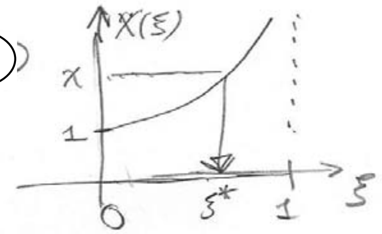
4.8)

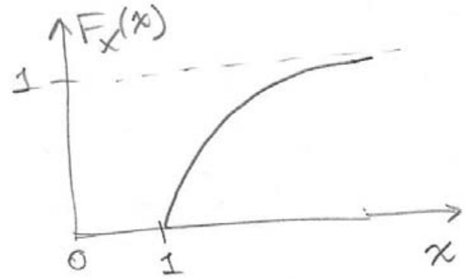


$$X(\xi) = \frac{1}{\sqrt{1-\xi^2}}$$

$$S_X = \{x : 1 \leq x < \infty\}$$

$$P[X(\xi) \leq x] = P\left[\frac{1}{\sqrt{1-\xi^2}} \leq x\right] = P\left[\frac{1}{1-\xi^2} \leq x^2\right]$$

$$= P\left[\frac{1}{x^2} \leq 1-\xi\right] = P\left[\xi \leq 1 - \frac{1}{x^2}\right] = 1 - \frac{1}{x^2}$$

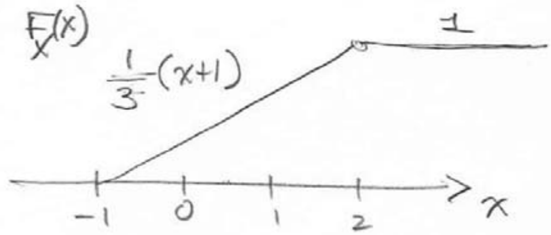


$$P[X > 10] = 1 - F_X(10)$$
$$= 1 - \left(1 - \frac{1}{10^2}\right) = \frac{1}{100}$$

$$P[5 < X < 7] = F_X(7) - F_X(5) = \left(1 - \frac{1}{49}\right) - \left(1 - \frac{1}{25}\right)$$
$$= \frac{1}{25} - \frac{1}{49} = 0.01959$$

$$P[X \leq 20] = 1 - \frac{1}{400} = 0.9975$$

4.11



$P[X<0] = F_X(0) = \frac{1}{3}$

$P\left[|X-\tfrac{1}{2}|<1\right] = P\left[-1 < X - \tfrac{1}{2} < 1\right] = P\left[-\tfrac{1}{2} < X < \tfrac{3}{2}\right]$

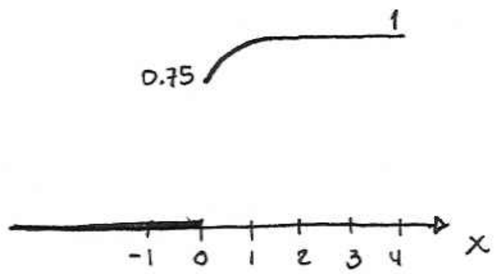
$= \frac{1}{3}\left(\frac{3}{2}+1\right) - \frac{1}{3}\left(-\frac{1}{2}+1\right)$

$= \frac{1}{3}\left(\frac{3}{2}+1+\frac{1}{2}-1\right) = \frac{2}{3}$

$P\left[X > -\tfrac{1}{2}\right] = 1 - P\left[X \le -\tfrac{1}{2}\right] = 1 - \frac{1}{3}\left(-\frac{1}{2}+1\right) = \frac{5}{6}$

4.13

a)



Mixed type random variable

b) $P[X \leq 2] = 1 - \frac{1}{4} e^{-2(2)}$

$= 0.9954$

$P[X = 0] = 1 - \frac{1}{4} e^{-2(0)}$

$= 0.75$

$P[X < 0] = 0$

$P[2 < X < 6] = P[X \leq 6] - P[X \leq 2]$

$= 1 - \frac{1}{4} e^{-2(6)} - 1 + \frac{1}{4} e^{-2(2)}$

$= 0.0046$

$P[X > 10] = 1 - P[X \leq 10]$

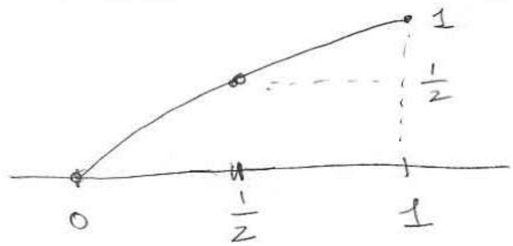
$= 1 - \left(1 - \frac{1}{4} e^{-2(10)}\right)$

