CHAPTER 2: Probability

Sample Space: 2.1

A random experiment is an experiment in which the outcome varies in an unpredictable fashion when the experiment is repeated under the same conditions.

Examples

- Experiment E_1: Toss a coin three times and note the sequence of heads and tails.
- Experiment E_2: Toss a coin three times and note the number of heads
- Experiment E_3: A block of information is transmitted repeatedly over a noisy channel until an error-free block arrives at the receiver. Count the number of transmissions required.
- Experiment E_4: Measure the lifetime of a given computer memory chip in a specified environment.

Since random experiments do not consistently yield the same result, it is necessary to determine what the set of possible results can be.

The sample space S of a random experiment is defined as the set of all possible outcomes.

Note: When we perform a random experiment, one and only one outcome occurs.

The samples spaces corresponding to the experiments in the last Example are given below:

- Experiment E_1: Toss a coin three times and note the sequence of heads and tails.

- Experiment E_2: Toss a coin three times and note the number of heads

- Experiment E_3: A block of information is transmitted repeatedly over a noisy channel until an error-free block arrives at the receiver. Count the number of transmissions required.

- Experiment E_4: Measure the lifetime of a given computer memory chip in a specified environment.

Random experiments involving the same experimental procedure may have different sample spaces as shown by Experiments E_1 and E_2.

A tree diagram:
Example: The three balls numbered 1 to 3 in an urn are drawn at random one at a time until the urn is empty. The sequence of the ball numbers is noted. Find the sample space.

Events: 2.2

We are usually not interested in the occurrence of specific outcomes, but rather on the occurrence of some event (i.e. whether the outcome satisfies certain conditions).

Example:
Experiment: Determine the value of a voltage waveform at time t_1
$S = (-\infty, \infty)$.
We might be interested in the event "voltage is negative" which corresponds to $(-\infty, 0)$.

The event occurs if and only if the outcome of the experiment is in this subset.

For this reason we define an **event** as a subset of S

Example:
Experiment E_3: A block of information is transmitted repeatedly over a noisy channel until an error-free block arrives at the receiver. Count the number of transmissions required.
$S_3 =$
$A =$ Fewer than 10 transmissions are required $=$

Ex. 10 on p. 30: An engineering firm is hired to determine if certain waterways in Virginia are safe for fishing. Samples are taken from three rivers.
(a) List the elements of a sample space S, using the letters F for "safe to fish" and N for "not safe to fish".
(b) List the elements of S corresponding to event E that at least two of the rivers are safe for fishing.
(c) Define an event that has as its elements the points {FFF, NFF, FFN, NFN}.

Set Operations

- The **union** of two events A and B, denoted by $A \cup B$, is defined as the set of outcomes that are either in A or in B, or both.

  The event $A \cup B$ occurs if either A, or B, or both A and B occur.

- The **intersection** of two events A and B, denoted by $A \cap B$, is defined as the set of outcomes that are common to A and B.

- The **complement** of an event A, denoted by A', is defined as the set of all outcomes not in A. The event A' occurs when the event A does not occur and vice versa.

Example: A deck of playing contains 52 cards. We perform the experiment of randomly selecting one card from the deck.

$$S = \text{The collection of all 52 cards}$$

Let us consider the following 4 events

$$A = \text{The card selected is the king of hearts}$$

$$B = \text{The card selected is a king}$$

$$C = \text{The card selected is a heart}$$

$$D = \text{The card selected is a face card}$$

How many outcomes comprising each of these four events?

Solution:
A:
B:
C:
D:

Determine D', $B \cap C$, $B \cup C$, $C \cap D$

Counting Sample Points: 2.3

The generalized multiplication rule: If an operation can be performed in n_1 ways, and if for each of these a second operation can be performed in n_2 ways, and for each of the first two a third operation can be performed in n_3 ways, and so forth, then the sequence of k operations can be performed in $n_1 n_2 \dots n_k$ ways.

Example: A six-sided die is tossed, a coin is flipped, and a card is selected at random from a deck of 52 distinct cards. Find the number of possible outcomes.

