DAR ES SALAAM INSTITUTE OF TECHNOLOGY

CALCULUS: DIFFERENTIATION AND INTEGRATION

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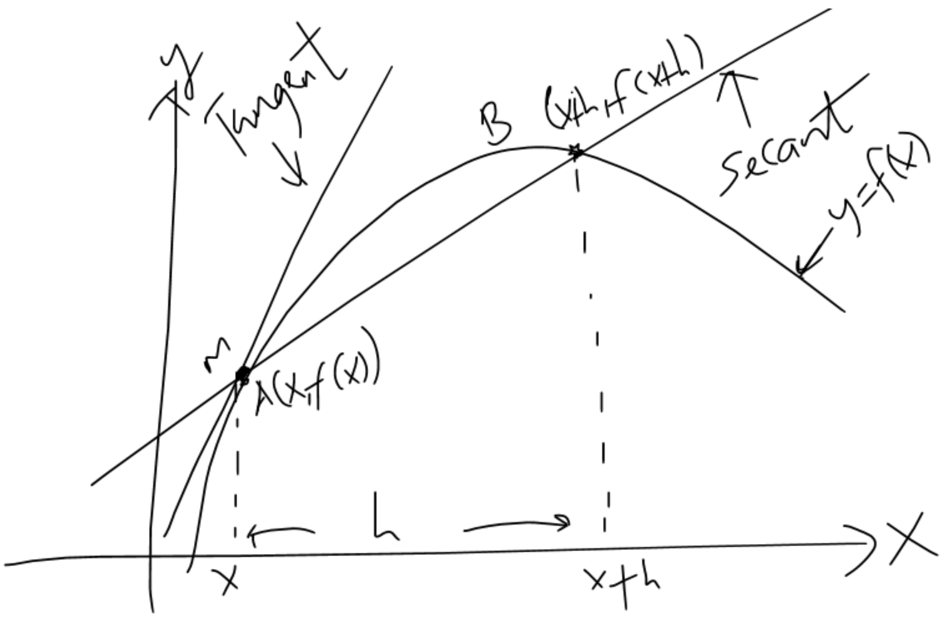
MODULE INSTRUCTIONS

1. 80% class attendance will be considered to sit for End semester Exam.
2. No make up Test or extra activity will be added for student who failed to attain minimum marks.
3. AI is strictly prohibited.
4. One or two individual assignment or group work.
5. Module registration start first day after open.

Differentiation

Consider the function $y = f(x)$ which is represented by the figure below



- **Tangent:** Is the line which touch a curve at only one point.
- **Secant line:** Is the line which cut or cross a curve at two distinct points.
- From the figure above a line touch M is called tangent and a line passing through points A and B is called Secant line.
- The slope of line $\overline{AB} = \frac{\triangle y}{\triangle x} = \frac{f(x+h)-f(x)}{x+h-x} = \frac{f(x+h)-f(x)}{h}$
- If we reduce the distance between x and $x+h$ the line AB become tangent at point M. The derivative $f'(x)$ of curve at M is equal to the slope of tangent at M.

– As $h \to 0$, the derivatives:

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right) \tag{1}$$

- Equation (1) is called Derivative from **First principle.**

Derivative Notations

- $y'(y \quad \text{prime}) = f'(x) \quad$ pronounce as f prime $= \frac{dy}{dx}(dydx) = \frac{d}{dx}(y)(ddx \quad \text{of} \quad y)$ all are called First derivatives
- $y'' = f''(x) = \frac{d^2y}{dx^2}$ all are called second derivatives.

Examples: Find the derivatives using first principle

a) Derivative of constant $f(x) = k$
Solution

$$f(x) = k \tag{2}$$
$$f(x+h) = k \tag{3}$$

Substitute Equations (2) and (3) into formula

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \tag{4}$$
$$= \lim_{h\to 0}\frac{k-k}{h}$$
$$f'(x) = \lim_{h\to 0}(0) = 0$$

Therefore Derivative of constant function is zero.

b) $f(x) = x^2$, at $x = 2$

$$f(x) = x^2 \quad \textbf{and} \quad f(x+h) = (x+h)^2$$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{x}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$f'(x) = 2x \quad \textbf{but} \quad x = 2$$

$$f'(2) = 2(2) = 4$$

c) $f(x) = x^3$

$$f(x) = x^3 \quad \textbf{and} \quad f(x+h) = (x+h)^3$$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + 3x^h + 3xh^2 + h^3 - x^3}{x}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \lim_{h \to 0} (3x^2 + 3xh + h^2)$$

$$f'(x) = 3x^2$$

d) $f(x) = \frac{1}{x}$, at point $x = 4$

$$
\begin{aligned}
f(x) &= \frac{1}{x} \quad \text{and} \quad f(x+h) = \frac{1}{x+h} \\
f'(x) &= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\
&= \lim_{h \to 0} \frac{x - (x+h)}{h(x(x+h))} \\
&= \lim_{h \to 0} \frac{-h}{h(x(x+h)} \\
&= \lim_{h \to 0} \frac{-1}{x(x+h)} \\
f'(x) &= \frac{-1}{x^2} \quad \text{but} \quad x = 4 \\
f'(x) &= \frac{-1}{16}
\end{aligned}
$$

e) $f(x) = \sqrt{x}$

$$
\begin{aligned}
f(x) &= \sqrt{x} \quad \text{and} \quad f(x+h) = \sqrt{x+h} \\
f'(x) &= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h \to 0} \frac{x + h - x}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\
f'(x) &= \frac{1}{2\sqrt{x}}
\end{aligned}
$$

Exercise

Use first principle to differentiate the following functions;

1. $f(x) = 4x^4 + 7x^3 + 5$

2. $f(x) = ax^2 + bx + c$

3. $y = kx^3$

4. $f(x) = \sqrt{1-2x}$

5. $f(x) = \frac{2x-1}{x-1}$

6. $f(x) = \frac{x^2-1}{x^2+1}$

7. $y = \frac{1}{\sqrt{x}}$

8. $f(x) = x^2 + 7x - 2$ at the point $(2, 16)$.

Differentiation Rules

1. Constant rule
   If c is any constant number, then $\frac{d}{dx}(c) = 0$

Proof left as class activity

2. The Power Rule
   If n is a positive integer and $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

   Proof. 1
   As we know that: $x^n - a^n = (x-a)(x^{n-1} + x^{n-1}a + \cdots + xa^{n-2} + a^{n-1})$
   Now, $f'(x) = \lim\limits_{x \to a} \frac{f(x)-f(a)}{(x-a)}$
   $= \lim\limits_{x \to a}(x^{n-1} + x^{n-1}a + \cdots + xa^{n-2} + a^{n-1})$
   $= a^{n-1} + a^{n-2}a + \cdots + aa^{n-2} + a^{n-1}$
   $=na^{n-1}$
   Thus, if $f(x) = x^n$ then $f'(x) = nx^{n-1.}$ □

   Proof. 2:
   $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim\limits_{h \to 0} \frac{(x+h)^n - x^n}{h}$
   But the Binomial expansion of $(x+h)^n = x^n + nx^{n-1}h +$

$\frac{n(n-1)x^{n-2}h^2}{2} + \cdots + h^n$

$f'(x) = \lim_{h\to 0} \frac{x^n + nx^{n-1}h + \frac{n(n-1)x^{n-2}h^2}{2} + \cdots + h^n - x^n}{h}$

$= \lim_{h\to 0} \frac{nx^{n-1}h + \frac{n(n-1)x^{n-2}h^2}{2} + \cdots + h^n}{h}$

$= \lim_{h\to 0} (nx^{n-1} + \frac{n(n-1)x^{n-2}h}{2} + \cdots + h^{n-1})$

$f'(x) = nx^{n-1}$ □

3. The constant Multiple Rule
   If c is a constant and f is a differentiable function, then $\frac{d}{dx}\big(cf(x)\big) = c\frac{d}{dx}\big(f(x)\big)$.

   Proof left as class activity

4. The Sum and Difference Rule
   If f and g are two differentiable functions then $\frac{d}{dx}\Bigg(f(x) \pm g(x)\Bigg) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$.

   Proof left as class activity

5. The product rule
   This rule is used to find the derivative of a product of two functions. Let $f(x) = u(x)v(x)$ then $h'(x) = v\frac{du}{dx} + u\frac{dv}{dx}$

   Proof. From the First principle

   $f'(x) = \lim_{h\to 0} \left(\frac{f(x+h) - f(x)}{h}\right)$

   $= \lim_{h\to 0} \left(\frac{u(x+h)v(x+h) - u(x)v(x)}{h}\right)$

   Add and Subtract $u(x+h)v(x)$

   $= \lim_{h\to 0} \left(\frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h}\right)$

   $= \lim_{h\to 0} \left(\frac{u(x+h)\big(v(x+h) - v(x)\big) + v(x)\big(u(x+h) - u(x)\big)}{h}\right)$

$$= \lim_{h\to 0} v(x) \lim_{h\to 0} \frac{\big(u(x+h)-u(x)\big)}{h} + \lim_{h\to 0} u(x+h) \lim_{h\to 0} \frac{\big(v(x+h)-v(x)\big)}{h}$$

$$f'(x) = v\frac{du}{dx} + u\frac{dv}{dx}$$ □

6. The Quotient rule:
   The quotient rule gives the derivatives of one function divided by another. Let $f(x) = \frac{u(x)}{v(x)}$ then $f'(x) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

   Proof. From First principle

   $$= \frac{\frac{u(x+h)}{v(x+h)} - \frac{u(x)}{v(x)}}{h}$$

   $$= \frac{u(x+h)v(x) - u(x)v(x+h)}{hv(x+h)v(x)}$$

   Add and subtract $u(x)v(x)$

   $$= \lim_{h\to 0}\left(\frac{u(x+h)v(x) - u(x)v(x) + u(x)v(x) - u(x)v(x+h)}{hv(x+h)v(x)}\right)$$

   $$= \frac{\lim_{h\to 0} v(x) \lim_{h\to 0}\left(\frac{u(x+h)-u(x)}{h}\right) - \lim_{h\to 0} u(x) \lim_{h\to 0}\left(\frac{v(x+h)-v(x)}{h}\right)}{\lim_{h\to 0}\big(v(x)\big) \lim_{h\to 0}\big(v(x+h)\big)}$$

   $$f'(x) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$ □

7. The chain rule

   - If $f(x)$ and $g(x)$ are both differentiable and $F(x) = f(g(x))$ is the composite function then F is differentiable and is given by $F''(x) = f'(g(x))g'(x)$
   - In Leibniz notation if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

   $$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \tag{5}$$

   Exercise 2

   1 Find the derivative of the function $f(x) = \frac{x}{x-3}$.

2 Use the product rule to find the derivative $h(x) = (2x^3 + x^6)x^9$

3 Differentiate $y = x^m(x^n + 1)$

4 Differentiate the function $y = 3x^4 + x^2$

5 Find the derivative of the function $y = \frac{1+\sqrt{x}}{\sqrt{x}}$

6 Find $\frac{dy}{dx}$ of: (a) $y = (5x^7 + 6x - 4)^8$ (b) $y = \sqrt[3]{x^2 + 4x + \frac{1}{x}}$ (C) $y = (x - \frac{3}{x^2})^2$.

