7.4

Equation 7.6 gives the probability per unit energy for finding an electron in a given energy state E at a given temperature T:

$$P(T,E) = \frac{1}{1+\exp\left\{(E-E_{\mathrm{F}})/kT\right\}},$$

where E_{F} is the Fermi energy and k is the Stefan-Boltzmann constant and equals $8.62\times10^{-5}\,\mathrm{eV\ K^{-1}}$.

The bandgap energy for silicon is the difference in energy between the top of the valence band and the bottom of the conduction band $E_{\mathrm{G}} = 1.12\,\mathrm{eV}$ (Table 7.4).

We know that the bandgap energy is defined in terms of the uppermost valence band energy E_{v} and the lowermost conduction band energy E_{c} as $E_{\mathrm{G}} = E_{\mathrm{c}} - E_{\mathrm{v}}$ and that $E_{\mathrm{F}} = \frac{1}{2}(E_{\mathrm{c}} + E_{\mathrm{v}})$, i.e., the average of the two energies.

We can re-arrange these equations to show that $E_{\mathrm{c}} = E_{\mathrm{F}} + \frac{1}{2}E_{\mathrm{G}}$ and $E_{\mathrm{v}} = E_{\mathrm{F}} - \frac{1}{2}E_{\mathrm{G}}$.

By taking the ratio of P at the two temperatures, we can estimate the relative probabilities:

$$P(E=E_{\mathrm{c}})/P(E=E_{\mathrm{v}}) = \frac{1+\exp\left(\left(E_{\mathrm{F}}-\frac{1}{2}E_{\mathrm{G}}-E_{\mathrm{F}}\right)/kT\right)}{1+\exp\left(\left(E_{\mathrm{F}}+\frac{1}{2}E_{\mathrm{G}}-E_{\mathrm{F}}\right)/kT\right)} = \frac{1+\exp\left(-\frac{1}{2}E_{\mathrm{G}}/kT\right)}{1+\exp\left(\frac{1}{2}E_{\mathrm{G}}/kT\right)}$$

