2023-04 数学分析 II 阶段练习 1 解答　　2023-04-19

1. 计算积分 (1).$\int_0^{\ln 2}\sqrt{e^x+1}dx$；

解 令$\sqrt{e^x+1}=t, x=\ln\left(t^2-1\right), dx=\frac{2t}{t^2-1}dt, x=0\leftrightarrow t=\sqrt{2}, x=\ln 2\leftrightarrow t=\sqrt{3}$.

$$\therefore I=\int_{\sqrt{2}}^{\sqrt{3}}\frac{2t^2}{t^2-1}dt=\int_{\sqrt{2}}^{\sqrt{3}}\left(2+\frac{2}{t^2-1}\right)dt=\left(2t+\ln\left|\frac{t-1}{t+1}\right|\right)\Bigg|_{\sqrt{2}}^{\sqrt{3}}$$

$$=2\left(\sqrt{3}-\sqrt{2}\right)+\ln\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)-\ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)=2\left(\sqrt{3}-\sqrt{2}\right)+2\ln\left[\left(\sqrt{3}-1\right)\left(\sqrt{2}+1\right)\right]-\ln 2.$$

(2). $\int_{\pi/4}^{\pi/3}\frac{dx}{2\tan^2 x+\sin^2 x}$.

解 $$I=\int_{\pi/4}^{\pi/3}\frac{dx}{2\tan^2 x+\sin^2 x}=\int_{\pi/4}^{\pi/3}\frac{\sec^2 x}{2\sec^2 x\tan^2 x+\tan^2 x}dx=\int_{\pi/4}^{\pi/3}\frac{1}{\tan^2 x\left(2\sec^2 x+1\right)}d\tan x$$

$$\overset{\tan x=t}{=\!=\!=}\int_1^{\sqrt{3}}\frac{1}{t^2\left(2t^2+3\right)}dt\overset{\sqrt{2}t=u}{=\!=\!=}\sqrt{2}\int_{\sqrt{2}}^{\sqrt{6}}\frac{1}{u^2\left(u^2+3\right)}du=\frac{\sqrt{2}}{3}\int_{\sqrt{2}}^{\sqrt{6}}\left(\frac{1}{u^2}-\frac{1}{u^2+3}\right)du$$

$$=\frac{\sqrt{2}}{3}\left(-\frac{1}{u}-\frac{1}{\sqrt{3}}\arctan\frac{u}{\sqrt{3}}\right)\Bigg|_{\sqrt{2}}^{\sqrt{6}}=\frac{3-\sqrt{3}}{9}+\frac{\sqrt{6}}{9}\left(\arctan\frac{\sqrt{6}}{3}-\arctan\sqrt{2}\right).$$

2. 试讨论反常积分的敛散性 (1). $\int_0^1\ln^2 x dx$； (2). $\int_0^{+\infty}\frac{1}{\left(1+x^2\right)^2}dx$.

解 (1). $$\int_0^1\ln^2 x dx\underset{x\to 0^+\Leftrightarrow t\to+\infty}{\overset{-\ln x=t\Leftrightarrow x=e^{-t}}{=\!=\!=\!=\!=\!=\!=\!=}}\int_{+\infty}^{0}t^2\left(-e^{-t}\right)dt=\int_0^{+\infty}t^2e^{-t}dt=-\left(t^2+2t+2\right)e^{-t}\Big|_0^{+\infty}=2 .$$

(2). $$\int_0^{+\infty}\frac{1}{\left(1+x^2\right)^2}dx\underset{x\to+\infty\Leftrightarrow t\to\frac{\pi}{2}}{\overset{x=\tan t}{=\!=\!=\!=\!=\!=\!=}}\int_0^{\pi/2}\frac{\sec^2 t}{\left(1+\tan^2 t\right)^2}dt=\int_0^{\pi/2}\cos^2 t dt=\frac{1}{2}\int_0^{\pi/2}\left(1+\cos 2t\right)dt=\frac{\pi}{4} .$$

3. 试证明$\int_0^{2\pi}\frac{\sin x}{x}dx>0$.

