Bifurcations in continuous time piecewise smooth dynamical systems

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A brief summary on PWS flows

A simple piecewise smooth function:

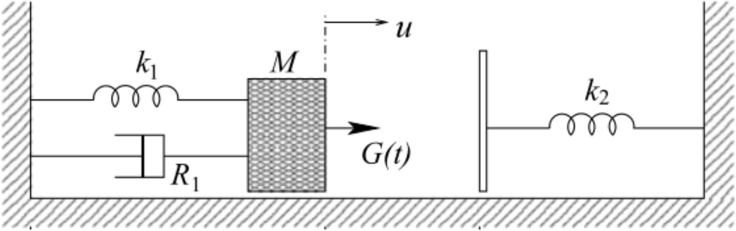
$$\dot{x} = \begin{cases} F_1(x) & : H(x) < 0 \\ F_2(x) & : H(x) > 0 \end{cases}$$

Switching manifold: $H(x) = 0$
The flows of F_1 and F_2 are $\varphi_1(x)$ and $\varphi_2(x)$ respectively, defined in respective regions and also in the neighbourhood of the switching manifold:

$$\dot{\varphi}_1 = F_1 \quad (1)$$

$$\dot{\varphi}_2 = F_2 \quad (2)$$

Figure : Example: Impact Oscillator



HOW TO ANALYZE NONLINEAR SYSTEMS?

Figure : Look at the vector field

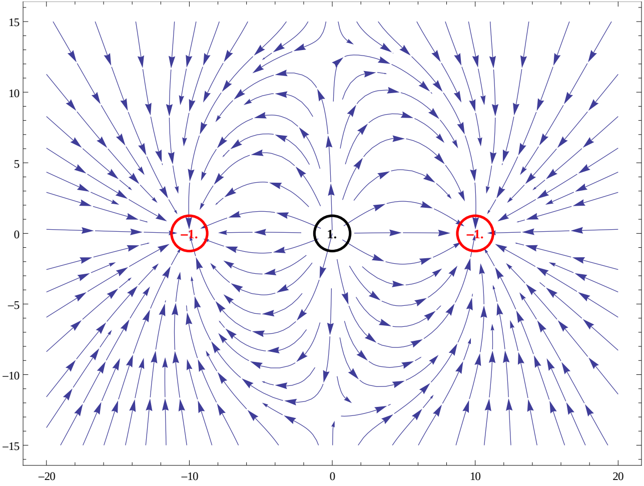
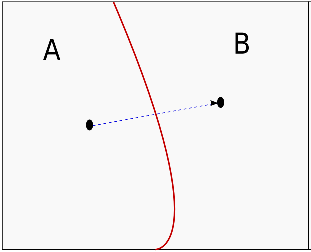
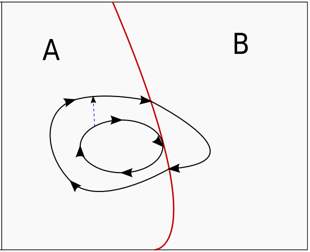


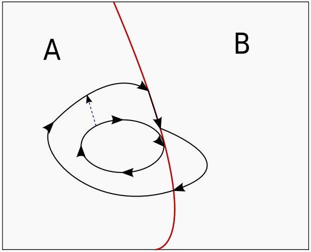
Figure : Possible scenarios



Bifurcation of equillibria



Grazing bifurcation



Sliding bifurcation

Choose a coordinate system such that:

$$\dot{\mathbf{x}} = \begin{cases} F_1(\mathbf{x}, \mu) & : x_n < 0 \\ F_2(\mathbf{x}, \mu) & : x_n > 0 \end{cases}$$

and $\mathbf{x} = \mathbf{0}$ is a grazing point.
$x_n == n$th component of $\mathbf{x}$.
Locally linearize:

$$\dot{\mathbf{x}} = \begin{cases} \mathbf{A_1}\mathbf{x} + \mathbf{B}\mu & : x_n < 0 \\ \mathbf{A_2}\mathbf{x} + \mathbf{B}\mu & : x_n < 0 \end{cases}$$

