17-02 复合函数微分法

一. 复合函数微分的链式法则

二. 一阶全微分形式不变性

上页 下页 返回

一.复合函数微分的链式法则

Th.17.5.若函数$u=\varphi(x), v=\psi(x)$在点x处都可导,函数$z=f(u,v)$在点(u,v)处具有连续的偏导数,则$z=f\big(\varphi(x),\psi(x)\big)$在点$x$处可导,且有全导数公式:

$$\frac{dz}{dx}=\frac{\partial z}{\partial u}\cdot\frac{du}{dx}+\frac{\partial z}{\partial v}\cdot\frac{dv}{dx}.$$

证明 设 x 获得增量 Δx,

则 $\Delta u=\varphi(x+\Delta x)-\varphi(x)$,

$\Delta v=\psi(x+\Delta x)-\psi(x)$.

$$\Delta z = f\left(u+\Delta u, v+\Delta v\right) - f(u,v)$$

$$= \left[f\left(u+\Delta u, v+\Delta v\right) - f\left(u, v+\Delta v\right) \right]$$

$$+ \left[f\left(u, v+\Delta v\right) - f(u,v) \right]$$

$$= f_u\left(u+\theta_1 \Delta u, v+\Delta v\right)\Delta u + f_v\left(u, v+\theta_2 \Delta v\right)\Delta v$$

$\left(\because z = f(u,v)\right.$在点$(u,v)$处有连续的偏导数,$\left.\right)$

$$= f_u\left(u,v\right)\Delta u + f_v\left(u,v\right)\Delta v + \varepsilon_1 \Delta u + \varepsilon_2 \Delta v$$

$$\varepsilon_i = \varepsilon_i\left(\Delta u, \Delta v\right), \lim_{\substack{\Delta u \to 0 \\ \Delta v \to 0}} \varepsilon_i = 0, 0 < \theta_i < 1, i = 1,2.$$

上页 下页 返回

$\because z = f(u,v)$在点(u,v)处有连续的偏导数,

$$\Delta z = \frac{\partial z}{\partial u}\Delta u + \frac{\partial z}{\partial v}\Delta v + \varepsilon_1 \Delta u + \varepsilon_2 \Delta v,$$

当$\Delta u \to 0, \Delta v \to 0$时有$\varepsilon_1 \to 0, \varepsilon_2 \to 0$,

$$\frac{\Delta z}{\Delta x} = \frac{\partial z}{\partial u}\cdot\frac{\Delta u}{\Delta x} + \frac{\partial z}{\partial v}\cdot\frac{\Delta v}{\Delta x} + \varepsilon_1 \frac{\Delta u}{\Delta x} + \varepsilon_2 \frac{\Delta v}{\Delta x},$$

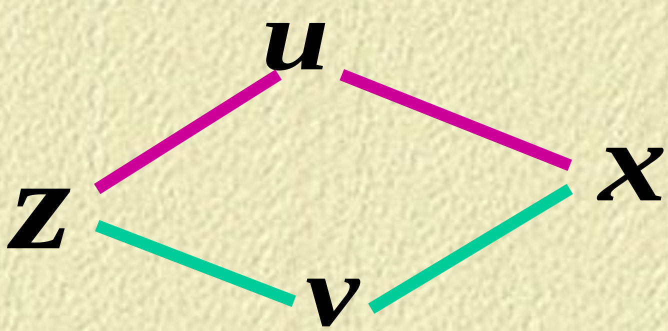
当 $\Delta x \to 0$时有$\Delta u \to 0, \Delta v \to 0$,

$$\lim_{\Delta x \to 0}\frac{\Delta u}{\Delta x} = \frac{du}{dx}, \lim_{\Delta x \to 0}\frac{\Delta v}{\Delta x} = \frac{dv}{dx},$$

$$\therefore \frac{dz}{dx} = \lim_{\Delta x \to 0}\frac{\Delta z}{\Delta x} = \frac{\partial z}{\partial u}\cdot\frac{du}{dx} + \frac{\partial z}{\partial v}\cdot\frac{dv}{dx}.$$

$$z = f(u,v), u = \varphi(x), v = \psi(x)$$

$$\Rightarrow \frac{dz}{dx} = \frac{\partial z}{\partial u}\cdot\frac{du}{dx} + \frac{\partial z}{\partial v}\cdot\frac{dv}{dx}.$$



