Solutions to HW4

Note: Most of these solutions were generated by R. D. Yates and D. J. Goodman, the authors of our textbook. I have added comments in italics where I thought more detail was appropriate.

Problem 2.2.6 ■

You are manager of a ticket agency that sells concert tickets. You assume that people will call three times in an attempt to buy tickets and then give up. You want to make sure that you are able to serve at least 95% of the people who want tickets. Let p be the probability that a caller gets through to your ticket agency. What is the minimum value of p necessary to meet your goal.

Problem 2.2.6 Solution

The probability that a caller fails to get through in three tries is $(1-p)^3$. To be sure that at least 95% of all callers get through, we need $(1-p)^3 \leq 0.05$. This implies $p = 0.6316$.

Problem 2.2.7 ■

In the ticket agency of Problem 2.2.6, each telephone ticket agent is available to receive a call with probability 0.2. If all agents are busy when someone calls, the caller hears a busy signal. What is the minimum number of agents that you have to hire to meet your goal of serving 95% of the customers who want tickets?

Problem 2.2.7 Solution

In Problem 2.2.6, each caller is willing to make 3 attempts to get through. An attempt is a failure if all n operators are busy, which occurs with probability $q = (0.8)^n$. Assuming call attempts are independent, a caller will suffer three failed attempts with probability $q^3 = (0.8)^{3n}$. The problem statement requires that $(0.8)^{3n} \leq 0.05$. This implies $n \geq 4.48$ and so we need 5 operators.

Problem 2.2.9 ♦

When someone presses "SEND" on a cellular phone, the phone attempts to set up a call by transmitting a "SETUP" message to a nearby base station. The phone waits for a response and if none arrives within 0.5 seconds it tries again. If it doesn't get a response after $n = 6$ tries the phone stops transmitting messages and generates a busy signal.

(a) Draw a tree diagram that describes the call setup procedure.

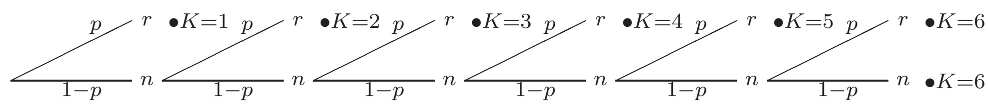
(b) If all transmissions are independent and the probability is p that a "SETUP" message will get through, what is the PMF of K, the number of messages transmitted in a call attempt?

(c) What is the probability that the phone will generate a busy signal?

(d) As manager of a cellular phone system, you want the probability of a busy signal to be less than 0.02 If $p = 0.9$, what is the minimum value of n necessary to achieve your goal?

Problem 2.2.9 Solution

(a) In the setup of a mobile call, the phone will send the "SETUP" message up to six times. Each time the setup message is sent, we have a Bernoulli trial with success probability p. Of course, the phone stops trying as soon as there is a success. Using r to denote a successful response, and n a non-response, the sample tree is



(b) We can write the PMF of K, the number of "SETUP" messages sent as

$$P_K(k) = \begin{cases} (1-p)^{k-1}p & k = 1, 2, \ldots, 5 \\ (1-p)^5 p + (1-p)^6 = (1-p)^5 & k = 6 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Note that the expression for $P_K(6)$ is different because $K = 6$ if either there was a success or a failure on the sixth attempt. In fact, $K = 6$ whenever there were failures on the first five attempts which is why $P_K(6)$ simplifies to $(1-p)^5$.

(c) Let B denote the event that a busy signal is given after six failed setup attempts. The probability of six consecutive failures is $P[B] = (1-p)^6$.

(d) To be sure that $P[B] \leq 0.02$, we need $p \geq 1 - (0.02)^{1/6} = 0.479$.

Problem 2.3.10 ■

A radio station gives a pair of concert tickets to the sixth caller who knows the birthday of the performer. For each person who calls, the probability is 0.75 of knowing the performer's birthday. All calls are independent.

(a) What is the PMF of L, the number of calls necessary to find the winner?

(b) What is the probability of finding the winner on the tenth call?

