Bifurcations in continuous time dynamical systems

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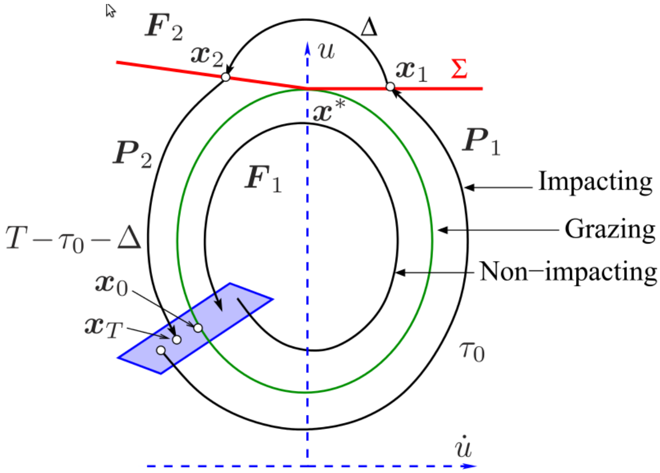
Grazing

Figure : Poincare section for grazing orbit

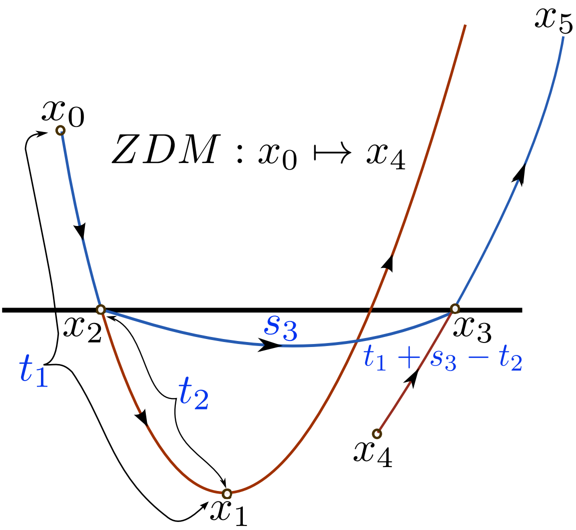


PRPPERTIES OF GRAZING ORBIT

Let the grazing take place at $x = 0$.

- $H(0) = 0$
- $\frac{dH}{dx} \neq 0$
- $\frac{d}{dt} H(\varphi_i(o,t))|_{t=0} = 0$
- $a_i := \frac{d^2}{dt^2} H(\varphi_i(o,t))|_{t=0} > 0$

Figure : What is ZDM?

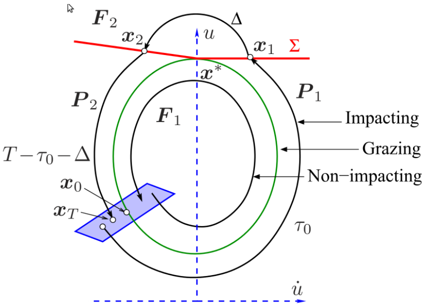


Why the ZDM?

Why ZDM?

Consider a Poincare section t mod $T = 0$. Then the Poincare map would be simplified to:

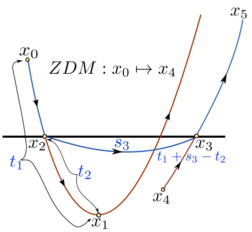
$$f(x_n) = \varphi_2(ZDM(\varphi_1(x_n, \tau_0)), T - \tau_0)$$



Does the ZDM exist?

Calculate t_1: Let $T_1(x)$ be the function solving:

$$
\begin{aligned}
E_1(x, T_1) &= 0 \\
\left.\frac{\partial H(\varphi_1(x,t))}{\partial t}\right|_{T_1(x)} &= 0 \\
\frac{dH}{dx}(\varphi_1(x,t))\frac{\partial \varphi_1(x,t)}{\partial t}\ |_{T_1(x)} &= 0 \\
\frac{dH}{dx}(\varphi_1(x,t))F_1(x, T_1(x)) &= 0
\end{aligned}
$$



and

$$T_1(0) = 0 \tag{1}$$

Question: Can such a function ever be found?

