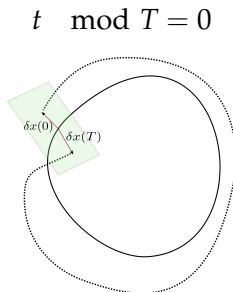


Bifurcations in continuous time dynamical systems

Debsankha Manik

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POINCARÉ SECTION FOR NON-AUTONOMOUS SYSTEMS



The Poincaré map:

$$x(T) = f(T)x(0)$$

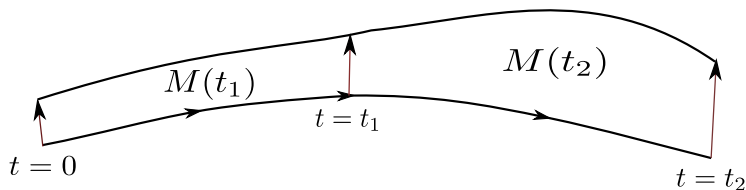
The Poincaré map can be locally linearized in the neighbourhood of a fixed point:

$$\delta x(T) = M(T)\delta x(0)$$

THE MONODROMY MATRIX

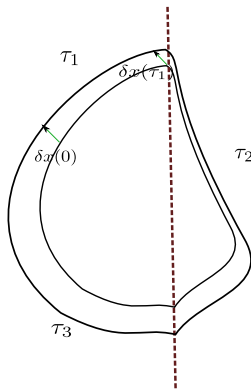
$$\delta x(T) = M(T)\delta x(0)$$

The eigenvalues of M determine the stability of the fixed point.



$\delta x(t_1 + t_2) = M(t_2)M(t_1)\delta x(0) = M(t_1 + t_2)\delta x(0)$ Clearly,
 $M(t_1 + t_2) = M(t_1 + t_2)$.

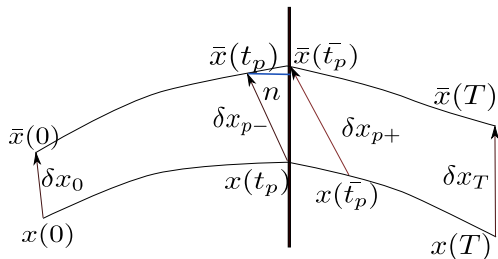
WHAT ABOUT BORDER COLLISION?



$$M \neq M(\tau_3)M(\tau_2)M(\tau_1)$$

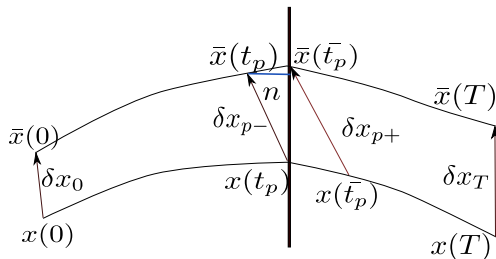
Although x and $x + \delta x$ start close by, they do not hit the boundary simultaneously. So some correction factor must be applied to M .

SALTATION MATRIX



We know that $M(T) \neq M(T - t_p)M(t_p)$. What is the correction factor?

SALTATION MATRIX

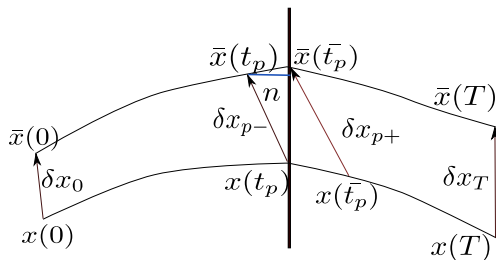


We know that $M(T) \neq M(T - t_p)M(t_p)$. What is the correction factor?

We need to find out S satisfying:

$$\delta x_{p+} = S \delta x_{p-}$$

Let $\delta t = \bar{t}_p - t_p$

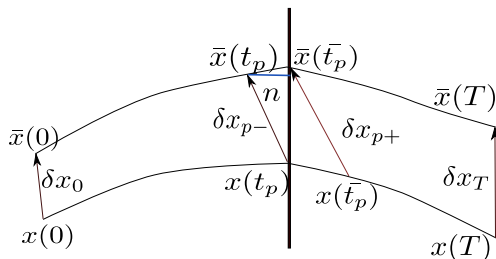


$$\bar{x}(\bar{t}_p) = \bar{x}(t_p) + f_{p-} \delta t \quad (1)$$

$$x(\bar{t}_p) = x(t_p) + f_{p+} \delta t \quad (2)$$

Subtracting:

$$\delta x_{p+} = \delta x_{p-} + (f_{p-} - f_{p+}) \delta t \quad (3)$$



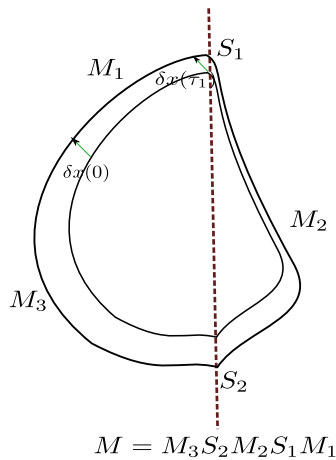
To evaluate δt , note that:

$$n^T f_{p-} \delta t = -n^T \delta x_{p-} \quad (4)$$

$$\delta t = -\frac{n^T \delta x_{p-}}{n^T f_{p-}} \quad (5)$$

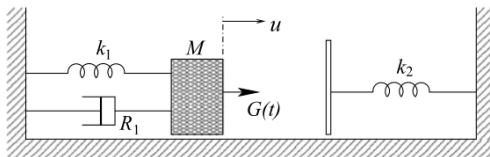
$$\delta x_{p+} = \delta x_{p-} + (f_{p+} - f_{p-}) \frac{n^T \delta x_{p-}}{n^T f_{p-}} \quad (6)$$

$$S = I + \frac{(f_{p+} - f_{p-}) n^T}{n^T f_{p-}} \quad (7)$$



SOFT IMPACT IN SIMPLE HARMONIC OSCILLATOR

Figure :



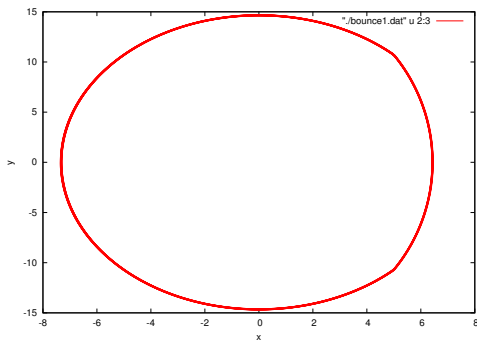
The non-forced case:

$$X = \begin{bmatrix} x \\ v \end{bmatrix} \quad (8)$$

$$F_1(X) = \begin{bmatrix} 0 & 1 \\ -k_1 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} \quad (9)$$

$$F_2(X) = \begin{bmatrix} 0 & 1 \\ -k_1 - k_2 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} \quad (10)$$

Figure : One trajectory



MONODROMY MATRIX

$$\begin{aligned}
 X(t) &= \exp \left(\begin{bmatrix} 0 & 1 \\ -k_1 & 0 \end{bmatrix} t \right) \begin{bmatrix} x(0) \\ v(0) \end{bmatrix} \\
 &= \begin{bmatrix} \text{Cos}(tw) & \frac{\text{Sin}(tw)}{w} \\ -w\text{Sin}(tw) & \text{Cos}(tw) \end{bmatrix} \begin{bmatrix} x(0) \\ v(0) \end{bmatrix}
 \end{aligned}$$

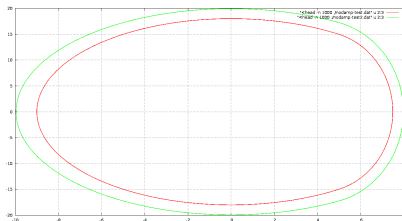
$$(k = w^2)$$

$$S = I + \frac{(f_{p+} - f_{p-})n^T}{n^T f_{p-}} \quad (11)$$

$$= I + \begin{bmatrix} 0 & 0 \\ -k_2 x & 0 \end{bmatrix} / v \quad (12)$$

MONODROMY MATRIX

Figure : Undamped case



$$M(T) = M(\tau_3) \cdot S_2 \cdot M'(\tau_2) \cdot S_1 \cdot M(\tau_1)$$

Now, $\det(M) = \det(M') = 1$

$\det(S_1) = \det(S_2) = 1$ as well.

Therefore all periodic orbits are neither attracting nor repelling

DRIVEN, DAMPED CASE

$$\begin{aligned}F_1 &= -k_1x - G_1v + F\cos(\omega t) \\F_2 &= -(k_1 + k_2)x - G_1v + F\cos(\omega t)\end{aligned}$$

The full solution:

$$x(t, x_0, v_0) = x_p(t) + x_h(t, x_0, v_0)$$

► Go to solution

Initial condition affects x_h *only*.

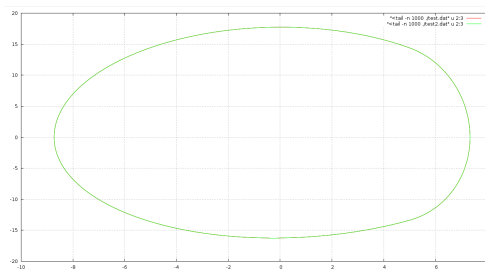
Therefore, $\delta X(t)$ is not dependent on the forcing function at all.

