

§ 17-04 中值定理 与多元函数的极值

- 一. 高阶偏导数
- 二. 中值定理
- 三. 二元函数的泰勒公式
- 四. 多元函数的无条件极值

一. 高阶偏导数

1. 高阶偏导数

函数 $z = f(x, y)$ 的二阶偏导数

$$\begin{cases} \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial^2 z}{\partial x^2} = f_{xx}(x, y) \\ \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}(x, y) \end{cases},$$

二阶混合偏导

$$\begin{cases} \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial^2 z}{\partial x \partial y} = f_{xy}(x, y) \\ \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial^2 z}{\partial y \partial x} = f_{yx}(x, y) \end{cases}.$$

类似地可以定义更高阶的偏导数.

例如, $z = f(x, y)$ 关于 x 的三阶偏导数为

$$\frac{\partial}{\partial x} \left(\frac{\partial^2 z}{\partial x^2} \right) = \frac{\partial^3 z}{\partial x^3},$$

$z = f(x, y)$ 关于 x 的 $n-1$ 阶偏导数, 再关于 y 的一阶偏导数为

$$\frac{\partial}{\partial y} \left(\frac{\partial^{n-1} z}{\partial x^{n-1}} \right) = \frac{\partial^n z}{\partial x^{n-1} \partial y},$$

习惯上, 二阶及二阶以上的偏导数统称为高阶偏导数.

例1. 设 $z = x^3 y^2 - 3xy^3 - xy$, 求二阶偏导.

解 $\frac{\partial z}{\partial x} = 3x^2 y^2 - 3y^3 - y$,

$$\frac{\partial z}{\partial y} = 2x^3 y - 9xy^2 - x;$$

$$\frac{\partial^2 z}{\partial x^2} = 6xy^2, \frac{\partial^2 z}{\partial y^2} = 2x^3 - 18xy;$$

$$\frac{\partial^2 z}{\partial x \partial y} = (3x^2 y^2 - 3y^3 - y)'_y = 6x^2 y - 9y^2 - 1,$$

$$\frac{\partial^2 z}{\partial y \partial x} = (2x^3 y - 9xy^2 - x)'_x = 6x^2 y - 9y^2 - 1.$$

例2. 设 $f(x, y) = \begin{cases} xy \frac{x^2 - y^2}{x^2 + y^2}, & x^2 + y^2 \neq 0 \\ 0, & x^2 + y^2 = 0 \end{cases}$, 求 $f_{xy}(0, 0)$ 和 $f_{yx}(0, 0)$.

解 $f_x(x, y) = \begin{cases} y \frac{x^4 + 4x^2y^2 - y^4}{(x^2 + y^2)^2}, & x^2 + y^2 \neq 0 \\ 0, & x^2 + y^2 = 0 \end{cases}$

$f_y(x, y) = \begin{cases} x \frac{x^4 + 4x^2y^2 - y^4}{(x^2 + y^2)^2}, & x^2 + y^2 \neq 0 \\ 0, & x^2 + y^2 = 0 \end{cases}$

$$\left. \begin{aligned} f_{xy}(0, 0) &= \lim_{\Delta y \rightarrow 0} \frac{f_x(0, \Delta y) - f_x(0, 0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{-\Delta y}{\Delta y} = -1 \\ f_{yx}(0, 0) &= \lim_{\Delta x \rightarrow 0} \frac{f_y(\Delta x, 0) - f_y(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta x}{\Delta x} = 1 \end{aligned} \right\}$$

*Q.*构成相同,次序不同的混合偏导数何时能够相等?

*Th.17.6.*若函数 $z = f(x, y)$ 的二阶混合偏导数 $f_{xy}(x, y), f_{yx}(x, y)$ 在点 (x, y) 处连续,则

$$f_{xy}(x, y) = f_{yx}(x, y).$$

该定理结论可以推广到更高阶的偏导数或多个自变量的情形.在许多问题中,混合偏导数是连续的,因而不考虑求导的次序,这给我们解题带来了便利.

例3.已知函数 $z = f(x, y)$ 满足 $\frac{\partial^2 z}{\partial x \partial y} = 4xy$,

且 $f_x(x, 0) = 3x^2$, $f(0, y) = y$, 求函数表达式.

$$\text{解 } \frac{\partial^2 z}{\partial x \partial y} = 4xy \Rightarrow \frac{\partial z}{\partial x} = 2xy^2 + C_1(x) = f_x(x, y),$$

$$\because f_x(x, 0) = 3x^2 \Rightarrow C_1(x) = 3x^2,$$

$$\frac{\partial z}{\partial x} = 2xy^2 + 3x^2 \Rightarrow z = x^2 y^2 + x^3 + C_2(y),$$

$$\text{由 } f(0, y) = y \Rightarrow C_2(y) = y,$$

$$\therefore f(x, y) = x^2 y^2 + x^3 + y.$$

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返回

(续) *Sec.1*例4.试问 $(1+x^2y)dx + (e^x - \sin y)dy$

是否是一个二元函数的全微分?

分析: 若 $(1+x^2y)dx + (e^x - \sin y)dy$

是函数 $z = f(x, y)$ 的全微分, 由 $Th.2$

——函数可微的必要条件知

$$\frac{\partial z}{\partial x} = 1 + x^2 y, \quad \frac{\partial z}{\partial y} = e^x - \sin y.$$

而一个函数的两个偏导数 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

之间是有着紧密的关系的...

解 若 $(1+x^2y)dx + (e^x - \sin y)dy$
是函数 $z = f(x, y)$ 的全微分, 由 *Th.2*
——函数可微的必要条件知

$$\frac{\partial z}{\partial x} = 1 + x^2 y, \quad \frac{\partial z}{\partial y} = e^x - \sin y \cdots \cdots (1)$$

$$\text{由 } \frac{\partial z}{\partial x} = 1 + x^2 y \Rightarrow z = x + \frac{1}{3} x^3 y + C(y),$$
$$\Rightarrow \frac{\partial z}{\partial y} = \frac{1}{3} x^3 + C'(y), \text{ 而这与(1)式相矛盾.}$$

\therefore 题设不可能是某函数的全微分.

