Chap18 隐函数定理及其应用

Sec.18.1 隐函数

一. 隐函数概念
二. 隐函数存在与可微性定理
三. 隐函数求导举例

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一. 隐函数概念

此前,我们所接触的函数,其表达式大多是自变量的某个算式,如

$$y = x + 1 \ , u = e^{xyz}\left(\sin xy + \sin yz + \sin zx\right)$$

这种形式的函数称为显函数.但在许多场合我们常会遇到另一种形式的函数,其自变量与因变量之间的对应法则是由一个方程式所决定的,这种形式的函数称为隐函数.

例如方程 $x^2+y^2=1\left(y>0\right)$就可以

解出函数 $x\in[-1,1], y=\sqrt{1-x^2}$.

而在方程 $x^2+y^2+z^2=xye^{z^2}\ \left(z>0\right)$中,

通过分析可以看到,有一组确定的值$\left(x,y\right)$,

就有唯一的z与之对应,即z是 x,y 的函数,

但是我们无法解出z作为 x,y 函数的显式

表达式.

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设$X \subset \mathbb{R}, Y \subset \mathbb{R}$,函数$F: X \times Y \to \mathbb{R}$,

对于方程 $F(x, y) = 0 \cdots\cdots\cdots\cdots\cdots\cdots (1)$

若存在集合$I \subset X$与$J \subset Y$,使得$\forall x \in I$,

恒有唯一确定的$y \in J$满足方程

$F(x, y) = 0$,则称由方程(1)确定了一个

隐函数$y = f(x), D_f = I, R_f \subset J$,

此时有$F\left(x, f(x)\right) \equiv 0, x \in I$.

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说明:

(1).我们要了解方程$F(x,y)=0$在什么条件下能够确定隐函数的存在性.

(2).所谓“方程$F(x,y)=0$在一定条件下确定一个隐函数$y=f(x)$”是指这种函数的存在性而不是给出函数的可操作性.如天体力学中著名的*Kepler*方程 $y-x-\varepsilon\sin y=0\left(0<\varepsilon<1\right)$确定了隐函数是存在的,但是却无法给出其表达式.

(3).在方程能确定一个隐函数时,我们需要了解其连续性,研究其可微性以及求导方法等.

二. 隐函数存在与可微性定理

Th.18.1.设函数$F(x,y)$满足

(1).在点$P_0\left(x_0,y_0\right)$的某邻域内有连续的偏导数;

(2).$F\left(x_0,y_0\right)=0$;

(3).$F_y\left(x_0,y_0\right)\neq 0$.

则方程$F(x,y)=0$在点x_0的某邻域内可唯一确定一个单值连续函数$y=f(x)$,满足条件$y_0=f\left(x_0\right)$,

且有连续的导数 $\dfrac{dy}{dx}=-\dfrac{F_x}{F_y}$.(隐函数求导公式)

定理证明从略,仅就求导公式推导如下:

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设$y=f(x)$是$F(x,y)=0$所确定的隐函数,

则 $F\left(x, f(x)\right) \equiv 0$

$\Downarrow$ 两边对x求导

$$\frac{\partial F}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial F}{\partial y}\cdot\frac{dy}{dx}=0 \text{ 即} F_x\cdot 1+F_y\cdot\frac{dy}{dx}=0$$

$\Downarrow$ 在$\left(x_0, y_0\right)$的某邻域内 $F_y \neq 0$

得 $$\frac{dy}{dx}=-\frac{F_x}{F_y}.$$

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说明:

(1).定理条件(2)被称为“相容性条件”.

(2).定理的条件是充分的,如$x^3 - y^3 = 0$在点$O(0,0)$处不满足条件(3),但能确定存在唯一的函数$y = x$. 条件(3)只是用来保证在$U\left(P_0\right)$内,F关于变量y是严格单调的,故定理条件是充分而非必要条件.

(3).定理条件(3)若变为$F_x\left(x_0, y_0\right) \neq 0$,那么相应地可确定一个单值连续函数 $x = g(y)$.

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隐函数存在定理与求导公式:

$Th.18.2.$设函数$F(x,y,z)$满足

(1).在点$P_0\left(x_0,y_0,z_0\right)$的某邻域内有连续的偏导数;

(2).$F\left(x_0,y_0,z_0\right)=0$;

(3).$F_z\left(x_0,y_0,z_0\right)\neq 0$.

则方程$F(x,y,z)=0$在点$\left(x_0,y_0\right)$的某邻域内可唯一确定一个单值连续函数$z=f(x,y)$,满足条件$z_0=f\left(x_0,y_0\right)$,且有连续的偏导数

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z},\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}.$$

设$z=f(x,y)$是$F(x,y,z)=0$所确定的隐函数，

则 $F\big(x,y,f(x,y)\big)\equiv 0$

$\Downarrow$ 两边对x求偏导数

$$\frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial x}+\frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial x}+\frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x}=0$$

即 $F_x\cdot 1+F_y\cdot 0+F_z\cdot\dfrac{\partial z}{\partial x}=0$

$\Downarrow$ 在$\big(x_0,y_0,z_0\big)$的某邻域内 $F_z\neq 0$

得 $\dfrac{\partial z}{\partial x}=-\dfrac{F_x}{F_z}$, 同理 $\dfrac{\partial z}{\partial y}=-\dfrac{F_y}{F_z}$.

三. 隐函数求导举例

例1.验证方程$\sin y+e^x-xy-1=0$在点$(0,0)$某邻域内可确定一个单值可导函数 $y=\varphi(x)$,并求出$\left.\frac{dy}{dx}\right|_{x=0}$,$\left.\frac{d^2y}{dx^2}\right|_{x=0}$.

解 令$F(x,y)=\sin y+e^x-xy-1$,则

(1).$F_x=e^x-y$, $F_y=\cos y-x$ 连续,

(2).$F(0,0)=0$,

(3).$F_y(0,0)=1\neq 0$,

由$Th.18.1.$知,在$x=0$ 的某邻域内方程可确定一个单值可导函数 $y=\varphi(x)$,且有

$$\frac{dy}{dx}=-\frac{F_x}{F_y}=\frac{y-e^x}{\cos y-x},\left.\frac{dy}{dx}\right|_{x=0}=\left.\frac{y-e^x}{\cos y-x}\right|_{\substack{x=0\\y=0}}=-1,$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{y-e^x}{\cos y-x}\right)$$

$$=\frac{\left(y'-e^x\right)\left(\cos y-x\right)-\left(y-e^x\right)\left(-\sin y\cdot y'-1\right)}{\left(\cos y-x\right)^2}$$

$x=0, y=0$ 时$y'=-1$, $\therefore \left.\frac{d^2y}{dx^2}\right|_{x=0}=-3.$

例2.设$\ln\sqrt{x^2+y^2}=\arctan\frac{y}{x}$,求$\frac{dy}{dx}$.

解　令$F(x,y)=\ln\sqrt{x^2+y^2}-\arctan\frac{y}{x}$,

则　$F_x(x,y)=\frac{x+y}{x^2+y^2}$,$F_y(x,y)=\frac{y-x}{x^2+y^2}$,

$$\therefore \frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{x+y}{y-x}\quad(y\neq x).$$

例2.设$\ln\sqrt{x^2+y^2}=\arctan\dfrac{y}{x}$,求$\dfrac{dy}{dx}$.