(c) What is the probability that the station will need nine or more calls to find a winner?

Problem 2.3.10 Solution

(a) We can view whether each caller knows the birthdate as a Bernoulli trial. As a result, L is the number of trials needed for 6 successes. That is, L has a Pascal PMF with parameters $p = 0.75$ and $k = 6$ as defined by Definition 2.8. In particular,

$$P_L(l) = \begin{cases} \binom{l-1}{5}(0.75)^6(0.25)^{l-6} & l = 6, 7, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(b) The probability of finding the winner on the tenth call is

$$P_L(10) = \binom{9}{5}(0.75)^6(0.25)^4 \approx 0.0876 \tag{2}$$

(c) The probability that the station will need nine or more calls to find a winner is

$$\begin{aligned} P[L \geq 9] &= 1 - P[L < 9] && (3) \\ &= 1 - P_L(6) - P_L(7) - P_L(8) && (4) \\ &= 1 - (0.75)^6[1 + 6(0.25) + 21(0.25)^2] \approx 0.321 && (5) \end{aligned}$$

Problem 2.3.11 ■

In a packet voice communications system, a source transmits packets containing digitized speech to a receiver. Because transmission errors occasionally occur, an acknowledgment (ACK) or a nonacknowledgment (NAK) is transmitted back to the source to indicate the status of each received packet. When the transmitter gets a NAK, the packet is retransmitted. Voice packets are delay sensitive and a packet can be transmitted a maximum of d times. If a packet transmission is an independent Bernoulli trial with success probability p, what is the PMF of T, the number of times a packet is transmitted?

Problem 2.3.11 Solution

The packets are delay sensitive and can only be retransmitted d times. For $t < d$, a packet is transmitted t times if the first $t-1$ attempts fail followed by a successful transmission on attempt t. Further, the packet is transmitted d times if there are failures on the first $d-1$ transmissions, no matter what the outcome of attempt d. So the random variable T, the number of times that a packet is transmitted, can be represented by the following PMF.

$$P_T(t) = \begin{cases} p(1-p)^{t-1} & t = 1, 2, \ldots, d-1 \\ (1-p)^{d-1} & t = d \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Problem 2.4.5 ■

At the One Top Pizza Shop, a pizza sold has mushrooms with probability $p = 2/3$. On a day in which 100 pizzas are sold, let N equal the number of pizzas sold before the first pizza with mushrooms is sold. What is the PMF of N? What is the CDF of N?

Problem 2.4.5 Solution

Since mushrooms occur with probability 2/3, the number of pizzas sold before the first mushroom pizza is $N = n < 100$ if the first n pizzas do not have mushrooms followed by mushrooms on pizza $n + 1$. Also, it is possible that $N = 100$ if all 100 pizzas are sold without mushrooms. the resulting PMF is

$$P_N(n) = \begin{cases} (1/3)^n(2/3) & n = 0, 1, \ldots, 99 \\ (1/3)^{100} & n = 100 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

For integers $n < 100$, the CDF of N obeys

$$F_N(n) = \sum_{i=0}^{n} P_N(i) = \sum_{i=0}^{n} (1/3)^i(2/3) = 1 - (1/3)^{n+1} \tag{2}$$

A complete expression for $F_N(n)$ must give a valid answer for every value of n, including non-integer values. We can write the CDF using the floor function $\lfloor x \rfloor$ which denote the largest integer less than or equal to X. The complete expression for the CDF is

$$F_N(x) = \begin{cases} 0 & x < 0 \\ 1 - (1/3)^{\lfloor x \rfloor + 1} & 0 \le x < 100 \\ 1 & x \ge 100 \end{cases} \tag{3}$$

Problem 2.4.8 ■

In Problem 2.2.9, find and sketch the CDF of N, the number of attempts made by the cellular phone for $p = 1/2$.

