Constraint Satisfaction Problems

Các bài toán thỏa mãn ràng buộc

Outline

- Constraint Satisfaction Problems (CSP)
- Backtracking search for CSPs
- Local search for CSPs

Constraint satisfaction problems (CSPs)

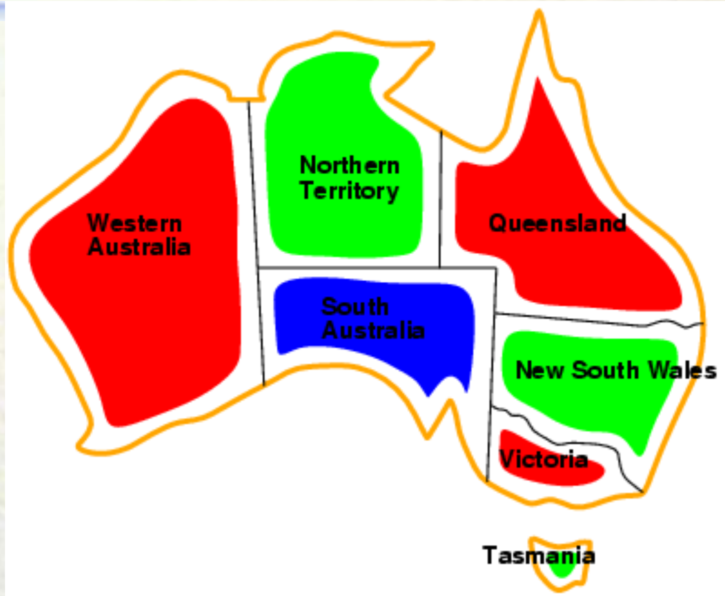
- Standard search problem:
  - state is a "black box“ – any data structure that supports successor function, heuristic function, and goal test
- CSP:
  - state is defined by variables X_i with values from domain D_i
  - goal test is a set of constraints specifying allowable combinations of values for subsets of variables.
  - Aim is to find an assignment of X_i from domain D_i in such a way that none of the constraints are violated.
- Simple example of a formal representation language
- Allows useful general-purpose algorithms with more power than standard search algorithms

Example: Map-Coloring

- Variables *WA, NT, Q, NSW, V, SA, T*
- Domains D_i = {red,green,blue}
- Constraints: adjacent regions must have different colors

- e.g., WA ≠ NT, or (WA,NT) in {(red,green),(red,blue),(green,red), (green,blue),(blue,red),(blue,green)}

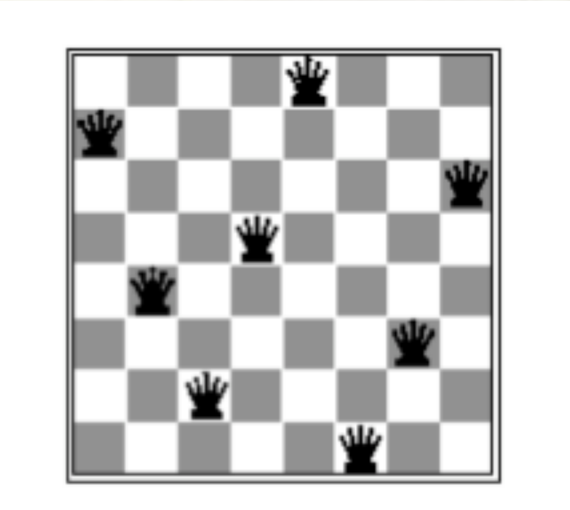
Example: Map-Coloring



- Solutions are complete and consistent assignments, e.g., WA = red, NT = green,Q = red,NSW = green,V = red,SA = blue,T = green

Example: n-queens puzzle

- Assume one queen in each column.
- Variables Q_1, ..Q_n.
- Domains D_i={1,..,n}
- Constraints
- $Q_i \neq Q_j$ (cannot be in the same row)
- $|Q_i-Q_j| \neq |i-j|$ (or same diagonal)