7 Find $\frac{dy}{dx}$ of $y = (x^2 - 3)(x + 1)^2$.

8 Find $\frac{dy}{dx}$ of a) $y = \frac{x^{\frac{3}{2}}}{1+x^{\frac{1}{2}}}$ (b) $y = \frac{x}{\sqrt{1+x^2}}$ (c) $y = \frac{1-x}{1+x}$ (d) $y = (1 + x^2)(1 - 2x^2)$ (e) $y = \frac{1-\sqrt{x}}{1+\sqrt{x}}$

9 Differentiate $n(x) = (2x^2 + 5)(x^2 - 1)$

10 Differentiate $r(x) = (x^2 - 3x + 1)(4x^2 - 1)$

11 Differentiate $v(x) = (2x - 1)^2(3x + 5)^4$

Derivative of Natural logarithm

Let $f(x) = \ln x$ then by first principle we need $f(x) = \ln x$ and $f(x+h) = \ln(x+h)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{\ln(x+h) - \ln x}{h}$$
$$= \lim_{h\to 0} \frac{\ln(1+\frac{h}{x})}{h}$$

Multiplying by x numerator and denominator

$$= \lim_{h\to 0} \frac{\ln(1+\frac{h}{x})^{\frac{h}{x}}}{x}$$

let $t = \frac{h}{x}$ and as $t \to 0$ then $(1+t)^t \approx e$ so we have

$$= \lim_{t\to 0} \frac{\ln e}{x} \quad \text{but} \quad \ln e = 1$$
$$= \lim_{t} \frac{1}{x}$$
$$f'(x) = \frac{1}{x}$$

Note: The derivative of $y = \ln x$ is $\frac{1}{x}$ and this lead to the following general formula for derivative of natural logarithmic:

$$y = \ln(f(x)) \quad \text{then} \quad y' = \frac{f'(x)}{f(x)}$$

Examples Find derivatives of the following functions

a) $\ln(x^2+1)$ (b) $\ln(\ln(\ln x))$ c) $\ln\left(\frac{x^4}{(3x-4)^2}\right)$

Differentiation of Trigonometric Functions

- For small angles some of trigonometric functions are approximated to the following:
  a) As $\theta \to 0, \quad \sin\theta \approx \theta$
  b) As $\theta \to 0, \quad \cos\theta \approx 1 - \frac{\theta^2}{2}$
  c) As $\theta \to 0, \quad \tan\theta \approx \theta$

1. Derivative of $y = \sin x$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{2\sin(\frac{x+h+x}{2})\cos(\frac{x+h-x}{2})}{h} \\
&= \lim_{h\to 0} \frac{\cos(x+\frac{h}{2})\sin(\frac{h}{2})}{\frac{h}{2}} \\
&= \lim_{h\to 0} \frac{\cos(x+\frac{h}{2}) \times \frac{h}{2}}{\frac{h}{2}} \\
&= \lim_{h\to 0} \cos\left(x+\frac{h}{2}\right) \\
\frac{d}{dx}&(\sin x) = \cos x
\end{aligned}
$$

2. Derivative of $y = \cos x$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} \\
&= \lim_{h\to 0} \frac{-2\sin(\frac{x+h+x}{2})\sin(\frac{x+h-x}{2})}{h} \\
&= \lim_{h\to 0} \frac{-\sin(x+\frac{h}{2})\sin(\frac{h}{2})}{\frac{h}{2}} \\
&= \lim_{h\to 0} \frac{-\sin(x+\frac{h}{2}) \times \frac{h}{2}}{\frac{h}{2}} \\
&= -\lim_{h\to 0} \sin\left(x + \frac{h}{2}\right) \\
&\frac{d}{dx}(\cos x) = -\sin x
\end{aligned}
$$

3. Derivative of $y = \tan x$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{\tan(x+h) - \tan x}{h} \\
&= \lim_{h\to 0} \frac{\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}}{h} \\
&= \lim_{h\to 0} \frac{\sin(x+h)\cos x - \cos(x+h)\sin x}{h\cos(x+h)\cos x} \\
&= \lim_{h\to 0} \frac{\sin(x+h-x)}{h\cos(x+h)\cos x} \\
&= \lim_{h\to 0} \frac{\sin(h)}{h\cos(x+h)\cos x} \\
&= \frac{\lim\limits_{h\to 0}\sin(h)}{h\lim\limits_{h\to 0}\cos(x+h)\lim\limits_{h\to 0}\cos x} \\
&= \frac{h}{h\cos x\cos x} \\
&= \frac{1}{\cos^2 x} \\
&\frac{d}{dx}(\tan x) = \sec^2 x
\end{aligned}
$$

Approach 2: Use quotient rule to show that $\frac{d}{dx}(\tan x) = \sec^2 x$

4. Derivative of $y = \cot x$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{\cot(x+h) - \cot x}{h} \\
&= \lim_{h\to 0} \frac{\frac{\cos(x+h)}{\sin(x+h)} - \frac{\cos x}{\sin x}}{h} \\
&= \lim_{h\to 0} \frac{\cos(x+h)\sin x - \sin(x+h)\cos x}{h\sin(x+h)\sin x} \\
&= \lim_{h\to 0} \frac{\sin(x-x-h)}{h\sin(x+h)\sin x} \\
&= \lim_{h\to 0} \frac{-\sin(h)}{h\cos(x+h)\cos x} \\
&= \frac{-\lim_{h\to 0}\sin(h)}{h\lim_{h\to 0}\sin(x+h)\lim_{h\to 0}\sin x} \\
&= \frac{-h}{h\sin x\sin x} \\
&= \frac{-1}{\sin^2 x} \\
\frac{d}{dx}&(\cot x) = -\csc^2 x
\end{aligned}
$$

Approach 2: use quotient rule to show that $\frac{d}{dx}(\cot x) = -\csc^2 x$

5. If $y = \sec x$ then the derivative is $\frac{d}{dx}(\sec x) = \sec x \tan x$.

   Proof left as class activity

6. Derivative of $y = \csc x$
   If $y = \csc x$ then the derivative is $\frac{d}{dx}(\csc x) = -\csc x \cot x$.

   Proof left as class activity

Differentiation of inverse of Trigonometric functions

1. Derivative $y = \sin^{-1} x$

$$\begin{aligned} x &= \sin y \\ 1 &= cosy\frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{1}{\cos y} \\ &= \frac{1}{\sqrt{1-\sin^2 y}} \\ \frac{dy}{dx} &= \frac{1}{\sqrt{1-x^2}} \end{aligned}$$

2. Derivative of $y = \cos^{-1} x$

$$\begin{aligned} x &= \cos y \\ 1 &= -\sin y\frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{-1}{\sin y} \\ &= \frac{-1}{\sqrt{1-\cos^2 y}} \\ \frac{dy}{dx} &= \frac{-1}{\sqrt{1-x^2}} \end{aligned}$$

3. Derivative of $y = \tan^{-1} x$

$$\begin{aligned} x &= \tan y \\ 1 &= \sec^2 y \frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{1}{\sec^2 y} \\ &= \frac{1}{1 + \tan^2 y} \\ \frac{dy}{dx} &= \frac{1}{1 + x^2} \end{aligned}$$

4. Derivative of $y = \cot^{-1} x$.

$$\begin{aligned} x &= \cot y \\ 1 &= -\csc^2 y \frac{dy}{dx} \\ \frac{dy}{dx} &= -\frac{1}{\csc^2 y} \\ \frac{dy}{dx} &= \frac{-1}{1 + \cot^2 y} \\ \frac{dy}{dx} &= \frac{-1}{1 + x^2} \end{aligned}$$

5. Derivative of $y = \csc^{-1} x$

$$\begin{aligned} x &= \csc y \\ 1 &= -\csc y \cot y \frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{1}{\csc y \sqrt{\csc^2 y - 1}} \\ \frac{dy}{dx} &= \frac{1}{x\sqrt{x^2 - 1}} \end{aligned}$$

Example Find derivative

1. $y = \sin 3x$, from first principle

$$= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$
$$= \lim_{h\to 0}\left(\frac{\sin(3x+3h)-\sin 3x}{h}\right)$$
$$= \lim_{h\to 0}\left(\frac{\sin 3x\cos 3h+\cos 3x\sin 3h-\sin 3x}{h}\right)$$

As $h \to 0$ then $\sin 3h \approx 3h$ and $\cos 3h \approx 1 - \dfrac{(3h)^2}{2}$

$$= \lim_{h\to 0}\left(\frac{\sin 3x(1-\frac{9h^2}{2})+3h\cos 3x-\sin 3x}{h}\right)$$
$$= \lim_{h\to 0}\left(\frac{\sin 3x-\frac{9h^2}{2}\sin 3x+3h\cos 3x-\sin 3x}{h}\right)$$
$$= \lim_{h\to 0}\left(\frac{-\frac{9h^2}{2}\sin 3x+3h\cos 3x}{h}\right)$$
$$= \lim_{h\to 0}\left(-\frac{9h}{2}\sin 3x+3\cos 3x\right)$$
$$= \lim_{h\to 0}\left(-\frac{9(0)}{2}\sin 3x+3\cos 3x\right)$$
$$\boxed{\frac{d}{dx}\left(\sin 3x\right) = 3\cos 3x}$$

2. $y = \sin(x+\sqrt{x})$

3. $y = \sqrt{\frac{1-\sin\theta}{1+\sin\theta}}$

4. $y = \tan\sqrt{x}$, from first principle.

5. $y = \sin^{-1}(\cos x)$.