解二 若原方程能够确定函数$y=y(x)$,

那么,在方程两边对x求导:

$$\left[\frac{1}{2}\ln\left(x^2+y^2\right)\right]'=\left[\arctan\frac{y}{x}\right]',$$

$$\therefore \frac{1}{2}\cdot\frac{2x+2y\cdot y'_x}{x^2+y^2}=\frac{1}{1+\left(\dfrac{y}{x}\right)^2}\cdot\frac{xy'_x-y}{x^2}$$

$$\therefore \frac{dy}{dx}=y'_x=-\frac{x+y}{y-x}\ (y\neq x).$$

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例3.设$x^2+y^2+z^2-4z=0$,求$\dfrac{\partial^2 z}{\partial x^2}$.

解 在方程两边对x求偏导:

则 $2x+2z\dfrac{\partial z}{\partial x}-4\dfrac{\partial z}{\partial x}=0 \rightarrow \dfrac{\partial z}{\partial x}=\dfrac{x}{2-z}$,

再在上式两边对x求偏导:

$$2+2\left(\frac{\partial z}{\partial x}\right)^2+2z\frac{\partial^2 z}{\partial x^2}-4\frac{\partial^2 z}{\partial x^2}=0,$$

$$\therefore \frac{\partial^2 z}{\partial x^2}=\frac{1+\left(\dfrac{\partial z}{\partial x}\right)^2}{2-z}=\frac{(2-z)^2+x^2}{(2-z)^3}.$$

解二 利用*Th*.18.2.的公式：

$F(x,y,z)=x^2+y^2+z^2-4z$,则

$F_x=2x, F_z=2z-4\neq 0$ 时, $\dfrac{\partial z}{\partial x}=-\dfrac{F_x}{F_z}=\dfrac{x}{2-z}$,

再对上述函数对x求偏导：

$$\frac{\partial^2 z}{\partial x^2}=\left(\frac{x}{2-z}\right)'_x$$

$$=\frac{(2-z)-x(-1)\dfrac{\partial z}{\partial x}}{(2-z)^2}=\frac{(2-z)^2+x^2}{(2-z)^3}.$$

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例4.设$z=f\left(x+y+z,xyz\right)$,求$\dfrac{\partial z}{\partial x},\dfrac{\partial x}{\partial y},\dfrac{\partial y}{\partial z}$.

分析:

将z看作x,y的函数,对x求偏导数得$\dfrac{\partial z}{\partial x}$;

将x看作y,z的函数,对y求偏导数得$\dfrac{\partial x}{\partial y}$;

将y看作z,x的函数,对z求偏导数得$\dfrac{\partial y}{\partial z}$.

设$z=f(x+y+z,xyz)$,求$\frac{\partial z}{\partial x},\frac{\partial x}{\partial y},\frac{\partial y}{\partial z}$.

解 令$F(x,y,z)=f(x+y+z,xyz)-z$,

将x,y,z看作是三个地位对等的自变量,求F_x时,有 $\frac{\partial y}{\partial x}=\frac{\partial z}{\partial x}=0.$

同理,求得F_y,F_z,所以,

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z},\frac{\partial z}{\partial y}=-\frac{F_y}{F_z},\frac{\partial x}{\partial y}=-\frac{F_y}{F_x},\cdots$$

设$z=f(x+y+z,xyz)$，求$\frac{\partial z}{\partial x},\frac{\partial x}{\partial y},\frac{\partial y}{\partial z}$.

解 令$F(x,y,z)=f(x+y+z,xyz)-z$，

又令$u=x+y+z, v=xyz$，则

$$F(x,y,z)=f(u,v)-z,\begin{cases}u=x+y+z\\ v=xyz\end{cases}$$

$$\frac{\partial F}{\partial x}=f_1+f_2yz,\quad \frac{\partial F}{\partial y}=f_1+f_2xz,$$

$$\frac{\partial F}{\partial z}=f_1+f_2xy-1, \therefore \frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=\cdots$$

注记

(1).在显函数$z=f(x,y)$的微分运算中,我们是将x与y理解为两个独立的自变量,因而$\dfrac{\partial x}{\partial y}=0=\dfrac{\partial y}{\partial x}$,所以偏导数记号与导数(==微商)记号不同.

(2).在方程$F(x,y)=0$确定的隐函数的微分运算中,可见x与y是两个有函数关系的变量,因而$\dfrac{dy}{dx}\cdot\dfrac{dx}{dy}=1$.

而在方程$F(x,y,z)=0$确定的隐函数的微分运算中,如果将z看作因变量,那么x与y就是两个独立的自变量,此时$\dfrac{\partial x}{\partial y}=0=\dfrac{\partial y}{\partial x}$.

(3).在方程$F(x,y,z)=0$确定的隐函数的微分运算中,如果要计算$\dfrac{\partial z}{\partial x}$:

法一:用两边求导的方法时,是将z看作自变量x的一元函数,而y相对于x就是常数,即$\dfrac{\partial y}{\partial x}=0.$

法二:在$F(x,y,z)$中,将 x,y,z 看作是三个地位对等的自变量,求F_x时有$\dfrac{\partial y}{\partial x}=\dfrac{\partial z}{\partial x}=0$.同理,求得$F_y,F_z$,所以,$\dfrac{\partial z}{\partial x}=-\dfrac{F_x}{F_z},\dfrac{\partial z}{\partial y}=-\dfrac{F_y}{F_z},\dfrac{\partial x}{\partial y}=-\dfrac{F_y}{F_x}\cdots$

(4).若$F(x,y,z)$有连续的偏导数,且F_x,F_y,F_z

都不为零,则 $\dfrac{\partial z}{\partial x}\cdot\dfrac{\partial x}{\partial z}=\left(-\dfrac{F_x}{F_z}\right)\cdot\left(-\dfrac{F_z}{F_x}\right)=1,$

这反映了互为反函数的两个函数的导数的倒数关系,与一元函数相同.

$$\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial y}\cdot\frac{\partial y}{\partial z}=\left(-\frac{F_x}{F_z}\right)\left(-\frac{F_y}{F_x}\right)\left(-\frac{F_z}{F_y}\right)=-1,$$

毕竟,多元函数的偏导数与一元函数导数还是有区别的.

练习:

1.对于理想气体的状态方程 $pV=RT$,

R是常数.$\dfrac{\partial p}{\partial V},\dfrac{\partial V}{\partial T},\dfrac{\partial T}{\partial p}$.

求由此可知有 $\dfrac{\partial p}{\partial V}\cdot\dfrac{\partial V}{\partial T}\cdot\dfrac{\partial T}{\partial p}=-1.$

2.设 $z^3-3xyz=a^3$,求$dz,\dfrac{\partial^2 z}{\partial x\partial y}$.

3. $\Phi(u,v)$有连续的偏导数,由$\Phi(cx-az,cy-bz)=0$

决定函数 $z=z(x,y)$,证明:$a\dfrac{\partial z}{\partial x}+b\dfrac{\partial z}{\partial y}=c.$

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练习参考解答：

1.对于理想气体的状态方程 $pV = RT$, R是常数.

求 $\frac{\partial p}{\partial V}, \frac{\partial V}{\partial T}, \frac{\partial T}{\partial p}$.由此可知有 $\frac{\partial p}{\partial V} \cdot \frac{\partial V}{\partial T} \cdot \frac{\partial T}{\partial p} = -1$.

