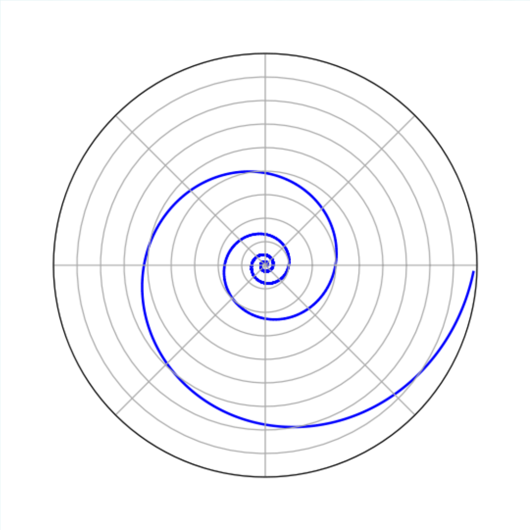
- Check your attendance – it matters!
- A pop quiz may pop at any time.

## Data Structures

- *discrete math*

# ITP20001/ECE 20010 DATA STRUCTURES



- *Logarithmic spiral*
- *miraculous spiral*
- *Spira mirabilis [Latin]*

### 1.1 Logarithms

- **Exponents:**
  - $X^Y$, or "**X to the $Y^{th}$ power**";
    X multiplied by itself Y times
- **Some useful identities:**
  - $X^0 = 1$, provided $x \neq 0$.
  - $X^A X^B = X^{A+B}$
  - $X^A / X^B = X^{A-B}$
  - $X^{-B} = \frac{1}{X^B}$
  - $X^{1/n} = \sqrt[n]{X}$
  - $X^N + X^N = 2X^N$
  - $2^N + 2^N = 2^{N+1}$

## 1.1 Logarithms

- Logarithms
  - *definition*: $X^A = B$ if and only if $log_X B = A$
  - *intuition*: $\boldsymbol{log_X B}$ means: the power $\boldsymbol{X}$ must be raised to, to get $\boldsymbol{B}$
  - This is the same as asserting $X^{log_X B} = B$.
- Examples:
  - $log_X 1 = 0$
  - $log_X X = 1$
  - $\log_2 16 = 4$
  - $\log_{10} 1000 = 3$

- Most people use base 10, written as $log_{10}$ or $log$, base $e$ or $ln$ .

- In computer science, we typically use base 2, written as $\boldsymbol{log_2}$ or $lg$. **Between us**, however, we simply use $\boldsymbol{log}$ instead of $log_2$ or $\boldsymbol{lg}$.

### 1.1 Logarithms

**Exercise:**

- How many bits does it take to encode $1{,}000{,}000$ different values?
  - Each bit can take on one of two values (0 or 1).
  - Therefore, $n$ bits can represent $2^n$ values. (ex. 8bits: 0~255)
  - So, encoding 1,000,000 values will require $log_2 n = 20$ bits.
    To be exact, $\lceil log_2(1{,}000{,}000) \rceil$ = 20. // “a little under 20”

1.1 Logarithms

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- ⌈ x ⌉ → **Ceiling** function: the smallest integer ≥ x.
  - Ex. $\lceil 2.3 \rceil = 3$ $\lceil -2.3 \rceil = -2$ $\lceil 2 \rceil = 2$
- ⌊ x ⌋ → **Floor** function: the largest integer ≤ x.
  - Ex. $\lfloor 2.7 \rfloor = 2$ $\lfloor -2.7 \rfloor = -3$ $\lfloor 2 \rfloor = 2$

### 1.1 Logarithms

**Powers of 2:**

- A bit is 0 or 1 (just two different "letters" or "symbols")
- A sequence of *n* bits can represent $2^n$ distinct things
  - For example, the numbers 0 through $2^n-1$
- $2^{10}$ is 1024 ("about a thousand", kilo in CSE speak)
- $2^{20}$ is "about a million", mega in CSE speak
- $2^{30}$ is "about a billion", giga in CSE speak

Java: an **`int`** is 32 bits and signed, so "max int" is "about 2 billion"

a **`long`** is 64 bits and signed, so "max long" is $2^{63}-1$

## 1.1 Logarithms

**Examples:**

- If we have an alphabetically **sorted list of 100 names,** how many records do we need to look at to find a given individual?
  - Since the list is sorted, we can use **binary search**.
  - Look at the middle element: if it’s after than the name we’re looking for, search the first half of the list. If it’s before the name we’re looking for, look at the second half of the list.
  - Each check cuts the size of the list in half;
    how many times can we do this?