6. $y = \cos^{-1}(x+10)$

7. $y = x\tan^{-1}x$

8. $y = \sin^2(x^2 + 7x)$

9. $y = \frac{\sin x}{2+3x}$

10. $y = t^2 \sin 3t$

11. $y = \cos 5x$, from first principle.

$$= \lim_{h\to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$$
$$= \lim_{h\to 0}\left(\frac{\cos(5x+5h) - \cos 3x}{h}\right)$$
$$= \lim_{h\to 0}\left(\frac{\cos 5x\cos 5h - \sin 5x\sin 5h - \cos 5x}{h}\right)$$

As $h \to 0$ then $\cos 5h \approx 1 - \dfrac{25h^2}{2}$ and $\sin 5h \approx 5h$

$$= \lim_{h\to 0}\left(\frac{\cos 5x(1 - \frac{25h^2}{2}) + 5h\sin 5x - \cos 5x}{h}\right)$$
$$= \lim_{h\to 0}\left(\frac{\cos 5x - \frac{25h^2}{2}\cos 5x - 5h\sin 5x - \cos 5x}{h}\right)$$
$$= \lim_{h\to 0}\left(\frac{-\frac{25h^2}{2}\cos 5x - 5h\sin 5x}{h}\right)$$
$$= \lim_{h\to 0}\left(-\frac{25h}{2}\cos 5x - 5\sin 5x\right)$$
$$= \lim_{h\to 0}\left(-\frac{25(0)}{2}\cos 5x - 5\sin 5x\right)$$
$$\boxed{\frac{d}{dx}\left(\cos 5x\right) = -5\sin 5x}$$

12. $y = (1 - \frac{1}{x})\tan x$

13. $y = \sec x \sin x$

14. $y = \sec 5x$, from first principle

15. $y = \cot 2x$

$$y = \cot 2x$$

Let $u = 2x \rightarrow \frac{du}{dx} = 2$

$$y = \cot u \rightarrow \frac{dy}{du} = -\csc^2 u = \csc^2 2x$$

By chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\boxed{\frac{dy}{dx} = -2\csc^2 2x}$$

16. $y = (1 + \sin x)^5$

$$y = (1 + \sin x)^5$$

Let $u = 1 + \sin x \rightarrow \frac{du}{dx} = \cos x$

$$y = u^5 \rightarrow \frac{dy}{du} = 5u^4 = 5(1 + \sin x)^4$$

By chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = 5(1 + \sin x)^4 \times \cos x$$

$$\boxed{\frac{dy}{dx} = 5\cos x(1 + \sin x)^4}$$

17. $y = x\sin 2x$

$$y = x\sin 2x$$
Let $u = x \to \frac{du}{dx} = 1$
$$v = \sin 2x \to \frac{dv}{dx} = 2\cos 2x$$
By Product rule
$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$
$$\boxed{\frac{dy}{dx} = \sin 2x + x\cos 2x}$$

18. Find $\frac{dy}{dx}$ if (a) $y = \sin^{-1}(\frac{a}{x})$ b) $y = \cos^{-1}(\frac{x}{5})$ c) $y = \cos^{-1}(x^2 - 1)$

19. Differentiate $x\tan^{-1}(mx)$, where m is a constant.

20. Differentiate $\tan(3x + \frac{\pi}{4})$

$$y = \tan(3x + \frac{\pi}{4})$$
Let $u = 3x + \frac{\pi}{4} \to \frac{du}{dx} = 3$
$$y = \tan u \to \frac{dy}{du} = \sec^2 u = \sec^2(3x + \frac{\pi}{4})$$
By Chain rule
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\boxed{\frac{dy}{dx} = 3\sec^2\left(3x + \frac{\pi}{4}\right)}$$

Differentiation of exponential function $y = a^x$

If $y = a^x$ then $\frac{dy}{dx} = a^x \ln a$

Proof

Given that

$$y = a^x \tag{6}$$

Apply $\ln$ both sides

$$\ln y = x \ln a \tag{7}$$

Differentiate Equation (7) both sides w.r.t x

$$\frac{1}{y}\frac{dy}{dx} = \ln a$$

$$\frac{dy}{dx} = y \ln a, \text{ but } y = a^x$$

$$\boxed{\frac{dy}{dx} = a^x \ln a}.$$

Examples Find slope of the following functions

1. $y = 3^{x+2}$

   Solution

   $$y = 3^{x+2}$$

   Apply $\ln$ both sides

   $$\ln y = \ln(3^{x+2})$$

   $$\ln y = (x+2)\ln 3 \tag{8}$$

   Differentiate Equation (8) both sides w.r.t x

   $$\frac{1}{y}\frac{dy}{dx} = \ln 3$$

   $$\frac{dy}{dx} = y \ln 3, \text{ but } y = 3^{x+2}$$

   $$\frac{dy}{dx} = 3^{x+2} \ln 3$$

2. $y = a^{\sin x}$

3. $f(x) = 10^{\sqrt{x^2+\sin x}}$

4. $y = 2^{\cos^2 x}$

5. $y = e^{2x}$

6. $y = e^{\tan x}$

7. $(2-x)^{\sqrt{x}}$

8. $(\sin x)^{\cos x}$

9. $(\sin^2 x)^{6x}$

Differentiation of Logarithm
If $y = \log_a x$ then $\frac{dy}{dx} = \frac{1}{x}\log_a e$
Proof

$y = \log_a x$
Exponential form
$$x = a^y \tag{9}$$
Apply $\ln$ from Equation (6) both sides
$$\ln x = \ln(a^y)$$
$$\ln x = y\ln a \tag{10}$$
Differentiate Equation (7) both sides w.r.t x
$$\frac{1}{x} = \frac{dy}{dx}\ln a \tag{11}$$
$\frac{dy}{dx} = \frac{1}{x\ln a}$, as we know that $\log_a x = \frac{\ln x}{\ln a}$ this means $\frac{1}{\ln a} = \log_a e$
$$\boxed{\frac{dy}{dx} = \frac{1}{x}\log_a e} \tag{12}$$

Example Find derivative

a) $y = \log_3(2x+1)$

b) $y = \log(3+\cos x)$

c) $y = \log \sin x$
Solution

$$y = \log \sin x$$
$$\sin x = 10^y$$

Apply $\ln$ both sides;

$$\ln \sin x = y \ln(10) \tag{13}$$

Differentiate Equation (13) both sides w.r.t x

$$\frac{\cos x}{\sin x} = \frac{dy}{dx} \ln 10$$
$$\frac{dy}{dx} = \frac{\cos x}{\sin x \ln 10}$$
$$\boxed{\frac{dy}{dx} = \cot x \log_{10} e}$$

d) $y = \log \sqrt{x}$

Implicit Differentiation

- We say that $y = f(x)$ is defined explicitly. However some functions are not explicitly defined, for example $x^2 + y^2 = 2xy$ is implicitly defined and we write because it is difficult to solve for explicitly.

- The derivative of an implicit function is obtained by differentiating both sides of the equation with respect to x and then solve or make subject $\frac{dy}{dx}$.
  Example Find $\frac{dy}{dx}$ a) $x^2 + y^2 = 2y$
  Solution

$$x^2 + y^2 = 2y \tag{14}$$

Differentiate Equation (14) both sides w.r.t x

$$2x + 2y\frac{dy}{dx} = 2\frac{dy}{dx}$$

$$2x = (2 - 2y)\frac{dy}{dx}$$

$$\frac{2x}{2(1-y)} = \frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{x}{(1-y)}}$$

(b) $x^2 + y^2 = 2xy$

Solution

$$x^2 + y^2 = 2xy \tag{15}$$

Differentiate Equation (15) both sides

$$2x + 2y\frac{dy}{dx} = 2y + 2x\frac{dy}{dx}$$

Make $\frac{dy}{dx}$ subject

$$(2y - 2x)\frac{dy}{dx} = 2y - 2x$$

$$\frac{dy}{dx} = \frac{2y - 2x}{2y - 2x}$$

$$\boxed{\frac{dy}{dx} = 1}$$

(c) $x^5 + 3x^2y^2 + 5x^4 = 12$

Solution

$$x^5 + 3x^2y^2 + 5x^4 = 12 \tag{16}$$

Differentiate Equation (16) throughout w.r.t x

$$5x^4 + 6xy^2 + 6x^2y\frac{dy}{dx} + 20x^3 = 0$$

$$6x^2y\frac{dy}{dx} = -(20x^3 + 6xy^2 + 5x^4)$$

$$\frac{dy}{dx} = \frac{-(20x^3 + 6xy^2 + 5x^4)}{6x^2y}$$

Exercise Find derivative

1. $\frac{1}{x} + \frac{1}{y} = 1$
2. $xy + 2x + 3x^22 = 4$
3. $x^2 + y^2 + xy = 0$
4. $y + \cos(x + y) = x$
5. $x^2y + xy^3 = 2x$
6. If $x^2 + y^2 - 2y\sqrt{1 + x^2} = 0$, show that $\frac{dy}{dx} = \frac{x}{\sqrt{1+x^2}}$.
7. If $3\sin xy + 4\cos xy = 7$, show that $\frac{dy}{dx} = -\frac{y}{x}$
8. If $\sqrt{1 - x^2} + \sqrt{1 - y^2} = a(x - y)$, prove that $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$.
9. $xy + \sin xy = 1$.

Higher derivatives

- Higher derivatives: these are derivatives higher than first derivative.
- Higher derivative is obtained by differentiating previous derivative.

