Investigating Piecewise Smooth Hybrid Systems

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HYBRID SYSTEMS

These are systems described partly by differential equations and partly by maps: a *hybrid* of continuous time and discrete time systems.

Examples:

- A bell.
- A typewriter.
- Walking motion.

Mathematical Definition

Piecewise smooth hybrid system

A system described by a set of ODE's and a set of **reset maps**:

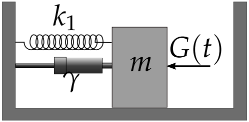
$$\dot{x} = F_i(x, \mu), \quad \forall x \in S_i \tag{1}$$

$$x \mapsto R_{ij}(x, \mu), \quad \forall x \in \Sigma_{ij} = \bar{S}_i \cup \bar{S}_j \tag{2}$$

is called a piecewise smooth hybrid system if all the R_i's, F_i's as well as the associated flows φ_i's are smooth in both x and the parameter μ in the appropriate regimes.

EXAMPLE: OSCILLATOR WITH HARD IMPACTS

Figure: Hard impacting oscillator



$$m\ddot{x} = -\gamma\dot{x} - k_1 x + G(t) \qquad \text{for } x < \sigma \qquad (3)$$

$$(x, v) \mapsto (x, -rv) \qquad \text{for } x = \sigma \qquad (4)$$

r is the coefficient of restitution, which is 1 for perfectly elastic collisions.

Bifurcations in Hybrid Systems

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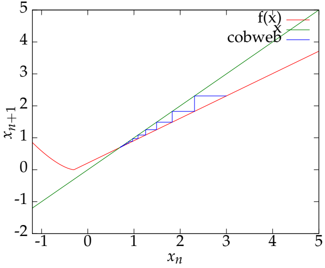
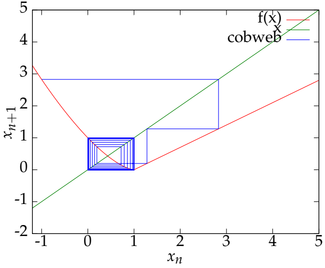


$\rightarrow$



GRAZING BIFURCATION OF LIMIT CYCLES

Figure: Grazing orbit

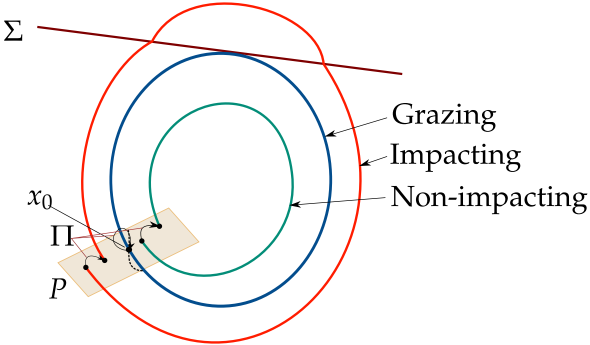
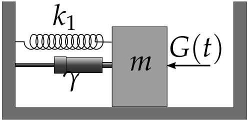


Figure: Hard impacting oscillator



$$m\ddot{x} = -\gamma\dot{x} - \omega_0^2 x + F\cos\omega t \qquad \text{for } x < \sigma \tag{5}$$

$$(x, v) \mapsto (x, -rv) \qquad \text{for } x = \sigma \tag{6}$$

If we forget the boundary for a moment, the equation of motion is a non-homogeneous ODE. As usual, its solution is a sum total of a *homogeneous solution* and a *particular solution*:

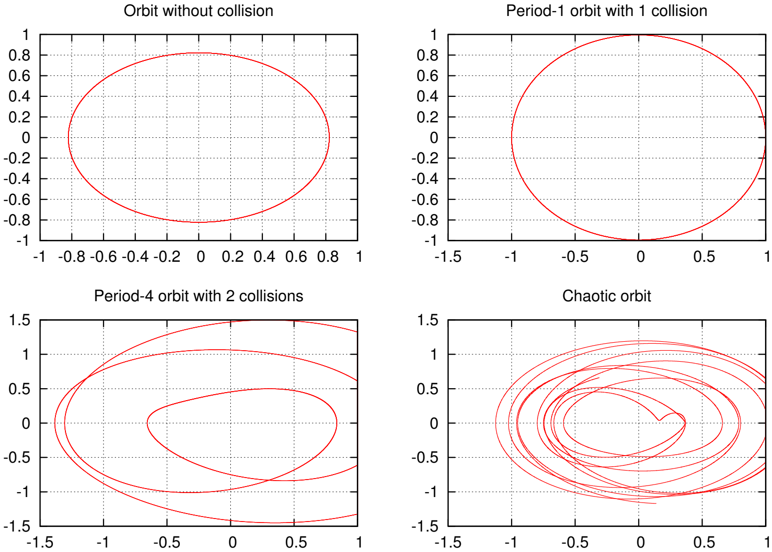
$$x(t) = x_p(t) + x_h(t) \tag{7}$$

$$x_p(t) = \frac{F/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}} cos(\omega t + tan^{-1}\frac{\omega\gamma}{\omega^2 - \omega_0^2}) \tag{8}$$

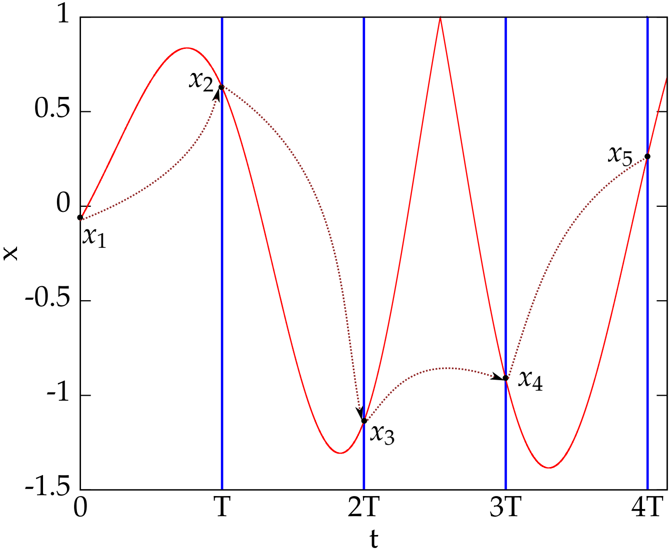
$$x_h(t) = \frac{e^{-\gamma t/2}}{\omega_g}\left\{(\omega_g \cos\omega_g t + \frac{\gamma}{2}\sin\omega_g t)x(0) + (\sin\omega_g t)v(0)\right\} \tag{9}$$

$$\omega_g = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \tag{10}$$

A FEW POSSIBLE TRAJECTORIES



Stroboscopic Poincaré map



Kundu and Banerjee [12, 11] investigated the grazing bifurcations in this system by deriving an approximate analytical expression for the stroboscopic Poincaré map near grazing.

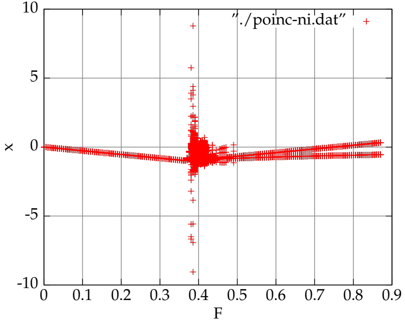
Their results:

- Unless $n = \frac{2\omega_g}{\omega_{forcing}} \in \mathbb{N}$, chaos immediately follows grazing of the steady state orbit.
- If $n \in \mathbb{N}$, no chaos after grazing.

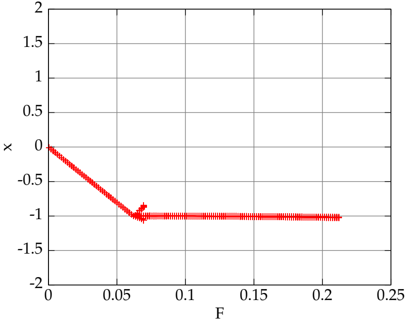
Experimental data:

- Chaos vanishes not only *at* $n \in \mathbb{N}$, but at small neighbourhoods around each $n \in \mathbb{N}$

Bifurcation digram for $n = 2.2$



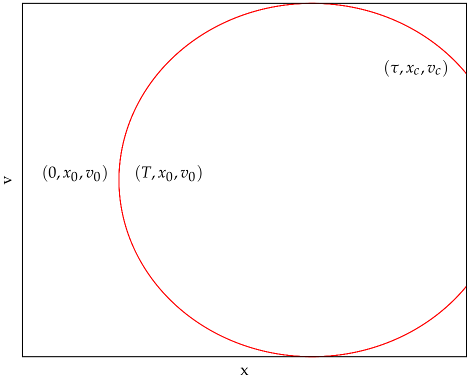
Bifurcation diagram for $n = 2$



Numerical computation of orbits

The bifurcation diagrams suggest that for $n \in \mathbb{N}$, a stable period-1 orbit emerges right after grazing, unlike non-integer n values.

