Bifurcations in continuous time dynamical systems

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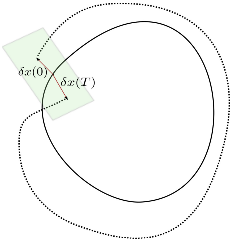
Poincare section for non-autonomous systems

$$t \mod T = 0$$



The Poincare map:

$$x(T) = f(T)x(0)$$

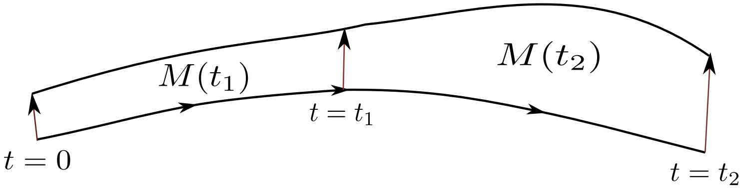
The Poincare map can be locally linearized in the neighbourhood of a fixed point:

$$\delta x(T) = M(T)\delta x(0)$$

The Monodromy Matrix

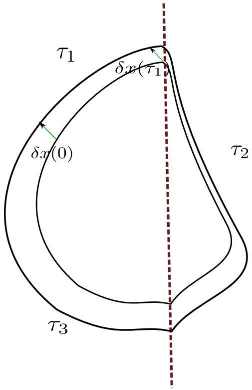
$$\delta x(T) = M(T)\delta x(0)$$

The eigenvalues of M determine the stability of the fixed point.



$\delta x(t_1 + t_2) = M(t_2)M(t_1)\delta x(0) = M(t_1 + t_2)\delta x(0)$ Clearly, $M(t_1 + t_2) = M(t_1 + t_2)$.

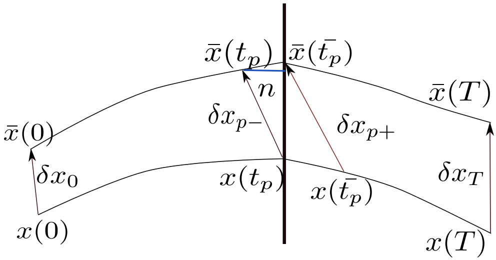
WHAT ABOUT BORDER COLLISION?



$$M \neq M(\tau_3)M(\tau_2)M(\tau_1)$$

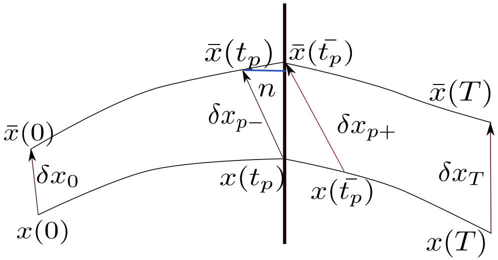
Although x and $x + \delta x$ start close by, they do not hit the boundary simultaneously. So some correction factor must be applied to M.

Saltation matrix



We know that $M(T) \neq M(T - t_p)M(t_p)$. What is the correction factor?

Saltation matrix

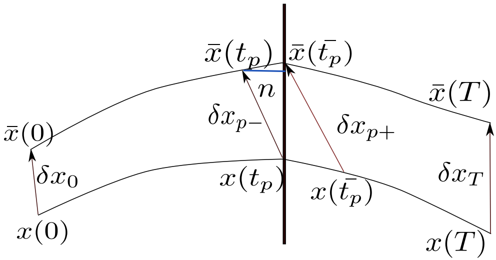


We know that $M(T) \neq M(T - t_p)M(t_p)$. What is the correction factor?

We need to find out S satisfying:

$$\delta x_{p+} = S\delta x(p-)$$

Let $\delta t = \bar{t_p} - t_p$

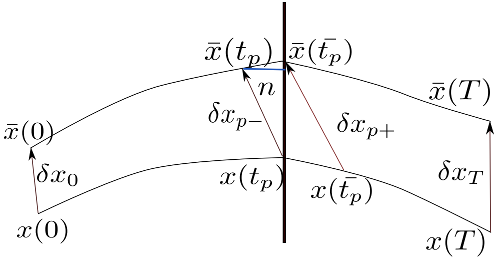


$$\bar{x}(\bar{t}_p) = \bar{x}(t_p) + f_{p-}\delta t \tag{1}$$

$$x(\bar{t}_p) = x(t_p) + f_{p+}\delta t \tag{2}$$

Subtracting:

$$\delta x_{p+} = \delta x_{p-} + (f_{p-} - f_{p+})\delta t \tag{3}$$



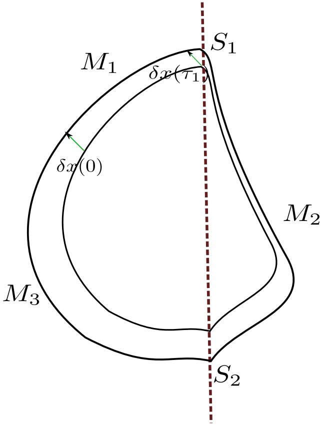
To evaluate δt, note that:

$$n^T f_{p-} \delta t = -n^T \delta x_{p-} \tag{4}$$

$$\delta t = -\frac{n^T \delta x_{p-}}{n^T f_{p-}} \tag{5}$$

$$\delta x_{p+} = \delta x_{p-} + (f_{p+} - f_{p-}) \frac{n^T \delta x_{p-}}{n^T f_{p-}} \tag{6}$$

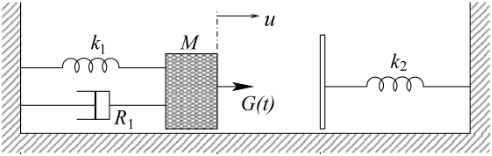
$$S = I + \frac{(f_{p+} - f_{p-}) n^T}{n^T f_{p-}} \tag{7}$$



$$M = M_3 S_2 M_2 S_1 M_1$$

SOFT IMPACT IN SIMPLE HARMONIC OSCILLATOR

Figure :



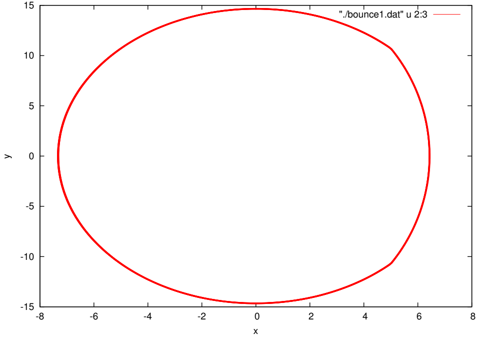
The non-forced case:

$$X = \begin{bmatrix} x \\ v \end{bmatrix} \tag{8}$$

$$F_1(X) = \begin{bmatrix} 0 & 1 \\ -k_1 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} \tag{9}$$

$$F_2(X) = \begin{bmatrix} 0 & 1 \\ -k_1 - k_2 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} \tag{10}$$

Figure : One trajectory



Monodromy Matrix

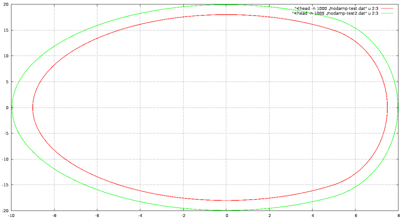
$$
\begin{aligned}
X(t) &= \exp\left(\begin{bmatrix} 0 & 1 \\ -k_1 & 0 \end{bmatrix} t\right)\begin{bmatrix} x(0) \\ v(0) \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{Cos}(tw) & \frac{\mathrm{Sin}(tw)}{w} \\ -w\mathrm{Sin}(tw) & \mathrm{Cos}(tw) \end{bmatrix}\begin{bmatrix} x(0) \\ v(0) \end{bmatrix}
\end{aligned}
$$

$(k = w^2)$

$$
\begin{aligned}
S &= I + \frac{(f_{p+} - f_{p-})n^T}{n^T f_{p-}} && (11) \\
&= I + \begin{bmatrix} 0 & 0 \\ -k_2 x & 0 \end{bmatrix} / v && (12)
\end{aligned}
$$

Monodromy Matrix

Figure : Undamped case



$$M(T) = M(\tau_3) \cdot S_2 \cdot M'(\tau_2) \cdot S_1 \cdot M(\tau_1)$$

Now, $det(M) = det(M') = 1$
$det(S_1) = det(S_2) = 1$ as well.
Therefore all periodic orbits are neither attracting nor repelling