Example Sudoku

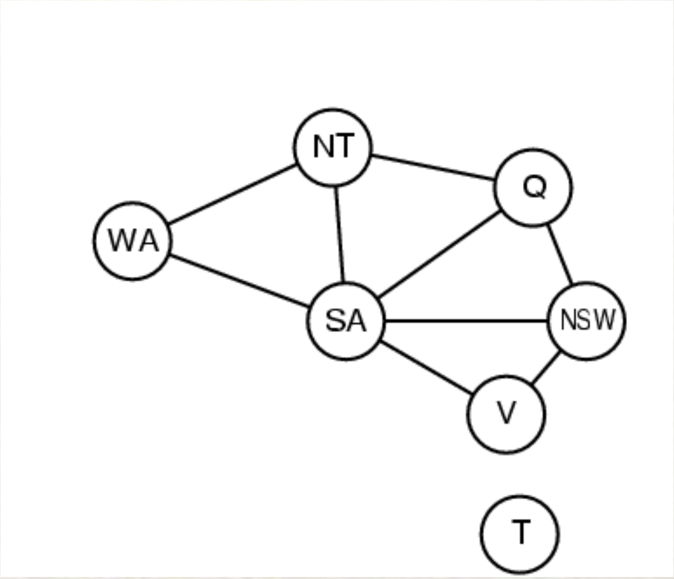
9				6				3
1		5		9	3	2		6
	4			5				9
8						4	7	1
		4	8	7				
7		2	6		1			8
2								
5				3	2		9	4
	8	7		1	6	3	5	

Real-world CSPs

- Assignment problems (e.g. who teaches what class)
- Timetabling problems (e.g. which class is offered when and where?)
- Hardware configuration
- Transport scheduling
- Factory scheduling

Constraint graph

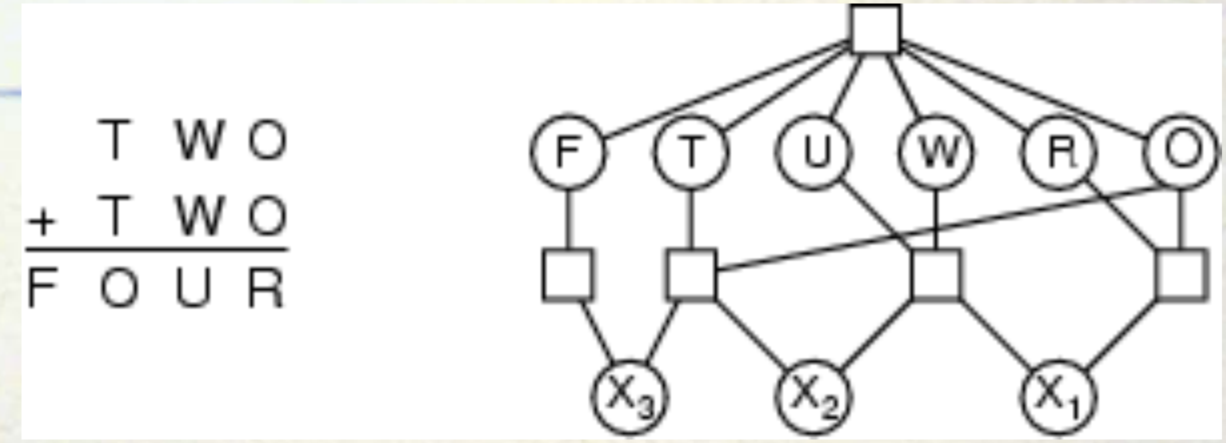
- Binary CSP: each constraint relates two variables
- Constraint graph: nodes are variables, arcs are constraints



Varieties of constraints

- Unary constraints involve a single variable,
  - e.g., SA ≠ green
- Binary constraints involve pairs of variables,
  - e.g., SA ≠ WA
- Higher-order constraints involve 3 or more variables,
  - e.g., cryptarithmetic column constraints
- Soft constraints (preferences)
  - 11am lecture is better than 8am lecture

Example: Cryptarithmetic



- Variables: $F\ T\ U\ W\ R\ O\ X_1\ X_2\ X_3$
- Domains: $\{0,1,2,3,4,5,6,7,8,9\}$
- Constraints: *Alldiff (F,T,U,W,R,O)*
  - $O + O = R + 10 \cdot X_1$
  - $X_1 + W + W = U + 10 \cdot X_2$
  - $X_2 + T + T = O + 10 \cdot X_3$
  - $X_3 = F$, $T \neq 0$, $F \neq 0$

Standard search formulation (incremental)

Let's start with the straightforward approach, then fix it

States are defined by the values assigned so far

- Initial state: the empty assignment { }
- Successor function: assign a value to an unassigned variable that does not conflict with current assignment
  → fail if no legal assignments
- Goal test: the current assignment is complete

1. This is the same for all CSPs
2. Every solution appears at depth *n* with *n* variables
   → use depth-first search
3. Path is irrelevant, so can also use complete-state formulation
4. $b = (n - \ell)d$ at depth ℓ, hence $n! \cdot d^n$ leaves (d: number of variable values)

Backtracking search

- Variable assignments are commutative, i.e.,

[WA = red then NT = green] same as [NT = green then WA = red]

- Only need to consider assignments to a single variable at each node
  - → b = d and there are d^n leaves
- Depth-first search for CSPs with single-variable assignments is called backtracking search
- Backtracking search is the basic uninformed algorithm for CSPs
- Can solve *n*-queens for $n \approx 25$

