ITP20001/ECE 20010 Data Structures

Chapter 5

- *introduction*
- *tree, binary tree, binary search tree*
- ***heaps data structure***
  - ***complete binary tree***
  - ***priority queues (Chapter 9)***
  - *binary heap and min-heap*
  - *max-heap demo*
  - *max-heap implementation*
  - *heap sort (Chapter 7)*

Chapter 5.2 Binary Trees – Properties

Definition: *A **full** binary tree* of $\boldsymbol{level\ k}$ is a binary tree having $\mathbf{2^k - 1}$ **nodes**, $k \geq 0$.

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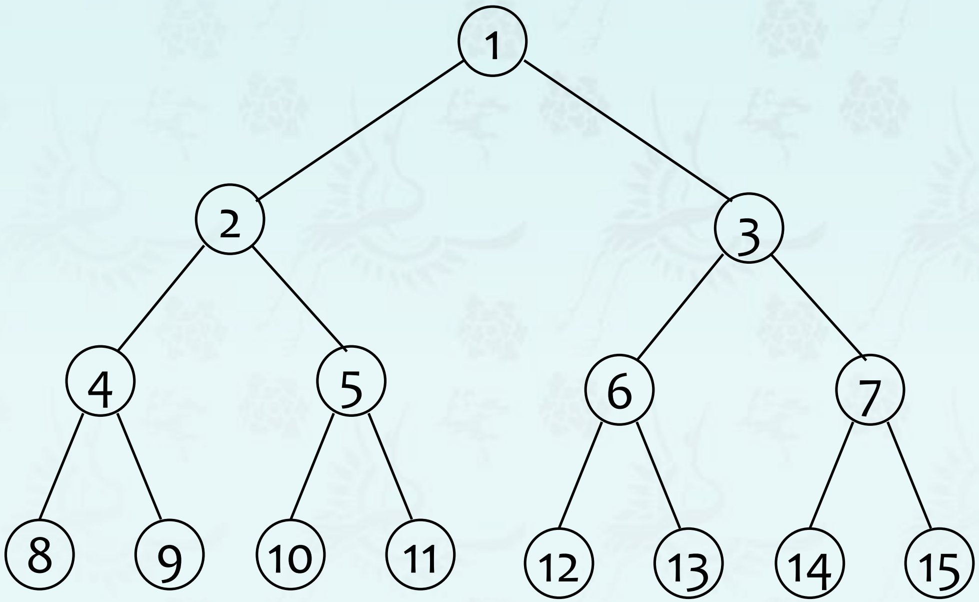

*A **full** binary tree*

Chapter 5.2 Binary Trees – Properties

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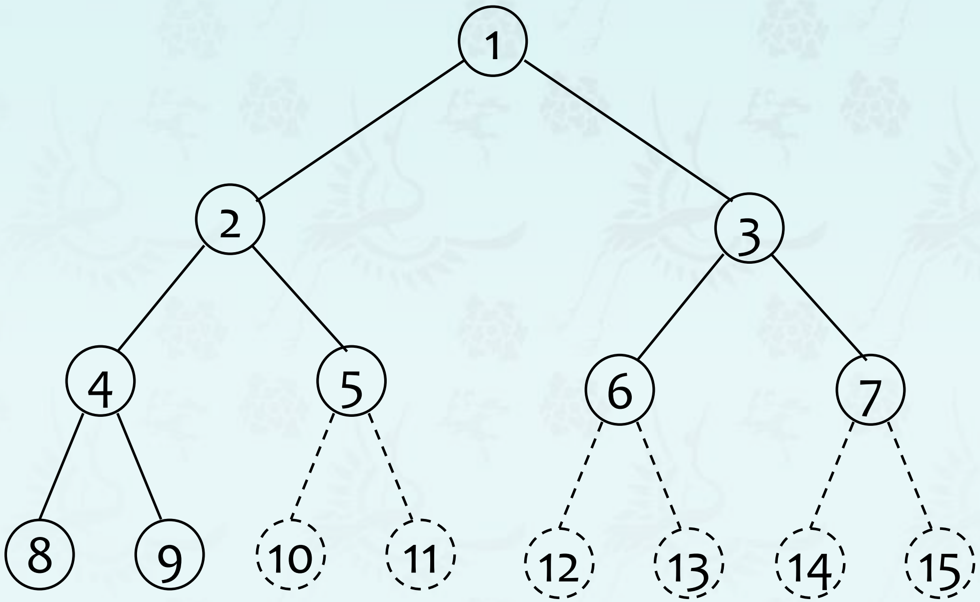
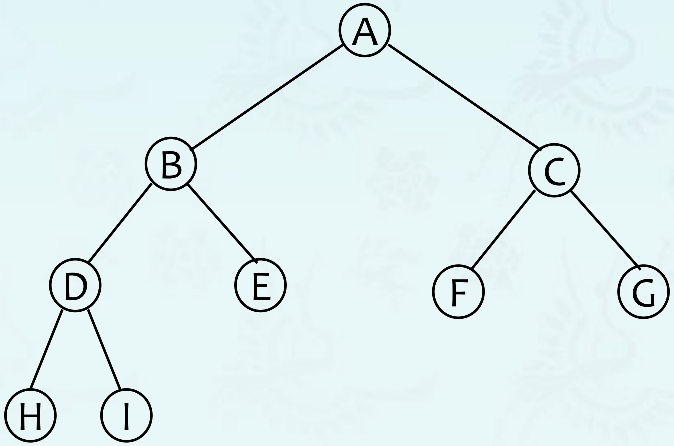


*A **complete** binary tree*

Chapter 5.2 Binary Trees – Array representation

Property: a **complete** binary tree with n nodes, any node index i, $1 \le i \le n$, we have

(1) $parent(i)$ is at $\boxed{\lfloor i/2 \rfloor}$ $if\ i\ \neq\ 1.$ $If\ i\ =\ 1,$ i is at the root and has no parent.
(2) $leftChild(i)$ is at $\mathbf{2i}$ $if\ \ 2i\ <=\ n.$ $If\ 2i\ >\ n,$ then i has no left child.
(3) $rightChild(i)$ is at $\mathbf{2i+1}$ $if\ \ 2i\ +\ 1\ \ <=\ n.$ $If\ 2i\ +\ 1\ >\ n,$ then i has no right child.



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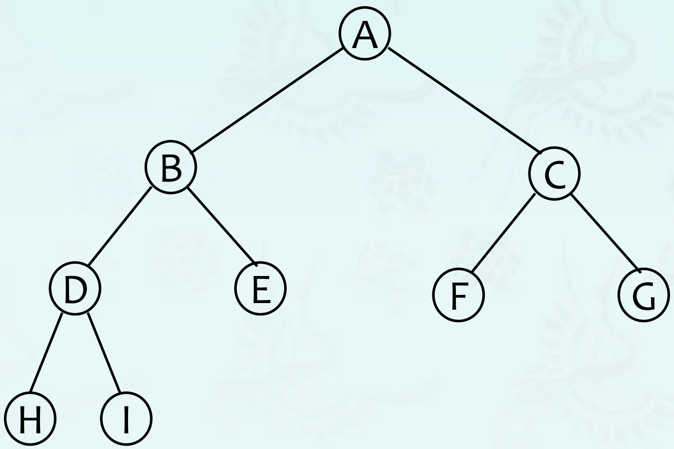

[0]	-
[1]	A
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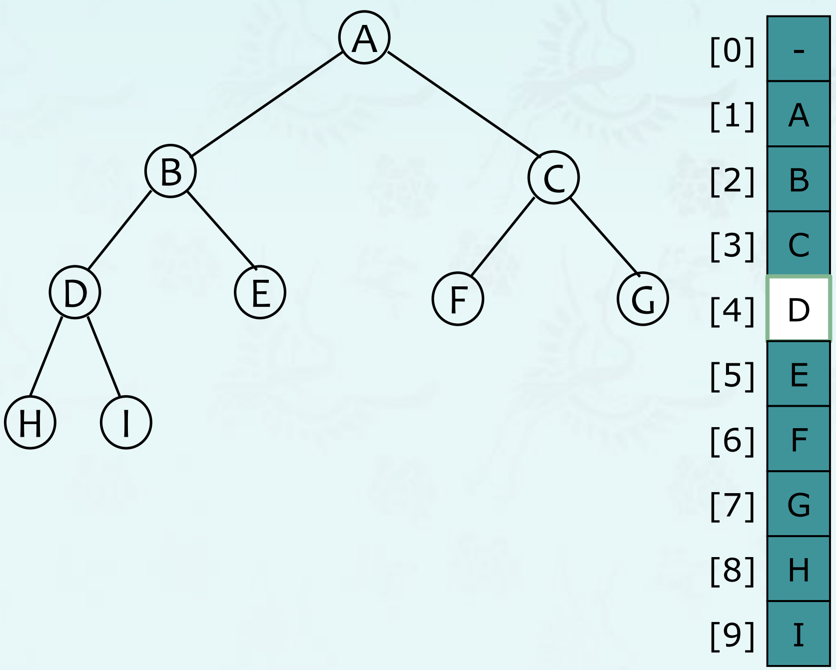

Example:
Find its parent, left child and right child at node D

Solution:

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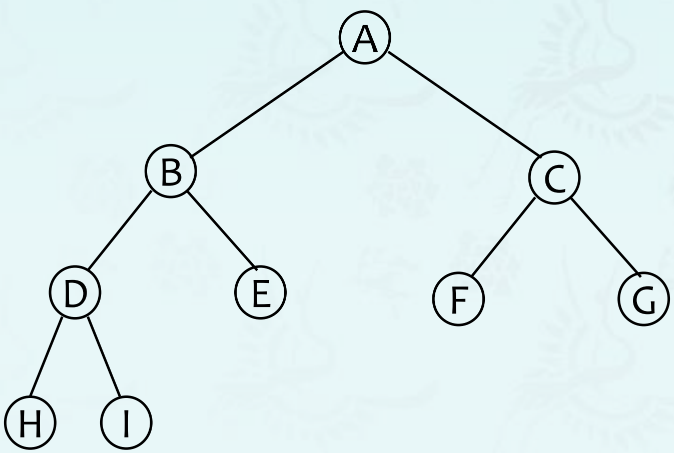

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Example:
Find its parent, left child and right child at node D

Solution:
$parent(i = 4)$ is at 4/2 = 2
$leftChild(4)$ is at 2x4 = 8
$rightChild(4)$ is at 2x4 + 1 = 9

How do you like this property of the tree?

Chapter 5.6 *A full binary tree in nature*

ITP20001/ECE 20010 Data Structures

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- Because it provides an efficient implementation for **priority queues.**

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Priority queues.

- Queues with priorities associated to.
- **Example:** A line waiting to be served at a bank and served FIFO except if a senior or a disabled person arrives in the line. They are served first. Seniors and disabled persons have higher priority than others.

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A typical ADT for Priority Queue

- Get the top priority element (min or max)
- Insert an element
- Delete the top priority element
- Decrease the priority of an element

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A typical ADT for Priority Queue

- Get the top priority element (min or max)
- Insert an element
- Delete the top priority element
- Decrease the priority of an element

- O(1)
- O(log n)
- O(log n)
- O(log n)

Priority queue applications

• Event-driven simulation.	[customers in a line, colliding particles]
• Numerical computation.	[reducing roundoff error]
• Data compression.	[Huffman codes]
• Graph searching.	[Dijkstra's algorithm, Prim's algorithm]
• Number theory.	[sum of powers]
• Artificial intelligence.	[A* search]
• Statistics.	[maintain largest M values in a sequence]
• Operating systems.	[load balancing, interrupt handling]
• Discrete optimization.	[bin packing, scheduling]
• Spam filtering.	[Bayesian spam filter]

Challenge: Find the largest **M** items in a stream of **N** items.

- Fraud detection: isolate $$ transactions.
- Hacking: KT's customer DB access by their sales agents
- File maintenance: find biggest files, directories, or emails.

N huge,
M large

Constraints: Not enough memory to store N items.

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%more trans.txt

Turing	6/17/1990	644.08
vonNeumann	3/26/2002	4121.85
Dijkstra	8/22/2007	2678.40
vonNeumann	1/11/1999	4409.74
Dijkstra	11/18/1995	837.42
Hoare	5/10/1993	3229.27
vonNeumann	2/12/1994	4732.35
Hoare	8/18/1992	4381.21
Turing	1/11/2002	66.10
Thompson	2/27/2000	4747.08
Turing	2/11/1991	2156.86
Hoare	8/12/2003	1025.70
vonNeumann	10/13/1993	2520.97
Dijkstra	9/10/2000	708.95
Turing	10/12/1993	3532.36
Hoare	2/10/2005	4050.20

%java TopM 5 < trans.txt

Thompson	2/27/2000	4747.08
vonNeumann	2/12/1994	4732.35
vonNeumann	1/11/1999	4409.74
Hoare	8/18/1992	4381.21
vonNeumann	3/26/2002	4121.85

Sort key

Challenge: Find the largest **M** items in a stream of **N** items.
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N huge,
M large

Order of growth of finding the largest M **in a stream of N items**

implementation	time	space
sort	N log N	N
binary heap	N log M	M
best in theory	N	M

Challenge: Find the largest **M** items in a stream of **N** items.
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N huge,
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Order of growth of finding the largest M **in a stream of N items**

implementation	insert	delete	min/max
unordered array	1	N	N
ordered array	N	1	1
goal	log N	log N	log N

Mission Impossible?

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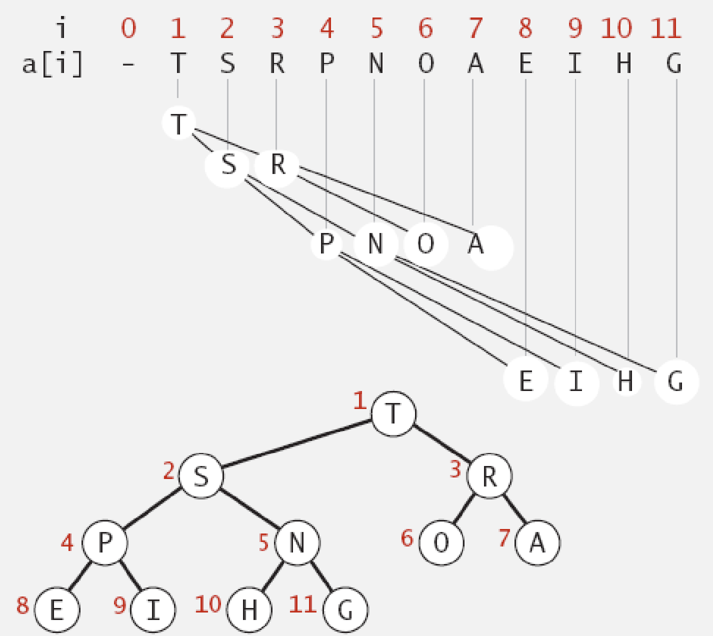
Binary heap: array representation of a **heap-ordered** complete binary tree

- **Properties:**

- Array representation



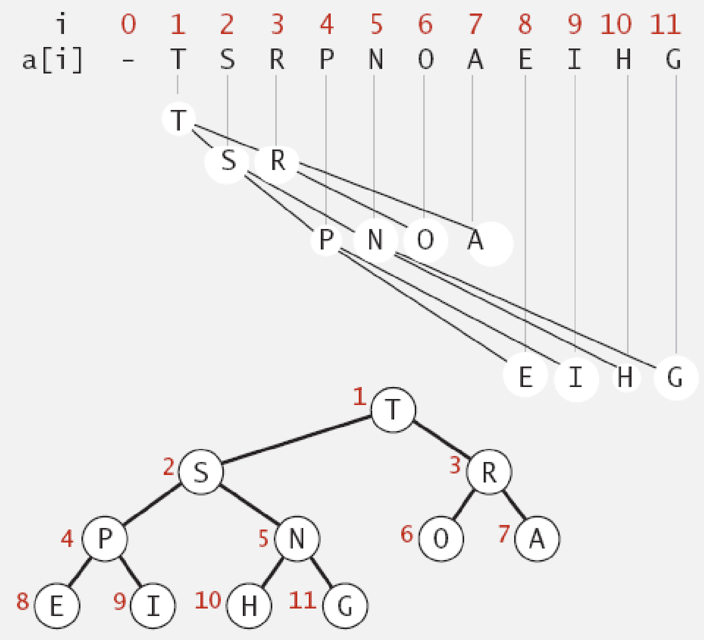
Heap representations

Chapter 5.6 Heaps & Priority Queues

Binary heap: array representation of a **heap-ordered** complete binary tree

- **Properties:**
  - **Heap-ordered:**
    Parent's key no smaller than children's keys. [maxheap]
  - **Heap-structure:**
    A complete binary tree

- Array representation



Heap representations

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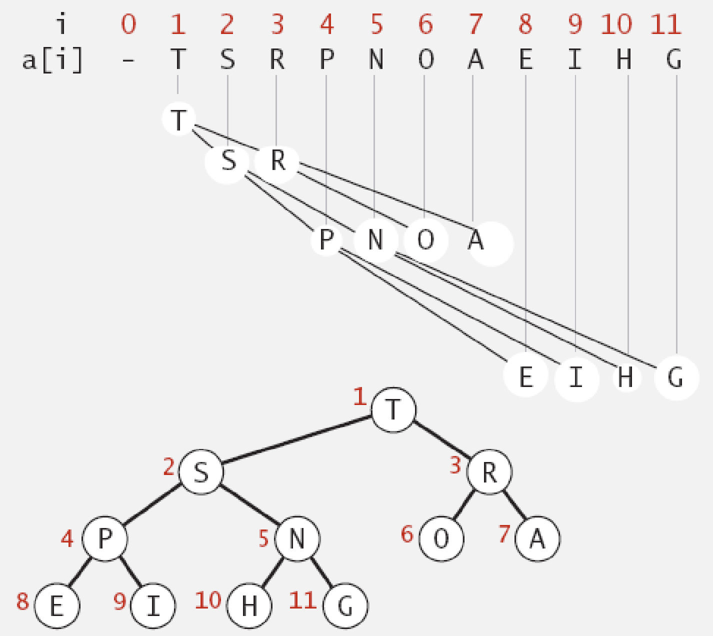
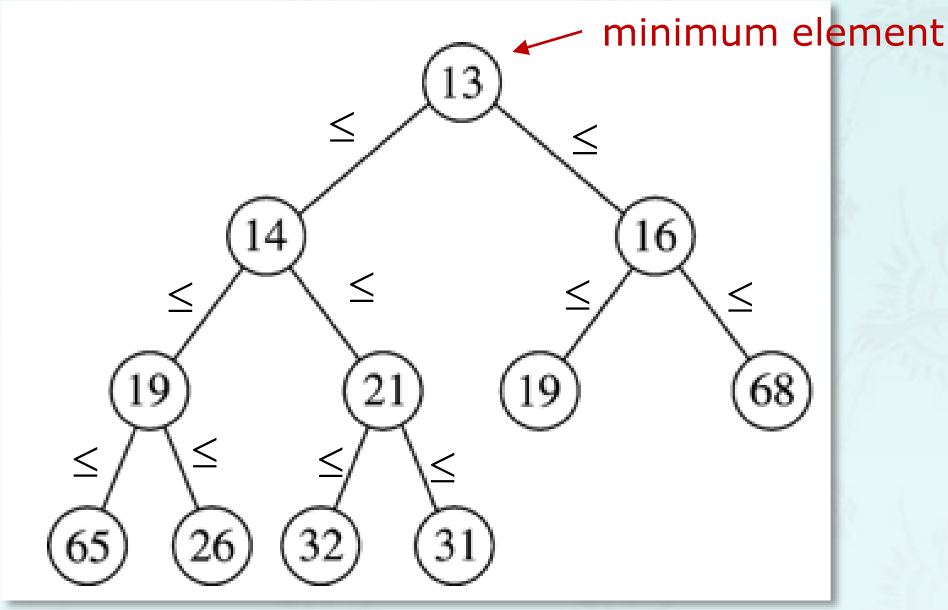
- **Properties:**
  - **Heap-ordered:**
    Parent's key no smaller than children's keys. [maxheap]
  - **Heap-structure:**
    A complete binary tree
- Array representation
  - Indices start at 1.
  - Take nodes in level order.
    - Parent at k is at k/2.
    - Children at k are at 2k and 2k+1.
  - No explicit links needed!