Where:

$$\mathbf{A_i} = \left.\frac{\partial \mathbf{F_i}}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{0}}$$

and $\mathbf{B} = \frac{\partial \mathbf{F_1}}{\partial \mu}\,|_{\mu=\mathbf{0}} = \frac{\partial \mathbf{F_2}}{\partial \mu}\,|_{\mu=\mathbf{0}}$ (Due to continuity).
Also, $\mathbf{A_1}$ and $\mathbf{A_2}$ can differ only in the $n-$th column (Again due to continuity).

Let: $A_1\mathbf{x_1^*} + \mathbf{B}\mu = \mathbf{0}$, $A_2\mathbf{x_2^*} + \mathbf{B}\mu = \mathbf{0}$.
Assuming A_i's are invertible:

$$\mathbf{x_i^*} = -\mathbf{A_i}^{-1}\mathbf{B}\mu = -\frac{\mathbf{adj}(\mathbf{A_i})}{\mathbf{det}(\mathbf{A_i})}\mathbf{B}\mu$$

The solutions exist iff:

$$x^*_{1_{n<0}} < 0, x^*_{2_n} > 0.$$

Now,

$$x^*_{1_k} = \frac{c^*_{1_k}}{det(\mathbf{A_1})}\mu, x^*_{2_k} = \frac{c^*_{2_k}}{det(\mathbf{A_2})}$$

Where,

$$c^*_{i_k} = [-adj(\mathbf{A_i})\mathbf{B}]_\mathbf{k} = [-\mathbf{adj}(\mathbf{A_i})_\mathbf{kj}\mathbf{B_j}]$$

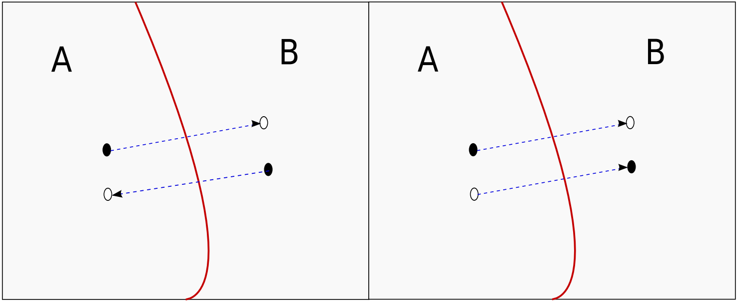
Because A_1 differs from A_2 only in $n-$th column, $adj(A_1)$ and $adj(A_2)$ shares a common $n-$th row, $c^*_{1_n} = c^*_{2_n} := C$

Condition for border crossing of equillibria

$$x^*_{1_n} = \frac{C}{det(\mathbf{A_1})}\mu, x^*_{2_k} = \frac{C}{det(\mathbf{A_2})}\mu$$

Cases:

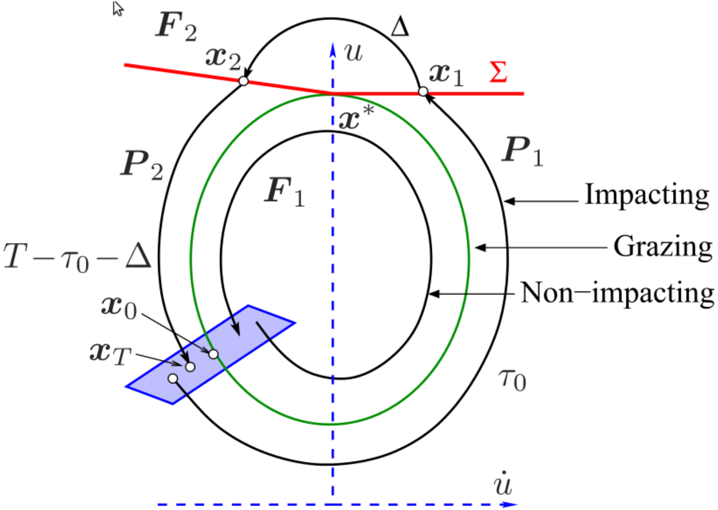
1. $det(\mathbf{A_1})det(\mathbf{A_1}) < 0$. $x^*_{1_n}$ and $x^*_{2_n}$ always have opposite signs.
2. $det(\mathbf{A_1})det(\mathbf{A_1}) > 0$. $x^*_{1_n}$ and $x^*_{2_n}$ always have same signs.



