Bifurcations in continuous time dynamical systems

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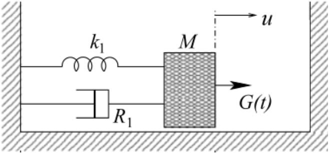
Hard impact in an oscillating system

Figure:



The equation of motion:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = F\cos\omega t \tag{1}$$

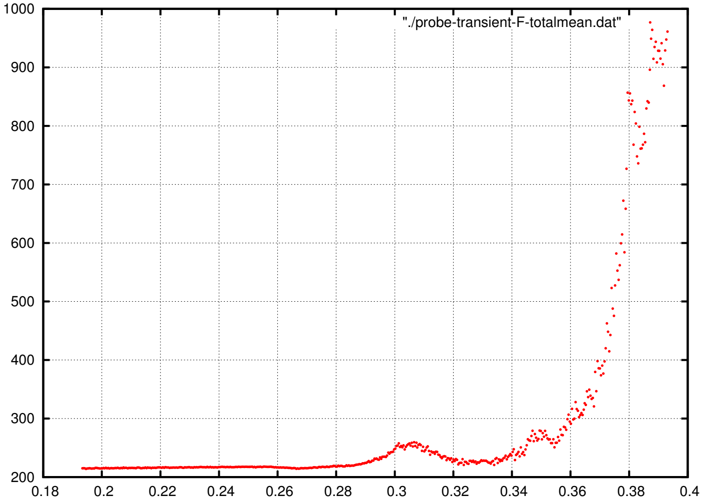
Switching manifold: If $x = \sigma$,

$$\begin{aligned} x &\mapsto x \\ v &\mapsto -v \end{aligned}$$

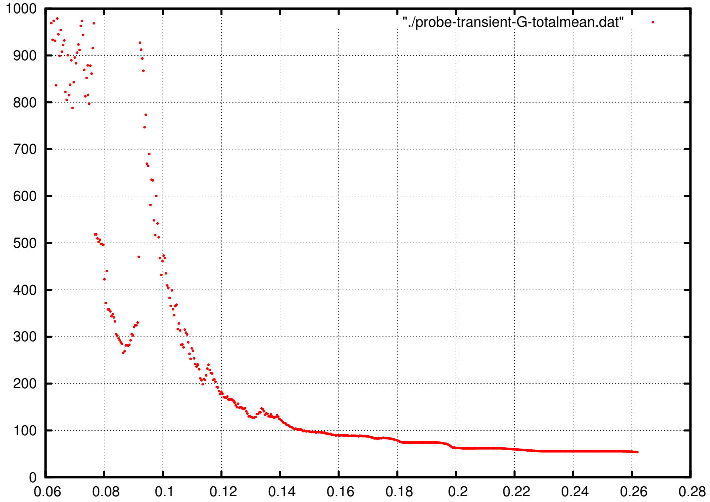
The solution to equation (1) is a sum of two parts: a *particular solution* that is independent of the initial conditions, and a *homogeneous solution* that is dependent on the initial conditions. To be more precise:

$$
\begin{aligned}
x_p(t) &= \frac{F}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}} cos(\omega t + tan^{-1}\frac{\omega\gamma}{\omega^2-\omega_0^2}) \\
x_h(t) &= \frac{e^{-\gamma t/2}}{\omega_g}\left\{(\omega_g \cos\omega_g t + \frac{\gamma}{2}\sin\omega_g t)x_0 + (\sin\omega_g t)v_0\right\} \\
\omega_g &= \sqrt{\omega_0^2-\frac{\gamma^2}{4}}
\end{aligned}
$$