- Second derivative is obtained by differentiate first derivative.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \tag{17}$$

Example Find $\frac{d^2y}{dx^2}$ of the following functions

a) $y = x^4 + 2x^3$

$$y = x^4 + 2x^3$$
$$\frac{dy}{dx} = 4x^3 + 6x^2$$
$$\boxed{\frac{d^2y}{dx^2} = 12x^2 + 12x.}$$

b) $y = e^{\cos x}$, you may use chain rule or apply natural log.

$$y = e^{\cos x}$$
$$\frac{dy}{dx} = e^{\cos x}\cos x$$
$$\boxed{\frac{d^2y}{dx^2} = e^{\cos x}\cos^2 x}$$

c) $y = \ln(2x)$

d) $y = \tan^{-1} x$

e) $y = \frac{6}{x^2}$ at $(2, -2)$

f) $y = \frac{x}{x^2+1}$

g) $y = e^{2x}$, you may use chain rule or apply natural log.

$$y = e^{2x}$$
$$\frac{dy}{dx} = 2e^{2x}$$
$$\boxed{\frac{dy}{dx} = 4e^{2x}}$$

h) $x^2 + y^3 - 4x = 8$

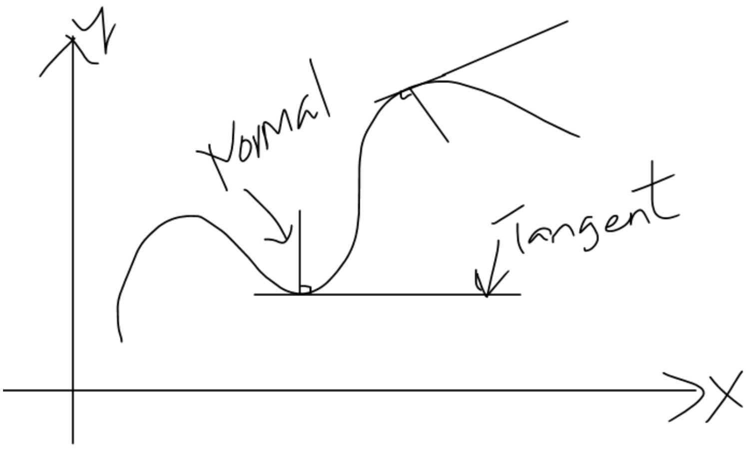
i) $y^2 + 5x^2y^3 + x^4 = 9$ at $(5, 1)$

j) $y = x^2e^{5x}$

k) $y = \sin(10x)$

Tangents and Normal

- Tangent line is perpendicular to the normal line.



- Gradient or slope of tangent at point of contact is equal to the derivative of function at a point.
- Since tangent and normal are perpendicular to each other, hence their slopes obeys the following identity:

$$\text{slope of tangent} \times \text{slope of normal} = -1.$$
$$M_t \times M_n = -1.$$

- Equation of tangent on a curve $f(x)$ may be obtained by following few steps below:
  1. Calculate derivative at a point of contact.
  2. Slope of tangent equal to derivative at point of contact (step 1).

3. Equation of tangent with slope and point at contact is given by: $y = m_t(x - x_o) + y_o$.

- Equation of normal on a curve $f(x)$ may be obtained as follows:
  1. Calculate slope of tangent at a point of contact.
  2. Use $m_t \times m_n = -1$ to obtain slope of normal line.
  3. Equation of normal with slope and point at intersection is given by: $y = m_n(x - x_o) + y_o$.

Exercise

1. Find the equation of the normal to $y = x^2 - 3x + 2$ which has a gradient $\frac{1}{2}$.

   $y = x^2 - 3x + 2$

   $\frac{dy}{dx} = 2x - 3$

   slope of tangent given slope of normal $\frac{1}{2}$

   $\frac{1}{2} \times m_t = -1$

   $m_t = -2, \rightarrow \frac{dy}{dx} = -2$ then $2x - 3 = -2 \rightarrow x = \frac{1}{2}, y = \frac{3}{4}$

   Equation of normal

   $y = m_n(x - x_o) + y_o$

   $y = \frac{1}{2}(x - \frac{1}{2}) + \frac{3}{4}$

   $4y = 2x + 1$

   $y = \frac{1}{2}x + \frac{1}{4}$ $y =$

2. Find the value of k for which $y = 2x + k$ is normal to $y = 2x^2 - 3$

3. Find the equation of the tangent to the curve $y = x^2 + 5x - 2$ at the point where this curve cuts the line $x = 4$.

4. Find the equation of the tangent to $y = (x-5)(2x+1)$ which is parallel to the $x-$ axis.

5. Find the equation of the tangent and normal to the curve $y = x^2 - 4x + 1$ at the point $(-2, 13)$.

6. Find the coordinates of the points where the curve $y = x^2 - x - 12$ cuts the $x-$axis and determine the gradient of $y = x^2 - x - 12$ at these points.

7. Find the coordinates of the point on the curve $y = x - x^2$ where the tangent is parallel to the line $2y + x - 3 = 0$.

8. Find the equation of the normal to the curve $y = \frac{1}{x}$ at $x = 2$, at which point the normal cuts the curve again?

$$y = \frac{1}{x} \tag{18}$$

$$\frac{dy}{dx} = -\frac{1}{x^2}$$

At $x = 2,\ m_t = \left.\frac{dy}{dx}\right|_{x=2} = -\frac{1}{2^2} = -\frac{1}{4}$

Slope of normal (m_n)

$$m_n \times m_t = -1$$

$$m_n = 4$$

Equation of normal given slope $m_n = 4$ and point$(2, \frac{1}{2})$

$$y = m_n(x - x_o) + y_o$$

$$y = 4(x-2) + \frac{1}{2} = 4x - \frac{15}{2}$$

$$y = 4x - \frac{15}{2} \tag{19}$$

Normal line cut again $y = \frac{1}{x}$ and $y = 4x - \frac{15}{2}$ at point $(-2, -0.5)$.

Increasing and Decreasing functions

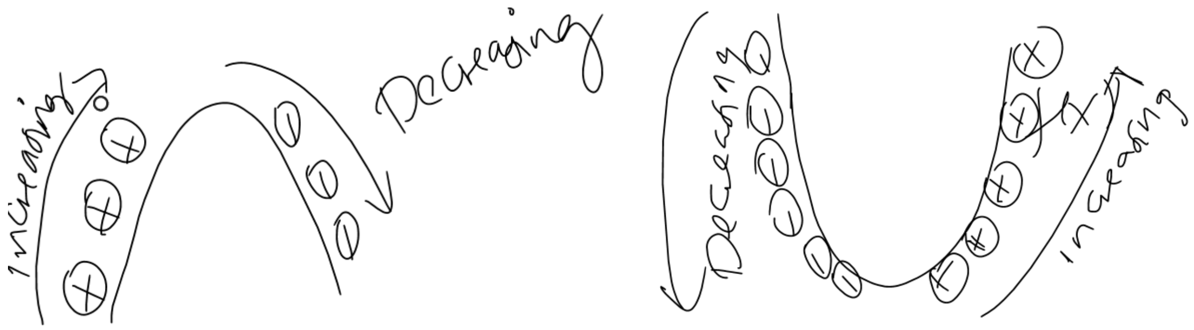
- A Turning or Stationary or Critical point: Is the point where the curve is changing direction. Left or right of the turning point the curve is either increasing or decreasing.

- A function $y = f(x)$ is said to be increasing in a given interval if and only if the slope is positive at each point or first derivative is positive at each point. That is

$$\frac{dy}{dx} > 0 \tag{20}$$

- A function $y = f(x)$ is said to be decreasing in a given interval if and only if the slope is negative at each point or first derivative is negative at each point. That is

$$\frac{dy}{dx} < 0 \tag{21}$$



- At turning point the slope or first derivative is zero ($\frac{dy}{dx} = 0$). Therefore, the curve is neither increasing nor decreasing.

Example Find intervals where the function is increasing or decreasing

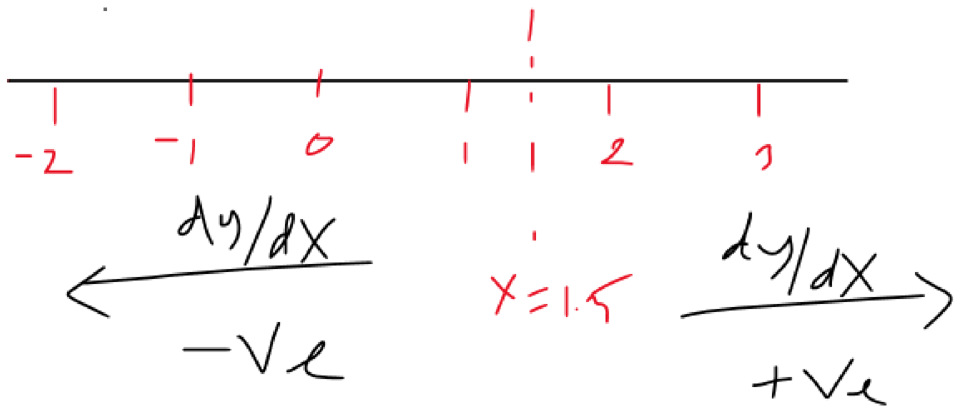
a) $f(x) = x^2 - 3x + 1.$

$$\frac{dy}{dx} = 2x - 3, \quad \text{at turning point } \frac{dy}{dx} = 0$$

$2x - 3 = 0$

$x = 1.5$

Using first derivative test left and right of $x = 1.5$



The function f increase at $x > 1.5$ and decrease at $x < 1.5$

b) $f(x) = x^3 - 9x^2 + 24x.$

c) $y = \frac{2}{5-3x}$

d) $y = \frac{x+2}{x+1}$

e) $y = \frac{1}{1+x^2}$

Example If $f(x) = \frac{1}{3}x^3 - x^2 + 2x$, show that f(x) is increasing for all values of x.

Concavity, Convexity and point of In-flexion

- At turning point a curve is either upward (open up) or downward (open down)
- If the curve is concave upward is also called convex.
- Other terms used are maxima for concave downward and minima for concave upward.

Second Derivative Test for Concavity, Convexity and in-flexion

- If $\frac{d^2y}{dx^2} < 0$ then the turning point is called Maxima or concave downward.
- If $\frac{d^2y}{dx^2} > 0$ then the turning point is called Minima or concave upward or convex.
- If $\frac{d^2y}{dx^2} = 0$ then the turning point is neither maxima nor minima and the point is called point of in flexion.
  Example Check the concavity of $y = \ln x$ at $x = 4$
  Solution

$$\begin{aligned} & y = \ln x \\ & \text{Differentiate w.r.t } x \\ & \frac{dy}{dx} = \frac{1}{x} \\ & \frac{d^2y}{dx^2} = -\frac{1}{x^2} \text{ at } x = 4 \\ & \frac{d^2y}{dx^2} = -\frac{1}{16} < 0 \end{aligned} \tag{22}$$

Therefore, since we obtain second derivative is negative, the curve concave downward.
Example Let $f(x) = \frac{1}{3}x^3 - 4x^2 + 15x$. On what intervals is f concave up? Justify.

Example What is the x-coordinate of the point of inflexion for $f(x) = \frac{1}{3}x^3 + 5x^2 + 24$.
Example Show that origin is the point of inflexion of the curve $y = x^{\frac{1}{3}}$.
Example At What intervals f is Concave for the curve $f(x) = x^3 - 3x^2 + 4x - 2$
Example Concavity for $f(x)$ where $f'(x) = x^4 + 4x^3 + 2x^2 + 12x + 1$.
ExampleThe curve C has equation $y = x^3 - 2x^2 - 4x + 5$

a) Show that C is concave on the interval $[-2, 0]$ and convex on the interval $[1, 3]$.

b) Find the coordinates of the point of inflexion.