Ex. 4 on p. 38: Students at a private liberal arts college are classified as being freshmen, sophomores, juniors, or seniors, and also according to whether they are male or female. Find the total number of possible classifications for the students of that college.

A permutation is an arrangement of all or part of a set of objects.
The number of permutations of n distinct objects is $n!$.

Example: Find the number of permutations of three distinct objects $\{1, 2, 3\}$.

Solution:

Example: What is the number of permutations of the three letters a, b, c if we take only two of the three at the time.

Solution:

The number of permutations of n distinct objects taken r at a time is

$$n^P r = n \cdot (n-1) \ldots \cdot (n-r+1) =$$

$$\frac{n \cdot (n-1) \ldots \cdot (n-r+1) \cdot (n-r)!}{(n-r)!} =$$

$$\frac{n!}{(n-r)!}$$

Example: Find the number of ways of selecting a president, a vice president, a secretary and a treasure in a club consisting of 10 persons.

Solution:

Combinations: In many problems we are interested in the number of ways of selecting r objects from n without regard to order. These selections are called combinations.

Example: What is the number of combinations of the three letters a, b, c if we take only two of the three at the time.

The number of combinations of n distinct objects taken r at a time is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Example: Among the seven nominees for two vacancies on a city council are 3 men and 4 women. In how many ways can these vacancies be filled?
(a) with any two of the seven nominees

(b) with any two of the four women

(c) with one of the men and one of the women

Probability of an event

Let E be a random experiment with sample space S. A probability law for the experiment E is a rule that assigns to each event A a number $p(A)$, called the probability of A, that satisfies the following axioms:

$$0 \leq P(A) \leq 1, \quad P(\Phi) = 0, \quad \text{and} \quad P(S) = 1$$

Furthermore if $A_1, A_2, A_3, \ldots$ is a sequence of mutually exclusive events then

$$P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + P(A_3) + \ldots$$

Example: Suppose that a coin is tossed three times. If we observe the sequence of heads and tails, then there are eight possible outcomes

$$S =$$

$$\{\text{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT}\}.$$

If we assume that the outcomes of S are equiprobable, then the probability of each of the eight elementary events is 1/8. Let A be the event of obtaining two heads in three tosses.

$$p(A) = p[\{\text{HHT, HTH, THH}\}] = \frac{3}{8}.$$

Theorem 2.9 on p. 41: If an experiment can result in any one of the N different equally likely outcome, and if exactly n of these outcomes correspond to event A, the the probability of event A is

$$P(A) = \frac{n}{N}$$

Example: A batch of 6 items contains 4 defective items. Suppose 3 items are selected at random and tested. What is the probability that exactly 2 of the items tested are defective?

Example:
A group of three undergraduate and five graduate students are available to fill certain student government posts. If four students are to be randomly selected from this group, find the probability that exactly two undergraduates will be among the four chosen.

Solution

Example: The probabilities that a secretary will make 0, 1, 2, 3, 4, or 5 or more mistake in typing a recent report are, respectively, 0.12, 0.25, 0.36, 0.14, 0.09, 0.04
Let $A =$ the secretary is making at most 2 mistakes.
Let $B =$ the secretary is making at least 4 mistakes.
Find $P(A \cup B)$

Solution

Additive Rules: 2.5

If A and B are any two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Example: For married couples living in a certain suburb, the probability that the husband will vote in a school board elections is 0.21, the probability that the wife will vote in the election is 0.28, and the probability that they will both vote is 0.15. What is the probability that at least one will vote.

Solution

Example: Disease I and II are prevalent among people in a certain population. It is assumed that 10% of the population will contract disease I sometime during their lifetime, 15% will contract disease II eventually, and 3% will contract both diseases. Find the probability that a randomly chosen person from this population will contract at least one disease.