Heap representations

min-heap example



- Duplicates are allowed
- No order implied for elements which do not share ancestor-descendant relationship

min-heap example

insertion:

- Insert a new element **while maintaining a heap-structure**
- Move the element up the heap **while not satisfying heap-ordered**

min-heap example

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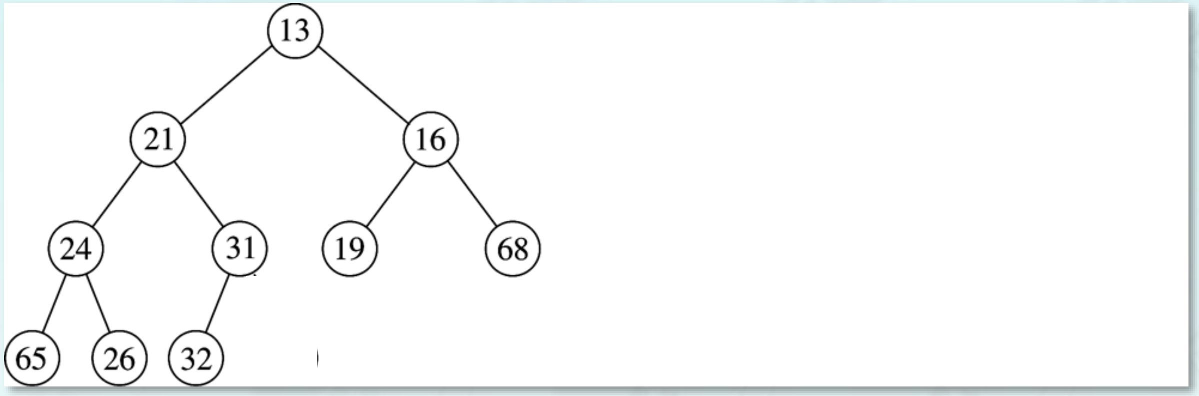
min-heap example

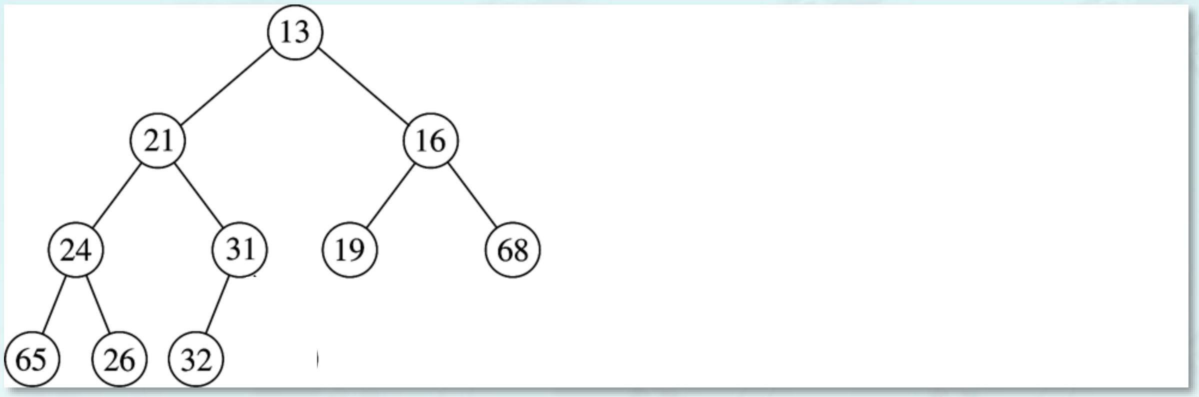
insertion: Insert a node 14



min-heap example

insertion: Insert a node 14

Where is an empty node to start?



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min-heap example

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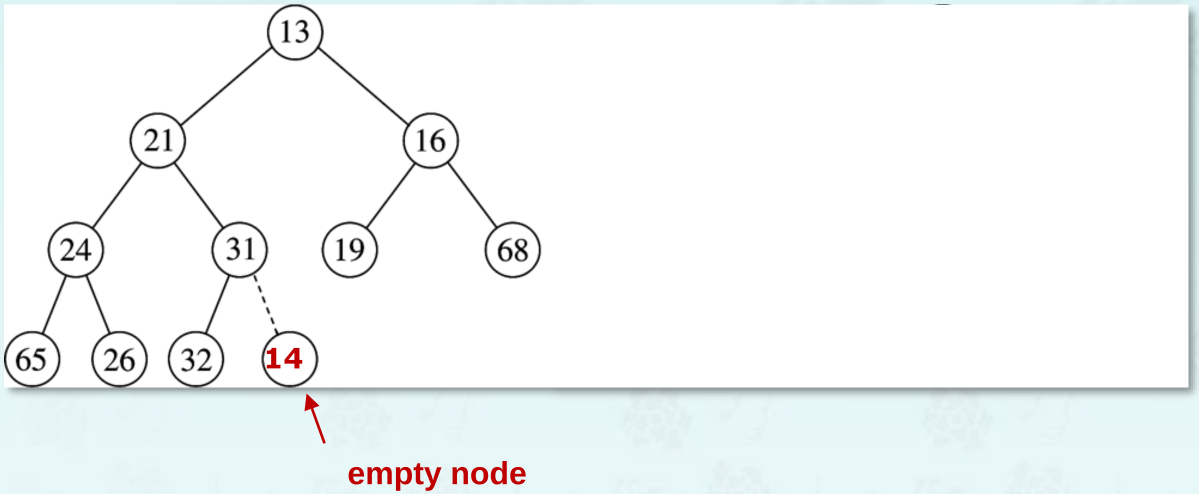

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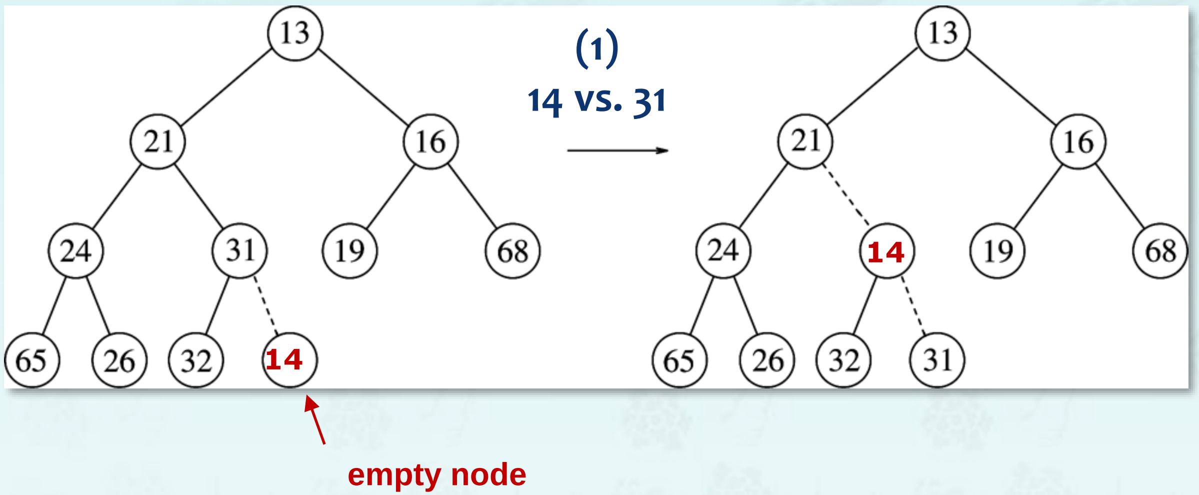

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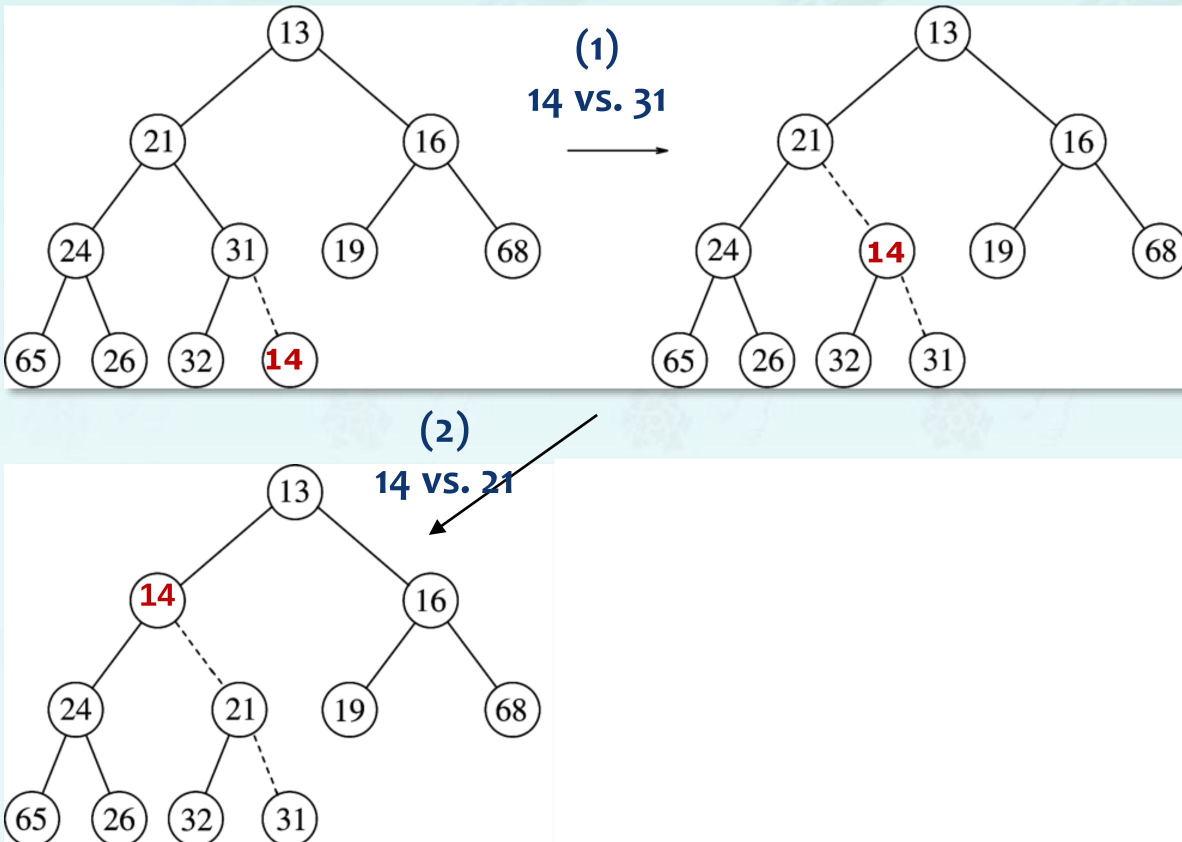

min-heap example

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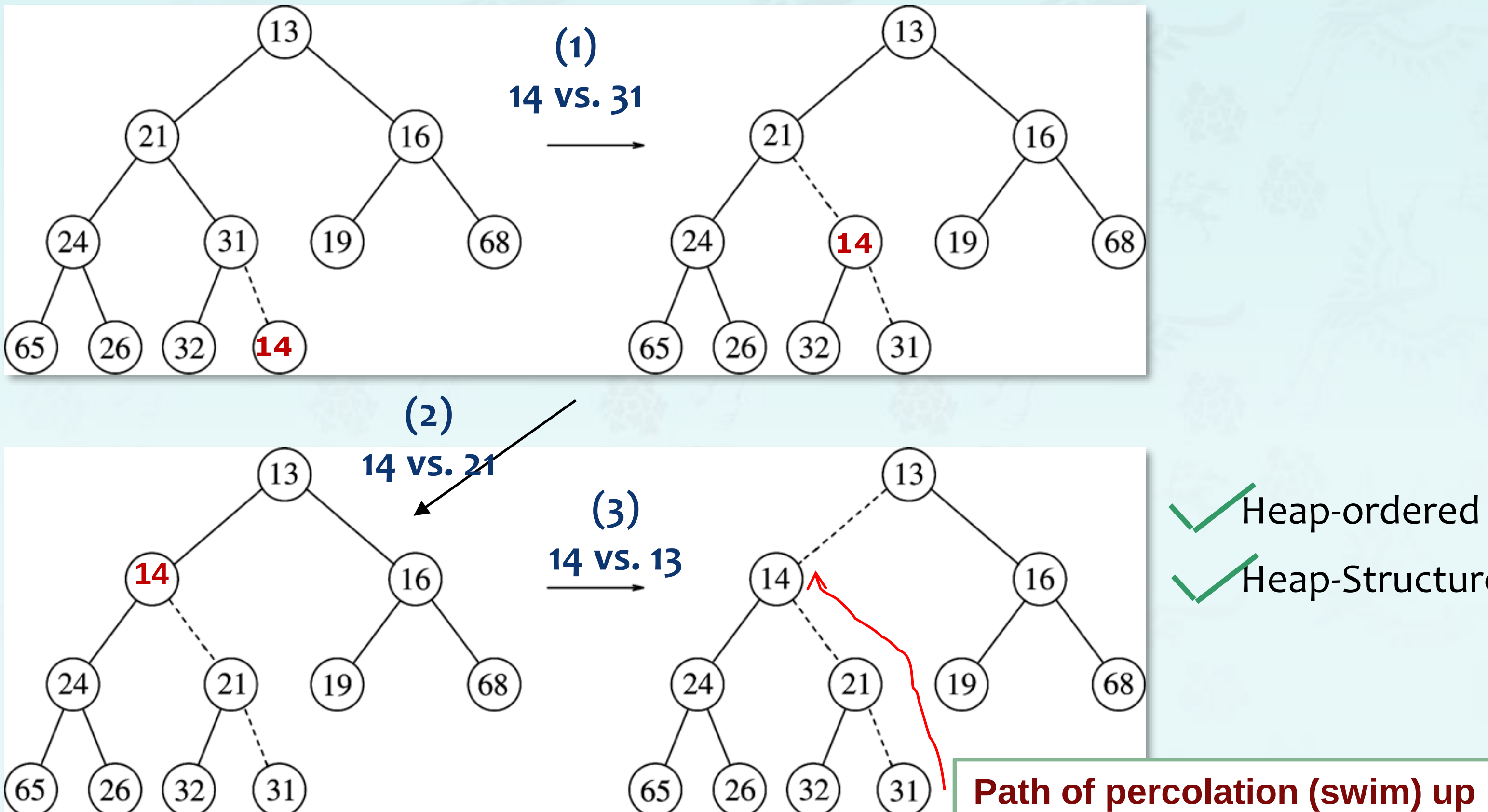


Heap-ordered

Heap-Structure

min-heap example

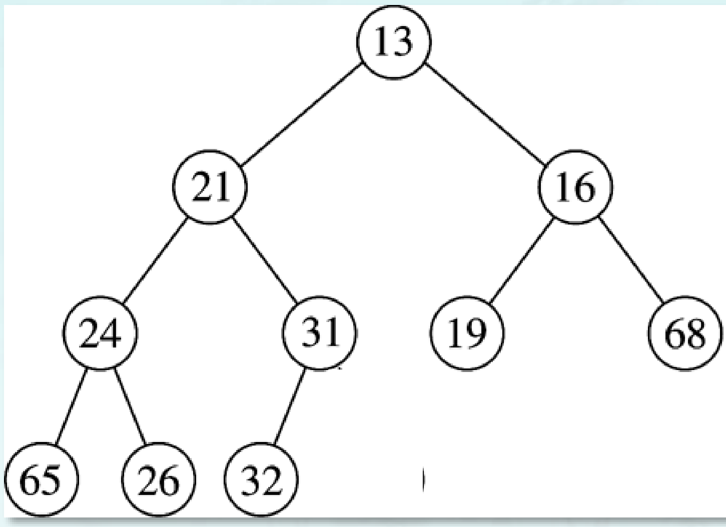
deletion: dequeue – delete the root

- Swap the root and the the last element.
- Heap decreases by one in size.
- ***Move down (sink) the root*** while not satisfying heap-ordered.
  - Minimum element is always at the root (by min-heap definition).

min-heap example

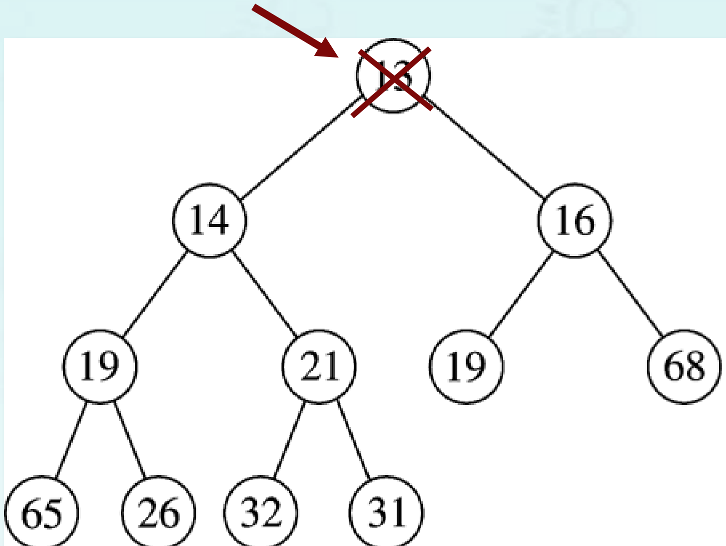
deletion: dequeue – delete the root

Which position of the node will be empty?