解二 若 $(1+x^2y)dx + (e^x - \sin y)dy$ 是函数 $z = f(x, y)$ 的全微分, 由可微的必要条件知

$$\frac{\partial z}{\partial x} = 1 + x^2 y, \quad \frac{\partial z}{\partial y} = e^x - \sin y$$

$$\Rightarrow \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial^2 z}{\partial x \partial y} = x^2, \quad \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial^2 z}{\partial y \partial x} = e^x,$$

而初等函数 x^2, e^x 在 \mathbb{R}^2 上有定义因而连续,

由 *Th.17.6* 知应有 $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$, 由此矛盾知

题设不可能是某函数的全微分.

例4*. 已知函数 $z = f\left(\sqrt{x^2 + y^2}\right)$ 满足 $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$, 其

中 f 有连续的二阶导数. 求 $z = f\left(\sqrt{x^2 + y^2}\right)$ 表达式.

解 记 $z = f(r)$, $r = \sqrt{x^2 + y^2}$, $\therefore \frac{\partial r}{\partial x} = \frac{x}{r}$,

$$\frac{\partial^2 r}{\partial x^2} = \frac{r - x \cdot r'_x}{r^2} = \frac{r^2 - x^2}{r^3},$$

$$\frac{\partial z}{\partial x} = f'(r) \cdot r'_x = f'(r) \cdot \frac{x}{r},$$

$$\frac{\partial^2 z}{\partial x^2} = f''(r) \cdot \left(\frac{x}{r}\right)^2 + f'(r) \cdot \frac{r^2 - x^2}{r^3},$$

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返回

$$\frac{\partial^2 z}{\partial x^2} = f''(r) \cdot \left(\frac{x}{r}\right)^2 + f'(r) \cdot \frac{r^2 - x^2}{r^3},$$

同理：
$$\frac{\partial^2 z}{\partial y^2} = f''(r) \cdot \left(\frac{y}{r}\right)^2 + f'(r) \cdot \frac{r^2 - y^2}{r^3}.$$

$$\therefore \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = f''(r) \cdot \left(\frac{x^2}{r^2} + \frac{y^2}{r^2}\right) + f'(r) \cdot \frac{2r^2 - x^2 - y^2}{r^3}$$

$$= f''(r) + \frac{1}{r} f'(r) = 0,$$

\therefore 函数 $z = f(r)$ 满足一个二阶可降阶的微分方程

$$rf''(r) + f'(r) = 0$$

对于特殊的二阶微分方程

$$rf''(r) + f'(r) = 0$$

即 $(rf'(r))'_r = 0 \Rightarrow rf'(r) = C_1,$

$$f'(r) = \frac{C_1}{r}, \therefore f(r) = C_1 \ln r + C_2,$$

C_1, C_2 是任意常数, 于是

得到函数 $z = f\left(\sqrt{x^2 + y^2}\right)$ 的表达式 .

例5. 已知函数 $z = z(x, y)$ 具有连续的二阶偏导数, 设 $\begin{cases} u = x + y \\ v = x - y \end{cases}$, 试将方程 $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$ 化为新坐标系中的形式, 并由此求解该偏微分方程.

解
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v},$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial u} \right) + \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial v} \right)$$

$$= \frac{\partial^2 z}{\partial u^2} \cdot \frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial u \partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial^2 z}{\partial v \partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial v^2} \cdot \frac{\partial v}{\partial x}$$

$$= \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v \partial u} + \frac{\partial^2 z}{\partial v^2}$$

$$= \frac{\partial^2 z}{\partial u^2} + 2 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2},$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v},$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial u} \right) - \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial v} \right)$$

$$= \frac{\partial^2 z}{\partial u^2} \cdot \frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial u \partial v} \cdot \frac{\partial v}{\partial y} - \frac{\partial^2 z}{\partial v \partial u} \cdot \frac{\partial u}{\partial y} - \frac{\partial^2 z}{\partial v^2} \cdot \frac{\partial v}{\partial y}$$

$$= \frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial u \partial v} - \frac{\partial^2 z}{\partial v \partial u} + \frac{\partial^2 z}{\partial v^2}$$

$$= \frac{\partial^2 z}{\partial u^2} - 2 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}.$$

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0 \Rightarrow \frac{\partial^2 z}{\partial u \partial v} = 0 \quad (P135/7)$$

$$\begin{cases} u = x + y \\ v = x - y \end{cases} \Leftrightarrow \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \text{记为 } U = AX,$$

矩阵A可逆,故自变量 x, y 相互独立 $\Leftrightarrow u, v$ 相互独立.

$$\therefore \frac{\partial^2 z}{\partial u \partial v} = 0 \Leftrightarrow \frac{\partial z}{\partial u} = g(u), z = \int g(u) du = G(u) + H(v),$$

\therefore 满足方程 $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$ 的解为

$$z = G(x + y) + H(x - y),$$

$G(s), H(t)$ 为任意的可导函数.

例6. 设 $z = f(x + y, xy)$, f 具有

至为重要

连续的二阶偏导数, 求: $\frac{\partial z}{\partial x}, \frac{\partial^2 z}{\partial x \partial y}$.

解 令 $u = x + y, v = xy$,

记 $f_1 = \frac{\partial f(u, v)}{\partial u}$, 同理 $f_2 = \frac{\partial f(u, v)}{\partial v}$,

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = f_1 + y f_2;$$

$$\frac{\partial^2 z}{\partial x \partial y} = (f_1 + y f_2)'_y = \frac{\partial f_1}{\partial y} + f_2 + y \frac{\partial f_2}{\partial y}$$

$$u = x + y, v = xy, \frac{\partial z}{\partial x} = f_1 + yf_2.$$

$$\frac{\partial^2 z}{\partial x \partial y} = (f_1 + yf_2)'_y = \frac{\partial f_1}{\partial y} + f_2 + y \frac{\partial f_2}{\partial y},$$

$$\frac{\partial f_1}{\partial y} = \frac{\partial f_1}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f_1}{\partial v} \cdot \frac{\partial v}{\partial y} = f_{11} + xf_{12},$$

记 $f_1 = \frac{\partial f(u, v)}{\partial u}$, f_1, f_2 仍是以 u, v 为中间变量, 以 x, y 为自变量的两个新的函数.

