

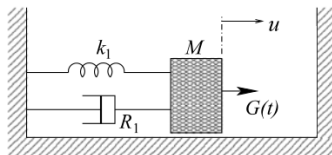
Bifurcations in continuous time dynamical systems

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HARD IMPACT IN AN OSCILLATING SYSTEM

Figure:



The equation of motion:

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = F \cos \omega t \quad (1)$$

Switching manifold: If $x = \sigma$,

$$x \mapsto x$$

$$v \mapsto -v$$

The solution to equation (1) is a sum of two parts: a *particular solution* that is independent of the initial conditions, and a *homogeneous solution* that is dependent on the initial conditions. To be more precise:

$$x_p(t) = \frac{F}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}} \cos(\omega t + \tan^{-1} \frac{\omega \gamma}{\omega^2 - \omega_0^2})$$

$$x_h(t) = \frac{e^{-\gamma t/2}}{\omega_g} \left\{ (\omega_g \cos \omega_g t + \frac{\gamma}{2} \sin \omega_g t) x_0 + (\sin \omega_g t) v_0 \right\}$$

$$\omega_g = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

Now, $x_h(t)$ decays exponentially with time. So, if the hard wall were not there, any arbitrary initial condition would have been attracted to the period-1 orbit $x_p(t)$.

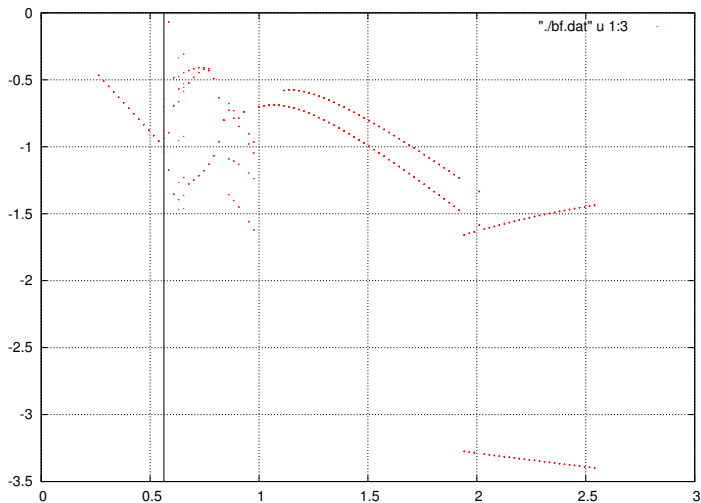
Therefore $x_p(t)$ is a limit cycle whose basin of attraction is the whole phase space (ignoring the wall).

However, we note here that the friction term γ dictates how quickly any arbitrary initial condition converges to this limit cycle.

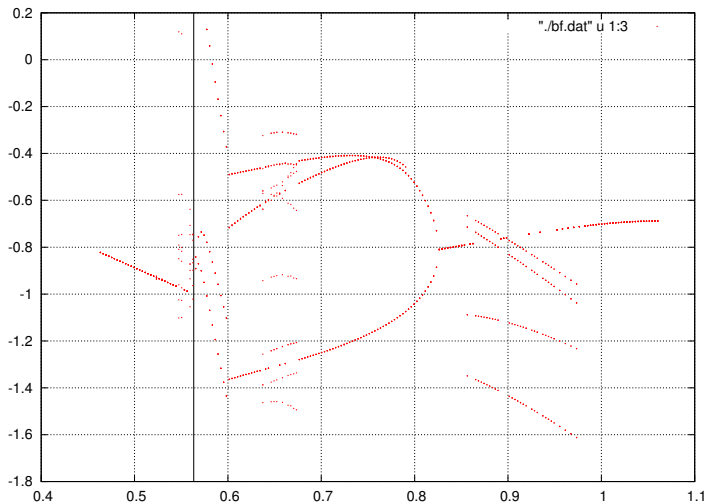
The limit cycle is basically a sinusoidal orbit with amplitude

$$\frac{F}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}} := F_g(\omega_0, \omega, \gamma, F)$$