解 $p = \frac{RT}{V}$,则 $\frac{\partial p}{\partial V} = -\frac{RT}{V^2}$,

$V = \frac{RT}{p}$,则 $\frac{\partial V}{\partial T} = \frac{R}{p}$,

$T = \frac{pV}{R}$,则 $\frac{\partial T}{\partial p} = \frac{V}{R}$.

$$\therefore \frac{\partial p}{\partial V} \cdot \frac{\partial V}{\partial T} \cdot \frac{\partial T}{\partial p} = -\frac{RT}{pV} = -1.$$

2.设$z^3-3xyz=a^3$,求dz,$\dfrac{\partial^2 z}{\partial x\partial y}$.

解法一：令$F(x,y,z)=z^3-3xyz-a^3$,

x,y,z是$F(x,y,z)$的三个地位对等的自变量.

求F_x时$F(x,y,z)$是x的一元函数,而y,z是常数,

$\therefore F_x=-3yz, F_y=-3xz, F_z=3z^2-3xy,$

$$\therefore \frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=\frac{yz}{z^2-xy}, \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=\frac{xz}{z^2-xy},$$

$$\therefore dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=\frac{z\left(ydx+xdy\right)}{z^2-xy}.$$

设$z^3-3xyz=a^3$,求$dz,\dfrac{\partial^2 z}{\partial x\partial y}$.

法二:理解为$z=z(x,y)$, x,y为自变量.

求$\dfrac{\partial z}{\partial x}$时将$z$看作是$x$的函数,而$y$是常数,

$$\therefore\quad 3z^2\cdot\frac{\partial z}{\partial x}-3y\left(z+x\cdot\frac{\partial z}{\partial x}\right)=0,$$

解得$\dfrac{\partial z}{\partial x}=\dfrac{yz}{z^2-xy}$,同理得$\dfrac{\partial z}{\partial y}=\dfrac{xz}{z^2-xy}$,

$$\therefore dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=\frac{z\left(ydx+xdy\right)}{z^2-xy}.$$

设$z^3-3xyz=a^3$,求$dz,\dfrac{\partial^2 z}{\partial x\partial y}$.

解法二的一个变化:

理解为$z=z(x,y)$, x,y为自变量.

$$d\left(z^3-3xyz\right)=d\left(a^3\right),$$

$$\therefore 3z^2dz-3\left(yzdx+xzdy+xydz\right)=0,$$

解得 $dz=\dfrac{z\left(ydx+xdy\right)}{z^2-xy}$,

$$\therefore \frac{\partial z}{\partial x}=\frac{yz}{z^2-xy},\frac{\partial z}{\partial x}=\frac{xz}{z^2-xy}.$$

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由 $$\frac{\partial z}{\partial x}=\frac{yz}{z^2-xy}.$$

则 $$\frac{\partial^2 z}{\partial x\partial y}=\left(\frac{\partial z}{\partial x}\right)'_y=\left(\frac{yz}{z^2-xy}\right)'_y$$ ← 视x为常数，z为y的函数.

$$=\frac{(yz)'_y\cdot\left(z^2-xy\right)-yz\cdot\left(z^2-xy\right)'_y}{\left(z^2-xy\right)^2}$$

$$=\cdots=\frac{z\left(z^4-2xyz^2-x^2y^2\right)}{\left(z^2-xy\right)^3}.$$

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3.$\Phi(u,v)$有连续的偏导数,由$\Phi(cx-az,cy-bz)=0$

决定函数$z=z(x,y)$,证明:$a\dfrac{\partial z}{\partial x}+b\dfrac{\partial z}{\partial y}=c.$

解法一:理解为 $z=z(x,y)$, x,y为自变量,

记$\Phi(cx-az,cy-bz)=\Phi(u,v)=0,$

在$\Phi(cx-az,cy-bz)=0$两边分别对x、y求导:

$$\therefore \frac{\partial \Phi}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial \Phi}{\partial v}\cdot\frac{\partial v}{\partial x}=0 \Rightarrow \Phi_1\left(c-a\frac{\partial z}{\partial x}\right)+\Phi_2\left(-b\frac{\partial z}{\partial x}\right)=0,$$

$$\frac{\partial \Phi}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial \Phi}{\partial v}\cdot\frac{\partial v}{\partial y}=0 \Rightarrow \Phi_1\left(-a\frac{\partial z}{\partial y}\right)+\Phi_2\left(c-b\frac{\partial z}{\partial y}\right)=0,$$

从中解出$\dfrac{\partial z}{\partial x}$、$\dfrac{\partial z}{\partial y}$代入,即得结论.

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解法二：视 $F(x,y,z)=\Phi\left(cx-az,cy-bz\right)$

$$\frac{\partial F}{\partial x}=\frac{\partial \Phi}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial \Phi}{\partial v}\cdot\frac{\partial v}{\partial x}=\Phi_1\cdot c,$$

$$\frac{\partial F}{\partial y}=\frac{\partial \Phi}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial \Phi}{\partial v}\cdot\frac{\partial v}{\partial y}=\Phi_2\cdot c,$$

$$\frac{\partial F}{\partial z}=\frac{\partial \Phi}{\partial u}\cdot\frac{\partial u}{\partial z}+\frac{\partial \Phi}{\partial v}\cdot\frac{\partial v}{\partial z}=\Phi_1\cdot\left(-a\right)+\Phi_2\cdot\left(-b\right),$$

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z},\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}\cdots$$

Sec.18.2 隐函数组

一.方程组所确定的隐函数组及其导数

二.反函数组和坐标变换

现在我们来回顾一下多元函数复合函数及隐函数微分的计算方法.

$Th.17.5'$.若函数$u=\varphi(x,y),v=\psi(x,y)$在点$(x,y)$处都可$\left(偏\right)$导,函数$z=f(u,v)$在点$(u,v)$处具有连续的偏导数,则$z=f\left(\varphi(x,y),\psi(x,y)\right)$在点$(x,y)$处可$\left(偏\right)$导,且有全导数公式:

$$\begin{cases}\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial x}+\dfrac{\partial z}{\partial v}\cdot\dfrac{\partial v}{\partial x}\\ \dfrac{\partial z}{\partial y}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial y}+\dfrac{\partial z}{\partial v}\cdot\dfrac{\partial v}{\partial y}\end{cases}.$$

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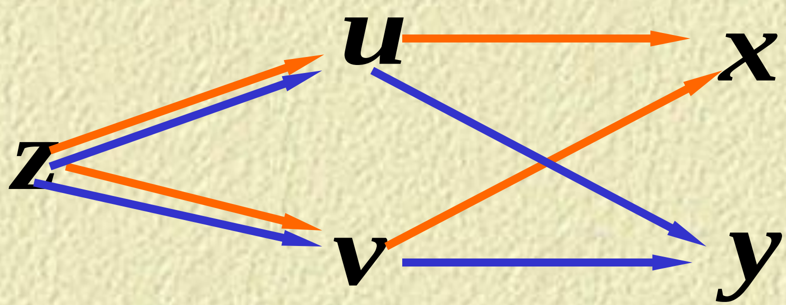
链式法则如图示



$$\left.\begin{aligned}\frac{\partial z}{\partial x}&=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x},\\ \frac{\partial z}{\partial y}&=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}.\end{aligned}\right\}\text{全导数}$$

隐函数存在定理与求导公式：

$Th.18.2.$设函数$F(x,y,z)$满足

(1).在点$P_0\left(x_0,y_0,z_0\right)$的某邻域内有连续的偏导数；

(2).$F\left(x_0,y_0,z_0\right)=0$;

(3).$F_z\left(x_0,y_0,z_0\right)\neq 0.$

则方程$F(x,y,z)=0$在点$\left(x_0,y_0\right)$的某邻域内可唯一确定一个单值连续函数$z=f(x,y)$，满足条件$z_0=f\left(x_0,y_0\right)$，且有连续的偏导数

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z},\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}.$$

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设$z = f(x, y)$是$F(x, y, z) = 0$所确定的隐函数,

则 $F\left(x, y, f(x, y)\right) \equiv 0$

$\Downarrow$ 两边对x求偏导数

$$\frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial x} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x} = 0$$

即 $F_x \cdot 1 + F_y \cdot 0 + F_z \cdot \dfrac{\partial z}{\partial x} = 0$

$\Downarrow$ 在$\left(x_0, y_0, z_0\right)$的某邻域内 $F_z \neq 0$

得 $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z}$, 同理 $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z}$.

