7.3

$\mathscr{L} = T - U = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - \frac{1}{2}k\left(x^2 + y^2\right)$
$\frac{\partial \mathscr{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{x}} \Rightarrow \boxed{-kx = m\ddot{x}}$, a harmonic oscillation in x. The solution for the y-coordinate is the same.

7.14

$\mathscr{L} = T - U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\omega^2 + mgx.$

($U = -mgx$ because the yo-yo starts at $x = 0$ and accelerates into positive x.)

We're told $I = \frac{1}{2}mR^2$ for a uniform disc, so $\mathscr{L} = \frac{1}{2}m\dot{x}^2 + \frac{1}{4}mR^2\omega^2 + mgx.$

Since the yo-yo is rolling without slipping, the rate at it falls must be equal to the rate at which it rolls, i.e.
$R\omega = \dot{x} \Rightarrow \mathscr{L} = \frac{1}{2}m\dot{x}^2 + \frac{1}{4}m\dot{x}^2 + mgx = \frac{3}{4}m\dot{x}^2 + mgx.$

Applying Euler-Lagrange: $mg = \frac{3}{2}m\ddot{x} \Rightarrow \boxed{\ddot{x} = \frac{2}{3}g}$.

7.17

The total length of the string connecting the masses $l = x + y = \text{const} \Rightarrow y = l - x$ and $\dot{y} = -\dot{x}$. Moreover, the pulley turns without slipping, i.e. $R\omega = \dot{x}$.

$$\mathscr{L} = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2 + \frac{1}{2}I\omega^2 + m_1gx + m_2gy = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}\left(\frac{I}{R^2}\right)\dot{x}^2 + m_1gx + m_2g(l - x)$$

$\Rightarrow \mathscr{L} = \frac{1}{2}\left[m_1 + m_2 + \frac{I}{R^2}\right]\dot{x}^2 + g(m_1 - m_2)x$. The constant l that appears in U can be ignored.

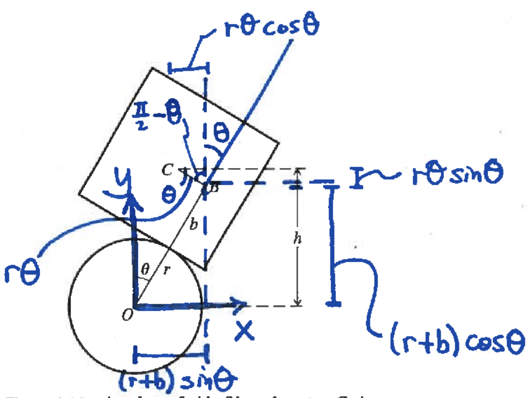
E-L gives: $\boxed{\ddot{x} = g\left(\frac{m_1 - m_2}{m_1 + m_2 + \frac{I}{R^2}}\right)}$, indicating a downward acceleration for m_1 smaller than gravity, which is what we expect: the gravitational pull on m_2 and angular inertia of the pulley should both act to slow m_1.

An interesting limit to consider: $\lim_{m_1 \to \infty} \ddot{x} = g$, i.e. for very large m_1, the gravitational pull on m_2 and angular inertia of the pulley are unimportant.

7.32

In [2]:
```python
from IPython.display import Image
Image(filename='cube_wheel.png', width='300')
```

Out[2]:



To work out the velocity of the cube's center of mass (CM), let's start with its position $\vec{R}_{\text{CM}}$ in Cartesian coordinates and then take a time derivative. The diagram above shows the x and y-positions as functions of θ.

$$\vec{R}_{\text{CM}} = [(r+b)\sin\theta - r\theta\cos\theta]\,\hat{x} + [(r+b)\cos\theta + r\theta\sin\theta]\,\hat{y}.$$

Then $\dot{\vec{R}}_{\text{CM}} = \left[(r+b)\cos\theta\,\dot{\theta} - r\dot{\theta}\cos\theta + r\theta\sin\theta\,\dot{\theta}\right]\hat{x} + \left[-(r+b)\sin\theta\,\dot{\theta} + r\dot{\theta}\sin\theta + r\theta\cos\theta\,\dot{\theta}\right]\hat{y}.$

The book suggests taking $\theta \ll 1$ to simplify things, so let's expand to second order in θ.

$$\dot{\vec{R}}_{\text{CM}} \approx \left[(r+b)\left(1-\tfrac{1}{2}\theta^2\right)\dot{\theta} - r\left(1-\tfrac{1}{2}\theta^2\right)\dot{\theta} + r\theta^2\,\dot{\theta}\right]\hat{x} + \left[-(r+b)\theta\,\dot{\theta} + r\theta\,\dot{\theta} + r\theta\,\dot{\theta}\right]\hat{y}.$$

The kinetic energy of CM motion to leading order is $\frac{1}{2}m\left(\dot{\vec{R}}_{\text{CM}}\cdot\dot{\vec{R}}_{\text{CM}}\right) \approx \frac{1}{2}mb^2\dot{\theta}^2$. The $\theta^2\dot{\theta}^2$ term is higher order and neglected.

