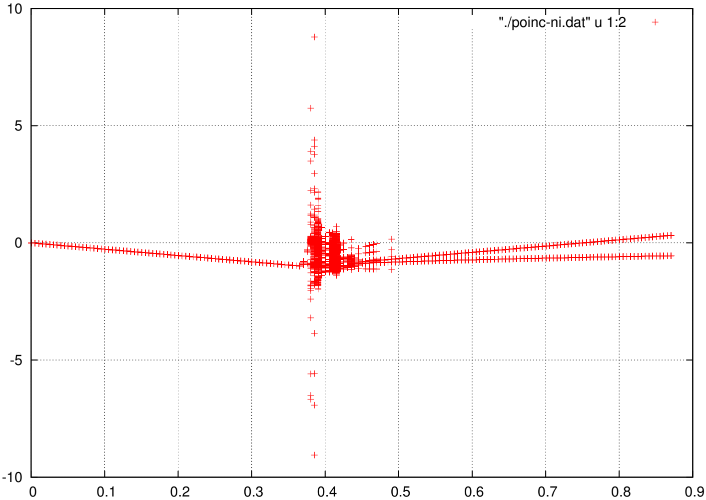
Bifurcations in continuous time dynamical systems

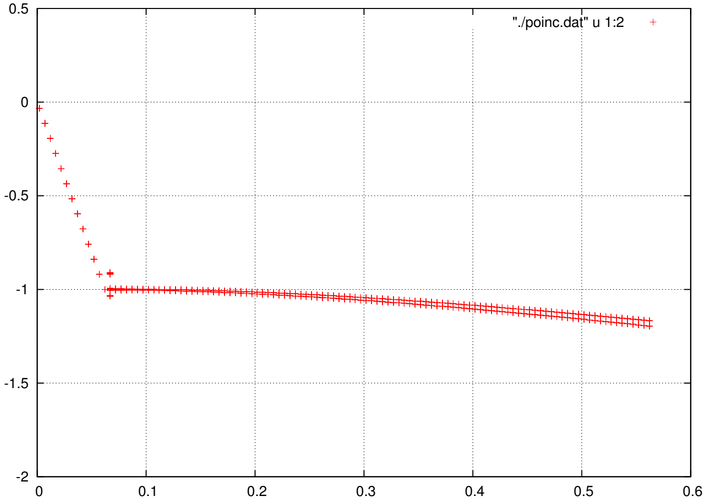
Debsankha Manik

April 4, 2013

Fixed points by Newton-Raphson Method

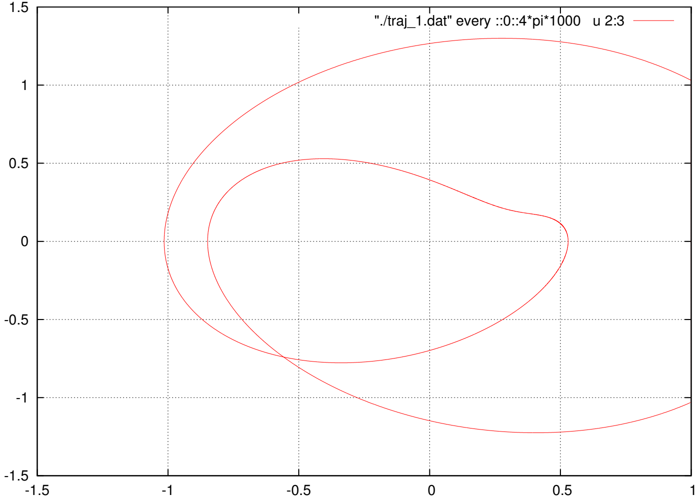
After grazing, a new period-1 orbit gains stability (sometimes after a short spell of chaos)





"./poinc.dat" u 1:2
0.5
0
-0.5
-1
-1.5
-2
0
0.1
0.2
0.3
0.4
0.5
0.6

Figure:



How to pinpoint these fixed points?

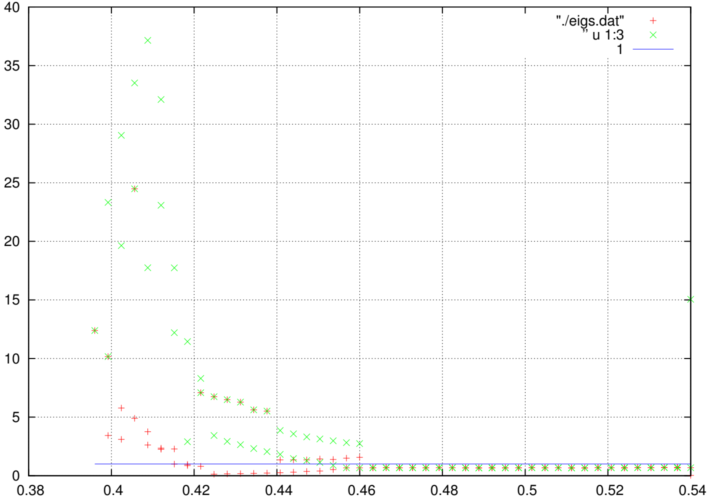
Define a 5-dim vector $y = (x_0, v_0, x_c, v_c, \tau)$.
And a system of equations:

$$\begin{aligned} G_{1,2}(y) &= \vec{x_1} - \varphi(\tau, 0, \vec{x_0}) = 0 \\ G_{3,4}(y) &= \vec{x_0} - \varphi(2T, \tau, \vec{x_1}) = 0 \\ G_5(y) &= x_1 - \sigma = 0 \end{aligned}$$

Then we can solve the equation $G(y) = 0$ using NR method:

$$y_{n+1} = y_n + J(y)^{-1} G(y)$$

Figure: Eigenvalues of the fixed points vs. bifurcation parameter



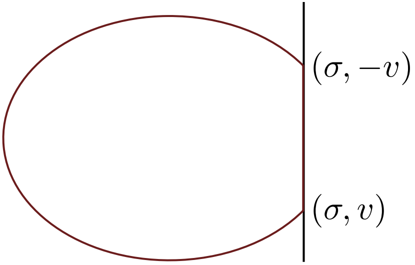
Problems

- Sometimes we get unphysical fixed points.
- We get 2 fixed points for each period-1 orbit
- Sometimes, the eigenvalues of those 2 fixed points are different!