$$\frac{\partial f_1}{\partial u} = f_{11}, \quad \frac{\partial f_1}{\partial v} = f_{12}, \quad \frac{\partial f_2}{\partial v} = f_{22}.$$

$$z = f(u, v), u = x + y, v = xy,$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = f_1 + yf_2;$$

$$\frac{\partial^2 z}{\partial x \partial y} = (f_1 + yf_2)'_y = \frac{\partial f_1}{\partial y} + f_2 + y \frac{\partial f_2}{\partial y},$$

$$\frac{\partial f_1}{\partial y} = \frac{\partial f_1}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f_1}{\partial v} \cdot \frac{\partial v}{\partial y} = f_{11} + xf_{12},$$

$$\frac{\partial f_2}{\partial y} = \frac{\partial f_2}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f_2}{\partial v} \cdot \frac{\partial v}{\partial y} = f_{21} + xf_{22},$$

f 有连续的二阶偏导数 $\Rightarrow f_{12} = f_{21} \cdots$

思考题 1.

1.(1). 验证函数 $u(x, y) = \ln \sqrt{x^2 + y^2}$ 满足

Laplace 方程 $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

1.(2). 记 $r = \sqrt{x^2 + y^2 + z^2}$, 证明函数 $u = \frac{1}{r}$ 满足

Laplace 方程 $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$

2. 设 $(axy^3 - y^2 \cos x)dx + (1 + by \sin x + 3x^2 y^2)dy$

是一个二元函数 $z = f(x, y)$ 的全微分. 试确定 a, b 的值, 并求出函数 $z = f(x, y)$.

思考题 1.参考解答

1.(1).验证函数 $u(x, y) = \ln \sqrt{x^2 + y^2}$ 满足

Laplace 方程 $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

解 $\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2}, \frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2},$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2 + y^2) - x \cdot 2x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2},$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2 + y^2) - y \cdot 2y}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}.$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{x^2 - y^2}{(x^2 + y^2)^2} = 0.$$

注意
对称性

1.(2).记 $r = \sqrt{x^2 + y^2 + z^2}$, 证明函数 $u = \frac{1}{r}$ 满足

Laplace 方程 $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$.

解 $\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r}, \frac{\partial u}{\partial x} = -\frac{r_x}{r^2} = -\frac{x}{r^3},$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{r^3 - x \cdot 3r^2 \cdot \frac{x}{r}}{r^6} = \frac{3x^2 - r^2}{r^5}.$$

由形式对称性得

$$\frac{\partial^2 u}{\partial y^2} = \frac{3y^2 - r^2}{r^5}, \frac{\partial^2 u}{\partial z^2} = \frac{3z^2 - r^2}{r^5},$$

代入 *Laplace* 方程, 结论成立.

注: 这里的函数 $u = \frac{1}{r}$ 常称为“位势函数”

2. 设 $(axy^3 - y^2 \cos x)dx + (1 + by \sin x + 3x^2 y^2)dy$ 是一个二元函数 $z = f(x, y)$ 的全微分. 试确定 a, b 的值, 并求出函数 $z = f(x, y)$.

解 $\because (axy^3 - y^2 \cos x)dx$
 $+ (1 + by \sin x + 3x^2 y^2)dy = dz,$

$$\therefore \begin{cases} \frac{\partial z}{\partial x} = axy^3 - y^2 \cos x \\ \frac{\partial z}{\partial y} = 1 + by \sin x + 3x^2 y^2 \end{cases},$$

$$(axy^3 - y^2 \cos x)dx + (1 + by \sin x + 3x^2 y^2)dy = dz,$$

$$\therefore \frac{\partial z}{\partial x} = axy^3 - y^2 \cos x, \frac{\partial z}{\partial y} = 1 + by \sin x + 3x^2 y^2,$$

$$z = \frac{1}{2}ax^2 y^3 - y^2 \sin x + C(y) \quad \Rightarrow$$

$$\frac{\partial z}{\partial y} = C'(y) - 2y \sin x + \frac{3}{2}ax^2 y^2 = 1 + by \sin x + 3x^2 y^2,$$

$$\therefore a = 2, b = -2, C'(y) = 1.$$

$$(2xy^3 - y^2 \cos x)dx + (1 - 2y \sin x + 3x^2 y^2)dy = dz,$$

$$\therefore \begin{cases} \frac{\partial z}{\partial x} = 2xy^3 - y^2 \cos x \cdots \cdots (1) \\ \frac{\partial z}{\partial y} = 1 - 2y \sin x + 3x^2 y^2 \cdots (2) \end{cases},$$

$$\text{由(1)得 } z = x^2 y^3 - y^2 \sin x + C(y) \cdots (3),$$

$$\text{对(3)计算 } \frac{\partial z}{\partial y} = 3x^2 y^2 - 2y \sin x + C'(y),$$

$$\text{与(2)比较得: } C'(y) = 1, \therefore C(y) = y + C,$$

$$\therefore z = x^2 y^3 - y^2 \sin x + y + C,$$

.....(其中C为任意常数).

或者,

$$(axy^3 - y^2 \cos x)dx + (1 + by \sin x + 3x^2 y^2)dy = dz,$$

$$\therefore \frac{\partial z}{\partial x} = axy^3 - y^2 \cos x, \frac{\partial z}{\partial y} = 1 + by \sin x + 3x^2 y^2,$$

$$\frac{\partial^2 z}{\partial x \partial y} = 3axy^2 - 2y \cos x, \frac{\partial^2 z}{\partial y \partial x} = by \cos x + 6xy^2,$$

由 *Th.17.6*: “二阶混合偏导函数若连续则相等”

$$\text{知 } 3axy^2 - 2y \cos x = by \cos x + 6xy^2,$$

$$\therefore a = 2, b = -2.$$

二. 二元函数的中值定理

2. 二元函数的微分中值定理

定理17.7(中值定理) 若函数 $z = f(x, y)$ 在凸开域 $D \subset \mathbb{R}^2$ 上连续, 在 D 内可微, 则 $\forall P(a, b)$,

$Q(a + h, b + k) \in D, \exists \theta \in (0, 1)$, 使得

$$f(a + h, b + k) - f(a, b) =$$

$$f_x(a + \theta h, b + \theta k)h + f_y(a + \theta h, b + \theta k)k.$$

证明: 设 $\Phi(t) = f(a + th, b + tk), t \in [0, 1]$,

则 $\Phi(t)$ 在 $[0, 1]$ 上满足 *Lagrange* 微分中值定理的条件,

证明 设 $\Phi(t) = f(a + th, b + tk), t \in [0, 1]$,
则 $\Phi(t)$ 在 $[0, 1]$ 上满足 *Lagrange* 微分中值
定理的条件, 所以 $\exists \theta \in (0, 1)$, 使得

$$\Phi(1) - \Phi(0) = \Phi'(\theta), \text{ 而}$$

$$\Phi'(\theta) = f_x(a + \theta h, b + \theta k)h + f_y(a + \theta h, b + \theta k)k.$$

由此可得推论：

若函数 $z = f(x, y)$ 在凸开域 D 内可微,

且 $f_x = f_y \equiv 0$, 则在区域 D 内 $f(x, y) \equiv C$.