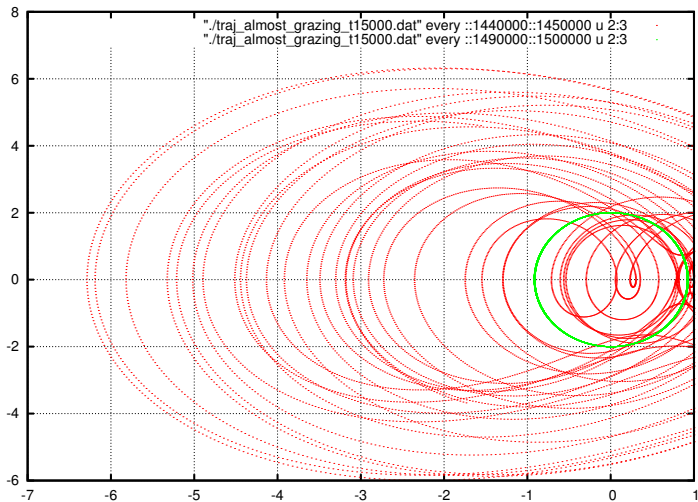
BIFURCATION W.R.T. F



A CLOSER LOOK



TRANSIENT BEHAVIOUR



Here it needs to be emphasized that this long-lived transient is an artifact of the nonlinearity of the system.

In this particular example, $\gamma = 0.062$. Therefore,
 $e^{-0.062 \times 1000/2} \approx 3.44e - 14$.

The system should have settled down to the period-1 limit cycle long ago.

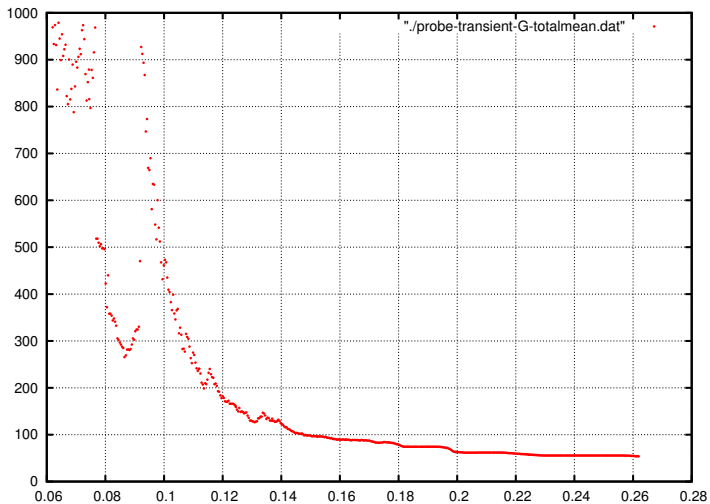
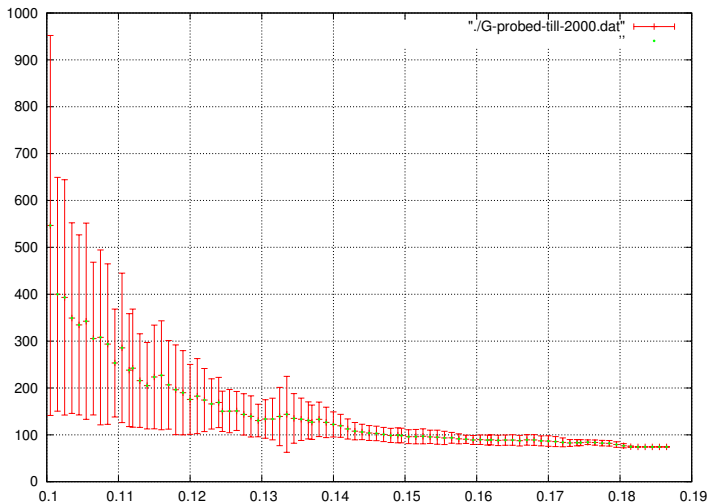
DEPENDENCE OF TRANSIENCE ON γ 

Figure: γ dependence with errorbars

POINTS TO NOTE:

- ① We let the system evolve and determined how much time it takes for an orbit to settle down to a periodic orbit. When no such orbit emerged till $t = 2000s$, we set the time to be $2000s$ itself. For each γ value, 300 points were taken and the averages plotted.
- ② We saw that the time taken to reach stability increased rapidly (insert analysis here), until $\gamma = 0.178$. After that the orbit seemed to settle on higher periodic orbits very quickly. Hence the dip in the plot. The periodicities detected were 3, 8 and 16. Then after a small gap, those higher periodic orbits seemed to get unstable. Hence the rise in stability time again.
- ③ The lowest γ was set to be the one where the unperturbed orbit (i.e. the particular solution) grazes the wall. So throughout the γ range, there exists a period-1 orbit.