$= 5.15 \times 10^{-10}$

4.16

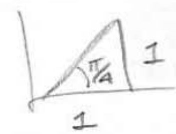
$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} + c \sin^2 \frac{\pi}{2} x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

$\sin^2 \frac{\pi}{2} x$



$\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

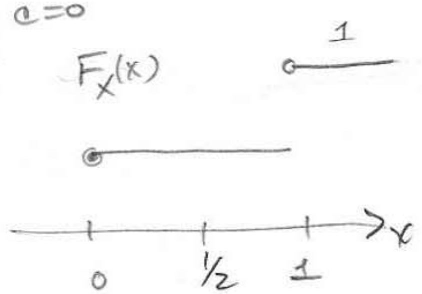
$\sin^2 \frac{\pi}{4} = \frac{1}{2}$



The constant c should be selected so properties of the cdf are satisfied: $c > 0$ so cdf is increasing

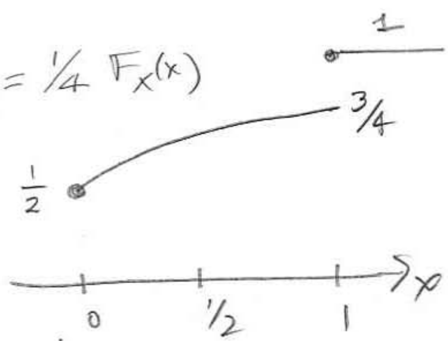
$c < \frac{1}{2}$ so cdf ≤ 1

$c = 0$



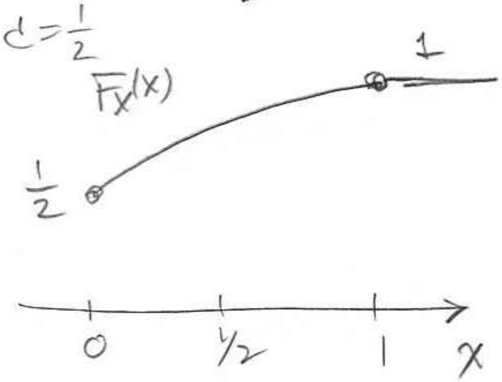
$P[X > 0] = 1 - P[X \le 0] = \frac{1}{2}$

$c = 1/4$



$P[X > 0] = \frac{1}{2}$

$c = \frac{1}{2}$



$P[X > 0] = \frac{1}{2}$

4.56

a) $E[Y] = 3E[X]+2$

$VAR[Y] = VAR[3X+2] = VAR[3X] = 9\,VAR[X]$

b) Laplacian R.V. $E[X]=0$

$VAR[X] = \frac{2}{\alpha^2}$

$E[Y] = 2$

$VAR[Y] = 9\left(\frac{2}{\alpha^2}\right) = \frac{18}{\alpha^2}$

c) Gaussian R.V. $E[X]=m$

$VAR[X]=\sigma^2$

$E[Y] = 3m+2$

$VAR[Y] = 9\sigma^2$

d) $E[X] = b\int_0^1 \cos(2\pi u)\,du = -b\sin(2\pi u)\Big|_0^1 = 0$

$$\begin{aligned} VAR[X] &= b^2\int_0^1 \cos^2(2\pi u)\,du \\ &= b^2\int_0^1 \tfrac{1}{2}du + \frac{b^2}{2}\int_0^1 \cos 4\pi u\,du \\ &= b^2\frac{1}{2} + b^2\left(\frac{1}{4\pi}\right)\left(-\sin 4\pi u\right)\Big|_0^1 \\ &= \frac{b^2}{2} \end{aligned}$$