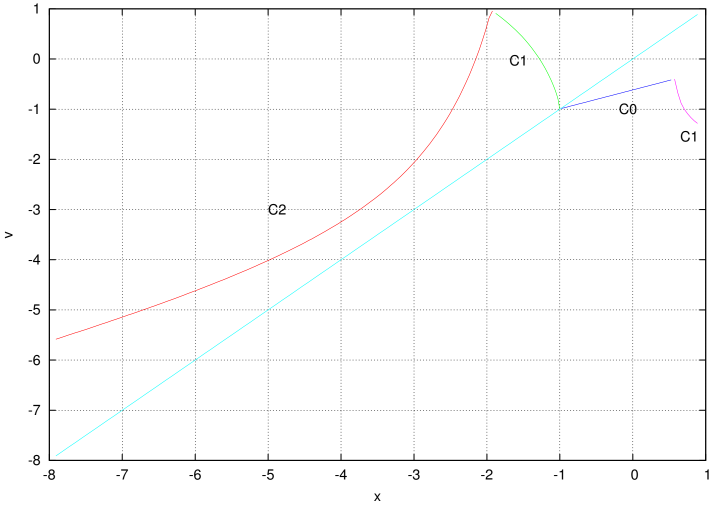
Dependence of transience on F



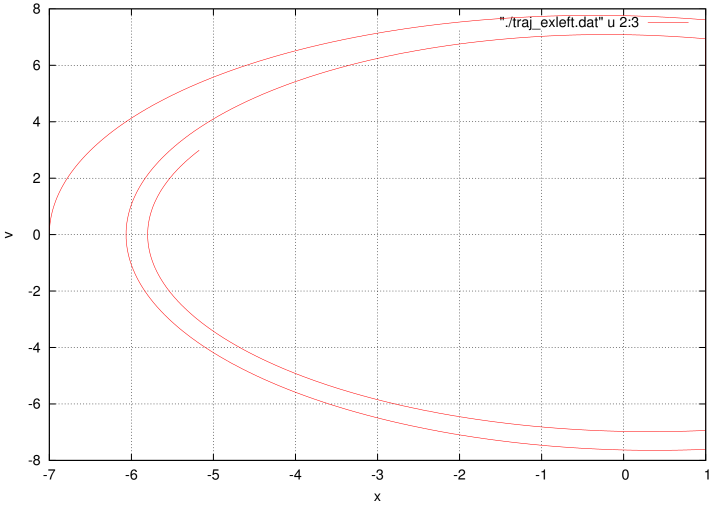
Dependence of transience on γ



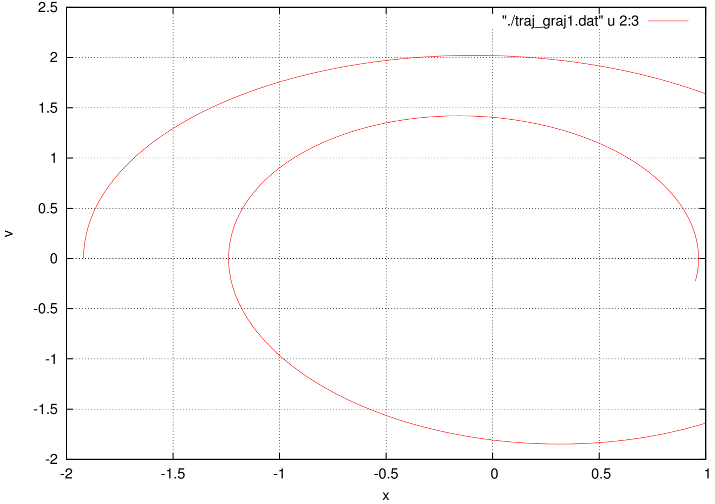
The Poincare map(a projection)