Impact map approach

C. Budd, F. Dux, A. Cliffe, The effect of frequency and clearance variations on single-degree-of-freedom impact oscillators, Journal of Sound and Vibration, Volume 184, Issue 3, 20 July 1995

Figure: Impact map in case of $PnC1$ orbit



$$M_I(\sigma, v) \mapsto (\sigma, -v)$$

DERIVATION

Budd et al did it for the without damping. Introduction of damping makes things more interesting, as we will see.

$$\begin{pmatrix} x^-(\tau) \\ v^-(\tau) \end{pmatrix} = \begin{pmatrix} x_p(\tau) \\ v_p(\tau) \end{pmatrix} + \begin{pmatrix} x_h(\tau) \\ v_h(\tau) \end{pmatrix} \tag{1}$$

$$\begin{pmatrix} x^+(\tau) \\ v^+(\tau) \end{pmatrix} = \begin{pmatrix} x_p(\tau) \\ v_p(\tau) \end{pmatrix} + \begin{pmatrix} x_h(\tau) \\ -v_h(\tau) - 2v_p(\tau) \end{pmatrix} \tag{2}$$

$$\begin{pmatrix} x^-(\tau + nT) \\ v^-(\tau + nT) \end{pmatrix} = \begin{pmatrix} x_p(\tau + nT) \\ v_p(\tau + nT) \end{pmatrix} + M(nT) \begin{pmatrix} x_h(\tau) \\ -v_h(\tau) - 2v_p(\tau) \end{pmatrix} \tag{3}$$

$$= \begin{pmatrix} x_p(\tau) \\ v_p(\tau) \end{pmatrix} + M(nT) \begin{pmatrix} x_h(\tau) \\ -v_h(\tau) - 2v_p(\tau) \end{pmatrix} \tag{4}$$

Provided a $PnC1$ orbit exists (stable or unstable),

$$M(nT)\begin{pmatrix} x_h(\tau) \\ -v_h(\tau) - 2v_p(\tau) \end{pmatrix} = \begin{pmatrix} x_h(\tau) \\ v_h(\tau) \end{pmatrix}$$

and

$$x_p(\tau) + x_h(\tau) = \sigma$$

We recall:

$$x_p(t) = A^1 cos(\omega t + tan^{-1}\frac{\omega\gamma}{\omega^2 - \omega_0^2})$$

$$\begin{aligned} v_p(t) &= -A\omega sin(\omega t + tan^{-1}\frac{\omega\gamma}{\omega^2 - \omega_0^2}) \\ &= \mp A\omega\sqrt{1 - \left(\frac{x_p}{A}\right)^2} \\ &= \mp A\omega\sqrt{1 - \left(\frac{\sigma - x_h}{A}\right)^2} \end{aligned}$$

[1] $A = F/\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$

The condition for existence of a $PnC1$ orbit:

$$M(nT)\begin{pmatrix} x \\ -v \pm 2\omega\sqrt{A^2-(\sigma-x_h)^2} \end{pmatrix} = \begin{pmatrix} x \\ v \end{pmatrix} \tag{5}$$

The uncertainty about the $\pm$ sign is slightly unsettling. There actually *isn't* any justification for choosing one sign over the other : it is a direct consequence of the fact that for the same position, the velocity of the oscillator can have two values equal in magnitude but with opposite signs.
Suppose

$$M(nT) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Equations for the fixed point

$$x = ax - bv \pm 2b\omega\sqrt{A^2 - (\sigma - x)^2} \tag{6}$$

$$v = cx - dv \pm 2d\omega\sqrt{A^2 - (\sigma - x)^2} \tag{7}$$

v can be easily eliminated:

$$\frac{x(1-a) + bv}{b} = \frac{v(1+d) - cx}{d}$$
$$bv = x(d - ad + bc)$$

Substituting in (6):

$$x(a - d + ad - bc) \pm 2b\omega\sqrt{A^2 - (\sigma - x)^2} = x$$
$$A^2 - (\sigma - x)^2 - \left\{\frac{(a - d + ad - bc)}{2b\omega}\right\}^2 x^2 = 0$$

Let

$$\alpha = \frac{(a - d + ad - bc)}{2b\omega}$$

Then:

$$\begin{aligned} A^2 - (\sigma - x)^2 \alpha^2 x^2 \\ x^2(\alpha^2 + 1) - 2\sigma x + (\sigma^2 - A^2) = 0 \end{aligned}$$

Therefore we have the solution:

$$x^* = \frac{\sigma \pm \sqrt{\sigma^2 - (\alpha^2 + 1)(\sigma^2 - A^2)}}{\alpha^2 + 1} \tag{8}$$

$$v^* = \frac{(d - ad + bc)x^*}{b} \tag{9}$$