对比.

定理17.3 设函数 $f(x, y)$ 在点 (x_0, y_0) 的某邻域内有偏导数,若点 (x, y) 属于该邻域,则 $\exists \xi = x_0 + \theta_1(x - x_0), \eta = y_0 + \theta_2(y - y_0)$, $0 < \theta_1 < 1, 0 < \theta_2 < 1$,使得

$$f(x, y) - f(x_0, y_0) = f_x(\xi, \eta)(x - x_0) + f_y(x_0, \eta)(y - y_0),$$

三. 二元函数的泰勒公式

3. 二元函数的Taylor定理

一元函数的Taylor公式

$$\begin{aligned} f(x) = & f(x_0) + f'(x_0)(x - x_0) \\ & + \frac{f''(x_0)}{2}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \\ & + \frac{f^{(n+1)}(x_0 + \theta(x - x_0))}{(n+1)!}(x - x_0)^{n+1} \quad (0 < \theta < 1). \end{aligned}$$

意义: 可用 n 次多项式来近似表达函数 $f(x)$, 且误差是当 $x \rightarrow x_0$ 时比 $(x - x_0)^n$ 高阶的无穷小.

Th.17.8(二元函数的带*Lagrange*型余项的*Taylor*定理)

设 $z = f(x, y)$ 在点 (x_0, y_0) 的某一邻域 $U(x_0, y_0)$ 内有 $n + 1$ 阶连续的偏导数, $\forall (x_0 + h, y_0 + k) \in U(x_0, y_0)$, 则有

$$\begin{aligned} f(x_0 + h, y_0 + k) &= f(x_0, y_0) + \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) f(x_0, y_0) \\ &+ \frac{1}{2!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^2 f(x_0, y_0) + \cdots + \frac{1}{n!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^n f(x_0, y_0) \\ &+ \frac{1}{(n+1)!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^{n+1} f(x_0 + \theta h, y_0 + \theta k), \quad (0 < \theta < 1) \end{aligned}$$

.....(1)

其中,记号 $\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x_0,y_0)$ 表示

$$hf_x(x_0,y_0)+kf_y(x_0,y_0),$$

$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x_0,y_0)$ 表示

$$h^2 f_{xx}(x_0,y_0)+2hkf_{xy}(x_0,y_0)+k^2 f_{yy}(x_0,y_0),$$

一般地, $\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^m f(x_0,y_0)$ 表示

$$\sum_{i=0}^m C_m^i h^i k^{m-i} \frac{\partial^m f}{\partial x^i \partial y^{m-i}} \Big|_{(x_0,y_0)}.$$

$n = 0$ 时(1)即为

$$\begin{aligned} f(x_0 + h, y_0 + k) &= f(x_0, y_0) \\ &\quad + hf_x(x_0 + \theta h, y_0 + \theta k) \\ &\quad + kf_y(x_0 + \theta h, y_0 + \theta k), \quad (0 < \theta < 1) \end{aligned}$$

上式称为二元函数的*Lagrange*微分中值公式,即前面的定理17.7.

Th.17.8的证明 设 $\Phi(t) = f(x_0 + th, y_0 + tk)$,

由Th.17.5(全导数公式)知 $\Phi(t)$ 在 $[0,1]$ 上满足Taylor th.条件,
于是有

$$\Phi(1) = \Phi(0) + \Phi'(0) + \frac{\Phi''(0)}{2!} + \cdots + \frac{\Phi^{(n)}(0)}{n!} + \frac{\Phi^{(n+1)}(\theta)}{(n+1)!}, \theta \in (0,1),$$

$$\text{由全导数公式得 } \Phi^{(m)}(t) = \frac{1}{m!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^m f(x_0 + th, y_0 + tk),$$
$$m \in \{1, 2, \dots, n, n+1\}$$

$$\text{于是 } \Phi^{(m)}(0) = \frac{1}{m!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^m f(x_0, y_0), m \in \{1, 2, \dots, n\}$$

$$\Phi^{(n+1)}(t) = \frac{1}{m!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^{n+1} f(x_0 + \theta h, y_0 + \theta k),$$

将此结果代入上式即得Taylor th.结论.

Th.17.9 (二元函数的带*Peano*型余项的*Taylor*定理)

设 $z = f(x, y)$ 在点 (x_0, y_0) 的某一邻域 $U(x_0, y_0)$ 内有
 n 阶连续的偏导数, $\forall (x_0 + h, y_0 + k) \in U(x_0, y_0)$, 则

当 $\rho = \sqrt{h^2 + k^2} \rightarrow 0$ 时有

$$f(x_0 + h, y_0 + k) =$$

$$= f(x_0, y_0) + \sum_{i=1}^n \frac{1}{i!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^i f(x_0, y_0) + o(\rho^n).$$

下面是 $Th.17.9$ (二元函数的带 $Peano$ 型余项的 $Taylor$ 定理)

我们常用到的形式: 设 $z = f(x, y)$ 在点 (x_0, y_0) 的某一邻域 $U(x_0, y_0)$ 内有二阶连续的偏导数,

$\forall (x_0 + h, y_0 + k) \in U(x_0, y_0)$, 则当 $\rho = \sqrt{h^2 + k^2} \rightarrow 0$ 时有

$$f(x_0 + h, y_0 + k) = f(x_0, y_0) + [f_x(x_0, y_0)h + f_y(x_0, y_0)k] \\ + \frac{1}{2!} [f_{xx}(x_0, y_0)h^2 + 2f_{xy}(x_0, y_0)hk + f_{yy}(x_0, y_0)k^2] + o(\rho^2)$$

