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EECS 445 HW2

(a) $p(t) = \phi^{t}(1-\phi)^{1-t}$

$$p(x|t=0) = \frac{1}{(2\pi)^{\frac{1}{2}}|\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(x-\mu_0)^T \Sigma^{-1}(x-\mu_0)\right)$$

$$p(x|t=1) = \frac{1}{(2\pi)^{\frac{1}{2}}|\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(x-\mu_1)^T \Sigma^{-1}(x-\mu_1)\right)$$

Want to show

$$p(t=1|x) = \frac{1}{1+\exp(-w^T x)}$$

Use log odds $\Rightarrow$

$$\log\left(\frac{p(C_1|x)}{p(C_2|x)}\right) = \log \frac{p(C_1|x)}{1-p(C_1|x)}$$

$$= \log \frac{\exp\left(-\frac{1}{2}(x-\mu_0)^T \Sigma^{-1}(x-\mu_0)\right)}{\exp\left(-\frac{1}{2}(x-\mu_1)^T \Sigma^{-1}(x-\mu_1)\right)} + \log \frac{p(C_1)}{p(C_2)}$$

$$= -\frac{1}{2}(x-\mu_0)^T\Sigma^{-1}(x-\mu_0) - \left\{-\frac{1}{2}(x-\mu_1)^T\Sigma^{-1}(x-\mu_1)\right\} + \log\frac{p(C_1)}{p(C_2)}$$

$$= (\mu_0-\mu_1)^T\Sigma^{-1}x - \frac{1}{2}\mu_0^T\Sigma^{-1}\mu_0 + \frac{1}{2}\mu_1^T\Sigma^{-1}\mu_1 + \log\frac{p(C_1)}{p(C_2)}$$

$$= (\Sigma^{-1}(\mu_0-\mu_1))^T x + w_0, \text{ where}$$

$$w_0 = -\frac{1}{2}\mu_0^T\Sigma^{-1}\mu_0 + \frac{1}{2}\mu_1^T\Sigma^{-1}\mu_1 + \log\frac{p(C_1)}{p(C_2)}$$

To get into form we want, construct $w = \begin{bmatrix} w_0 \\ \Sigma^{-1}(\mu_0-\mu_1)^T \end{bmatrix}$ and

1(a) cont

$x' = \begin{bmatrix} 1 \\ x \end{bmatrix}$, s.t our solution is of the form

$$P(C_1 | x) = \frac{1}{1+\exp(-w^T x')}, \text{ where } w = \begin{bmatrix} w_0 \\ \Sigma^{-1}(\mu_0 - \mu_1)^T \end{bmatrix} \text{ and}$$

$$x' = \begin{bmatrix} 1 \\ x \end{bmatrix}.$$

1 (b) First step is to write $\ell(w)$

$$\ell(w) = \log \prod_{i=1}^{N} p(x^{(i)} | t^{(i)}) \, p(t^{(i)})$$

$$= \sum_{i=1}^{N} \log \{ p(x^{(i)} | t^{(i)}) \} p(t^{(i)}) \text{, split using indicators.}$$

$$= \sum_{i=1}^{N} \log \{ p(t^{(i)}) \, p(x^{(i)} | t=1)^{1\{t=1\}} \, p(x^{(i)} | t=0)^{1\{t=0\}} \}$$

$$= \sum_{i=1}^{N} \log(p(t^{(i)})) + \log(p(x^{(i)} | t=1)^{1\{t=1\}} + \log(p(x^{(i)} | t=0)^{1\{t=0\}} \}$$

$$= \sum_{i=1}^{N} t^{(i)} \log(\phi) + (1-t^{(i)}) \log(1-\phi) + 1\{t=1\} \log(p(x^{(i)} | t=1)) + 1\{t=0\} \log(p(x^{(i)} | t=0))$$

We will use this ℓ in our subsequent derivatives $\Rightarrow$

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b(i) $\nabla_\phi \ell(w) = \sum_{i=1}^{N} t^{(i)}\left(\frac{1}{\phi}\right) + (1-t^{(i)})\left(\frac{-1}{1-\phi}\right)$

$$= \sum_{i=1}^{N} \frac{t^{(i)}(1-\phi) - \phi(1-t^{(i)})}{\phi(1-\phi)} = \sum_{i=1}^{N} \frac{t^{(i)} - \phi}{\phi(1-\phi)}$$

