Reviewer Discrete 2
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## Exercises

1. In how many different ways can five elements be selected in order from a set with three elements when repetition is allowed?
2. In how many different ways can five elements be selected in order from a set with five elements when repetition is allowed?
3. How many strings of six letters are there?
4. Every day a student randomly chooses a sandwich for lunch from a pile of wrapped sandwiches. If there are six kinds of sandwiches, how many different ways are there for the student to choose sandwiches for the seven days of a week if the order in which the sandwiches are chosen matters?
5. How many ways are there to assign three jobs to five employees if each employee can be given more than one job?
6. How many ways are there to select five unordered elements from a set with three elements when repetition is allowed?
7. How many ways are there to select three unordered elements from a set with five elements when repetition is allowed?
8. How many different ways are there to choose a dozen donuts from the 21 varieties at a donut shop?
9. A bagel shop has onion bagels, poppy seed bagels, egg bagels, salty bagels, pumpernickel bagels, sesame seed bagels, raisin bagels, and plain bagels. How many ways are there to choose
   a) six bagels?
   b) a dozen bagels?
   c) two dozen bagels?
   d) a dozen bagels with at least one of each kind?
   e) a dozen bagels with at least three egg bagels and no more than two salty bagels?
10. A croissant shop has plain croissants, cherry croissants, chocolate croissants, almond croissants, apple croissants, and broccoli croissants. How many ways are there to choose

Answers:

1. $3^5 = 243$
2. $5^5 = 3125$
3. $26^6 = 308{,}915{,}776$
4. $6^7 = 279{,}936$
5. $5^3 = 125$
6. $\binom{3+5-1}{5} = \binom{7}{5} = 21$
7. $\binom{5+3-1}{3} = \binom{7}{3} = 35$
8. $\binom{21+12-1}{12} = \binom{32}{12} = 225{,}792{,}840$
9. (8 bagel types)
   - a) $\binom{8+6-1}{6} = \binom{13}{6} = 1{,}716$
   - b) $\binom{8+12-1}{12} = \binom{19}{12} = 50{,}388$
   - c) $\binom{8+24-1}{24} = \binom{31}{7} = 2{,}629{,}575$
   - d) at least one of each: $\binom{4+8-1}{7} = \binom{11}{7} = 330$
   - e) at least 3 egg and ≤2 salty: $\binom{16}{7} - \binom{13}{7} = 11{,}440 - 1{,}716 = 9{,}724$
10. (6 croissant types, choosing a dozen)
    $\binom{6+12-1}{12} = \binom{17}{5} = 6{,}188$

1. What is the probability that a card selected at random from a standard deck of 52 cards is an ace?
2. What is the probability that a fair die comes up six when it is rolled?
3. What is the probability that a randomly selected integer chosen from the first 100 positive integers is odd?
4. What is the probability that a randomly selected day of a leap year (with 366 possible days) is in April?
5. What is the probability that the sum of the numbers on two dice is even when they are rolled?
6. What is the probability that a card selected at random from a standard deck of 52 cards is an ace or a heart?
7. What is the probability that when a coin is flipped six times in a row, it lands heads up every time?
8. What is the probability that a five-card poker hand contains the ace of hearts?
9. What is the probability that a five-card poker hand does not contain the queen of hearts?
10. What is the probability that a five-card poker hand contains the two of diamonds and the three of spades?
11. What is the probability that a five-card poker hand contains the two of diamonds, the three of spades, the six of hearts, the ten of clubs, and the king of hearts?
12. What is the probability that a five-card poker hand contains exactly one ace?
13. What is the probability that a five-card poker hand contains at least one ace?
14. What is the probability that a five-card poker hand contains cards of five different kinds?
15. What is the probability that a five-card poker hand contains two pairs (that is, two of each of two different kinds and a fifth card of a third kind)?
16. What is the probability that a five-card poker hand contains a flush, that is, five cards of the same suit?
17. What is the probability that a five-card poker hand contains a straight, that is, five cards that have consecutive kinds? (Note that an ace can be considered either the lowest card of an A-2-3-4-5 straight or the highest card of a 10-J-Q-K-A straight.)
18. What is the probability that a five-card poker hand contains a straight flush, that is, five cards of the same suit of consecutive kinds?

*19. What is the probability that a five-card poker hand contains cards of five different kinds and does not contain a flush or a straight?