## 1.1 Logarithms

**Examples:**

- Let’s suppose that we begin with a value $N$, divide it by 2, then the result that we divide it by 2, and so on, until reaching 1 or less.
  - N, N/2, N/4, … … , 4, 2, 1

- **Question:** How many times did we divide before reaching 1 or less?
  - Think of it from the other direction: How many times do I have to multiply by 2 to reach N?
  - 1, 2, 4, … ., N/4, N/2, N<br>Call this **k** number of times, then $N = 2^{\boldsymbol{k}}$ , or $\boldsymbol{k} = \lg(N)$.

- **Exercise:** How is this related to the idea of binary search?
  - (I leave this one for you to think about it, as an exercise.)

1.1 Logarithms

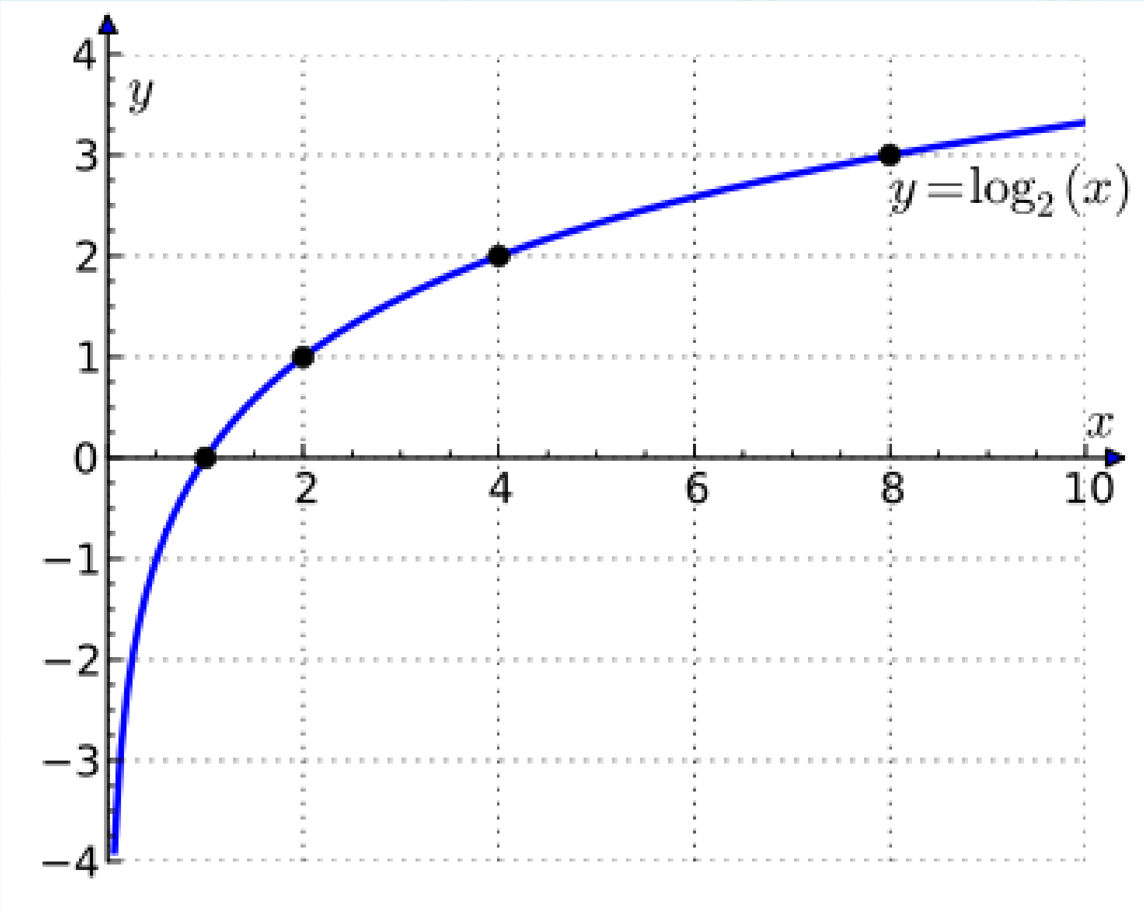
## Logarithmic Operators:

- $\log ab = \log a + \log b$
- $\log \frac{a}{b} = \log a - \log b$
- $\log a^b = b \log a$
- $log_a n = \frac{log_b n}{log_b a} = \frac{\log n}{\log a}$ (this is used to change bases.)
- $log_a a = 1, for\ all\ a > 0$
- $log_a 1 = 0, for\ all\ a > 0$

- Evaluate $log_4 4 + log_2 2 + log_{10} 1$
- Evaluate $log_2 \frac{1}{2}$
- Plot $y = log_2 x$

1.1 **Logarithms**

- Plot $y = log_2 x$



### 1.1 Logarithms

**Logarithmic Operators:**

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- Evaluate $log_4 2 + log_4 32$
  - $log_4 2 + log_4 32 = log_4 2 \, * 32 = log_4 64 = (log_4 4^3) = 3$
- Evaluate $log_2 400$ (all most all calculators don't do base 2…. wow!)
  - $log_2 400 = \frac{log_e 400}{log_e 2} = \frac{5.991}{0.69} = 8.68$
- What is x?
  - $2^x = 2^{10} + 2^{10}$
  - x lg 2 = lg ( 2*2^10 ), x = lg 2 + lg 2^10 , x = 1 + 10 = 11

### 1.1 Logarithms

## Logarithmic Operators:

- Logarithms can also be very useful for comparing very large or small numbers.
- **Exercise**: $10^{100} > 2^{256}$ is true or not?
  Neither number can be calculated directly without risking overflow.
- Hint: Since the logarithm function is monotonically increasing, if $a < b$, then $\log a < \log b$ .

## 1.1 Logarithms

**Exercise:** Compute the order of growth rate b in $T(n) \cong a\, n^b$ of the running time as a function of $n$ using Selection Sort of which the time complexity is $O(n^2)$.

| n | time |
|---|---|
| 100 | 0.000023 |
| 200 | 0.000079 |
| 300 | 0.000173 |
| 400 | 0.000299 |
| 500 | 0.000477 |
| 600 | 0.000660 |
| 700 | 0.000904 |
| 800 | 0.001174 |
| 900 | 0.001468 |
| 1000 | 0.001818 |

As input size changes. the *growth rate* **b** of the execution time would be

$$\left(\frac{n_2}{n_1}\right)^{\boldsymbol{b}} = \frac{t_2}{t_1}$$

When input size increases twofold , it would be

$$(2)^{\boldsymbol{b}} = \frac{t_2}{t_1}$$

In case of the $O(n^2)$ algorithm, the growth rate should be close to 2.0.

Let′s pick up $n_1$ = 500 to $n_2$ = 1000, *then* $t_1$ = 0.000477 to $t_2$ = 0.001818, respectively.

The growth rate of this algorithm is

$$b = \log\left(\frac{t_2}{t_1}\right) = \log\left(\frac{0.001818}{0.000477}\right) = \log(3.81) = \mathbf{1.93}$$

### 1.1 Logarithms

**Exercise:** Compute the order of *growth rate b* in $T(n) \cong a\, n^b$ of the running time as a function of $n$ using Selection Sort of which the time complexity is $O(n^2)$.

