LIMITS AND CONTINUITIES OF FUNCTION

An Informal Definition of Limit

The limit of the function $f(x)$ as x approaches $'a'$ without actually reaching it is L, denoted by $\lim_{x\to a} f(x) = \text{L}$.

METHODS USED TO EVALUATE/FIND LIMITS

To evaluate limit of the given function one can simply use one of the following methods:

- Direct Substitution method
- Factoring and cancelling method
- Rationalizing method
- L'Hôpital's rule

Direct Substitution method

Direct substitution method is used when the function is defined at the value $'a'$. Normally this works for all continuous function at $'a'$.

Factoring and cancelling method

This method is used when the function involves algebraic expression that can be factored out and cancel out the common factor(s) to simplify the expression before evaluating the limit.

Rationalizing method

The method is used when the function is in the radical form and is not defined $'a'$.

L'Hôpital's rule

The L'Hôpital's rule is used when we want to evaluate limits involving indeterminate forms, such as $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0-\infty$ and $\infty-\infty$.

The L'Hôpital's rule is given by $\lim_{x\to a}\left[\frac{f(\text{x})}{g(x)}\right] = \frac{\lim_{x\to a} f'(x)}{\lim_{x\to a} g'(x)}$, where $g(x) \neq 0$ and $g'(x) \neq 0$. The L'Hôpital's method need as to differentiate the numerator and denominator separately and then evaluate the limit of their ratio

Example

Evaluate each of the following:

(i) $\lim_{x \to 3}(x^2 - 3x + 2)$

(ii) $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$

(iii) $\lim_{h \to 0} \frac{\sqrt{4 + h} - 2}{h}$

(iv) $\lim_{x \to 0^+} \frac{\sin x}{\sqrt{x}}$

Practice:

Find the limit of each of the following:

1. $\lim_{x \to -1} \frac{1}{2x - 5}$
2. $\lim_{x \to 4} \frac{\sqrt{x} - 2}{4 - x}$
3. $\lim_{h \to 0} \frac{(2 + \mathrm{h})^4 - 16}{h}$
4. Evaluate $\lim_{x \to 0} \left[\frac{e^{5x} - 5x - 1}{\sin 4x \sin 3x} \right]$

Show that:

1. $\lim_{x \to 0} \frac{\sin x}{x} = 1$
2. $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$
3. $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$
4. $\lim_{x \to 1} \frac{x - 1}{\ln x} = 1$

EXISTANCE OF LIMIT

The limit is said to exist if and only if $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = \mathrm{L}$.

In other words:

If $\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$, then the limit does not exist.

The limit $\lim_{x \to a^+} f(x)$ is a one-sided limit and is read as:

- The limit $f(x)$ approaches $'a'$ from right hand side.
- The symbol $x \to a^{+}$ is mathematically captured by the sign $x \geq 0$.

The limit $\lim_{x \to a^{-}} f(x)$ is also a one-sided limit and is read as:

- The limit $f(x)$ approaches $'a'$ from left hand side.
- The symbol $x \to a^{-}$ is mathematically captured by the sign $\backslash x \leq 0$.

Example:

Determine whether the $\lim_{x \to 0} f(x)$ exist, where $f(x) = \begin{cases} x^2 + 2\cos x + 1 \text{ for } x < 0 \\ \text{Sec } x - 4 \text{ for } x \geq 0 \end{cases}$

Practice:

1. **Let** $f(x) = \begin{cases} 3x - 1, & x < 0 \\ 0, & x = 0 \\ 2x + 5, & x > 0 \end{cases}$ evaluate:

   a) $\lim_{x \to 0^{+}} f(x)$

   b) $\lim_{x \to 0^{-}} f(x)$

   c) State whether $\lim_{x \to 0} f(x)$ exist

   d) $\lim_{x \to -3} f(x)$

   e) $\lim_{x \to 2} f(x)$

CONTINUITY OF A FUNCTION

A function f is continuous at $x = a$ if and only if:

(i) $f(\text{a})$ is defined.

(ii) $\lim_{x \to a} f(x)$ exist.

(iii) $\lim_{x \to a} f(x) = f(a)$

Example

Determine whether $f(x) = \begin{cases} 3x - 2 & \text{if } x < 3 \\ x^4 - 74 & \text{if } x \geq 3 \end{cases}$ is continuous at $x = 3$.

Practice:

1. Find the value of c such that the function $f(x)=\begin{cases} x+2 & x\le -1 \\ c+2 & x>-1 \end{cases}$ is continuous at $x=-1$.
2. Is the function $f(x)=\begin{cases} 2x & \text{for } x=0 \\ \mathrm{x}^2+3 & \text{for } x\neq 0 \end{cases}$ continuous at $x=0$? Justify your answer.
3. Find the value of c so that $f(x)$ is continuous, where $f(x)=\begin{cases} cx^2+2x, & x\le 2 \\ x^3-cx, & x>2 \end{cases}$
4. The function $f(x)=\begin{cases} x^3 & \text{if } x<\text{-}1 \\ ax+b & \text{if } \text{-}1\le x<1 \\ x^2+2 & \text{if } x\ge 1 \end{cases}$ is continuous for all x. Find the values of a and b.

LIMITS AT INFINITY

The limit of the function $f(x)$ approaches the limit L as x approaches the limit positive or negative infinity and is denoted as $\lim_{x\to\infty} f(x)$. As x become very large, then $\frac{1}{x}=0$

Example

Evaluate $\lim_{x\to\infty}\frac{3x^2-x-2}{5x^2+4x+1}$.

Practice:

1. Evaluate $\lim_{x\to\infty}\frac{2x^4-3x^2+1}{6x^4+x^3-3x}$
2. $\lim_{x\to\infty}\sqrt{x^2+x}-x$