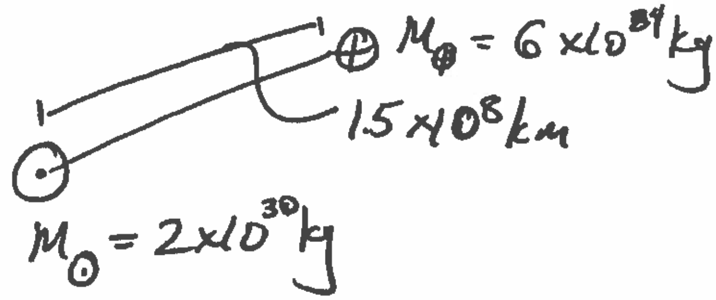
PHYS 341 – Mechanics – HW3 Sol'ns

3.16

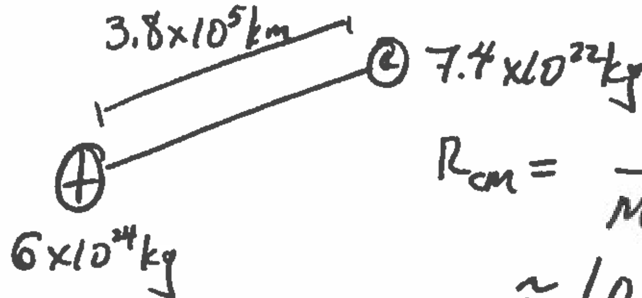


$$R_{cm} = \frac{M_\odot(0) + M_\oplus(1.5\times10^8\,\text{km})}{M_\odot + M_\oplus}$$

$$\simeq (3\times10^{-6})(1.5\times10^8\,\text{km}) = \boxed{4.5\times10^2\,\text{km from Sun's center}}$$

$R_\odot = 7\times10^5\,\text{km} > R_{cm}$, so the CM lies inside the Sun.

3.17

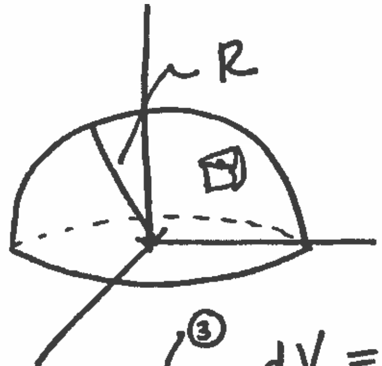


$$R_{cm} = \frac{M_☾}{M_\oplus + M_☾}(3.8\times10^5\,\text{km})$$

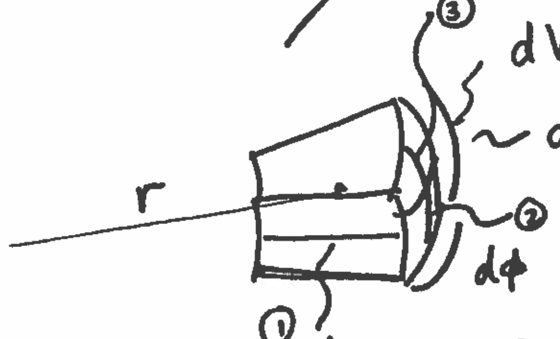
$$\simeq (0.01)(3.8\times10^5\,\text{km}) = 3.8\times10^3\,\text{km}$$

$\simeq$ 60% $\cdot R_\oplus$ from Earth's center.

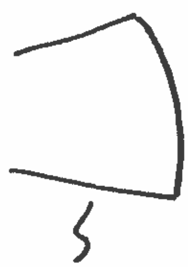
3.22

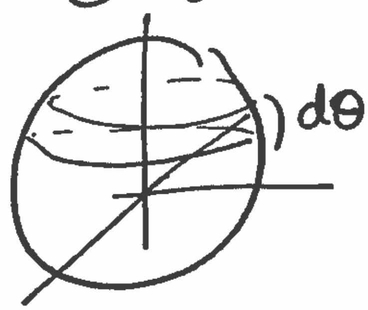


$dV = dx\,dy\,dz$



① $= dr$



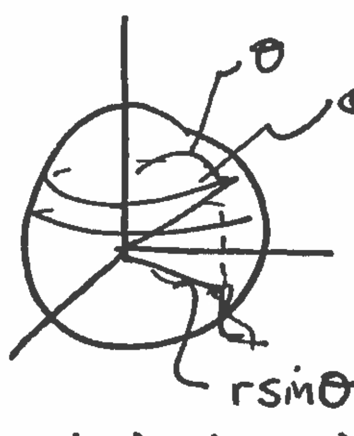


② $= (r + \frac{1}{2}dr) \cdot d\theta \simeq r\,d\theta$ ✓

Define angle!