Problem 2.4.8 Solution

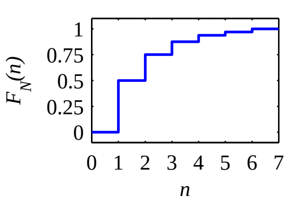
From Problem 2.2.9, the PMF of the number of call attempts is

$$P_N(n) = \begin{cases} (1-p)^{k-1}p & k = 1, 2, \ldots, 5 \\ (1-p)^5 p + (1-p)^6 = (1-p)^5 & k = 6 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

For $p = 1/2$, the PMF can be simplified to

$$P_N(n) = \begin{cases} (1/2)^n & n = 1, 2, \ldots, 5 \\ (1/2)^5 & n = 6 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

The corresponding CDF of N is



$$F_N(n) = \begin{cases} 0 & n < 1 \\ 1/2 & 1 \le n < 2 \\ 3/4 & 2 \le n < 3 \\ 7/8 & 3 \le n < 4 \\ 15/16 & 4 \le n < 5 \\ 31/32 & 5 \le n < 6 \\ 1 & n \ge 6 \end{cases} \tag{3}$$

Problem 2.6.5 ■

A source wishes to transmit data packets to a receiver over a radio link. The receiver uses error detection to identify packets that have been corrupted by radio noise. When a packet is received error-free, the receiver sends an acknowledgment (ACK) back to the source. When the receiver gets a packet with errors, a negative acknowledgment (NAK) message is sent back to the source. Each time the source receives a NAK, the packet is retransmitted. We assume that each packet transmission is independently corrupted by errors with probability q.

(a) Find the PMF of X, the number of times that a packet is transmitted by the source.

(b) Suppose each packet takes 1 millisecond to transmit and that the source waits an additional millisecond to receive the acknowledgment message (ACK or NAK) before retransmitting. Let T equal the time required until the packet is successfully received. What is the relationship between T and X? What is the PMF of T?

Problem 2.6.5 Solution

(a) The source continues to transmit packets until one is received correctly. Hence, the total number of times that a packet is transmitted is $X = x$ if the first $x - 1$ transmissions were in error. Therefore the PMF of X is

$$P_X(x) = \begin{cases} q^{x-1}(1-q) & x = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(b) The time required to send a packet is a millisecond and the time required to send an acknowledgment back to the source takes another millisecond. Thus, if X transmissions of a packet are needed to send the packet correctly, then the packet is correctly received after $T = 2X - 1$ milliseconds. Therefore, for an odd integer $t > 0$, $T = t$ iff $X = (t + 1)/2$. Thus,

$$P_T(t) = P_X((t+1)/2) = \begin{cases} q^{(t-1)/2}(1-q) & t = 1, 3, 5, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Problem 2.6.6 ■

Suppose that a cellular phone costs \$20 per month with 30 minutes of use included and that each additional minute of use costs \$0.50. If the number of minutes you use in a month is a geometric random variable M with expected value of $E[M] = 1/p = 30$ minutes, what is the PMF of C, the cost of the phone for one month?

Problem 2.6.6 Solution

The cellular calling plan charges a flat rate of \$20 per month up to and including the 30th minute, and an additional 50 cents for each minute over 30 minutes. Knowing that the time you spend on the phone is a geometric random variable M with mean $1/p = 30$, the PMF of M is

$$P_M(m) = \begin{cases} (1-p)^{m-1}p & m = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The monthly cost, C obeys

$$P_C(20) = P[M \leq 30] = \sum_{m=1}^{30} (1-p)^{m-1}p = 1 - (1-p)^{30} \tag{2}$$

When $M \geq 30$, $C = 20 + (M - 30)/2$ or $M = 2C - 10$. Thus,

$$P_C(c) = P_M(2c - 10) \qquad c = 20.5, 21, 21.5, \ldots \tag{3}$$

The complete PMF of C is

$$P_C(c) = \begin{cases} 1 - (1-p)^{30} & c = 20 \\ (1-p)^{2c-10-1}p & c = 20.5, 21, 21.5, \ldots \end{cases} \tag{4}$$

Problem 2.7.5 ■

For the cellular phone in Problem 2.6.6, express the monthly cost C as a function of M, the number of minutes used. What is the expected monthly cost $E[C]$?