Backtracking search

```
function BACKTRACKING-SEARCH( csp) returns a solution, or failure
   return RECURSIVE-BACKTRACKING({}, csp)
function RECURSIVE-BACKTRACKING( assignment,csp) returns a solution, or failure
   if assignment is complete then return assignment
   var ← SELECT-UNASSIGNED-VARIABLE( Variables[csp], assignment, csp)
   for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
      if value is consistent with assignment according to Constraints[csp] then
         add { var = value } to assignment
         result ← RECURSIVE-BACKTRACKING(assignment, csp)
         if result ≠ failue then return result
         remove { var = value } from assignment
   return failure
```

Backtracking example

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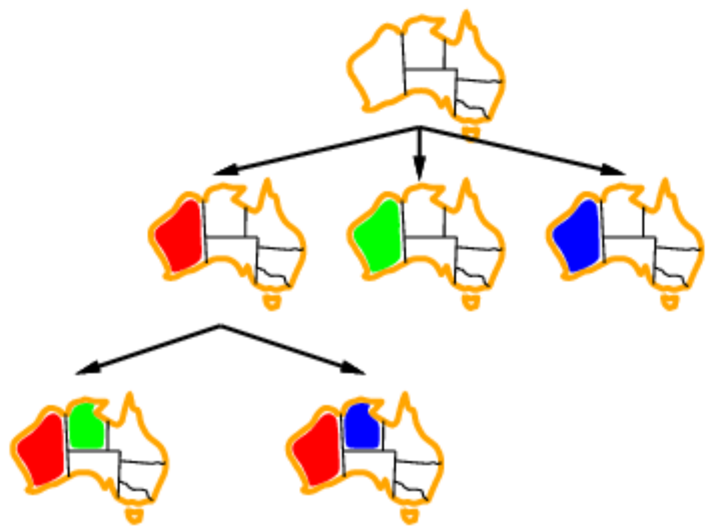
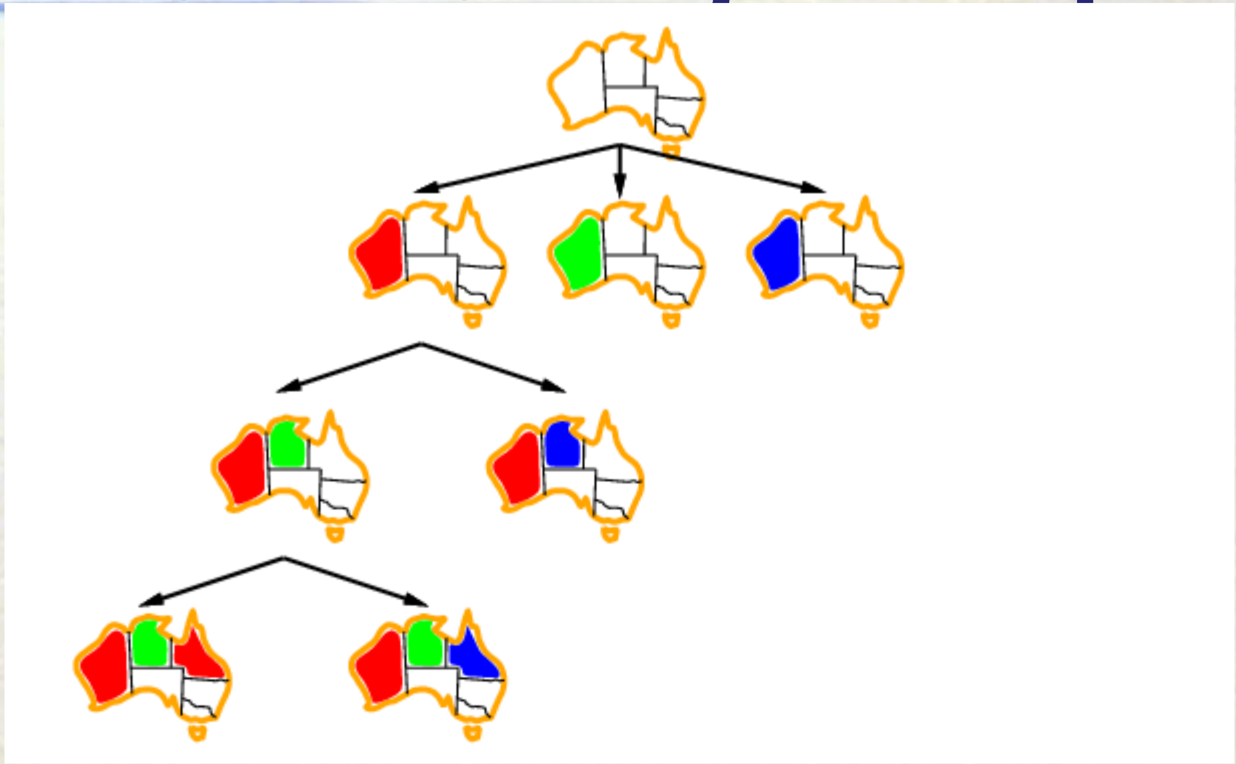


Backtracking example



Improving backtracking efficiency

- General-purpose methods can give huge gains in speed:
  - Which variable should be assigned next?
  - In what order should its values be tried?
  - Can we detect inevitable failure early?