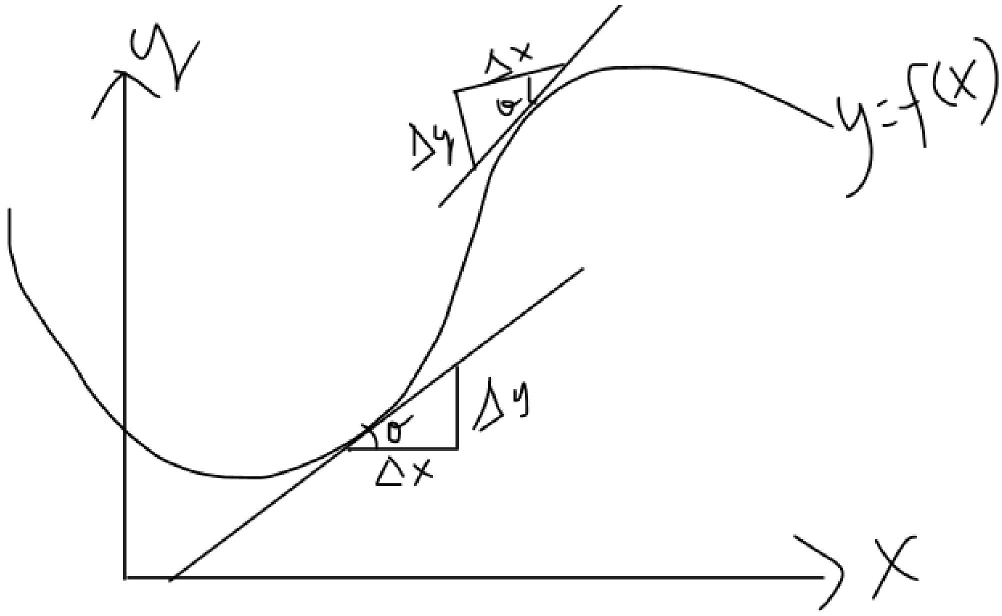
Curvature and Radius of curvature.

- Curvature: Is the measure of how quickly the curve turning at a point. Sharp peak indicate faster turn of a curve, medium peak indicate widely turn of curve.
- Equation $y = f(x)$, the tangent line turns at a certain rate. This rate of turning is the curvature k defined by

$$k = \left| \frac{f''(x)}{\left(1 + [f'(x)]^2\right)^{\frac{3}{2}}} \right| \tag{23}$$

- If value of k is large enough it shows that the curve turn sharply at tangent point.
- If the value of k is very small it show that widely turn of the curve.
- If the value of k is zero it show that no turn of the curve at all, hence its straight line.

Proof: Consider the sketch of $y = f(x)$ below



From Pythagoras Theorem

$$\triangle s^2 = \triangle x^2 + \triangle y^2 \tag{24}$$

Divide by $\triangle x^2$ Equation (8) both sides we get:

$$\left(\frac{\triangle s}{\triangle x}\right)^2 = 1 + \left(\frac{\triangle y}{\triangle x}\right)^2$$

$$\frac{\triangle s}{\triangle x} = \sqrt{1 + \left(\frac{\triangle y}{\triangle x}\right)^2} \tag{25}$$

As $\triangle x \to 0$ then $\frac{\delta s}{\triangle x} \approx \frac{ds}{dx}$ and $\frac{\triangle y}{\triangle x} \approx \frac{dy}{dx}$,therefore Equation (9) is transformed as:

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \tag{26}$$

But, $\frac{dy}{dx} = \tan\theta$, second derivative we get

$$\frac{d^2y}{dx^2} = \sec^2\theta \frac{d\theta}{dx}$$
$$\frac{d\theta}{dx} = \frac{\frac{d^2y}{dx^2}}{\sec^2\theta}$$
$$= \frac{\frac{d^2y}{dx^2}}{1+\tan^2\theta}$$
$$\frac{d\theta}{dx} = \frac{\frac{d^2y}{dx^2}}{1+(\frac{dy}{dx})^2} \tag{27}$$

The curvature k

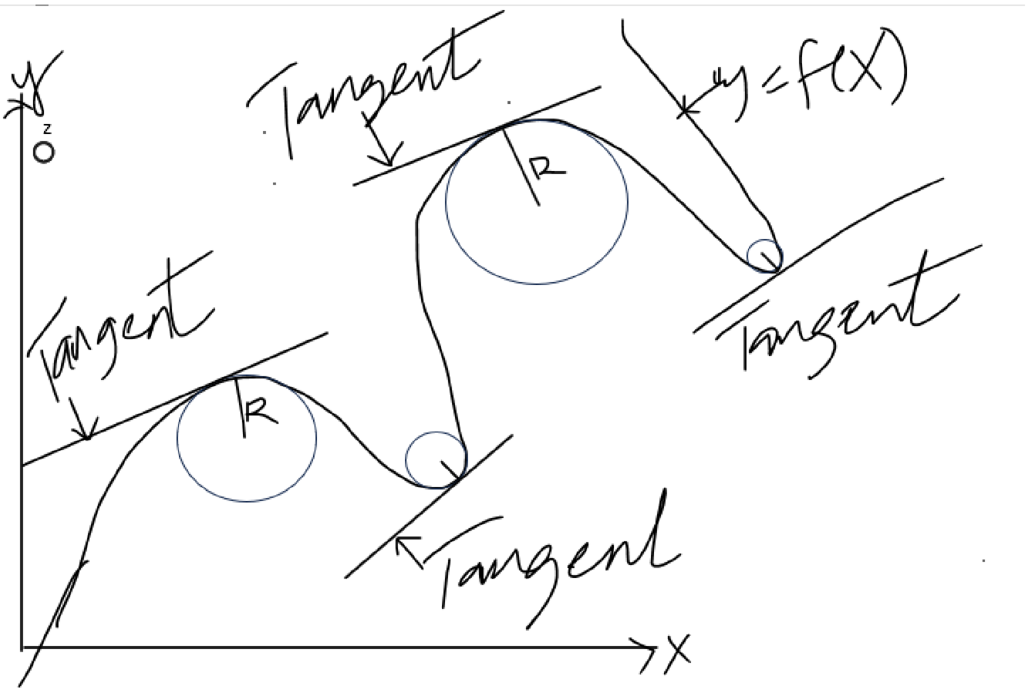
$$k = \left|\frac{d\theta}{ds}\right| = \left|\frac{d\theta}{dx} \div \frac{ds}{dx}\right|$$

$$k = \frac{|f''(x)|}{\left(1+[f'(x)]^2\right)^{\frac{3}{2}}}. \tag{28}$$

Example Find the curvature of the equation $y = x^2$ at point (-2,2).
Example Find the curvature of $y = (a^2 - x^2)^{\frac{1}{2}}$.
Example What is the curvature of the curve $x^2 + y^2 = 2$ at the point $(1, 1)$.

Radius of curvature

- Radius of curvature: Is the distance from a fixed point of curvature to the tangent point of curvature.



- Large value of k,Sharp turn of a curve, hence, small radius of curvature.
- Small value of k, widely turn of a curve, hence large radius of curvature.
- Relationship between curvature and radius of curvature from an observation on figure above;

$$R = \frac{1}{k} \tag{29}$$

Example Find the radius of curvature of the curve $\sqrt{x} + \sqrt{y} = 1$ at the point $(\frac{1}{4}, \frac{1}{4})$

$$\sqrt{x}+\sqrt{y}=1$$
$$\frac{1}{2}x^{-\frac{1}{2}}+\frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx}=0$$
$$\frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx}=-\frac{1}{2}x^{-\frac{1}{2}}$$
$$\frac{dy}{dx}=-\frac{\sqrt{y}}{\sqrt{x}} \tag{30}$$
$$\left.\frac{dy}{dx}\right|_{(1,1)}=-1$$

To obtain second derivative, we differentiate again Equation (21)

$$\frac{d^2y}{dx^2}=-\frac{\frac{1}{2}\sqrt{x}y^{-\frac{1}{2}}\frac{dy}{dx}-\frac{1}{2}\sqrt{y}x^{-\frac{1}{2}}}{x}$$
$$\left.\frac{d^2y}{dx^2}\right|_{(1,1)}=-\frac{\frac{1}{2}\sqrt{x}y^{-\frac{1}{2}}\frac{dy}{dx}-\frac{1}{2}\sqrt{y}x^{-\frac{1}{2}}}{x}$$
$$\left.\frac{d^2y}{dx^2}\right|_{(1,1)}=1 \tag{31}$$

Substitute Equation (22) into the formula

$$k=\frac{\left(1+[f'(x)]^2\right)^{\frac{3}{2}}}{|f''(x)|}.$$
$$k=\frac{\sqrt{2^3}}{1}$$
$$k=\sqrt{8}$$
$$R=\frac{1}{k}$$
$$R=\frac{1}{\sqrt{8}}\text{units}$$

Exercise

1. Find the radius of curvature of $y = e^x$ at $(0, 1)$.
2. Find the radius of curvature of the curve $xy = 30$ at the point $(3, 10)$.
3. What is the radius of curvature at $(2, -1)$ to the curve $y = x^4 - 8x - 42$.

INTEGRATION

- **Definition** Integration: Is the reverse process of differentiation.

- Integration is also called anti derivative. Example when $f(x) = x^2$ is differentiated with respect to x is $f'(x) = 2x$. The reverse process of $f'(x) = 2x$, it is clear will be $f(x) = x^2$.

- **Theorem:** If F is an anti-derivative of f on an interval I, then the most general anti-derivative of f on I is $F(x) + C$

- $\int f(x)dx$ is called the integral of $f(x)$ with respect to x, $\int$ is integral sign or summation sign, $f(x)$ is called Integrand.