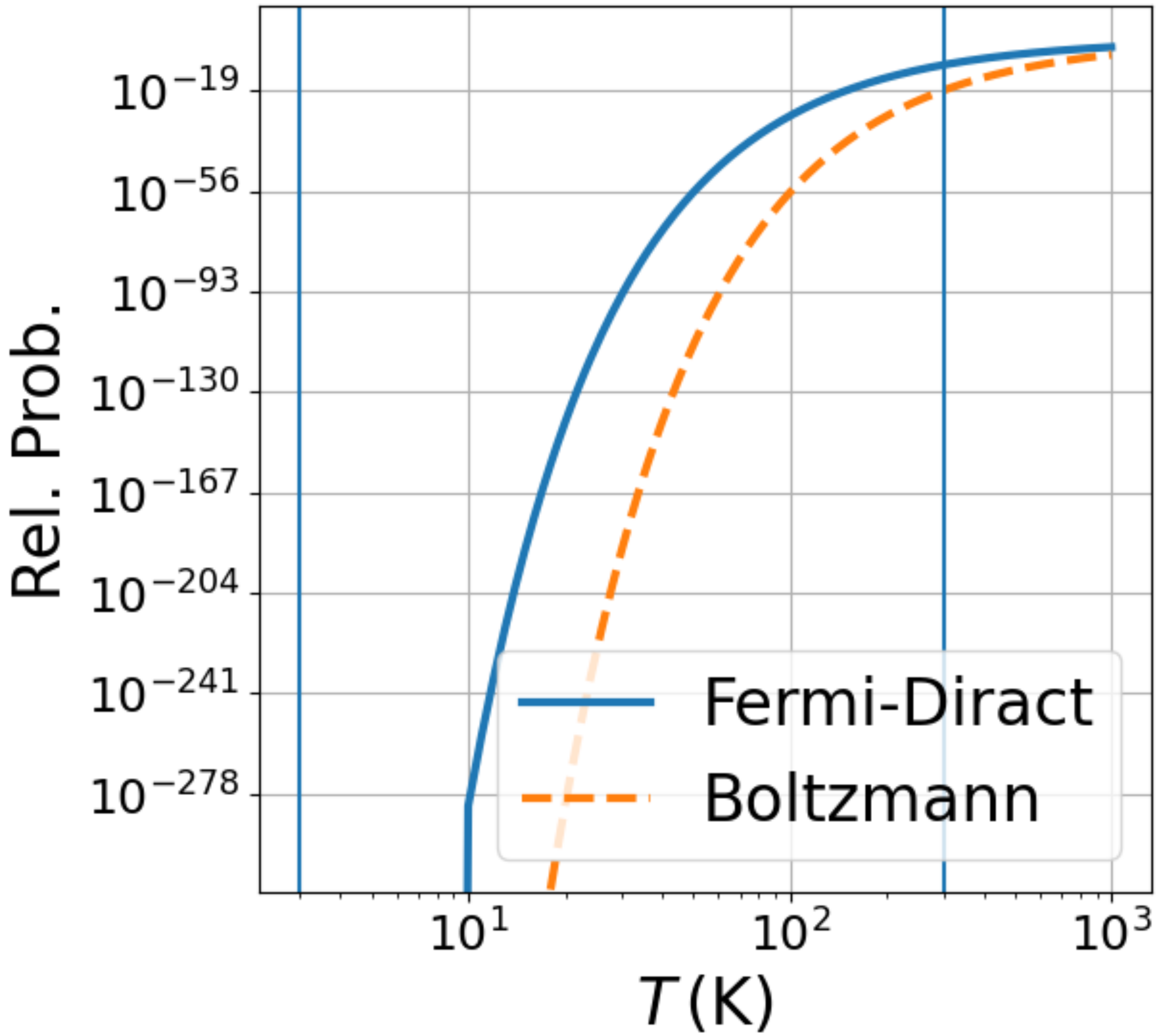
Let's make a plot to compare the prediction from Fermi-Dirac statistics (Equation 7.6) to that of Boltzmann (Equation 7.3).

In [11]:
```
%matplotlib inline

import matplotlib.pyplot as plt
import numpy as np

StefanBoltzmann_const = 8.62e-5 # eV/K

def FermiDirac_probability(T, EG):

    return 1./(1. + np.exp(0.5*EG/StefanBoltzmann_const/T))
```

```
def Boltzmann_probability(T, EG, gi=1, gj=1):
    return gi/gj*np.exp(-0.5*EG/StefanBoltzmann_const/T)

temperatures = np.linspace(3., 1000., 1000)
FD_prob = FermiDirac_probability(temperatures, 1.12)/FermiDirac_probability(temperatures, -1.12)
B_prob = Boltzmann_probability(temperatures, 1.12)/Boltzmann_probability(temperatures, -1.12)


fig = plt.figure(figsize=(6, 6))
ax = fig.add_subplot(111)

ax.loglog(temperatures, FD_prob, lw=3, label="Fermi-Diract")
ax.loglog(temperatures, B_prob, lw=3, ls='--', label="Boltzmann")

ax.grid(True)
ax.legend(loc='best', fontsize=24)
ax.tick_params(labelsize=18)
ax.set_xlabel(r'$T\,\left( {\rm K} \right)$', fontsize=24)
ax.set_ylabel("Rel. Prob.", fontsize=24)
ax.axvline(3.)
ax.axvline(300.)
```

```
/var/folders/qn/shk5dvhn3mb9twv7bvyjp5nm0000gn/T/ipykernel_23852/3703470191.py:10: RuntimeWarning: overflow
encountered in exp
  return 1./(1. + np.exp(0.5*EG/StefanBoltzmann_const/T))
/var/folders/qn/shk5dvhn3mb9twv7bvyjp5nm0000gn/T/ipykernel_23852/3703470191.py:13: RuntimeWarning: overflow
encountered in exp
  return gi/gj*np.exp(-0.5*EG/StefanBoltzmann_const/T)
```

Out[11]:
```
<matplotlib.lines.Line2D at 0x7fc889693550>
```



You can see that, even for $T \to 1000\,\mathrm{K}$, there is a very small probability to occupy the conduction band.