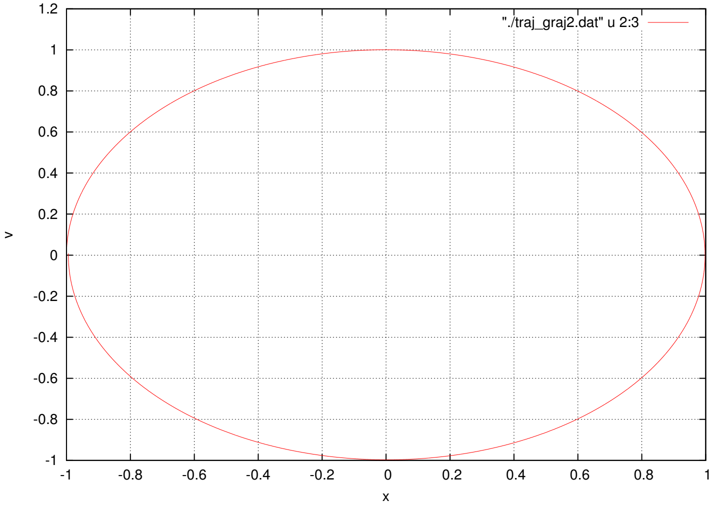
2 COLLISIONS



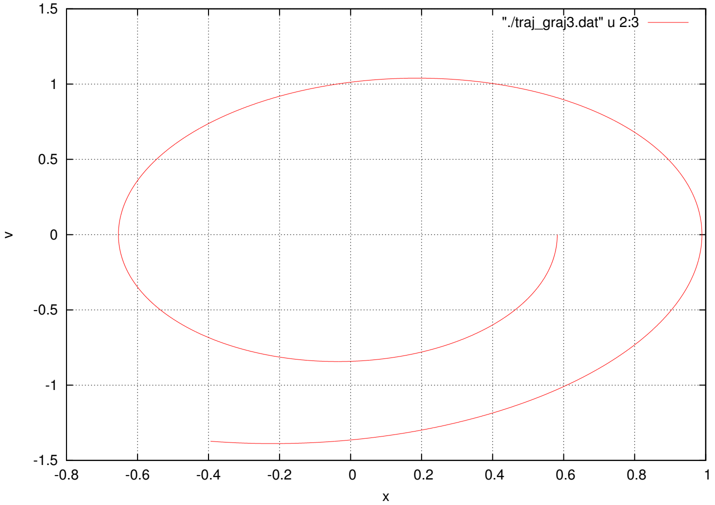
Grazing 1



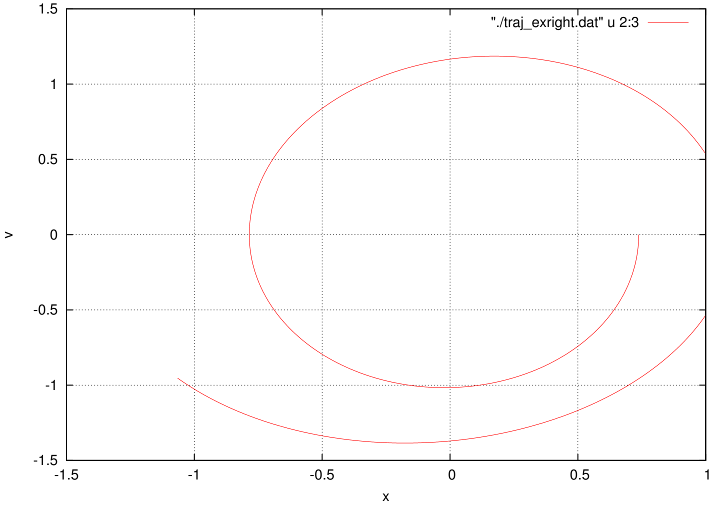
GRAZING 2


"./traj_graj2.dat" u 2:3
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
v
x

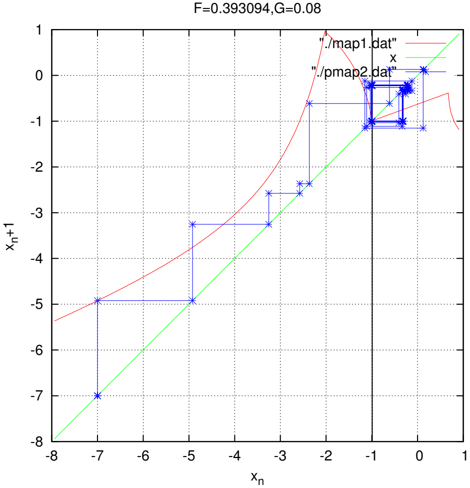
Grazing 3



1 COLLISION



DRAWBACKS OF THE PROJECTION



- Even before the stable orbit grazes, we see higher periodic orbts. Previously we thought it is due to a temporary lock-in to a pseudo periodic orbit due to intremittency.
- We saw that the cause is not intermittency, but rather the periods getting locked on to what looks like a genuine periodic orbit. We saw period 3.
- At the same time, the period-1 fixed point exists and is stable, just the basin of attraction not filling up the whole phase space.
- How does the basin boundary look?
- We tried to plot it by taking many initial conditions and following the trajectories to 'paint' the phase space.
- It does not work. Since the system is non-autonomous, later trajectories overwrite the colors painted by previous trajectories.

