

3.27 $E[X] = \sum_{j=1}^{\infty} j P[X=j] = \sum_{j=1}^{\infty} j \frac{c}{j^2} = c \sum_{j=1}^{\infty} \frac{1}{j} = \infty$

mean does not exist.

$E[X^2] = \sum_{j=1}^{\infty} j^2 \frac{c}{j^2} = c \sum_{j=1}^{\infty} 1 = \infty$

none of the moments exist

This pmf decays sufficiently fast that probabilities add to 1, but too slowly for moments to exist.

3.28 $E[Y] = -1 \cdot \frac{1}{10} + 0 \cdot \frac{2}{10} + 1 \cdot \frac{3}{10} + 2 \cdot \frac{4}{10} = \frac{10}{10} = 1$

$E[Y^2] = 1 \cdot \frac{1}{10} + 1 \cdot \frac{3}{10} + 4 \cdot \frac{4}{10} = \frac{20}{10} = 2$

$\text{VAR}[Y] = 2 - 1^2 = 1.$

4.84

$$F_Y(y) = F_X\left(\frac{y-2}{3}\right)$$

$$f_Y(y) = \frac{1}{3} f_X\left(\frac{y-2}{3}\right)$$

• X is Laplacian

$$F_Y(y) = \begin{cases} \frac{1}{2} e^{\alpha(\frac{y-2}{3})} & y \leq 2 \\ 1 - \frac{1}{2} e^{-\alpha(\frac{y-2}{3})} & y \geq 2 \end{cases}$$

$$f_Y(y) = \frac{1}{3} \frac{\alpha}{2} e^{-\alpha|\frac{y-2}{3}|}$$

• X is Gaussian

$$F_Y(y) = \Phi\left(\frac{\frac{y-2}{3} - m}{\sigma}\right) = \Phi\left(\frac{y - (2+3m)}{3\sigma}\right)$$

$$f_Y(y) = \frac{1}{3\sigma\sqrt{2\pi}} e^{-\left(\frac{y-2}{3} - m\right)^2 / 2\sigma^2} = \frac{1}{3\sigma\sqrt{2\pi}} e^{-(y - (2+3m))^2 / 2(3\sigma)^2}$$

• $X = b \cos(2\pi U)$

$$F_Y(y) = \begin{cases} 0 & y < 2-3b \\ \frac{1}{\pi} \sin^{-1}\left(\frac{y-2}{3b}\right) + \frac{1}{\pi} \sin^{-1}\left(-\frac{1}{b}\right) & 2-3b \leq y \leq 3b+2 \\ 1 & y \geq 3b+2 \end{cases}$$

$$f_Y(y) = \frac{1}{3} \frac{1}{\pi b \sqrt{1 - \left(\frac{y-2}{3b}\right)^2}} \quad 2-3b \leq y < 3b+2$$

4.85

X : Gaussian, $Y = aX + b$, a linear combination of X .

Y is also Gaussian

$$E[Y] = aE[X] + b = am + b = m'$$

$$\text{Var}[Y] = a^2 \text{Var}[X] = a^2 \alpha^2 = \alpha'^2$$

$$a = \alpha' / \alpha, \quad b = m' - am = m' - m\alpha' / \alpha$$

4.100

$$Y = X/n$$

$$E[Y] = E[X]/n = np/n = p$$

$$VAR[Y] = VAR[X]/n^2 = npq/n^2 = pq/n, q = 1 - p$$

$$P\{|Y - p| > a\} \leq \frac{\sigma^2}{a^2} = \frac{pq}{na^2}$$

$$\text{as } n \rightarrow \infty \quad P\{|Y - p| > a\} \rightarrow 0 \text{ for any fixed } a > 0$$