上页 下页 返回

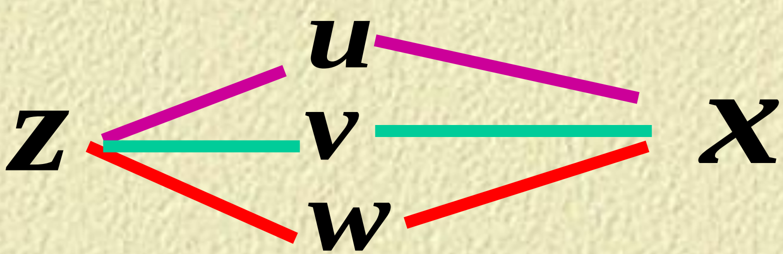
$$z = f(u,v),\ u = \varphi(x),\ v = \psi(x)$$

$$\Rightarrow \frac{dz}{dx} = \frac{\partial z}{\partial u}\cdot\frac{du}{dx} + \frac{\partial z}{\partial v}\cdot\frac{dv}{dx}.$$

该结论可推广至多个中间变量的情形,如

$$z = f(u,v,w),\ u = \varphi_1(x),\ v = \varphi_2(x),\ w = \varphi_3(x),$$

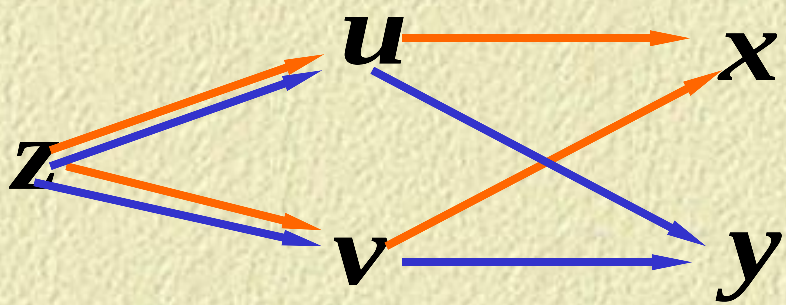
$$\Rightarrow \frac{dz}{dx} = \frac{\partial z}{\partial u}\cdot\frac{du}{dx} + \frac{\partial z}{\partial v}\cdot\frac{dv}{dx} + \frac{\partial z}{\partial w}\cdot\frac{dw}{dx}.$$



$Th.17.5'$.若函数$u=\varphi(x,y),v=\psi(x,y)$在点$(x,y)$处都可(偏)导,函数$z=f(u,v)$在点$(u,v)$处具有连续的偏导数,则$z=f\left(\varphi(x,y),\psi(x,y)\right)$在点$(x,y)$处可(偏)导,且有全导数公式:

$$\begin{cases}\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial x}+\dfrac{\partial z}{\partial v}\cdot\dfrac{\partial v}{\partial x}\\[2ex]\dfrac{\partial z}{\partial y}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial y}+\dfrac{\partial z}{\partial v}\cdot\dfrac{\partial v}{\partial y}\end{cases}.$$

链式法则如图示 (轨道图)

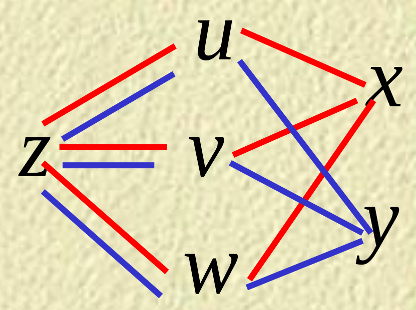


$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x},$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}.$$

类似地再推广：若函数$u=\varphi_1(x,y), v=\varphi_2(x,y)$, $w=\varphi_3(x,y)$在点(x,y)处都可(偏)导,函数$z=f(u,v,w)$在点(u,v,w)处具有连续的偏导数,则$z=f\big(\varphi_1(x,y),\varphi_2(x,y),\varphi_3(x,y)\big)$在点$(x,y)$处可(偏)导,且有

$$\begin{cases}\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial x}+\dfrac{\partial z}{\partial v}\cdot\dfrac{\partial v}{\partial x}+\dfrac{\partial z}{\partial w}\cdot\dfrac{\partial w}{\partial x}\\[2ex]\dfrac{\partial z}{\partial y}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial y}+\dfrac{\partial z}{\partial v}\cdot\dfrac{\partial v}{\partial y}+\dfrac{\partial z}{\partial w}\cdot\dfrac{\partial w}{\partial y}\end{cases}.$$



上页 下页 返回

例1.求幂指函数$y=(\sin x)^{x^2}\ (x\in(0,\pi))$的导数.