Problem 2.7.5 Solution

As a function of the number of minutes used, M, the monthly cost is

$$C(M) = \begin{cases} 20 & M \leq 30 \\ 20 + (M-30)/2 & M \geq 30 \end{cases} \tag{1}$$

The expected cost per month is

$$E[C] = \sum_{m=1}^{\infty} C(m) P_M(m) = \sum_{m=1}^{30} 20 P_M(m) + \sum_{m=31}^{\infty} (20 + (m-30)/2) P_M(m) \tag{2}$$

$$= 20 \sum_{m=1}^{\infty} P_M(m) + \frac{1}{2} \sum_{m=31}^{\infty} (m-30) P_M(m) \tag{3}$$

Since $\sum_{m=1}^{\infty} P_M(m) = 1$ and since $P_M(m) = (1-p)^{m-1}p$ for $m \geq 1$, we have

$$E[C] = 20 + \frac{(1-p)^{30}}{2} \sum_{m=31}^{\infty} (m-30)(1-p)^{m-31}p \tag{4}$$

Making the substitution $j = m - 30$ yields

$$E[C] = 20 + \frac{(1-p)^{30}}{2} \sum_{j=1}^{\infty} j(1-p)^{j-1}p = 20 + \frac{(1-p)^{30}}{2p} \tag{5}$$

Problem 2.7.6 ■

A new cellular phone billing plan costs \$15 per month plus \$1 for each minute of use. If the number of minutes you use the phone in a month is a geometric random variable with mean $1/p$, what is the expected monthly cost $E[C]$ of the phone? For what values of p is this billing plan preferable to the billing plan of Problem 2.6.6 and Problem 2.7.5?

Problem 2.7.6 Solution

Since our phone use is a geometric random variable M with mean value $1/p$,

$$P_M(m) = \begin{cases} (1-p)^{m-1}p & m = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

For this cellular billing plan, we are given no free minutes, but are charged half the flat fee. That is, we are going to pay 15 dollars regardless and \$1 for each minute we use the phone. Hence $C = 15 + M$ and for $c \geq 16$, $P[C = c] = P[M = c - 15]$. Thus we can construct the PMF of the cost C

$$P_C(c) = \begin{cases} (1-p)^{c-16}p & c = 16, 17, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Since $C = 15 + M$, the expected cost per month of the plan is

$$E[C] = E[15 + M] = 15 + E[M] = 15 + 1/p \tag{3}$$

In Problem 2.7.5, we found that that the expected cost of the plan was

$$E[C] = 20 + [(1-p)^{30}]/(2p) \tag{4}$$

In comparing the expected costs of the two plans, we see that the new plan is better (i.e. cheaper) if

$$15 + 1/p \leq 20 + [(1-p)^{30}]/(2p) \tag{5}$$

A simple plot will show that the new plan is better if $p \leq p_0 \approx 0.2$.

Problem 2.8.5 ■

Let X have the binomial PMF

$$P_X(x) = \binom{4}{x}(1/2)^4.$$

(a) Find the standard deviation of the random variable X.

(b) What is $P[\mu_X - \sigma_X \leq X \leq \mu_X + \sigma_X]$, the probability that X is within one standard deviation of the expected value?

Problem 2.8.5 Solution

(a) The expected value of X is

$$E[X] = \sum_{x=0}^{4} x P_X(x) = 0\binom{4}{0}\frac{1}{2^4} + 1\binom{4}{1}\frac{1}{2^4} + 2\binom{4}{2}\frac{1}{2^4} + 3\binom{4}{3}\frac{1}{2^4} + 4\binom{4}{4}\frac{1}{2^4} \quad (1)$$

$$= [4 + 12 + 12 + 4]/2^4 = 2 \quad (2)$$

The expected value of X^2 is

$$E\left[X^2\right] = \sum_{x=0}^{4} x^2 P_X(x) = 0^2\binom{4}{0}\frac{1}{2^4} + 1^2\binom{4}{1}\frac{1}{2^4} + 2^2\binom{4}{2}\frac{1}{2^4} + 3^2\binom{4}{3}\frac{1}{2^4} + 4^2\binom{4}{4}\frac{1}{2^4} \quad (3)$$