Set to 0, want numerator $= 0$.

$$\sum_{i=1}^{N} t^{(i)} - \phi = 0 \Rightarrow \sum_{i=1}^{N} t^{(i)} - N\phi = 0 \Rightarrow \phi = \frac{1}{N}\sum_{i=1}^{N} 1\{t^{(i)}=1\}$$

$\boxtimes$

(ii) $\nabla_{\mu_0} \ell(w) = \nabla_{\mu_0} \sum_{i=1}^{N} -\frac{1}{2} 1\{t^{(i)}=0\}\left[(x^{(i)}-\mu_0)^T \Sigma^{-1} (x^{(i)}-\mu_0)\right]$

$$= \nabla_{\mu_0} -\frac{1}{2} \sum_{i=1}^{N} 1\{t^{(i)}=0\} \Sigma^{-1} (x^{(i)}-\mu_0)^T (x^{(i)}-\mu_0)$$

$$= \nabla_{\mu_0} -\frac{1}{2} \sum_{i=1}^{N} 1\{t^{(i)}=0\} \Sigma^{-1} (x^{(i)}-\mu_0)^2$$

$$= \cancel{-}\frac{1}{\cancel{2}} \sum_{i=1}^{N} 1\{t^{(i)}=0\} \cancel{\Sigma^{-1}}\, \cancel{2}(x^{(i)}-\mu_0)\cancel{(-1)}$$

$$\Rightarrow \sum_{i=1}^{N} 1\{t^{(i)}=0\}(x^{(i)}-\mu_0) = 0$$

$$\sum_{i=1}^{N} 1\{t^{(i)}=0\} x^{(i)} - \sum_{i=1}^{N} 1\{t^{(i)}=0\}\mu_0 = 0$$

$$\mu_0 = \frac{\sum_{i=1}^{N} 1\{t^{(i)}=0\} x^{(i)}}{\sum_{i=1}^{N} 1\{t^{(i)}=0\}}$$

$\boxtimes$

(iii) $\nabla_{\mu_1} \ell(w) = \nabla_{\mu_1} \sum_{i=1}^{N} \frac{-1}{2} 1\{t^{(i)}=1\}\left[(x^{(i)}-\mu_1)^T \Sigma^{-1} (x^{(i)}-\mu_1)\right]$

$$= \nabla_{\mu_1} \frac{-1}{2} \sum_{i=1}^{N} 1\{t^{(i)}=1\} \Sigma^{-1} (x^{(i)}-\mu_1)^2$$

$$= \cancel{-\frac{1}{2}} \sum_{i=1}^{N} 1\{t^{(i)}=1\} \cancel{\Sigma^{-1}} (x^{(i)}-\mu_1)\cancel{(-1)}$$

$$\Rightarrow \sum_{i=1}^{N} 1\{t^{(i)}=1\}(x^{(i)}-\mu_1) = 0$$

$$\sum_{i=1}^{N} 1\{t^{(i)}=1\}(x^{(i)}-\mu_1) = 0 \Rightarrow \sum_{i=1}^{N} 1\{t^{(i)}=1\}x^{(i)} - \sum_{i=1}^{N} 1\{t^{(i)}=1\}\mu$$

$$\mu_1 = \sum_{i=1}^{N} \frac{1\{t^{(i)}=1\}x^{(i)}}{\sum_{i=1}^{N} 1\{t^{(i)}=1\}} \qquad \boxtimes$$