解 令$y=u^v$, $\begin{cases} u=\sin x \\ v=x^2 \end{cases}$,用全导数公式:

则 $$\frac{dy}{dx}=\frac{\partial y}{\partial u}\cdot\frac{du}{dx}+\frac{\partial y}{\partial v}\cdot\frac{dv}{dx}$$

$$=v\cdot u^{v-1}\cdot u'_x+u^v\ln u\cdot v'_x$$

$$=\cdots$$

所以为什么这种函数要称为“幂指函数”!

上页 下页 返回

例2.设$z=e^u\sin v$,而$\begin{cases} u=xy \\ v=x+y \end{cases}$,求:$\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}$.

解 $$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$=e^u\sin v\cdot y+e^u\cos v\cdot 1=e^u\left(y\sin v+\cos v\right),$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$=e^u\sin v\cdot x+e^u\cos v\cdot 1=e^u\left(x\sin v+\cos v\right).$$

设$z = e^u \sin v$,而$\begin{cases} u = xy \\ v = x + y \end{cases}$,

记$z = e^u \sin v = f(u,v)$,则

$$\frac{\partial z}{\partial u} = e^u \sin v = f_u(u,v),$$

$$\frac{\partial z}{\partial v} = e^u \cos v = f_v(u,v),$$

$f_u(u,v)$, $f_v(u,v)$仍是以u,v为中间变量,以x, y为自变量的两个新的函数.

上页 下页 返回

例3.设f有连续的一阶偏导数,求：$\dfrac{\partial z}{\partial x}$.

(1).$z = f(x+y, xy)$;

(2).$z = f\big(\varphi(x,y), x, y\big)$,$\varphi$可微.

解(1).令$u = x+y, v = xy$,

记$f_1 = \dfrac{\partial f(u,v)}{\partial u}, f_2 = \dfrac{\partial f(u,v)}{\partial v}$,

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = f_1 + y f_2.$$

$$(2).z = f\left(\varphi(x,y),x,y\right),\begin{cases} u=\varphi(x,y) \\ v=x \\ w=y \end{cases},$$

$$令 z = f(u,v,w)$$

$$\therefore \frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial z}{\partial w}\cdot\frac{\partial w}{\partial x}$$

$$=f_1\cdot\frac{\partial u}{\partial x}+f_2\cdot 1+f_3\cdot 0,\ \frac{\partial u}{\partial x}=\varphi_x(x,y),$$

$$\Rightarrow \frac{\partial z}{\partial x}=f_1\cdot\frac{\partial u}{\partial x}+f_2=f_1\cdot\varphi_x+f_2$$

上页 下页 返回

$$(2).z = f\left(\varphi(x,y),x,y\right) = f(u,v,w),$$

$$u = \varphi(x,y), v = x, w = y,$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial z}{\partial w}\cdot\frac{\partial w}{\partial x}\cdots(1)$$

按理,$v = x, \dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x}$,但需注意到,(1)

式左边的$\dfrac{\partial z}{\partial x}$是全部,右边的$\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x}$

是部分,要注意两者的区别,所以有

的书上记之为$\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x} = \dfrac{\partial f}{\partial x}$.

举例说明：

将$z=\sqrt{x^2+y^2}+e^x+y$ 视为

$$z=f(u,x,y)=\sqrt{u}+e^x+y, u=x^2+y^2,$$
$$v=x$$

$$\therefore \frac{\partial z}{\partial x}=\left(\sqrt{x^2+y^2}+e^x+y\right)'_x,$$

而$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}=\left(\sqrt{u}+e^x+y\right)'_x=e^x,$

此处将u, x, y看作地位对等的

都是自变量.