min-heap example

deletion: dequeue – delete the root

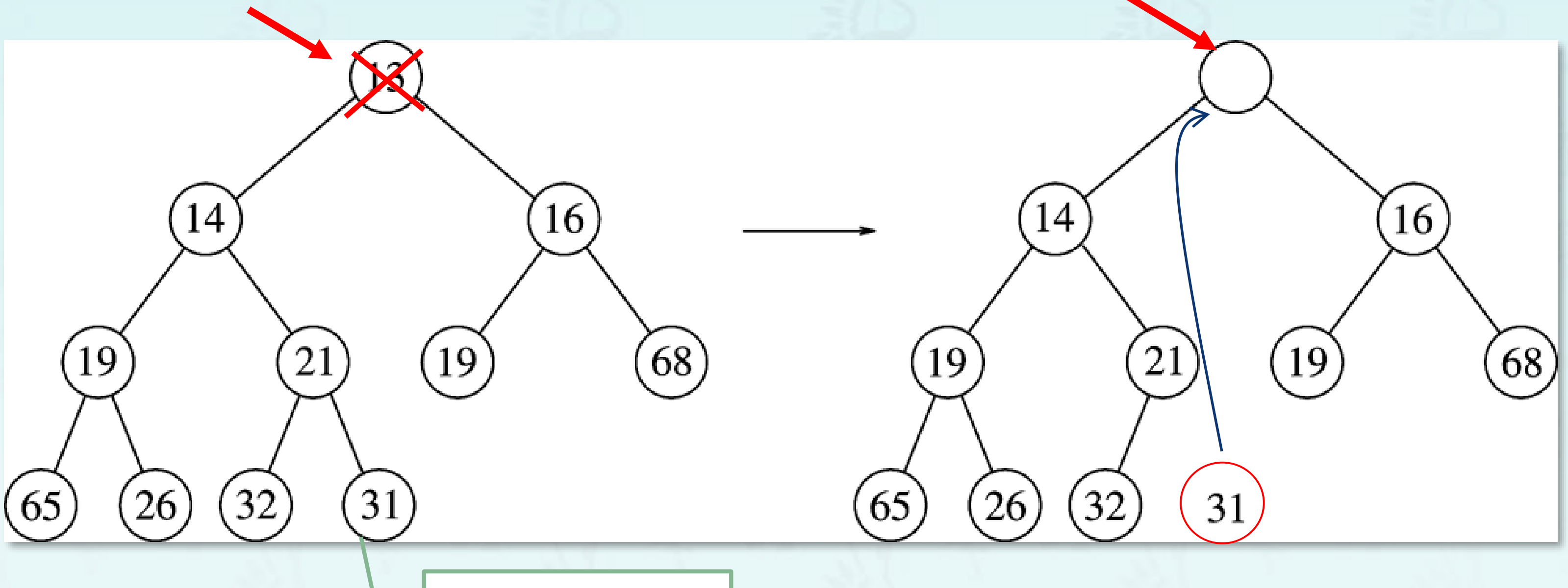


Make this position empty

min-heap example

deletion: dequeue – delete the root

Copy 31 temporarily here.

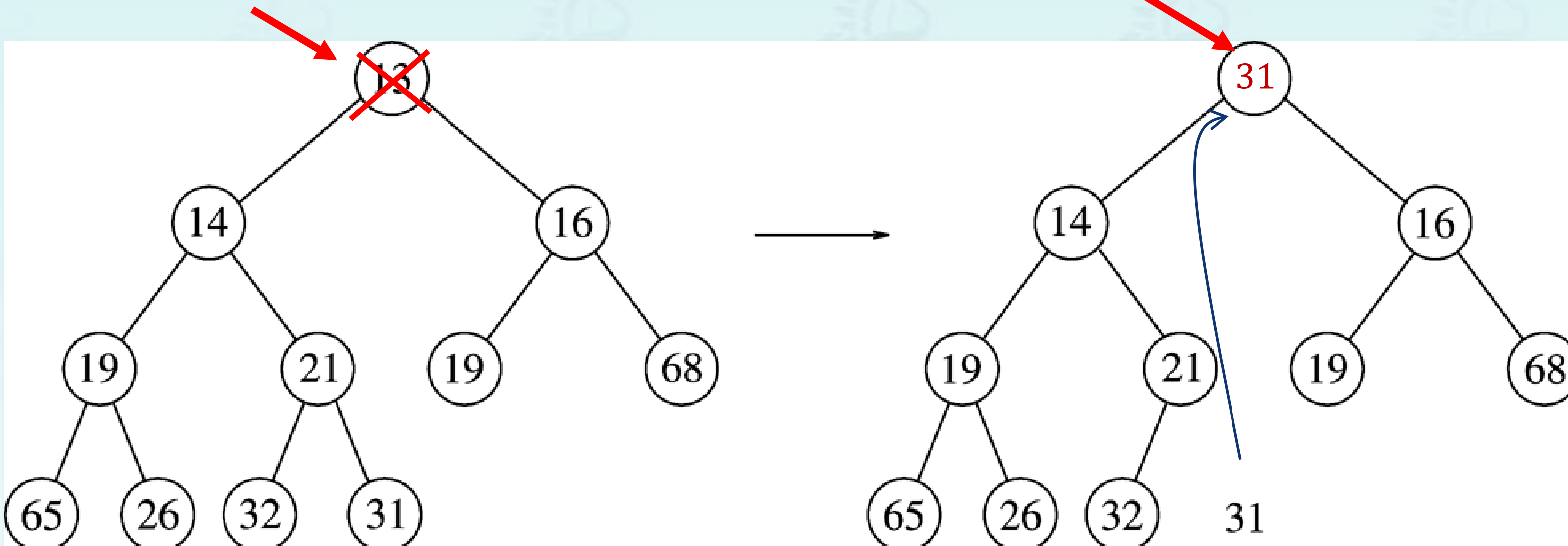


Make this position empty

min-heap example

deletion: dequeue – delete the root

Copy 31 temporarily here and ask **heap-ordered?**



Make this position empty

Is 31 > min(14,16)?

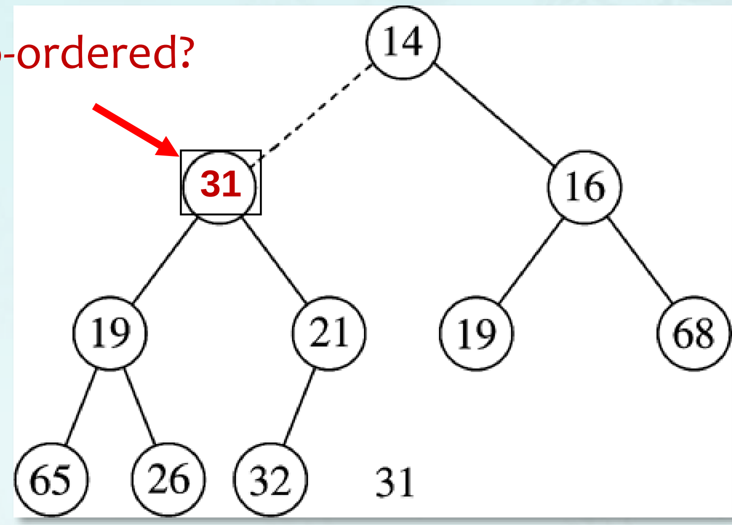
- Yes - swap 31 with min(14,16)

sink…

min-heap example

deletion: dequeue – delete the root

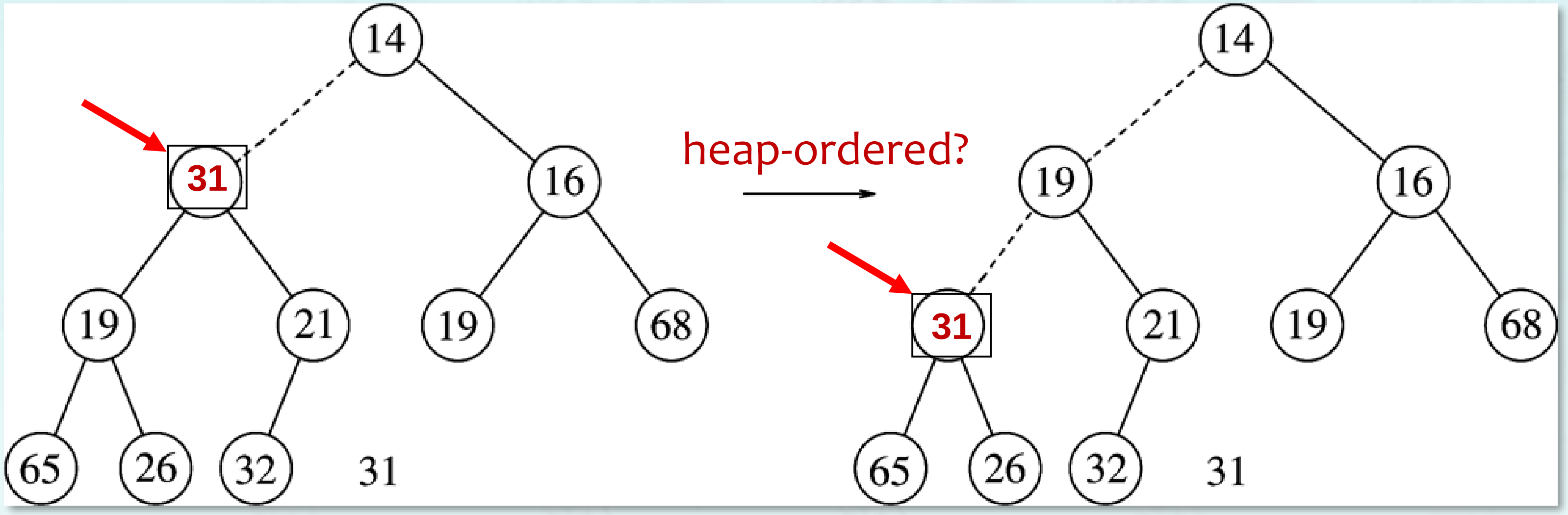
heap-ordered?



Is 31 > min(19,21)?
- Yes - swap 31 with min(19,21)

min-heap example

deletion: dequeue – delete the root



Is 31 > min(19,21)?
- Yes - swap 31 with min(19,21)

Is 31 > min(65,26)?
- Yes - swap 31 with min(65,26)

min-heap example

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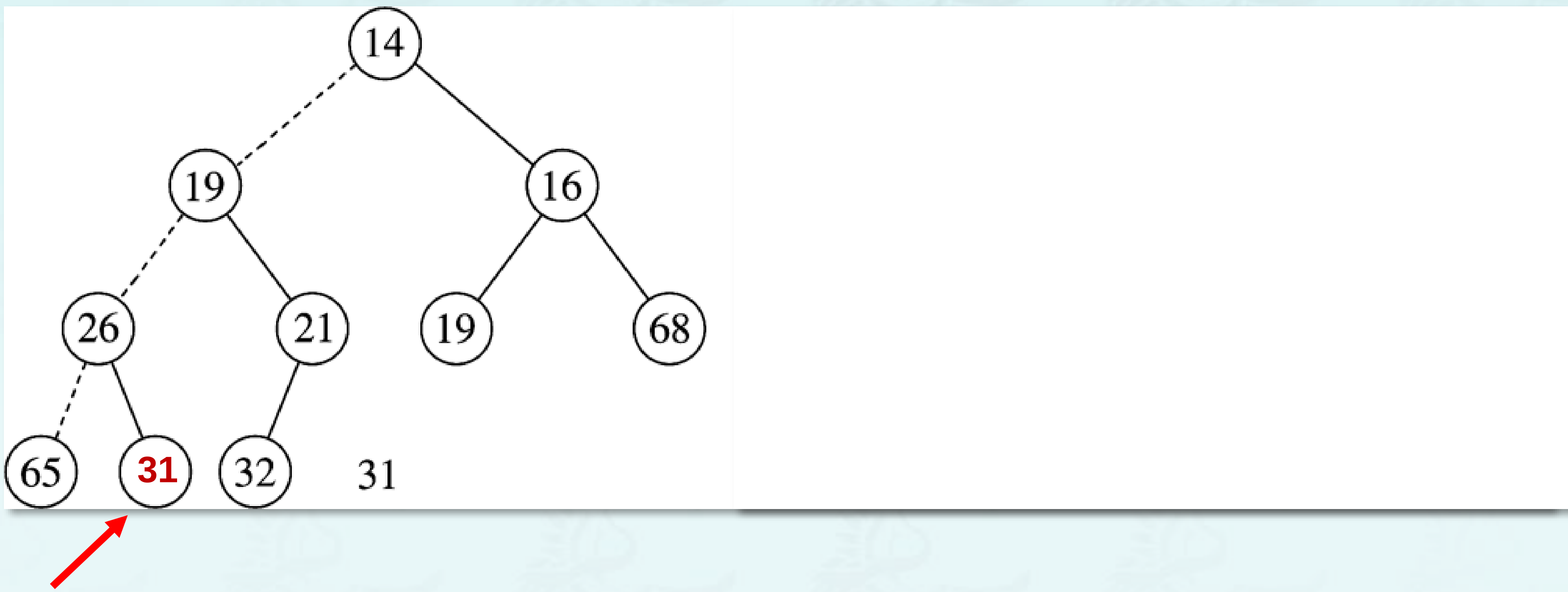

min-heap example

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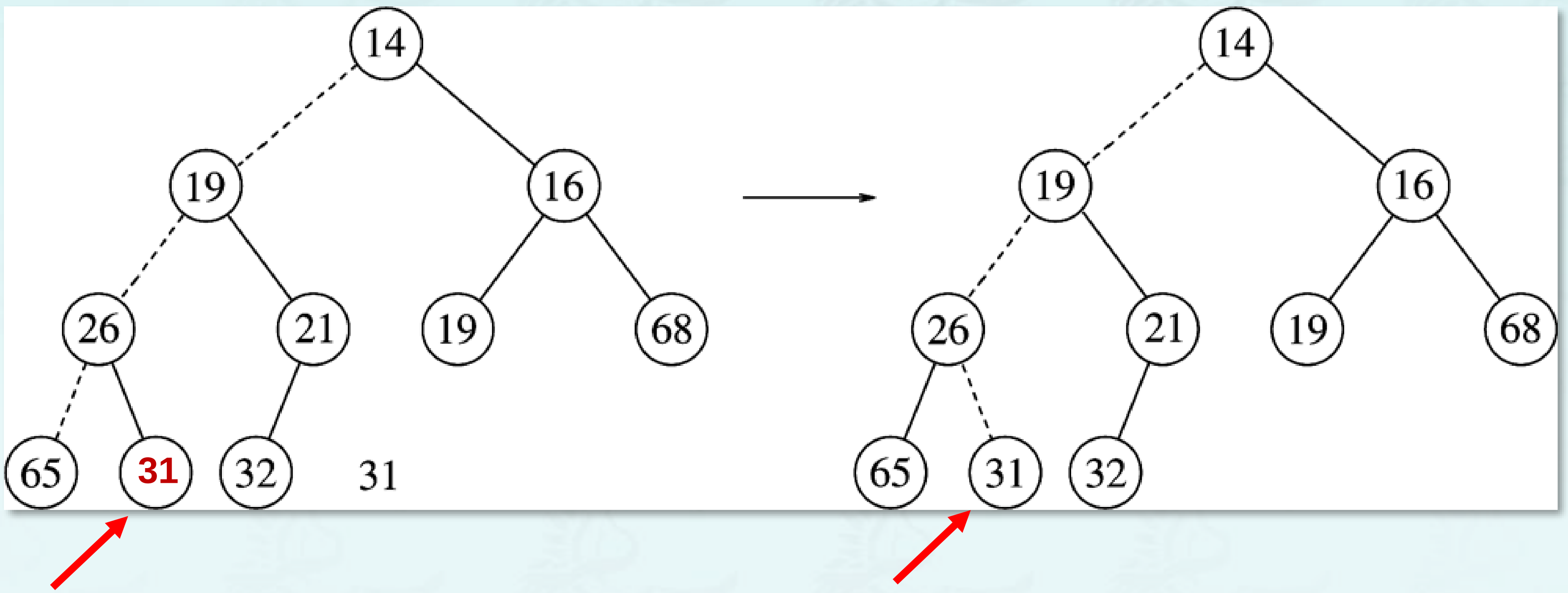

Heap-ordered

Heap-structure

Binary heap operations time complexity:

- Level of heap is $\lfloor \log_2 N \rfloor$
- insert: O(log N) for each insert
  - In practice, expect less
- delete: O(log N) // deleting root node in min/max heap
- decreaseKey: O(log N)
- increaseKey: O(log N)
- remove: O(log N) // removing a node in any location

Binary heap operations time complexity with N items:

Implementation	Insert	Delete	max
Unordered array	1	N	N
Ordered array	N	1	1
Binary heap	log N	log N	1

Mission Completed

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max-heap Demo

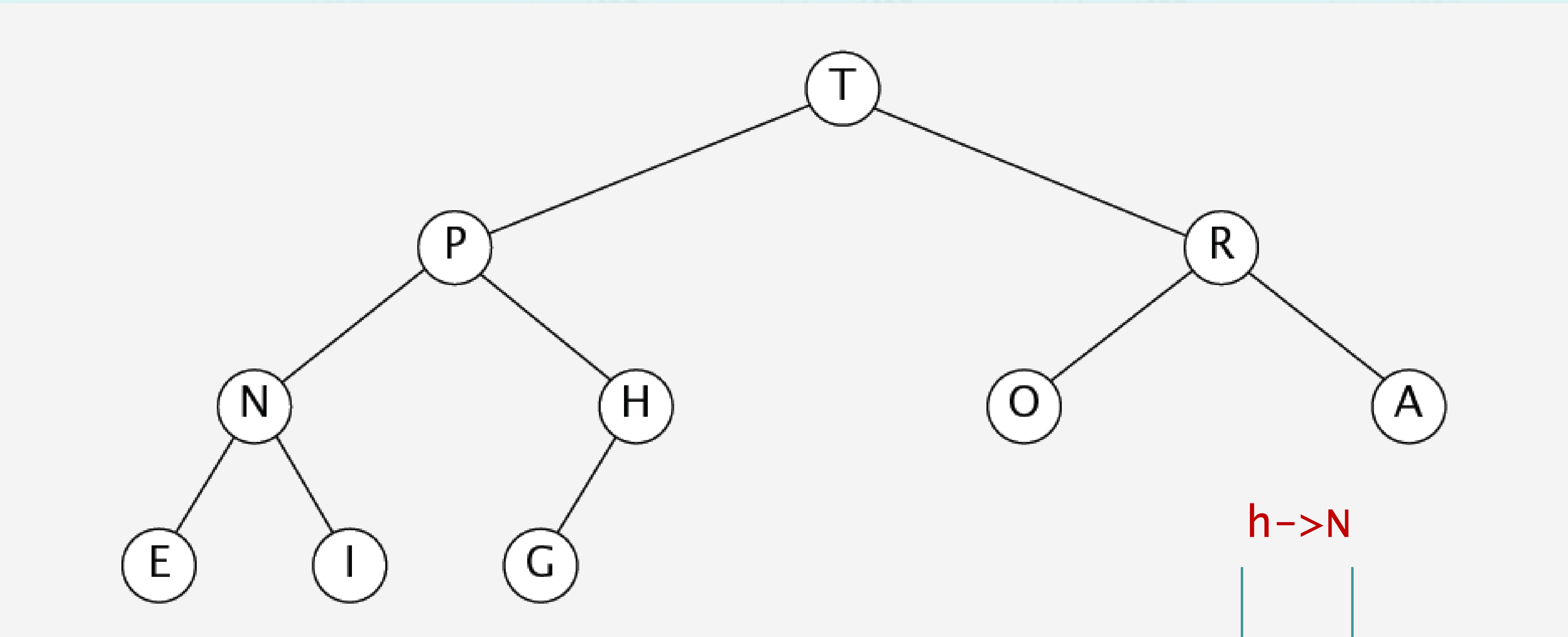
- **Insert**: Add node at end, then swim it up.

