Reviewer Discrete 2
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1. $3^5 = 243$
2. $5^5 = 3125$
3. $26^6 = 308{,}915{,}776$
4. $6^7 = 279{,}936$
5. $5^3 = 125$
6. $\binom{3+5-1}{5} = \binom{7}{5} = 21$
7. $\binom{5+3-1}{3} = \binom{7}{3} = 35$
8. $\binom{21+12-1}{12} = \binom{32}{12} = 225{,}792{,}840$
9. (8 bagel types)

- a) $\binom{8+6-1}{6} = \binom{13}{6} = 1{,}716$
- b) $\binom{8+12-1}{12} = \binom{19}{12} = 50{,}388$
- c) $\binom{8+24-1}{24} = \binom{31}{7} = 2{,}629{,}575$
- d) at least one of each: $\binom{4+8-1}{7} = \binom{11}{7} = 330$
- e) at least 3 egg and ≤2 salty: $\binom{16}{7} - \binom{13}{7} = 11{,}440 - 1{,}716 = 9{,}724$

10. (6 croissant types, choosing a dozen)
$\binom{6+12-1}{12} = \binom{17}{5} = 6{,}188$

1. $4/52 = 1/13 \approx 0.07692$
2. $1/6 \approx 0.1667$
3. $50/100 = 1/2$
4. $30/366 = 5/61 \approx 0.08197$
5. $1/2$
6. Ace or heart: $(4 + 13 - 1)/52 = 16/52 = 4/13 \approx 0.3077$
7. $(1/2)^6 = 1/64 \approx 0.015625$
8. Contains the ace of hearts: $\binom{51}{4}/\binom{52}{5} = 5/52 \approx 0.09615$
9. Does **not** contain the queen of hearts: $\binom{51}{5}/\binom{52}{5} = 47/52 \approx 0.90385$
10. Contains (♦2 and ♠3): $\binom{50}{3}/\binom{52}{5} = \mathbf{5/663} \approx 0.007541$
11. Exactly the five listed cards: $1/\binom{52}{5} = 1/2{,}598{,}960 \approx 3.848 \times 10^{-7}$
12. Exactly one ace: $\dfrac{\binom{4}{1}\binom{48}{4}}{\binom{52}{5}} = \mathbf{3243/10829} \approx 0.29947$
13. At least one ace: $1 - \dfrac{\binom{48}{5}}{\binom{52}{5}} = \mathbf{18472/54145} \approx 0.34116$
14. Five different ranks (no pair): $\dfrac{\binom{13}{5}\,4^5}{\binom{52}{5}} = \mathbf{2112/4165} \approx 0.50708$
15. Two pairs: $\dfrac{\binom{13}{2}\,[\binom{4}{2}]^2 \cdot 11 \cdot 4}{\binom{52}{5}} = \mathbf{198/4165} \approx 0.04754$

$\binom{52}{5}$

16. Flush (any five same suit, incl. straight flush): $\dfrac{4\binom{13}{5}}{\binom{52}{5}} = \mathbf{33/16660} \approx 0.001981$
17. Straight (five consecutive ranks, incl. straight flush): $\dfrac{10 \cdot 4^5}{\binom{52}{5}} = \mathbf{128/32487} \approx 0.003940$
18. Straight flush (incl. royal): $\dfrac{40}{\binom{52}{5}} = \mathbf{1/64974} \approx 1.539 \times 10^{-5}$
19. Five different ranks **and not** a flush or straight ("high-card" hands):
$$\frac{\binom{13}{5}4^5 - 10 \cdot 4^5 - 4\binom{13}{5} + 40}{\binom{52}{5}} = \mathbf{1277/2548} \approx 0.50118$$
20. Royal flush: $\dfrac{4}{\binom{52}{5}} = \mathbf{1/649740} \approx 1.539 \times 10^{-6}$
21. Six rolls, never even (all odd): $(3/6)^6 = (1/2)^6 = 1/64$
22. Integer $1..100$ divisible by 3: $\lfloor 100/3 \rfloor / 100 = 33/100 = 0.33$
23. Integer $1..100$ divisible by 5 or 7: $(20 + 14 - 2)/100 = \mathbf{32/100} = 8/25 = 0.32$

1. A biased coin where heads is three times as likely as tails.
Let $P(T) = x$. Then $P(H) = 3x$ and $3x + x = 1 \Rightarrow x = 1/4$.
So $P(T) = 1/4,\ P(H) = 3/4$.

2. Loaded die where a 3 is three times as likely as each of the other five faces.
Let each of the other faces have probability p. Then $P(3) = 3p$ and $5p + 3p = 1 \Rightarrow 8p = 1 \Rightarrow p = 1/8$.
So $P(3) = 3/8$ and each of the others $= 1/8$.

3. Biased die: rolling a 2 or a 4 is three times as likely as rolling each of the other four numbers, and 2 and 4 are equally likely.
Let each "other" face have probability p. Then $P(2) = P(4) = 3p$. Total: $2(3p) + 4p = 10p = 1 \Rightarrow p = 1/10$.
So $P(2) = P(4) = 3/10$, each other face $= 1/10$.

5. Pair of dice: on the first die $P(4) = 2/7$ and all other faces $= 1/7$; on the second die $P(3) = 2/7$ and all other faces $= 1/7$.
We want $P(\text{sum} = 7)$. Possible ordered pairs giving 7: $(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)$.
Compute each:

- $P(1,6) = (1/7)(1/7) = 1/49$
- $P(2,5) = 1/49$
- $P(3,4) = (1/7)(1/7) = 1/49$
- $P(4,3) = (2/7)(2/7) = 4/49$
- $P(5,2) = 1/49$
- $P(6,1) = 1/49$

Sum $= \dfrac{1+1+1+4+1+1}{49} = \dfrac{9}{49}$.

6. Random permutation of $\{1, 2, 3\}$ (6 equally likely permutations).

a. Event "1 precedes 3": by symmetry half the permutations have 1 before 3 ⇒ $P = 1/2$.
b. Event "3 precedes 1": likewise $P = 1/2$.
c. Event "3 precedes 1 and 3 precedes 2" (i.e. 3 is before both others): 3 must be in the first position. Number of permutations with 3 first = $2! = 2$. So $P = 2/6 = 1/3$.

7. Random permutation of $\{1, 2, 3, 4\}$ (24 equally likely permutations).

a. "1 precedes 4": by symmetry $P = 1/2$.
b. "4 precedes 1": $P = 1/2$.
c. "4 precedes 1 and 4 precedes 2" (4 is before both 1 and 2): among the three elements $\{4, 1, 2\}$ the probability that 4 is the earliest is $1/3$. So $P = 1/3$.
d. "4 precedes 1, 2, and 3" (4 is before all other three): probability that 4 is the earliest of the 4 elements $=$ $1/4$.
e. "4 precedes 3 and 2 precedes 1": these are independent relative-order constraints on two disjoint pairs, so $P = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. (Equivalently count: among 24 permutations exactly 6 satisfy both constraints.)