$$= [4 + 24 + 36 + 16]/2^4 = 5 \quad (4)$$

The variance of X is

$$\text{Var}[X] = E\left[X^2\right] - (E[X])^2 = 5 - 2^2 = 1 \quad (5)$$

Thus, X has standard deviation $\sigma_X = \sqrt{\text{Var}[X]} = 1$.

(b) The probability that X is within one standard deviation of its expected value is

$$P[\mu_X - \sigma_X \le X \le \mu_X + \sigma_X] = P[2 - 1 \le X \le 2 + 1] = P[1 \le X \le 3] \quad (6)$$

This calculation is easy using the PMF of X.

$$P[1 \le X \le 3] = P_X(1) + P_X(2) + P_X(3) = 7/8 \quad (7)$$

Problem 2.8.6 ■

The binomial random variable X has PMF

$$P_X(x) = \binom{5}{x}(1/2)^5.$$

(a) Find the standard deviation of X.

(b) Find $P[\mu_X - \sigma_X \le X \le \mu_X + \sigma_X]$, the probability that X is within one standard deviation of the expected value.

Problem 2.8.6 Solution

(a) The expected value of X is

$$E[X] = \sum_{x=0}^{5} x P_X(x) \quad (1)$$

$$= 0\binom{5}{0}\frac{1}{2^5} + 1\binom{5}{1}\frac{1}{2^5} + 2\binom{5}{2}\frac{1}{2^5} + 3\binom{5}{3}\frac{1}{2^5} + 4\binom{5}{4}\frac{1}{2^5} + 5\binom{5}{5}\frac{1}{2^5} \quad (2)$$

$$= [5 + 20 + 30 + 20 + 5]/2^5 = 5/2 \quad (3)$$

The expected value of X^2 is

$$E\left[X^2\right] = \sum_{x=0}^{5} x^2 P_X(x) \tag{4}$$

$$= 0^2\binom{5}{0}\frac{1}{2^5} + 1^2\binom{5}{1}\frac{1}{2^5} + 2^2\binom{5}{2}\frac{1}{2^5} + 3^2\binom{5}{3}\frac{1}{2^5} + 4^2\binom{5}{4}\frac{1}{2^5} + 5^2\binom{5}{5}\frac{1}{2^5} \tag{5}$$

$$= [5 + 40 + 90 + 80 + 25]/2^5 = 240/32 = 15/2 \tag{6}$$

The variance of X is

$$\text{Var}[X] = E\left[X^2\right] - (E[X])^2 = 15/2 - 25/4 = 5/4 \tag{7}$$

By taking the square root of the variance, the standard deviation of X is $\sigma_X = \sqrt{5/4} \approx 1.12$.

(b) The probability that X is within one standard deviation of its mean is

$$P[\mu_X - \sigma_X \le X \le \mu_X + \sigma_X] = P[2.5 - 1.12 \le X \le 2.5 + 1.12] \tag{8}$$

$$= P[1.38 \le X \le 3.62] \tag{9}$$

$$= P[2 \le X \le 3] \tag{10}$$

By summing the PMF over the desired range, we obtain

$$P[2 \le X \le 3] = P_X(2) + P_X(3) = 10/32 + 10/32 = 5/8 \tag{11}$$

Problem 2.9.5 •

In Problem 2.8.6, find $P_{X|B}(x)$, where the condition $B = \{X \ge \mu_X\}$. What are $E[X|B]$ and $\text{Var}[X|B]$?