For the cube's potential energy, we have $U = mgh = mg\,[(r+b)\cos\theta + r\theta\sin\theta] \approx mg\left[(r+b)\left(1-\tfrac{1}{2}\theta^2\right) + r\theta^2\right] = \frac{1}{2}mg\,(r-b)\,\theta^2 +$ (an ignorable constant).

Our Lagrangian is $\mathscr{L} = T - U =$ (kinetic energy of cube CM) + (rotational kinetic energy of cube) - (gravitational potential of cube), so

$$\mathscr{L} = \tfrac{5}{6}mb^2\dot{\theta}^2 - \tfrac{1}{2}mg\,(r-b)\,\theta^2.$$

E-L: $-mg\,(r-b)\,\theta = \frac{5}{3}mb^2\ddot{\theta} \Rightarrow \ddot{\theta} = -\frac{3g(r-b)}{5b^2}\theta$, which is the harmonic oscillator equation with an angular frequency $\boxed{\omega_0 = \sqrt{\dfrac{3g\,(r-b)}{5b^2}}}$.

7.34

(a) Take the spring length to be L. We need to figure out the relationship between the velocity $\dot{x}_s$ of a tiny length dx_s of the spring and the distance x_s of that length from the end of the spring fixed to the wall. Keep in mind x_s represents only distances measured from the fixed end of the spring, NOT the position of the free end of the spring x.

Since the fixed end of the spring is fixed, we require $\dot{x}_s(x_s = 0) =$. The free end of the spring moves with velocity $\dot{x}_s(x_s = x) = \dot{x}$. The requirement for uniform stretching suggest a linear relationship between $\dot{x}_s$ and x_s, and with the above boundary conditions, we have $\dot{x}_s = \dot{x}\left(\frac{x_s}{L}\right)$.

The kinetic energy of each tiny length is $dT = \frac{1}{2}dm\,\dot{x}_s^2$, so the whole spring's kinetic energy is the related integral, where dm is the mass of the tiny length and $dm = \frac{M}{L}dx_s$. So $T = \int \frac{1}{2}dm\,\dot{x}_s^2 = \frac{1}{2}\frac{M}{L}\dot{x}^2 \int_{x_s=0}^{L}\left(\frac{x_s}{L}\right)^2 dx_s = \boxed{\frac{1}{6}M\dot{x}^2}$. The factor $\frac{1}{3}$ comes in because the entire spring is not in motion.

Including the cart, the Lagrangian for the system is now $\mathscr{L} = \frac{1}{2}m\dot{x}^2 + \frac{1}{6}M\dot{x}^2 - \frac{1}{2}kx^2$.

(b) Applying E-L: $\left(m + \frac{1}{3}M\right)\ddot{x} = -kx \Rightarrow \boxed{\omega_0 = \sqrt{\frac{k}{m + M/3}}}$.

7.41

The bead executes motion in three dimensions (ρ, z, and ϕ), giving three velocities $\dot{\rho}$, $\dot{z}$, and $\rho\omega$. Thus, $T = \frac{1}{2}m\left(\dot{\rho}^2 + \dot{z}^2 + \rho^2\omega^2\right)$. However, the bead is constrained to move along the wire such that $z = k\rho^2$, so $\dot{z} = 2k\rho\dot{\rho}$. Thus, $T = \frac{1}{2}m\left(\dot{\rho}^2 + (2k\rho\dot{\rho})^2 + \rho^2\omega^2\right)$. The potential energy $U = mgz = mgk\rho^2$, giving $\boxed{\mathscr{L} = \frac{1}{2}m\left(\dot{\rho}^2 + (2k\rho\dot{\rho})^2 + \rho^2\omega^2\right) - mgk\rho^2}$.

Applying E-L and doing some algebra gives $\left(4k^2\rho^2 + 1\right)\ddot{\rho} = -\left(2gk + 4k^2\dot{\rho}^2 - \omega^2\right)\rho$

Stability requires that small displacements or velocities from equilibrium result in accelerations back toward the equilibrium. $\rho = 0$ seems an obvious point of equilibrium since there $\ddot{\rho} = 0$. Any other value of ρ would result in a non-zero $\ddot{\rho}$ in general. So take $\rho = \delta\rho$ and $\dot{\rho} = \delta\dot{\rho}$, and $\delta\rho, \delta\dot{\rho} \ll 1$.

Taking only the leading order terms in the E-L equation gives $\ddot{\rho} \approx -\left(2gk - \omega^2\right)\rho$, so equilibrium requires the term in parentheses on the right hand side is positive, i.e. $\boxed{2gk \geq \omega^2}$. The corresponding natural frequency of the system is $\omega_0 = \sqrt{2gk - \omega^2}$.