7.5

We're told to assume the electrical conductivity depends on the number density of electrons, which we can estimate as a function of bandgap energy E_G and temperature T using Equation 7.9:

$$n_N = AT^{3/2} e^{-\frac{E_G}{kT}}.$$

For silicon, $E_G = 1.12\,\mathrm{eV}$ and for germanium, $E_G = 0.67\,\mathrm{eV}$ (Table 7.4).

So we're asked to compare n_N for $T = 40\,\mathrm{K}$ and for $T = 40\,\mathrm{K} + 1\,\mathrm{K}$ for these two semiconductors. This is a very small difference $\Delta T/T = 1/40$, so let's Taylor-expand n_N about small ΔT:

$$(T + \Delta T)^{3/2} = T^{3/2}\left(1 + \frac{\Delta T}{T}\right)^{3/2} \approx T^{3/2}\left(1 + \frac{3\Delta T}{2T}\right)$$

$$\exp\left(-\frac{E}{k\left(T + \Delta T\right)}\right) = \exp\left(-\frac{E}{kT}\left(1 + \frac{\Delta T}{T}\right)^{-1}\right) \approx \exp\left(-\frac{E}{kT}\left(1 - \frac{\Delta T}{T}\right)\right) = \exp\left(-\frac{E}{kT}\right)\,\exp\left(\frac{E}{kT}\frac{\Delta T}{T}\right)$$

$$\Rightarrow T^{3/2} e^{-\frac{E}{kT}}\left(1 + \left(\frac{3\Delta T}{2T}\right)\right)\exp\left(\frac{E}{kT}\frac{\Delta T}{T}\right).$$

The term on the left outside the parentheses is just number density of electrons at temperature T, so taking the ratio gives