或表示为

$$f(x_0 + h, y_0 + k) = f(x_0, y_0) + (f_x(x_0, y_0), f_y(x_0, y_0)) \begin{pmatrix} h \\ k \end{pmatrix} \\ + \frac{1}{2!} (h, k) \begin{pmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{xy}(x_0, y_0) & f_{yy}(x_0, y_0) \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} + o(\rho^2)$$

下面是 $Th.17.9$ (二元函数的带 $Peano$ 型余项的 $Taylor$ 定理)

我们常用到的形式: 设 $z = f(x, y)$ 在点 (x_0, y_0) 的某一邻域
 $U(x_0, y_0)$ 内有二阶连续的偏导数, 记 $X_0 = (x_0, y_0)$,

$X = (x_0 + h, y_0 + k), \forall X \in U(X_0), \Delta X = (h, k),$

则当 $\rho = \sqrt{h^2 + k^2} = \|\Delta X\| \rightarrow 0$ 时有

$$f(X) = f(X_0) + \nabla f(X_0) \cdot \Delta X + \frac{1}{2!} \Delta X \cdot H_{f(X_0)} \cdot \Delta X^T + o(\|\Delta X\|^2)$$

$$\nabla f(X_0) = (f_x, f_y)_{X_0} = \text{grad} f|_{X_0}, \leftarrow \text{梯度}$$

$$H_{f(X_0)} = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}_{X_0} \leftarrow \text{Hesse 矩阵} \\ \text{(Hessian Matrix)}$$

$\Delta X \cdot H_{f(X_0)} \cdot \Delta X^T$ 是一个实的二次型

四.多元函数的(无条件)极值

设函数 $z = f(x, y)$ 在点 $P_0(x_0, y_0)$ 的某邻域 $U(P_0)$ 内有定义, $\forall (x, y) \in U(P_0)$ 有

$$f(x, y) \leq (\geq) f(x_0, y_0),$$

则称函数在点 P_0 处取得极大(小)值.

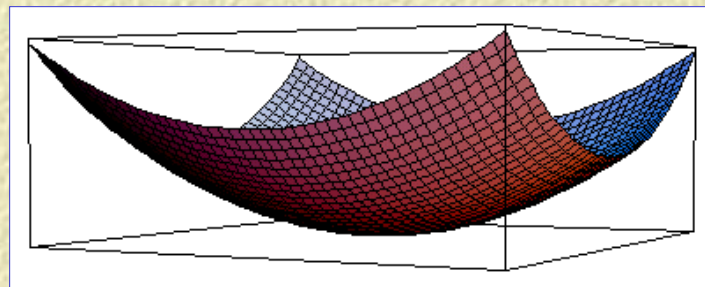
极大值, 极小值统称为极值.

使函数取得极值的点称为函数的极值点.

观察函数在 $(0,0)$ 处的极值情况.

(1). $z = x^2 + 2y^2$

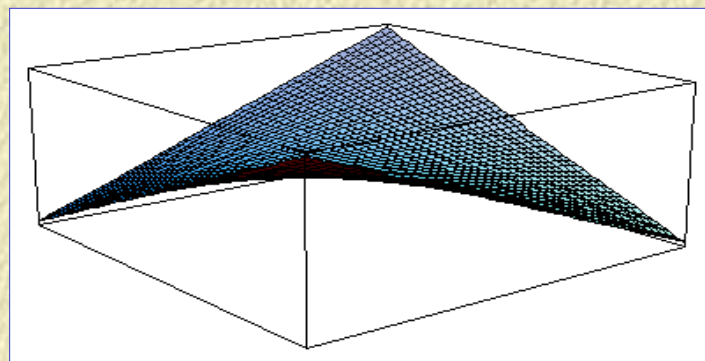
在 $(0,0)$ 处有极小值.



(1)

(2). $z = x^2 - y^2$

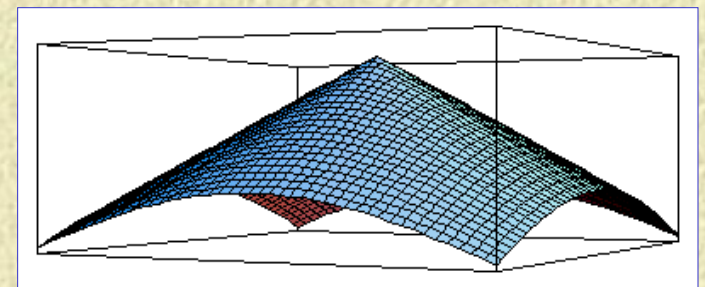
在 $(0,0)$ 处无极值.



(2)

(3). $z = -\sqrt{x^2 + y^2}$

在 $(0,0)$ 处有极大值.



(3)

4.多元函数取得极值的条件

Th.17.10.(必要条件) 设函数 $z = f(x, y)$ 在点 (x_0, y_0) 处有偏导数, 且在该点处取得极值, 则有 $f_x(x_0, y_0) = 0, f_y(x_0, y_0) = 0$.

证明 不妨设 $z = f(x, y)$ 在点 (x_0, y_0) 处有极大值, 则对于 (x_0, y_0) 的某邻域内的任意一点 (x, y) 都有 $f(x, y) \leq f(x_0, y_0)$.

故当 $y = y_0, x \neq x_0$ 时有 $f(x, y_0) \leq f(x_0, y_0)$,
即 $f(x, y_0)$ 在 $x = x_0$ 处取得极大值,

$$\therefore f_x(x_0, y_0) = 0.$$

同样地,有 $f_y(x_0, y_0) = 0$.

Th.17.10.(必要条件)的推广:

函数 $u = f(x, y, z)$ 在点 (x_0, y_0, z_0) 处有偏导数,
且在该点处取得极值,则有 $f_x(x_0, y_0, z_0) = 0$,
 $f_y(x_0, y_0, z_0) = 0, f_z(x_0, y_0, z_0) = 0$.

思考题：

若函数 $f(x, y_0)$ 在点 $x = x_0$ 处取得极值，

函数 $f(x_0, y)$ 在点 $y = y_0$ 处也取得极值.