$(\text{angle}) = \dfrac{(\text{arc})}{(\text{tot. arc})}$

③



$(\text{circum.}) = 2\pi(r\sin\theta)$

So $(\text{arc length}) = d\phi(r\sin\theta)$

③ $= r\sin\theta \cdot d\phi$

$\Rightarrow dV = (dr) \cdot (r\,d\theta) \cdot (r\sin\theta\,d\phi)$

$= r^2 dr \sin\theta\,d\theta \cdot d\phi$ ✓

(3.22) Cont.

The x and y positions of CM are both zero by symmetry. So we just need to calculate $\langle z \rangle_M$.

Intuitively, we expect $\langle z \rangle_M$ to be a bit less than R since more mass is located near the x-y plane.

$$\langle z \rangle_M = \int_V z \cdot \rho \cdot dV = \rho \int_{r=0}^{R} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} r\cos\theta \cdot r^2 dr \sin\theta \, d\theta \, d\phi$$

$$= 2\pi\rho \cdot \int_{r=0}^{R} r^3 dr \cdot \int_{\theta=0}^{\pi/2} \cos\theta \sin\theta \, d\theta$$

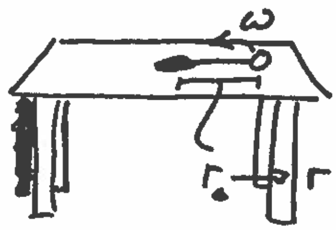
$$= \frac{1}{2}\pi\rho \cdot R^4 \cdot \int_{\theta=0}^{\pi/2} \cos\theta \sin\theta \, d\theta; \text{ Take } u \equiv \sin\theta$$

$$du = \cos\theta \cdot d\theta$$

$$\Rightarrow \frac{1}{2}\pi\rho R^4 \int_{u=0}^{1} u \cdot du = \frac{1}{4}\pi\rho R^4 ; M = \frac{2}{3}\pi R^3 \cdot \rho$$

$$\Rightarrow \boxed{\langle z \rangle_M = \frac{3}{8} \cdot R}$$

3.25



Since the string exerts a radial tension $\perp$ to the particle's linear velocity, changing the string's length doesn't change the mass's ang. mom.

$$\vec{L} = \vec{r} \times \vec{p} \ ; \quad \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}, \text{ but } \vec{F} \parallel \vec{r},$$

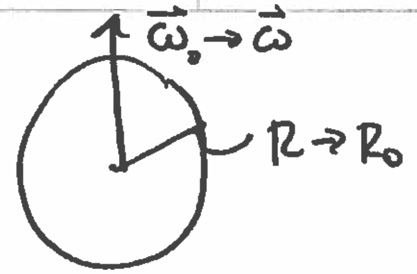
$$\text{so} \quad \vec{r} \times \vec{F} = 0 = \dot{\vec{L}} \Rightarrow \vec{L} = \text{const}$$

$L_0 = r_0 \cdot m v_T^{(0)}$ where $v_T^{(0)}$ is the initial tangential velocity

$$v_T^{(0)} = r_0 \cdot \omega_0 \Rightarrow L_0 = m r_0^2 \omega_0 = m r^2 \omega$$

$$\Rightarrow \boxed{\omega = \left(\frac{r_0}{r}\right)^2 \omega_0}$$

3.29



Ang. mom. is constant by assumption.