Let's pick up

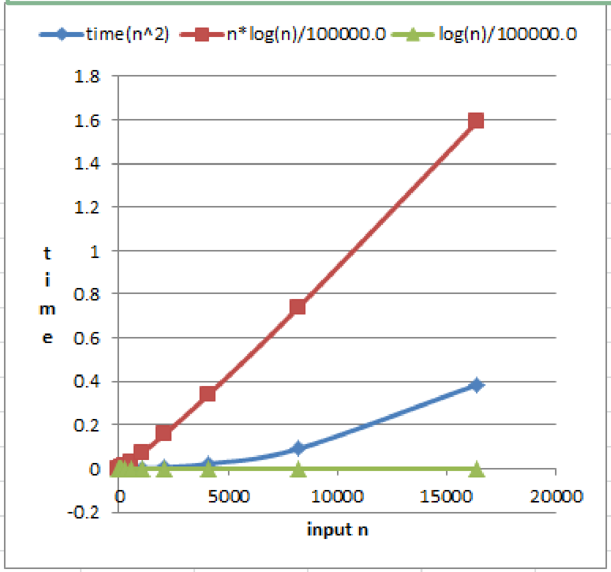
$n_1$ = 4096, $n_2$ = 8192, *then* $t_1$ = 0.022022, $t_2$ = 0.09075, respectively.

The ***measured*** *growth rate b* of the algorithm is

$$b = \log\left(\frac{t_2}{t_1}\right) = \log\left(\frac{0.09075}{0.022022}\right) = \log(4.12) = \mathbf{2.04}$$

F22

| | A | B | C | D |
|---|---|---|---|---|
| 1 | n | time(n^2) | n*log(n)/100000.0 | log(n)/100000.0 |
| 2 | | SelectionSort | QuickSort(average) | ? |
| 3 | 1 | 0 | 0 | 0 |
| 4 | 2 | 0 | 0.000014 | 0.000007 |
| 5 | 4 | 0 | 0.000055 | 0.000014 |
| 6 | 8 | 0 | 0.000166 | 0.000021 |
| 7 | 16 | 0 | 0.000444 | 0.000028 |
| 8 | 32 | 0.000002 | 0.001109 | 0.000035 |
| 9 | 64 | 0.000006 | 0.002662 | 0.000042 |
| 10 | 128 | 0.000023 | 0.006211 | 0.000049 |
| 11 | 256 | 0.000089 | 0.014196 | 0.000055 |
| 12 | 512 | 0.000364 | 0.03194 | 0.000062 |
| 13 | 1024 | 0.001427 | 0.070978 | 0.000069 |
| 14 | 2048 | 0.005464 | 0.156152 | 0.000076 |
| 15 | 4096 | 0.022022 | 0.340696 | 0.000083 |
| 16 | 8192 | 0.09075 | 0.738174 | 0.00009 |
| 17 | 16384 | 0.384667 | 1.589913 | 0.000097 |
| 18 | | | | |



**2. Summations**

- In analyzing a program's performance, we'll need to add up the number of times an operation is taken.
- It is typically written as:

a closed form

$$\sum_{i=1}^{n} f(i)$$

which is equivalent to $f(1) + f(2) + \ ... + f(n)$

- **Example:**
  - summation of 1 … 10?
  - summation of 1 … n?

**2. Summations - Arithmetic Sum**

- We will be particularly interested in the following sum:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

- We can easily see the solution if we add the sequence twice with one of the sequences written in inverse order.

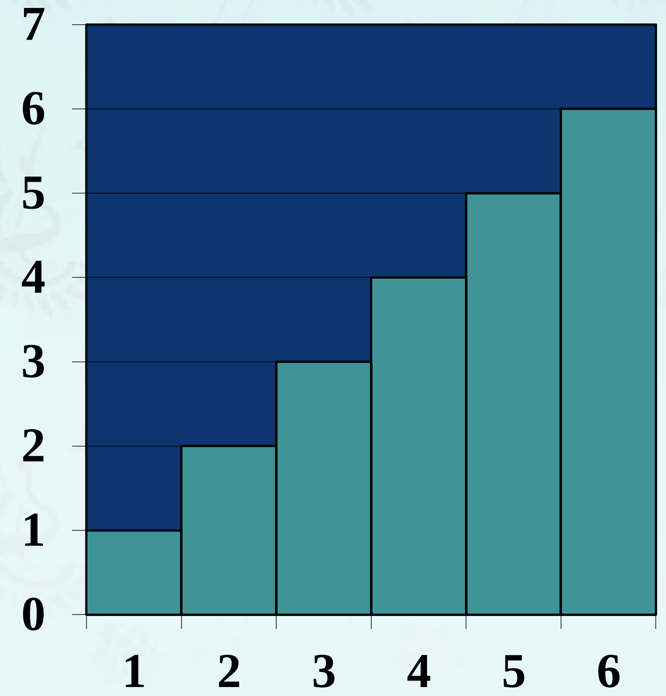
$$S = 1 + 2 + 3 + \ldots\ldots + (n-1) + n$$
$$S = n + (n-1) + (n-2) + \ldots\ldots + 2 + 1$$