The collision detector

The previous strategy of determining the basin boundary of period-1 orbit did not work due to the extra dimension: time.

But we know that Poincáre sections can be used to take care of extra dimensions.

Also, what if we don't look for "basin boundary" as such, but some area in the $v \times x$ space which merely *guarantees* a period one orbit? In other words, a sufficient, but not necessary condition for period-1 orbit, which has the added advantage of being more tractable.

A SUFFICIENT CONDITION

Consider an initial condition:[1]

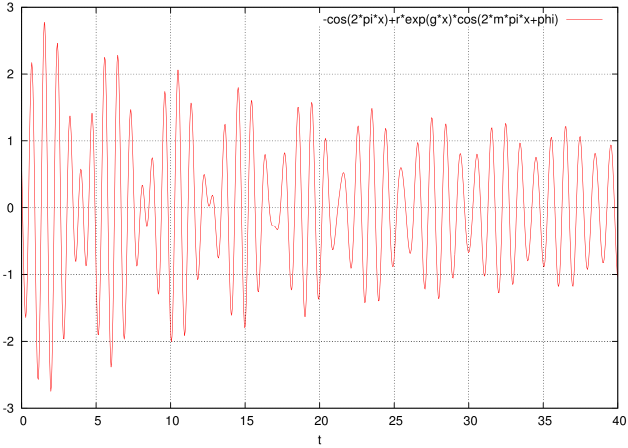
$$\vec{x_p}(0) = (-A, 0)$$
$$\vec{x_h}(0) = (x_0, v_0)$$

Where $A = \frac{F}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}}$
Starting from this point, the solution is, as usual:

$$x(t) = -A\cos\omega t + \frac{e^{-\gamma t/2}}{\omega_g}\left\{(\omega_g \cos\omega_g t + \frac{\gamma}{2}\sin\omega_g t)x_0 + (\sin\omega_g t)v_0\right\}$$

[1] Taking a slice through the time dimension

Akin to decaying beats

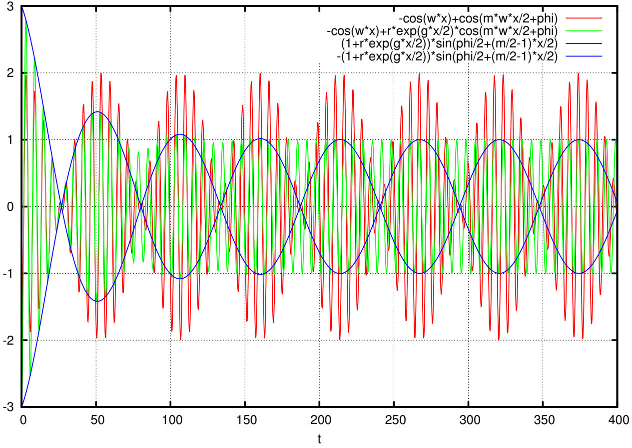


A little tidying up

$$
\begin{aligned}
x(t) &= -A\cos\omega t + \frac{e^{-\gamma t/2}}{\omega_g}\left\{(\omega_g \cos\omega_g t + \frac{\gamma}{2}\sin\omega_g t)x_0 + (\sin\omega_g t)v_0\right\} \\
&= -A\cos\omega t + e^{-\gamma t/2} B\cos\left(\omega_g t + C\right) \\
C &= -tan^{-1}\frac{\frac{\gamma}{2\omega_g}x_0 + \frac{v_0}{\omega_g}}{x_0} \\
B &= \sqrt{x_0^2 + \left(\frac{\gamma}{2\omega_g}x_0 + \frac{v_0}{\omega_g}\right)^2}
\end{aligned}
$$

PREDICTING COLLISIONS

In general, the sum of two different sine waves differing both in amplitude and frequency cannot be further simplified. However, an approximate envelope can be calculated.