Most constrained variable
Biến bị ràng buộc nhiều nhất

- Most constrained variable: choose the variable with the fewest legal values

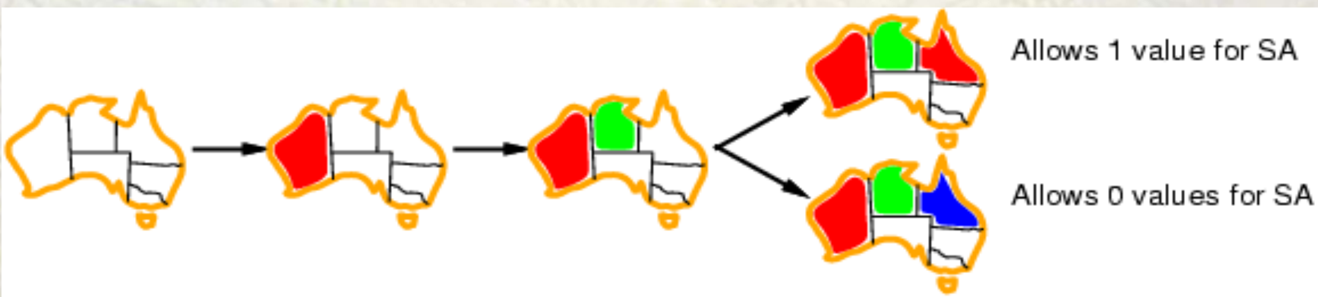
- a.k.a. minimum remaining values (MRV) heuristic

Most constraining variable
Biến ràng buộc nhiều nhất

- Tie-breaker among most constrained variables
- Most constraining variable (degree heuristic):
  - choose the variable with the most constraints on remaining variables

Least constraining value
Giá trị ràng buộc ít nhất

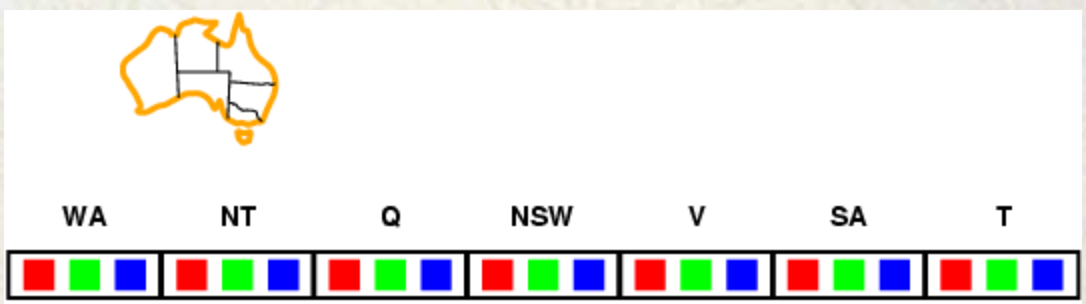
- Given a variable, choose the least constraining value:
  - the one that rules out the fewest values in the remaining variables



- Combining these heuristics makes 1000 queens feasible

Forward checking
Kiểm tra trước

- Idea:
  – Keep track of remaining legal values for unassigned variables
  – Terminate search when any variable has no legal values



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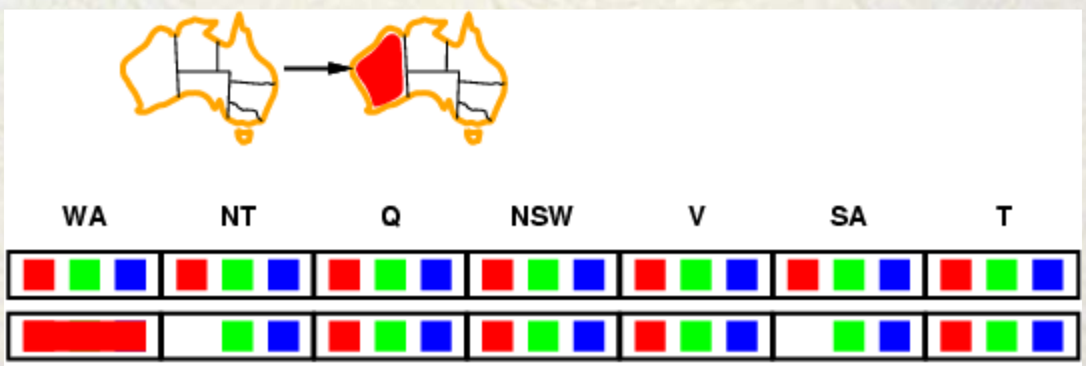