- By adding two sequences, each of the pairs adds to $(n + 1)$. There are $n$ of them.
- $2S = (1+n) + (1+n) + (1+n) + \ldots\ldots + (n+1) + (n+1)$
- $= n(n+1)$
- Therefore, $S = n(n+1)/2$.

**2. Summations - Arithmetic Sum**

- There is a simple visual proof of this fact for the following sum:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$



**2. Summations - Arithmetic Sum**

- Some common summations and their closed-form solutions:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{2n^3 + 3n^2 + n}{6}$$

**2. Summations - Geometric Sum**

We are also be interested in the sum, it is so called **geometric** sum;

$$\sum_{i=0}^{n} a^i = 1 + a^1 + a^2 + \ \dots \dots + a^{n-1} + a^n$$

- **Proof:** Let's use $S$ to denote the sum:
- $\boldsymbol{S} = \ 1 + a^1 + a^2 + \ \dots \dots + a^{n-1} + a^n$
- $\boldsymbol{aS} = \qquad a^1 + a^2 + \ \dots \dots + a^{n-1} + a^n + a^{n+1}$

  $= \ S + \ a^{n+1} \ - 1$

From $aS = S + \ a^{n+1} \ - 1$**, we solve for** $\boldsymbol{S}$**, obtaining:**

$$\boldsymbol{S} \ = \sum_{i=0}^{n} a^i = \frac{a^{n+1} \ - \ 1}{a \ - 1}$$

**2. Summations - Geometric Sum**

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$$\sum_{i=0}^{n} a^i = 1 + a^1 + a^2 + \ \ldots\ldots + a^{n-1} + a^n = \frac{a^{n+1} - 1}{a - 1}$$

**Exercise:** $\sum_{i=0}^{n-1} a^i =$

**Exercise:** $\sum_{i=0}^{n} 2^i =$

### 2. Summations - Geometric Sum

Infinite geometric series….

$$\sum_{i=0}^{n} a^i = 1 + a^1 + a^2 + \ \ldots\ldots + a^{n-1} + a^n = \sum_{i=0}^{n} a^i = \frac{a^{n+1} - 1}{a - 1}$$

When $\boldsymbol{a} < \mathbf{1}$ **and** $\boldsymbol{n}$ **goes to infinity**, the sum becomes

$$\sum_{i=0}^{\infty} a^i = \frac{\mathbf{1}}{\mathbf{1} - \boldsymbol{a}}$$

**Exercise:**

Compute the infinite “geometric” series intuitively and using arithmetic sum.

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ \ldots\ldots$$

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Infinite geometric series….

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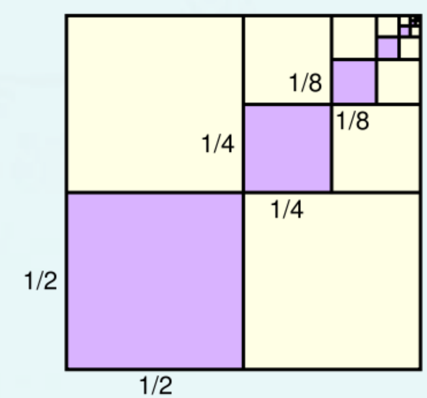
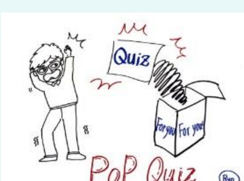
$$\sum_{i=0}^{\infty} a^i = \frac{\mathbf{1}}{\mathbf{1} - \boldsymbol{a}}$$

**Exercise**: Compute the sum of the areas of the purple squares.
Intuitively? Using arithmetic sum?

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} \dots\dots$$



## 2. Summations - Arithmetic Sum

Infinite geometric series….