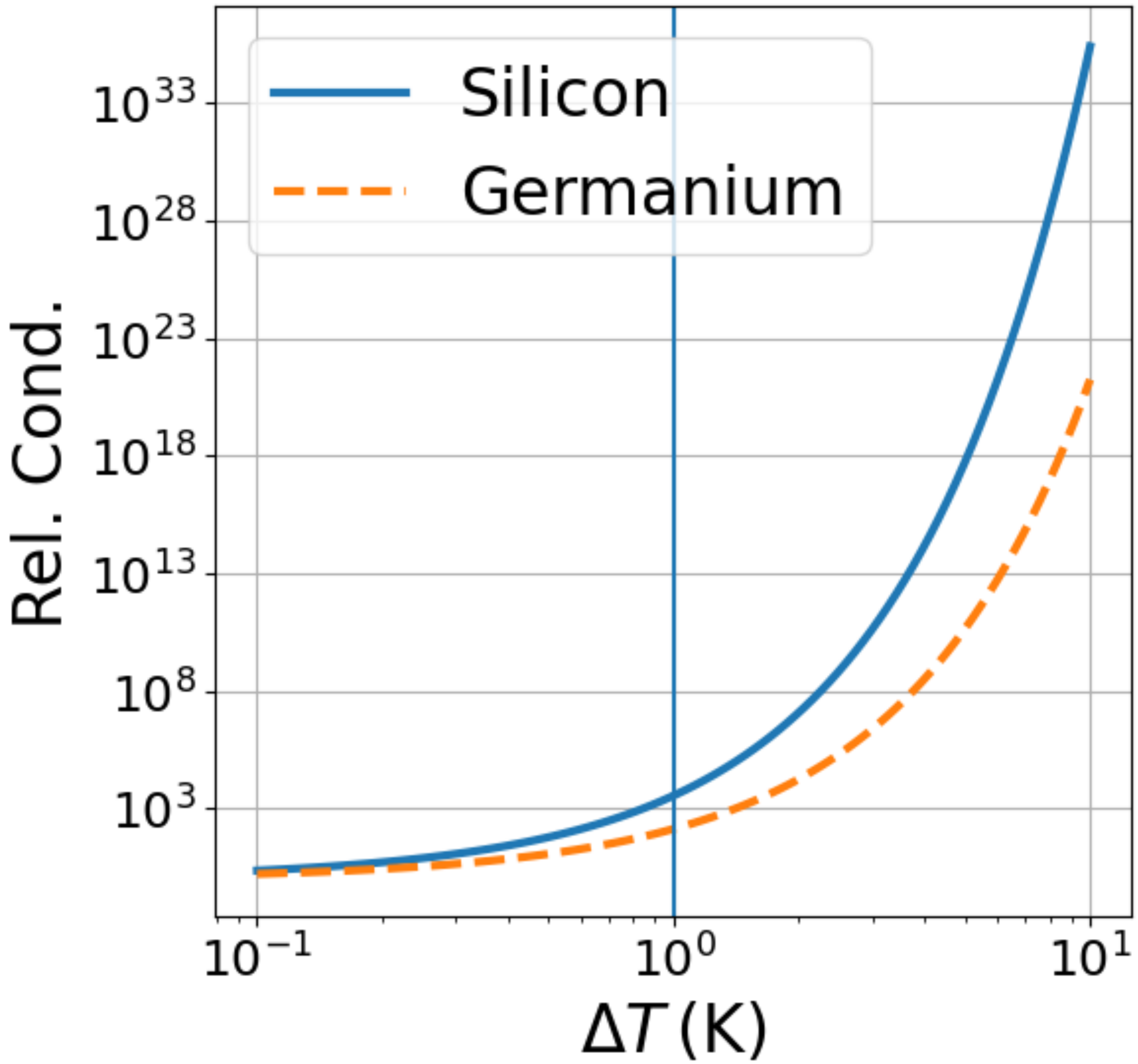
$$\frac{n_N(T + \Delta T)}{n_N(T)} \approx \left(1 + \left(\frac{3\Delta T}{2T}\right)\right)\exp\left(\frac{E}{kT}\frac{\Delta T}{T}\right).$$

Again, we can plot this expression.

In [1]:
```
%matplotlib inline

import matplotlib.pyplot as plt
import numpy as np

StefanBoltzmann_const = 8.62e-5 # eV/K

def relative_conductivity(Delta_T, E, T=40.):
```

```
    return (1. + 3*Delta_T/2/T)*np.exp(E/StefanBoltzmann_const/T*Delta_T/T)

Delta_T = 10**np.linspace(-1, 1, 100)
E_Si = 1.12
E_Ge = 0.67
Si_prob = relative_conductivity(Delta_T, E_Si)
Ge_prob = relative_conductivity(Delta_T, E_Ge)

fig = plt.figure(figsize=(6, 6))
ax = fig.add_subplot(111)

ax.loglog(Delta_T, Si_prob, lw=3, label="Silicon")
ax.loglog(Delta_T, Ge_prob, lw=3, ls='--', label="Germanium")

ax.grid(True)
ax.legend(loc='best', fontsize=24)
ax.tick_params(labelsize=18)
ax.set_xlabel(r'$\Delta T\,\left( {\rm K} \right)$', fontsize=24)
ax.set_ylabel("Rel. Cond.", fontsize=24)
ax.axvline(1.)
```

Out[1]:
```
<matplotlib.lines.Line2D at 0x7f7b60793b50>
```



7.6

The quantum efficiency q is a measure of how readily photons are converted into current by the CCD, and it depends on the surface reflectivity R, the absorption coefficient α, and the layer thickness z.

Equation 7.11 tells us how R depends on the index of refraction:

$$R = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2,$$

where n_1 is the index for air (≈ 1) and n_2 is the index for silicon (we're told $n_2 = 4$ at $500\,\mathrm{nm}$).