Answer: Implicit function theorem

Given any equation $f(x, y) = 0$, an explicit function $y = y(x)$ can always be found in the neighbourhood of a point (x_0, y_0) satisfying $f(x_0, y_0) = 0$ if:

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} \neq 0$$

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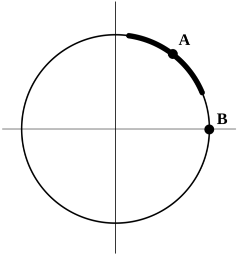
In other words, $y = y(x)$ can always be found if $y = y(x)$ exists!



$$f(x,y) = x^2 + y^2 - 1$$

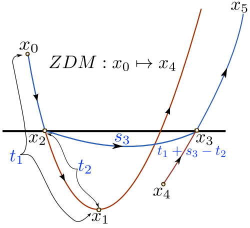
$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = 2y_0$$

Now, in case of (1), $T_1(x)$ will be found if:

$$\frac{\partial E_1}{\partial t}(0,0) = \frac{\partial^2}{\partial t^2} H(\varphi_i(x,t))|_{0,0} = a_i \neq 0$$

But that was the last criterion of our grazing orbit!

So, $T_1(x)$ can be found and $H_{min}(x) = H(\varphi_1(x_0, T_1(x_0))) = H(x_1)$ is the minimum value of H that can possibly be found in a travectory starting with x_0 (Completely disregarding the boundary.)



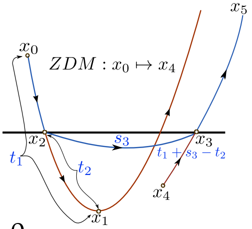
Calculate t_2: Can we do the same trick and extract t_2?.

Attempt 1: $H(\varphi_1(x_1, -T_2(x_1))) = 0$ close to $T_2(0) = 0$.

IFT not applicable because there are multiple values of T_2.

Attempt 2: Solve for $E_2(x, y, T_2)$ given by:

$$T_2\sqrt{\frac{H(\varphi_1(x, T_1(x) - T_2)) - H(\varphi_1(x, T_1(x)))}{T_2^2}} - y = 0$$



Around $(0, 0, 0)$.
$t_2 = T_2(x_0, \sqrt{(} - H_{min}(x_0)))$

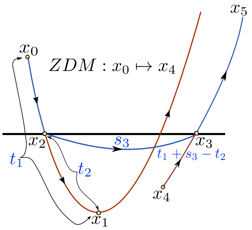
Calculate s_3:

$$E_3(x, S_3) = \frac{H(\varphi_2(x, s_3)) - H(x)}{s_3} = 0$$

Near $(0, 0)$.
$s_3 = S_3(x_2)$

(Again manipulated to remove one root)

This proves that ZDM exists.



Actually calculating it

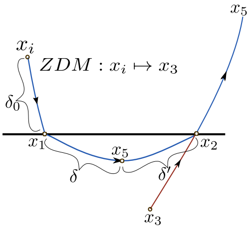
First intersection:

$$x_1 \approx x_i + \frac{\partial \varphi_1(x,t)}{\partial t} \Big|_{(x_i,0)} \delta_0$$
$$= x_i + F_1(x_i)\delta_0$$

Upto 1st order:

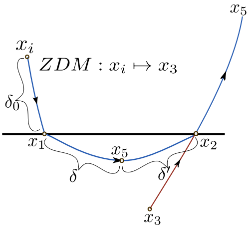
$$x_1 = x_i + F_1^* \delta_0 \qquad (2)$$

$F^* := F(x)|_{x^*}$



Calculate δ:

$$
\begin{aligned}
H(x_1) &= H(\varphi(x_5, -\delta)) \\
&\approx H(\varphi_2(x_5, 0)) + \frac{\partial}{\partial t} H(\varphi_2(t, x_5)) \mid_{t=0} (-\delta) \\
&+ \frac{\partial^2}{\partial t^2} H(\varphi_2(t, x_5)) \mid_{t\approx 0} \frac{\delta^2}{2} \\
&\approx H(x_5) + \frac{\partial^2}{\partial t^2} H(\varphi_2(t, 0)) \mid_{t\approx 0} \frac{\delta^2}{2} \\
&\approx H(x_5) + a_2^* \frac{\delta^2}{2}
\end{aligned}
$$