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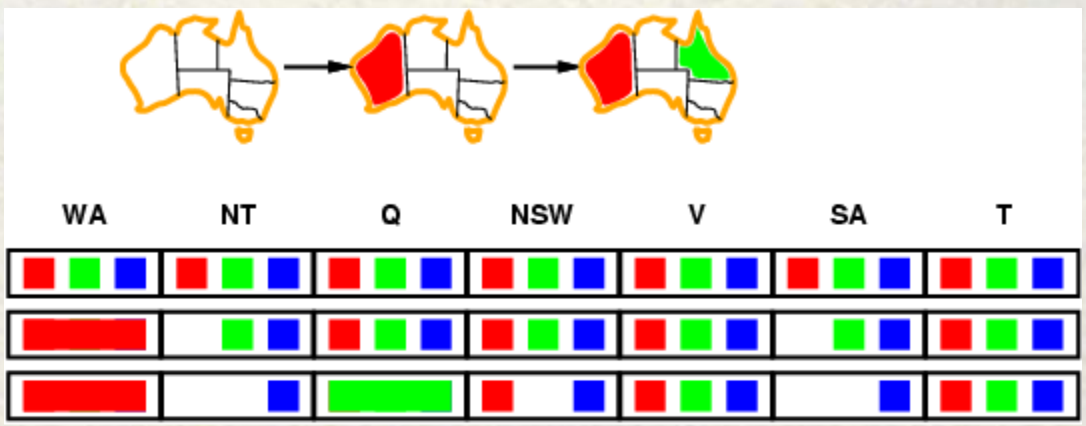

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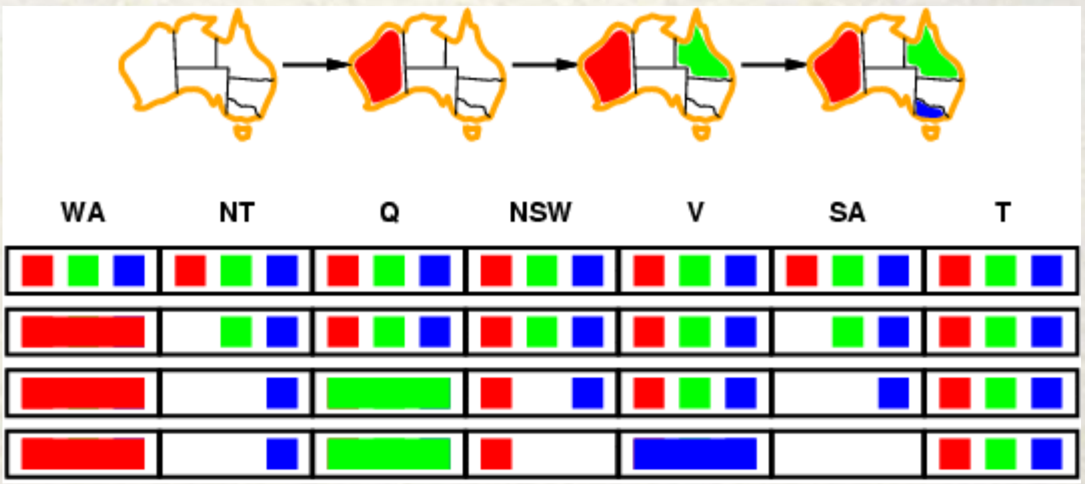
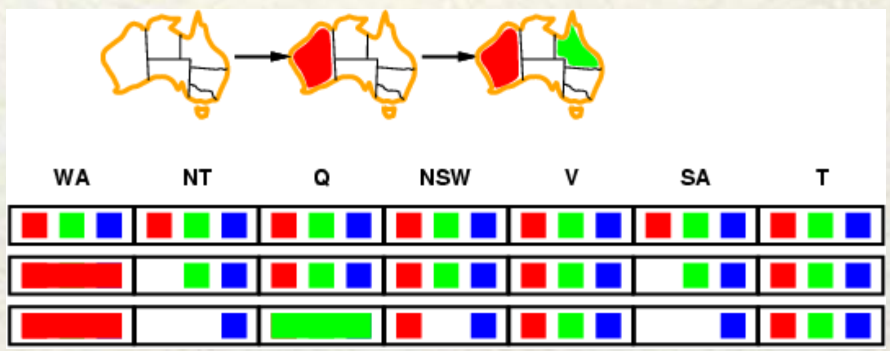


