Homework 4

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Handed out: 2015 Feb 1

Due: 2015 Feb 6

1. 4.2
2. 4.4
3. 4.22
4. 4.23
5. 4.30
6. 4.35

PHYS 341 - Mechanics — HW4 Sol'ns 2015 Feb. 8

(4.2) (a) $W = \int \vec{F}\cdot d\vec{r} = \int_O^Q F_x\,dx + \int_Q^P F_y\,dy$

From O to Q, $y=0$.

$$\int_0^Q F_x\,dx = \int_{x=0}^{1} x^2\,dx = \frac{1}{3}$$

$$\int_Q^P F_y\,dy = \int_{y=0}^{1} [2x\cdot y]_{x=1}\,dy = \int_{y=0}^{1} 2y\cdot dy = 1$$

$$\Rightarrow \boxed{\int_{(a)} \vec{F}\cdot d\vec{r} = \frac{4}{3}}$$

(b) $\int_O^P (F_x\,dx + F_y\,dy)$, with $y = x^2$, $dy = 2x\,dx$

$$\Rightarrow \int_O^P (\ ) = \int_{x=0}^{1} (x^2\cdot dx + 2x\cdot x^2\cdot 2x\,dx)$$

$$= \int_{x=0}^{1} (x^2 + 4x^4)\,dx = \left[\frac{1}{3}x^3 + \frac{4}{5}x^5\right]_{x=0}^{1}$$

$$= \boxed{\frac{17}{15}}$$

4.2 (c) $x = t^3$ $\quad\quad y = t^2$

$dx = 3t^2 dt \quad\quad dy = 2t \cdot dt$

[redacted]

$$\int_{(c)} \vec{F} \cdot d\vec{r} = \int_{t=0}^{1} \left(t^6 \cdot 3t^2 dt + 2t^3 \cdot t^2 \cdot 2t dt \right)$$

$$= \int_{t=0}^{1} \left(3t^8 + 4t^6 \right) dt = \left[\frac{3}{9} t^9 + \frac{4}{7} t^7 \right]_0^1$$

$$= \frac{3}{9} + \frac{4}{7} = \boxed{\frac{57}{63}}$$

(4.4) (a) $L = \text{const.}$, ang. mom. conserved

$$m\omega_0 r_0^2 = m\omega r^2 \Rightarrow \boxed{\omega = \left(\frac{r_0}{r}\right)^2 \omega_0}$$

(b) $F_r = m(\ddot{r} - r\dot{\phi}^2) \leftarrow$ The only force that can do work.

So $dW = F_r \cdot dr = F_r \dot{r}\, dt$

Now $\dot{r}$ is small always, so $\ddot{r}$ is also small.

$$\Rightarrow dW = m(\ddot{r} - r\dot{\phi}^2)\dot{r}\, dt \simeq -mr\dot{\phi}^2 \dot{r}\, dt$$

$$\simeq -mr\dot{\phi}^2 dr = -mr\omega^2 dr$$

So total work done from r_0 to r

$$-\int_{r'=r_0}^{r} mr'\omega^2 dr' = -\int_{r_0}^{r} mr'\left(\frac{r_0}{r'}\right)^4 \omega_0^2\, dr'$$

$$= -mr_0^4\omega_0^2 \int_{r'=r_0}^{r} (r')^{-3} dr' = \frac{1}{2} m r_0^4 \omega_0^2 \left(r^{-2} - r_0^{-2}\right)$$

$$= \boxed{\frac{1}{2} m r_0^2 \omega_0^2 \left[\left(\frac{r_0}{r}\right)^2 - 1\right] \simeq \Delta W}$$

(c) $\Delta T = \frac{1}{2} m (v^2 - v_0^2) = \frac{1}{2} m \left[(r\omega)^2 - (r_0\omega_0)^2\right]$

$$= \frac{1}{2} m \left[r^2 \left(\frac{r_0}{r}\right)^4 \omega_0^2 - r_0^2\omega_0^2\right] = \boxed{\frac{1}{2} m r_0^2 \omega_0^2 \left[\left(\frac{r_0}{r}\right)^2 - 1\right]} = \Delta W$$

(4.22) $\vec{F} = \frac{\gamma \hat{r}}{r^2}$, [redacted]

For $(\nabla \times \vec{F})_{\hat{r}}$

$$= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta \cdot \cancelto{0}{F_\phi}) - \frac{\partial}{\partial\phi}\cancelto{0}{F_\theta}\right]$$

$$(\nabla \times \vec{F})_{\hat{\theta}} = \left[\frac{1}{r\sin\theta}\frac{\partial \cancelto{0}{F_r}}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(r\cancelto{0}{F_\phi})\right]$$

$$(\nabla \times \vec{F})_{\hat{\phi}} = \frac{1}{r}\left[\frac{\partial}{\partial r}(r\cancelto{0}{F_\theta}) - \cancelto{0}{\frac{\partial}{\partial\theta}F_r}\right]$$

$\Rightarrow \nabla \times \vec{F} = 0.$

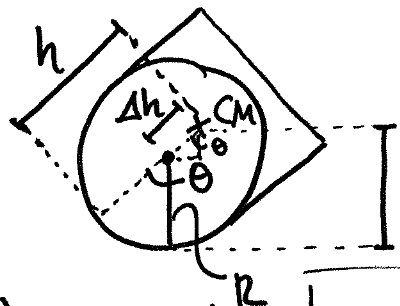
4.23 (a) $\vec{F} = k(x, 2y, 3z)$; $\nabla\times\vec{F} = 0$ because each component only depends on the corresponding coordinate.