$$L_0 = I_0\omega_0 = I\cdot\omega$$

If radius is R with density ρ,

$$M = \rho\cdot\frac{4\pi}{3}R^3$$

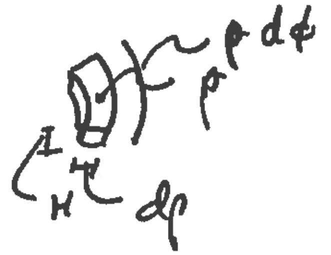
So $I = \frac{2}{5}MR^2 = \frac{2}{5}\rho\cdot\frac{4\pi}{3}R^5$

$$\Rightarrow \frac{2}{5}\cancel{\rho}\frac{4\pi}{3}R_0^5\cdot\omega_0 = \frac{2}{5}\cancel{\rho}\frac{4\pi}{3}R^5\omega$$

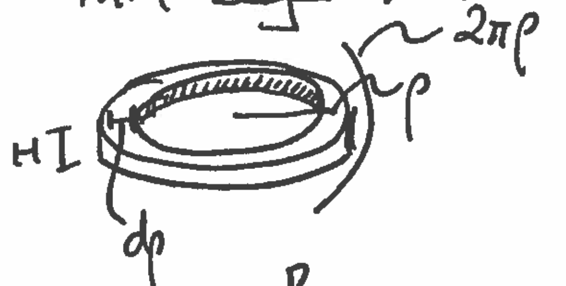
$$\Rightarrow \boxed{\omega = \left(\frac{R_0}{R}\right)^5\omega_0}$$

So if $R = 2R_0$, $\omega = \left(\frac{1}{2}\right)^5\omega_0 = \boxed{\frac{1}{32}\omega_0}$

3.31 $I = \sum m\rho^2 \rightarrow \int dm \cdot \rho^2$



Since every mass element at a ρ value for any ϕ-value is at the same radial distance from the axis, we can divide the disc into very thin washers



$dm = \lambda \cdot 2\pi\rho \cdot H \cdot d\rho$

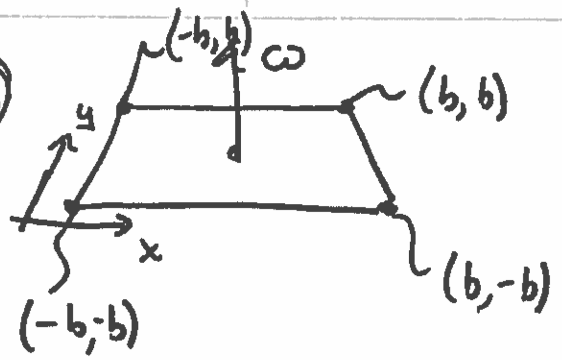
where λ is the mass per washer.

$$\Rightarrow I = \int_{\rho=0}^{R} \lambda \cdot 2\pi\rho H \, d\rho \cdot \rho^2 = 2\pi\lambda H \int_{\rho=0}^{R} \rho^3 \, d\rho$$

$$= 2\pi\lambda H \, \tfrac{1}{4} R^4 \quad ; \quad M = \pi R^2 H \cdot \lambda,$$

$$\Rightarrow I = (\pi R^2 \cdot H\lambda) \cdot \tfrac{1}{2} R^2 = \boxed{\tfrac{1}{2} M R^2}$$

3.33



$$I = \int dm \cdot r^2 \quad , \quad dm = \rho \cdot dx\,dy \quad r^2 = x^2 + y^2$$

$$\Rightarrow I = \int_{x=-b}^{b} \int_{y=-b}^{b} \rho\, dx\, dy \,(x^2 + y^2)$$

$$= \rho \int_{x=-b}^{b} \left[y \cdot x^2 + \tfrac{1}{3} y^3 \right]_{y=-b}^{b} dx = \rho \int_{x=-b}^{b} \left(2b \cdot x^2 + \tfrac{2}{3} b^3 \right) dx$$

$$= \rho \left[2b \tfrac{1}{3} x^3 + \tfrac{2}{3} b^3 \cdot x \right]_{x=-b}^{b} = \rho \frac{8}{3} b^4$$

$$M = \rho \cdot (2b)^2 = \rho \cdot 4b^2 \Rightarrow I = (\rho \cdot 4b^2) \cdot \tfrac{2}{3} b^2$$

$$\Rightarrow \underline{\underline{I = \tfrac{2}{3} M b^2}}$$