解 $\because 0$点是函数$\frac{\sin x}{x}$在$\left[0,2\pi\right]$上的可去间断点,$\therefore\frac{\sin x}{x}$在$\left[0,2\pi\right]$上可积.

$$\therefore\int_0^{2\pi}\frac{\sin x}{x}dx=\int_0^{\pi}\frac{\sin x}{x}dx+\int_{\pi}^{2\pi}\frac{\sin x}{x}dx\overset{x-\pi=t}{=\!=\!=}\int_0^{\pi}\frac{\sin x}{x}dx+\int_0^{\pi}\frac{\sin(\pi+t)}{\pi+t}dt=\int_0^{\pi}\frac{\sin x}{x}dx-\int_0^{\pi}\frac{\sin t}{\pi+t}dt$$

$$=\int_0^{\pi}\frac{\sin x}{x}dx-\int_0^{\pi}\frac{\sin x}{\pi+x}dx=\int_0^{\pi}\frac{\pi\sin x}{x\left(\pi+x\right)}dx,$$

在$\left(0,\pi\right]$上$\frac{\pi\sin x}{x(\pi+x)}\ge 0$,连续又不恒等于0, $\therefore\int_0^{\pi}\frac{\pi\sin x}{x\left(\pi+x\right)}dx>0.$

4. 设函数$f(x),g(x)$在$[a,b]$上连续,且同是单调增加的,求证:

$$\int_a^b f(x)dx\int_a^b g(x)dx \le (b-a)\int_a^b f(x)g(x)dx\ .$$

解 $f,g\in C[a,b]$,记$\Phi(u)=(u-a)\int_a^u f(x)g(x)dx-\int_a^u f(x)dx\int_a^u g(x)dx, u\in[a,b]$,

则$\Phi(u)$在$[a,b]$上连续,$\Phi(a)=0$.

$$\Phi'(u)=\int_a^u f(x)g(x)dx+(u-a)f(u)g(u)-f(u)\int_a^u g(x)dx-g(u)\int_a^u f(x)dx$$

$$=\int_a^u f(x)g(x)dx+\int_a^u f(u)g(u)dx-\int_a^u f(u)g(x)dx-\int_a^u f(x)g(u)dx$$

$$=\int_a^u[f(x)g(x)+f(u)g(u)-f(u)g(x)-f(x)g(u)]dx=\int_a^u[f(x)-f(u)][g(x)-g(u)]dx$$

$a\le x\le u\le b$,由条件知$[f(x)-f(u)][g(x)-g(u)]\ge 0,\therefore \int_a^u[f(x)-f(u)][g(x)-g(u)]dx\ge 0.$

$\Rightarrow b\ge a$ 时有$\Phi(b)\ge\Phi(a)=0.$

法二 $\forall x,y\in[a,b]$,有$[f(x)-f(y)][g(x)-g(y)]\ge 0,\therefore \int_a^b[f(x)-f(y)][g(x)-g(y)]dx\ge 0.$

即$\int_a^b[f(x)g(x)+f(y)g(y)-f(y)g(x)-f(x)g(y)]dx\ge 0,$

$\int_a^b f(x)g(x)dx+f(y)g(y)(b-a)-f(y)\int_a^b g(x)dx-g(y)\int_a^b f(x)dx\ge 0,$

$\int_a^b\left[\int_a^b f(x)g(x)dx+f(y)g(y)(b-a)-f(y)\int_a^b g(x)dx-g(y)\int_a^b f(x)dx\right]dy\ge 0,\quad\longrightarrow$

$(b-a)\int_a^b f(x)g(x)dx+(b-a)\int_a^b f(y)g(y)dy-\int_a^b f(y)dy\int_a^b g(x)dx-\int_a^b f(x)dx\int_a^b g(y)dy\ge 0.$

积分值无关乎积分变量之符号. 结论成立.

5. 求由曲线$y=\sin x, x\in[0,\pi]$与x轴围成图形分别绕x轴, y轴旋转一周所成立体的体积 .

解 $V_x=\pi\int_0^\pi \sin^2 xdx=\frac{1}{2}\pi\int_0^\pi(1-\cos 2x)dx=\frac{1}{2}\pi\left(x-\frac{1}{2}\sin 2x\right)\Big|_0^\pi=\frac{1}{2}\pi^2.$

$V_y=2\pi\int_0^\pi x\sin xdx=2\pi(\sin x-x\cos x)\Big|_0^\pi=2\pi^2.$

6. $a>0$,求由曲线$(x^2+y^2)^2=a^3y$ 所围成的平面图形绕y 轴旋转一周所成的立体的体积 .

解 不假思索地套用公式 $V=\pi\int_0^a x^2dy=\pi\int_0^a\left(\sqrt{a^3y}-y^2\right)dy=\frac{1}{3}\pi a^3,$

其中曲线$(x^2+y^2)^2=a^3y$ 与y轴的交点为$O(0,0),A(0,a)$.简单明了!

7. 求由椭球面 $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ 所围椭球体的体积．

解 用平面$x=x\left(|x|<a\right)$截椭球面得椭圆,将此椭圆向坐标面yOz作正投影,即得

$$\frac{y^2}{b^2\left(1-\dfrac{x^2}{a^2}\right)}+\frac{z^2}{c^2\left(1-\dfrac{x^2}{a^2}\right)}=1.$$

此椭圆的面积为 $A(x)=\pi\left(b\sqrt{1-\dfrac{x^2}{a^2}}\right)\left(c\sqrt{1-\dfrac{x^2}{a^2}}\right)=\dfrac{\pi bc}{a}\left(a^2-x^2\right),x\in[-a,a]$.