Let $H(x_5) = H_{min}(x_i) = -y^2$:

$$\delta = y\sqrt{\frac{2}{a_2^*}} \tag{3}$$

Calculate ZDM: Using identical arguments,

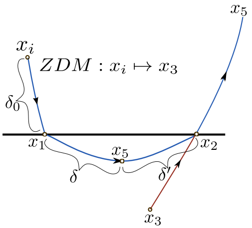
$\delta' = y\sqrt{\frac{2}{a_2^*}}$

Let $\delta + \delta' = \delta_1$

Then:

$$x3 \approx x_2 - F_1(x_2)\delta_1 + \dot{F}_1(x_2)\frac{\delta_1^2}{2} - \ddot{F}_1(x_2)\frac{\delta_1^3}{6} \tag{4}$$

$$x1 \approx x_2 - F_2(x_2)\delta_1 + \dot{F}_2(x_2)\frac{\delta_1^2}{2} - \ddot{F}_2(x_2)\frac{\delta_1^3}{6} \tag{5}$$



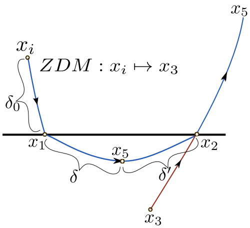
$$x_3 = x_1 - (F_1 - F_2)(x_2)\delta_1 + (\dot{F}_1 - \dot{F}_2)(x_2)\frac{\delta_1^2}{2} - (\ddot{F}_1 - \ddot{F}_2)(x_2)\frac{\delta_1^3}{6} \tag{6}$$

A system with discontinuous first derivative:

$$F_1(x) = F(x) \tag{7}$$
$$F_2(x) = F(x) + GH(x) \tag{8}$$

Then, ignoring the spatial derivatives of $H(x)$:

$$\begin{aligned}
(F_1 - F_2)(x_2) &\approx (F_1 - F_2)(x^*) \\
&= GH(0) \\
&= 0 \\
(\dot{F}_1 - \dot{F}_2)(x_2) &= G\dot{H}(0) \\
&= 0 \\
(\ddot{F}_1 - \ddot{F}_2)(x_2) &= G\ddot{H}(0) \\
&= Ga^*
\end{aligned}$$



$$x_3 = x_i + \frac{8}{3} G \sqrt{\frac{2}{a^*}} y(x_i)^3$$

General form of ZDM

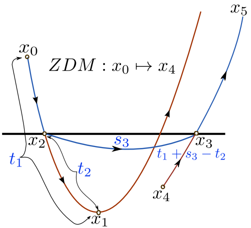
Evaluating t_2

$$0 = H(\varphi_1(x_1, t_2))$$

$$0 = H(\varphi_1(x_1, t_2)) - (y^2 + H(x_1)) - \mathcal{L}_{F_1} H(x_1) t_2$$

$$0 \approx H(x_1) + \mathcal{L}_{F_1} H(x_1) t_2 + \mathcal{L}_{F_1}^2 H(x_1) \frac{t_2^2}{2} + \mathcal{L}_{F_1}^3 H(x_1) \frac{t_2^3}{6} - y^2 - H(x_1) - \mathcal{L}_{F_1} H(x_1) t_2$$

$$0 \approx \mathcal{L}_{F_1}^2 H(x_1) \frac{t_2^2}{2} + \mathcal{L}_{F_1}^3 H(x_1) \frac{t_2^3}{6} - y^2$$