20. What is the probability that a five-card poker hand contains a royal flush, that is, the 10, jack, queen, king, and ace of one suit?
21. What is the probability that a fair die never comes up an even number when it is rolled six times?
22. What is the probability that a positive integer not exceeding 100 selected at random is divisible by 3?
23. What is the probability that a positive integer not exceeding 100 selected at random is divisible by 5 or 7?
24. Find the probability of winning a lottery by selecting the correct six integers, where the order in which these integers are selected does not matter, from the positive integers not exceeding
    **a)** 30. **b)** 36. **c)** 42. **d)** 48.
25. Find the probability of winning a lottery by selecting the correct six integers, where the order in which these integers are selected does not matter, from the positive integers not exceeding
    **a)** 50. **b)** 52. **c)** 56. **d)** 60.
26. Find the probability of selecting none of the correct six integers in a lottery, where the order in which these integers are selected does not matter, from the positive integers not exceeding
    **a)** 40. **b)** 48. **c)** 56. **d)** 64.
27. Find the probability of selecting exactly one of the correct six integers in a lottery, where the order in which these integers are selected does not matter, from the positive integers not exceeding
    **a)** 40. **b)** 48. **c)** 56. **d)** 64.
28. In a superlottery, a player selects 7 numbers out of the first 80 positive integers. What is the probability that a person wins the grand prize by picking 7 numbers that are among the 11 numbers selected at random by a computer.
29. In a superlottery, players win a fortune if they choose the eight numbers selected by a computer from the positive integers not exceeding 100. What is the probability that a player wins this superlottery?

1. $4/52 = 1/13 \approx 0.07692$
2. $1/6 \approx 0.1667$
3. $50/100 = 1/2$
4. $30/366 = 5/61 \approx 0.08197$
5. $1/2$
6. Ace or heart: $(4 + 13 - 1)/52 = 16/52 = 4/13 \approx 0.3077$
7. $(1/2)^6 = 1/64 \approx 0.015625$
8. Contains the ace of hearts: $\binom{51}{4}/\binom{52}{5} = 5/52 \approx 0.09615$
9. Does **not** contain the queen of hearts: $\binom{51}{5}/\binom{52}{5} = 47/52 \approx 0.90385$
10. Contains (♦2 and ♠3): $\binom{50}{3}/\binom{52}{5} = \mathbf{5/663} \approx 0.007541$
11. Exactly the five listed cards: $1/\binom{52}{5} = 1/2{,}598{,}960 \approx 3.848 \times 10^{-7}$
12. Exactly one ace: $\dfrac{\binom{4}{1}\binom{48}{4}}{\binom{52}{5}} = \mathbf{3243/10829} \approx 0.29947$
13. At least one ace: $1 - \dfrac{\binom{48}{5}}{\binom{52}{5}} = \mathbf{18472/54145} \approx 0.34116$
14. Five different ranks (no pair): $\dfrac{\binom{13}{5}\,4^5}{\binom{52}{5}} = \mathbf{2112/4165} \approx 0.50708$
15. Two pairs: $\dfrac{\binom{13}{2}\,[\binom{4}{2}]^2 \cdot 11 \cdot 4}{\binom{52}{5}} = \mathbf{198/4165} \approx 0.04754$

$\binom{52}{5}$

16. Flush (any five same suit, incl. straight flush): $\dfrac{4\binom{13}{5}}{\binom{52}{5}} = \mathbf{33/16660} \approx 0.001981$
17. Straight (five consecutive ranks, incl. straight flush): $\dfrac{10 \cdot 4^5}{\binom{52}{5}} = \mathbf{128/32487} \approx 0.003940$
18. Straight flush (incl. royal): $\dfrac{40}{\binom{52}{5}} = \mathbf{1/64974} \approx 1.539 \times 10^{-5}$
19. Five different ranks **and not** a flush or straight ("high-card" hands):
$$\frac{\binom{13}{5}4^5 - 10 \cdot 4^5 - 4\binom{13}{5} + 40}{\binom{52}{5}} = \mathbf{1277/2548} \approx 0.50118$$
20. Royal flush: $\dfrac{4}{\binom{52}{5}} = \mathbf{1/649740} \approx 1.539 \times 10^{-6}$
21. Six rolls, never even (all odd): $(3/6)^6 = (1/2)^6 = 1/64$
22. Integer $1..100$ divisible by 3: $\lfloor 100/3 \rfloor / 100 = 33/100 = 0.33$
23. Integer $1..100$ divisible by 5 or 7: $(20 + 14 - 2)/100 = \mathbf{32/100} = 8/25 = 0.32$

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## Exercises

1. What probability should be assigned to the outcome of heads when a biased coin is tossed, if heads is three times as likely to come up as tails? What probability should be assigned to the outcome of tails?
2. Find the probability of each outcome when a loaded die is rolled, if a 3 is twice as likely to appear as each of the other five numbers on the die.
3. Find the probability of each outcome when a biased die is rolled, if rolling a 2 or rolling a 4 is three times as likely as rolling each of the other four numbers on the die and it is equally likely to roll a 2 or a 4.
4. Show that conditions (*i*) and (*ii*) are met under Laplace's definition of probability, when outcomes are equally likely.
5. A pair of dice is loaded. The probability that a 4 appears on the first die is $2/7$, and the probability that a 3 appears on the second die is $2/7$. Other outcomes for each die



appear with probability $1/7$. What is the probability of 7 appearing as the sum of the numbers when the two dice are rolled?