问：函数 $f(x, y)$ 在点 (x_0, y_0) 处一定取得极值么？

A : No!

直觉上就是不能“以偏概全”。

思考题解答：

No! 如 $f(x, y) = x^2 - y^2$,

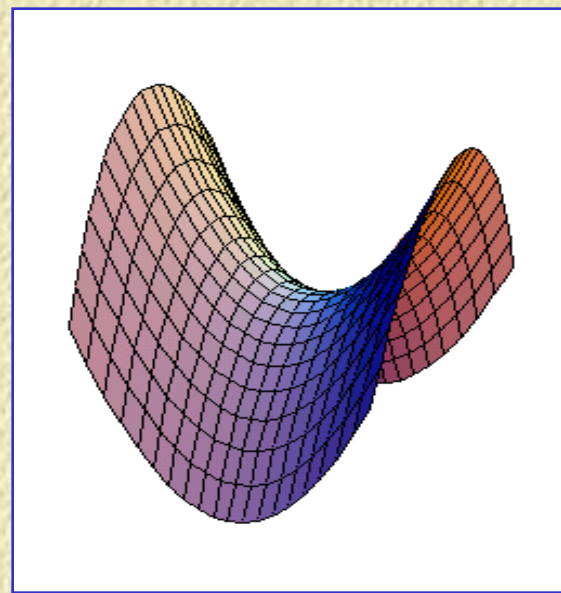
当 $x = 0$ 时 $f(0, y) = -y^2$

在 $(0, 0)$ 处取得极大值;

当 $y = 0$ 时 $f(x, 0) = x^2$ 在 $(0, 0)$ 处取得极小值;

但 $f(x, y) = x^2 - y^2$ 在 $(0, 0)$ 处取不到极值.

点 $(0, 0, 0)$ 是马鞍面 $z = x^2 - y^2$ 的鞍点, 是曲面的不稳定的平衡点.



凡使得函数的一阶偏导数同时为零的点称为函数的驻点.

需注意, 驻点 \neq 极值点.

如 $(0,0)$ 点是函数 $z = x^2 - y^2$ 的驻点但不是极值点, 而点 $(0,0)$ 是函数 $z = -\sqrt{x^2 + y^2}$ 的极值点但不是驻点.

Q: 如何判定一个驻点是函数的极值点?

定理17.11.(充分条件) 设函数 $f(x, y)$ 在驻点 (x_0, y_0) 的某邻域内有连续的一阶和二阶偏导数.

令 $f_{xx}(x_0, y_0) = A, f_{xy}(x_0, y_0) = B, f_{yy}(x_0, y_0) = C.$

则函数 $f(x, y)$ 在点 (x_0, y_0) 处取得极值的情况如下:

(1). $AC - B^2 > 0$ 时有极值: $A > 0$ 时有极小值,

$A < 0$ 时有极大值.

(2). $AC - B^2 < 0$ 时一定没有极值.

(3). $AC - B^2 = 0$ 时极值情况不确定.

设 $P_0(x_0, y_0)$ 为函数 f 的驻点,

Hessian matrix

Hesse 矩阵

$$H_{f(P_0)} = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}_{P_0} = \begin{pmatrix} A & B \\ B & C \end{pmatrix},$$

记 $|H| = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = AC - B^2$. 则

- (1). 当 $|H| > 0$ 时, 函数 f 在 (x_0, y_0) 处取极值,
且 $A > 0$, 函数取极小值, $A < 0$, 函数取极大值.
- (2). $|H| < 0$ 时, 函数 f 在 (x_0, y_0) 处不取极值.
- (3). $|H| = 0$ 时, 函数 f 在 (x_0, y_0) 处的极值
情况无法确定.

极值充
分条件
的定理
要利用
二元函
数的泰
勒公式
来证明.

极值判断(充分条件)定理-对比:

二元: $f_x(x_0, y_0) = 0, f_y(x_0, y_0) = 0.$

$$f_{xx}(x_0, y_0) = A, f_{xy}(x_0, y_0) = B,$$

$$f_{yy}(x_0, y_0) = C.$$

当 $AC - B^2 > 0$ 时有极值:

$A > 0 (< 0)$ 时有极小(大)值.

一元: $f'(x_0) = 0, f''(x_0) = A,$

则当 $A \neq 0$ 时有极值:

$A > 0 (< 0)$ 时有极小(大)值.

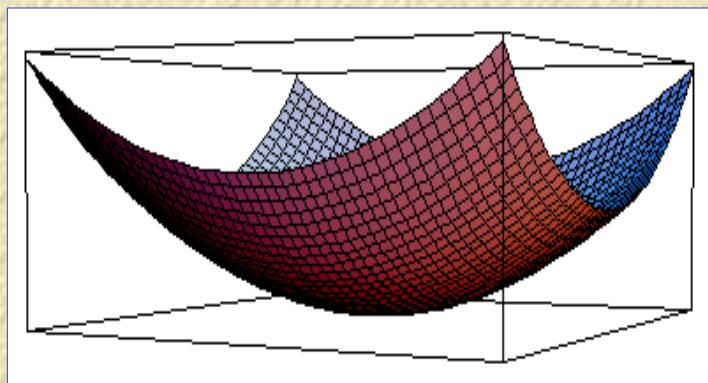
通过具体例子帮助记住定理结论.

(1). $z = x^2 + 2y^2$ 在驻点 $(0,0)$ 处取得极小值, 且有

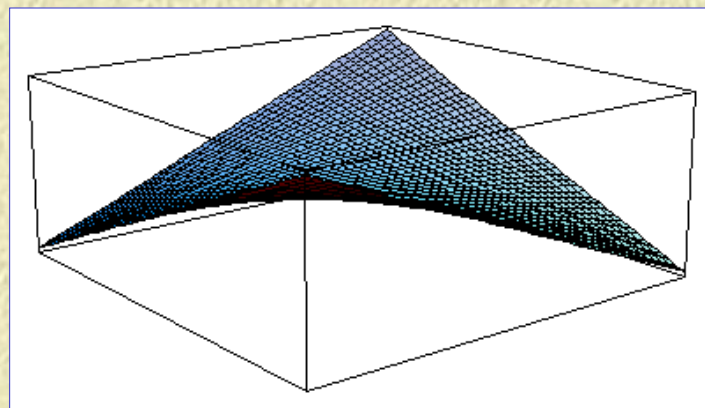
$$A = 2, B = 0, C = 4, AC - B^2 = 8 > 0.$$

(2). $z = x^2 - y^2$ 在驻点 $(0,0)$ 处无极值,

$$A = 2, B = 0, C = -2, AC - B^2 = -4 < 0.$$



(1)



(2)



例7.求 $f(x, y) = x^3 - y^3 + 3x^2 + 3y^2 - 9x$
的极值.