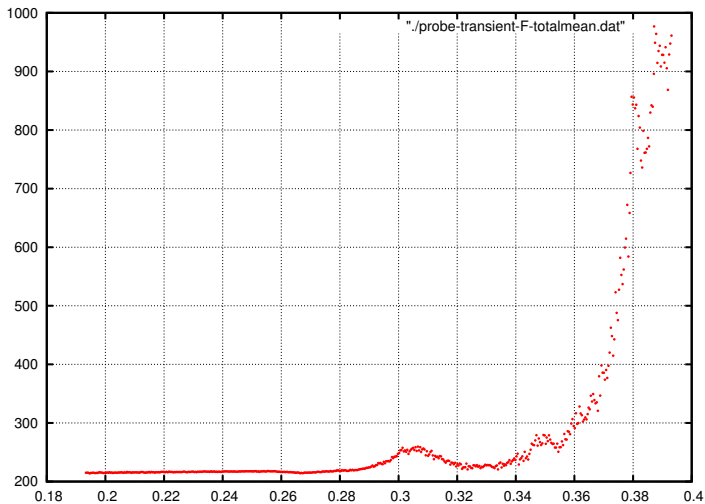
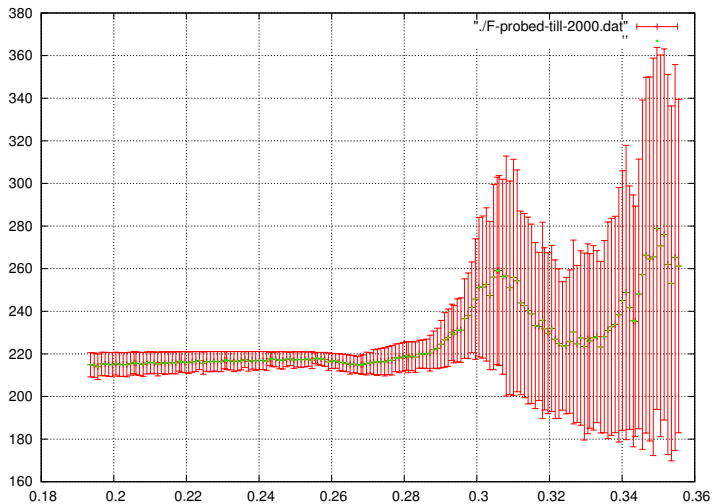
DEPENDENCE OF TRANSIENCE ON F 

Figure: F dependence With errorbars



POINTS TO NOTE:

- ① We use same procedure as in the case of γ .
- ② Time taken to reach stability increases rapidly (analysis here) as F approaches grazing value.
- ③ There is a small bump in the range $0.281 < F < 0.326$.
Although there is no emergence of higher periodic orbit.
- ④ Remark: You must set higher time cutoff.

IN EXISTING LITERATURE

Celso Grebogi, Edward Ott, James A. Yorke ,“ Fractal Basin Boundaries, Long-Lived Chaotic Transients, and Unstable-Unstable Pair Bifurcation”, Phys. Rev. Lett. 50, 935–938 (1983).

Their results:

Just after chaotic orbit vanishes due to boundary crisis in a 2-D map, chaotic transients can be very long lived. Average lifetime decays as:

$$\ln \tau \sim \exp (\alpha - \alpha_*)^{-1/2}$$

(However, no proof given)

Celso Grebogi, Edward Ott, James A. Yorke, Critical Exponent of Chaotic Transients in Nonlinear Dynamical Systems, Phys. Rev. Lett. 57, 1284–1287 (1986):

Their results:

- ❶ For many initial conditions, $\tau \sim e^{-\lambda(\tau-T)}/T$.
- ❷ $\tau \sim (\alpha - \alpha_*)^{-\gamma}$, γ the “critical exponent” depends on the system.
- ❸ Derives expression for γ in terms of the eigenvalues of the fixed points at collision.

Is our system of this type or the previous (exponentially decaying) type?