$E[Y]=2$

$VAR[Y] = \frac{9b^2}{2}$

4.57

$$\begin{aligned} \mathcal{E}[X^n] &= \int_0^1 x^n dx = \frac{x^{n+1}}{n+1}\Big|_0^1 = \frac{1}{n+1} \\ \mathcal{E}[Y^n] &= \frac{1}{b-a}\int_a^b y^n dy = \frac{1}{b-a}\left[\frac{b^{n+1}-a^{n+1}}{n+1}\right] \end{aligned}$$

4.62 a)

$$\begin{aligned} P[X \le x] &= 1 - e^{-\lambda x} = \frac{r}{100} \\ 1 - \frac{r}{100} &= e^{-\lambda x} \\ \Rightarrow \pi(r) &= x = -\frac{1}{\lambda}\ln\left(1 - \frac{r}{100}\right) = \frac{1}{\lambda}\ln\left(\frac{100}{100-r}\right) \end{aligned}$$

$$\pi(90) \cong \frac{23}{\lambda} \qquad \pi(95) \approx \frac{3}{\lambda} \qquad \pi(99) \approx \frac{4.6}{\lambda}$$

b) $P[X \le x] = 1 - Q\left(\frac{x}{\sigma}\right)$ Using Tables 4.2 and 4.3:

$$1 - Q\left(\frac{x}{\sigma}\right) = 0.90 \quad \Rightarrow \tfrac{x}{\sigma} = 1.28 \quad \Rightarrow \pi(90) = 1.28\sigma$$
$$1 - Q\left(\frac{x}{\sigma}\right) = 0.95 \quad \Rightarrow \tfrac{x}{\sigma} \approx 1.5 \quad \Rightarrow \pi(95) \approx 1.5\sigma$$
$$1 - Q\left(\frac{x}{\sigma}\right) = 0.99 \quad \Rightarrow \tfrac{x}{\sigma} \approx 2.33 \quad \Rightarrow \pi(99) \approx 2.33\sigma$$

4.63

a) $P[X>4] = 1-F_X(4) = 1-\Phi\left(\frac{4-5}{4}\right) = 1-\Phi\left(-\frac{1}{4}\right) = \Phi\left(\frac{1}{4}\right) = 0.598$

$P[X\geq 7] = 1-F_X(7) = 1-\Phi\left(\frac{7-5}{4}\right) = 1-\Phi\left(\frac{1}{2}\right) = 0.308$

$P[6.72<X<10.16] = \Phi\left(\frac{10.16-5}{4}\right) - \Phi\left(\frac{6.72-5}{4}\right) = \Phi(1.29)-\Phi(0.43) = 0.235$

$P[2<X<7] = \Phi\left(\frac{7-5}{4}\right) - \Phi\left(\frac{2-5}{4}\right) = \Phi\left(\frac{1}{2}\right) - \Phi\left(-\frac{3}{4}\right) = 0.465$

$P[6\leq X\leq 8] = \Phi\left(\frac{8-5}{4}\right) - \Phi\left(\frac{6-5}{4}\right) = \Phi\left(\frac{3}{4}\right) - \Phi\left(\frac{1}{4}\right) = 0.175$

b) $P[X<a] = 0.8869$

$\Phi\left(\frac{a-5}{4}\right) = 0.8869 = 1-Q(x)$

$Q(x) = 0.1131 \rightarrow x = 1.2 = \frac{a-5}{4} \rightarrow a = 9.8$

c) $P[X>b] = 1-\Phi\left(\frac{b-5}{4}\right) = 0.11131$

$Q(x) = 0.11131 \rightarrow x = 1.2 = \frac{b-5}{4} \rightarrow b = 9.8$

d) $P[13<X\leq c] = 0.0123$

$\Phi\left(\frac{c-5}{4}\right) - \Phi\left(\frac{13-5}{4}\right) = \Phi\left(\frac{c-5}{4}\right) - \Phi(2) = 0.0123$

$\Phi\left(\frac{c-5}{4}\right) = 0.0123 + 0.9772 = 0.9895$

$Q\left(\frac{c-5}{4}\right) = 0.0105 \rightarrow x = 2.3 = \frac{c-5}{4} \rightarrow c = 14.2$

