Bifurcations in continuous time dynamical systems

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September 14, 2012

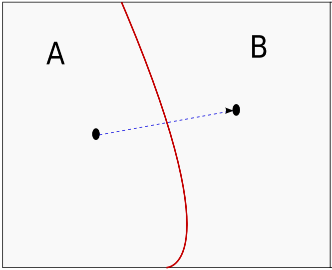
DEFINITIONS

A simple piecewise smooth function:

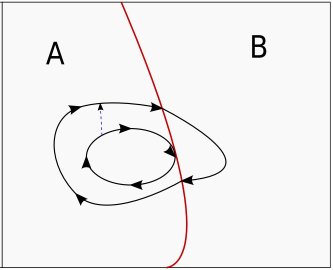
$$\dot{\mathbf{x}} = \begin{cases} F_1(\mathbf{x}) & : H(\mathbf{x}) < \mathbf{0} \\ F_2(\mathbf{x}) & : H(\mathbf{x}) > \mathbf{0} \end{cases}$$

$H(\mathbf{x})$ =The boundary.
The flows of F_1 and F_2 are $\varphi_1(x)$ and $\varphi_2(x)$ respectively, defined in respective regions and also in the neighbourhood of the boundary.

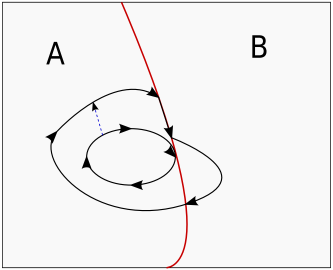
Figure : Possible scenarios



Bifurcation of equillibria



Grazing bifurcation



Choose a coordinate system such that:

$$\dot{\mathbf{x}} = \begin{cases} F_1(\mathbf{x}) & : x_n < 0 \\ F_2(\mathbf{x}) & : x_n > 0 \end{cases}$$

and $\mathbf{x} = \mathbf{0}$ is a grazing point.
$x_n ==$ nth component of $\mathbf{x}$.
Locally linearize:

$$\dot{\mathbf{x}} = \begin{cases} \mathbf{A_1 x} + \mathbf{B}\mu & : x_n < 0 \\ \mathbf{A_2 x} + \mathbf{B}\mu & : x_n < 0 \end{cases}$$

Where:

$$\mathbf{A_i} = \frac{\partial \mathbf{F_1}}{\partial \mathbf{x}}_{\mathbf{x=0}}$$

and $\mathbf{B} = \frac{\partial \mathbf{F_1}}{\partial \mu}_{\mu=\mathbf{0}} = \frac{\partial \mathbf{F_2}}{\partial \mu}_{\mu=\mathbf{0}}$ (Due to continuity).
Also, $\mathbf{A_1}$ and $\mathbf{A_2}$ can differ only in the $n-$th column (Again due to continuity).

Let:
$A_1\mathbf{x_1^*} + \mathbf{B}\mu = \mathbf{0}, A_2\mathbf{x_2^*} + \mathbf{B}\mu = \mathbf{0}.$
Assuming A_i's are invertible:

$$\mathbf{x_i^*} = -\mathbf{A_i}^{-\mathbf{1}}\mathbf{B}\mu = -\frac{\mathbf{adj}(\mathbf{A_i})}{\mathbf{det}(\mathbf{A_i})}\mathbf{B}\mu$$

The solutions exist iff:

$$x^*_{1_{n<0}} < 0, x^*_{2_n} > 0.$$

Now,

$$x^*_{1_k} = \frac{c^*_{1_k}}{det(\mathbf{A_1})}\mu, x^*_{2_k} = \frac{c^*_{2_k}}{det(\mathbf{A_2})}$$

Where,

$$c^*_{i_k} = [-adj(\mathbf{A_i})\mathbf{B}]_{\mathbf{k}}$$

Because A_1 differs from A_2 only in $n-$th column, $c^*_{1_n} = c^*_{2_n} := C$

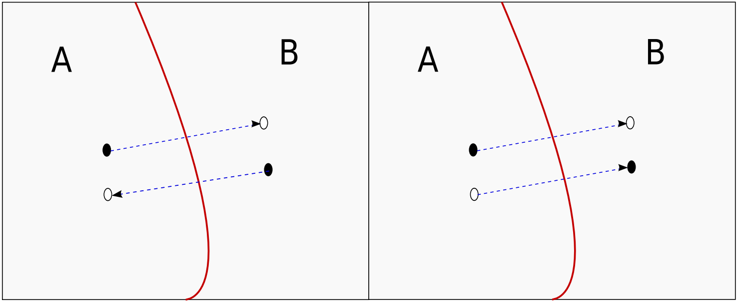
Condition for border crossing of equillibria

Note:

$$x_{1_n}^* = \frac{c}{det(\mathbf{A_1})}\mu, x_{2_k}^* = \frac{c}{det(\mathbf{A_2})}\mu$$

Cases:

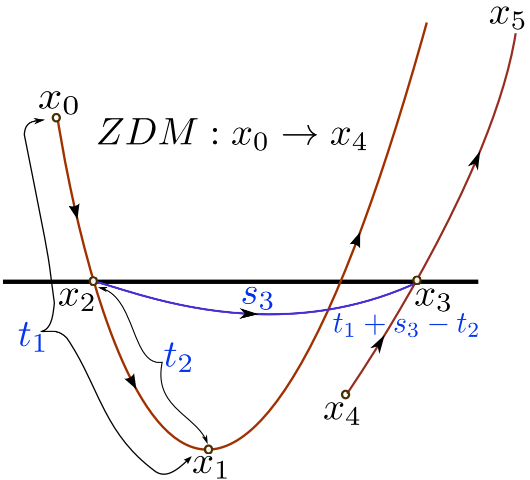
1. $det(\mathbf{A_1})det(\mathbf{A_1}) < 0$. $x_{1_n}^*$ and $x_{2_n}^*$ always have opposite signs.
2. $det(\mathbf{A_1})det(\mathbf{A_1}) > 0$. $x_{1_n}^*$ and $x_{2_n}^*$ always have same signs.



Case 1 Case 2

Grazing

Figure :



How to get the ZDM?

Why ZDM?

Consider a Poincare section $tmodT = 0$. Then the Poincare map would be simplified to:

$$f(x_n) = \varphi_2(ZDM(\varphi_1(x_n, \tau)), T - \tau)$$

- t_1 == time for x_0 to x_1
- t_2 == time for x_2 to x_1
- s_3 == time for x_2 to x_3

1. Solve for: $T_1(x)$ given by:

$$E_1(x, T_1) = \frac{dH}{dt}|_{t=T_1} = 0, T_1(0) = 0.$$

$t_1 = T_1(x_0)$

2. Solve for $E_2(x, y, T_2)$ given by:

$$T_2\sqrt{\frac{H(\varphi_1(x, T_1(x) - T_2)) - H(\varphi_1(x, T_1(x)))}{T_2^2}} - y = 0$$

1. Solve for

$$E_3(x, S_3) = \frac{H(\varphi_2(x, s_3)) - H(x)}{s_3} = 0$$

$s_3 = S_3(x_2)$