Now the job is simple:

1. Consider the $x_0 \times v_0$ space.
2. For any point in the space, using the envelope, predict if there will be any future collision or not.
3. Points for which there is no further collision should form a closed area in the whole space.
4. The area must shrink as we approach grazing.
5. Correlate the area with transient lifetime.

The trajectory:

$$x(t) = -A\cos\omega t + e^{-\gamma t/2} B\cos\left(\omega_g t + C\right) \tag{2}$$

back Will have an envelope:[2]

$$E(t) = \left\{A + Be^{-\gamma t/2}\right\} \sin\left(\frac{C + (\omega_g - \omega)t}{2}\right) \tag{3}$$

If $\omega \approx \omega_g$ Other cases

The next peak of the envelope occurs at

$$t_c = \begin{cases} \frac{\pi - C}{\omega_g - \omega} & \text{if } \omega_g > \omega \\ \frac{\pi + C}{\omega - \omega_g} & \text{if } \omega_g < \omega \end{cases} \tag{4}$$

And has height:

$$E_m = A + Be^{-\gamma t_c/2} \tag{5}$$

[2]There are some aberrations in the first bulge of the envelope

Some definitions

Definition

E_m = height of the next peak of the envelope.

Definition

x_m = height of the next peak of the trajectory [a]

[a] Except in some cases, $x_m = E_m$

Definition

$\mu_{xv} = \{(x, v) \in \mathbb{X} \times \mathbb{V} : x_m(x, v) < \sigma\}$

Some Useful Inequalities

$$t_c \leq \frac{3\pi/2}{|\omega_g - \omega|} \tag{6}$$

$$x_m \geq E_m \tag{7}$$

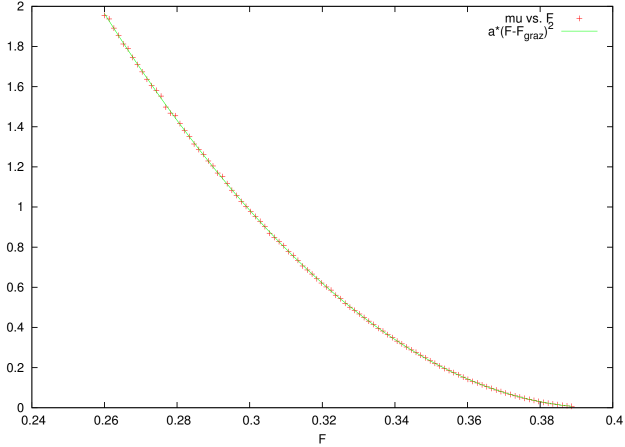
$$E_m \geq A + Be^{-\gamma \frac{3\pi/4}{|\omega_g - \omega|}} \tag{8}$$

$$B \geq max\left\{|x_0|, \left|\frac{\gamma}{2\omega_g}x_0 + \frac{v_0}{\omega_g}\right|\right\} \tag{9}$$

$$\mu_{xv} \leq \left\{(x, v) : B \leq (\sigma - A)e^{\gamma \frac{3\pi/4}{|\omega_g - \omega|}} = \chi\right\} \tag{10}$$

$$\leq \left\{(x, v) : -\chi \leq x \leq \chi, -\chi\omega_g - \frac{\gamma x}{2} \leq v \leq \chi\omega_g - \frac{\gamma x}{2}\right\} \tag{11}$$