If A and A' are complementary events, then

$$P(A) + P(A') = 1$$

Ex. 6 on p. 46: From past experience a stockbroker believes that under present economic conditions a customer will invest in tax-free bonds with a probability of 0.6, will invest in mutual funds with a probability of 0.3, and will invest in both tax-free bonds and mutual funds, with a probability of 0.15. At this time, find the probability that a customer will invest
(a) in either tax-free bonds or mutual funds;
(b) in neither tax-free bonds nor mutual funds.

Solution

Consider the events
B: customer invests in tax free bonds
M: customer invests in mutual funds

Conditional Probability: 2.6

Example:
The following table contains counts (in thousands) of persons aged 16 to 24 who are enrolled in school classified by gender and employment status.

	Employed	Unemployed	Not in labor force	Total
Male	3927	520	4611	9058
Female	4313	446	4357	9116
Total	8240	966	8968	18174

(a) Randomly choose a person aged 16 to 24 who is enrolled in school. What is the probability the the person is employed?

(b) Now we are told that the person is female. What is the probability the person is employed, given the information that the person is female?

Quite often we are interested in determining whether two events, A and B, are related in the sense that knowledge about the occurrence of one, say B, alters the likelihood of occurrence of the other, A. This requires that we find the **conditional probability**, $P(A|B)$, of event A given that event B has occurred. The conditional probability is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{for} \quad P(B) > 0$$

Example
An experiment is conducted to examine the relationship between cigarette smoking and cancer. The individuals are randomly selected from the male population of a certain section of the United States. The results are summarized as follows:

	Developing Cancer	
Smoker	Yes	No
Yes	0.05	0.20
No	0.03	0.72

Find $P(\text{developing cancer}|\text{smoker})$

Example: Call a household prosperous if its income exceeds $\$100,000$. Call the household educated if the household completed college. Select an American household at random, and let A be the event that the selected household is prosperous and let B be the event that it is educated. According to the Current Population Survey, $P(A) = 0.134$, $P(B) = 0.254$, $P(A \cap B) = 0.080$.

(a) Find the conditional probability that a household is educated, given that it is prosperous.
(b) Find the conditional probability that a household is prosperous, given that it is educated.

The multiplicative rule:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{for} \quad P(B) > 0 \Rightarrow$$

$$P(A \cap B) = P(A|B)P(B)$$

Example:
A focus group of 10 consumers has been selected to view a new TV commercial. After the viewing, 2 members of the focus group will be randomly selected and asked to answer detailed questions about the commercial. The group contain 4 men and 6 women. What is the probability that the 2 chosen to answer questions will both be women?

P(first person is female) =

P(second person is female | first person is female) =

p(both people are female) =

Example: A box has three tickets, colored red, white and blue. Two tickets will be drawn at random without replacement. What is the chance of drawing the red and then the white?

Example: If we randomly pick two television tubes in succession from a shipment of 240 television tubes of which 15 are defective, what is the probability that they will both be defective?

Solution:

The multiplication rule can be extended:

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2)$$

Example A box of fuses contains 20 fuses, of which 5 are defective. If 3 of the fuses are selected at random, what is the probability that all three fuses are defective?

Solution:

Example: Only 5% of male high school basketball, baseball, and football players go on to play at the college level. Of these, only 1.7% enter major league professional sports. About 40% of the athletes who compete in college and then reach the pros have a career of more than 3 years. What is the probability that a high school athlete competes in college and then goes on to have a pro career of more than 3 years?

Solution
Let
$A = \{\text{competes in college}\}$
$B = \{\text{competes professionally}\}$
$C = \{\text{pro career longer than 3 years}\}$
Then

We now know how to compute conditional probabilities from unconditional probabilities.

Independence of events

Two events A and B are independent if

$$P(A|B) = P(A) \quad \text{and} \quad P(B|A) = P(B)$$

which implies that

$$P(A \cap B) = P(A)P(B)$$

Example: Call a household prosperous if its income exceeds $100,000. Call the household educated if the household completed college. Select an American household at random, and let A be the event that the selected household is prosperous and let B be the event that it is educated. According to the Current Population Survey, $P(A) = 0.134$, $P(B) = 0.254$, $P(A \cap B) = 0.080$. Are events A and B independent?