4.64

$$
\begin{aligned}
Q(-x) &= \frac{1}{\sqrt{2\pi}}\int_{-x}^{\infty} e^{t^2/2}dt = 1-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-x} e^{-t^2/2}dt \\
&= 1-\frac{1}{\sqrt{2\pi}}\int_{\infty}^{x} e^{-t'^2/2}(-dt') \text{ where } t'=-t \\
&= 1-\frac{1}{\sqrt{2\pi}}\int_{x}^{\infty} e^{-t'^2/2}dt' = 1-Q(x)
\end{aligned}
$$

4.65) To generate Q values

```
> x = [0:0.1:10];
> format short e
> 1-normal_cdf(x)
```

To generate $Q(x_k) = 10^{-k}$

$$10^{-k} = Q(x_k) = 1 - Q(-x) = F_X(-x_k)$$

$$x_k = -F_X^{-1}(10^{-k})$$

```
> k = [1:1:10];
> p = ones(1,10)/10;
> p2 = p.^k;
> -normal_inv(p2)
```

4.66

$$P[X < m] = P[X \leq m] = \Phi\left(\frac{n-m}{\sigma}\right) = \Phi(0) = \frac{1}{2}$$

$$\begin{aligned} P[|X-m| > k\sigma] &= 1 - P[-k\sigma + m \leq X \leq m + k\sigma] \\ &= 1 - \left(\Phi\left(\frac{m+k\sigma-m}{\sigma}\right) - \Phi\left(\frac{m-k\sigma-m}{\sigma}\right)\right) \\ &= \underbrace{1-\Phi(k)}_{Q(k)} + \underbrace{\Phi(-k)}_{Q(k)} \\ &= \quad Q(k) \quad + \quad Q(k) \quad = 2Q(k) \end{aligned}$$

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	from 4.2
$2Q(k)$	0.318	$4.56(10^{-2})$	$4.05(10^{-3})$	$6.34(10^{-5})$	$5.74(10^{-7})$	Table ~~3.3~~

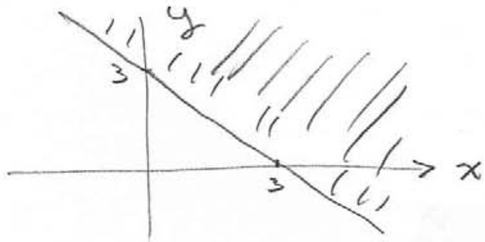
$$P[X > m + k\sigma] = Q\left(\frac{m+k\sigma-m}{\sigma}\right) = Q(k)$$

	$k=1.28$	$k=3.09$	$k=4.26$	$k=5.20$	from 4.3
$Q(k)$	$\approx 10^{-1}$	$\approx 10^{-3}$	$\approx 10^{-5}$	$\approx 10^{-7}$	Table ~~3.4~~