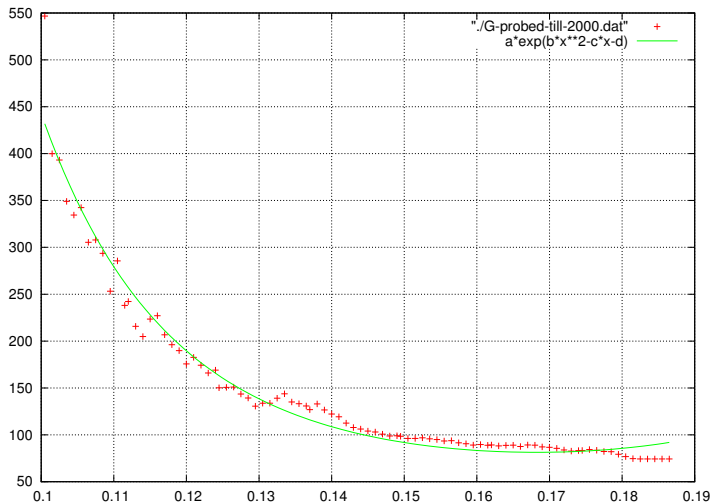
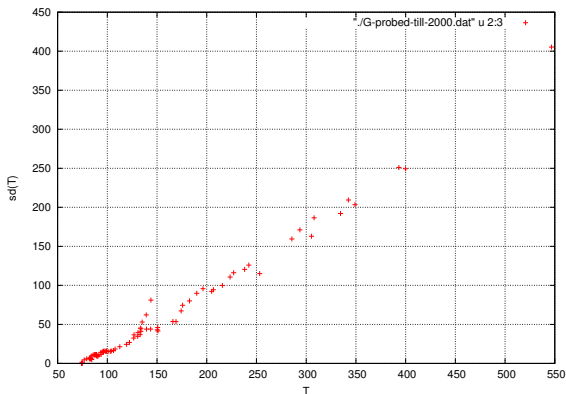
Figure: τ vs. γ 

Figure: Standard deviation of τ distribution vs. the mean

τ 's could be exponentially distributed: for exponential distribution, mean \sim standard deviation

INTERMITTENCY: SIMILAR PHENOMENA?

Yves Pomeau and Paul Manneville, "Intermittent Transition to Turbulence in Dissipative Dynamical Systems" , Commun. Math. Phys. 74, 189—197 (1980)

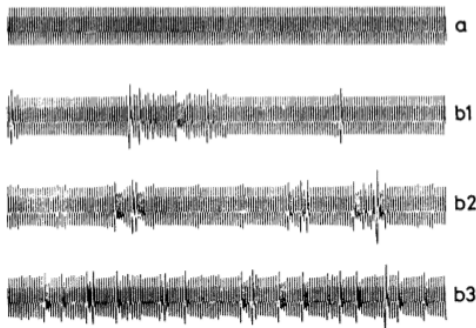
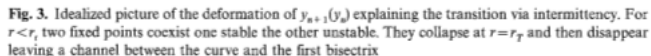


Fig. 1a and b. Time record of one coordinate (z) in the Lorenz model. **a** Stable periodic motion for $r=166$. **b** Above the threshold the oscillations are interrupted by bursts which become more frequent as r is increased



To Do

- ❶ Transience lifetime analysis does not seem to be covered in existing literature for either piecewise smooth systems or continuous time systems. Although [2] mention that their method is valid for differential equations, no further work seems to be published.
- ❷ Can we transform the problem of hard collision to a suitable poincare map and check if it's a boundary crisis that's happening at grazing? If yes, existing work predicts power law or exponential law for decay. How to explain the peak in the τ vs. F plot?

PROBING 'EM HARDER WRT G

- At first glance, bumps in τ seems to coincide with the appearance of higher periodic orbits in the bifurcation diagram.
- But later we plotted only τ 's for period one orbit and still saw the bumps.
- Does this mean some other period 1 orbits are appearing with high transient stability?
- Are the higher periodic orbits at all real? Might they not be the artifact of our finite ε for determining periodic orbits?
- If they are indeed real, could it happen that some points which would ultimately go to a period 1 orbit are getting trapped for a long time in a transient orbit (à la intermittency)?



H. Kantz, P. Grassberger, Repellers, semi-attractors, and long-lived chaotic transients, *Physica D: Nonlinear Phenomena*, Volume 17, Issue 1, August 1985, Pages 75-86, ISSN 0167-2789, 10.1016/0167-2789(85)90135-6. (Contains a proof of exponential decay)



Celso Grebogi, Edward Ott, James A. Yorke, Fractal Basin Boundaries, Long-Lived Chaotic Transients, and Unstable-Unstable Pair Bifurcation, *Phys. Rev. Lett.* 50, 935–938 (1983) (Chaotic transients associated with the coalescence of unstable-unstable pair of fixed pts are shown to be extraordinarily long-lived.)



Celso Grebogi, Edward Ott, James A. Yorke, Critical Exponent of Chaotic Transients in Nonlinear Dynamical Systems, *Phys. Rev. Lett.* 57, 1284–1287 (1986) (The average lifetime of a chaotic transient versus a system parameter is studied for the case wherein a chaotic attractor is converted into a chaotic transient upon collision with its basin boundary)



Yves Pomeau and Paul Manneville, Intermittent Transition to Turbulence in Dissipative Dynamical Systems, *Commun. Math. Phys.* 74, 189—197 (1980) (Describes 3 types of intermittency: long transient almost-periodic behaviour)