Solution:

A question from a test
Three radar sets, operating independently, are set to detect nay aircraft flying through a certain area. Each set has a probability of 0.02 of failing to detect a plane in its area. If an aircraft enters the area, what is the probability that it goes undetected?

(a) $(0.02)^3$
(b) $1 - (0.98)^3$
(c) $(0.98)^3$
(d) $1 - (0.02)^3$

What conditions should three events A, B, and C satisfy in order for them to be independent? First, they should be pairwise independent, that is,

$$P(A \cap B) = P(A)P(B), \quad P(A \cap C) = P(A)P(C),$$

$$P(B \cap C) = P(B)P(C).$$

In addition, knowledge of the joint occurrence of any two, say A and B, should not affect the probability of the third, that is,

$$P(C|A \cap B) = P(C)$$

In order for this to hold, we must have

$$P(C|A \cap B) = \frac{P(A \cap B \cap C)}{P(A \cap B)} = P(C).$$

Thus in turn implies that we must have

$$P(A \cap B \cap C) = P(A \cap B)P(C) = P(A)P(B)P(C)$$

Example: It is known that a patient will respond to treatment of a particular disease with a probability equal to 0.9. If 3 patients are treated independently, find the probability that
(a) None of the patient will respond.
(b) At least one will respond

Ex. 22 on p. 56: Suppose an electrical system is given in the diagram below. What is the probability that the system works?

More about the multiplication rule

$$P(A \cap B) = P(A|B)P(B)$$

Example: According to the Arizona Chapter of the American Lung Association, 7.0% of the population has lung disease. Of those having lung disease, 90% are smokers; of those not having lung disease, 25.3% are smokers.
(a)Determine the probability that a randomly selected person is a smoker and has lung disease.
(b) Determine the probability that a randomly selected person is a smoker and has no lung disease.

Solution

$$S = \{\text{the person selected is a smoker}\}$$

$$L = \{\text{the person selected has lung disease}\}$$

The theorem of total probability

$$P(A) = p((E \cap A) \cup (E' \cap A)) = p(E \cap A) + P(E' \cap A) =$$
$$= P(E)P(A|E) + P(E')P(A|E')$$

Example: According to the Arizona Chapter of the American Lung Association, 7.0% of the population has lung disease. Of those having lung disease, 90% are smokers; of those not having lung disease, 25.3% are smokers. Determine the probability that a person selected randomly is a smoker.

Solution:

Generalization of the theorem of total probability:
Example 2.38: In a certain assembly plant, three machines, B_1, B_2 and B_3, make 30%, 45% and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective. Now,suppose a finished product is randomly selected. What is the probability that it is defective?

Solution: Consider the following events:
A: the product is defective,
B_i the product is made by machine B_i, $i = 1, 2, 3$.

Bayes' Rule

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

It is common to think of Bayes rule in terms of updating our belief about a hypothesis A in the light of a new evidence B.

Example 1: Suppose we are interested in diagnosing cancer in patients who visit a chest clinic.
Let C represent the event "Person has cancer"
Let S represent represent the event "Person is a smoker"

- We know the probability of the prior event $P(C) = 0.1$ on the basis of past data (10% of the patients entering the clinic turn out to have cancer).

- We know $P(S|C)$ by checking from our record the proportion of smokers among those diagnosed with cancer. Suppose $P(S|C) = 0.8$

- We know $P(S|C')$ by checking from our record the proportion of smokers among those not diagnosed with cancer. Suppose $P(S|C') = 0.466$

$P(S) = (0.8)(0.1) + (0.466)(0.9) = 0.5$
$P(C|S) = \frac{(0.8)(0.1)}{0.5} = 0.16$
Thus, in light of the evidence that the person is a smoker we revise our prior probability from 0.1 to a posterior probability of 0.16. This is a significance increase, but it is still unlikely that the person has cancer.

Example 2:
You ask your neighbor to water a sickly plane while you are on vacation. Without water it will die with probability .8; with water it will die with probability .15. You are 90 percent certain that your neighbor will remember to water the plant.