一.方程组所确定的隐函数组及其导数

隐函数存在定理可以推广至方程组情形.

以两个方程确定两个隐函数的情形为例,即

$$\begin{cases} F(x, y, u, v) = 0 \\ G(x, y, u, v) = 0 \end{cases} \Rightarrow \begin{cases} u = u(x, y) \\ v = v(x, y) \end{cases}.$$

如果方程组确定了以u,v为x, y的隐函数组,

用两边求导法直接求出u,v对x, y的偏导数,

然后,解方程组就可以得到

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \cdots$$

例1. 设 $\begin{cases} xu - yv = 0 \\ yu + xv = 1 \end{cases}, \left(x^2 + y^2 \neq 0\right)$

求: $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$.

解 题目意为视u,v为 x,y 的二元函数,

方程组两边对x求导,并移项可得 $\begin{cases} x\dfrac{\partial u}{\partial x} - y\dfrac{\partial v}{\partial x} = -u \\ y\dfrac{\partial u}{\partial x} + x\dfrac{\partial v}{\partial x} = -v \end{cases}$,

当 $x^2 + y^2 \neq 0$ 时可以解得

$$\frac{\partial u}{\partial x} = -\frac{xu + yv}{x^2 + y^2}, \frac{\partial v}{\partial x} = -\frac{xv - yu}{x^2 + y^2}.$$

方程组$\begin{cases} xu - yv = 0 \\ yu + xv = 1 \end{cases}$两边分别对$x, y$求导,

$$\begin{cases} x\dfrac{\partial u}{\partial x} + u - y\dfrac{\partial v}{\partial x} = 0 \\ y\dfrac{\partial u}{\partial x} + x\dfrac{\partial v}{\partial x} + v = 0 \end{cases}, \quad \begin{cases} x\dfrac{\partial u}{\partial y} - y\dfrac{\partial v}{\partial y} - v = 0 \\ y\dfrac{\partial u}{\partial y} + u + x\dfrac{\partial v}{\partial y} = 0 \end{cases},$$

当 $x^2 + y^2 \neq 0$ 时可以解得

$$\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}; \quad \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}.$$

隐函数组存在定理：

以两个方程确定两个隐函数的情形为例,即

$$\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases} \Rightarrow \begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}.$$

由F,G的偏导数组成的行列式

$$J=\frac{\partial(F,G)}{\partial(u,v)}=\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}$$

称为F,G对变量u,v的雅可比$(Jacobi)$行列式.

$$\begin{pmatrix} F_u & F_v \\ G_u & G_v \end{pmatrix}$$

称为F,G对变量u,v的雅可比$(Jacobi)$矩阵.

$Th.18.3.$设函数$F(x,y,u,v),G(x,y,u,v)$满足

(1).在点$P_0(x_0,y_0,u_0,v_0)$的某邻域内有连续的偏导数;

(2).$F(x_0,y_0,u_0,v_0)=0,G(x_0,y_0,u_0,v_0)=0;$

相容性

(3).$J\big|_{P_0}=\left.\dfrac{\partial(F,G)}{\partial(u,v)}\right|_{P_0}\neq 0.$

则方程组$F(x,y,u,v)=0,G(x,y,u,v)=0$在点$(x_0,y_0)$的某邻域内可唯一确定一组单值连续函数

$$u=u(x,y),v=v(x,y),$$

满足条件 $u_0=u(x_0,y_0),v_0=v(x_0,y_0).$

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且有偏导数公式如下：

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = -\begin{pmatrix} F_u & F_v \\ G_u & G_v \end{pmatrix}^{-1} \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix},$$

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix},$$

$$J = \frac{\partial(F,G)}{\partial(u,v)} = \det\begin{pmatrix} F_u & F_v \\ G_u & G_v \end{pmatrix} \neq 0$$

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现在我们推导偏导数公式如下：

设 $\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases} \Rightarrow \begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$，即

$$\begin{cases} F\big(x,y,u(x,y),v(x,y)\big)\equiv 0 \\ G\big(x,y,u(x,y),v(x,y)\big)\equiv 0 \end{cases}$$ 方程两边对 x 求导

则方程两边对x求导(全导数公式!)

$$\begin{cases} \dfrac{\partial F}{\partial x}\cdot\dfrac{\partial x}{\partial x}+\dfrac{\partial F}{\partial y}\cdot\dfrac{\partial y}{\partial x}+\dfrac{\partial F}{\partial u}\cdot\dfrac{\partial u}{\partial x}+\dfrac{\partial F}{\partial v}\cdot\dfrac{\partial v}{\partial x}=0 \\ \dfrac{\partial G}{\partial x}\cdot\dfrac{\partial x}{\partial x}+\dfrac{\partial G}{\partial y}\cdot\dfrac{\partial y}{\partial x}+\dfrac{\partial G}{\partial u}\cdot\dfrac{\partial u}{\partial x}+\dfrac{\partial G}{\partial v}\cdot\dfrac{\partial v}{\partial x}=0 \end{cases}$$

$$\begin{cases} \dfrac{\partial F}{\partial x}\cdot\dfrac{\partial x}{\partial x}+\dfrac{\partial F}{\partial y}\cdot\dfrac{\partial y}{\partial x}+\dfrac{\partial F}{\partial u}\cdot\dfrac{\partial u}{\partial x}+\dfrac{\partial F}{\partial v}\cdot\dfrac{\partial v}{\partial x}=0 \\ \dfrac{\partial G}{\partial x}\cdot\dfrac{\partial x}{\partial x}+\dfrac{\partial G}{\partial y}\cdot\dfrac{\partial y}{\partial x}+\dfrac{\partial G}{\partial u}\cdot\dfrac{\partial u}{\partial x}+\dfrac{\partial G}{\partial v}\cdot\dfrac{\partial v}{\partial x}=0 \end{cases}$$

$$\therefore \begin{cases} F_x+F_u\cdot\dfrac{\partial u}{\partial x}+F_v\cdot\dfrac{\partial v}{\partial x}=0 \\ G_x+G_u\cdot\dfrac{\partial u}{\partial x}+G_v\cdot\dfrac{\partial v}{\partial x}=0 \end{cases}$$

$$\therefore \begin{pmatrix} F_u & F_v \\ G_u & G_v \end{pmatrix}\begin{pmatrix} u_x \\ v_x \end{pmatrix}=-\begin{pmatrix} F_x \\ G_x \end{pmatrix}\cdots\cdots(1)$$