上页 下页 返回

注1 : $Th.17.5$ 中外层函数f要求是可微而不仅仅是可偏导,由下面的例子可知f可微的条件是需要的.

例如, $z = f(u,v) = \begin{cases} \dfrac{2u^2v}{u^2+v^2}, u^2+v^2 \neq 0 \\ 0 \ , \ u^2+v^2 = 0 \end{cases}$, 易知

$f_u(0,0) = f_v(0,0) = 0$,但是f在点$(0,0)$处不可微.

取$u = v = x$,则 $z = x, z'_x = 1$,不过在$x = 0$ 时,

$$\left.\left(\frac{\partial z}{\partial u}\cdot\frac{du}{dx}+\frac{\partial z}{\partial v}\cdot\frac{dv}{dx}\right)\right|_{x=0} = 0 \neq \left.\frac{dz}{dx}\right|_{x=0}$$

所以,多元函数的情况比一元函数要复杂得多.

注2：$Th.17.5'$.若函数$u=\varphi(x,y),v=\psi(x,y)$在点$(x,y)$处都可(偏)导,函数$z=f(u,v)$在点$(u,v)$处具有连续的偏导数,则$z=f\left(\varphi(x,y),\psi(x,y)\right)$在点$(x,y)$处可(偏)导,且有全导数公式：

$$\left(\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}\right)=\left(\frac{\partial z}{\partial u},\frac{\partial z}{\partial v}\right)\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}.$$

二. 一阶全微分形式不变性

设函数$z=f(u,v)$有连续的偏导数,则

有全微分 $dz=\frac{\partial z}{\partial u}du+\frac{\partial z}{\partial v}dv.$

又若$u=\varphi(x,y),v=\psi(x,y)$可微,则有

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy.$$

这就是说,无论视函数z是自变量x,y的函数还是中间变量u,v的函数,其全微分的结果形式上是完全一样的.

我们称此为全微分的形式不变性.

上页 下页 返回

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy =$$

$$\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\right)dx + \left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\right)dy$$

$$= \frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right) + \frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right)$$

$$= \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv$$

例4.设$z = f(x-y, x^2y)$, f具有连续的一阶偏导数,求:dz.

解:设$\begin{cases} u = x - y \\ v = x^2 y \end{cases}$,记$f_1 = \dfrac{\partial f(u,v)}{\partial u}$, $f_2 = \dfrac{\partial f(u,v)}{\partial v}$,

$$\frac{\partial z}{\partial x} = f_1 + 2xyf_2, \frac{\partial z}{\partial y} = -f_1 + x^2 f_2,$$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$= \left(f_1 + 2xyf_2\right)dx + \left(-f_1 + x^2 f_2\right)dy$$

一阶全微分形式不变性

设 $u = x - y, v = x^2 y$,

记 $f_1 = \dfrac{\partial f(u,v)}{\partial u}, f_2 = \dfrac{\partial f(u,v)}{\partial v}$,

$$\frac{\partial z}{\partial x} = f_1 + 2xyf_2, \frac{\partial z}{\partial y} = -f_1 + x^2 f_2,$$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \left(f_1 + 2xyf_2\right)dx + \left(-f_1 + x^2 f_2\right)dy$$

$$= f_1\left(dx - dy\right) + f_2\left(2xydx + x^2dy\right)$$

$$= f_1 d\left(x - y\right) + f_2 d\left(x^2 y\right) = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv.$$

思考练习

1.设$z = f\left(x^2 - y^2, 2xy\right)$, f有连续的一阶偏导数,求dz.

上页 下页 返回

1.设$z = f\left(x^2 - y^2, 2xy\right)$, f有连续的一阶偏导数,求dz.

解 记 $x^2 - y^2 = u, 2xy = v, z = f\left(u,v\right)$,

$$dz = f_1 du + f_2 dv = f_1 d\left(x^2 - y^2\right) + f_2 d\left(2xy\right)$$

$$= f_1\left(2xdx - 2ydy\right) + f_2\left(2ydx + 2xdy\right),$$

$$= 2\left(xf_1 + yf_2\right)dx + 2\left(xf_2 - yf_1\right)dy.$$