Bifurcations in continuous time dynamical systems

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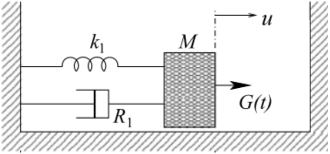
Hard impact in an oscillating system

Figure:



The equation of motion:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = F\cos\omega t \tag{1}$$

Switching manifold: If $x = \sigma$,

$$\begin{aligned} x &\mapsto x \\ v &\mapsto -v \end{aligned}$$

The solution to equation (1) is a sum of two parts: a *particular solution* that is independent of the initial conditions, and a *homogeneous solution* that is dependent on the initial conditions. To be more precise:

$$\begin{aligned} x_p(t) &= \frac{F}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}} cos(\omega t + tan^{-1}\frac{\omega\gamma}{\omega^2-\omega_0^2}) \\ x_h(t) &= \frac{e^{-\gamma t/2}}{\omega_g}\left\{(\omega_g \cos\omega_g t + \frac{\gamma}{2}\sin\omega_g t)x_0 + (\sin\omega_g t)v_0\right\} \\ \omega_g &= \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \end{aligned}$$

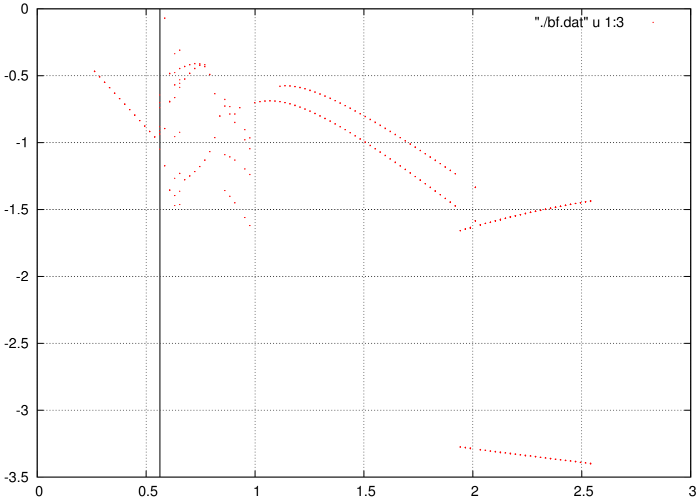
Now, $x_h(t)$ decays exponentially with time. So, if the hard wall were not there, any arbitrary initial condition would have been attracted to the period-1 orbit $x_p(t)$.

Therefore $x_p(t)$ is a limit cycle whose basin of attraction is the whole phase space (ignoring the wall).

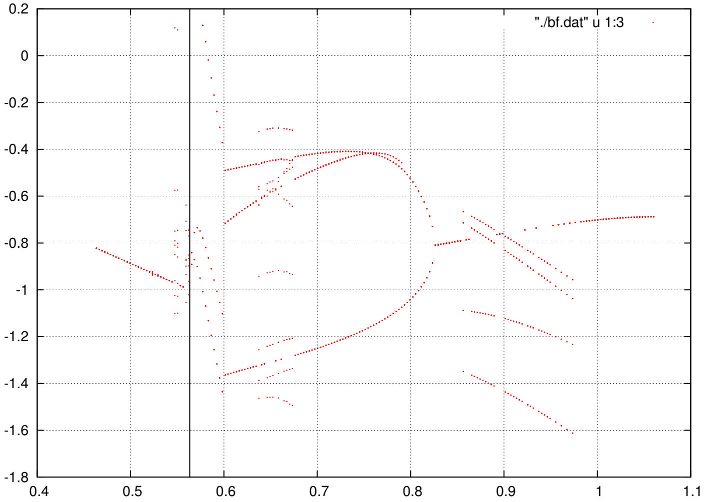
However, we note here that the friction term γ dictates how quickly any arbitrary initial condition converges to this limit cycle.

The limit cycle is basically a sinusoidal orbit with amplitude $\frac{F}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}} := F_g(\omega_0, \omega, \gamma, F)$

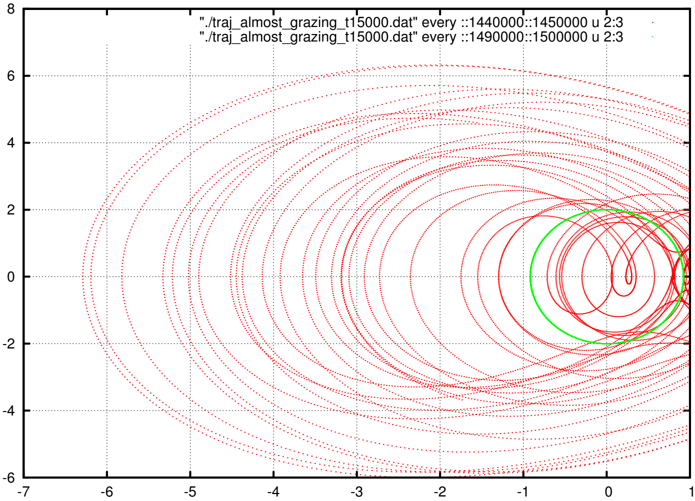
BIFURCATION W.R.T. F



A closer look



Transient behaviour



Here it needs to be emphasized that this long-lived transient is an artifact of the nonlinearity of the system.