$$\begin{cases} F\big(x,y,u(x,y),v(x,y)\big)\equiv 0 \\ G\big(x,y,u(x,y),v(x,y)\big)\equiv 0 \end{cases}$$ 方程两边对y求导

$$\begin{cases} F_y+F_u\dfrac{\partial u}{\partial y}+F_v\dfrac{\partial v}{\partial y}=0 \\ G_y+G_u\dfrac{\partial u}{\partial y}+G_v\dfrac{\partial v}{\partial y}=0 \end{cases}$$

$$\therefore \begin{pmatrix} F_u & F_v \\ G_u & G_v \end{pmatrix}\begin{pmatrix} u_y \\ v_y \end{pmatrix}=-\begin{pmatrix} F_y \\ G_y \end{pmatrix}\cdots\cdots(2)$$

$$\begin{cases} F\left(x,y,u(x,y),v(x,y)\right)\equiv 0 \\ G\left(x,y,u(x,y),v(x,y)\right)\equiv 0 \end{cases},$$

$$\therefore \begin{pmatrix} F_u & F_v \\ G_u & G_v \end{pmatrix}\begin{pmatrix} u_x \\ v_x \end{pmatrix} = -\begin{pmatrix} F_x \\ G_x \end{pmatrix} \cdots\cdots (1)$$

$$\begin{pmatrix} F_u & F_v \\ G_u & G_v \end{pmatrix}\begin{pmatrix} u_y \\ v_y \end{pmatrix} = -\begin{pmatrix} F_y \\ G_y \end{pmatrix} \cdots\cdots (2)$$

(1),(2)两式结合起来就是

$$\begin{pmatrix} F_u & F_v \\ G_u & G_v \end{pmatrix}\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = -\begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}$$

$A\beta_1=\gamma_1,$

$A\beta_2=\gamma_2,$

$A\in\mathbb{M}_{m\times n},$

$\beta_i\in\mathbb{M}_{n\times 1},$

$\gamma_i\in\mathbb{M}_{m\times 1},$

$(A\beta_1,A\beta_2)=(\gamma_1,\gamma_2)$

$A(\beta_1,\beta_2)=(\gamma_1,\gamma_2)$

$\leftrightarrow AB=C.$

$$\begin{pmatrix} F_u & F_v \\ G_u & G_v \end{pmatrix}\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = -\begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}$$

函数F,G对于变量u,v的*Jacobi*行列式

$$J = \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix} \neq 0,$$ 意味着矩阵 $$\begin{pmatrix} F_u & F_v \\ G_u & G_v \end{pmatrix}$$ 可逆,

$$\therefore \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = -\begin{pmatrix} F_u & F_v \\ G_u & G_v \end{pmatrix}^{-1}\begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}$$

再看例 1.设$\begin{cases} xu - yv = 0 \\ yu + xv = 1 \end{cases}$, $\left(x^2 + y^2 \neq 0\right)$

求:$\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$.

解 方程组$\begin{cases} xu - yv = 0 \\ yu + xv = 1 \end{cases}$两边分别对$x, y$求导,

$$\begin{cases} x\dfrac{\partial u}{\partial x} + u - y\dfrac{\partial v}{\partial x} = 0 \\ y\dfrac{\partial u}{\partial x} + x\dfrac{\partial v}{\partial x} + v = 0 \end{cases}, \quad \begin{cases} x\dfrac{\partial u}{\partial y} - y\dfrac{\partial v}{\partial y} - v = 0 \\ y\dfrac{\partial u}{\partial y} + u + x\dfrac{\partial v}{\partial y} = 0 \end{cases},$$

$$\Rightarrow \begin{pmatrix} x & -y \\ y & x \end{pmatrix}\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = -\begin{pmatrix} u & -v \\ v & u \end{pmatrix},$$

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$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}=-\begin{pmatrix} u & -v \\ v & u \end{pmatrix}$$

$$J=\begin{vmatrix} x & -y \\ y & x \end{vmatrix}=x^2+y^2\neq 0,\therefore\begin{pmatrix} x & -y \\ y & x \end{pmatrix}\text{为可逆阵}$$

$$\therefore\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}=-\begin{pmatrix} x & -y \\ y & x \end{pmatrix}^{-1}\begin{pmatrix} u & -v \\ v & u \end{pmatrix},$$

将这一结果利用矩阵的求逆和乘法,可以求得与前面一样的结果。

例2.设$\begin{cases} u^2 - v + x = 0 \\ u + v^2 - y = 0 \end{cases}$ 在点$P_0(-3,3,2,1)$的邻域内确定函数

$$\begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases},\text{求}:\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial v}{\partial x},\frac{\partial v}{\partial y}.$$

解 令$\begin{cases} F(x,y,u,v) = u^2 - v + x, \\ G(x,y,u,v) = u + v^2 - y, \end{cases}$

则 $$\frac{\partial(F,G)}{\partial(u,v)} = \begin{vmatrix} 2u & -1 \\ 1 & 2v \end{vmatrix} = 4uv + 1,$$

$$J\big|_{P_0} = \left.\frac{\partial(F,G)}{\partial(u,v)}\right|_{P_0} = 9 \neq 0$$

$$\begin{cases} F\left(P_0\right)=F\left(-3,3,2,1\right)=0 \\ G\left(P_0\right)=G\left(-3,3,2,1\right)=0 \end{cases}$$

所以,问题符合隐函数组定理的条件,在点$P_0\left(-3,3,2,1\right)$的邻域内确定了隐函数组的存在性.

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令$\begin{cases} u^2 - v + x = 0 \\ u + v^2 - y = 0 \end{cases}$，

对x求导 $\begin{cases} 2u \cdot u_x - v_x + 1 = 0 \\ u_x + 2v \cdot v_x - 0 = 0 \end{cases}$，

对y求导 $\begin{cases} 2u \cdot u_y - v_y + 0 = 0 \\ u_y + 2v \cdot v_y - 1 = 0 \end{cases}$，

$$\therefore \begin{pmatrix} 2u & -1 \\ 1 & 2v \end{pmatrix} \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

令$\begin{cases} u^2 - v + x = 0 \\ u + v^2 - y = 0 \end{cases}$,

$$\begin{pmatrix} 2u & -1 \\ 1 & 2v \end{pmatrix}\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

当$\begin{pmatrix} 2u & -1 \\ 1 & 2v \end{pmatrix}$为可逆矩阵时,

有$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} 2u & -1 \\ 1 & 2v \end{pmatrix}^{-1}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

同样,若 $\begin{cases} u^2 - v + x = 0 \\ u + v^2 - y = 0 \end{cases}$ 确定函数 $\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$

对u求导 $\begin{cases} 2u + 0 + x_u = 0 \\ 1 + 0 - y_u = 0 \end{cases}$,

对v求导 $\begin{cases} 0 - 1 + x_v = 0 \\ 0 + 2v - y_v = 0 \end{cases}$,

$$\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2u & -1 \\ 1 & 2v \end{pmatrix}$$

此处有 $\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} 2u & -1 \\ 1 & 2v \end{pmatrix}^{-1} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2u & -1 \\ 1 & 2v \end{pmatrix}$$

$$\therefore \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \cdot \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

真正错综复杂

注意,此处没有 $\dfrac{\partial u}{\partial x} \cdot \dfrac{\partial x}{\partial u} = 1,$

而是 $\dfrac{\partial u}{\partial x} \cdot \dfrac{\partial x}{\partial u} + \dfrac{\partial u}{\partial y} \cdot \dfrac{\partial y}{\partial u} = 1,$

又是一个前所未见的结论！

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二.反函数组和坐标变换

一般地,$\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases}$,

当$\dfrac{\partial(F,G)}{\partial(u,v)}=\det\begin{pmatrix} F_u & F_v \\ G_u & G_v \end{pmatrix}\neq 0$时,

则由方程组可确定函数$\begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$,

同样,当$\dfrac{\partial(F,G)}{\partial(x,y)}=\det\begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}\neq 0$时,

则可由方程组确定函数$\begin{cases} x=x(u,v) \\ y=y(u,v) \end{cases}$.