Problem 2.9.5 Solution

The probability of the event B is

$$P[B] = P[X \ge \mu_X] = P[X \ge 3] = P_X(3) + P_X(4) + P_X(5) \tag{1}$$

$$= \frac{\binom{5}{3} + \binom{5}{4} + \binom{5}{5}}{32} = 21/32 \tag{2}$$

The conditional PMF of X given B is

$$P_{X|B}(x) = \begin{cases} \frac{P_X(x)}{P[B]} & x \in B \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \binom{5}{x}\frac{1}{21} & x = 3, 4, 5 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

The conditional first and second moments of X are

$$E[X|B] = \sum_{x=3}^{5} x P_{X|B}(x) = 3\binom{5}{3}\frac{1}{21} + 4\binom{5}{4}\frac{1}{21} + 5\binom{5}{5}\frac{1}{21} \tag{4}$$

$$= [30 + 20 + 5]/21 = 55/21 \tag{5}$$

$$E\left[X^2|B\right] = \sum_{x=3}^{5} x^2 P_{X|B}(x) = 3^2\binom{5}{3}\frac{1}{21} + 4^2\binom{5}{4}\frac{1}{21} + 5^2\binom{5}{5}\frac{1}{21} \tag{6}$$

$$= [90 + 80 + 25]/21 = 195/21 = 65/7 \tag{7}$$

The conditional variance of X is

$$\text{Var}[X|B] = E\left[X^2|B\right] - (E[X|B])^2 = 65/7 - (55/21)^2 = 1070/441 = 2.43 \tag{8}$$

Problem 2.9.6 ■

Select integrated circuits, test them in sequence until you find the first failure, and then stop. Let N be the number of tests. All tests are independent with probability of failure $p = 0.1$. Consider the condition $B = \{N \geq 20\}$.

(a) Find the PMF $P_N(n)$.

(b) Find $P_{N|B}(n)$, the conditional PMF of N given that there have been 20 consecutive tests without a failure.

(c) What is $E[N|B]$, the expected number of tests given that there have been 20 consecutive tests without a failure?

Problem 2.9.6 Solution

(a) Consider each circuit test as a Bernoulli trial such that a failed circuit is called a success. The number of trials until the first success (i.e. a failed circuit) has the geometric PMF

$$P_N(n) = \begin{cases} (1-p)^{n-1}p & n = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(b) The probability there are at least 20 tests is

$$P[B] = P[N \geq 20] = \sum_{n=20}^{\infty} P_N(n) = (1-p)^{19} \tag{2}$$

Note that $(1-p)^{19}$ is just the probability that the first 19 circuits pass the test, which is what we would expect since there must be at least 20 tests if the first 19 circuits pass. The conditional PMF of N given B is

$$P_{N|B}(n) = \begin{cases} \frac{P_N(n)}{P[B]} & n \in B \\ 0 & \text{otherwise} \end{cases} = \begin{cases} (1-p)^{n-20}p & n = 20, 21, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

(c) Given the event B, the conditional expectation of N is

$$E[N|B] = \sum_n n P_{N|B}(n) = \sum_{n=20}^{\infty} n(1-p)^{n-20} p \tag{4}$$

Making the substitution $j = n - 19$ yields

$$E[N|B] = \sum_{j=1}^{\infty} (j+19)(1-p)^{j-1} p = 1/p + 19 \tag{5}$$

We see that in the above sum, we effectively have the expected value of $J + 19$ where J is geometric random variable with parameter p. This is not surprising since the $N \geq 20$ iff we observed 19 successful tests. After 19 successful tests, the number of additional tests needed to find the first failure is still a geometric random variable with mean $1/p$.

Problem 2.9.7 ■

Every day you consider going jogging. Before each mile, including the first, you will quit with probability q, independent of the number of miles you have already run. However, you are sufficiently decisive that you never run a fraction of a mile. Also, we say you have run a marathon whenever you run at least 26 miles.

(a) Let M equal the number of miles that you run on an arbitrary day. What is $P[M > 0]$? Find the PMF $P_M(m)$.

(b) Let r be the probability that you run a marathon on an arbitrary day. Find r.

(c) Let J be the number of days in one year (not a leap year) in which you run a marathon. Find the PMF $P_J(j)$. This answer may be expressed in terms of r found in part (b).