We used a method devised by Ma et al.[13] for checking if that indeed is the case.

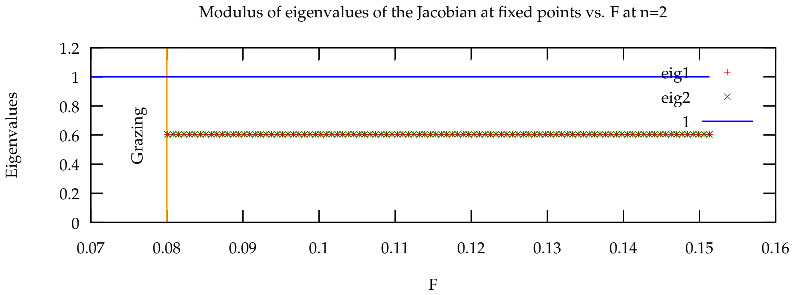


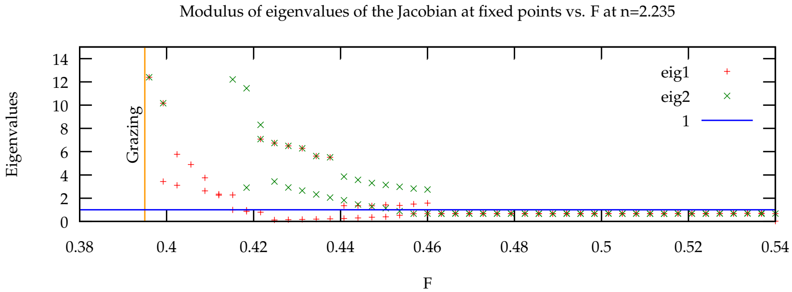
Define $y = (x_0, v_0, x_c, v_c, \tau)$
Then we get a system of equations $\mathbf{G}(y) = \mathbf{0}$ where:

$$\begin{aligned} G_{1,2}(y) &= \vec{x_c} - \varphi(\tau, 0, \vec{x_0}) = 0 \\ G_{3,4}(y) &= \vec{x_0} - \varphi(2T, \tau, \vec{x_c}) = 0 \\ G_5(y) &= x_1 - \sigma = 0 \end{aligned}$$

Since this is a set of 5 equations in 5 unknowns, we can solve the equation $G(y) = 0$ using Newton-Raphson method:

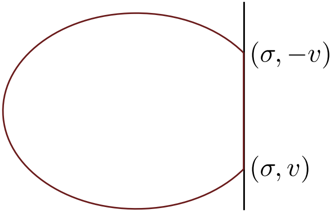
$$y_{n+1} = y_n + J(y)^{-1} G(y)$$


Modulus of eigenvalues of the Jacobian at fixed points vs. F at n=2
Eigenvalues
F
Grazing
eig1
eig2
1
0
0.2
0.4
0.6
0.8
1
1.2
0.07
0.08
0.09
0.1
0.11
0.12
0.13
0.14
0.15
0.16


Modulus of eigenvalues of the Jacobian at fixed points vs. F at n=2.235
Eigenvalues
F
Grazing
eig1
eig2
1
0
2
4
6
8
10
12
14
0.38
0.4
0.42
0.44
0.46
0.48
0.5
0.52
0.54

Analytical approach

Figure: Impact map in case of $PnC1$ orbit



$$M_I(\sigma, v) \mapsto (\sigma, -v)$$

$$M_I : \begin{pmatrix} x \\ v \end{pmatrix} \mapsto M(mT) \begin{pmatrix} x \\ -v \pm 2\omega\sqrt{A^2 - (\sigma - x)^2} \end{pmatrix} \tag{11}$$

$$M(t) = \frac{e^{-\gamma t/2}}{\omega_g} \begin{pmatrix} \omega_g \cos\omega_g t + \frac{\gamma}{2}\sin\omega_g t & \sin\omega_g t \\ -k\sin\omega_g t & \omega_g\cos\omega_g t - \frac{\gamma}{2}\sin\omega_g t \end{pmatrix} \tag{12}$$