T	P	R	N	H	O	A	E	I	G	

max-heap Demo

- **Insert**: Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

Heap ordered



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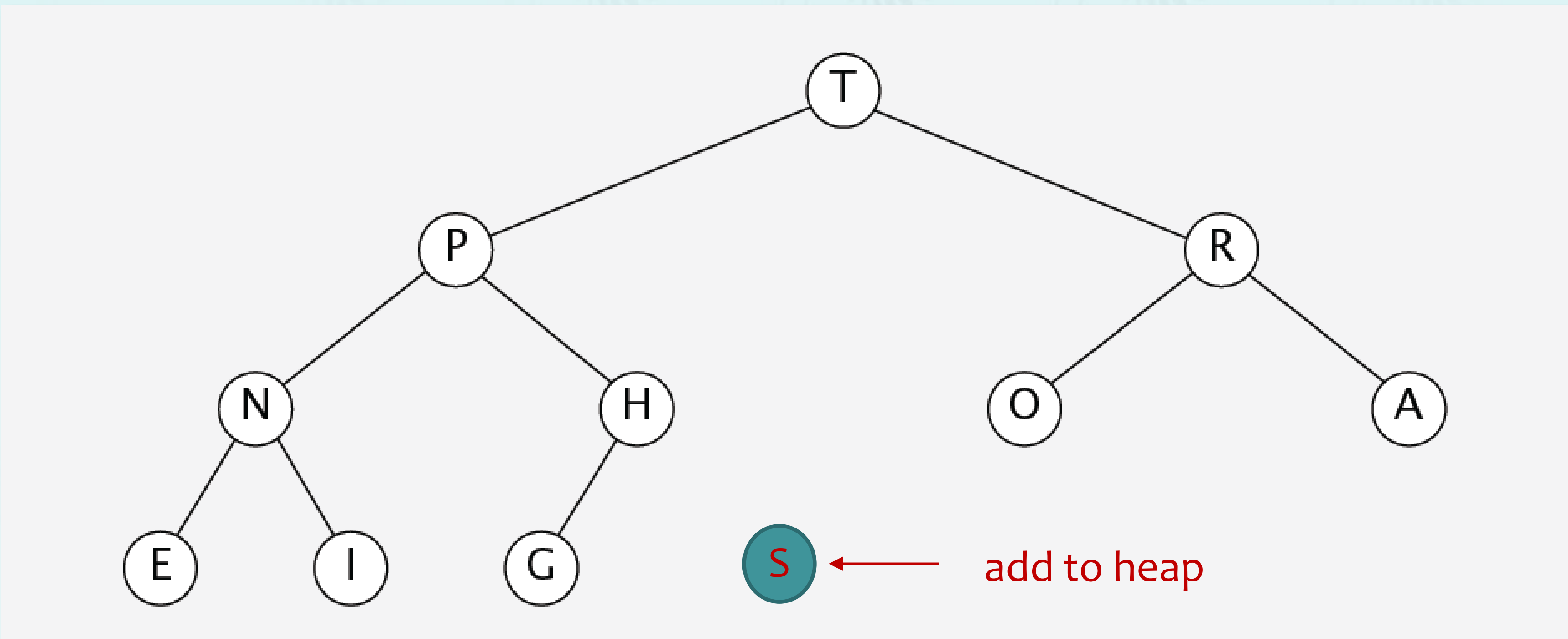
insert S



T	P	R	N	H	O	A	E	I	G	

h->N

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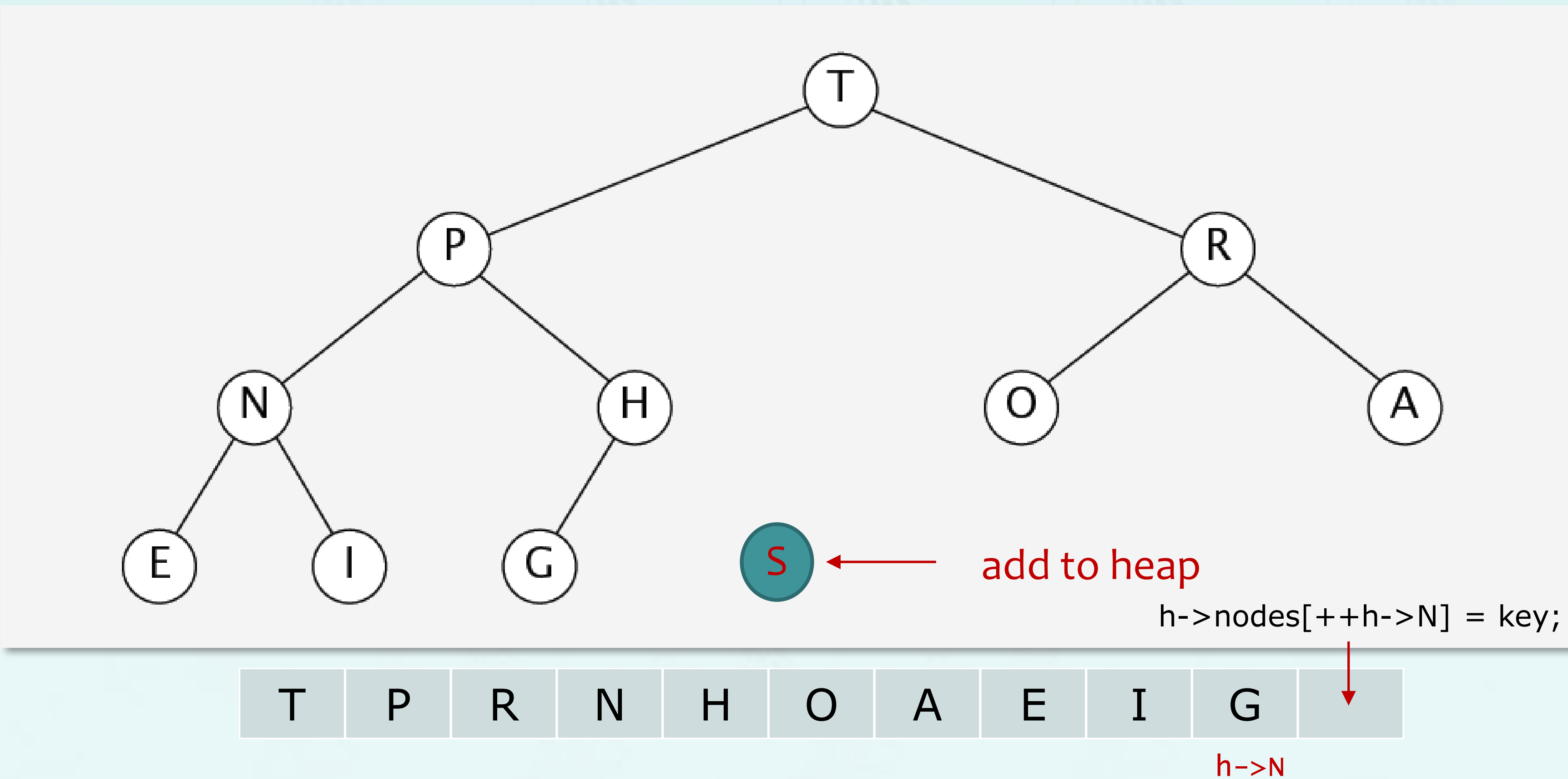

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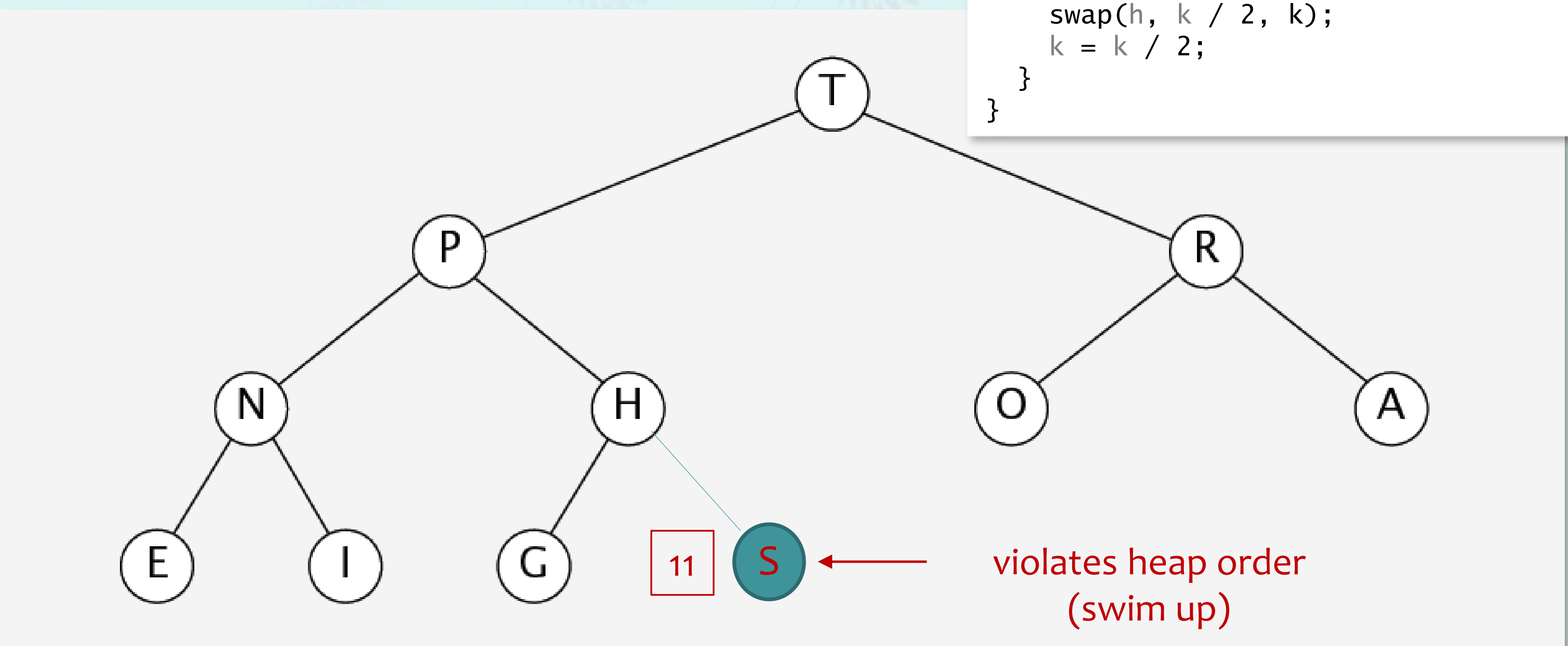
```
void swim(heap h, int k) {
  while (k > 1 && less(h, k / 2, k)) {
    swap(h, k / 2, k);
    k = k / 2;
  }
}
```

insert S



T	P	R	N	H	O	A	E	I	G	S

11

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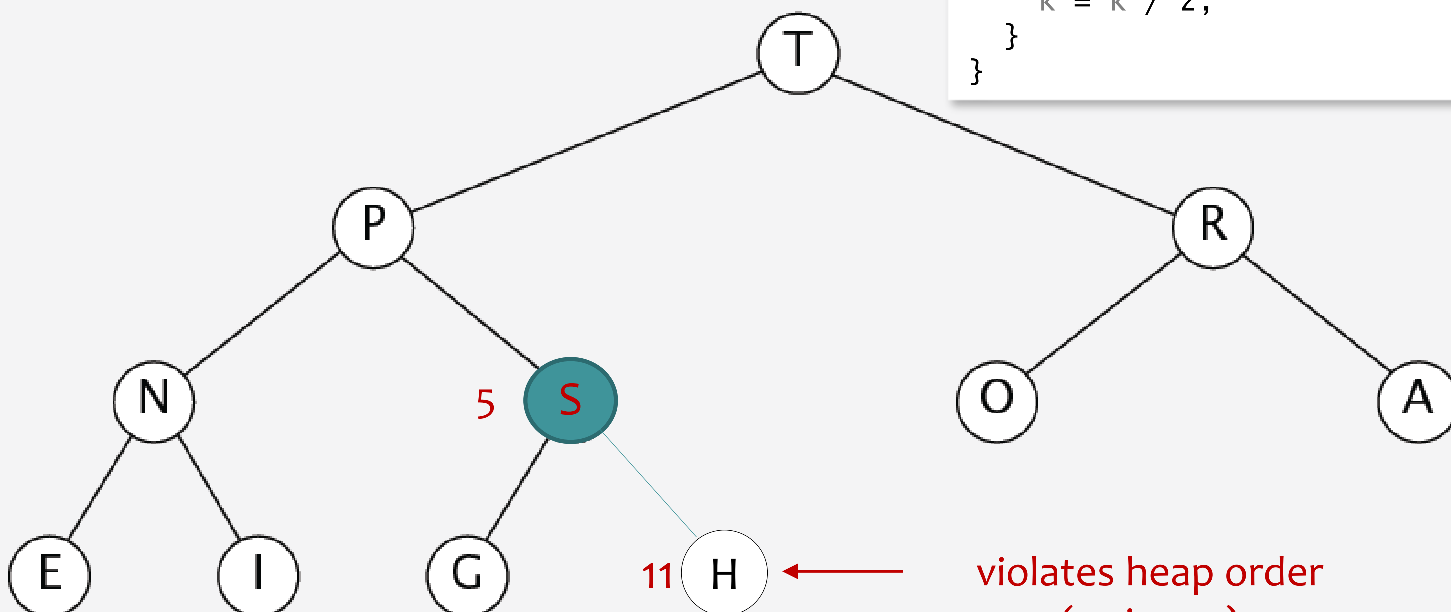

violates heap order
(swim up)

T	P	R	N	S	O	A	E	I	G	H
				5						11

max-heap Demo

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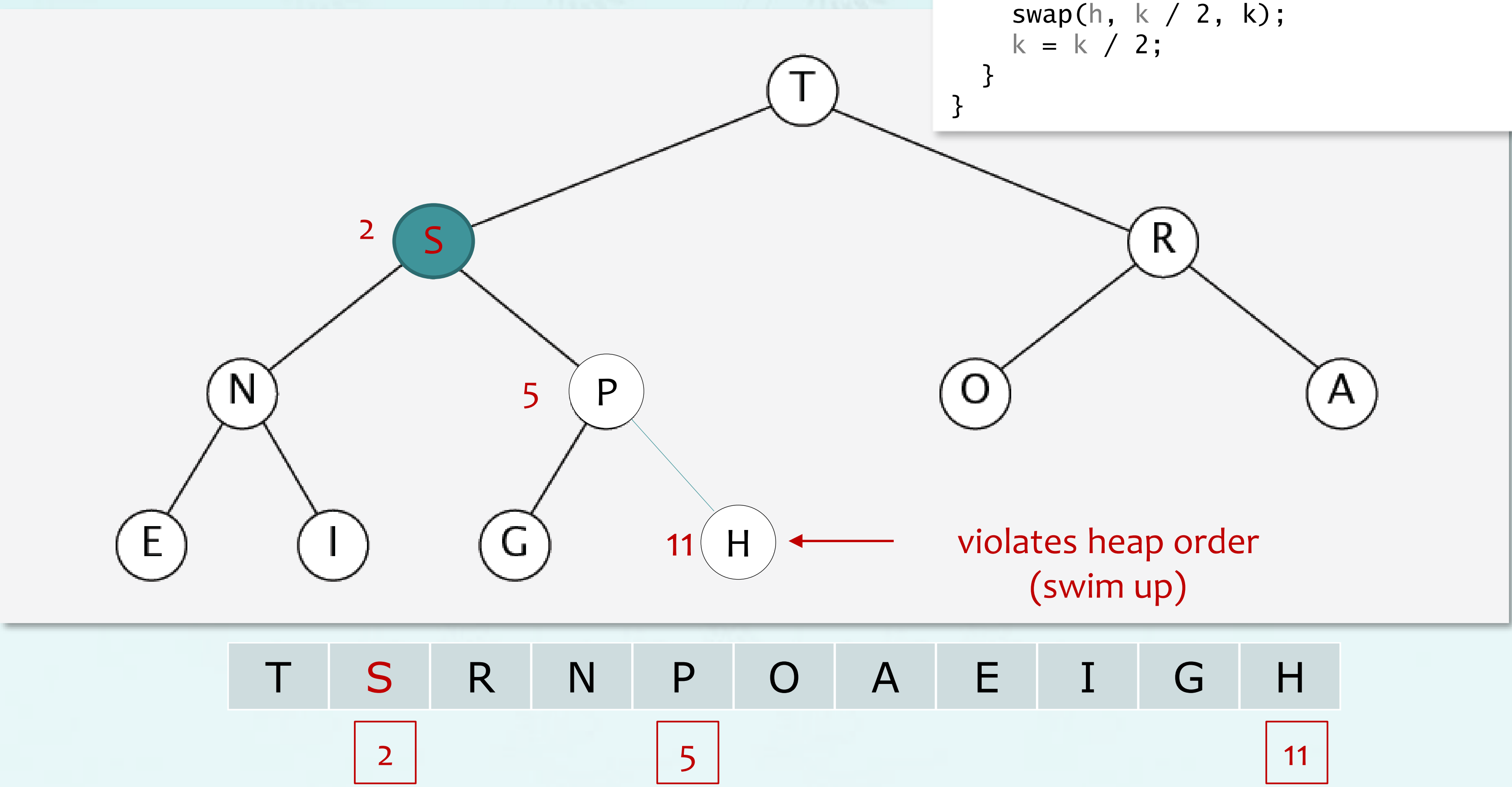