In this particular example, $\gamma = 0.062$. Therefore, $e^{-0.062 \times 1000/2} \approx 3.44e - 14$.

The system should have settled down to the period-1 limit cycle long ago.

DEPENDENCE OF TRANSIENCE ON γ

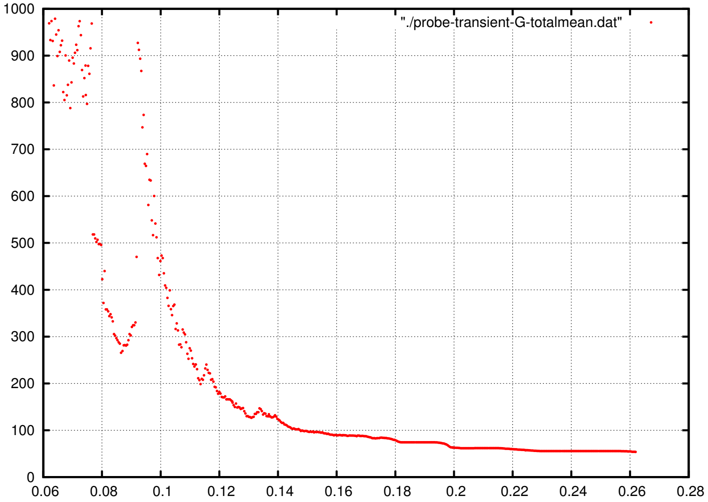
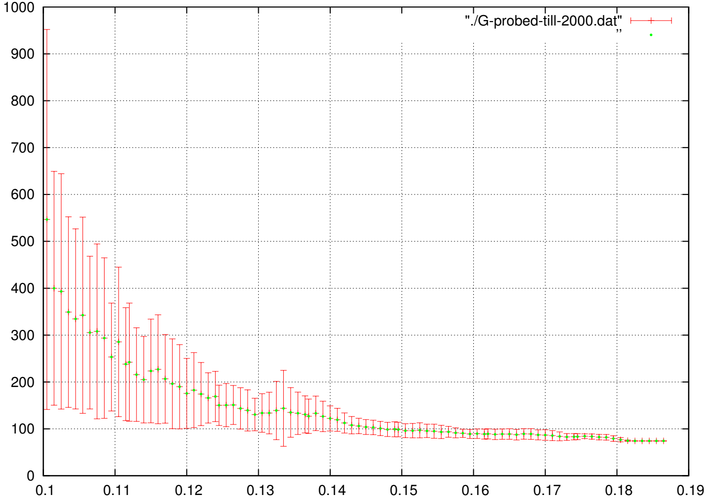


Figure: γ dependence with errorbars



Points to note:

1. We let the system evolve and determined how much time it takes for an orbit to settle down to a periodic orbit. When no such orbit emerged till $t = 2000s$, we set the time to be $2000s$ itself. For each γ value, 300 points were taken and the averages plotted.
2. We saw that the time taken to reach stability increased rapidly (insert analysis here), until $\gamma = 0.178$. After that the orbit seemed to settle on higher periodic orbits very quickly. Hence the dip in the plot. The periodicities detected were $3, 8$ and 16. Then after a small gap, those higher periodic orbits seemed to get unstable. Hence the rise in stability time again.
3. The lowest γ was set to be the one where the unpurturbed orbit (i.e. the particular solution) grazes the wall. So throughout the γ range, there exists a period-1 orbit.