μ_{xv} Vs. F



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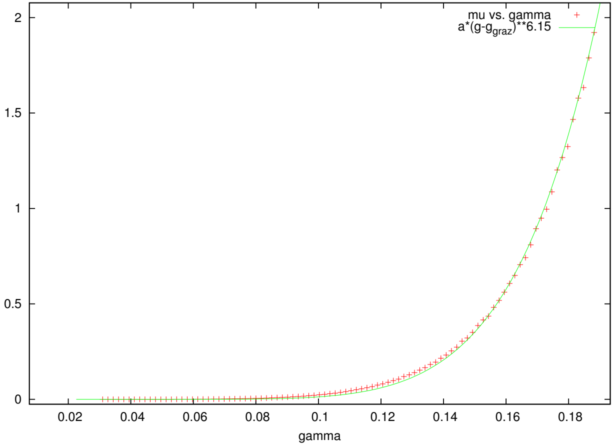
μ_{xv} vs. γ



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Celso Grebogi, Edward Ott, James A. Yorke, Critical Exponent of Chaotic Transients in Nonlinear Dynamical Systems, Phys. Rev. Lett. 57, 1284–1287 (1986):

Their results:

- $\tau \sim (\alpha - \alpha_*)^{-\gamma}$, γ the "critical exponent" depends on the system.
- Derives expression for γ in terms of the eigenvalues of the fixed points at collision.

Drawing parallels between the behaviour of μ_{xv} and the previous result, we can surmise:

$$\mu_{xv} \sim \frac{1}{\tau}$$

An approximate expression for the no-collision area μ_{xv}

So far, we have calculated μ_{xv} by means of a Monte Carlo integration: we chose N initial points, evaluated the number n of them that does not lead to collision and equated $\frac{n}{N}$ to μ_{xv}. Now we'll try an analytical approach.

$$\mu_{xv} = \int_{A+B(x,v)e^{-\gamma t_c(x,v)/2}<\sigma} dxdv$$

Now we recall (4)

$$t_c = \begin{cases} \frac{\pi - C}{\omega_g - \omega} & \text{if } \omega_g > \omega \\ \frac{\pi + C}{\omega - \omega_g} & \text{if } \omega_g < \omega \end{cases} \tag{12}$$

Where $C = -tan^{-1}\frac{\frac{\gamma}{2\omega_g}x_0+\frac{v_0}{\omega_g}}{x_0}$.
Since C is a rapidly varying function of both x and v and has range $\{-\pi/2, \pi/2\}$, we can replace $C \approx 0$ without too much error.
Therefore

$$\begin{aligned}\mu_{xv} &\sim \int_{A+B(x,v)e^{-\gamma\pi/(2|\omega_g-\omega|)}<\sigma} dxdv \\ &= \int_{B(x,v)<(\sigma-A)e^{\gamma\pi/(2|\omega_g-\omega|)}} dxdv\end{aligned}$$

Now we recall that $B = \sqrt{x_0^2 + \left(\frac{\gamma}{2\omega_g}x_0 + \frac{v_0}{\omega_g}\right)^2}$.
So our integral is now of the form: $\int_{x^2+(ax+by)^2<\chi^2} dxdv$

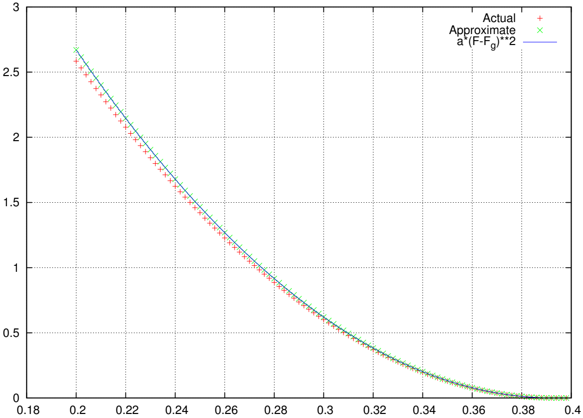
$$\int_{x^2+(ax+by)^2<\chi^2} dxdy$$
$$= \int_{-\chi}^{\chi} dx \int_{x^2+(ax+by)^2<\chi^2} dy$$
$$= \int_{-\chi}^{\chi} \frac{\sqrt{(2abx)^2 - 4b^2(x^2(1+a^2) - \chi^2)}}{b^2} dx$$
$$= \frac{1}{b^2} \int_{-\chi}^{\chi} 2b\sqrt{\chi^2 - x^2} dx$$
$$= \frac{2}{b} \int_{-\chi}^{\chi} \sqrt{\chi^2 - x^2} dx$$

