a smaller plane is usually helpful. This gives strong evidence that the answer is $\frac{1}{2}$, and indeed this is the answer. To prove this, suppose you were assigned to seat A and the last passenger boarding was assigned to seat B. At some point, one of the first 136 passengers is going to randomly choose either seat A or seat B. It's 50/50 whether seat A or seat B will be chosen. Whichever of A or B that is not chosen during the boarding process will be remaining for the last passenger when she boards the plane. So the probability of seat B remaining at the end of the boarding process is exactly $\frac{1}{2}$.

27. (a) is $\frac{1}{3}$ whereas (b) is $\frac{1}{2}$. (a) gives no additional information whereas (b) does, becuase there was a chance that you would have drawn your own name, but you didn't. This is a very subtle point—it can often be drawn out by using a larger example, say 100 prisoners where you fish 98 names out of the jailer's pocket.

Simpson's Paradox example:

In all of these scenarios, we want to maximize the probability of picking a green ball.

Game #1:
Hat A has 5 green and 6 red
Hat B has 3 green and 4 red
We pick hat A since $\frac{5}{11} > \frac{3}{7}$

Game #2:
Hat C has 6 green and 3 red
Hat D has 9 green and 5 red
We pick hat C since $\frac{6}{9} > \frac{9}{14}$

Game #3:
Combine Hat A and Hat C into Hat E
Combine Hat B and Hat D into Hat F
"Logic" tells us we want Hat E, since Hat A "beats" Hat B and Hat C "beats" Hat D. But in fact...
Hat E has 11 green and 9 red
Hat F has 12 green and 9 red
Clearly Hat F is better!