With these numbers, we get

$$R = \left(\frac{1 - 4.4}{1 + 4.4}\right)^2 = 0.40,$$

which means 40% of the incident energy is reflected, leaving 60% to be absorbed the CCD.

Next, we need to determine the absorption coefficient for silicon at 500 nm. Figure 7.10 gives that information. The figure suggests $\alpha = 10^4\,\mathrm{cm}^{-1}$. (Unfortunately, Chromey has a typo in the figure caption. You can see a similar plot with the right units here - https://www.pveducation.org/pvcdrom/materials/optical-properties-of-silicon.)

So we can estimate the required thickness z to achieve a given quantum efficiency q using

$$q = (1 - R)\,e^{-\alpha z} \Rightarrow z = \frac{\ln\left(\frac{1-R}{q}\right)}{\alpha} = \frac{\ln\left(\frac{0.6}{0.4}\right)}{\left(10^4\,\mathrm{cm}^{-1}\right)} \approx \boxed{405\,\mathrm{nm}}.$$

8.1

We're told we start out with a quantum efficiency of 40% for our detector.

Referring to Problem 7.6 above, we can cast the quantum efficiency as

$$q = (1 - R)\,e^{-\alpha z}.$$

We are told that the reflectivity of the detector is reduced from 30% to 5%. Since nothing else about the detector has changed, we can calculate the ratio of the new to the old efficieny as

$$\frac{q'}{q} = \frac{1-R'}{1-R},$$

where the primed quantities represent the new values.

Solving for q' gives

$$q' = q\left(\frac{1-R'}{1-R}\right) = (0.4)\left(\frac{1-0.05}{1-0.3}\right) = \boxed{0.54}.$$

8.2

Equation 8.5 tells us

$$\mathrm{DQE} = \frac{(\mathrm{SNR}_{\mathrm{out}})^2}{(\mathrm{SNR}_{\mathrm{perfect}})^2} = \frac{(\mathrm{SNR}_{\mathrm{out}})^2}{N_{\mathrm{in}}}.$$

We are told that the measurement involves 10^4 photons, which will take as N_{in}. So next we'll need the signal-to-noise ratio for the voltages, $\mathrm{SNR}_{\mathrm{out}}$.

Our average voltage (the signal) is

$$S = \frac{(113+120+115)\ \mathrm{mV}}{3} = 116\,\mathrm{mV}.$$

And the noise will be the standard deviation of the voltage measurements:

$$\sigma = 3\,\mathrm{mV}.$$

So now we can write our DQE as

$$\mathrm{DQE} = \frac{\left(\frac{116\ \mathrm{mV}}{3\ \mathrm{mV}}\right)^2}{10^4} \approx \boxed{0.15}.$$

8.3

The detector in this problem has a quantum efficiency of q. That means that, for every N_{in} photon that strikes the surface of the detector, $N_{\mathrm{detect}} = qN_{\mathrm{in}}$ are actually detected (and $(1-q)\,N_{\mathrm{in}}$ are NOT detected).

The detector is also said to have a quantum yield of y with an associated uncertainty $\sigma(y)$. That means that each of the $N_{\mathrm{detect}} = qN_{\mathrm{in}}$ induces $N_{\mathrm{events}} = yN_{\mathrm{detect}}$ events that are counted up by the detector.

So, as described by Equation 2.31 back in Chapter 2, the uncertainty $\sigma(N_{\mathrm{events}})$ associated with the events is given by

$$\sigma^2(N_{\mathrm{events}}) = \left(\frac{\partial N_{\mathrm{events}}}{\partial N_{\mathrm{detect}}}\right)^2 \sigma^2\left(N_{\mathrm{detect}}\right) + \left(\frac{\partial N_{\mathrm{events}}}{\partial y}\right)^2 \sigma^2\left(y\right) = y^2 N_{\mathrm{detect}} + N_{\mathrm{detect}}^2 \sigma^2(y),$$

taking Poisson error bars for N_{detect} (i.e., $\sigma^2\left(N_{\mathrm{detect}}\right) = N_{\mathrm{detect}}$).