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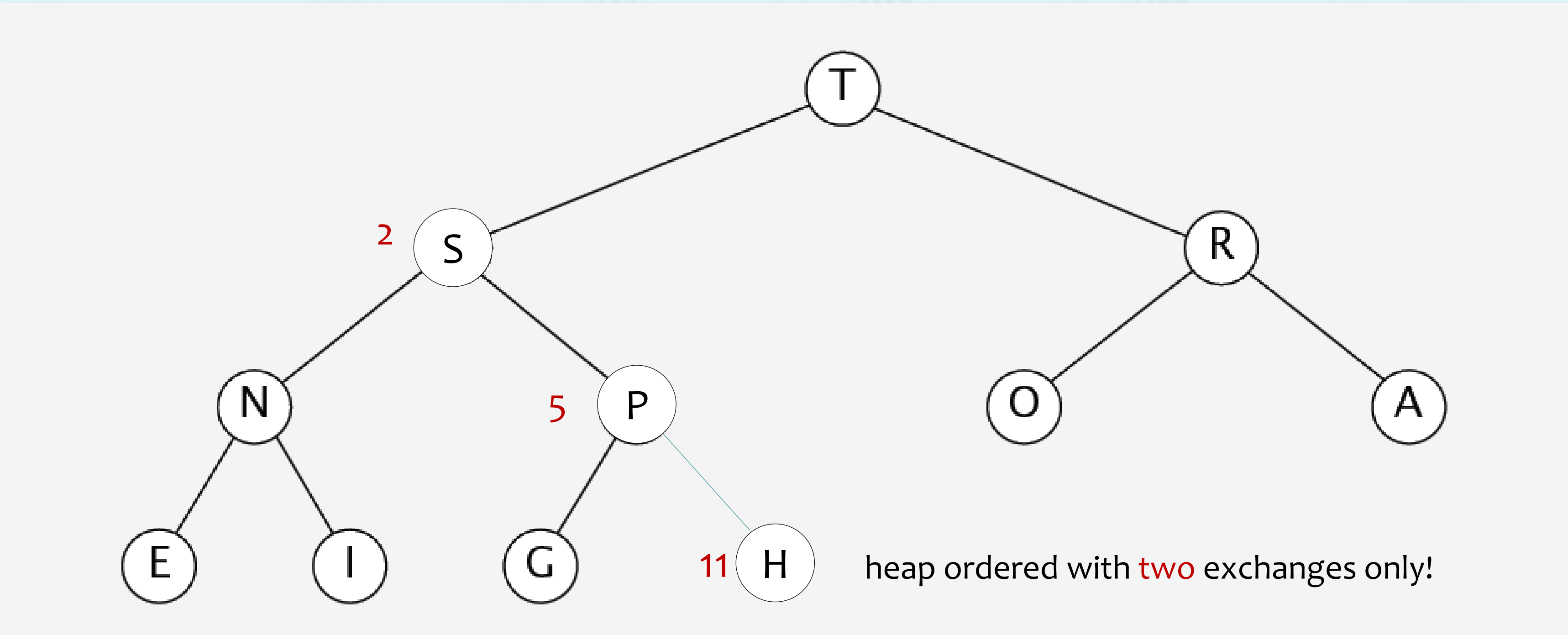
heap ordered



T	S	R	N	P	O	A	E	I	G	H
	2			5						11

max-heap Demo

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

remove the maximum(root)

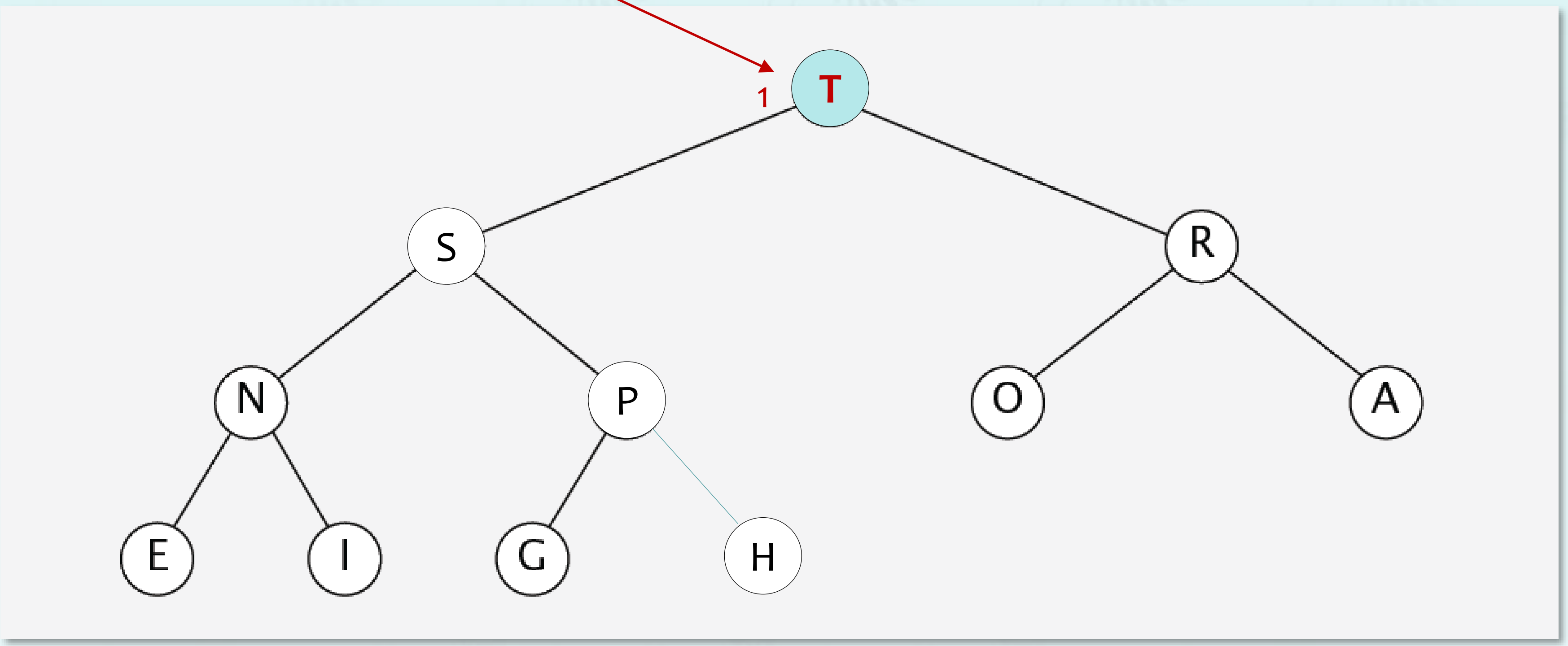
T	S	R	N	P	O	A	E	I	G	H

1

max-heap Demo

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

remove the maximum(root)



T	S	R	N	P	O	A	E	I	G	H
1										

max-heap Demo

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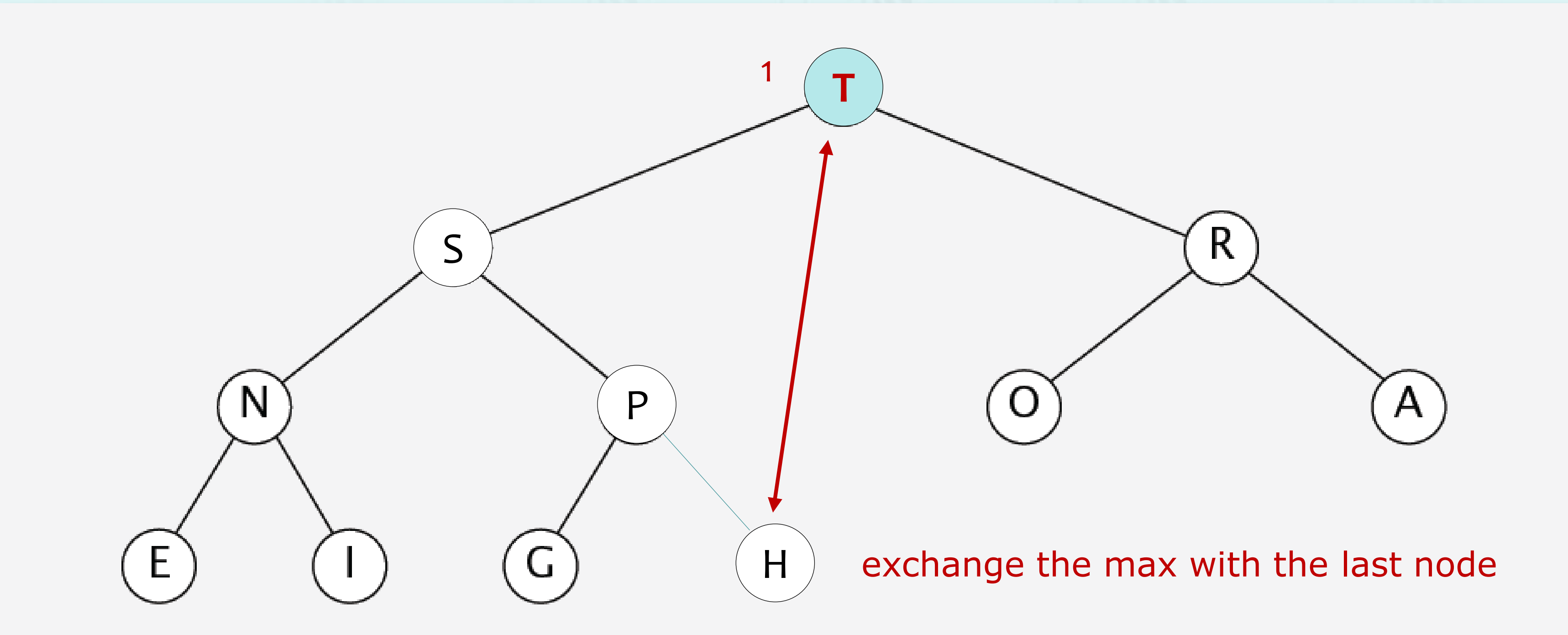
remove the maximum(root)



T	S	R	N	P	O	A	E	I	G	H

1

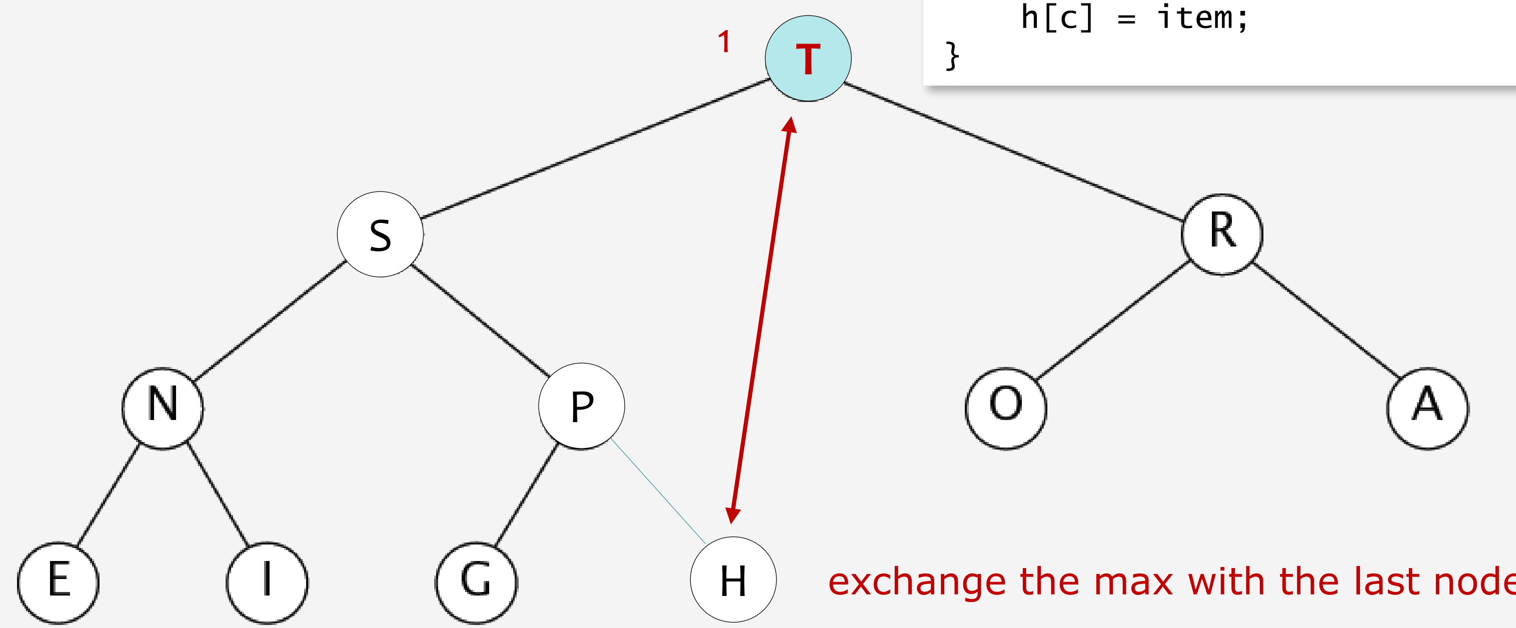
h->N

max-heap Demo

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

```
void swap(heap h, int p, int c) {
    Key item = h[p];
    h[p] = h[c];
    h[c] = item;
}
```

remove the maximum(root)



T	S	R	N	P	O	A	E	I	G	H

1

h->N

max-heap Demo

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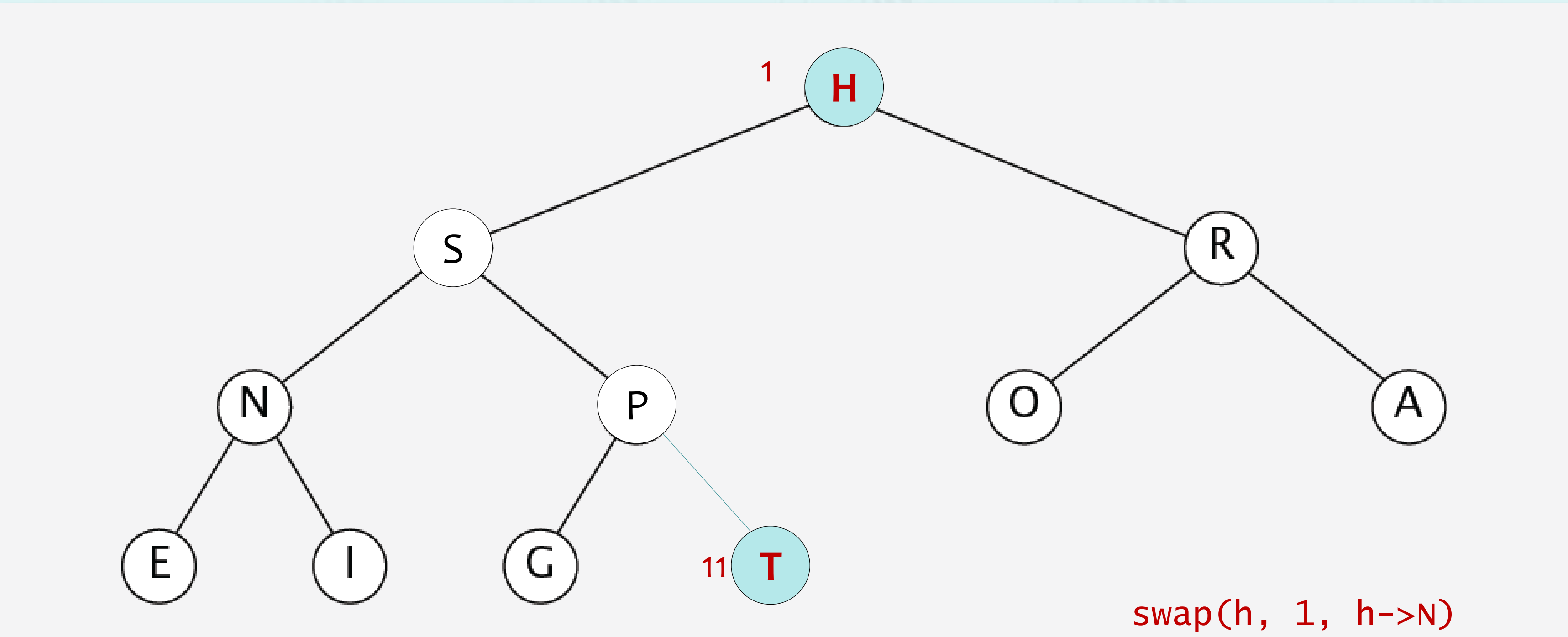
remove the maximum(root)



H	S	R	N	P	O	A	E	I	G	T
1										11

max-heap Demo

- **Insert:** Add node at end, then swim it up.
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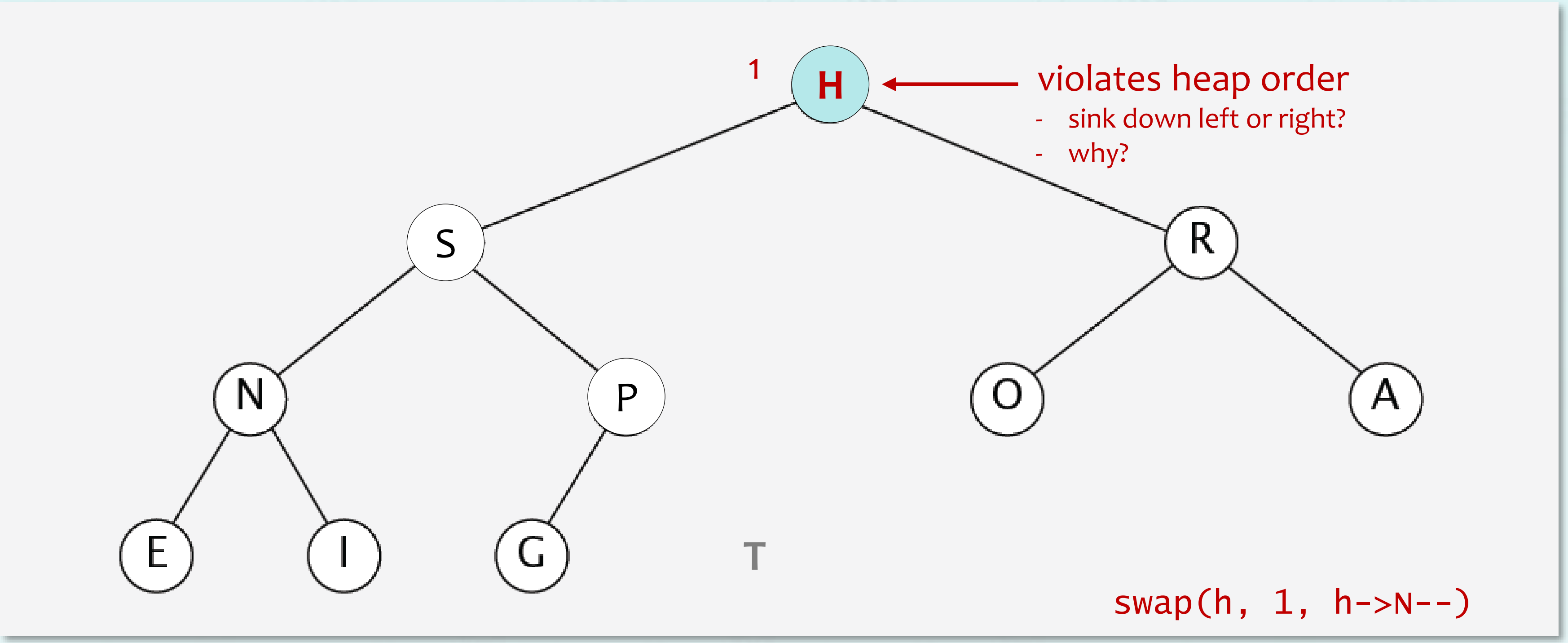
remove the maximum(root)