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$$\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases},$$

当$\det\begin{pmatrix} F_u & F_v \\ G_u & G_v \end{pmatrix} \neq 0$时，有$\begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$,

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = -\begin{pmatrix} F_u & F_v \\ G_u & G_v \end{pmatrix}^{-1} \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}$$

当$\det\begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} \neq 0$时，$\begin{cases} x=x(u,v) \\ y=y(u,v) \end{cases}$,

$$\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = -\begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}^{-1} \begin{pmatrix} F_u & F_v \\ G_u & G_v \end{pmatrix}$$

这就是由 $\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases}$ 确定的互逆关系的反函数组

$$\begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases} \quad \& \quad \begin{cases} x=x(u,v) \\ y=y(u,v) \end{cases},$$

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = -\begin{pmatrix} F_u & F_v \\ G_u & G_v \end{pmatrix}^{-1} \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}$$

$$\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = -\begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}^{-1} \begin{pmatrix} F_u & F_v \\ G_u & G_v \end{pmatrix}$$

$$\therefore \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \cdot \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases} \leftrightarrow \begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases} \& \begin{cases} x=x(u,v) \\ y=y(u,v) \end{cases},$$

$$\therefore \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \cdot \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

必须注意到,在隐函数组中没有 $\dfrac{\partial u}{\partial x} \cdot \dfrac{\partial x}{\partial u} = 1,$

而是 $\dfrac{\partial u}{\partial x} \cdot \dfrac{\partial x}{\partial u} + \dfrac{\partial u}{\partial y} \cdot \dfrac{\partial y}{\partial u} = 1,$ 切勿与一元函数中相关结论混淆!同时, $\dfrac{\partial u}{\partial x} \cdot \dfrac{\partial x}{\partial v} + \dfrac{\partial u}{\partial y} \cdot \dfrac{\partial y}{\partial v} = 0.$

例3. 计算极坐标变换的反变换的导数. $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

解 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$,方程两边对 x 求导,

$$\begin{cases} 1 = \dfrac{\partial r}{\partial x}\cos\theta + r\cdot(-\sin\theta)\dfrac{\partial\theta}{\partial x} \\ 0 = \dfrac{\partial r}{\partial x}\sin\theta + r\cdot\cos\theta\dfrac{\partial\theta}{\partial x} \end{cases}$$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r,$$

$J = r \neq 0$时,可解得$\dfrac{\partial r}{\partial x}$与$\dfrac{\partial\theta}{\partial x}$.

$$\begin{cases} \dfrac{\partial r}{\partial x}\cos\theta + r\cdot(-\sin\theta)\dfrac{\partial \theta}{\partial x} = 1 \\ \dfrac{\partial r}{\partial x}\sin\theta + r\cdot\cos\theta\dfrac{\partial \theta}{\partial x} \quad = 0 \end{cases}$$

即$$\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}\begin{pmatrix} \dfrac{\partial r}{\partial x} \\ \dfrac{\partial \theta}{\partial x} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$,方程两边对 y 求导,

$$\begin{cases} 0 = \dfrac{\partial r}{\partial y}\cos\theta + r\cdot(-\sin\theta)\dfrac{\partial \theta}{\partial y} \\ 1 = \dfrac{\partial r}{\partial y}\sin\theta + r\cdot\cos\theta\dfrac{\partial \theta}{\partial y} \end{cases}$$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r,$$

$J = r \neq 0$时,可解得 $\dfrac{\partial r}{\partial y}$ 与 $\dfrac{\partial \theta}{\partial y}$.

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$$\begin{cases} \dfrac{\partial r}{\partial y}\cos\theta + r\cdot(-\sin\theta)\dfrac{\partial \theta}{\partial y} = 0 \\ \dfrac{\partial r}{\partial y}\sin\theta + r\cdot\cos\theta\dfrac{\partial \theta}{\partial y} = 1 \end{cases}$$

即$$\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}\begin{pmatrix} \dfrac{\partial r}{\partial y} \\ \dfrac{\partial \theta}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\left.\begin{aligned}\begin{pmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{pmatrix}\begin{pmatrix}r_x\\ \theta_x\end{pmatrix}=\begin{pmatrix}1\\ 0\end{pmatrix}\\ \begin{pmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{pmatrix}\begin{pmatrix}r_y\\ \theta_y\end{pmatrix}=\begin{pmatrix}0\\ 1\end{pmatrix}\end{aligned}\right\}\Rightarrow$$

$$\begin{pmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{pmatrix}\begin{pmatrix}r_x & r_y\\ \theta_x & \theta_y\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}\begin{pmatrix} r_x & r_y \\ \theta_x & \theta_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r \neq 0\text{时},$$

$$\begin{pmatrix} r_x & r_y \\ \theta_x & \theta_y \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}^{-1}$$

$$= \frac{1}{r}\begin{pmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}, \quad \frac{\partial(x,y)}{\partial(r,\theta)} = r \neq 0\text{时},$$

$$\begin{pmatrix} r_x & r_y \\ \theta_x & \theta_y \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}^{-1}$$

$$= \frac{1}{r}\begin{pmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$\therefore$ 在实平面上除了极点也就是直角坐标系的原点外的其他地方,直角坐标与极坐标之间一一对应 ！

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \frac{\partial(x,y)}{\partial(r,\theta)} = r \neq 0\text{时},$$

$$\begin{pmatrix} r_x & r_y \\ \theta_x & \theta_y \end{pmatrix} = \frac{1}{r}\begin{pmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\begin{cases} \dfrac{\partial r}{\partial x} = \cos\theta = \dfrac{x}{\sqrt{x^2+y^2}} \\ \dfrac{\partial\theta}{\partial x} = \dfrac{-1}{r}\sin\theta = \dfrac{-y}{x^2+y^2} \end{cases},$$

同样,$\dfrac{\partial r}{\partial y} = \dfrac{y}{\sqrt{x^2+y^2}}, \dfrac{\partial\theta}{\partial y} = \dfrac{x}{x^2+y^2}$

思考与练习

设$y = y(x), z = z(x)$是由方程 $z = xf\left(x + y\right)$

与$F\left(x, y, z\right) = 0$所确定的函数，求$\dfrac{dz}{dx}$.