6. What is the probability of these events when we randomly select a permutation of $\{1, 2, 3\}$?
   **a)** 1 precedes 3.
   **b)** 3 precedes 1.
   **c)** 3 precedes 1 and 3 precedes 2.
7. What is the probability of these events when we randomly select a permutation of $\{1, 2, 3, 4\}$?
   **a)** 1 precedes 4.
   **b)** 4 precedes 1.
   **c)** 4 precedes 1 and 4 precedes 2.
   **d)** 4 precedes 1, 4 precedes 2, and 4 precedes 3.
   **e)** 4 precedes 3 and 2 precedes 1.

14. Use mathematical induction to prove the following generalization of Bonferroni's inequality:

$$p(E_1 \cap E_2 \cap \cdots \cap E_n) \geq p(E_1) + p(E_2) + \cdots + p(E_n) - (n - 1),$$

where $E_1, E_2, \ldots, E_n$ are $n$ events.

15. Show that if $E_1, E_2, \ldots, E_n$ are events from a finite sample space, then

$$p(E_1 \cup E_2 \cup \cdots \cup E_n) \leq p(E_1) + p(E_2) + \cdots + p(E_n).$$

This is known as **Boole's inequality**.

16. Show that if $E$ and $F$ are independent events, then $\overline{E}$ and $\overline{F}$ are also independent events.
17. If $E$ and $F$ are independent events, prove or disprove that $\overline{E}$ and $F$ are necessarily independent events.

**1.** A biased coin where heads is three times as likely as tails.
Let $P(T) = x$. Then $P(H) = 3x$ and $3x + x = 1 \Rightarrow x = 1/4$.
So $P(T) = 1/4,\ P(H) = 3/4$.

**2.** Loaded die where a 3 is three times as likely as each of the other five faces.
Let each of the other faces have probability $p$. Then $P(3) = 3p$ and $5p + 3p = 1 \Rightarrow 8p = 1 \Rightarrow p = 1/8$.
So $P(3) = 3/8$ and each of the others $= 1/8$.

**3.** Biased die: rolling a 2 or a 4 is three times as likely as rolling each of the other four numbers, and 2 and 4 are equally likely.
Let each "other" face have probability $p$. Then $P(2) = P(4) = 3p$. Total: $2(3p) + 4p = 10p = 1 \Rightarrow p = 1/10$.
So $P(2) = P(4) = 3/10$, each other face $= 1/10$.

**5.** Pair of dice: on the first die $P(4) = 2/7$ and all other faces $= 1/7$; on the second die $P(3) = 2/7$ and all other faces $= 1/7$.
We want $P(\text{sum} = 7)$. Possible ordered pairs giving 7: $(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)$.
Compute each:

- $P(1,6) = (1/7)(1/7) = 1/49$
- $P(2,5) = 1/49$
- $P(3,4) = (1/7)(1/7) = 1/49$
- $P(4,3) = (2/7)(2/7) = 4/49$
- $P(5,2) = 1/49$
- $P(6,1) = 1/49$

Sum $= \dfrac{1+1+1+4+1+1}{49} = \dfrac{9}{49}$.

**6.** Random permutation of $\{1,2,3\}$ (6 equally likely permutations).

a. Event "1 precedes 3": by symmetry half the permutations have 1 before 3 ⇒ $P = 1/2$.
b. Event "3 precedes 1": likewise $P = 1/2$.
c. Event "3 precedes 1 and 3 precedes 2" (i.e. 3 is before both others): 3 must be in the first position. Number of permutations with 3 first = $2! = 2$. So $P = 2/6 = 1/3$.

**7.** Random permutation of $\{1,2,3,4\}$ (24 equally likely permutations).

a. "1 precedes 4": by symmetry $P = 1/2$.
b. "4 precedes 1": $P = 1/2$.
c. "4 precedes 1 and 4 precedes 2" (4 is before both 1 and 2): among the three elements $\{4,1,2\}$ the probability that 4 is the earliest is $1/3$. So $P = 1/3$.
d. "4 precedes 1, 2, and 3" (4 is before all other three): probability that 4 is the earliest of the 4 elements $=$ $1/4$.
e. "4 precedes 3 and 2 precedes 1": these are independent relative-order constraints on two disjoint pairs, so $P = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. (Equivalently count: among 24 permutations exactly 6 satisfy both constraints.)