DEPENDENCE OF TRANSIENCE ON F

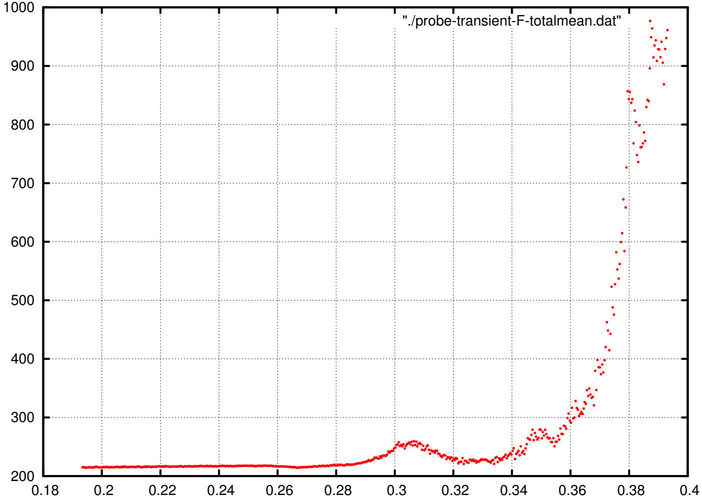
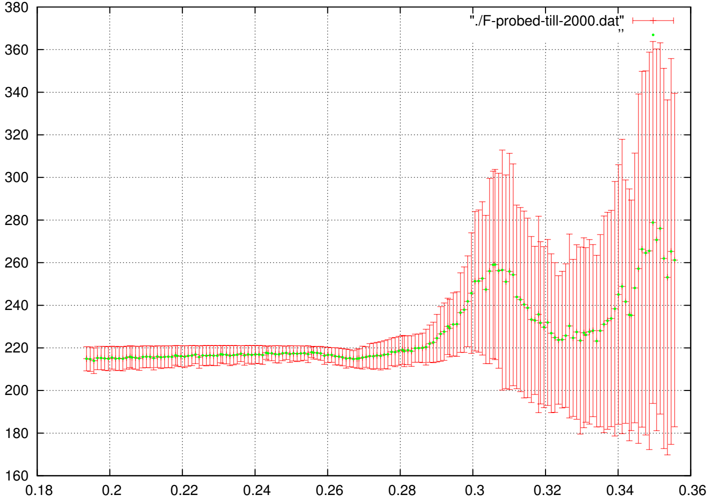


Figure: F dependence With errorbars



POINTS TO NOTE:

1. We use same procedure as in the case of γ.
2. Time taken to reach stability increases rapidly (analysis here) as F approaches grazing value.
3. There is a small bump in the range $0.281 < F < 0.326$. Although there is no emergence of higher periodic orbit.
4. Remark: You must set higher time cutoff.

IN EXISTING LITERATURE

Celso Grebogi, Edward Ott, James A. Yorke ,“ Fractal Basin Boundaries, Long-Lived Chaotic Transients, and Unstable-Unstable Pair Bifurcation”, Phys. Rev. Lett. 50, 935–938 (1983).

Their results:

Just after chaotic orbit vanishes due to boundary crisis in a 2-D map, chaotic transients can be very long lived. Average lifetime decays as:

$$\ln \tau \sim \exp\left(\alpha - \alpha_*\right)^{-1/2}$$

(However, no proof given)

Celso Grebogi, Edward Ott, James A. Yorke, Critical Exponent of Chaotic Transients in Nonlinear Dynamical Systems, Phys. Rev. Lett. 57, 1284–1287 (1986):

Their results:

1. For many initial conditions, $\tau \sim e^{-\lambda(\tau - T)}/T$.
2. $\tau \sim (\alpha - \alpha_*)^{-\gamma}$, γ the "critical exponent" depends on the system.
3. Derives expression for γ in terms of the eigenvalues of the fixed points at collision.

Is our system of this type or the previous (exponentially decaying) type?

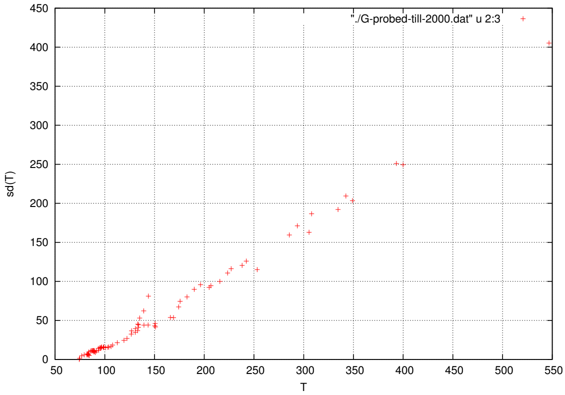


Figure: Standard deviation of τ distribution vs. the mean

τ's could be exponentially distributed: for exponential distribution, mean $\sim$ standard deviation

INTERMITTENCY: SIMILAR PHENOMENA?