(a) What is the probability that the plant will be alive when you return?

(b) If it is dead, what is the probability your neighbor forgot to water it?

Example3: According to the Arizona Chapter of the American Lung Association, 7.0% of the population has lung disease. Of those having lung disease, 90% are smokers; of those not having lung disease, 25.3% are smokers. Determine the probability that a randomly selected smoker has lung disease.

Solution:

Ex 15. on p. 62: A certain federal agency employs three consulting firms (A, B, and C) with probabilities 0.40, 0.35, and 0.25, respectively. From past experience it is known that the probability of cost overruns for the firms are 0.05, 0.03, and 0.15 respectively. Suppose a cost overrun is experienced by the agency.
(a) What is the probability that the consulting firm involved is company C?
(b) What is the probability that it is company A?

We now know how to use Bayes' rule to compute the probability that an event is due to a particular cause.

More

1. A history class contain 7 male students and 5 female students. Find the number of ways that the class can elect:
(a) a class representative
(b) two class representatives, one male and one female
(c) a president and a vice-president.

2. A class contains 10 students with 6 men and 4 women. Find the number of ways:
(a) a 4-member committee can be selected from the students,
(b) a 4-member committee with 2 men and 2 women can be selected.
(c) the class can select a president, vice-president, treasurer, and secretary.

3. In a large city, 70% of the households receive a daily newspaper, and 50% of those who receive a daily newspaper have a television set. What is the probability that a randomly selected household will be one that receives a daily newspaper and has a television set?

4. A bin contains 100 balls, of which 25 are red, 40 are white, and 35 are black. If two balls are selected from the bin without replacement, what is the probability that one will be red and one will be white?

5. There are 90 applications for a job with the news department of a television station. Some of them are college graduates and some are not, some of them have at least three years' experience and some have not, with the exact breakdown being:

	College graduates	Not College graduates
At least three years' experience	18	9
Lee than three years' experience	36	27

In the order in which the applicants are interviewed by the station manager is random, G is the event that the first applicant interviewed is a college graduate, and T is the event that the first applicant interviewed has at least three years' experience, determine

$$P(G), \qquad P(T') \qquad P(G \cap T) \qquad P(T|G) \qquad P(G'|T')$$

Questions from Test 1, Spring 2005

1. A lot of 10 components contains 3 that are defective. Two components are drawn at random and tested. Let A be the that the first component drawn is defective and let B be the event that the second component is defective.
   (a) Find $P(A \cap B)$
   (b) Find $P(B|A)$
   (c) Find $P(A' \cap B)$
   (d) Are A and B independent? Explain.
   (e) Are A and B mutually exclusive? Explain.
2. Amy commutes to work by two different routes A and B. If she comes home by route A, then she will be home no later than 6 P.M. with probability 0.8, but if she comes home by route B, then she will be home no later 6 P.M. with probability 0.7. In the past, the proportion of times that Amy chose route A is 0.4.
   (a) What proportion of times is Amy home no later than 6 P.M?
   (b) If Amy is home after 6 P.M. today, what is the probability that she took route B?
3. Let E_1 denote the event that s structural component fails during a test and E_2 denote the event that the component shows some strain, but does not fail. Given $P(E_1) = 0.15$ and $P(E_2) = 0.30$,
   (a) What is the probability that a structural component does not fail during a test?
   (b) What is the probability that a component neither fails or shows strain during a test
4. A system contains two components, C and D, connected in parallel. Assume C and D function independently. For the system to function, either C or D must function.
   (a) If the probability that C fails is 0.08 and the probability that D fails is 0.12, find the probability that the system function.
   (b) If both C and D have probability p of failing, what must the value of p be so that the probability the system function is 0.99?
5. Suppose that the probability that a person picked at random has lung cancer is 0.035 and the probability that the person has lung cancer and is a heavy smoker is 0.014. Given that someone picked at random has lung cancer, what is the probability that the person is a heavy smoker?