5.8

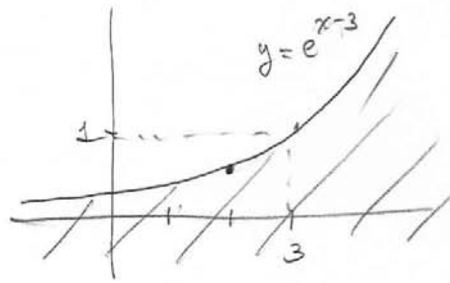
a) $\{X+Y>3\} = \{Y>3-X\}$

Not product form



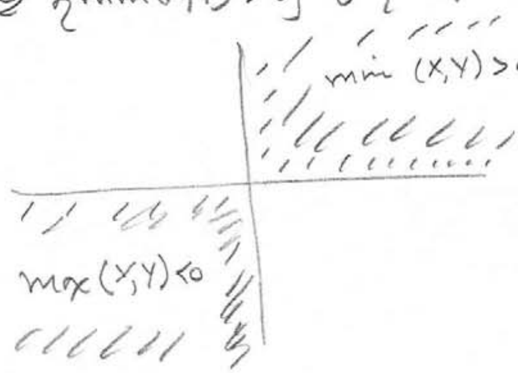
b) $\{e^X > Ye^3\}$

$= \{Y < e^{X-3}$



Not product form

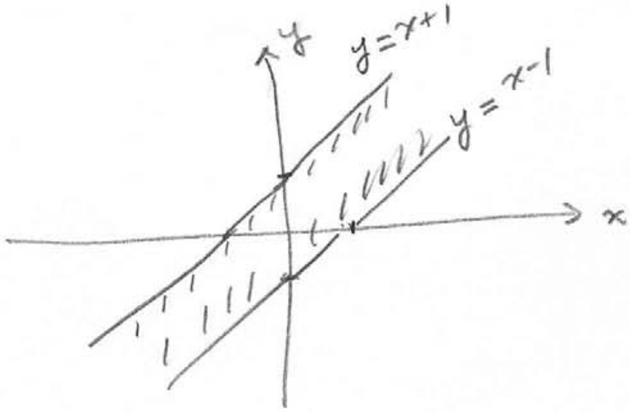
c) $\{\min(X,Y) > 0\} \cup \{\max(X,Y) < 0\}$



Not product form but union of product form

d) $\{|X-Y| \geq 1\} = \{-1 \leq X-Y \leq 1\}$

$= \{Y-1 \leq X \leq Y+1\}$



Not product form

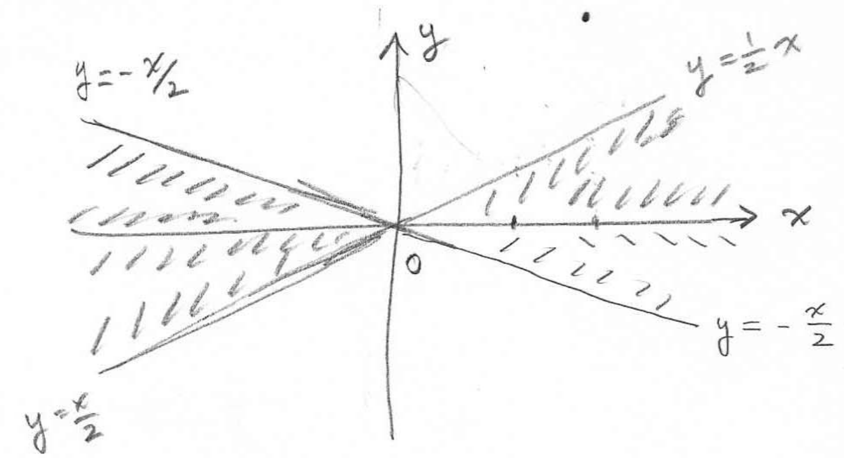
(e) $\{ |X/Y| > 2 \}$

$-\frac{x}{y} > 2 \iff -x/2 > y$

$x/y > 2 \iff \frac{1}{2}x > y$

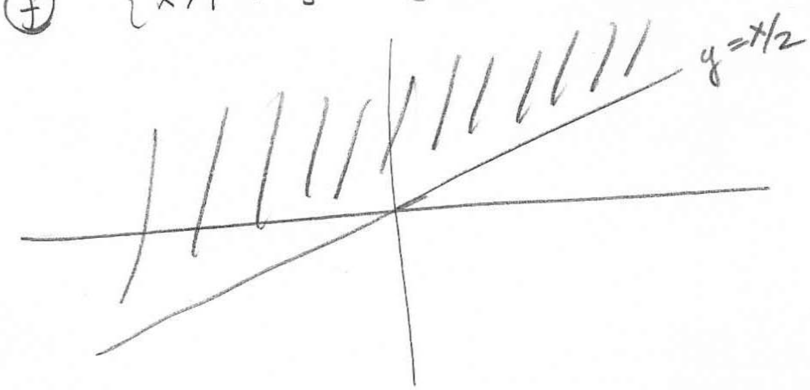
$x/y > 2 \iff \frac{x}{2} < y$

$-x/y > 2 \iff -\frac{x}{2} < y$



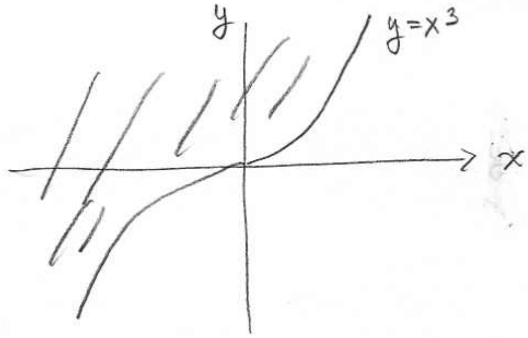
Not product form.