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When $a < 1$ **and** $n$ **goes to infinity,** the sum becomes

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**Exercise**: Compute the sum of the areas of the purple squares.
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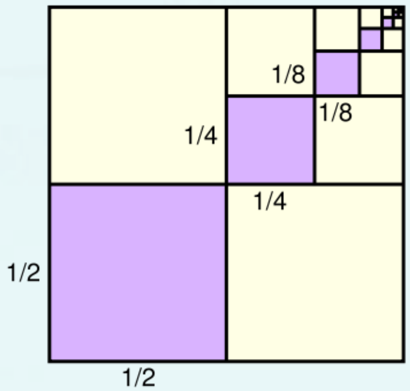
$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} \;\ldots\ldots$$



3. **Mathematical Proofs**

- The three most common forms of proof are:
  1. Direct proof (sometimes called a constructive proof)
  2. Indirect proof or proof by contradiction
  3. **Inductive proof**
     - It is very similar to recursion.
     - You establish a **base case** that is proved directly.
     - This is followed by an **inductive step** which shows how the **hypothesis** holds for **larger cases**.

3. **Mathematical Proofs**

- **Inductive proof**
  - In data structures and algorithms, we often want to prove that something holds over a range of values

  - The **base case** will prove the theorem for the initial $c$ values.
  - The **inductive step** will show that, if the theorem holds for $\boldsymbol{n-1}$, then it holds for $\boldsymbol{n}$.
    **Alternatively,** you may assume that it is true for $n$ which is the inductive hypothesis first and then prove it for $\boldsymbol{n+1}$. or for **2n**.

This is so called the inductive hypothesis

3. **Mathematical Proofs**

**Inductive proof Example:** prove $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

- **Base case:** Let $n = 1$. $\frac{1(1+1)}{2} = \frac{2}{2} = 1.$
- **Inductive step:** We state the **inductive hypothesis** for $\boldsymbol{n-1}$ that:

$$\sum_{i=1}^{n-1} i = \frac{(n-1)((n-1)+1)}{2} = \frac{(n-1)(n)}{2}$$

- Assuming this is true, adding the **$\boldsymbol{n}$th term** yields:

$$\sum_{i=1}^{n} i = \frac{(n-1)(n)}{2} + n = \frac{(n-1)(n)}{2} + \frac{2n}{2} = \frac{n(n+1)}{2}$$

Therefore, we have proved it.

### 3. Mathematical Proofs

**Inductive proof Example:** prove $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

- **Base case:** Let $n = 1$. $\frac{1(1+1)}{2} = \frac{2}{2} = 1$.
- **Inductive step:** Assume that it holds for $n$, then for $2n$; that is :

$$\sum_{i=1}^{2n} i = \frac{2n(2n+1)}{2} = \boldsymbol{n(2n+1)}$$

Using the **induction hypothesis** that the left side above can be rewritten and rearranged algebraically:

$$\sum_{i=1}^{2n} i = \frac{n(n+1)}{2} + [(n+1) + (n+2) + \ \dots + (n+n)]$$

$$= \frac{n(n+1)}{2} + n * n + \frac{n(n+1)}{2}$$

$$= n(n+1) + n * n$$

$$= \boldsymbol{n(2n+1)}$$

Therefore, we have proved it.

## 3. Mathematical Proofs

**Inductive proof Exercise:** prove $\sum_{i=1}^{n} i^2 = \frac{2n^3+3n^2+n}{6}$

- **Base case:** Let $n = 1$. $\frac{2+3+1}{6} = 1.$
- **Inductive step:** We state the **inductive hypothesis** for $\boldsymbol{n-1}$ that:

$$\sum_{i=1}^{n-1} i^2 = \frac{2(n-1)^3+3(n-1)^2+(n-1)}{6}$$

- Assuming this is true, adding the ***n*th term** yields:

$$\sum_{i=1}^{n-1} i^2 + n^2 = \frac{2(n-1)^3+3(n-1)^2+(n-1)}{6} + \frac{6n^2}{6}$$

$$= \frac{2n^3+3n^2+n}{6}$$

Therefore, we have proved it.

Exercise: Prove that $8^n - 3^n$ is divisible by 5 for all $n \geq 1$.

Summary

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