Moreover:

$$X_h(t) = O \begin{bmatrix} e^{\lambda_+ t} & 0 \\ 0 & e^{\lambda_- t} \end{bmatrix} O^{-1}$$

Where $\lambda_{\pm} = \frac{-G \pm \sqrt{G^2 - 4k}}{2}$

Therefore, $\det(M) = \exp(-G)$



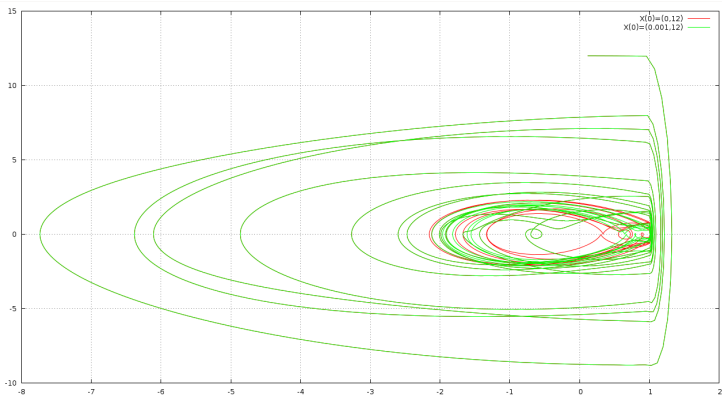
As before,

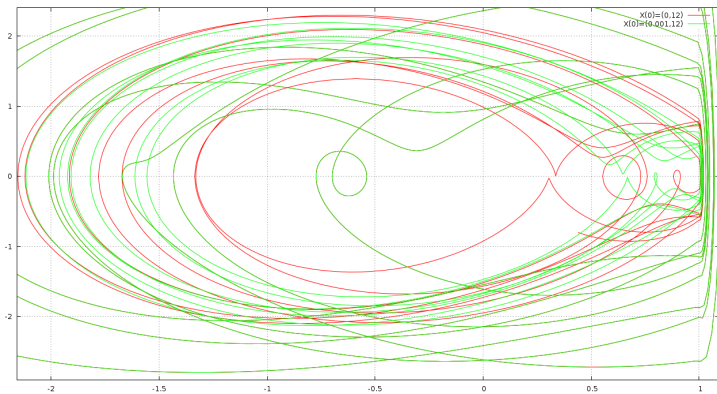
$$M(T) = M(\tau_3) \cdot S_2 \cdot M'(\tau_2) \cdot S_1 \cdot M(\tau_1)$$

$$\det(M), \det(M') < 1$$

$$\det(S_1) = \det(S_2) = 1, \text{ as before.}$$

$$\text{Therefore } \det(M_{total}) < 1$$





BEHAVIOUR NEAR GRAZING

We have seen previously that the ZDM for this kind of system is given by:

$$x' = x + \frac{8}{3}G\sqrt{\frac{2}{a^*}}y(x_i)^3$$

Here $G = -K2(x - \sigma)$

Let's derive the stroboscopic poincare section: $T = 2n\pi/W$.

There will be a grazing periodic orbit if the boundary is placed at the natural amplitude of the system (Which depends on the forcing function).

Let's start from $(-x_{max}, 0)$ i.e. the left extreme position.

Then the grazing occurs at time $T/2$.

Now let's start from $(-x_{max} + \delta x, \delta v)$.

Then the system hits the boundary at time $T/2 + \delta t$ where:

$$\delta t = (\beta \cot \frac{\beta T}{2} - \alpha) \frac{\delta x}{x(|\alpha|^2 + |\beta|^2)} + \frac{\delta v}{x(|\alpha|^2 + |\beta|^2)}$$

Therefore the Poincare map will be given by:

$$M(T/2 - \delta, DM(M(T/2 + \delta, \bar{X})))$$

This simplifies to:

$$X_n + 1 = X_n + E_h(T)X_n + E_n(T/2 - \delta t)\bar{N}$$

Where

$$\bar{N} = \frac{8}{3} \sqrt{\frac{2}{a^*}} (x_{max} - \sigma)^3 \begin{bmatrix} 0 \\ k2/m \end{bmatrix}$$

$$\ddot{x} + G\dot{x} + kx = F\cos(\omega t)$$

$$x_p(t) = \frac{F}{(k - \omega^2)^2 + \omega^2 G^2} \cos(\omega t + \tan^{-1} \frac{\omega G}{\omega^2 - k})$$

$$x_h(t) = O \begin{bmatrix} e^{\lambda_+ t} & 0 \\ 0 & e^{\lambda_- t} \end{bmatrix} O^{-1}$$

$$O = \begin{bmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{bmatrix}$$

$$\lambda_{\pm} = \frac{-G \pm \sqrt{G^2 - 4k}}{2}$$