To calculate DQE, we need $(SNR)^2_{\mathrm{out}}$ and $(SNR)^2_{\mathrm{perfect}}$. The latter is easy:

$$(SNR)^2_{\mathrm{perfect}} = N_{\mathrm{in}}.$$

The former is

$$(SNR)^2_{\mathrm{out}} = \frac{y^2 N_{\mathrm{detect}}^2}{y^2 N_{\mathrm{detect}} + N_{\mathrm{detect}}^2 \sigma^2(y)} = \frac{N_{\mathrm{detect}}^2}{N_{\mathrm{detect}} + N_{\mathrm{detect}}^2 \left(\frac{\sigma(y)}{y}\right)^2}$$

Now to calculate DQE:

$$\mathrm{DQE} = \frac{\frac{N_{\mathrm{detect}}^2}{N_{\mathrm{detect}} + N_{\mathrm{detect}}^2 \left(\frac{\sigma(y)}{y}\right)^2}}{N_{\mathrm{in}}}$$

But remember $N_{\mathrm{detect}} = qN_{\mathrm{in}}$, so

$$\mathrm{DQE} = \frac{\frac{q^2 N_{\mathrm{in}}^2}{qN_{\mathrm{in}} + q^2 N_{\mathrm{in}}^2 \left(\frac{\sigma(y)}{y}\right)^2}}{N_{\mathrm{in}}} = \frac{q^2 N_{\mathrm{in}}}{qN_{\mathrm{in}} + q^2 N_{\mathrm{in}}^2 \left(\frac{\sigma(y)}{y}\right)^2} = \boxed{\frac{q}{1 + qN_{\mathrm{in}} \left(\frac{\sigma(y)}{y}\right)^2}}$$

So you can see that, if $\sigma(y) = 0$, then $\mathrm{DQE} = q$. However, since all numbers are positive, if $\sigma(y) > 0$, then $\mathrm{DQE} < q$.

8.4

We'll need to recall the definition of QE (Equation 8.1):

$$\mathrm{QE} = \frac{N_{\mathrm{detect}}}{N_{\mathrm{in}}}.$$

We're told that $\mathrm{QE} = 0.9$, so if 1000 photons are incident in the 1-second exposure, we'll actually detect $(0.9) \times 1000 = 900$ of them.

Then recall the definition of DQE (Equation 8.5):

$$\mathrm{DQE} = \frac{(\mathrm{SNR})_{\mathrm{out}}^2}{(\mathrm{SNR})_{\mathrm{perfect}}^2}.$$

For our detector we have a signal $S = 900$ photons and sources of noise including the dark current $\left(1\ \mathrm{photon\,s^{-1}}\right) \times (1\,\mathrm{s}) = 1\ \mathrm{photon}$ and a read noise of 3 photons.

Therefore, our out SNR is

$$(\mathrm{SNR})_{\mathrm{out}} = \frac{900}{\sqrt{(900+1+3)+1+3}} \approx 30.$$

A perfect detector would have

$$(\mathrm{SNR})_{\mathrm{perfect}} = \sqrt{1000} \approx 32.$$

So

$$\mathrm{DQE} = \frac{30^2}{32^2} = \boxed{0.89}.$$

For the 4-second exposure, we have all the same numbers except now the dark current is $\left(1\ \mathrm{photon}\,\mathrm{s}^{-1}\right) \times (400\,\mathrm{s}) = 400\ \mathrm{photon}.$

So

$$(\mathrm{SNR})_{\mathrm{out}} = \frac{900}{\sqrt{(900+400+3)+400+3}} \approx 22.$$

And

$$\mathrm{DQE} = \frac{22^2}{32^2} = \boxed{0.47}.$$