7.4

Equation 7.6 gives the probability per unit energy for finding an electron in a given energy state E at a given temperature T:

$$P(T, E) = \frac{1}{1 + \exp\left\{ \left(E - E_{\rm F}\right)/kT \right\}},$$

where $E_{\rm F}$ is the Fermi energy and k is the Stefan-Boltzmann constant and equals $8.62\times10^{-5}\,{\rm eV\ K^{-1}}$.

The bandgap energy for silicon is the difference in energy between the top of the valence band and the bottom of the conduction band $E_{\rm G} = E - E_{\rm F} = 1.12\,{\rm eV}$ (Table 7.4).

By taking the ratio of P at the two temperatures, we can estimate the relative probabilities:

$$P(E = E_{\rm G})/P(E = E_{\rm F}) = \frac{\frac{1}{1 + \exp\left\{ \left(E - E_{\rm F}\right)/kT \right\}}}{1/2}.$$

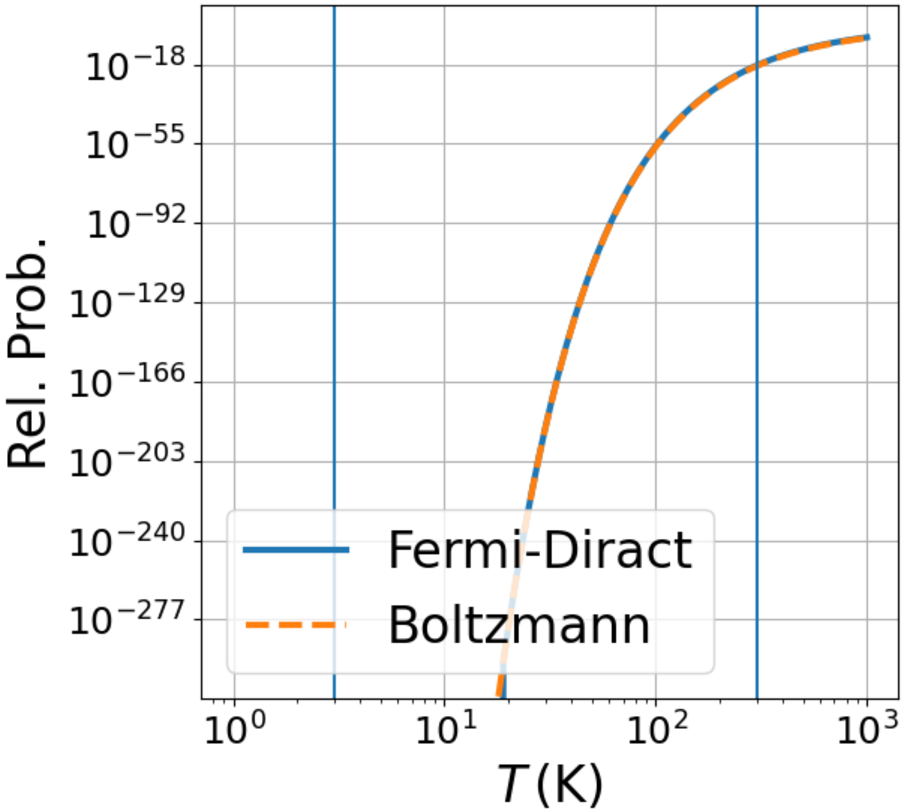
Let's make a plot to compare the prediction from Fermi-Dirac statistics (Equation 7.6) to that of Boltzmann (Equation 7.3).

In [11]:

```
%matplotlib inline

import matplotlib.pyplot as plt
import numpy as np

StefanBoltzmann_const = 8.62e-5 # eV/K

def FermiDirac_probability(T, E=1.12):

    return 2./(1. + np.exp(E/StefanBoltzmann_const/T))

def Boltzmann_probability(T, E=1.12, gi=1, gj=1):
    return gi/gj*np.exp(-E/StefanBoltzmann_const/T)

temperatures = np.linspace(1., 1000., 1000)
FD_prob = FermiDirac_probability(temperatures)
B_prob = Boltzmann_probability(temperatures)

fig = plt.figure(figsize=(6, 6))
ax = fig.add_subplot(111)

ax.loglog(temperatures, FD_prob, lw=3, label="Fermi-Diract")
ax.loglog(temperatures, B_prob, lw=3, ls='--', label="Boltzmann")

ax.grid(True)
ax.legend(loc='best', fontsize=24)
ax.tick_params(labelsize=18)
ax.set_xlabel(r'$T\,\left( {\rm K} \right)$', fontsize=24)
ax.set_ylabel("Rel. Prob.", fontsize=24)
ax.axvline(3.)
ax.axvline(300.)
```

```
/var/folders/0p/vptds8v9203dlqkw0pv01n100000gr/T/ipykernel_50529/2356649113.py:10: RuntimeWarning: overflow encountered in exp
  return 2./(1. + np.exp(E/StefanBoltzmann_const/T))
```

Out[11]: <matplotlib.lines.Line2D at 0x7fefd9a44700>



You can see that, even for $T \to 1000\,\mathrm{K}$, there is a very small probability to occupy the conduction band.

