Bifurcations in continuous time dynamical systems

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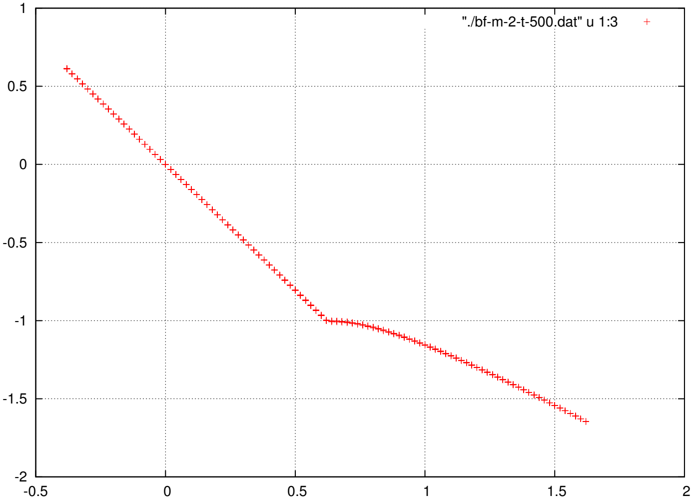
VANISHING CHAOS

When $\frac{2\omega_g}{\omega}$ is an integer, no chaos is observed at grazing:



Suppose the parameters of the system are set such that $\frac{F}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}} = F_g(\omega_0, \omega, \gamma, F) = \sigma$. This constitutes a steady state grazing orbit.

Suppose we look at stroboscopic time slices such that the grazing orbit grazes the boundary at $t = \tau$.

Now we perturb the system slightly so that the system hits the boundary with some non zero velocity at $t = \tau + \delta t$

$$\begin{aligned}
\begin{pmatrix} x(0) \\ v(0) \end{pmatrix} &= \begin{pmatrix} x_p(0) \\ v_p(0) \end{pmatrix} + \begin{pmatrix} x_h(0) \\ v_h(0) \end{pmatrix} \\
\begin{pmatrix} x(\tau+\delta t) \\ v(\tau+\delta t) \end{pmatrix} &= \begin{pmatrix} x_p(\tau+\delta t) \\ v_p(\tau+\delta t) \end{pmatrix} + \begin{pmatrix} x_h(\tau+\delta t) \\ v_h(\tau+\delta t) \end{pmatrix}
\end{aligned}$$

After collision:

$$\begin{aligned}
\begin{pmatrix} x \\ v \end{pmatrix} &= \begin{pmatrix} x_p(\tau+\delta t) \\ -v_p(\tau+\delta t) \end{pmatrix} + \begin{pmatrix} x_h(\tau+\delta t) \\ -v_h(\tau+\delta t) \end{pmatrix} \\
&= \begin{pmatrix} x_p(\tau+\delta t) \\ v_p(\tau+\delta t) \end{pmatrix} + \begin{pmatrix} x_h(\tau+\delta t) \\ v_h(\tau+\delta t) \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 \\ -2v_h(\tau+\delta t) - 2v_p(\tau+\delta t) \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\begin{pmatrix} x(T) \\ v(T) \end{pmatrix} &= \vec{x_p}(T) + M(T-\tau-\delta t)\left\{ \begin{pmatrix} x_h(\tau+\delta t) \\ v_h(\tau+\delta t) \end{pmatrix} \right. \\
&\quad \left. + \begin{pmatrix} 0 \\ -2v_h(\tau+\delta t) - 2v_p(\tau+\delta t) \end{pmatrix} \right\} \\
&= \vec{x_p}(T) + M(T)\vec{x_h}(0) \\
&\quad + M(T-\tau-\delta t) \begin{pmatrix} 0 \\ -2v_p(\tau+\delta t) - 2v_h(\tau+\delta t) \end{pmatrix}
\end{aligned}$$

Therefore we have our Poincare map:

$$\vec{x'}(T) = M(T)\vec{x'}(0) + M(T - \tau - \delta t) \begin{pmatrix} 0 \\ -2v_p(\tau + \delta t) - 2v_h(\tau + \delta t) \end{pmatrix} \quad (1)$$

back $\left\{ \vec{x'}(t) = \vec{x}(t) - \vec{x_p}(t) \right\}$

If we can find expressions for $\delta t, v_p, v_h$ in terms of $x(0), v(0)$, the job will be done.