H	S	R	N	P	O	A	E	I	G	T

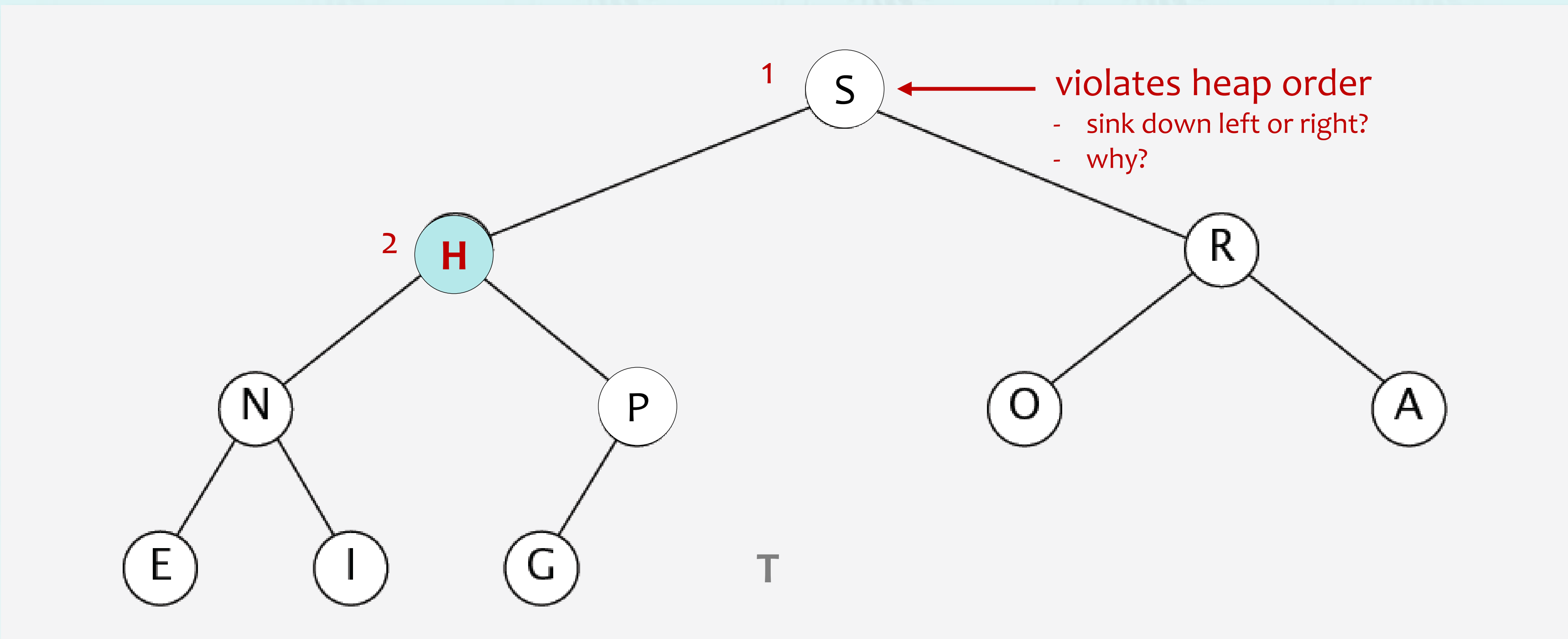
1

heap size decreased by one

max-heap Demo

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

remove the maximum(root)



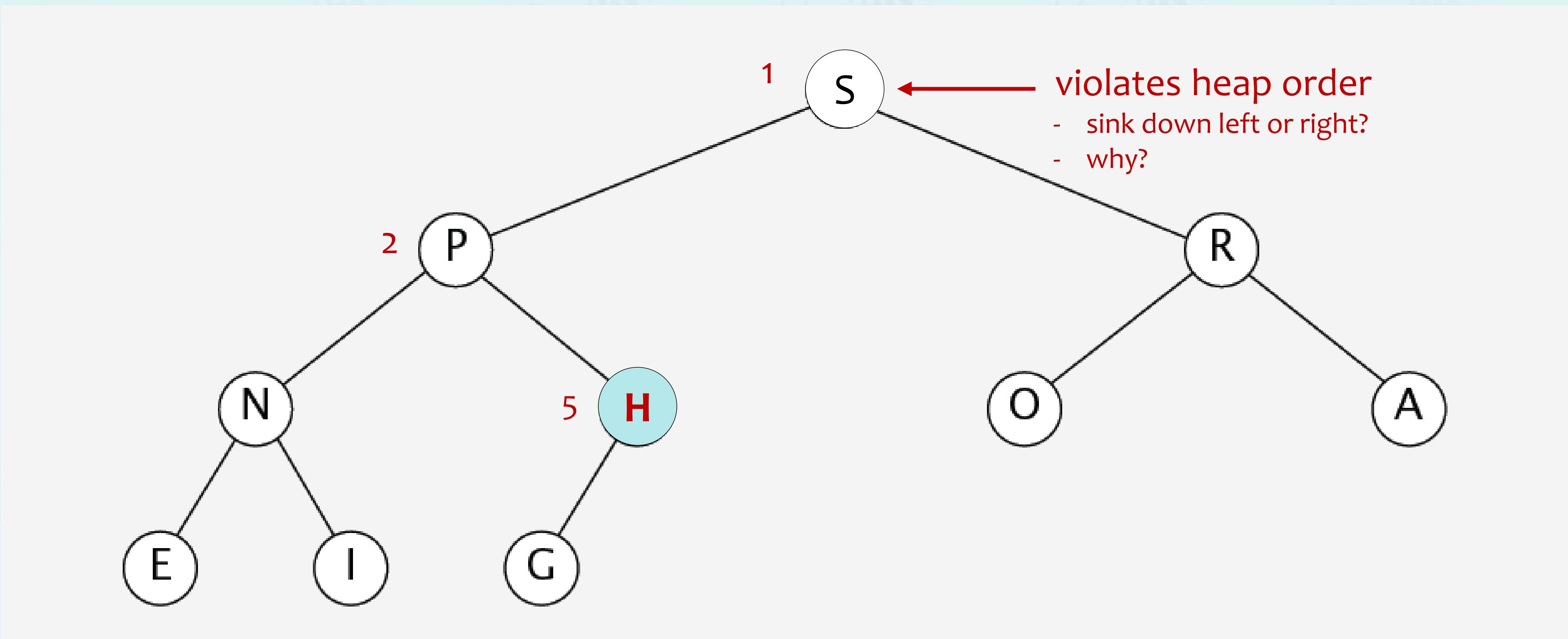
S	H	R	N	P	O	A	E	I	G	T

1 2

max-heap Demo

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

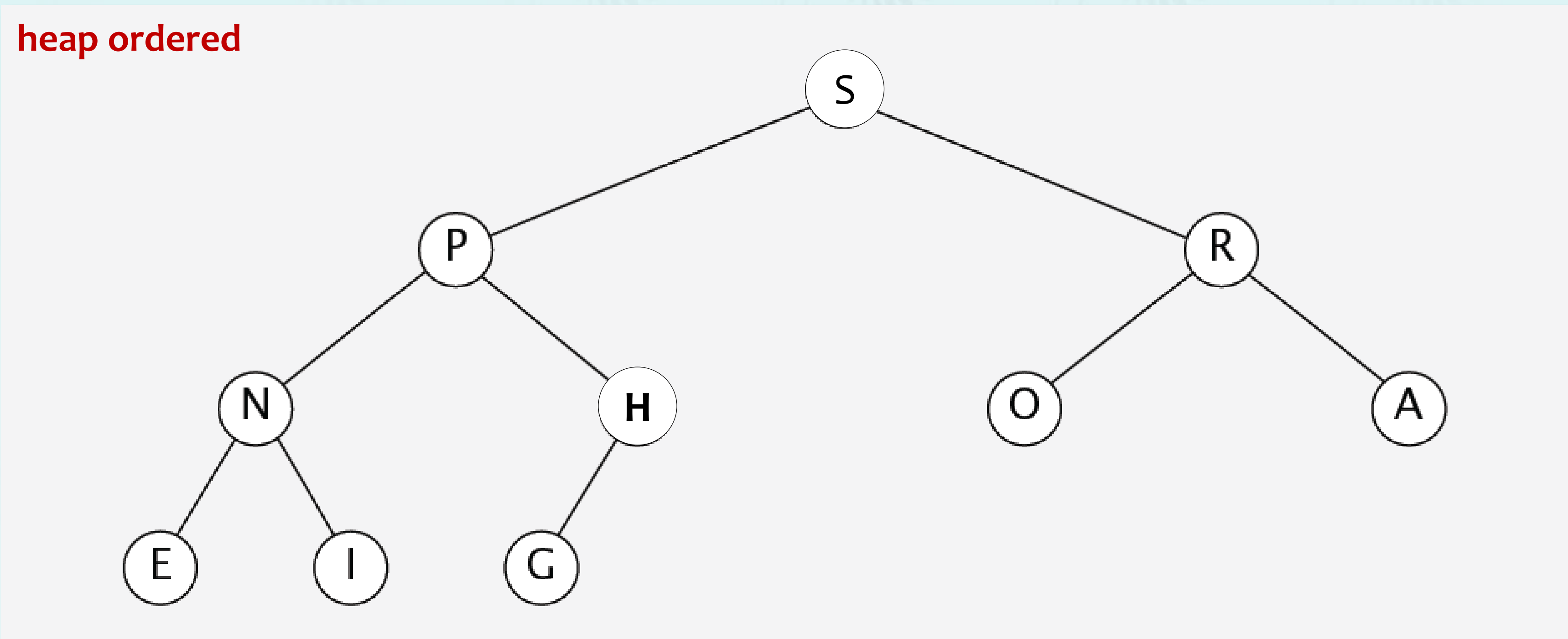
remove the maximum(root)



S	P	R	N	H	O	A	E	I	G	
1	2			5						

max-heap Demo

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.



S	P	R	N	H	O	A	E	I	G	

max-heap Demo

- **Insert:** Add node at end, then swim it up.
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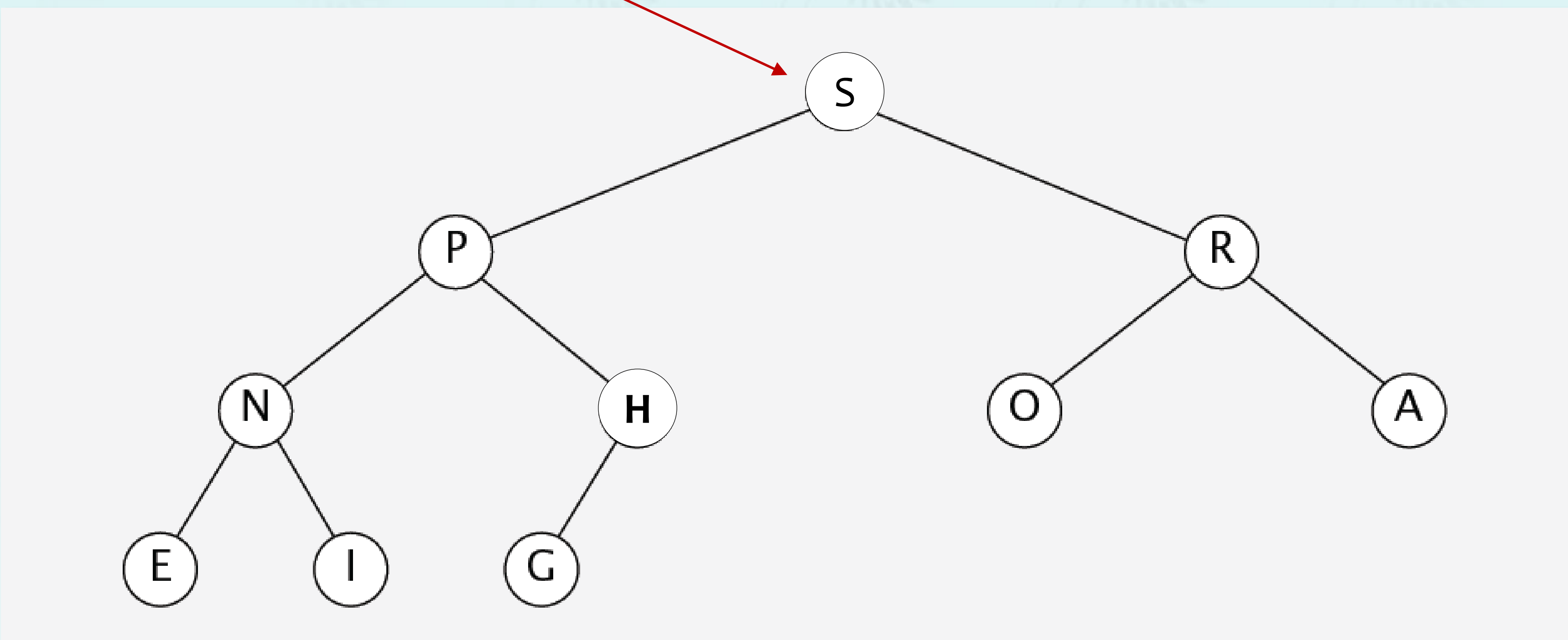
remove the maximum(root)



S	P	R	N	H	O	A	E	I	G	

max-heap Demo

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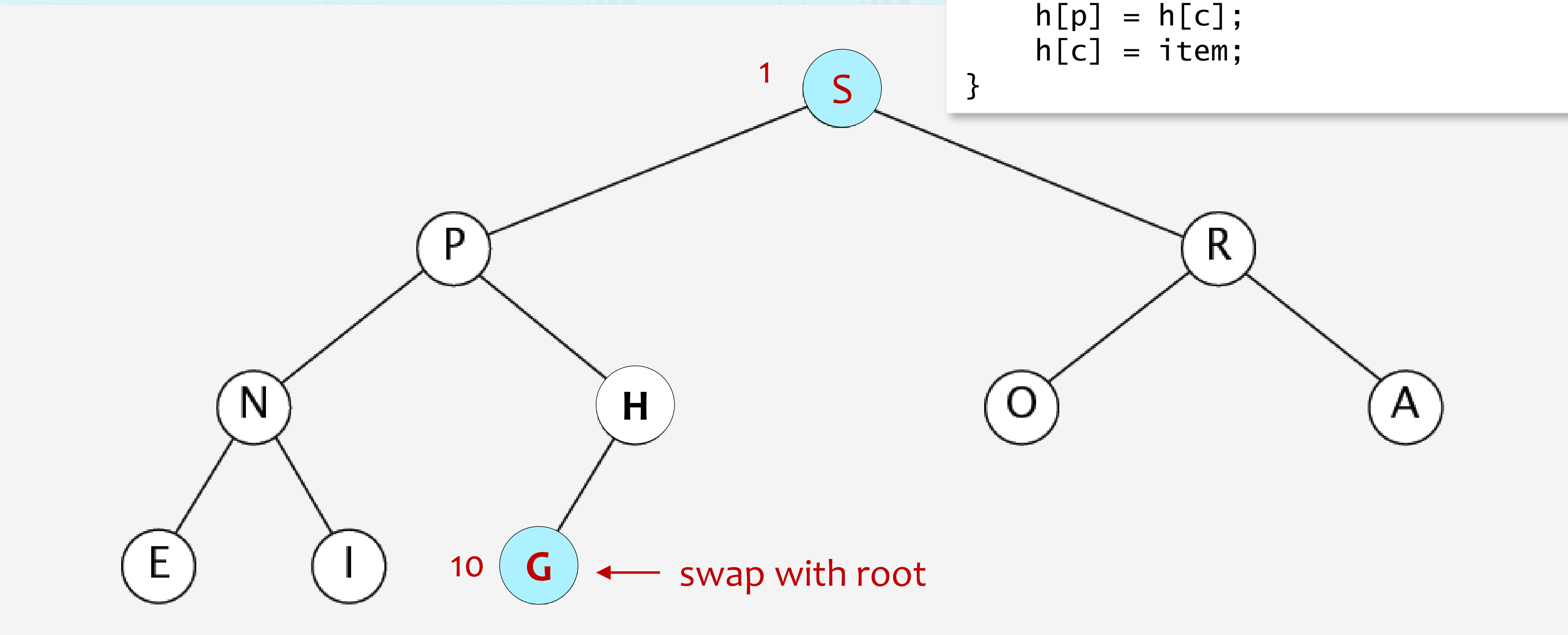
}
```

remove the maximum(root)



S	P	R	N	H	O	A	E	I	G	
1									10 h->N	

max-heap Demo

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