7.5

We're told to assume the electrical conductivity depends on the number density of electrons, which we can estimate as a function of bandgap energy E_G and temperature T using Equation 7.9:

$$n_N = A T^{3/2} e^{-\frac{E_\mathrm{G}}{kT}}.$$

For silicon, $E_\mathrm{G} = 1.12\,\mathrm{eV}$ and for germanium, $E_\mathrm{G} = 0.67\,\mathrm{eV}$ (Table 7.4).

So we're asked to compare n_N for $T = 40\,\mathrm{K}$ and for $T = 40\,\mathrm{K} + 1\,\mathrm{K}$ for these two semiconductors. This is a very small difference $\Delta T/T = 1/40$, so let's Taylor-expand n_N about small ΔT:

$$(T+\Delta T)^{3/2} = T^{3/2}\left(1+\frac{\Delta T}{T}\right)^{3/2} \approx T^{3/2}\left(1+\frac{3\Delta T}{2T}\right)$$

$$\exp\left(-\frac{E}{k\,(T+\Delta T)}\right) = \exp\left(-\frac{E}{kT}\left(1+\frac{\Delta T}{T}\right)^{-1}\right) \approx \exp\left(-\frac{E}{kT}\left(1-\frac{\Delta T}{T}\right)\right) = \exp\left(-\frac{E}{kT}\right)\ \exp\left(\frac{E}{kT}\frac{\Delta T}{T}\right)$$

$$\approx \exp\left(-\frac{E}{kT}\right)\left(1+\frac{E}{kT}\frac{\Delta T}{T}\right)$$

$$\Rightarrow T^{3/2}e^{-\frac{E}{kT}}\left(1+\left(\frac{3}{2}+\frac{E}{kT}\right)\frac{\Delta T}{T}\right).$$

The term on the left outside the parentheses is just number density of electrons at temperature T, so taking the ratio gives

$$\frac{n_N(T+\Delta T)}{n_N(T)} \approx \left(1+\left(\frac{3}{2}+\frac{E}{kT}\right)\frac{\Delta T}{T}\right).$$

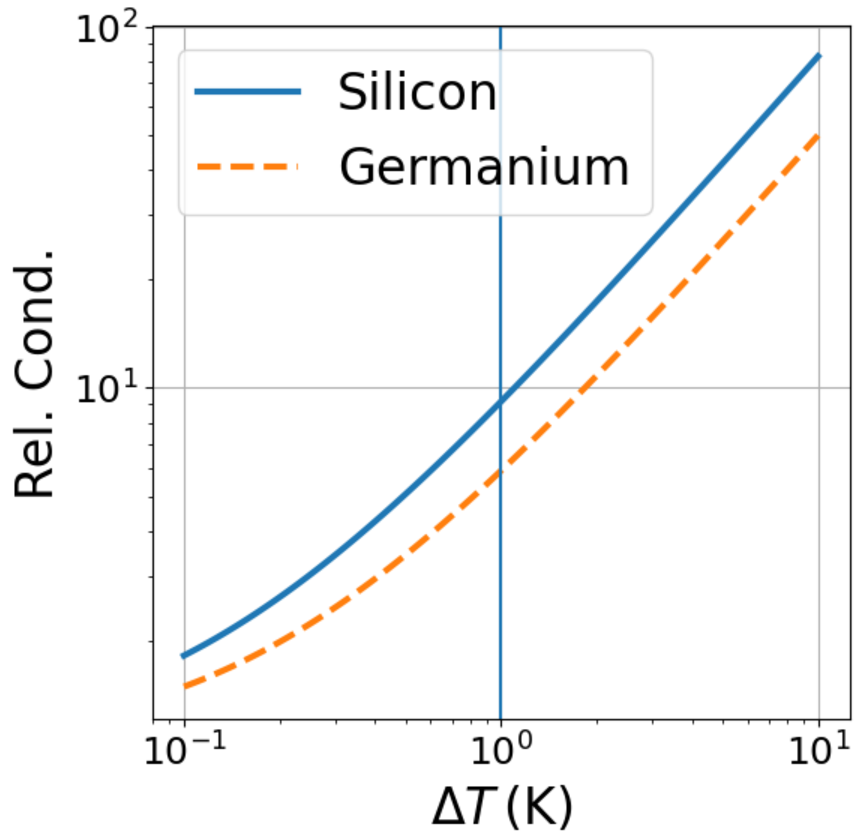
Again, we can plot this expression.