(d) Define $K = M - 26$. Let A be the event that you have run a marathon. Find $P_{K|A}(k)$.

Problem 2.9.7 Solution

(a) The PMF of M, the number of miles run on an arbitrary day is

$$P_M(m) = \begin{cases} q(1-q)^m & m = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

And we can see that the probability that $M > 0$, is

$$P[M > 0] = 1 - P[M = 0] = 1 - q \tag{2}$$

(b) The probability that we run a marathon on any particular day is the probability that $M \geq 26$.

$$r = P[M \geq 26] = \sum_{m=26}^{\infty} q(1-q)^m = (1-q)^{26} \tag{3}$$

(c) We run a marathon on each day with probability equal to r, and we do not run a marathon with probability $1-r$. Therefore in a year we have 365 tests of our jogging resolve, and thus 365 chances to run a marathon. So the PMF of the number of marathons run in a year, J, can be expressed as

$$P_J(j) = \begin{cases} \binom{365}{j} r^j (1-r)^{365-j} & j = 0, 1, \ldots, 365 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

(d) The random variable K is defined as the number of miles we run above that required for a marathon, $K = M - 26$. Given the event, A, that we have run a marathon, we wish to know how many miles in excess of 26 we in fact ran. So we want to know the conditional PMF $P_{K|A}(k)$.

$$P_{K|A}(k) = \frac{P[K = k, A]}{P[A]} = \frac{P[M = 26 + k]}{P[A]} \tag{5}$$

Since $P[A] = r$, for $k = 0, 1, \ldots,$

$$P_{K|A}(k) = \frac{(1-q)^{26+k} q}{(1-q)^{26}} = (1-q)^k q \tag{6}$$

The complete expression of for the conditional PMF of K is

$$P_{K|A}(k) = \begin{cases} (1-q)^k q & k = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

Problem 2.9.8 ■

In the situation described in Example 2.29, the firm sends all faxes with an even number of pages to fax machine A and all faxes with an odd number of pages to fax machine B.

(a) Find the conditional PMF of the length X of a fax, given the fax was sent to A. What are the conditional expected length and standard deviation?

(b) Find the conditional PMF of the length X of a fax, given the fax was sent to B and had no more than six pages. What are the conditional expected length and standard deviation?

Problem 2.9.8 Solution

Recall that the PMF of the number of pages in a fax is

$$P_X(x) = \begin{cases} 0.15 & x = 1, 2, 3, 4 \\ 0.1 & x = 5, 6, 7, 8 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) The event that a fax was sent to machine A can be expressed mathematically as the event that the number of pages X is an even number. Similarly, the event that a fax was sent to B is the event that X is an odd number. Since $S_X = \{1, 2, \ldots, 8\}$, we define the set $A = \{2, 4, 6, 8\}$. Using this definition for A, we have that the event that a fax is sent to A is equivalent to the event $X \in A$. The event A has probability

$$P[A] = P_X(2) + P_X(4) + P_X(6) + P_X(8) = 0.5 \tag{2}$$

Given the event A, the conditional PMF of X is

$$P_{X|A}(x) = \begin{cases} \frac{P_X(x)}{P[A]} & x \in A \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 0.3 & x = 2, 4 \\ 0.2 & x = 6, 8 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

The conditional first and second moments of X given A is

$$E[X|A] = \sum_x x P_{X|A}(x) = 2(0.3) + 4(0.3) + 6(0.2) + 8(0.2) = 4.6 \tag{4}$$

$$E\left[X^2|A\right] = \sum_x x^2 P_{X|A}(x) = 4(0.3) + 16(0.3) + 36(0.2) + 64(0.2) = 26 \tag{5}$$