Recall:

$$x_p(\tau) = \frac{F}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}} := x_m$$

$$v_p(\tau) = 0$$

We have:

$$\begin{aligned}
x_h(\tau+\delta t) + x_p(\tau+\delta t) &= \sigma \\
x_h(\tau) + \dot{x}(\tau)\delta t + x_m \cos\omega\delta t &= \sigma \\
x_h(\tau) + \dot{x}(\tau)\delta t + x_m\left(1 - \frac{(\omega\delta t)^2}{2}\right) &= \sigma
\end{aligned}$$

The root of this equation gives the value of δt (If the roots happen to be complex, it means no collision will take place)

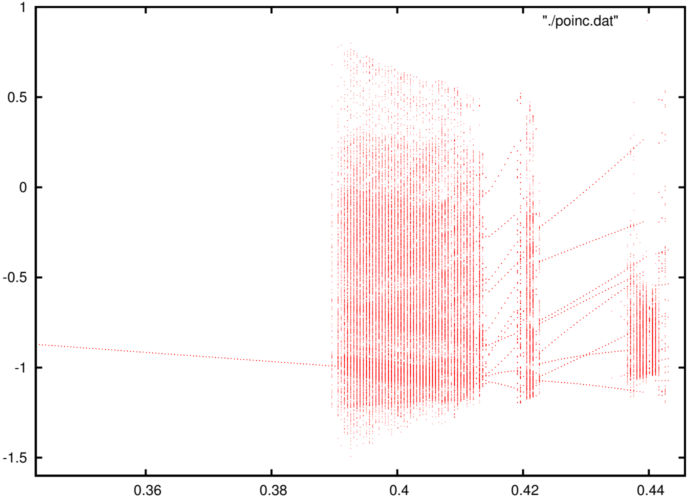
v_h and v_p are calculated in a similar fashion:

$$v_p(\tau + \delta t) = -x_m \omega \sin \omega \delta t$$

$$\begin{aligned} v_h(\tau + \delta t) &= v_h(\tau) + \dot{v_h}(\tau)\delta t \\ &= v_h(\tau) + \delta t(-\gamma v_h(\tau) - w_0^2 x_h(\tau)) \end{aligned}$$

This map should show all the bifurcations shown by the continuous time system.

Figure:



$$M(t) = \frac{e^{-\gamma t/2}}{\omega_g} \begin{pmatrix} \omega_g \cos \omega_g t + \frac{\gamma}{2} \sin \omega_g t & \sin \omega_g t \\ -k \sin \omega_g t & \omega_g \cos \omega_g t - \frac{\gamma}{2} \sin \omega_g t \end{pmatrix} \quad (2)$$

$(\omega_g = \frac{\sqrt{4k-\gamma^2}}{2})$ Back

We derived an exact expression for the poincare map sometime ago. Exact formula Let the parameters be such that the stable orbit grazes slightly: $x_m(\{parameters\}) = \sigma + \varepsilon$.
Now start from an initial condition such that $|x'(0)| = 0$. (Exactly on the stable orbit).
Then we have:

$$x_m cos\left(\omega\tau + tan^{-1}\frac{\omega\gamma}{\omega^2 - \omega_0^2}\right) = \sigma$$

$$\omega\tau + tan^{-1}\frac{\omega\gamma}{\omega^2 - \omega_0^2} = cos^{-1}\left(1 + \frac{\varepsilon}{\sigma}\right)^{-1}$$

$$\approx 2n\pi + \sqrt{\frac{2\varepsilon}{\sigma}}$$

$$\tau \approx \frac{1}{\omega}\left(2\pi + \sqrt{\frac{2\varepsilon}{\sigma}} - tan^{-1}\frac{\omega\gamma}{\omega^2 - \omega_0^2}\right)$$

Now suppose $0 < |x'(0)| << 1$. Let the time of collision $t_c = \tau + \delta t$

$$x_m cos\left(\omega(\tau+\delta t) + tan^{-1}\frac{\omega\gamma}{\omega^2-\omega_0^2}\right) + x_h(\tau+\delta t) = \sigma$$

$$x_m cos\left(\left(\omega\tau + tan^{-1}\frac{\omega\gamma}{\omega^2-\omega_0^2}\right) + \omega\delta t\right) + x_h(\tau+\delta t) = \sigma$$

$$\sigma - \omega x_m \delta t sin\left(\omega\tau + tan^{-1}\frac{\omega\gamma}{\omega^2-\omega_0^2}\right) + x_h(\tau+\delta t) = \sigma$$

$$-\omega\delta t x_m\sqrt{1-(\sigma/x_m)^2} + x_h(\tau+\delta t) = 0$$

$$\frac{x_h(\tau)}{\omega\sqrt{x_m^2-\sigma^2} - v_h(\tau)} = \delta t$$

$$
\begin{aligned}
v_p(\tau + \delta t) &= -\omega x_m sin\left(\omega(\tau + \delta t) + tan^{-1}\frac{\omega\gamma}{\omega^2 - \omega_0^2}\right) \\
&= -\omega x_m sin\left(2\pi + \sqrt{\frac{2\varepsilon}{\sigma}} - tan^{-1}\frac{\omega\gamma}{\omega^2 - \omega_0^2} + \omega\delta t + tan^{-1}\frac{\omega\gamma}{\omega^2 -
\right. \\
&= -\omega x_m sin\left(\omega\delta t + \sqrt{\frac{2\varepsilon}{\sigma}}\right)
\end{aligned}
$$