Constraint propagation

- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:

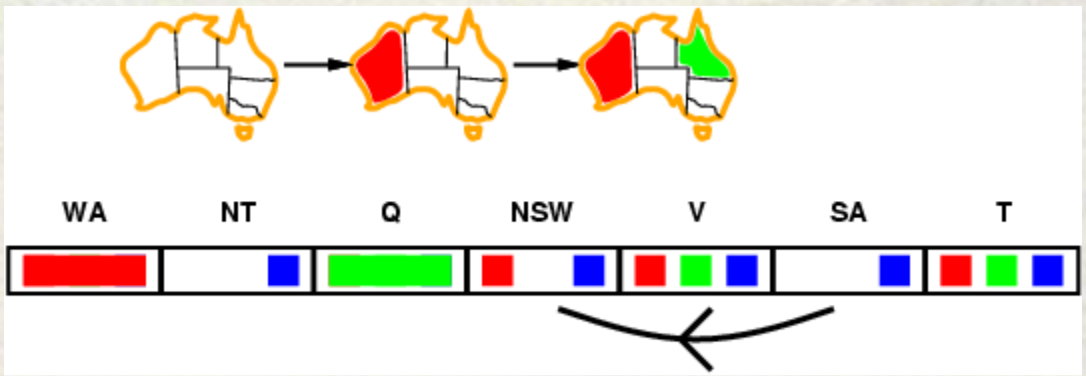


- NT and SA cannot both be blue!
- Constraint propagation repeatedly enforces constraints locally

Arc consistency

- Simplest form of propagation makes each arc consistent
- $X \rightarrow Y$ is consistent iff

  for every value x of X there is some allowed y



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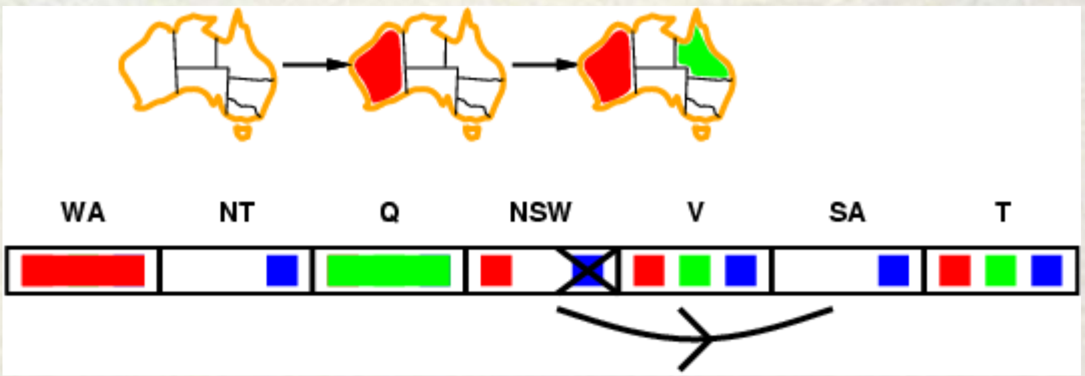

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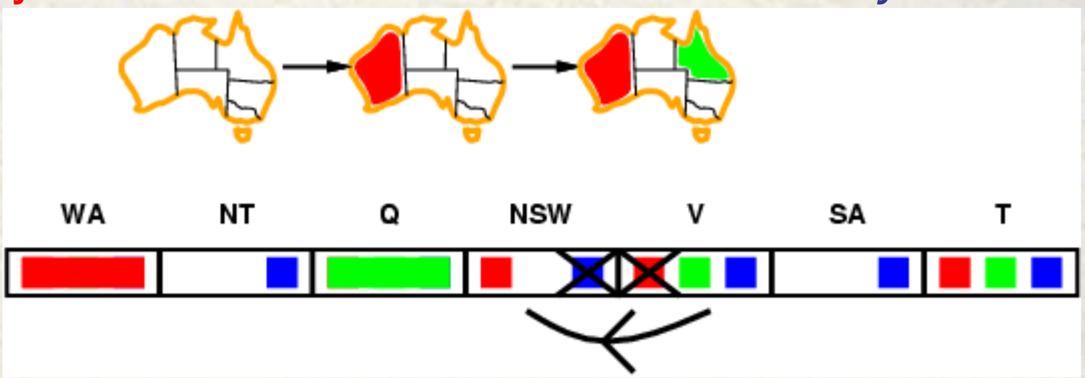
- *X* → *Y* is consistent iff