Yves Pomeau and Paul Manneville, “Intermittent Transition to Turbulence in Dissipative Dynamical Systems” , Commun. Math. Phys. 74, 189—197 (1980)

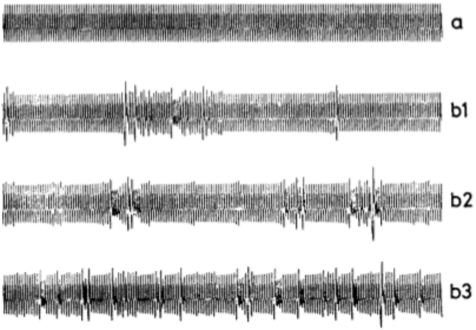


Fig. 1a and b. Time record of one coordinate (z) in the Lorenz model. **a** Stable periodic motion for $r=166$. **b** Above the threshold the oscillations are interrupted by bursts which become more frequent as r is increased

EXPLANATION

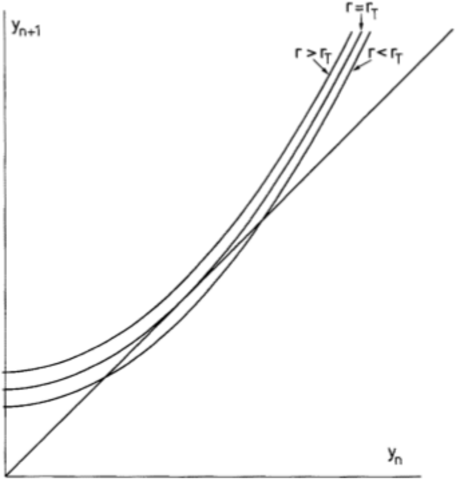
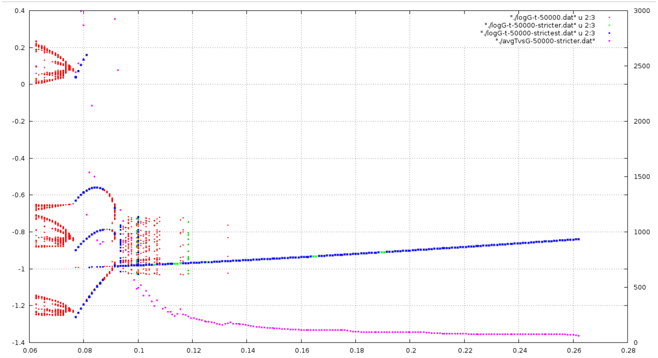


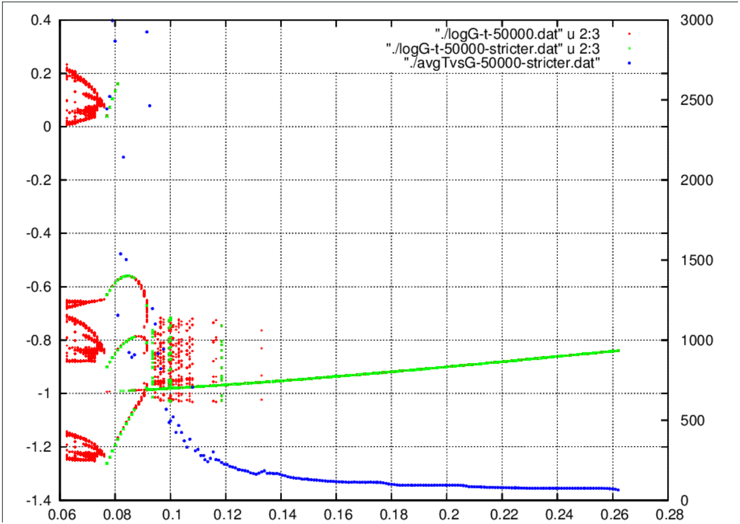
Fig. 3. Idealized picture of the deformation of $y_{n+1}(y_n)$ explaining the transition via intermittency. For $r<r_t$ two fixed points coexist one stable the other unstable. They collapse at $r=r_T$ and then disappear leaving a channel between the curve and the first bisectrix

To Do

1. Transience lifetime analysis does not seem to be covered in existing literature for either piecewise smooth systems or continuous time systems. Although [2] mention that their method is valid for differential equations, no further work seems to be published.
2. Can we transform the problem of hard collision to a suitable poincare map and check if it's a boundary crisis that's happening at grazing? If yes, existing work predicts power law or exponential law for decay. How to explain the peak in the τvs.F plot?