解 第一步 求驻点.

解方程组
$$\begin{cases} f_x(x, y) = 3x^2 + 6x - 9 = 0 \\ f_y(x, y) = -3y^2 + 6y = 0 \end{cases}$$

解得 $x = 1$ 或 3 , $y = 0$ 或 2 .

\therefore 驻点有 $(1, 0), (1, 2), (3, 0), (3, 2)$.

第二步 判别.

求二阶偏导数 $f_{xx} = 6x + 6 \cdots A$,

$$f_{xy} = 0 \cdots B, f_{yy} = -6y + 6 \cdots C$$

项目 驻点	$A = f_{xx}$	$B = f_{xy}$	$C = f_{yy}$	$AC - B^2$	极值情况
$(-3, 0)$	-12	0	6	< 0	不取极值
$(-3, 2)$	$-12 < 0$	0	-6	> 0	$\max f(x, y) = f(-3, 2)$
$(1, 0)$	$12 > 0$	0	6	> 0	$\min f(x, y) = f(1, 0)$
$(1, 2)$	12	0	-6	< 0	不取极值

以例说明定理17.10的条件是充分而非必要的.
(不要求掌握)

e.g.讨论函数 $z = x^3 + y^3$ 与 $z = (x^2 + y^2)^2$

在点 $(0,0)$ 处的极值情况.

解 显然 $(0,0)$ 都是它们的驻点,且在 $(0,0)$ 处都有 $AC - B^2 = 0$.

$z = x^3 + y^3$ 在点 $(0,0)$ 的邻域内的取值可能为:正/零/负,

$\therefore (0,0)$ 处 $z = x^3 + y^3$ 不取极值.

而当 $x^2 + y^2 \neq 0$ 时, $z = (x^2 + y^2)^2 > z|_{(0,0)} = 0$,

$\therefore z(0,0) = (x^2 + y^2)^2|_{(0,0)} = 0$ 为极小值.

一个多元函数极值判断的结论,在实际问题中的方法简单、但极其有用的应用:

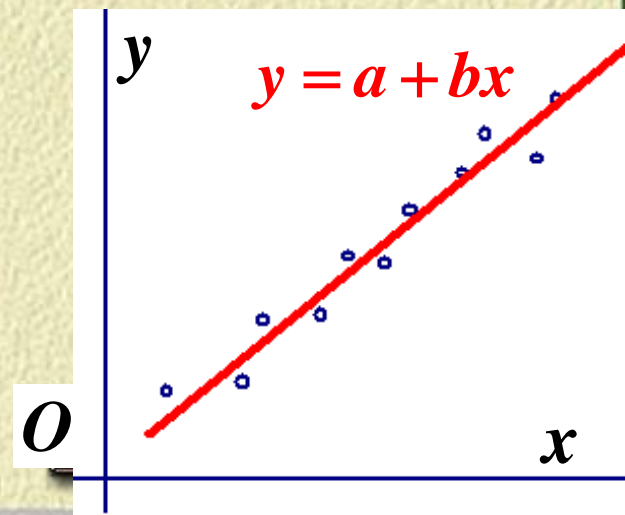
……最小二乘法

…寻找经验公式.

例8.计量经济学中的线性回归就是要找经验公式.设某一问题中,我们由抽样调查得到的数据 $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ 可以认为,内生变量 x 与变量 y 是一个线性函数关系, $y = a + bx$,那么由最小二乘法知,系数 a, b 是根据

$$\min L(a, b) = \sum_{i=1}^n (a + bx_i - y_i)^2$$

来确定的. (不要求掌握)



最小二乘法,就是确定系数 a, b ,使得

$$L(a, b) = \sum_1^n (a + bx_i - y_i)^2 \text{取值最小.}$$

$$\therefore \begin{cases} \frac{\partial L}{\partial a} = 0 \\ \frac{\partial L}{\partial b} = 0 \end{cases}, \text{即} \begin{cases} \sum_1^n (a + bx_i - y_i) = 0 \\ \sum_1^n (a + bx_i - y_i) x_i = 0 \end{cases}.$$

$$\begin{cases} na + \left(\sum_1^n x_i \right) b = \sum_1^n y_i \\ \left(\sum_1^n x_i \right) a + \left(\sum_1^n x_i^2 \right) b = \sum_1^n x_i y_i \end{cases}, \dots \dots$$

定理17.11.(极值存在充分条件)的解释:

$f(x, y)$ 在驻点 (x_0, y_0) 的某邻域内有连续的二阶偏导数.

令 $f_{xx}(x_0, y_0) = A, f_{xy}(x_0, y_0) = B, f_{yy}(x_0, y_0) = C.$

则函数 $f(x, y)$ 在点 (x_0, y_0) 处取得极值的情况如下:

(1). $AC - B^2 > 0$ 时有极值: $A > 0$ 时有极小值, $A < 0$ 时有极大值.

(2). $AC - B^2 < 0$ 时一定没有极值.

(3). $AC - B^2 = 0$ 时极值情况不确定.