  for every value *x* of *X* there is some allowed *y*



- If *X* loses a value, neighbors of *X* need to be rechecked

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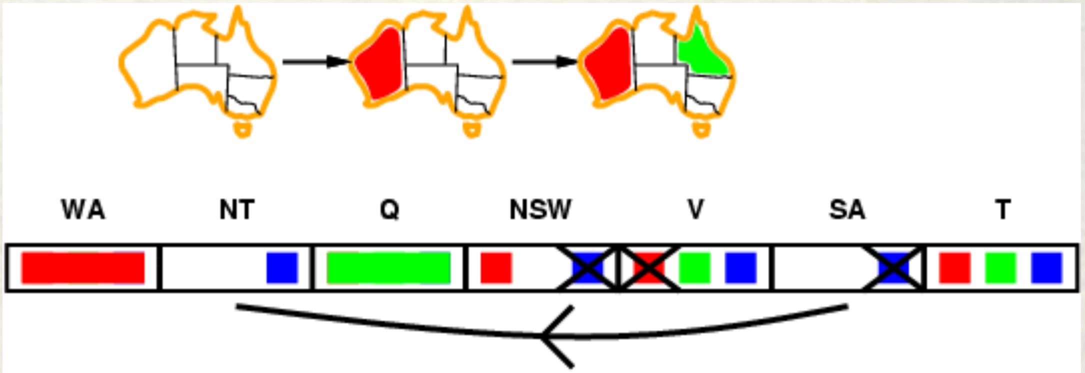


- If X loses a value, neighbors of X need to be rechecked
- Arc consistency detects failure earlier than forward checking
- Can be run as a preprocessor or after each assignment

Arc consistency algorithm AC-3

```
function AC-3( csp) returns the CSP, possibly with reduced domains
   inputs: csp, a binary CSP with variables {X1, X2, ..., Xn}
   local variables: queue, a queue of arcs, initially all the arcs in csp

   while queue is not empty do
      (Xi, Xj) ← REMOVE-FIRST(queue)
      if RM-INCONSISTENT-VALUES(Xi, Xj) then
         for each Xk in NEIGHBORS[Xi] do
            add (Xk, Xi) to queue
```

```
function RM-INCONSISTENT-VALUES( Xi, Xj) returns true iff remove a value
   removed ← false
   for each x in DOMAIN[Xi] do
      if no value y in DOMAIN[Xj] allows (x,y) to satisfy constraint(Xi, Xj)
         then delete x from DOMAIN[Xi];  removed ← true
   return removed
```

- Time complexity: $O(n^2d^3)$

Special constraints

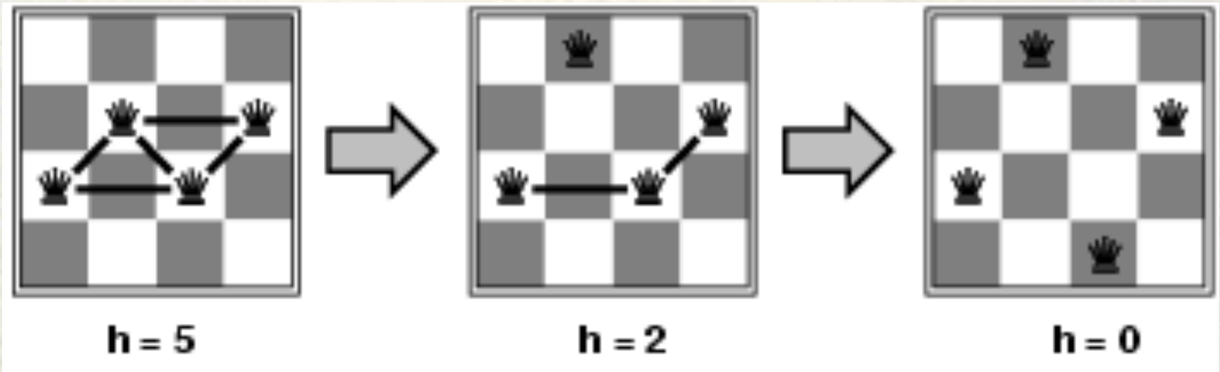
- Arc-consistency does miss some cases
- Example:
  - {WA=red, NSW=red}
  - AC-3: Domain for SA, NT, Q : {green, blue}
  - *Alldiff* constraint is violated as number of values is less than number of variables.