```
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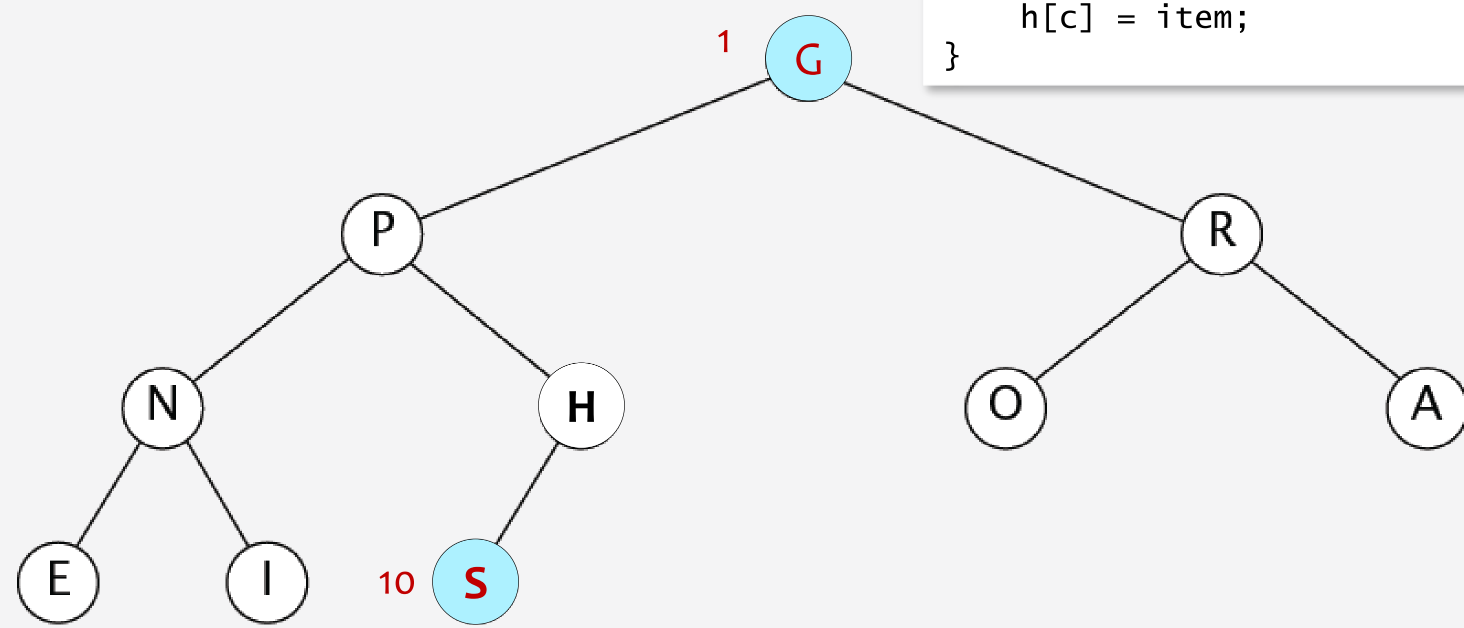
    Key item = h[p];
    h[p] = h[c];
    h[c] = item;
}
```

remove the maximum(root)

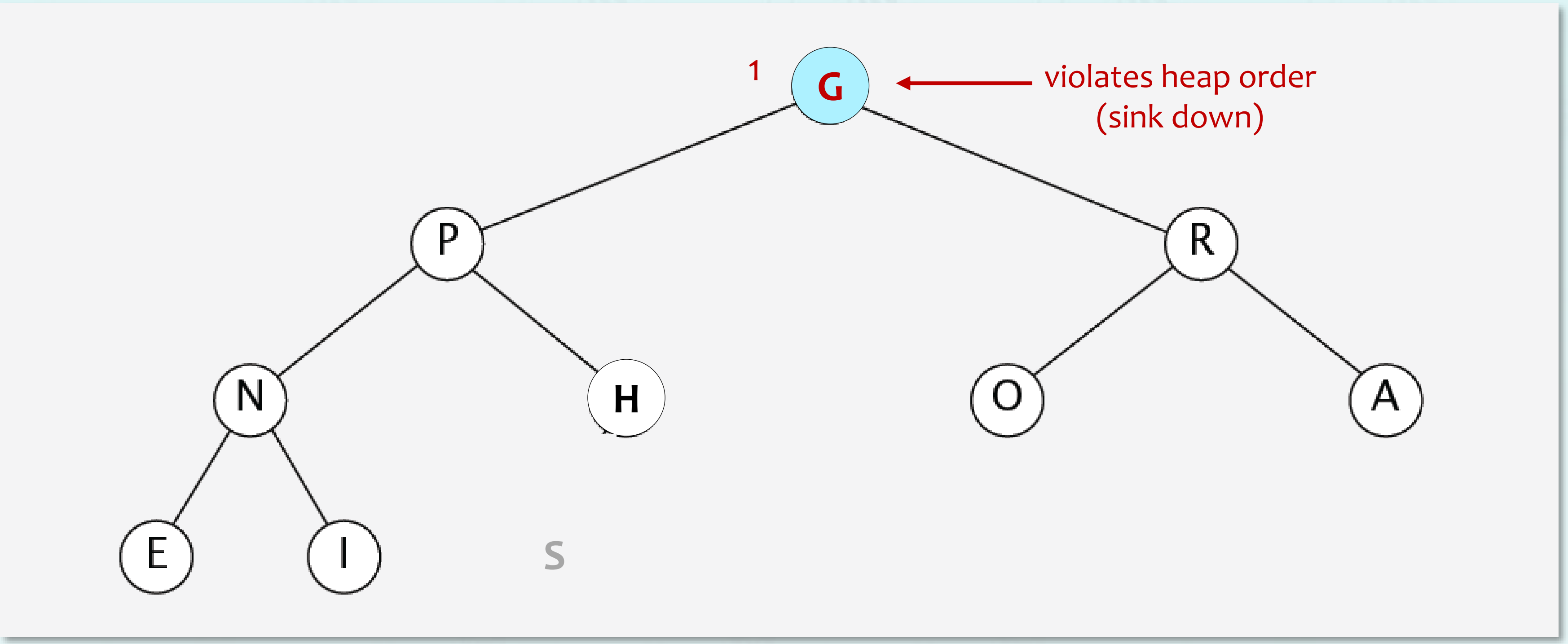


G	P	R	N	H	O	A	E	I	S	
1									10	

max-heap Demo

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

remove the maximum(root)



G	P	R	N	H	O	A	E	I	S	

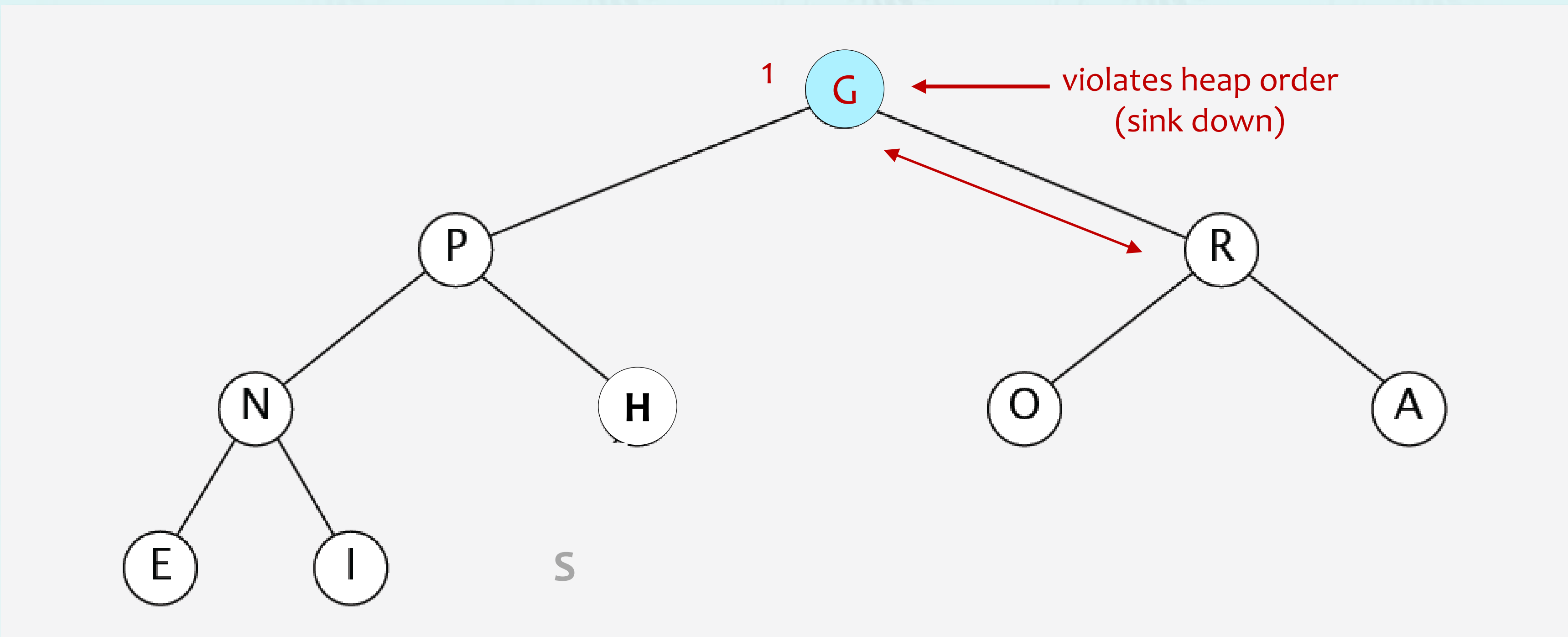
1

heap size decreased by one

max-heap Demo

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

remove the maximum(root)

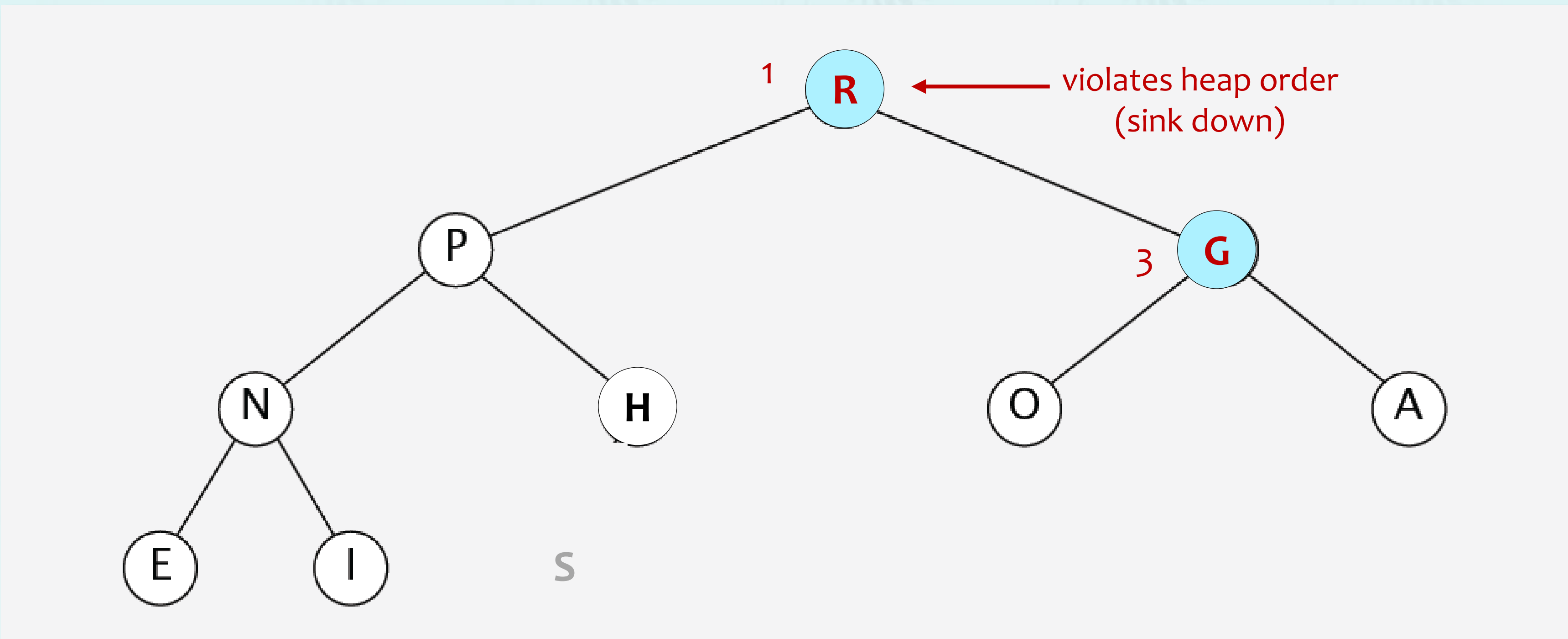


G	P	R	N	H	O	A	E	I	S	
1								N=9		

max-heap Demo

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

remove the maximum(root)

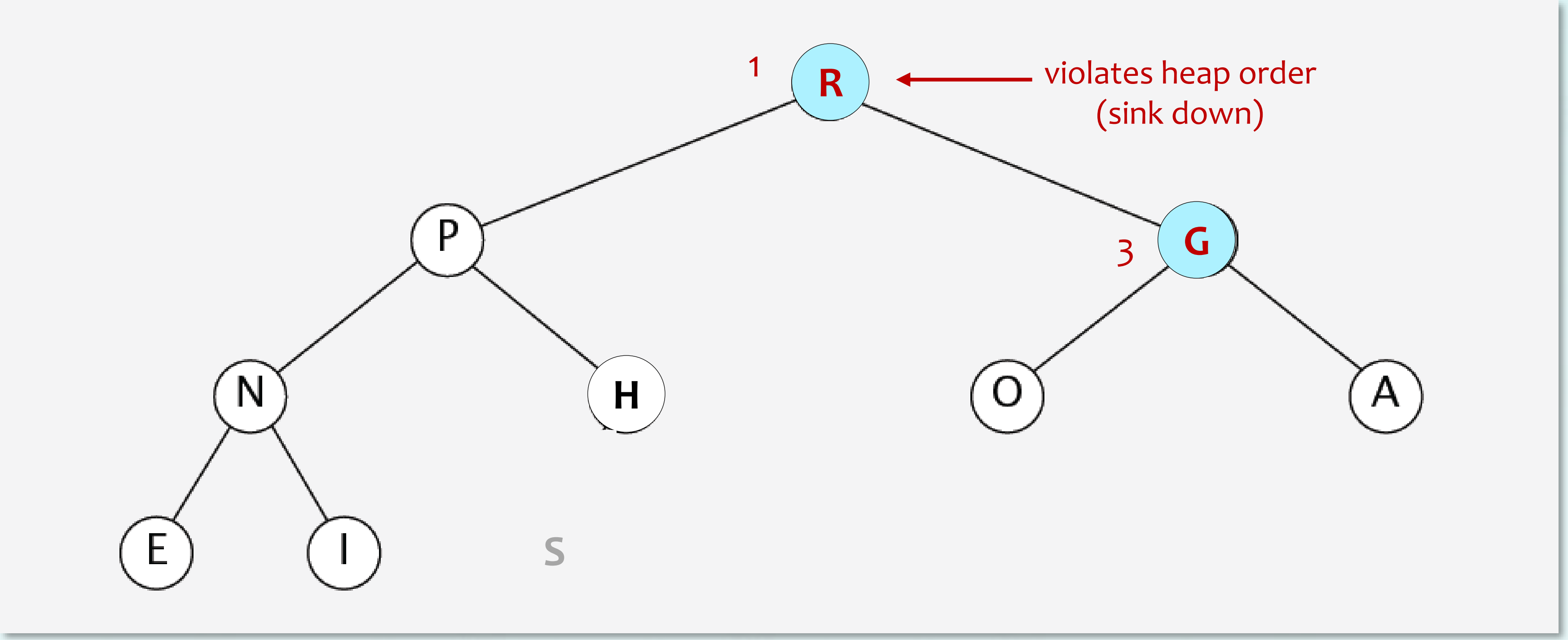


R	P	G	N	H	O	A	E	I	S	
1		3						N=9		

max-heap Demo

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

remove the maximum(root)

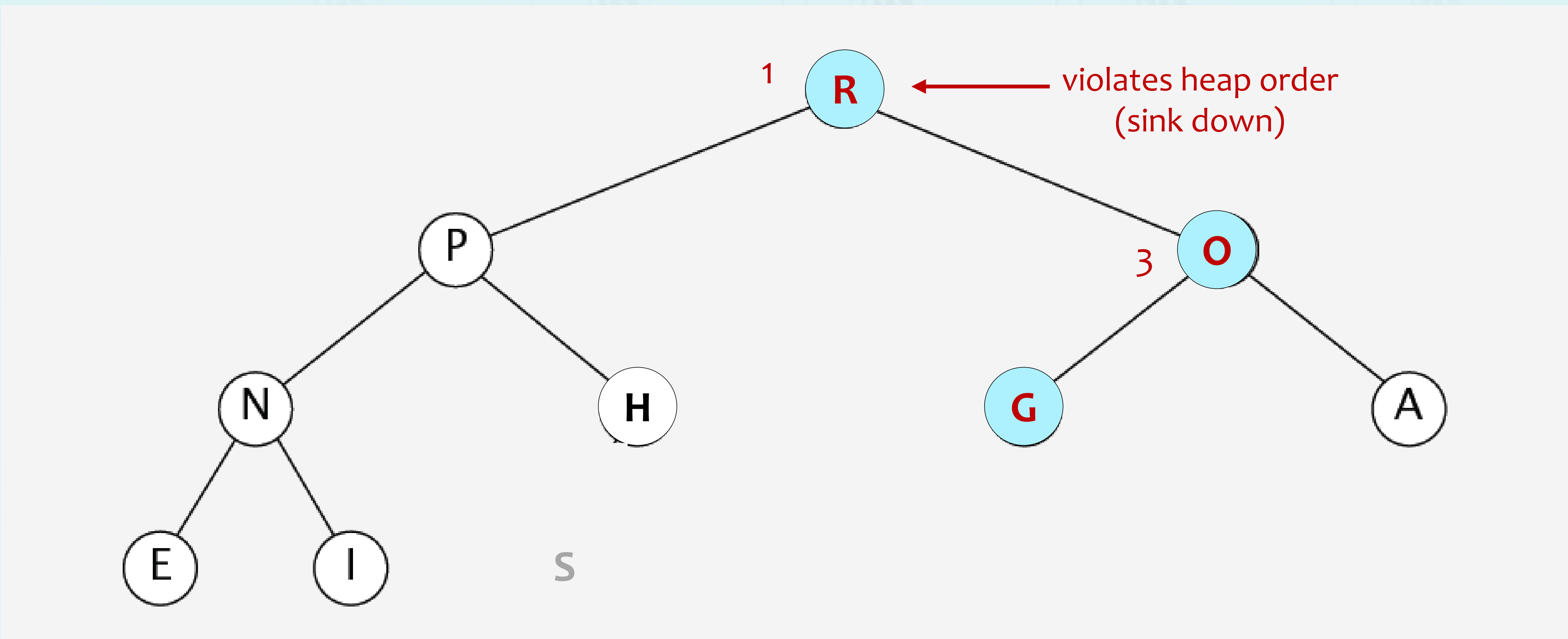


R	P	G	N	H	O	A	E	I	S	
1		3						N=9		

max-heap Demo

- **Insert**: Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

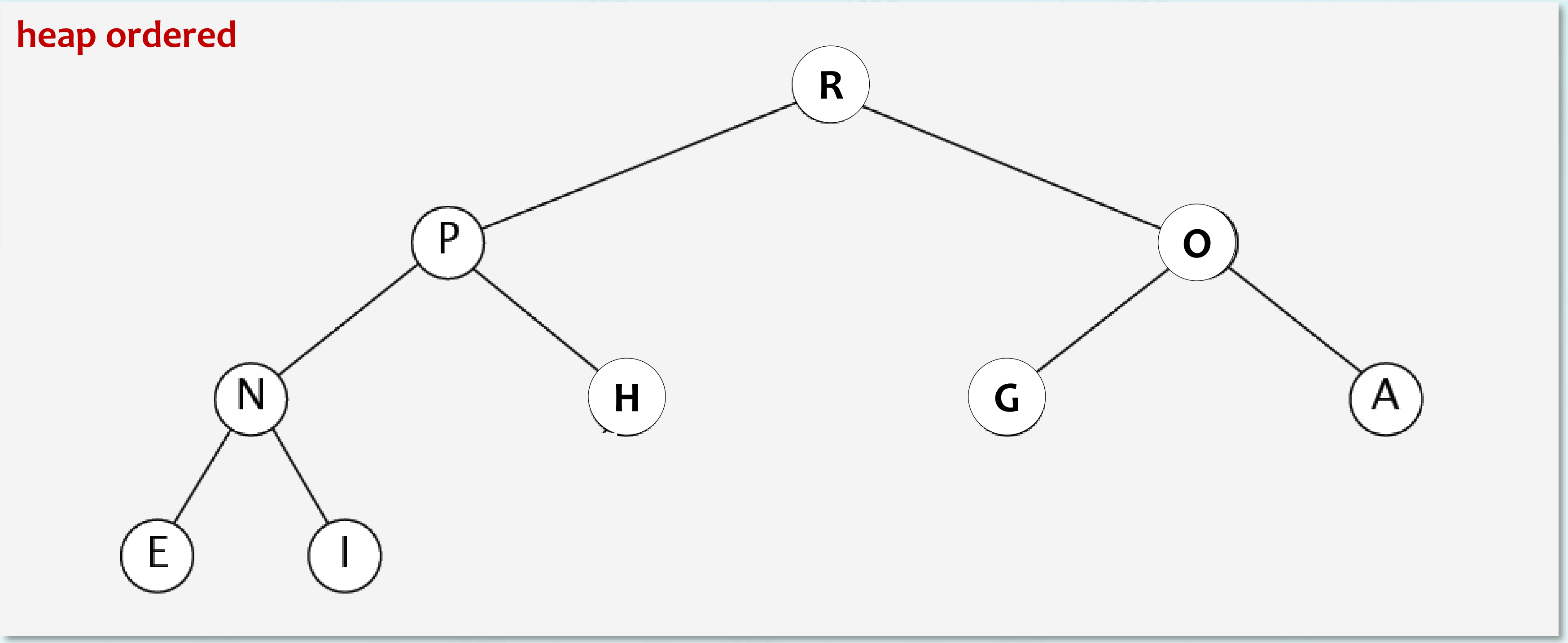
remove the maximum(root)



R	P	O	N	H	G	A	E	I	S	
1		3			6			N=9		

max-heap Demo

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.



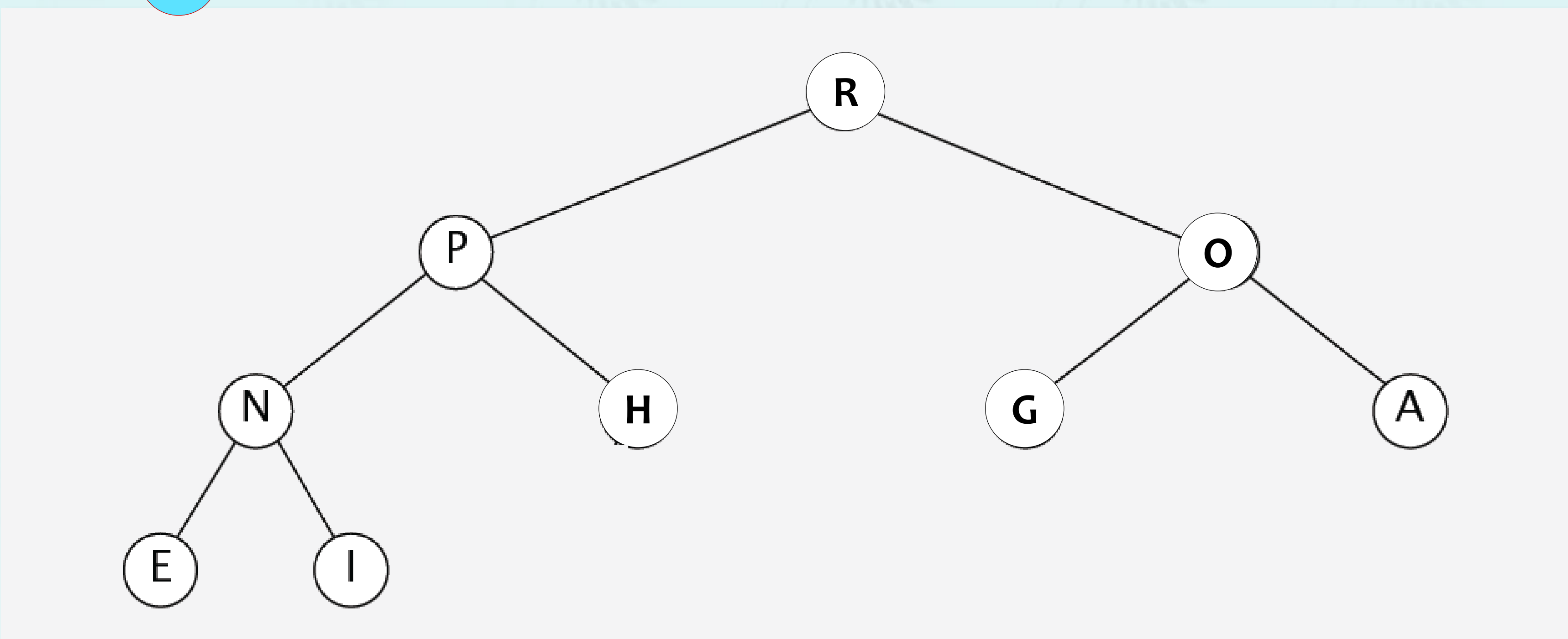
R	P	O	N	H	G	A	E	I		

N=9

max-heap Demo

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

insert S

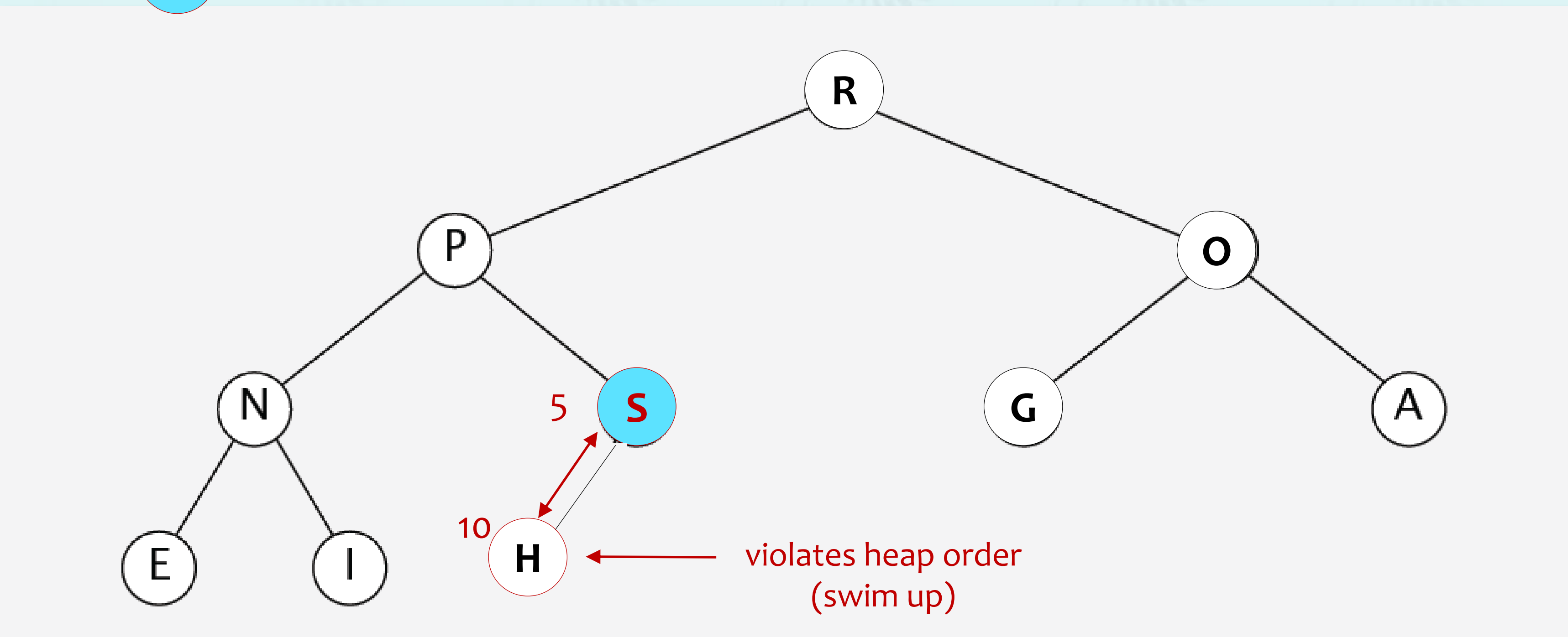