Case 1 Case 2

Grazing orbits

Figure :



GRAZING ORBITS

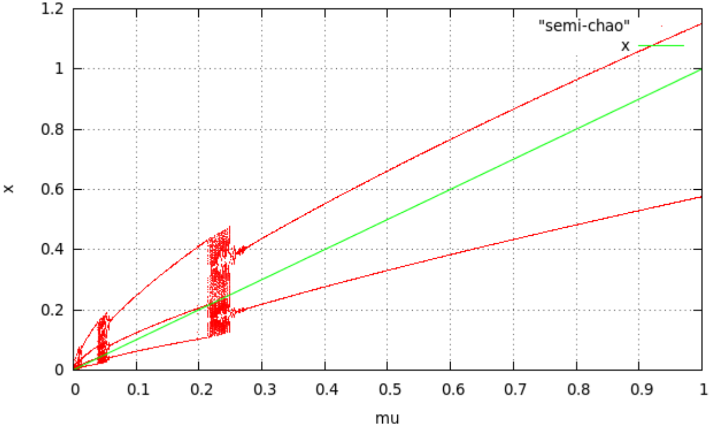
Case 1: $F_1(x) \neq F_2(x)$ at switching manifold: The Poincare map (given by Nordmak et al[4], using the ZDM formalism; also by Molenaar, without ZDM) is of the form:

$$\left.\begin{aligned} x_{n+1} &= ax_n + y_n + \rho \\ y_{n+1} &= -bx_n \end{aligned}\right\} \qquad x \leq 0$$

$$\left.\begin{aligned} x_{n+1} &= -c\sqrt{x_n} + y_n + \rho \\ y_{n+1} &= -dx_n \end{aligned}\right\} \qquad x > 0$$

- Jacobian of the system is singular.
- Infinite stretching of the phase space.
- In some cases, period-adding bifurcation occurs.
- This singularity affects only the trace of J, not the determinant. [3]

Figure : Bifurcation in a map with square root singularity



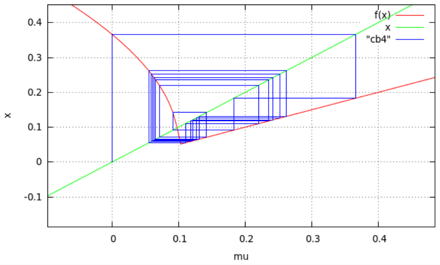
Case 2: $F_1(x) = F_2(x)$ at switching manifold: The Poincare map (given by Dankowicz and Nordmak) is of the form:

$$x_{n+1} = \begin{cases} ax_n + \rho & x \leq 0 \\ ax_n - bx_n^{3/2} + \rho & x > 0 \end{cases}$$

- Jacobian of the system is continuous.
- No chance of border collision bifurcation (Elaborate later).
- Both trace and determinant of J vary continuously.

DISCONTINUOUS JACOBIAN IS NECESSARY FOR PERIOD DOUBLING ON BC

Figure :



Suppose for the parameter value $\mu = 0$, the fixed point of the left hand size map crosses the boundary $x = 0$. Suppose a period doubling occurs.