(f) $\{ X/Y < 2 \} = \{ X/2 < Y \}$



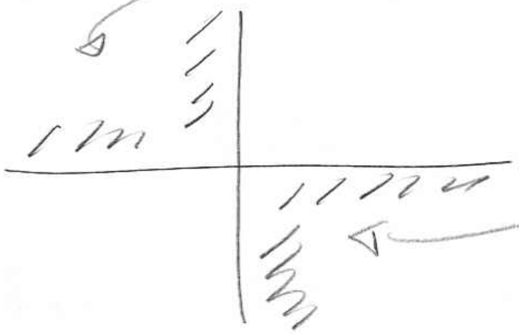
Not product form.

(g) $\{X^3 > Y\}$



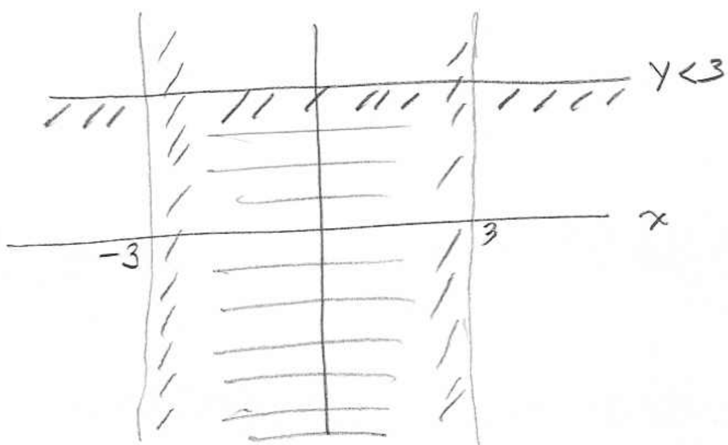
Not product form.

(h) $\{XY < 0\} = \{X < 0, Y > 0\} \cup \{X > 0, Y < 0\}$



Not product form but union of product form.

(i) $\max\{(|X|, Y) < 3\} = \max\{(-X, X, Y) < 3\}$
$= \{-X < 3, X < 3, Y < 3\}$



product form

5.11

(i)

X \ Y	−1	0	1	
−1	1/6	1/6	0	1/3
0	0	0	1/3	1/3
1	1/6	1/6	0	1/3
	1/3	1/3	1/3	

$P[X=i]=\frac{1}{3} \quad i\in\{-1,0,1\}$

$P[Y=i]=\frac{1}{3} \quad i\in\{-1,0,1\}$

$P[X>0]=\frac{1}{3}$

$P[X\geq Y]=\frac{1}{2}$

$P[X=-Y]=1/6$

(ii)

X \ Y	−1	0	1	
−1	1/9	1/9	1/9	1/3
0	1/9	1/9	1/9	1/3
1	1/9	1/9	1/9	1/3
	1/3	1/3	1/3	

$P[X=i]=\frac{1}{3} \quad i\in\{-1,0,1\}$

$P[Y=i]=\frac{1}{3} \quad i\in\{-1,0,1\}$

$P[X>0]=\frac{1}{3}$

$P[X\geq Y]=\frac{2}{3}$

$P[X=-Y]=\frac{1}{3}$

(iii)

X \ Y	−1	0	1	
−1	1/3	0	0	1/3
0	0	1/3	0	1/3
1	0	0	1/3	1/3
	1/3	1/3	1/3	

$P[X=i]=\frac{1}{3} \quad i\in\{-1,0,1\}$

$P[Y=i]=\frac{1}{3} \quad i\in\{-1,0,1\}$

$P[X>0]=\frac{1}{3}$

$P[X\geq Y]=1$

$P[X=-Y]=1/3$

Three different joint pmf's have the same marginal pmf's.

Events that involve joint behavior have different probabilities.