SERIES SOLUTION OF ODEs

A 2nd order linear DEs with constant coefficients admit a solution in the form of an exponential function.
This is because the derivatives of exponential functions are also exponential function, the DE reduces to an algebraic equation.

→ If the coefficients of DEs are not constants, the solution is not of exponential form.

→ Here we develop methods of solving ODEs with variable coefficients.

Consider the 2nd order ODE below

$$a_2(x)\,y'' + a_1(x)\,y' + a_0(x)\,y = a(x) \qquad \text{——— (i)}$$

Where $a_2(x) \neq 0$

Eqn (1) can be written as

$$y'' + \frac{a_1(x)}{a_2(x)}\,y' + \frac{a_0(x)}{a_2(x)}\,y = \frac{a(x)}{a_2(x)}$$

Let $\frac{a_1(x)}{a_2(x)}$ be P_1, $\frac{a_0}{a_2}$ be P_2 and $\frac{a}{a_2}$ be b.

then we have

$$y'' + P_1 y' + P_2 y = b. \qquad \text{——— (ii)}$$

Note

If $b = 0$, then eqn(ii) is homogeneous DE, its solution is given by $y = y_h + y_p$

for y_h — homogeneous solution

y_p — particular Integral

The particular integral is usually obtained by the method of variation of parameters. (Assignments).

⇒ The solution of the homogeneous DEs in the neighbourhood of a point x_0 is assumed to be given by a power series in $(x - x_0)$.

⇒ The form of this series depends on the nature of the point x_0

⇒ The point x_0 is an <u>ordinary point</u> if both P_1 and P_2 are analytic at the point x_0

<u>Note</u>

(i) The function $f(x)$ is analytic at x_0 if its Taylor series about x_0 exists.

(ii) That is to say $f(x)$ can be represented by the series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \quad \text{for } a_n = \frac{f^{(n)}(x_0)}{n!}$$

⇒ If one or both coefficients P_1 & P_2 are not analytic at x_0, then x_0 is a <u>singular point</u>

<u>Note</u>

From eqn $P_1 = \frac{a_1}{a_2}$ and $P_2 = \frac{a_0}{a_2}$, if $a_2(x_0) = 0$ and either a_1 and a_0 are non-zero, then x_0 is a singular point

<u>Example</u>

Analyze the following equations for ordinary and singular points:

(i) $y'' + xy' + (x^2 - 4)y = 0$

(ii) $(x-1)y'' + xy' + \frac{1}{x}y = 0$

(iii) $x^2(x-2)^2 y'' + 2(x-2)y' + (x+1)y = 0$

Soln

(i) Putting it in standard form, we have

$$y'' + xy' + (x^2-4)y = 0$$

$\Rightarrow P_1 = x$, and $P_2 = x^2-4$. and $b = 0$

They analytic everywhere and all points are ordinary points.

(ii) Putting in standard form, we have

$$y'' + \frac{x}{x-1}y' + \frac{1}{x(x-1)}y = 0$$

$$\Rightarrow P_1 = \frac{x}{x-1}, \quad P_2 = \frac{1}{x(x-1)} \quad \text{and } b = 0$$

P_1 is not analytic at $x=1$ and P_2 is not analytic at $x=0$ and at $x=1$.

The singular points are therefore at $x=0$. and at $x=1$, and all other points are ordinary points.

(iii) Standard form

$$y'' + \frac{2(x-2)}{x^2(x-2)^2}y' + \frac{x+1}{x^2(x-2)^2}y = 0$$

$$P_1 = \frac{2}{x^2(x-2)}, \quad P_2 = \frac{x+1}{x^2(x-2)^2} \quad \text{and } b = 0$$

Singular points are at $x=0$ and at $x=2$

$\Rightarrow$ If x_0 is a singular point of the DE and $(x-x_0)P_1$ and $(x-x_0)^2P_2$ are both analytic at x_0, then x_0 is a regular singular point of the DE.

$\Rightarrow$ If not, x_0 is an Irregular Singular point.

Example

Determine which of the singular points in example above are regular singular points, assuming $x_0 = 0$

soln

(i) All points are ordinary points

(ii) $(x-x_0)P_1 = xP_1 = \frac{x^2}{x-1}$

$(x-x_0)^2 P_2 = x^2 P_2 = \frac{x}{x-1}$

Both $(x-x_0)P_1$ and $(x-x_0)^2 P_2$ are analytic at $x=0$ and therefore $x_0 = 0$ is a regular singular point

(iii) $(x-x_0)P_1 = xP_1 = \frac{2}{x(x-2)}$

$(x-x_0)^2 P_2 = x^2 P_2 = \frac{x+1}{(x-2)^2}$

xP_1 is not analytic at $x=0$, and therefore $x_0 = 0$ is an Irregular singular point

Question

Legendre's equation has the form

$$(1-x^2)y'' - 2xy' + L(L+1)y = 0$$

Where L is a constant. Show that $x=0$ is an ordinary point and $x = \pm 1$ are regular singular points of this equation

soln

Divide by $1-x^2$