It can be argued from IFT that $t_2(x_1, y)$ can be expressed as apower series:

$$t_2 = A(x_1) y + B(x_1) y^2 + O(y^3)$$

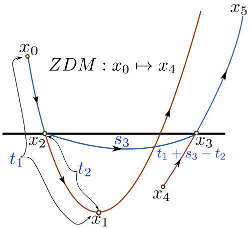
General form of ZDM

The coefficients A, B are easily computed:

$$t_2(x_1, y) = -\sqrt{\frac{2}{\mathcal{L}^2_{F_1} H(x_1)}}y - \frac{1}{3}\frac{\mathcal{L}^3_{F_1} H(x_1)}{(\mathcal{L}^2_{F_1} H(x))^2}y^2 + O(y^3) \tag{9}$$

There are multiple solutions, use the constraint $t_2 < 0$ to eliminate spurious ones

Evaluating s_3



$$\begin{aligned} H(\varphi_2(x_2, s_3)) &= 0 \\ \mathcal{L}_{F_2} H(x_2) s_3 + \mathcal{L}^2_{F_2} \frac{s_3^2}{2} &\approx 0 \end{aligned}$$

Now put $x_2 = \varphi_1(x_1, t_2)$ and Taylor expand:

General form of ZDM

$$\mathcal{L}_{F_2}H(x_1)s_3 - C(x_1)\mathcal{L}_{F_1}\mathcal{L}_{F_2}H(x_1)ys_3 + \frac{1}{2}\mathcal{L}_{F_1}^2 H(x_1)s_3^2 + O(s_3^3) = 0$$

Where $C(x) = \sqrt{\frac{2}{\mathcal{L}_{F_1}^2 H(x)}}$.

General form of ZDM

$$\mathcal{L}_{F_2}H(x_1)s_3 - C(x_1)\mathcal{L}_{F_1}\mathcal{L}_{F_2}H(x_1)ys_3 + \frac{1}{2}\mathcal{L}_{F_1}^2 H(x_1)s_3^2 + O(s_3^3) = 0$$

Where $C(x) = \sqrt{\frac{2}{\mathcal{L}_{F_1}^2 H(x)}}$.
Note that $\mathcal{L}_{F_2}H(x_1)$ should be zero because $\mathcal{L}_{F_1}H(x_1) = 0$ and to prevent sliding the lie derivatives of the two flows must have same sign in close neighbourhoods of the boundary.

General form of ZDM

$$\mathcal{L}_{F_2}H(x_1)s_3 - C(x_1)\mathcal{L}_{F_1}\mathcal{L}_{F_2}H(x_1)ys_3 + \frac{1}{2}\mathcal{L}_{F_1}^2H(x_1)s_3^2 + O(s_3^3) = 0$$

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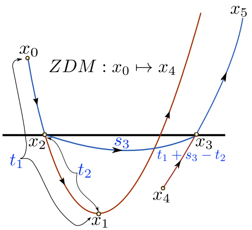
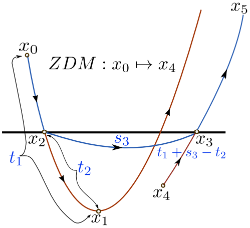
Now the non-zero solution for t_3 is easily obtained:



$$s_3(x_1, y) = 2\frac{\mathcal{L}_{F_1}\mathcal{L}_{F_2}H(x_1)}{\mathcal{L}_{F_2}^2H(x_1)}C(x_1)y + O(y^2) \tag{10}$$

General form of ZDM

$$
\begin{aligned}
ZDM(x_1) &= \varphi_1(\varphi_2(\varphi_1(x_1, t_2), s_3), -t_2 - s_3) \\
&= \varphi_1(\varphi_2(x_1 + F_1 t_2, s_3), -t_2 - s_3) + O(s_3^2) \\
&= \varphi_1(x_1 + F_1 t_2 + F_2 s_3, -t_2 - s_3) + O((s_3, t_2)^2) \\
&= x_1 + F_1 t_2 + F_2 s_3 - F_1(t_2 + s_3) + O((s_3, t_2)^2) \\
&= x_1 + s_3(F_2 - F_1)(x_1) + O((s_3, t_2)^2) \\
&\approx x_1 + 2\frac{\mathcal{L}_{F_1}\mathcal{L}_{F_2}H(x_1)}{\mathcal{L}^2_{F_2}H(x_1)} y \sqrt{\frac{2}{\mathcal{L}^2_{F_1}H(x)}}(F_2 - F_1)(x_1)
\end{aligned}
$$

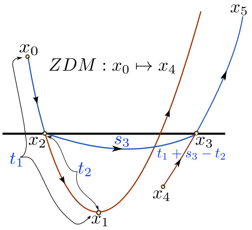


Where $y = \sqrt{-H(x_1)}$. But in general, we need $ZDM(x_0)$.