于是,所求椭球体体积为 $V=\int_{-a}^{a}A(x)dx=\dfrac{\pi bc}{a}\int_{-a}^{a}\left(a^2-x^2\right)dx=\dfrac{4}{3}\pi abc$．

8. 求由曲线 $x^2+xy+y^2=1$ 所围成的平面图形的面积．

解 $x^2+xy+y^2=1$ 即 $\dfrac{1}{4}x^2+xy+y^2=1-\dfrac{3}{4}x^2$, $\left(\dfrac{x}{2}+y\right)^2=1-\dfrac{3}{4}x^2$,

$\therefore y_{上}=\sqrt{1-\dfrac{3}{4}x^2}-\dfrac{x}{2}$,$y_{下}=-\sqrt{1-\dfrac{3}{4}x^2}-\dfrac{x}{2}$,$x\in\left[-\dfrac{2}{\sqrt3},\dfrac{2}{\sqrt3}\right]$.

$$\therefore\ A=\int_{-2/\sqrt3}^{2/\sqrt3}\left(y_{上}-y_{下}\right)dx=2\int_{-2/\sqrt3}^{2/\sqrt3}\sqrt{1-\frac{3}{4}x^2}dx\ \overset{\frac{\sqrt3}{2}x=\sin t}{\underset{x=\pm\frac{2}{\sqrt3}\Leftrightarrow t=\pm\frac{\pi}{2}}{=\!=\!=\!=\!=\!=}}\ \frac{4}{\sqrt3}\int_{-\pi/2}^{\pi/2}\cos^2 t dt=\frac{2\pi}{\sqrt3}.$$

法二 令$x=r\cos\theta,y=r\sin\theta,x^2+xy+y^2=1$,即 $r^2=\dfrac{1}{1+\sin\theta\cos\theta}$, $\theta\in[0,2\pi]$．

$$\therefore\ A=\frac{1}{2}\int_0^{2\pi}r^2d\theta=\frac{1}{2}\int_0^{2\pi}\frac{1}{1+\frac{1}{2}\sin2\theta}d\theta\overset{2\theta=t}{=\!=\!=}\frac{1}{2}\int_0^{4\pi}\frac{1}{2+\sin t}dt=\int_0^{2\pi}\frac{1}{2+\sin t}dt=\cdots=\frac{2\pi}{\sqrt3}.$$

法三 令$\begin{cases}x=-u\cos\dfrac{\pi}{4}+v\sin\dfrac{\pi}{4}=\dfrac{1}{\sqrt2}(-u+v)\\ y=\ u\sin\dfrac{\pi}{4}+v\cos\dfrac{\pi}{4}=\dfrac{1}{\sqrt2}(u+v)\end{cases}$, 即是将坐标系$xOy$ 绕原点O逆时针旋转$\dfrac{\pi}{4}$角成坐标系uOv.

$x^2+xy+y^2=1$,即 $\dfrac{u^2}{2}+\dfrac{v^2}{\frac{2}{3}}=1$，由椭圆面积公式得$A=\pi\cdot\sqrt2\cdot\sqrt{\dfrac{2}{3}}=\dfrac{2\pi}{\sqrt3}$．

9. 设在长 $2a\ (a>0)$、质量 M 的均匀细杆的垂直平分线上距离细杆 a 处放置一质量为 m 的质点，求此两者间的引力．

解　在 $[-a,a]$ 中取小区间 $\Delta=[x,x+dx]$，一小段细杆 Δ 与质点 m 间引力 $dF=G\dfrac{m\cdot\dfrac{M}{2a}\cdot dx}{(a^2+x^2)}$，

$$(dF)_x=G\frac{m\cdot\dfrac{M}{2a}\cdot dx}{(a^2+x^2)}\cdot\frac{x}{\sqrt{a^2+x^2}},\quad (dF)_y=G\frac{m\cdot\dfrac{M}{2a}\cdot dx}{(a^2+x^2)}\cdot\frac{a}{\sqrt{a^2+x^2}},\text{于是}, \int_{-a}^{a}(dF)_x=0,$$

$$F_y=\int_{-a}^{a}(dF)_y=GmM\int_0^a\frac{1}{\sqrt{(a^2+x^2)^3}}dx=GmM\int_0^{\pi/4}\frac{a\sec^2 t}{(a\sec t)^3}dt=\frac{GmM}{a^2}\int_0^{\pi/4}\cos t\,dt=\frac{GmM}{a^2\sqrt{2}}.$$