In [20]:

```python
%matplotlib inline

import matplotlib.pyplot as plt
import numpy as np

StefanBoltzmann_const = 8.62e-5 # eV/K

def relative_conductivity(Delta_T, E, T=40.):

    return (1. + (3./2 + E/StefanBoltzmann_const/T)*Delta_T/T)

Delta_T = 10**np.linspace(-1, 1, 100)
E_Si = 1.12
E_Ge = 0.67
Si_prob = relative_conductivity(Delta_T, E_Si)
Ge_prob = relative_conductivity(Delta_T, E_Ge)

fig = plt.figure(figsize=(6, 6))
ax = fig.add_subplot(111)

ax.loglog(Delta_T, Si_prob, lw=3, label="Silicon")
ax.loglog(Delta_T, Ge_prob, lw=3, ls='--', label="Germanium")

ax.grid(True)
ax.legend(loc='best', fontsize=24)
ax.tick_params(labelsize=18)
ax.set_xlabel(r'$\Delta T\,\left( {\rm K} \right)$', fontsize=24)
ax.set_ylabel("Rel. Cond.", fontsize=24)
ax.axvline(1.)
```

Out[20]:

```
<matplotlib.lines.Line2D at 0x7fefdac7dbe0>
```



7.6

The quantum efficiency q is a measure of how readily photons are converted into current by the CCD, and it depends on the surface reflectivity R, the absorption coefficient α , and the layer thickness z.

Equation 7.11 tells us how R depends on the index of refraction:

$$R = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2,$$

where n_1 is the index for air (≈ 1) and n_2 is the index for silicon (we're told $n_2 = 4$ at $500\,\mathrm{nm}$).

With these numbers, we get

$$R = \left(\frac{1-4}{1+4}\right)^2 = 0.36,$$

which means 36\% of the incident energy is reflected, leaving 74\% to be absorbed the CCD.

Next, we need to determine the absorption coefficient for silicon at 500 nm. Figure 7.10 gives that information. The figure suggests $\alpha = 10^4\,\mathrm{cm}^{-1}$. (Unfortunately, Chromey has a typo in the figure caption. You can see a similar plot with the right units here - https://www.pveducation.org/pvcdrom/materials/optical-properties-of-silicon.)

So we can estimate the required thickness z to achieve a given quantum efficiency q using

$$q = (1-R)\,e^{-\alpha z} \Rightarrow z = \frac{\ln\left(\frac{1-R}{q}\right)}{\alpha} = \frac{\ln\left(\frac{0.76}{0.4}\right)}{\left(10^4\,\mathrm{cm}^{-1}\right)} \approx \boxed{600\,\mathrm{nm}}.$$

8.1

We're told we start out with a quantum efficiency of 40\% for our detector.

Referring to Problem 7.6 above, we can cast the quantum efficiency as

$$q = (1-R)\,e^{-\alpha z}.$$

We are told that the reflectivity of the detector is reduced from 30\% to 5\%. Since nothing else about the detector has changed, we can calculate the ratio of the new to the old efficieny as

$$\frac{q'}{q} = \frac{1-R'}{1-R},$$

where the primed quantities represent the new values.

Solving for q' gives

$$q' = q\left(\frac{1-R'}{1-R}\right) = (0.4)\left(\frac{1-0.05}{1-0.3}\right) = \boxed{0.54}.$$

8.2

Equation 8.5 tells us

$$\mathrm{DQE} = \frac{\left(\mathrm{SNR}_{\mathrm{out}}\right)^2}{\left(\mathrm{SNR}_{\mathrm{perfect}}\right)^2} = \frac{\left(\mathrm{SNR}_{\mathrm{out}}\right)^2}{N_{\mathrm{in}}}.$$

We are told that the measurement involves 10^4 photons, which will take as N_{in}. So next we'll need the signal-to-noise ratio for the voltages, $\mathrm{SNR}_{\mathrm{out}}$.

Our average voltage (the signal) is

$$S = \frac{(113+120+115)\ \mathrm{mV}}{3} = 116\,\mathrm{mV}.$$

And the noise will be the standard deviation of the voltage measurements:

$$\sigma = 3\,\mathrm{mV}.$$

So now we can write our DQE as

$$\mathrm{DQE} = \frac{\left(\frac{116\,\mathrm{mV}}{3\,\mathrm{mV}}\right)^2}{10^4} \approx \boxed{0.15}.$$

8.3

The detector in this problem has a quantum efficiency of q. That means that, for every N_{in} photon that strikes the surface of the detector, $N_{\mathrm{detect}} = qN_{\mathrm{in}}$ are actually detected (and $(1-q)\,N_{\mathrm{in}}$ are NOT detected).