4.101

$$Y = \frac{1}{n} \sum_{i=1}^n X_i$$

$$E[Y] = \frac{1}{n} \sum_i E[X_i] = E[X]$$

$$Var[Y] = \frac{1}{n^2} Var\left[\sum_i X_i\right] = \frac{1}{n^2} \cdot \frac{n}{\lambda^2} = \frac{1}{n\lambda^2}$$

$$P\{|Y - E[X]| > a\} = P\{|Y - E[Y]| > a\} \\ \leq \frac{1}{n\lambda^2 a^2}$$

$$\text{as } n \rightarrow \infty \quad P\{|Y - E[X]| > a\} \rightarrow 0$$

$$(5.20) (b) F_X(x) = F_{XY}(x, \infty)$$

$$= \begin{cases} 1 - \frac{1}{x^2} & , x > 1 \\ 0 & , \text{otherwise} \end{cases}$$

$$F_Y(y) = F_{XY}(\infty, y)$$

$$= \begin{cases} 1 - \frac{1}{y^2} & , y > 1 \\ 0 & , \text{otherwise} \end{cases}$$

$$(c) P\{X < 3, Y \leq 5\}$$

$$= F_{XY}(3, 5)$$

$$= (1 - \frac{1}{9})(1 - \frac{1}{25})$$

$$= \frac{64}{75}$$

$$P\{X > 4, Y \leq 3\}$$

$$= 1 - F_{XY}(4, \infty) - F_{XY}(\infty, 3) + F_{XY}(4, 3)$$

$$= 1 - (1 - \frac{1}{16}) - (1 - \frac{1}{9}) + (1 - \frac{1}{16})(1 - \frac{1}{9})$$

$$= 1 - \frac{15}{16} - \frac{8}{9} + \frac{5}{6}$$

$$= \frac{1}{144}$$

8.4 Confidence Intervals

8.39

The i th measurement is $X_i = m + N_i$ where $\mathcal{E}[N_i] = 0$ and $\text{VAR}[N_i] = 10$. The sample mean is $M_{100} = 100$ and the variance is $\sigma = \sqrt{10}$.

Eqn. 5.37 with $z_{\alpha/2} = 1.96$ gives

$$\left(100 - \frac{1.96\sqrt{10}}{\sqrt{30}}, 100 + \frac{1.96\sqrt{10}}{\sqrt{30}}\right) = (98.9, 101.1)$$

8.40

5.32 The width of the confidence interval given by Eqn. 5.37 is

$$\left(M_n + \frac{z_{\alpha/2}\sigma}{\sqrt{n}}\right) - \left(M_n - \frac{z_{\alpha/2}\sigma}{\sqrt{n}}\right) = \frac{2z_{\alpha/2}\sigma}{\sqrt{n}}$$

a) For 95% confidence intervals $z_{\alpha/2} = 1.96$, so ($\sigma = 1$)

$$\text{width of interval} = \frac{2(1.96)}{\sqrt{n}} = \begin{cases} 1.96 & n = 4 \\ 0.98 & n = 16 \\ 0.29 & n = 100 \end{cases}$$

b) For 99% confidence intervals $z_{\alpha/2} = 2.576$ so

$$\text{width of interval} = \frac{2(2.576)}{\sqrt{n}} = \begin{cases} 2.576 & n = 4 \\ 1.288 & n = 16 \\ 0.515 & n = 100 \end{cases}$$

8.41

$$\text{5.33 } M_n = 223 \quad V_N^2 = 100 \quad n = 225 \\ \Rightarrow V_n = 10$$

6 Assuming that individual lifetimes are Gaussian RV's, Eqn. 5.43 with $n = \infty$

$$\left(M_n - \frac{z_{\alpha/2,\infty}V_n}{\sqrt{n}}, M_n + \frac{z_{\alpha/2,\infty}V_n}{\sqrt{n}}\right) = \left(223 - \frac{1.96(10)}{\sqrt{225}}, 223 + \frac{1.96(10)}{\sqrt{225}}\right) \\ = (222, 224)$$

41 5

From Eqn 8.59 the confidence interval for the sample variance is

$$\left[\frac{224(100)}{\chi^2_{0.025, 224}}, \frac{224(100)}{\chi^2_{0.975, 224}}\right] = \left[\frac{224(100)}{267.35}, \frac{224(100)}{184.44}\right] \\ = [83.785, 121.45]$$