The fixed points and the Jacobian

Fixed points

$$x^* = \frac{\sigma \pm \sqrt{\sigma^2 - (\alpha^2+1)(\sigma^2 - A^2)}}{\alpha^2+1} \tag{13}$$

$$v^* = \frac{(d - ad + bc)x^*}{b} \tag{14}$$

$$M(mT) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{15}$$

$$\alpha = \frac{(a - d + ad - bc - 1)}{2b\omega} \tag{16}$$

The Jacobian

$$J(x,v) = M(mT)\begin{pmatrix} 1 & 0 \\ -v \pm 2\omega\frac{\sigma - x}{\sqrt{A^2-(\sigma-x)^2}} & -1 \end{pmatrix} \tag{17}$$

Figure: Eigenvalues of J at fixed points Vs. F for $n = 2.01, 2.1, 2.2$

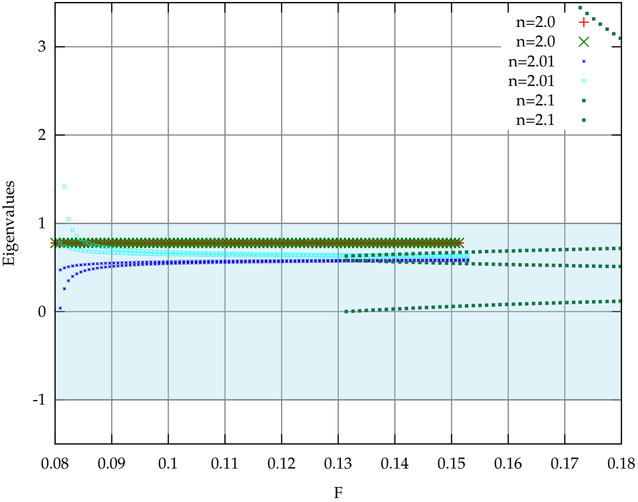


Figure: Eigenvalues of J at fixed points at grazing Vs. n

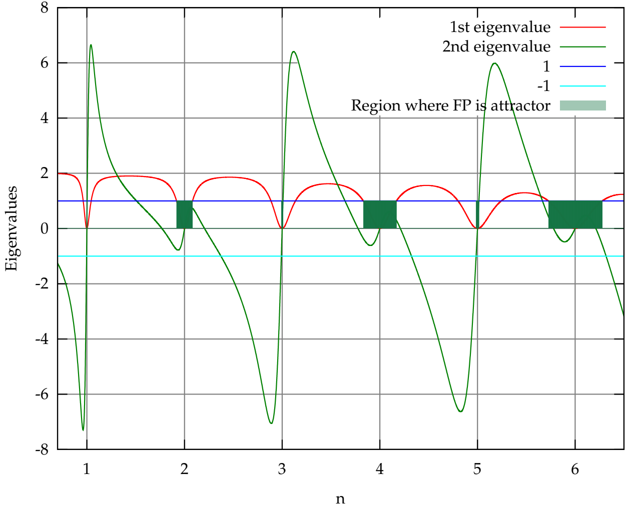


Figure: Average transient lifetime Vs. F in the hard impacting oscillator

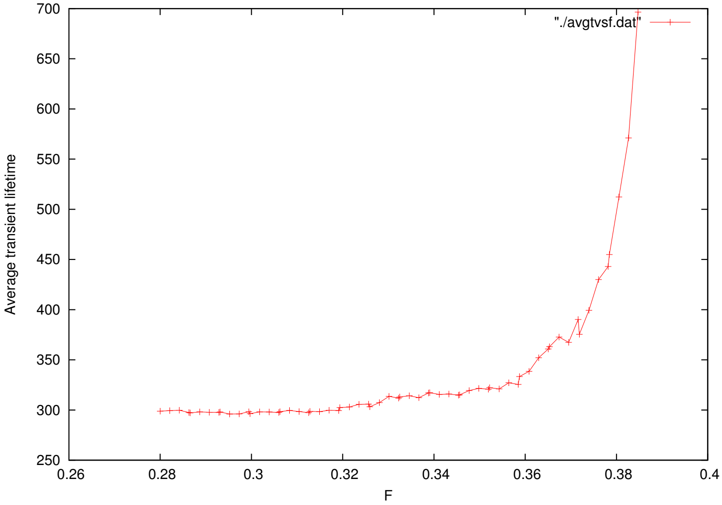
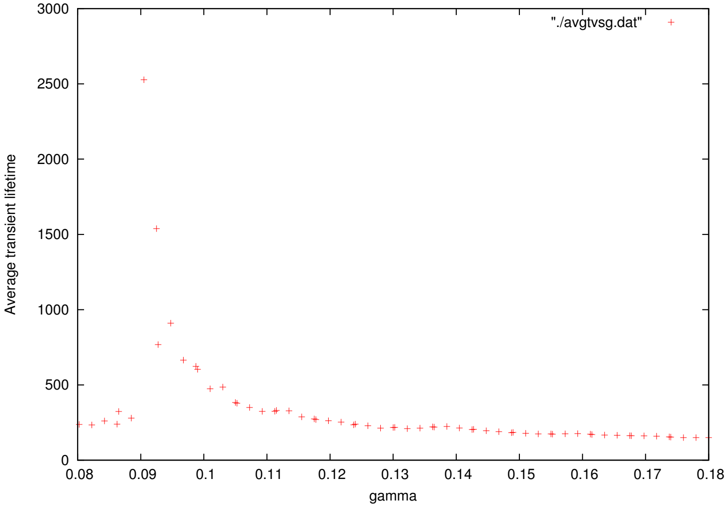