据Th.17.9 (二元函数的带Peano型余项的Taylor定理)

设 $z = f(x, y)$ 在点 (x_0, y_0) 的某一邻域 $U(x_0, y_0)$ 内有二阶连续的偏导数, $\forall (x_0 + h, y_0 + k) \in U(x_0, y_0)$, 则当 $\rho = \sqrt{h^2 + k^2} \rightarrow 0$ 时有

$$f(x_0 + h, y_0 + k) = f(x_0, y_0) + [f_x(x_0, y_0)h + f_y(x_0, y_0)k] \\ + \frac{1}{2!} [f_{xx}(x_0, y_0)h^2 + 2f_{xy}(x_0, y_0)hk + f_{yy}(x_0, y_0)k^2] + o(\rho^2)$$

由于 (x_0, y_0) 是函数 $f(x, y)$ 的驻点, 故 $f_x(x_0, y_0) = f_y(x_0, y_0) = 0.$

$$\therefore f(x_0 + h, y_0 + k) = f(x_0, y_0) + \frac{1}{2!}(h, k) \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}_{(x_0, y_0)} \begin{pmatrix} h \\ k \end{pmatrix} + o(\rho^2).$$

由二次型的知识可得

$$(h, k) \neq 0 \text{ 时 } (h, k) \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}_{(x_0, y_0)} \begin{pmatrix} h \\ k \end{pmatrix} > 0 \Leftrightarrow$$

$$\text{Hesse 矩阵 } H_{f(x_0)} = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}_{(x_0, y_0)} = \begin{pmatrix} A & B \\ B & C \end{pmatrix} \text{ 为正定矩阵, } \Leftrightarrow$$

而 $AC - B^2 > 0, A > 0$.

由于 $\rho = \sqrt{h^2 + k^2} \rightarrow 0$ 时 $o(\rho^2) \rightarrow 0$, 所以当 $(h, k) \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} > 0$

时存在某邻域 $U(x_0, y_0)$, 在该邻域内 $f(x_0 + h, y_0 + k) \geq f(x_0, y_0)$.

故在 (x_0, y_0) 处 $AC - B^2 > 0, A > 0$ 时函数取得极小值.

$$\therefore f(x_0 + h, y_0 + k) = f(x_0, y_0) + \frac{1}{2!}(h, k) \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}_{(x_0, y_0)} \begin{pmatrix} h \\ k \end{pmatrix} + o(\rho^2).$$

由二次型的知识可得

$$(h, k) \neq 0 \text{ 时 } (h, k) \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}_{(x_0, y_0)} \begin{pmatrix} h \\ k \end{pmatrix} < 0 \Leftrightarrow$$

$$\text{Hesse 矩阵 } H_{f(x_0)} = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}_{(x_0, y_0)} = \begin{pmatrix} A & B \\ B & C \end{pmatrix} \text{ 为负定矩阵, } \Leftrightarrow$$

而 $AC - B^2 > 0, A < 0$.

由于 $\rho = \sqrt{h^2 + k^2} \rightarrow 0$ 时 $o(\rho^2) \rightarrow 0$, 所以当 $(h, k) \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} < 0$

时存在某邻域 $U(x_0, y_0)$, 在该邻域内 $f(x_0 + h, y_0 + k) \leq f(x_0, y_0)$.

故在 (x_0, y_0) 处 $AC - B^2 > 0, A < 0$ 时函数取得极大值.

5.多元连续函数的最值

与一元函数相类似，我们可以利用函数的极值来求函数的最大值和最小值.

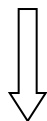
求最值的一般方法:

将函数在 D 内的所有驻点处的函数值及在 D 的边界上的最大值和最小值相互比较，其中最大者即为最大值，最小者即为最小值.

最值应用问题

依据

函数 f 在闭域上连续



函数 f 在闭域上可达到最值

最值可疑点 { 驻点
边界上的最值点

例9. 求函数 $z = f(x, y) = x^3 - 4x^2 + 2xy - y^2$ 在闭区域 $D = [-5, 5] \times [-1, 1]$ 上的最大值与最小值.

解 $z = f(x, y) = x^3 - 4x^2 + 2xy - y^2$

(1). 先求得 f 的驻点 $P_1(0, 0) \in D, P_2(2, 2) \notin D$

(2). $P_1(0, 0)$ 处, $H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} < 0, f(0, 0) = \max f(x).$

(3). ∂D 上, $x = 5, y \in [-1, 1], f(5, y) = 25 + 10y - y^2 \uparrow$

$x = -5, y \in [-1, 1], f(-5, y) = -225 - 10y - y^2 \downarrow$

$y = 1, x \in [-5, 5], f(x, 1) = x^3 - 4x^2 + 2x - 1$, 驻点 $x = \frac{4 \pm \sqrt{10}}{3}$,

$y = -1, x \in [-5, 5], f(x, -1) = x^3 - 4x^2 - 2x - 1$, 驻点 $x = \frac{4 \pm \sqrt{22}}{3}$,

$$(4). f(0,0) = 0, f(5,-1) = 14, f(5,1) = 34,$$

$$f(-5,-1) = -216, f(-5,1) = -236,$$

$$f\left(\frac{4+\sqrt{10}}{3}, 1\right) \approx -5.42, f\left(\frac{4-\sqrt{10}}{3}, 1\right) \approx -0.73,$$

$$f\left(\frac{4+\sqrt{22}}{3}, -1\right) \approx -16.05, f\left(\frac{4-\sqrt{22}}{3}, -1\right) \approx -0.76.$$

所以,函数 f 在区域 D 上的最大值、最小值分别为

$$f(5,1) = 34, f(-5,1) = -236.$$

要特别注意,当二元函数在某区域内部只有一个极值点 P 并且是极小值点时,该点函数值未必是函数在闭区域上的最小值.这是二元函数与一元函数的极值、最值问题中的一大差别,尤需注意!

例 10. 某厂要用铁板制作一个容积为 8m^3 的有盖长方体水箱,问水箱的尺寸应如何安排,能使用料最省?

解 设水箱的长、宽分别为 x 、 $y\text{m}$,则高为 $\frac{8}{xy}\text{m}$,

则水箱(忽略厚度)所用材料的面积为

$$A = 2\left(xy + y \cdot \frac{8}{xy} + x \cdot \frac{8}{xy}\right) = 2\left(xy + \frac{8}{x} + \frac{8}{y}\right), \begin{cases} x > 0 \\ y > 0 \end{cases}.$$

$$\text{令} \begin{cases} A_x = 0 \\ A_y = 0 \end{cases}, \text{即} \left(y - \frac{8}{x^2}\right) = \left(x - \frac{8}{y^2}\right) = 0 \text{ 得驻点}(2, 2).$$

根据问题的实际意义知在定义域内函数 A 的最小值一定存在,因此断定此唯一驻点就是最小值点,即水箱的长、宽、高均为 2m 时,水箱所用材料最省.