The conditional variance and standard deviation are

$$\text{Var}[X|A] = E\left[X^2|A\right] - (E[X|A])^2 = 26 - (4.6)^2 = 4.84 \tag{6}$$

$$\sigma_{X|A} = \sqrt{\text{Var}[X|A]} = 2.2 \tag{7}$$

(b) Let the event B' denote the event that the fax was sent to B and that the fax had no more than 6 pages. Hence, the event $B' = \{1, 3, 5\}$ has probability

$$P\left[B'\right] = P_X(1) + P_X(3) + P_X(5) = 0.4 \tag{8}$$

The conditional PMF of X given B' is

$$P_{X|B'}(x) = \begin{cases} \frac{P_X(x)}{P[B']} & x \in B' \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 3/8 & x = 1, 3 \\ 1/4 & x = 5 \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

Given the event B', the conditional first and second moments are

$$E\left[X|B'\right] = \sum_x x P_{X|B'}(x) = 1(3/8) + 3(3/8) + 5(1/4) + = 11/4 \tag{10}$$

$$E\left[X^2|B'\right] = \sum_x x^2 P_{X|B'}(x) = 1(3/8) + 9(3/8) + 25(1/4) = 10 \tag{11}$$

The conditional variance and standard deviation are

$$\text{Var}[X|B'] = E\left[X^2|B'\right] - (E\left[X|B'\right])^2 = 10 - (11/4)^2 = 39/16 \tag{12}$$

$$\sigma_{X|B'} = \sqrt{\text{Var}[X|B']} = \sqrt{39}/4 \approx 1.56 \tag{13}$$

Problem 2.10.6 ■

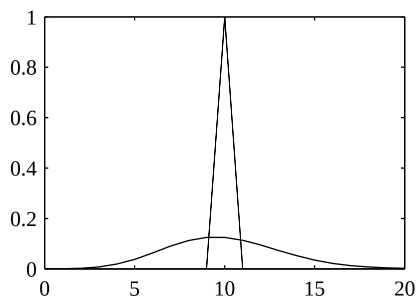
Test the convergence of Theorem 2.8. For $\alpha = 10$, plot the PMF of K_n for $(n,p) = (10,1)$, $(n,p) = (100,0.1)$, and $(n,p) = (1000,0.01)$ and compare against the Poisson (α) PMF.

Problem 2.10.6 Solution

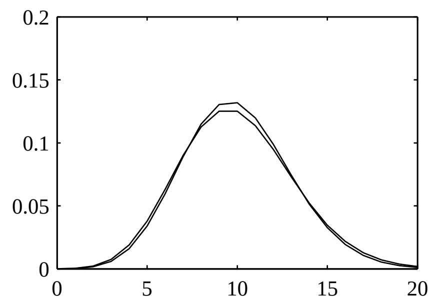
We can compare the binomial and Poisson PMFs for $(n,p) = (100,0.1)$ using the following MATLAB code:

```
x=0:20;
p=poissonpmf(100,x);
b=binomialpmf(100,0.1,x);
plot(x,p,x,b);
```

For $(n,p) = (10,1)$, the binomial PMF has no randomness. For $(n,p) = (100,0.1)$, the approximation is reasonable:

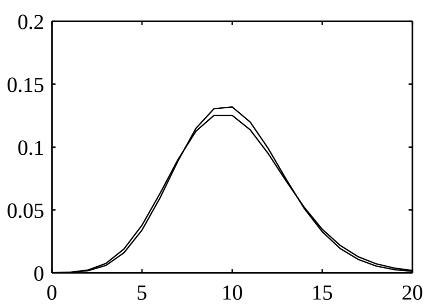


(a) $n = 10, p = 1$

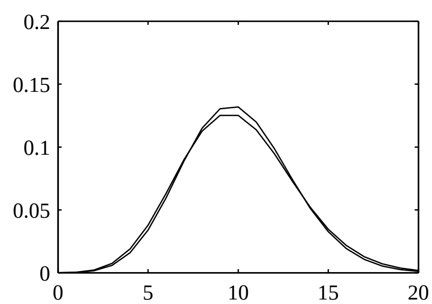


(b) $n = 100, p = 0.1$

Finally, for $(n,p) = (1000,0.01)$, and $(n,p) = (10000,0.001)$, the approximation is very good:



(a) $n = 1000, p = 0.01$



(b) $n = 10000, p = 0.001$