Figure: Average transient lifetime Vs. γ in the hard impacting oscillator



Grebogi et al.[9] predicted:

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These long lived transients are *artefacts of the nonlinearity* due to the impacts.
Without impacts, transient decays as

$$x_h(t) = \frac{e^{-\gamma t/2}}{\omega_g} \left\{ (\omega_g \cos \omega_g t + \frac{\gamma}{2} \sin \omega_g t) x_0 + (\sin \omega_g t) v_0 \right\} \tag{19}$$

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In that case, τ would vary inversely with γ and would not depend on F at all.
Therefore, we need to concentrate on the impacts.

Consider an initial condition:

$$\vec{x}_p(0) = (-A, 0)$$
$$\vec{x}_h(0) = (x_0, v_0)$$

Where $A = \frac{F/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}}$

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Where $A = \frac{F/m}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}}$
Starting from this point, the solution till the next collision is, as per (7):

$$\begin{aligned} x(t) &= -A\cos\omega t + \frac{e^{-\gamma t/2}}{\omega_g}\left\{(\omega_g\cos\omega_g t + \frac{\gamma}{2}\sin\omega_g t)x_0 + (\sin\omega_g t)v_0\right\} \\ &= -A\cos\omega t + e^{-\gamma t/2}B\cos\left(\omega_g t + C\right) \end{aligned}$$

$$C = -tan^{-1}\frac{\frac{\gamma}{2\omega_g}x_0 + \frac{v_0}{\omega_g}}{x_0}$$

$$B = \sqrt{x_0^2 + \left(\frac{\gamma}{2\omega_g}x_0 + \frac{v_0}{\omega_g}\right)^2}$$

Figure: An envelope of (7) for $\omega_g \approx \omega$

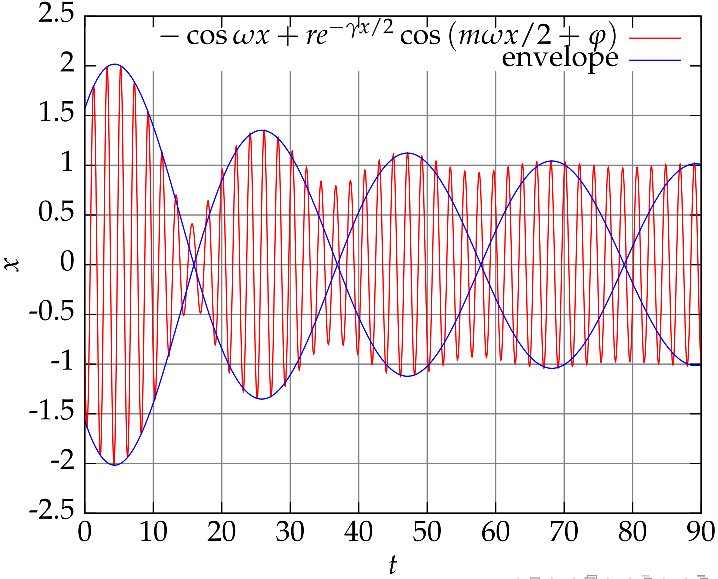
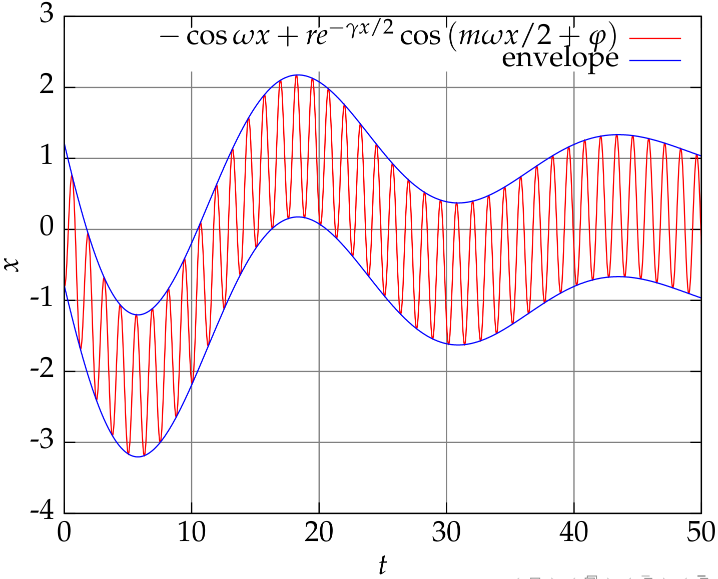