7.2 The Sample Mean and the Laws of Large Numbers

7.15
$$P\left[\left|\frac{N(t)}{t} - \lambda\right| \geq \varepsilon\right] = P[|N(t) - \lambda t| \geq \varepsilon t]$$

$$\leq \frac{\text{VAR}[N(t)]}{(\varepsilon t)^2} \quad \text{by Chebyshev Inq.}$$

$$= \frac{\lambda t}{\varepsilon^2 t^2} = \frac{\lambda}{\varepsilon^2 t}$$

7.16
$$p = \frac{2}{10}$$

$$P[|f_A(n) - p| < \varepsilon] \geq 1 - \frac{p(1-p)}{n\varepsilon^2} = 0.95$$

letting $p = \frac{2}{10}$, $\varepsilon = \frac{1}{50} \Rightarrow n = 8000$

7.17
$$M_{100} = \frac{1}{100}(X_1 + \dots + X_{100}) = \frac{1}{100}S_{100}$$

$$\mu = E[X] = \frac{1+2+\dots+6}{6} = 3.5$$

$$\sigma_X^2 = \frac{1}{6}(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2) - (3.5)^2 = 2.91667$$

$$P[300 < S_{100} < 400] = \left[3 < \frac{S_{100}}{100} < 4\right]$$

$$= P[-.5 < M_{100} - 3.5 < .5]$$

$$= P[|M_{100} - 3.5| < .5]$$

$$\geq 1 - \frac{2.92}{100 \left(\frac{1}{2}\right)^2} = 0.416$$

7.25

$$\mathcal{E}[X_i] = \frac{1}{\lambda} = 36 \quad \text{VAR}(X_i) = \frac{1}{\lambda^2} = 36^2$$

$$S = X_1 + \dots + X_{16} \quad \mathcal{E}[S] = 16(36) \quad \text{VAR}(S) = 16(36)^2$$

$$\begin{aligned} P[S < 600] &= P\left[\frac{S - 16(36)}{4(36)} < \frac{600 - 16(36)}{4(36)}\right] \\ &\cong 1 - Q\left(\frac{1}{6}\right) = 0.5692 \end{aligned}$$

7.26

$$\begin{aligned} \mathcal{E}[S_n] &= n\mathcal{E}[X_i] = n \cdot 1 = n \\ \text{VAR}[S_n] &= n\sigma_{x_i}^2 = n \cdot 1^2 = n \end{aligned}$$

Assuming S_n approximately Gaussian:

$$P[S_n > 15] = P\left[\frac{S_n - n}{\sqrt{n}} > \frac{15 - n}{\sqrt{n}}\right] \approx Q\left(\frac{15 - n}{\sqrt{n}}\right) = 0.99$$

From Table 3.4

$$\frac{15 - n}{\sqrt{n}} = -2.3263$$

$$\Rightarrow n - 2.3263\sqrt{n} - 15 = 0 \Rightarrow n = 27.04$$

\Rightarrow by 28 pens

7.30

Total error is

$$S_{64} = X_1 + X_2 + \dots + X_{64}$$

where X_i uniform is $[-\frac{1}{2}, \frac{1}{2}]$

$$E[X_i] = 0 \quad \text{VAR}[X_i] = \frac{1}{12}$$

$$P[S_{64} > 4] = P\left[\frac{S_{100}}{\sqrt{\frac{64}{12}}} > \frac{4}{\sqrt{\frac{64}{12}}}\right] \approx Q\left(\frac{4.16}{1.7321}\right) = 1.79(10^{-2})$$

7.31

$X_i \sim \text{Bernoulli}(1/2)$ $X_i = \begin{cases} 1 & \text{head} \\ 0 & \text{tail} \end{cases}$

$$S = \sum_{i=1}^{100} X_i \quad E[S] = 50, \quad \text{VAR}[S] = 25$$

$$a) P\{S_{100} \geq 91\} \leq \left(\frac{\frac{1}{2}}{\left(\frac{91}{100}\right)^{\frac{91}{100}} \left(\frac{9}{100}\right)^{\frac{9}{100}}}\right)^{100} = 1.0886 \times 10^{-17}$$

$$\text{using CLT: } P\{S_{100} \geq 91\} = Q\left(\frac{91-50}{5}\right) = Q(8.2) = 1.1 \times 10^{-16}$$

$$b) P\{S_{1000} \geq 651\} \leq \left(\frac{\frac{1}{2}}{\left(\frac{651}{1000}\right)^{\frac{651}{1000}} \left(\frac{349}{1000}\right)^{\frac{349}{1000}}}\right)^{1000} = 7.6332 \times 10^{-21}$$

$$\text{using CLT: } P\{S_{1000} \geq 651\} = Q\left(\frac{651-500}{\sqrt{250}}\right) = Q(9.5501) = 6.47 \times 10^{-22}$$