However, we have disregarded one very important restriction: $x < \sigma$, due to the hard wall.
So, actually:

$$\mu_{xv} = 2\omega_g \int_{-\chi}^{max\{\chi,1\}} \sqrt{\chi^2 - x^2} dx$$

Comparing the analytical result with Monte Carlo simulation

Figure: μ_{vx} vs. F



Attempt at justification of the envelope

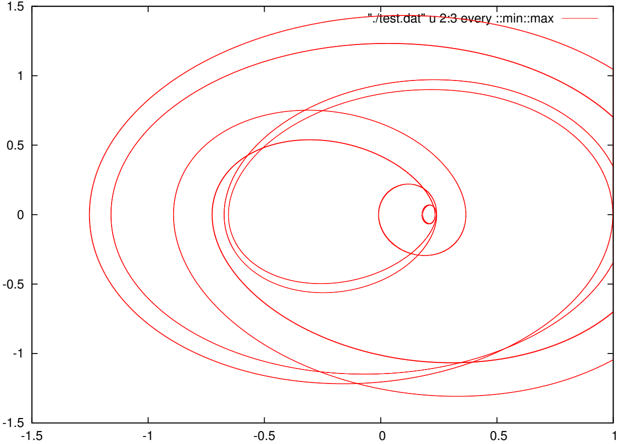
Recall that our calculation, both analytical and numerical, has started off with an assumption. Envelope
Now we'll try to justify.

$$
\begin{aligned}
&|f(t)cos(\omega_1 t) + g(t)cos(\omega_2 t + C)| \\
\leq& |(f(t) + g(t))(cos(\omega_1 t) + cos(\omega_2 t + C))| \\
=& 2\left|\{f(t) + g(t)\} sin\left(\frac{(\omega_1 - \omega_2)t + C}{2}\right)\right|\left|sin\left(\frac{(\omega_1 + \omega_2)t + C}{2}\right)\right|
\end{aligned}
$$

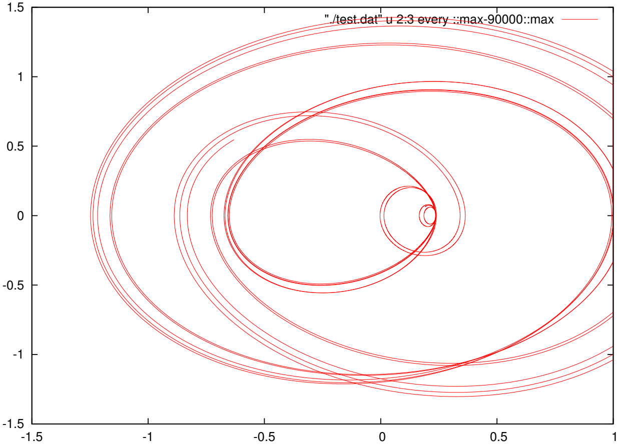
Now since the last term is much more rapidly varying than the second term, we replace it with its average over a full cycle without incurring too much error:

$$
\begin{aligned}
=& 2\left|\{f(t) + g(t)\} sin\left(\frac{(\omega_1 - \omega_2)t + C}{2}\right)\right| \frac{2}{\pi}\int_0^{\pi/2} cos(t)dt \\
=& 2\left|\{f(t) + g(t)\} sin\left(\frac{(\omega_1 - \omega_2)t + C}{2}\right)\right| \frac{2}{\pi}
\end{aligned}
$$