- Eg $\int 2xdx = x^2 + C, \quad \int 3x^2dx = x^3 + C.$

- There are two types of Integration 1) Infinite integral and 2) definite Integral

- **Infinite integral:** Is an integral which does not involve limit eg $\int f(x)dx = F(x) + C$

- **Definite Integral**: Is an integral which involve limits $\int_a^b f(x)dx = F(b) - F(a)$

Methods of Integration

1. **Power rule of integration**

   Let $f(x) = x^n$ be power function, where n is any integer then integration of $f(x)$ w.r.t x is given by;

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \tag{32}$$

   Where

   c is called Integral constant

   Example: Integrate the following

   a) $\int x dx = \frac{x^{1+1}}{1+1} + c = \frac{x^2}{2} + c$

   b) $\int_0^2 x^2 dx = \left[\frac{x^{2+1}}{2+1}\right]_0^2 = \frac{x^3}{3} = \frac{2^3}{3} - \frac{0^3}{3} = \frac{8}{3}$

   c) $\int x^3 dx = \frac{x^{3+1}}{3+1} + c = \frac{x^4}{4} + c$

   d) $\int 2dx = \int 2x^0 dx = 2\frac{x^{0+1}}{0+1} + c = 2x + c$

   e) $\int x^5 dx = \frac{x^6}{6} + c$

   Exercise:Integrate the following

   a) $\int \frac{dx}{x^2}$

   b) $\int \sqrt{x} dx$

   c) $\int \frac{dx}{\sqrt{x}}$

2. **Sum and Difference rule of Integration**

   If f and g are two integrable functions then

$$\int \big(f(x) \pm g(x)\big) dx = \int f(x) dx \pm \int g(x) dx \tag{33}$$

   Example Integrate the following

   a) $\int (x^2 + x) dx = \int x^2 dx + \int x dx = \frac{x^3}{3} + \frac{x^2}{2} + c$

b) $\int(x-3)dx=\int xdx-\int 3dx=\frac{x^2}{2}-3x+c$

c) $\int(x^4-6x^2+2x+7)dx$

Standard Integrals

a) $\int \sin xdx=-\cos x+c$

b) $\int \cos xdx=\sin x+c$

c) $\int \csc^2 xdx=-\cot x+c$

d) $\int \sec^2 xdx=\tan x+c$

e) $\int \sec xdx=\ln|\sec x+\tan x|+c$

f) $\int \csc x\cot xdx=-\csc x+c$

g) $\int e^x dx=e^x+c$

h) $\int \frac{1}{x}dx=\ln|x|+c$

3. **Integration by Substitution**

3.1 The form $\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b)+c$, $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b)+c$, $\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b)+c$ where $a \neq 0$ and $b=0$.

Example Integrate

i. $\int 4\sin 3x dx$

ii. $\int \cos 2x dx$

iii. $\int \sec^2(2x+5)dx$

iv. $\int 14\sin(-7x-5)dx$

3.2 The form $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b}+c$, where $a \neq 0$ and $b=0$. Integral can be evaluated, let $u = ax+b$, then differentiate w.r.t x

Example: Integrate the following

i. $\int e^{4x}dx$

Solution

$$I = \int e^{4x}dx \tag{34}$$

$$\text{Let } u = 4x \tag{35}$$

$$\frac{du}{dx} = 4$$

$$\frac{du}{4} = dx \tag{36}$$

Substitute Equations (33) and (34) into 32)

$$I = \int e^u \frac{du}{4}$$

$$\int e^{4x}dx = \frac{1}{4}e^{4x}+c$$

ii. $\int e^{5x-3}dx$

iii. $\int e^{\frac{1}{3}x}dx$

3.3 The form $\int \frac{dx}{ax+b} = \frac{1}{a}\ln(ax+b)+c$ where $a \neq 0$ and $b = 0$.

i. $\int \frac{dx}{4x+3}$

ii. $\int \frac{dx}{x-1}$

iii. $\int \frac{dx}{4(x+1)}$

iv. $\int \frac{-5dx}{10x-400}$

4. **Integration by substitution (Existence of function and its Derivative)**

When $\int (f(x))^n f'(x)dx$ is recognized, we let $u = f(x)$ so that $du = f'(x)dx$

$$\int u^n du = \frac{u^{n+1}}{n+1} + c$$

$$\text{Thus, } \int f(x)^n f'(x)dx = \frac{f(x)^{n+1}}{n+1} + c$$

Example Integrate the following

a) $\int \frac{x^2}{\sqrt{x^3+1}}dx$

b) $\int \frac{x+1}{x^2+2x+40}dx$

c) $\int (x^2+1)^{\frac{3}{2}}xdx$

d) $\int \sin x\sqrt{1-\cos x}dx$

e) $\int x^4\cos(6x^5+1)dx$

f) $\int (x^2-1)^5(2x-1)dx$

g) $\int x^2\sqrt{1+x^3}dx$

5. **Integration by Parts**
Integral of two different function where other methods failed, integration by parts can be used

$$\int u(x)dv = uv - \int vdu \tag{37}$$

Proof

From the product rule

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$d(uv) = vdu + udv$$

Integrate both sides

$$\int d(uv) = \int vdu + \int udv$$

$$uv = \int vdu + \int udv$$

$$\boxed{\int vdu = uv - \int udv}$$

- To use this method, you have to identify which function is more easy to differentiate and integrate.
- The easiest function to be differentiated, you let u and integrated $\frac{dv}{dx} = intagrand$.

- Trick to identify function use **ILATE:** I-Inverse,L-Logarithm, A-algebra,T-Trigonometric, E-Exponent.

I

ILATE

D

- From right to left functions are more differentiable and vice-versa. Eg T and E , T is simple to differentiate $(u = T)$ and $dv = Edx$ or I and L, I is simple to differentiate $(u = I)$ and $dv = Ldx$

  Example:Integrate the following

  a) $\int 3te^{2t}dt$

  b) $\int x\cos x dx$

  c) $\int x^2 \sin x dx$

  d) $\int 5xe^{4x}dx$

  e) $\int x\ln x dx$

  f) $\int xe^{2x}dx$

  g) $\int \frac{4x}{e^{3x}}dx$

  h) $\int 5\theta\cos 2\theta d\theta$

  i) $\int \sqrt{x}\ln x$

  Qn: Prove that $\int e^{ax}\cos bx dx = \frac{e^{ax}}{a^2+b^2}(b\sin bx + a\cos bx) + c$

6. **Integration by partial fraction**

   - Partial Fractions are simple fractions obtained by splitting complex fractions.

- **Case I:** When the denominator is the product of distinct linear functions

$$\int \frac{dx}{ax+b)(cx+d)} = \int \frac{Adx}{ax+b} + \int \frac{Bdx}{cx+d} \tag{38}$$

Example Integrate

a) $\int \frac{dx}{(x+1)(x+2)}$

b) $\int \frac{x+5}{x^2+x-2}dx$

c) $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx$

d) $\int \frac{dx}{x^2-a^2}$

- **Case II:** When the denominator is product of linear factors, some which repeated.

$$\int \frac{dx}{(ax+b)(cx+d)^n} = \int \frac{A}{ax+b}dx + \int \frac{B}{cx+d}dx + \cdots + \int \frac{Z}{(cx+d)^n}dx$$

Example Integrate

a) $\int \frac{x^2+2x+3}{x(x+1)^2}dx$

b) $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}dx$

c) $\int \frac{5x^2-2x-19}{(x+3)(x-1)^2}dx$

d) $\int \frac{3x^2+16x+15}{(x+3)^3}dx$

e) $\int \frac{18+21x-x^2}{(x-5)(x+2)^2}dx$

f) $\int \frac{4x-3}{(x+1)^2}dx$

- **Case III:** When the denominator consists of irreducible quadratic factors.

$$\int \frac{dx}{(ax+b)(cx^2+b)} = \int \frac{A}{ax+b}dx + \int \frac{Bx+C}{cx^2+d}dx \tag{39}$$

Example Integrate

a) $\int \frac{x+5}{(x-2)(x^2+1)} dx$

b) $\int \frac{2x^3+3x^2+2x+1}{(2x+1)(x^2+2)} dx$

c) $\int \frac{10}{(x^2+9)(x-1)} dx$

d) $\int \frac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)} dx$

e) $\int \frac{x^2+x+1}{(x^2+1)^2} dx$

f) $\int \frac{2x^2}{(x^3+1)^2} dx$

7. **Integration by Trigonometric Substitutions**

- **Case I:** Sine substitution, integrals of the form;

$$\int \frac{dx}{\sqrt{a^2 - x^2}} \tag{40}$$

$$\text{You let } \; x = a \sin \theta \tag{41}$$

- **Case II:** Cosine Substitution, integrals of the form;

$$\int \frac{-dx}{\sqrt{a^2 - x^2}} \tag{42}$$

$$\text{You let } \; x = a \cos \theta \tag{43}$$

- **Case III:** Tangent Substitution, integrals of the form;

$$\int \frac{dx}{a^2 + x^2} \tag{44}$$

$$\text{You let } \; x = a \tan \theta \tag{45}$$

Example: Integrate

a) $\int \frac{dx}{x^2+25}$

b) $\int \frac{dx}{x^2+2}$

c) $\int \frac{dx}{x^2+6x+16}$

d) $\int \frac{1}{\sqrt{4-x^2}} dx$

e) $\int \sqrt{36 - x^2}dx$

f) $\int x^2\sqrt{1 - x^2}dx$

g) $\int \frac{x^2}{4-x^2}dx$

h) $\int \left(\frac{7}{\sqrt{1+x^2}} + \frac{\sqrt{1-2x^2}}{4}\right)dx$

8. **Integration by reduction formula**

- Reduction formula $I_n = \int x^n e^{-x}dx$

$$I_n = -x^n e^{-x} + I_{n-1} + C, \quad n \geq 1 \tag{46}$$

- Reduction formula $I_n = \int \sec^n dx$

$$I_n = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1} I_{n-2} + C. \qquad (47)$$

Proof

$$= \int \sec^{n-2} x \sec^2 x dx$$

Let $u = \sec^{n-2} \to du = (n-2)\sec^{n-3} \sec x \tan x dx$

$$\int dv = \int \sec^2 x dx$$

$v = tanx$

Use integration by Parts

$$= uv - \int vdu$$

$$= \sec^{n-2} \tan x - \int \tan x((n-2)\sec^{n-3} \sec x \tan x dx)$$

$$= \sec^{n-2} \tan x - (n-2)\int \tan^2 x \sec^{n-2} dx$$

$$= \sec^{n-2} \tan x - (n-2)\int (\sec^2 - 1)\sec^{n-2} dx$$

$$= \sec^{n-2} \tan x - (n-2)\left[\int \sec^n x dx - \int \sec^{n-2} dx\right]$$

$$I_n = \sec^{n-2} \tan x - (n-2) I_n + (n-2) I_{n-2}$$

$$I_n(n-1) = \sec^{n-2} \tan x + (n-2) I_{n-2}$$

$$\boxed{I_n = \frac{1}{n-1}\sec^{n-2} \tan x + \frac{n-2}{n-1} I_{n-2} + C}$$

Example: Show that $\int \sec^4 x dx = \frac{1}{5}\sec^4 x \tan x + \frac{4}{15}\sec^2 x \tan x + \frac{8}{15}\tan x + D$.