DRIVEN, DAMPED CASE

$$F_1 = -k_1 x - G_1 v + Fcos(\omega t)$$
$$F_2 = -(k_1 + k_2)x - G_1 v + Fcos(\omega t)$$

The full solution:

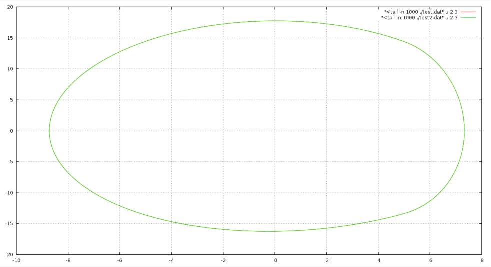
$$x(t, x_0, v_0) = x_p(t) + x_h(t, x_0, v_0)$$

▸ Go to solution

Initial condition affects x_h *only*.
Therefore, $\delta X(t)$ is not dependent on the forcing function at all.
Moreover:

$$X_h(t) = O \begin{bmatrix} e^{\lambda_+ t} & 0 \\ 0 & e^{\lambda_- t} \end{bmatrix} O^{-1}$$

Where $\lambda_\pm = \frac{-G \pm \sqrt{G^2 - 4k}}{2}$
Therefore, $det(M) = exp(-G)$



As before,

$$M(T) = M(\tau_3) \cdot S_2 \cdot M'(\tau_2) \cdot S_1 \cdot M(\tau_1)$$

$det(M), det(M') < 1$
$det(S_1) = det(S_2) = 1$, as before.
Therefore $det(M_{total}) < 1$

But that implies there can be no chaos in this kind of systems.

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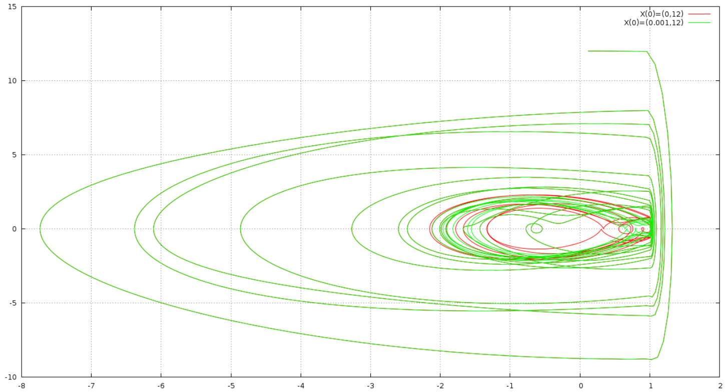
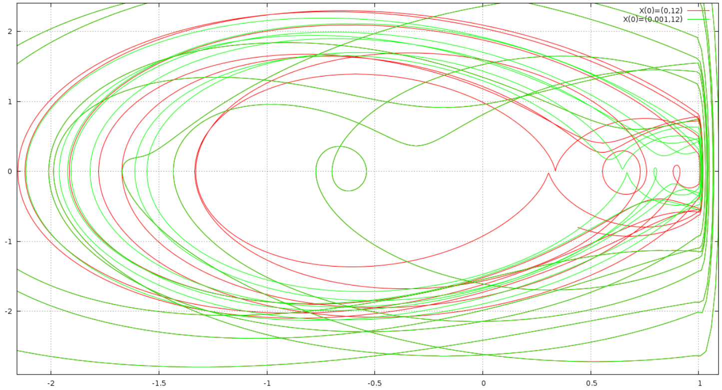
It is not true:




X(0)=(0,12)
X(0)=(0.001,12)
2
1
0
-1
-2
-2
-1.5
-1
-0.5
0
0.5
1

$$\begin{aligned}
\ddot{x} + G\dot{x} + kx &= Fcos(\omega t) \\
x_p(t) &= \frac{F}{(k-\omega^2)^2 + \omega^2 G^2} cos(\omega t + tan^{-1}\frac{\omega G}{\omega^2 - k}) \\
x_h(t) &= O \begin{bmatrix} e^{\lambda_+ t} & 0 \\ 0 & e^{\lambda_- t} \end{bmatrix} O^{-1} \\
O &= \begin{bmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{bmatrix} \\
\lambda_\pm &= \frac{-G \pm \sqrt{G^2 - 4k}}{2}
\end{aligned}$$

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