R	P	O	N	H	G	A	E	I		

max-heap Demo

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

insert S

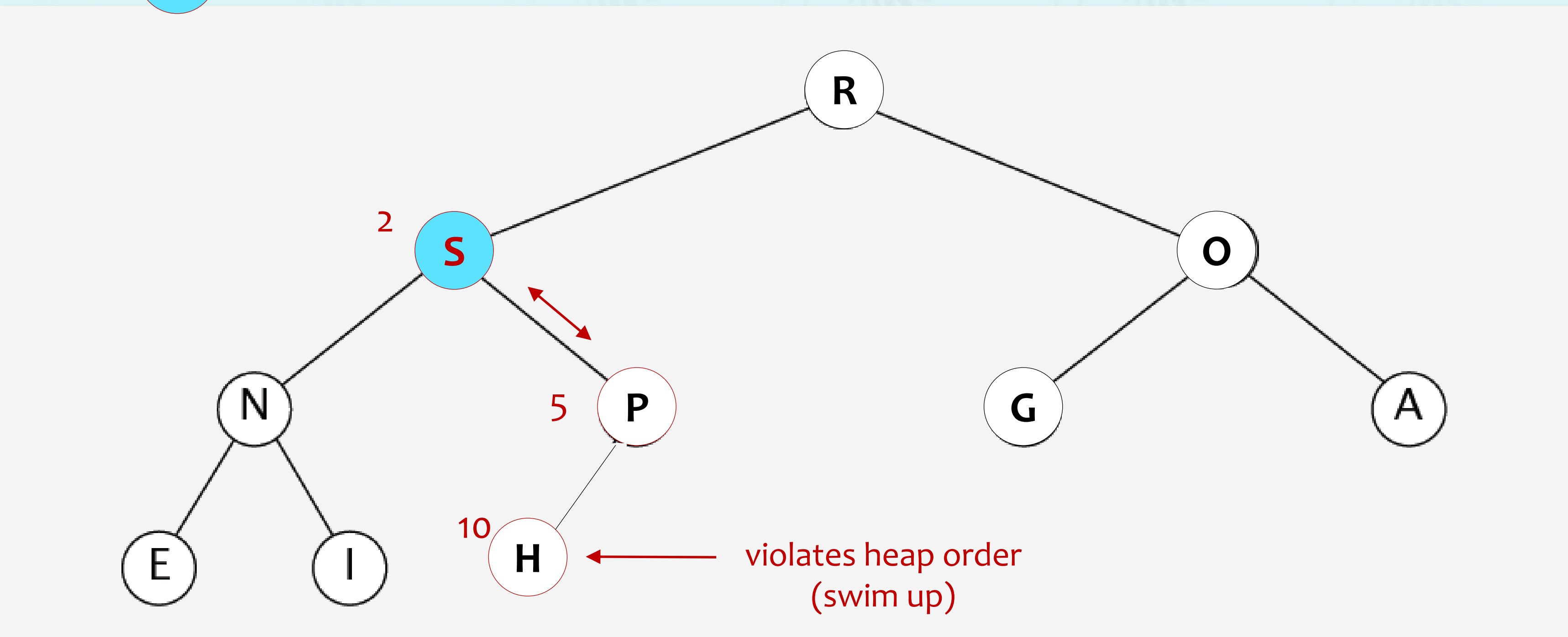


R	P	O	N	S	G	A	E	I	H	
				5					N=10	

max-heap Demo

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

insert S

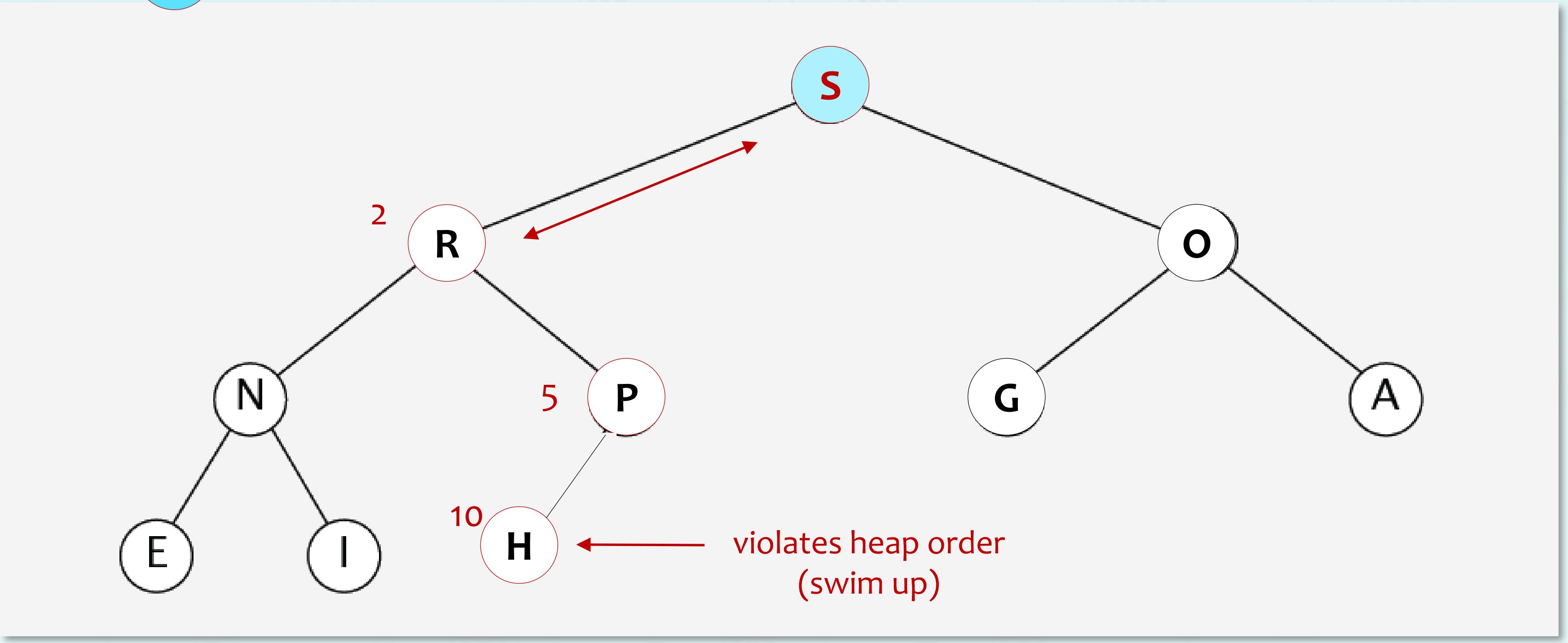


R	S	O	N	P	G	A	E	I	H	
	2			5					N=10	

max-heap Demo

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

insert S



S	R	O	N	P	G	A	E	I	H	
1	2			5					N=10	

max-heap Demo

- **Insert:** Add node at end, then swim it up.
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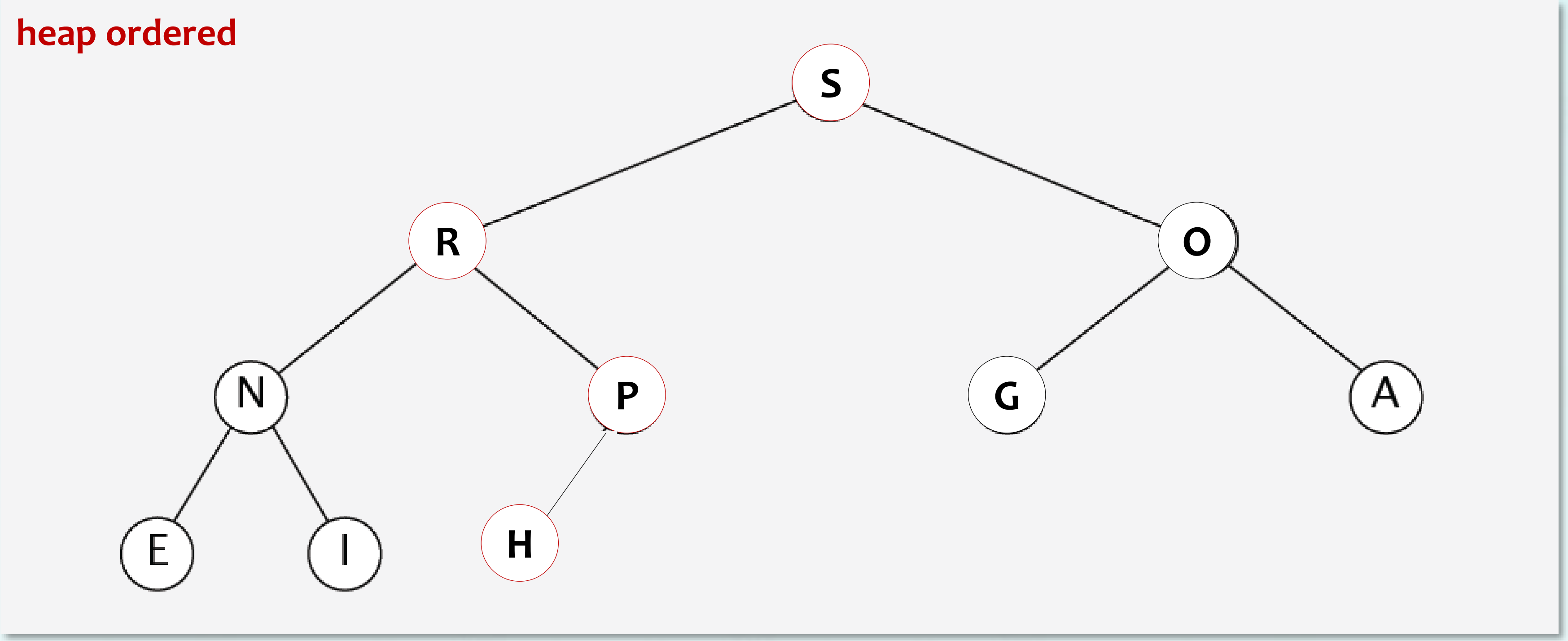

S	R	O	N	P	G	A	E	I	H	

Binary heap operations time complexity with N items:

- Level of heap is $\lfloor \log_2 N \rfloor$
- insert: O(log N) for each insert
  - In practice, expect less
- delete: O(log N) // deleting root node in min/max heap
- decreaseKey: O(log N)
- increaseKey: O(log N)
- remove: O(log N) // removing a node in any location

Binary heap operations time complexity with N items:

Implementation	Insert	Delete	max
Unordered array	1	N	N
Ordered array	N	1	1
Binary heap	log N	log N	1

Mission Completed

ITP20001/ECE 20010 Data Structures

Chapter 5

- *introduction*
- *tree, binary tree, binary search tree*
- *heaps data structure*
  - *complete binary tree*
  - *priority queues*
  - *binary heap and min-heap*
  - ***max-heap demo***
  - ***max-heap implementation***
  - *heap sort*

Chapter 7