The detector is also said to have a quantum yield of y with an associated uncertainty $\sigma(y)$. That means that each of the $N_{\mathrm{detect}} = qN_{\mathrm{in}}$ induces $N_{\mathrm{events}} = yN_{\mathrm{detect}}$ events that are counted up by the detector.

So, as described by Equation 2.31 back in Chapter 2, the uncertainty $\sigma(N_{\mathrm{events}})$ associated with the events is given by

$$\sigma^2(N_{\mathrm{events}}) = \left(\frac{\partial N_{\mathrm{events}}}{\partial N_{\mathrm{detect}}}\right)^2 \sigma^2\left(N_{\mathrm{detect}}\right) + \left(\frac{\partial N_{\mathrm{events}}}{\partial y}\right)^2 \sigma^2\left(y\right) = y^2 N_{\mathrm{detect}} + N_{\mathrm{detect}}^2 \sigma^2(y),$$

taking Poisson error bars for N_{detect} (i.e., $\sigma^2\left(N_{\mathrm{detect}}\right) = N_{\mathrm{detect}}$).

To calculate DQE, we need $(SNR)^2_{\mathrm{out}}$ and $(SNR)^2_{\mathrm{perfect}}$. The latter is easy:

$$(SNR)^2_{\mathrm{perfect}} = N_{\mathrm{in}}.$$

The former is

$$(SNR)^2_{\mathrm{out}} = \frac{y^2 N_{\mathrm{detect}}^2}{y^2 N_{\mathrm{detect}} + N_{\mathrm{detect}}^2 \sigma^2(y)} = \frac{N_{\mathrm{detect}}^2}{N_{\mathrm{detect}} + N_{\mathrm{detect}}^2 \left(\frac{\sigma(y)}{y}\right)^2}$$

Now to calculate DQE:

$$\mathrm{DQE} = \frac{\frac{N_{\mathrm{detect}}^2}{N_{\mathrm{detect}} + N_{\mathrm{detect}}^2 \left(\frac{\sigma(y)}{y}\right)^2}}{N_{\mathrm{in}}}$$

But remember $N_{\text{detect}} = qN_{\text{in}}$, so

$$\text{DQE} = \frac{\frac{q^2 N_{\text{in}}^2}{qN_{\text{in}} + q^2 N_{\text{in}}^2 \left(\frac{\sigma(y)}{y}\right)^2}}{N_{\text{in}}} = \frac{q^2 N_{\text{in}}}{qN_{\text{in}} + q^2 N_{\text{in}}^2 \left(\frac{\sigma(y)}{y}\right)^2} = \boxed{\frac{q}{1 + qN_{\text{in}} \left(\frac{\sigma(y)}{y}\right)^2}}$$

So you can see that, if $\sigma(y) = 0$, then $\text{DQE} = q$. However, since all numbers are positive, if $\sigma(y) > 0$, then $\text{DQE} < q$.

8.4

We'll need to recall the definition of QE (Equation 8.1):

$$\text{QE} = \frac{N_{\text{detect}}}{N_{\text{in}}}.$$

We're told that $\text{QE} = 0.9$, so if 1000 photons are incident in the 1-second exposure, we'll actually detect $(0.9) \times 1000 = 900$ of them.

Then recall the definition of DQE (Equation 8.5):

$$\text{DQE} = \frac{(\text{SNR})_{\text{out}}^2}{(\text{SNR})_{\text{perfect}}^2}.$$

For our detector we have a signal $S = 900$ photons and sources of noise including the dark current $\left(1 \text{ photon s}^{-1}\right) \times (1\,\text{s}) = 1$ photon and a read noise of 3 photons.

Therefore, our out SNR is

$$(\text{SNR})_{\text{out}} = \frac{900}{\sqrt{(900 + 1 + 3) + 1 + 3}} \approx 30.$$

A perfect detector would have

$$(\text{SNR})_{\text{perfect}} = \sqrt{1000} \approx 32.$$

So

$$\text{DQE} = \frac{30^2}{32^2} = \boxed{0.89}.$$

For the 4-second exposure, we have all the same numbers except now the dark current is $\left(1 \text{ photon s}^{-1}\right) \times (400\,\text{s}) = 400$ photon.

So

$$(\mathrm{SNR})_{\mathrm{out}} = \frac{900}{\sqrt{(900+400+3)+400+3}} \approx 22.$$

And

$$\mathrm{DQE} = \frac{22^2}{32^2} = \boxed{0.47}.$$