(b) $\vec{F} = k(y, x, 0)$; $\nabla\times\vec{F} = k\begin{pmatrix}\partial x\\ \partial y\\ \partial z\end{pmatrix}\times\begin{pmatrix}y\\ x\\ 0\end{pmatrix} = k\begin{pmatrix}0\\ 0\\ \frac{\partial y}{\partial y}-\frac{\partial x}{\partial x}=0\end{pmatrix}$

(c) $\vec{F} = k(-y, x, 0)$; $\nabla\times\vec{F} = k\begin{pmatrix}0\\ 0\\ -2\end{pmatrix}$

[redacted]

(4.30) (a)



$$h' = R + \Delta h \cos\theta$$

$$\Delta h = h - R$$

$$\Rightarrow h' = R + (h-R)\cos\theta$$

$$U(\theta) = mgh' = \boxed{mg\left[R + (h-R)\cos\theta\right]}$$

(b) For stable equilibrium at $\theta = 0$

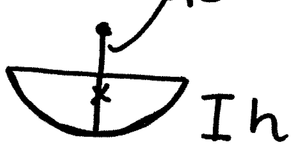
$$\frac{dU}{d\theta} = 0 \quad \& \quad \frac{d^2U}{d\theta^2} > 0$$

$$\left.\frac{dU}{d\theta}\right|_{\theta=0} = -mg(h-R)\sin\theta\Big|_{\theta=0} = 0. \quad \checkmark$$

$$\left.\frac{d^2U}{d\theta^2}\right|_{\theta=0} = -mg(h-R)\cos\theta\Big|_{\theta=0} = -mg(h-R) > 0$$

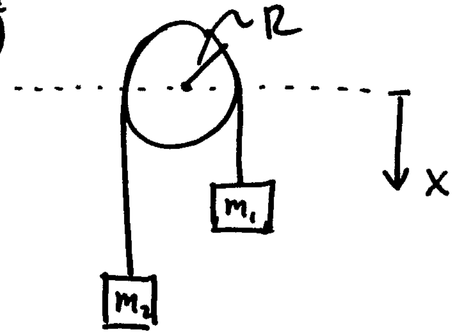
$$\Rightarrow h - R < 0 \Rightarrow \boxed{h < R}$$

which means the toy looks like



More boat shaped, assuming uniform density.

4.35



(a) $E = T + U$

$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}I\omega^2$

$U = -m_1 g x_1 - m_2 g x_2$

But total length of string ℓ' is fixed

$\ell' = x_1 + \frac{1}{2}2\pi R + x_2 \Rightarrow \ell = \ell' - \pi R = x_1 + x_2$

$\dot{\ell} = 0 = \dot{x}_1 + \dot{x}_2 \Rightarrow \dot{x}_2 = -\dot{x}_1$

And pulley turns without slipping

$R\omega = +\dot{x}_1 \Rightarrow T = \frac{1}{2}\left(m_1 + m_2 + \frac{I}{R^2}\right)\dot{x}_1^2$

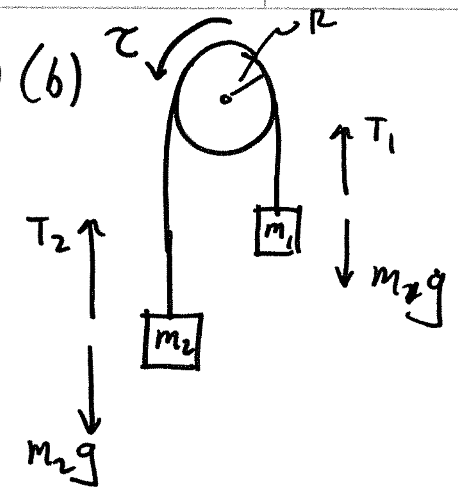
$U = -m_1 g x_1 - m_2 g(\ell - x_1) = -(m_1 - m_2)g x_1 - m_2 g \ell$

Since we're only ever interested in changes in U, the last term, $-m_2 g \ell$, can be dropped.

$\Rightarrow U = -(m_1 - m_2)g x_1$

$$\boxed{E = \frac{1}{2}\left(m_1 + m_2 + \frac{I}{R^2}\right)\dot{x}_1^2 - (m_1 - m_2)g x_1}$$

4.35 (b)



$$m_1 \ddot{x}_1 = -m_1 g + T_1 \Rightarrow T_1 = m_1(g+\ddot{x}_1)$$

$$m_2 \ddot{x}_2 = -m_2 g + T_2 \Rightarrow T_2 = m_2(g+\ddot{x}_2) = m_2(g-\ddot{x}_1)$$

$$(\text{torque}) = \tau = +I\dot{\omega} = (T_1 - T_2)R$$

$$R\omega = +\dot{x}_1 \Rightarrow \dot{\omega} = +\frac{\ddot{x}_1}{R}$$

$$+I\dot{\omega} = \left[m_1(g+\ddot{x}_1) - m_2(g-\ddot{x}_1)\right]\cdot R$$

$$\Rightarrow +\frac{I}{R^2}\ddot{x}_1 = m_1(g+\ddot{x}_1) - m_2(g-\ddot{x}_1)$$

$$\Rightarrow \underline{\underline{\left(\frac{I}{R^2} + m_1 + m_2\right)\ddot{x}_1}} = +(m_1 - m_2)g$$

AND $$\frac{dE}{dt} = 0 = \left(m_1 + m_2 + \frac{I}{R^2}\right)\ddot{x}_1 \cdot \dot{x}_1 - (m_1 - m_2)g\dot{x}_1$$

$$\Rightarrow \left(m_1 + m_2 + \underline{\underline{\frac{I}{R^2}\right)\ddot{x}_1 =}} (m_1 - m_2)g$$