解 在各方程两边分别对x求导：

$$\begin{cases} z' = f + xf'\cdot\left(1 + y'\right) \\ F_x + F_y\cdot y' + F_z\cdot z' = 0 \end{cases}$$

$$\therefore \begin{cases} -xf'\cdot y' + z' = f + xf' \\ F_y\cdot y' + F_z\cdot z' = -F_x \end{cases}.$$

$$z = xf\left(x+y\right),\ F\left(x,y,z\right)=0$$

$$\begin{cases} z' = f + xf'\cdot\left(1+y'\right) \\ F_x + F_y\cdot y' + F_z\cdot z' = 0 \end{cases}$$

$$\therefore \begin{cases} -xf'\cdot y' + z' = f + xf' \\ F_y\cdot y' + F_z\cdot z' = -F_x \end{cases}$$

$$\frac{dz}{dx} = \frac{\begin{vmatrix} -x f' & f + x f' \\ F_y & -F_x \end{vmatrix}}{\begin{vmatrix} -x f' & 1 \\ F_y & F_z \end{vmatrix}} = \frac{\left(f + xf'\right)F_y - xf'\cdot F_x}{F_y + xf'\cdot F_z}$$

$$\left(F_y + xf'\cdot F_z \neq 0\right)$$

Sec.18.3 多元函数微分在几何中的应用

一. 空间曲线的切线

二. 曲面的切平面与法线

0. 复习　在$Euclidean$空间$\mathbb{R}^3$中,

(I).向量$\alpha=(a_1,b_1,c_1)\neq 0,\beta=(a_2,b_2,c_2)\neq 0$,

$\alpha // \beta \Leftrightarrow \alpha,\beta$ 线性相关,即$\exists l,k\ (l^2+k^2\neq 0)$,

$$l\alpha+k\beta=0 \Leftrightarrow \frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}.$$

过点$M(x_0,y_0,z_0)$,以向量$\vec{\tau}=(a,b,c)\neq 0$

为方向向量的直线方程为

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}\ .$$

在$Euclidean$空间$\mathbb{R}^3$中,

(II).非零向量$\alpha=\left(a_1,b_1,c_1\right),\beta=\left(a_2,b_2,c_2\right)$,

$\alpha\perp\beta$,即α,β垂直或曰正交

$\Leftrightarrow$向量α,β的内积$\left(\alpha,\beta\right)$有$\left(\alpha,\beta\right)=0$,

即 $a_1a_2+b_1b_2+c_1c_2=0$

过点$M\left(x_0,y_0,z_0\right)$,以向量$\vec{n}=\left(a,b,c\right)\neq 0$

为法向量的平面方程为

$$a\left(x-x_0\right)+b\left(y-y_0\right)+c\left(z-z_0\right)=0$$

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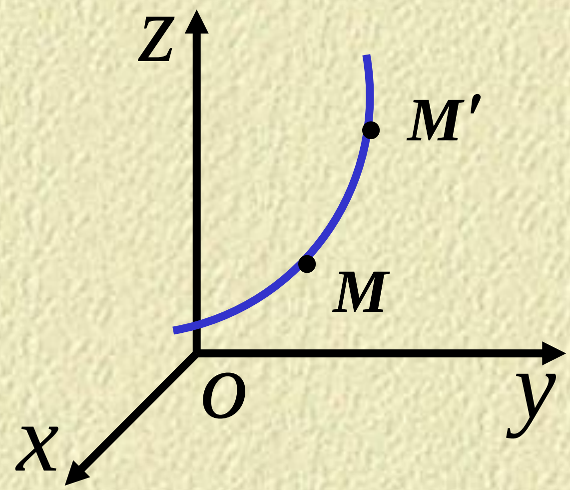
一. 空间曲线的切线

设空间曲线Γ: $\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$ (1)



(1)中的三个函数都可导,

设 $M\left(x_0, y_0, z_0\right)$,对应于 $t = t_0$,

$M'\left(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z\right)$

对应于 $t = t_0 + \Delta t$.

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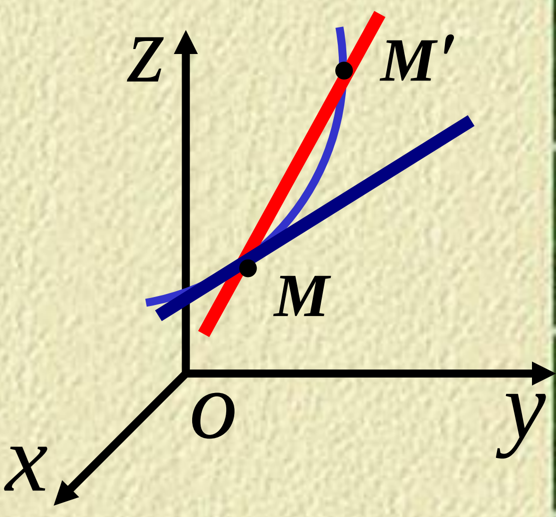
割线MM'的方程为

$$\frac{x-x_0}{\Delta x}=\frac{y-y_0}{\Delta y}=\frac{z-z_0}{\Delta z}$$



考察割线趋近于极限位置

——切线的过程,上式两

边同时除以Δt,

$$\frac{x-x_0}{\Delta x/\Delta t}=\frac{y-y_0}{\Delta y/\Delta t}=\frac{z-z_0}{\Delta z/\Delta t},$$

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当$M' \to M$,即$\Delta t \to 0$时，

曲线在点M处的切线方程为

$$\frac{x-x_0}{x'(t_0)} = \frac{y-y_0}{y'(t_0)} = \frac{z-z_0}{z'(t_0)}.$$

***Def*. 切向量：切线的方向向量是曲线的切向量.**

$$\vec{\tau} = \left(x'(t_0), y'(t_0), z'(t_0)\right)$$

***Def*. 法平面：过切点且与切线垂直的平面是曲线的法平面.**

$$x'(t_0)\left(x-x_0\right) + y'(t_0)\left(y-y_0\right) + z'(t_0)\left(z-z_0\right) = 0$$

特别地,空间曲线方程为$\begin{cases} y = y(x) \\ z = z(x) \end{cases}$,

在$M\left(x_0, y_0, z_0\right)$处,切线方程为

$$\frac{x - x_0}{1} = \frac{y - y_0}{y'(x_0)} = \frac{z - z_0}{z'(x_0)},$$

法平面方程为

$$\left(x - x_0\right) + y'(x_0)\left(y - y_0\right) + z'(x_0)\left(z - z_0\right) = 0.$$

例1.证明:若曲线在每一点处的法平面都过一定点,则该曲线必是一条球面曲线.

证明:设曲线$\Gamma \quad x = x(t), y = y(t), z = z(t)$.

Γ上点$M(x, y, z)$,对应$x = x(t), \cdots$.

曲线的切向量$\vec{\tau} = \left(x'(t), y'(t), z'(t) \right)$

点M处的法平面:

$$x'(t)(X - x) + y'(t)(Y - y) + z'(t)(Z - z) = 0,$$

上页 下页 返回

证　明：设曲线Γ　$x = x(t), y = y(t), z = z(t)$.

Γ上点$M(x, y, z)$，对应$x = x(t), \cdots$.