Nature of long lived transients

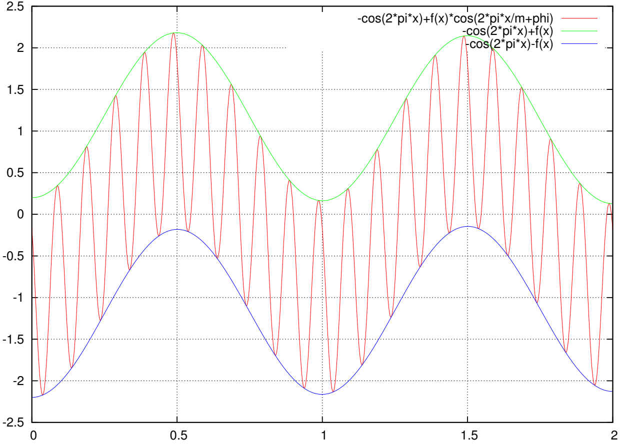


NATURE OF LONG LIVED TRANSIENTS



$\omega >> \omega_g$

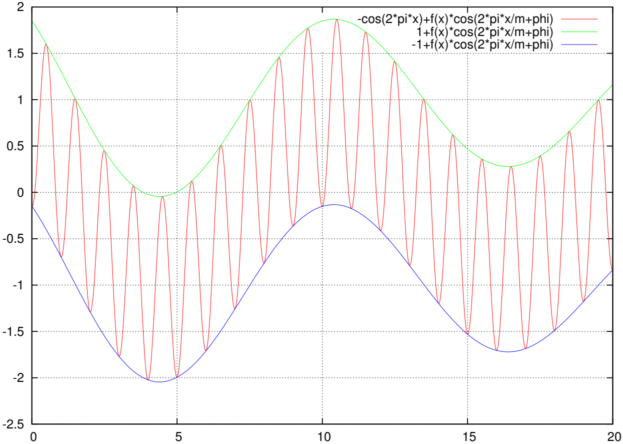
$$x(t) = -A\cos\omega t + e^{-\gamma t/2} B \cos\left(\omega_g t + C\right) \quad (13)$$
$$E(t) = A + e^{-\gamma t/2} B \cos\left(\omega_g t + C\right) \quad (14)$$



$\omega << \omega_g$

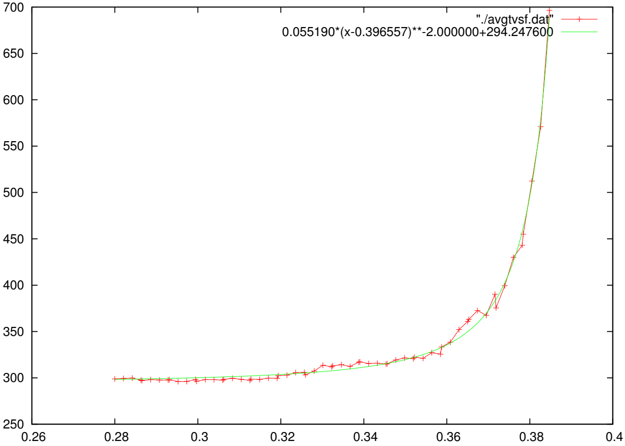
$$x(t) = -A\cos\omega t + e^{-\gamma t/2} B \cos\left(\omega_g t + C\right) \quad (15)$$

$$E(t) = -A\cos\omega t + e^{-\gamma t/2} B \quad (16)$$



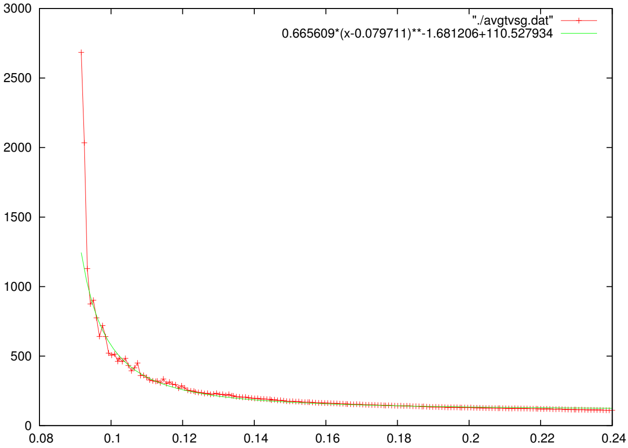
MATCHING WITH LIFETIME DATA

Figure: Transient lifetime vs. F



μ_{vx} vs. F

Figure: Transient lifetime vs. γ



μ_{vx} vs. γ

Figure: Transient lifetime vs. r