Example Use reduction formula integrate $\int \sec^2 x dx$

- Reduction formula $I_n = \int \cos^n x dx$

$$= \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2} + C. \tag{48}$$

Proof

$$= \int \cos^{n-1} x \cos x dx$$

Let $u = \cos^{n-1} x \to du = (n-1)\cos^{n-2} x \sin x dx$

$$\int dv = \int \cos x dx \to v = \sin x$$

Use integration by Parts

$$= uv - \int v du$$

$$= \cos^{n-1} x \sin x - \int \sin x((n-1)\cos^{n-2} x \sin x dx)$$

$$= \cos^{n-1} x \sin x - (n-1)\int \sin^2 x \cos^{n-1} x dx$$

$$= \cos^{n-1} x \sin x - (n-1)\int (1-\cos^2 x)\cos^{n-2} x dx$$

$$= \cos^{n-1} x \sin x - (n-1)\left[\int \cos^n x dx - \int \cos^{n-2} x dx\right]$$

$$I_n = \cos^{n-1} x \sin x - (n-1)I_n + (n-1)I_{n-2}$$

$$nI_n = \cos^{n-1} x \sin x + (n-1)I_{n-2}$$

$$\boxed{I_n = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-2}{n} I_{n-2} + C}$$

Example Use reduction formula integrate $\int \cos^2 x dx$
Example Use reduction formula integrate $\int \cos^3 x dx$
Example Use reduction formula integrate $\int \cos^5 x dx$

- Reduction formula for $I_n = \int \sin^n x dx$

$$= -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n} I_{n-2} + C \qquad (49)$$

Proof

$$= \int \sin^{n-1} x \sin x dx$$

Let $u = \sin^{n-1} \to du = (n-1)\sin^{n-2} x \cos x dx$

$$\int dv = \int \sin x dx \to v = -\cos x$$

Use integration by Parts

$$= uv - \int v du$$

$$= -\sin^{n-1} x\cos x + \int \cos x((n-1)\sin^{n-2} x \cos x dx)$$

$$= -\sin^{n-1} x\cos x + (n-1)\int \cos^2 x \sin^{n-2} x dx$$

$$= -\sin^{n-1} x\cos x + (n-1)\int (1-\sin^2 x)\sin^{n-2} x dx$$

$$= -\sin^{n-1} x\cos x + (n-1)\left[\int \sin^n x dx - \int \sin^{n-2} x dx\right]$$

$$I_n = -\sin^{n-1} x\cos x + (n-1)I_n + (n-1)I_{n-2}$$

$$nI_n = -\sin^{n-1} x\cos x - (n-1)I_{n-2}$$

$$\boxed{I_n = -\frac{1}{n}\sin^{n-1}\cos x + \frac{n-1}{n} I_{n-2} + C}$$

Example Use reduction formula integrate $\int \sin^2 x dx$
Example Use reduction formula integrate $\int \sin^3 x dx$
Example Use reduction formula integrate $\int \sin^4 x dx$

- Reduction formula for $I_n = \int \tan^n x dx$

$$= \frac{1}{n-1}\tan^{n-1} x - I_{n-2} + C \tag{50}$$

Proof

$$= \int \tan^{n-2} x \tan^2 x dx$$
$$= \int \tan^{n-2} x(\sec^2 x - 1)dx$$
$$= \int \tan^{n-2} x \sec^2 x dx - I_{n-2}$$

Now consider integral of $\int \tan^{n-2} x \sec^2 x dx$

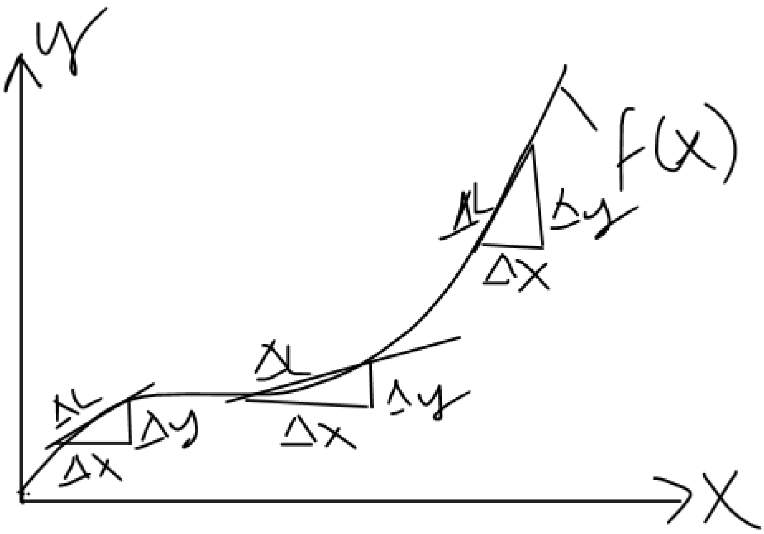
Let $u = \tan x \rightarrow dx = \dfrac{du}{\sec^2 x}$

$$= \int u^{n-2} \sec^2 x \frac{du}{\sec^2 x}$$
$$= \int u^{n-2} du$$
$$= \frac{u^{n-1}}{n-1} - I_{n-2}$$
$$\boxed{I_n = \frac{1}{n-1}\tan^{n-1} x - I_{n-2} + C}$$

Example Use reduction formula integrate $\int \tan^2 x dx$.
Example Use reduction formula integrate $\int \tan^4 x dx$.

9. **Application of Integration**

1. Length of a curve (arc Length) in Cartesian form.



By Pythagras Theorem

$\triangle \ell^2 = \triangle x^2 + \triangle y^2$

Divide by$\triangle x^2$throughout

$\dfrac{\triangle \ell^2}{\triangle x^2} = 1 + \dfrac{\triangle y^2}{\triangle x^2}$

$$\triangle \ell = \sqrt{1 + \left(\frac{\triangle y}{\triangle x}\right)^2}\,\triangle x \qquad (51)$$

As $\triangle x \to 0,$ Equation (49) becomes:

$d\ell = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}\,dx$

Total length of whole curve $f(x)$

$$\boxed{\ell = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx}$$

Example Find the length of the arc in first quadrant of the curve $y = 2x^{\frac{3}{2}}$, from $x = 0$ to $x = \frac{1}{3}$.

$$y = 2x^{\frac{3}{2}} \tag{52}$$

Differentiate Equation (50) w.r.t x

$$\frac{dy}{dx} = 3x^{\frac{1}{2}}$$

Substitute into formula

$$\ell = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx, \quad a = 0, \quad b = \frac{1}{3}$$

$$\ell = \int_0^{\frac{1}{3}} \sqrt{1 + (3x^{\frac{1}{2}})^2} dx$$

$$\ell = \int_0^{\frac{1}{3}} \sqrt{1 + 9x} dx$$

Use Calculator

$\ell = 0.5185$ units

Example Find the length of the curve $y = \frac{1}{2}x^2$ from $x = 0$ to $x = 1$.

$$y = \frac{1}{2}x^2 \tag{53}$$

Differentiate Equation (50) w.r.t x

$$\frac{dy}{dx} = x$$

Substitute into formula

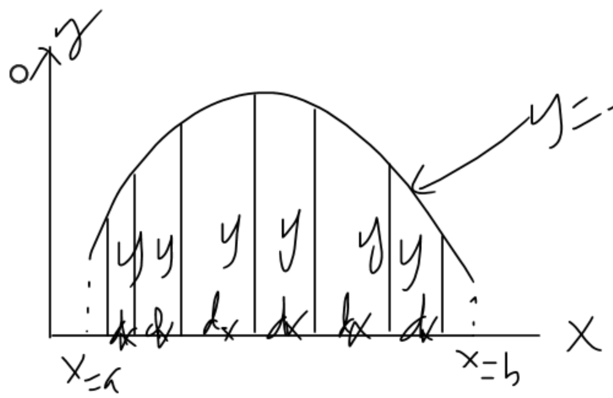
$$\ell = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx, \quad a = 0, \quad b = 1$$

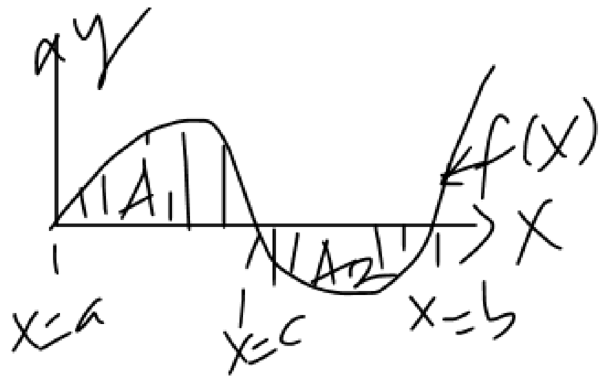
$$\ell = \int_0^1 \sqrt{1 + x^2} dx$$

$$\ell = 1.1478 \text{ units} \tag{54}$$

Example Find the length of the curve $6xy = 3 + x^4$ between the points whose abscissa are 1 and 3.
Example Find the length of the curve $y = \frac{1}{2}(e^x + e^{-x})$ between $x = 0$ to $x = \ln 2$.