Now the fixed point of both the maps are at δ^2, say. Then:

$$\begin{aligned}
f_2(f_1(-\epsilon^2)) &= -\epsilon^2 \\
f_2(\delta^2 + \dot{f}_1(\delta^2)(\delta^2 + \epsilon^2)) &= -\epsilon^2 \\
\delta^2 + \dot{f}_2(\delta^2)\dot{f}_1(\delta^2)(\delta^2 + \epsilon^2)) &= -\epsilon^2 \\
\dot{f}_2(\delta^2)\dot{f}_1(\delta^2) &= -1 \\
\dot{f}_2(0)\dot{f}_1(0) &\approx -1
\end{aligned}$$

Therefore the discontinuity is evident.

How to understand a border collision bifurcation

What do we do in smooth bifurcations?

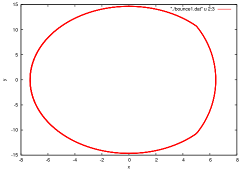
1. Take Poincare section.
2. Calculate fixed points.
3. Calculate jacobian and its eigenvalues. [If the absolute values of the eigenvalues are < 1, we will see attractive behaviour, otherwise reoulsive.]

What's the problem now?
-There's no easy way to get the Poincare map in a closed form.

However, we can still compute eigenvalues by searching for a direction which remains unchanged in each stroboscopic slice.
-But we need to find out the fixed points first.

A BORDER CROSSING ORBIT

Figure :



Supposing the orbit crosses the border only once.
Then finding out the orbit boils down to:

$$x_1 = \varphi_1(x_0, 0, \tau_0) \tag{3}$$

$$H(x_1) = 0 \tag{4}$$

$$x_2 = \varphi_2(x_1, \tau_0, \tau_1) \tag{5}$$

$$H(x_2) = 0 \tag{6}$$

$$x_0 = \varphi_1(x_2, \tau_1, T - \tau_0 - \tau_1) \tag{7}$$

This is a set of $3n+2$ equations in $3n+2$ unknowns, so this can be tackled with standard Newton's method of root finding:[3]

$$y_{n+1} = \frac{G(y_n)}{J(\bar{y}_n)}$$

Here $y := \{x_0, x_1, x_2, \tau_0, \tau_1\}$.

Theory of Border Collision Bifurcations

For a PWS system, we can now:

1. Find out all the periodic orbits. But finding out a T_j orbit involves inverting a $(2j+1)n + 2 * j$ dimensional matrix. Since best known matrix inversion algorithms are $O(n^2)$, it may not be feasible to do this for orbits that cross the border many times.

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3. Then we apply standard results regarding the stability of fixed learnt in case of smooth bifurcations to explain the phenomenon.

The locally linear map will be, in 2-D:

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{cases} \begin{bmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 0 \end{bmatrix} & x \leq 0 \\ \begin{bmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 0 \end{bmatrix} & x > 0 \end{cases}$$

The parameter space of $\delta_L, \delta_R, \tau_L, \tau_R$ can be divided into regions exhibiting different behaviours on border collision. [1]

[1] Soumitro Banerjee, Priya Ranjan, and Celso Grebogi. Bifurcations in two-dimensional piecewise smooth maps - theory and applications in switching circuits, 2000.

[2] M. Di Bernardo, F. Garofalo, L. Iannelli, and F. Vasca. Bifurcations in piecewise-smooth feedback systems. *International Journal of Control*, 75(16-17):1243–1259, 2002.

[3] Yue Ma, James Ing, Soumitro Banerjee, Marian Wiercigroch, and Ekaterina Pavlovskaia. The nature of the normal form map for soft impacting systems. *International Journal of Non-Linear Mechanics*, 43(6):504 – 513, 2008. ¡ce:title¿Non-linear Dynamics of Engineering Systems¡/ce:title¿.

[4] A.B. Nordmark. Non-periodic motion caused by grazing incidence in an impact oscillator. *Journal of Sound and Vibration*, 145(2):279 – 297, 1991.