γ-DEPENDENCE REVISITED





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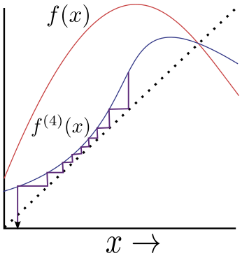
"./logG-t-50000-stricter.dat" u 2:3
"./avgTvsG-50000-stricter.dat"
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
3000
2500
2000
1500
1000
500
0
0.06
0.08
0.1
0.12
0.14
0.16
0.18
0.2
0.22
0.24
0.26
0.28

CONCLUSIONS:

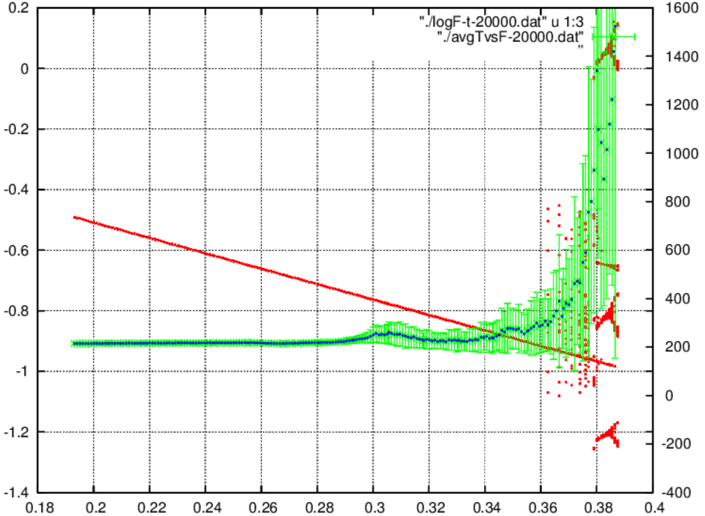
- At first glance, bumps in τ seem to coincide with the appearance of higher periodic orbits in the bifurcation diagram.
- But later we plotted only τ's for period one orbit and still saw the bumps.
- The higher periodic orbits prove to be not real at all: they are merely artifacts of our finite ε for determining periodic orbits.
- Could it happen that some points which would ultimately go to a period 1 orbit are getting trapped for a long time in a transient orbit (á la intermittency)? In that case, should a new fixed point not be born for some parameter value?
- If yes, what's the way to verify? Will Newton-Raphson work?

INTERMITTENCY



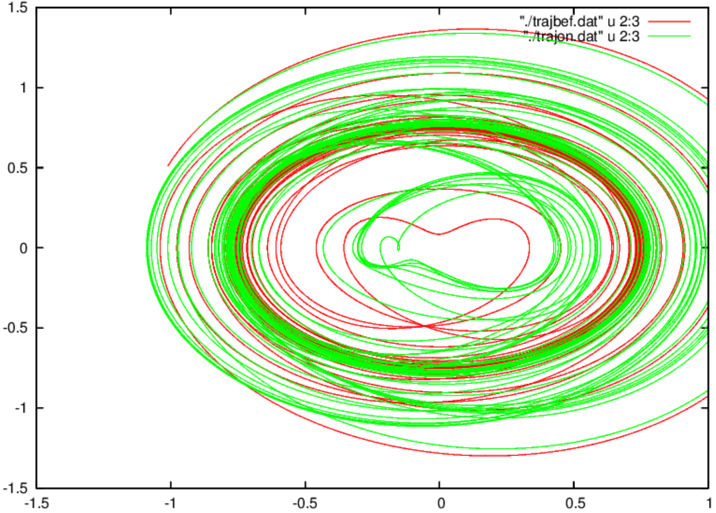
F DEPENDENCE REVISITED

Figure:



TWO ORBITS:

Figure:



- H. Kantz, P. Grassberger, Repellers, semi-attractors, and long-lived chaotic transients, Physica D: Nonlinear Phenomena, Volume 17, Issue 1, August 1985, Pages 75-86, ISSN 0167-2789, 10.1016/0167-2789(85)90135-6. (Contains a proof of exponential decay)

- Celso Grebogi, Edward Ott, James A. Yorke , Fractal Basin Boundaries, Long-Lived Chaotic Transients, and Unstable-Unstable Pair Bifurcation, Phys. Rev. Lett. 50, 935–938 (1983) (Chaotic transients associated with the coalescence of unstable-unstable pair of fixed pts are shown to be extraordinarily long-lived.)

- Celso Grebogi, Edward Ott, James A. Yorke, Critical Exponent of Chaotic Transients in Nonlinear Dynamical Systems, Phys. Rev. Lett. 57, 1284–1287 (1986) (The average lifetime of a chaotic transient versus a system parameter is studied for the case wherein a chaotic attractor is converted into a chaotic transient upon collision with its basin boundary)

- Yves Pomeau and Paul Manneville, Intermittent Transition to Turbulence in Dissipative Dynamical Systems, Commun. Math. Phys. 74, 189—197 (1980) (Describes 3 types of intermittency: long transient almost-periodic behaviour)