2. **Area under the curve**
   Consider the sketch of $f(x)$ below,





- The area under the curve is obtained by cutting $y = f(x)$ into small rectangles, each rectangle has length y and width dx.
- Area of each rectangle is given $A_1 = f(x)dx, A_2 = f(x)dx \cdots A_n = f(x)dx$. Sum of all areas it gives approximated area under curve $f(x)$ is given by;

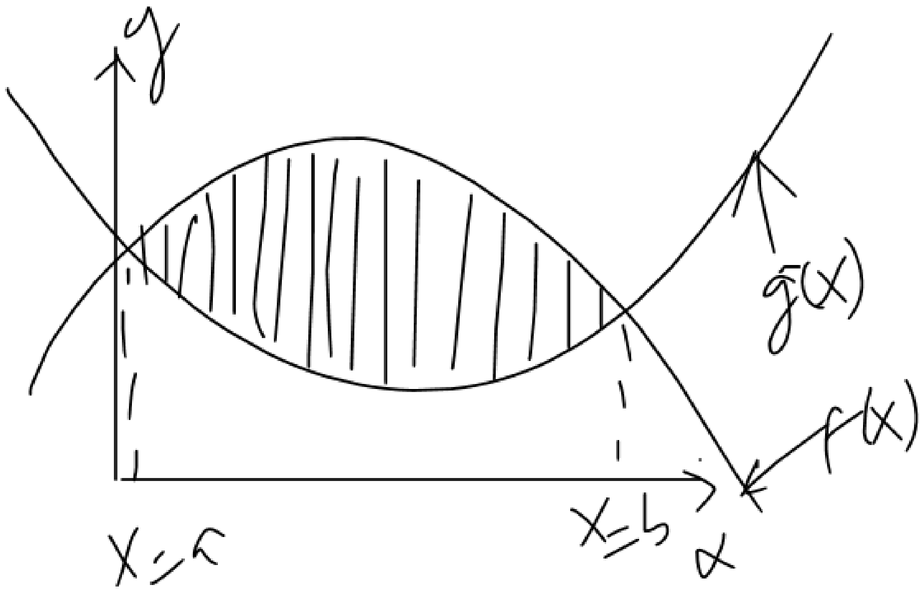
$$\boxed{A = \int_a^b f(x)dx} \tag{55}$$

- Area below axis is negative, so it taken as absolute value. Therefore, from figure(2) total area is given:

$$\boxed{A = \int_a^c f(x)dx + \left| \int_c^b f(x)dx \right|} \tag{56}$$

3. **Area between two curves**

consider the area enclosed between two functions $f(x)$ and $g(x)$ from $x = a$ to $x = b$ below;

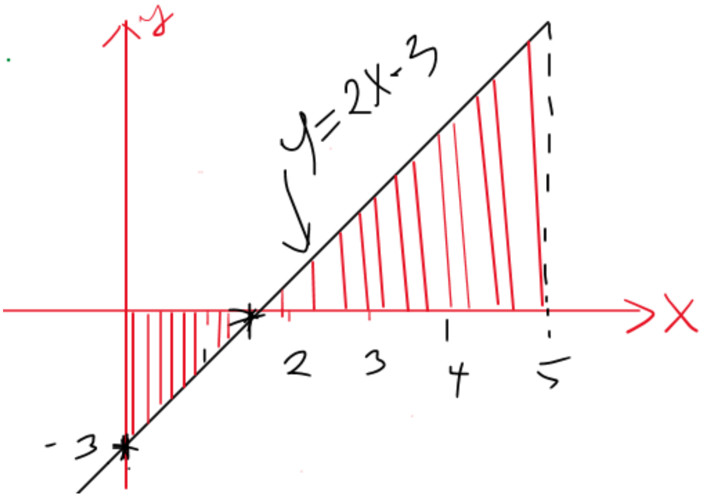


$$A = \int_a^b (F_{\text{Above}} - F_{\text{below}})dx$$

$$\boxed{A = \int_a^b (f(x) - g(x))dx} \qquad (57)$$

Example Find the area under the curve $y = x^2 + 4$ bounded by the lines $x = 2, x = 4$ and the x-axis.

Example Find the area enclosed by the line $y = 2x - 3$ and the x-axis between $x = 0$ and $x = 5$.
Solution
First sketch given function



From 0 to 1.5 area is below and from 1.5 to 5 area is above:

$$A = \left| \int_0^{1.5} f(x)dx \right| + \int_{1.5}^{5} f(x)dx$$

$$A = \left| \int_0^{1.5} (2x-3)dx \right| + \int_{1.5}^{5} (2x-3)dx$$

$$A = \left| -\frac{9}{4} \right| + \frac{49}{4}$$

$A = 14.5$ square units

Example In a school playground, a curved slide for the children to play is represented by the function $y = \sin x$ between $x = 0$ and $x = 2\pi$, where x represents the horizontal distance along the slide and y represents a vertical height. calculate the total area under the curve of the slides shape.
Example Calculate the area enclosed between the

curve $f(x) = -x^2 - 2x$ and the x-axis.

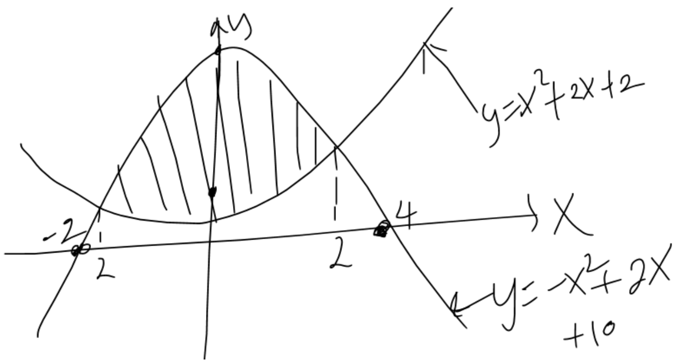
Example: Find the area enclosed by the curves $f(x) = x^2 - 1$ and $g(x) = x + 1$.

Example Find the area enclosed between the curve $y = (4 - x)^2$ and the x-axis from $x = 0$ to $x = 5$.

Example Find the area of the region enclosed by the curves $y = x^2$ and $y = 2x - x^2$

Example Find the area bounded by the curves $y = 3e^{2x}$ and $y = 3e^{-x}$ and the ordinates at $x = -1$ and $x = 2$.

Example Find the area enclosed between the curves representing the functions $f(x) = x^2+2x+2$ and $f(x) = -x^2 + 2x + 10$.



$$A = \int_{-2}^{2} \Big(f(x)_{above} - f(x)_{below}\Big)dx$$

$$A = \int_{-2}^{2} (-x^2 + 2x + 10 - (x^2 + 2x + 2))dx$$

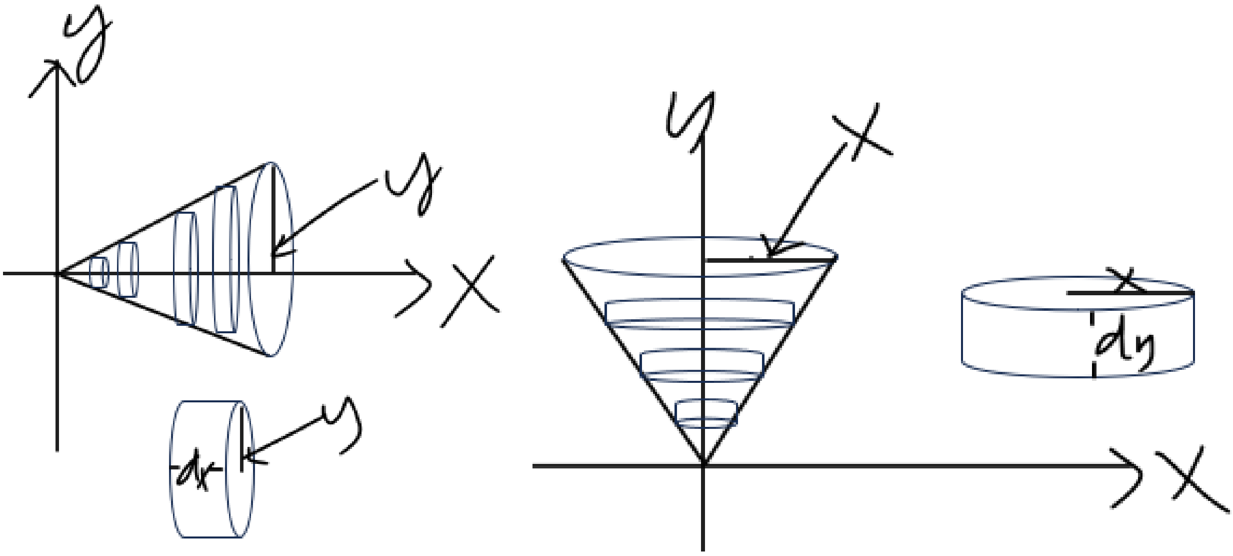
$$A = \int_{-2}^{2} (-2x^2 + 8)dx$$

$$A = 21.3 \text{ square units}$$

Example Determine the area enclosed between the curve $y = (x-1)^2$ and the straight line $y = x + 1$.

4. **Volume of revolution**

- Revolution: means rotation about axis.
- Let consider rotation of $f(x)$ about x and $y-$ axis as follows in figures below:



- The volume of formed shape after rotation is obtained by cutting its shape into small cylinders, each cylinder has radius and height.
- About $x-$ axis, the volume of each cylinder is given $v_1 = \pi y^2 dx, v_2 = \pi y^2 dx \cdots v_n = \pi y^2 dx$. Sum of all volume it gives approximated volume of revolution is given;

$$\boxed{V = \pi \int_a^b y^2 dx} \tag{58}$$

- About $y-$ axis, the volume of each ring is given $v_1 = \pi x^2 dy, v_2 = \pi x^2 dy \cdots v_n = \pi x^2 dy$. Sum of all volume it gives approximated volume of revolution and is given:

$$\boxed{V = \pi \int_a^b x^2 dy} \qquad (59)$$

Example Find the volume of a solid generated by rotating of the curve $f(x) = \sqrt{3x - x^2}$ from $x = 0$ to $x = 3$ along the x-axis to 2 significant figures.
Solution

$$y = \sqrt{3x - x^2}, \quad \text{from} \quad x = 0, x = 3$$
$$V = \pi \int_a^b y^2 dx$$
$$V = \pi \int_0^3 \left(\sqrt{3x - x^2}\right)^2 dx$$
$$V = \pi \int_0^3 \sqrt{3x - x^2} dx$$
$$V = 3.5\pi \quad \text{cubic units}$$

Example Find the volume of a solid obtained after revolution of the curve $y = x^2$ from $y = 0$ to $y = 4$ about y-axis.
Solution

$$y = x^2, \quad \text{from} \quad y = 0, y = 4$$
$$V = \pi \int_a^b x^2 dy, \quad \text{but} \quad x^2 = y$$
$$V = \pi \int_0^4 y dy$$
$$V = 8\pi \quad \text{cubic units}$$

Example Find the volume of a solid obtained after rotating the region bounded by the curve $y = \sqrt{x}, x = 0$ and $x = 4$ about the x-axis.
Example Find the volume of a solid obtained after revolution bounded by the curve $y = \sqrt{x^2 + 5}$ from $x = 0$ to $x = 4$ rotated about the x-axis.
Example Find the volume of a solid generated when the region between the curve $y = 4 - x^2$ and the line $y = 0$ and $y = 4$ is rotated about y-axis.