Figure: An envelope of (7) for $\omega_g \gg \omega$



Measuring how "collision-prone" the system is for a certain fixed parameter value:

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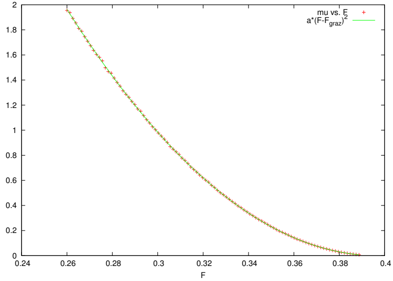
3. Points for which there is no further collision should form a closed area in the whole space μ_{xv}
4. See how this area shrinks as we approach grazing.

$$\mu_{xv} \approx 2\omega_g \int_{-\chi}^{max\{\chi,\sigma\}} \sqrt{\chi^2 - x^2} dx \tag{20}$$

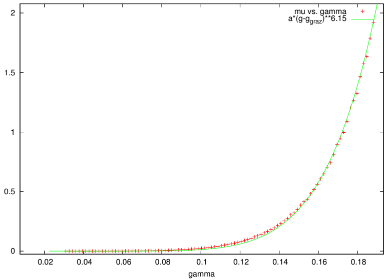
$$\chi = (\sigma - A) e^{\gamma t_c / 2} \tag{21}$$

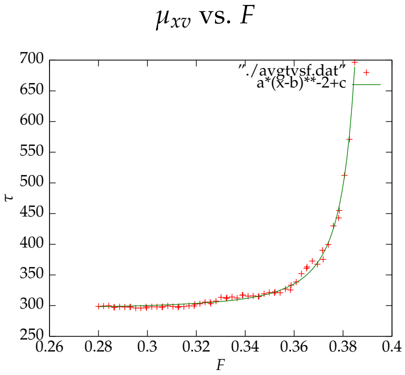
We hope $\tau \propto \frac{1}{\mu_{xv}}$

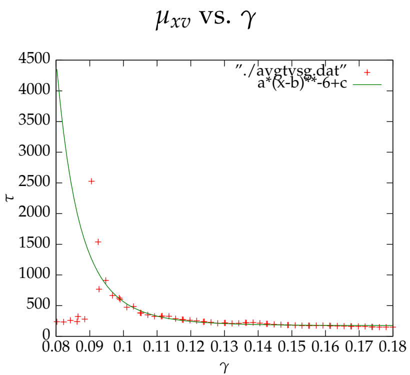
μ_{xv} vs. F



μ_{xv} vs. γ




μ_{xv} vs. F
"./avgtvsf.dat"
a*(x-b)**-2+c
700
650
600
550
500
450
400
350
300
250
0.26
0.28
0.3
0.32
0.34
0.36
0.38
0.4
τ
F


μ_{xv} vs. γ
"./avgtvsg.dat"
a*(x-b)**-6+c
4500
4000
3500
3000
2500
2000
1500
1000
500
0
0.08
0.09
0.1
0.11
0.12
0.13
0.14
0.15
0.16
0.17
0.18
τ
γ

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