(iv) $\cancel{\nabla_{\Sigma} \ell(w) = \sum_{i=1}^{N} (x}$

Observations: $\Sigma = \sigma^2 \quad |\Sigma|^{\frac{1}{2}} = \sigma$

consolidate indicator variables terms into $t^{(i)}$

$$\nabla_{\Sigma} \ell(w) = \sum_{i=1}^{N} \nabla \log(p(x^{(i)}|t^{(i)}))$$

$$= \sum_{i=1}^{N} \nabla_{\sigma} \log\left(\frac{1}{(2\pi)^{\frac{1}{2}}\sigma}\right) + \log\left(\exp\left((x-\mu_t)^T \Sigma^{-1} (x-\mu_t)\right)\right)$$

$$= \sum_{i=1}^{N} \nabla_{\sigma} \log\left(\frac{1}{(2\pi)^{\frac{1}{2}}}\right) + \log\left(\frac{1}{\sigma}\right) + \cancel{\tfrac{1}{2}}\Sigma^{-1} (x-\mu_t)^T(x-\mu_t))$$

continued $\Longrightarrow$

$$= \cancel{\sum_{i=1}^{N} - \frac{1}{\sigma} + (x-\mu_t)^T \frac{1}{\sigma^3}}$$

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$$\sum_{i=1}^{N} \left(-\Sigma \cancel{\phi} (x^{(i)} - \mu_t)^{\cancel{T}} (x^{(i)} - \mu_t)^T \right) = 0$$

$$\Rightarrow \cancel{\Delta\Sigma} - N\Sigma + \sum_{i=1}^{N} (x^{(i)} - \mu_t)^{\cancel{T}} (x^{(i)} - \mu_t)^T) = 0$$

$$\Sigma = \frac{1}{N} \sum_{i=1}^{N} (x^{(i)} - \mu_t)(x^{(i)} - \mu_t)^T \qquad \boxtimes$$

2(a) Wts

$$\nabla_{W_m} \ell(w) = \sum_{i=1}^{N} \phi(x^{(i)}) \left[1(t^{(i)} = m) - p(t^{(i)} = m \mid x^{(i)}) \right]$$

Start derivation.

$$\nabla_{W_m} \ell = \sum_{i=1}^{N} \sum_{m=1}^{k} \phi(x^{(i)}) 1\{t^{(i)} = m\} - \nabla_{W_m} 1(t^{(i)} = m) \log \left(\sum_{j=1}^{k} \exp(w^T{}_j \phi(x^{(i)})) \right]$$

$$= \sum_{i=1}^{N} \sum_{m=1}^{k} \phi(x^{(i)}) 1\{t^{(i)} = m\} - \nabla_{W_m} (1(t^{(i)} = m) \log \sum_{j=1}^{k} \exp(w^T{}_j \phi(x^{(i)}))$$

$$\Downarrow$$

$$u = \sum_{j=1}^{k} \exp(w^T{}_j \phi(x^{(i)})), \quad \nabla_{W_j} (\log \sum_{j=1}^{k} \exp(w^T \phi(x^{(i)}))$$

$$= \nabla_{W_j} \log(u)$$

$$\Rightarrow \frac{1}{u} \nabla_{W_j} u = \frac{1}{\sum_{j=1}^{k} \exp(w_j^T \phi(x^{(i)}))}$$

$$\Rightarrow \nabla_{W_j} u = \nabla_{W_j} \left(\sum_{k=1}^{k} \exp(W_k{}^T \phi(x^{(i)}) \right)$$

$$= \exp(w_j{}^T \phi(x)) \cdot \phi(x)$$

$$\Rightarrow \sum_{i=1}^{N} \sum_{k=1}^{k} \phi(x^{(i)}) 1\{t^{(i)} = m\} - \phi(x^{(i)}) \left(\frac{\exp(w_m{}^T \phi(x^{(i)}))}{\sum_{j=1}^{k} \exp(w_j{}^T \phi(x^{(i)}))} \right)$$

$$= \sum_{i=1}^{N} \sum_{k=1}^{k} \phi(x^{(i)}) \left(1\{t^{(i)} = m\} - p(t^{(i)} = m \mid x^{(i)}) \right)$$

We memove k-itorsten when we only consider w_m, so

$$\nabla_{W_m} \ell(w) = \sum_{i=1}^{N} \phi(x^{(i)}) \left[1\{t^{(i)} = m\} - p(t^{(i)} = m \mid x^{(i)}) \right] \quad \boxtimes$$