Local search for CSPs

- Local search or iterative improvement.
- Hill-climbing, simulated annealing typically work with "complete" states, i.e., all variables assigned
- To apply to CSPs:
  - allow states with unsatisfied constraints
  - operators reassign variable values
- Variable selection: randomly select any conflicted variable
- Value selection by min-conflicts (mâu thuẫn ít nhất) heuristic:
  - choose value that violates the fewest constraints
  - i.e., hill-climb with *h(n)* = total number of violated constraints

Example: 4-Queens

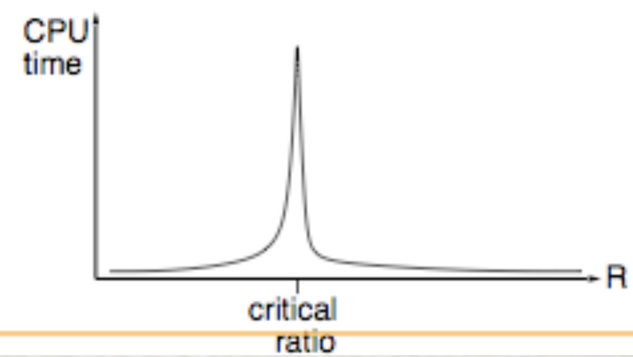
- States: 4 queens in 4 columns ($4^4 = 256$ states)
- Actions: move queen in column
- Goal test: no attacks
- Evaluation: *h(n)* = number of attacks



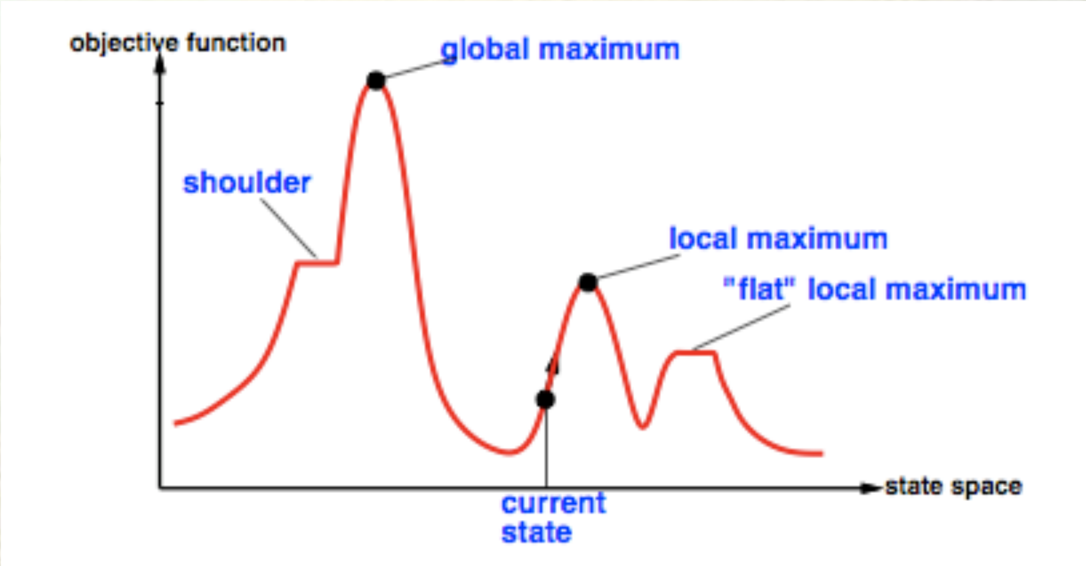
Phase transition in CSP' s

- Given random initial state, can solve *n*-queens in almost constant time for arbitrary *n* with high probability (e.g., *n* = 10,000,000)
- In general, randomly-generated CSP tend to be easy if there are very few or very many constraints. They become extra hard in a narrow range of the ratio:

$$R = \frac{\text{number of constraints}}{\text{number of variables}}$$



Flat regions and local optima



- Sometimes, have to go sideways or even backwards in order to make progress towards the actual solution.

Simulated Annealing

- Stochastic hill climbing based on difference between evaluation of previous state (h_0) and new state (h_1).
- If $h_1 < h_0$, definitely make the change.
- Otherwise, make the change with probability:

  $e^{-(h1-h0)/T}$,T is a “temperature” parameter
- Reduces to ordinary hill climbing when T=0.
- Become totally random search as T-> ∞
- We gradually decrease the value of T during the search.

Summary

- CSPs are a special kind of problem:
  - states defined by values of a fixed set of variables
  - goal test defined by constraints on variable values
- Backtracking = depth-first search with one variable assigned per node
- Variable ordering and value selection heuristics help significantly
- Forward checking prevents assignments that guarantee later failure
- Constraint propagation (e.g., arc consistency) does additional work to constrain values and detect inconsistencies
- Iterative min-conflicts is usually effective in practice
- Simulated Annealing can help to escape from local optima.

References

- Artificial Intelligence, A modern approach. Chapter 5.