General form of ZDM

Have to solve for t_1:

$$\begin{aligned}\mathcal{L}_{F_1} H(\varphi_1(x_0, t_1)) &= 0 \\ \mathcal{L}_{F_1} H(x_0) + \mathcal{L}^2_{F_1} H(x_0) t_1 &= 0 \\ \rightarrow t_1 = -\frac{\mathcal{L}_{F_1} H(x_0)}{\mathcal{L}^2_{F_1} H(x_0)}\end{aligned}$$



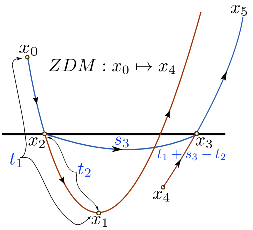
$$\begin{aligned}H_{min}(x) &= H(\varphi_1(x, t_1(x))) && (11) \\ &= H(x) + \mathcal{L}_{F_1} H(x) t_1(x) && (12) \\ &= H(x) - \frac{(\mathcal{L}_{F_1} H(x_0))^2}{\mathcal{L}^2_{F_1} H(x_0)} && (13)\end{aligned}$$

General form of ZDM

$$\begin{aligned} ZDM(x_0) =& \varphi_1(\varphi_2(\varphi_1(x_0, t_1 + t_2), s_3), -t_1 - t_2 - s_3) \\ \approx& x_0 + 2\frac{\mathcal{L}_{F_1}\mathcal{L}_{F_2}H(x_1)}{\mathcal{L}^2_{F_2}H(x_1)} y \sqrt{\frac{2}{\mathcal{L}^2_{F_1}H(x)}}(F_2 - F_1)(x_1) \end{aligned}$$

Where $y = \sqrt{-H(x) + \frac{(\mathcal{L}_{F_1}H(x_0))^2}{\mathcal{L}^2_{F_1}H(x_0)}}$.

Since we are keeping terms only upto the 1st order of the small parameter(s_3), we can evaluate all functions at the grazing point x^* instead of x_1, provided $|x^* - x_1| \approx 0$



THE CASE OF DISCONTINUOUS FIRST DERIVATIVE

What if $(F_2 - F_1)x^* = 0$**?** Then of course, the leading order term in the discontinuity map will be *2nd order* in the small parameters t_1, t_2, s_3. Consequentially, we must evaluate those parameters upto 2*nd* order themselves.

THE CASE OF DISCONTINUOUS FIRST DERIVATIVE

We will see that it will involve leading order discontinuity term of order $O(y^3) = O(|x|^3/2)$.
The procedure is identical, only upto higher order:

$$ZDM(x_0) = \varphi_1(\varphi_2(\varphi_1(x_0, t_1 + t_2), s_3), -t_1 - t_2 - s_3)$$

$$\varphi_1(x_0, t_1 + t_2) \approx x_0 + F_1(x_0)(t_1 + t_2) + \frac{dF_1(x_0)}{dt}(t_1 + t_2) + \frac{d^2F_1(x_0)}{dt^2}(t_1 + t_2)^2/2 + \frac{d^3F_1(x_0)}{dt^3}(t_1 + t_2)^3/6$$

Ultimately we get: $ZDM(x_0) = P(x_0, t_1, t_2, s_3) + O(t_1^4, t_2^4, s_3^4)$.
Next, we need to re-evaluate t_1, t_2 upto order y^3 and replace in the formula for ZDM. That completes the job.

Poincare map

$$\vec{X_{n+1}} = N_2 \cdot ZDM \cdot N_1 \vec{X_n}$$