曲线的切向量$\vec{\tau} = \left(x'(t), y'(t), z'(t) \right)$

点M处的法平面：

$$x'(t)(X - x) + y'(t)(Y - y) + z'(t)(Z - z) = 0,$$

设曲线的法平面过定点(a, b, c)，

则$x'(t)(a - x) + y'(t)(b - y) + z'(t)(c - z) = 0$，

则有$\dfrac{d}{dt}\left[(a - x)^2 + (b - y)^2 + (c - z)^2 \right] = 0$，

$$\therefore (x - a)^2 + (y - b)^2 + (z - c)^2 = C\left(> 0 \right). \otimes$$

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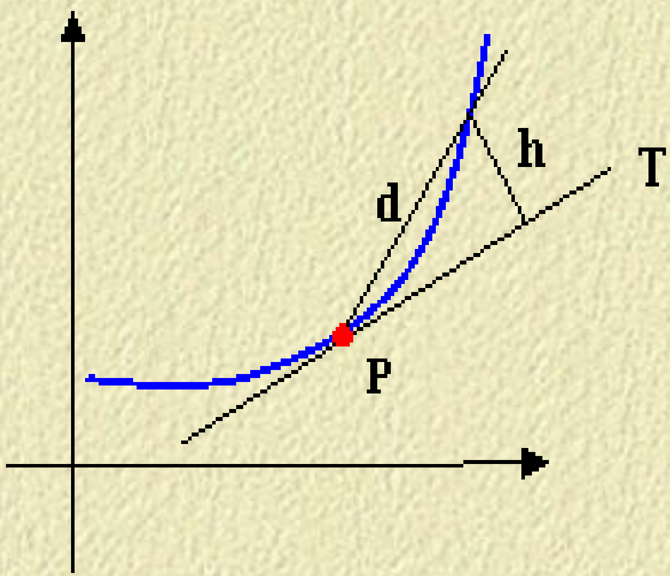
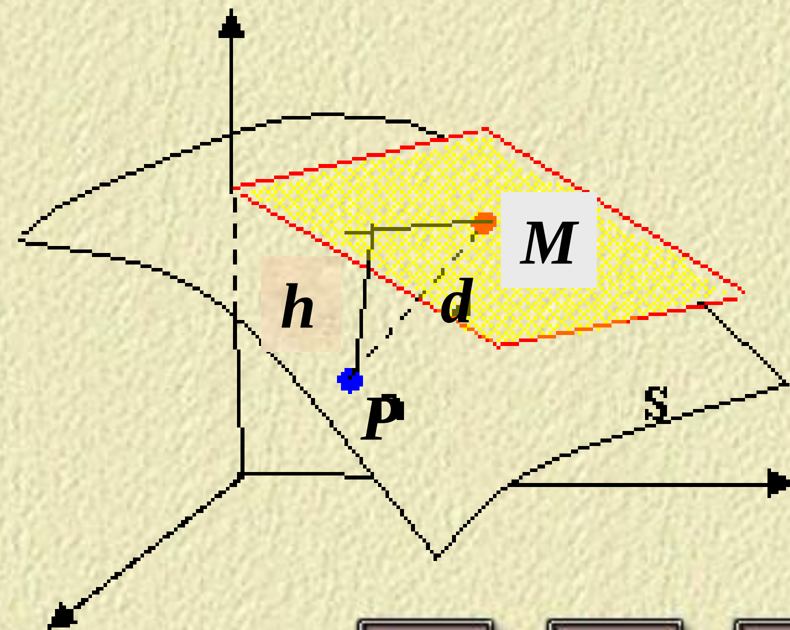
二. 曲面的切平面与法线

在17章**可微性的几何意义与应用**中，我们已经用多元函数的微分描述了空间曲面的切平面。

回顾一下前面介绍的切平面的定义之一





定义(切平面) 设M是曲面S上一点,H为通过M的一个平面,曲面S上的动点P到M和到平面H的距离分别为d和h,当P在S上以任何方式趋于M时,恒有$h/d\to 0$,则称平面H为曲面S在点M处的切平面,M为切点.

定理17.4 曲面$z=f(x,y)$在点$M(x_0,y_0,z_0)$,

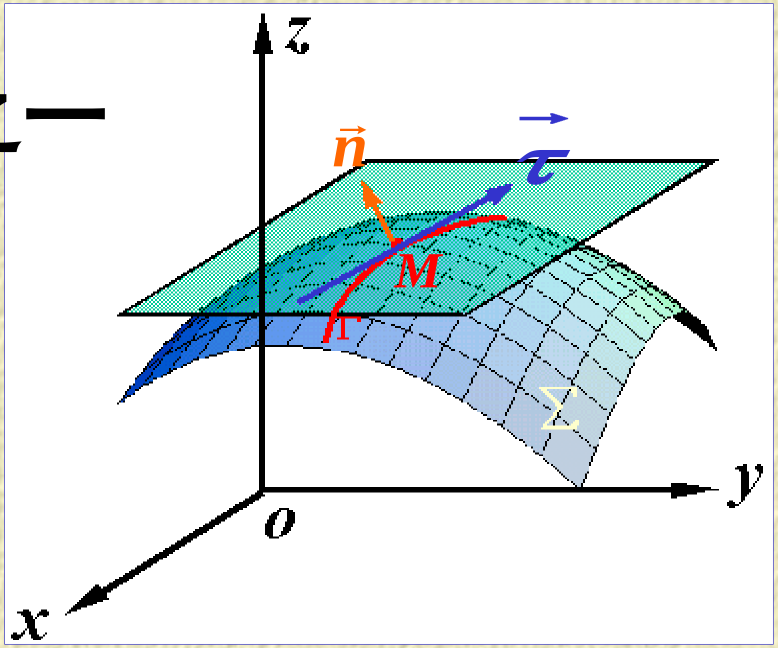
$z_0=f(x_0,y_0)$处存在不平行于z轴的切平面

$\Leftrightarrow$ 函数$z=f(x,y)$在点$P_0(x_0,y_0)$处可微.

可微性的几何意义

$\boxed{\text{函数可微}} \Rightarrow \boxed{\text{曲面有切平面}} \Rightarrow \boxed{\text{曲面光滑}}$

空间曲面的切平面定义之一



1. 设曲面方程

为$z = f(x, y)$,则

曲面在 点M处

的切平面方程为

$$f_x(x_0, y_0)\left(x - x_0\right) + f_y(x_0, y_0)\left(y - y_0\right) = z - z_0,$$

曲面在点M处的法向量为$\vec{n} = \pm\left(f_x(x_0, y_0), f_y(x_0, y_0), -1\right),$

曲面在点M处的法线方程为

$$\frac{x - x_0}{f_x(x_0, y_0)} = \frac{y - y_0}{f_y(x_0, y_0)} = \frac{z - z_0}{-1}.$$

若α,β,γ表示曲面的法向量的方向角,假定法向量的方向是向上的,即它与 z 轴的正向的夹角γ是锐角,则法向量的方向余弦为

$$\cos\alpha=\frac{-f_x}{\sqrt{1+f_x^2+f_y^2}},\cos\beta=\frac{-f_y}{\sqrt{1+f_x^2+f_y^2}},$$

$$\cos\gamma=\frac{1}{\sqrt{1+f_x^2+f_y^2}}.$$

$$f_x=f_x(x_0,y_0)\,f_y=f_y(x_0,y_0)$$

曲面在点M处的单位法向量为

$$\overrightarrow{n^o}=(\cos\alpha,\cos\beta,\cos\gamma)$$

全微分的几何意义

因为曲面在M处的切平面方程为

$$\boxed{z-z_0}=\boxed{f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)}$$

切平面上点的竖坐标的增量

函数$z=f(x,y)$在点(x_0,y_0)的全微分

$z=f(x,y)$在(x_0,y_0)的全微分，表示曲面$z=f(x,y)$在点(x_0,y_0,z_0)处的切平面上的点的竖坐标的增量.

全微分在近似计算中的应用

$$\Delta z \approx dz = f_x(x,y)\Delta x + f_y(x,y)\Delta y.$$

也就是

$$z - z_0 \approx f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$$

其几何直观的理解就是在切点的邻近，我们用切面片去近似替代曲面片。

以直代曲

空间曲面的切平面定义之二

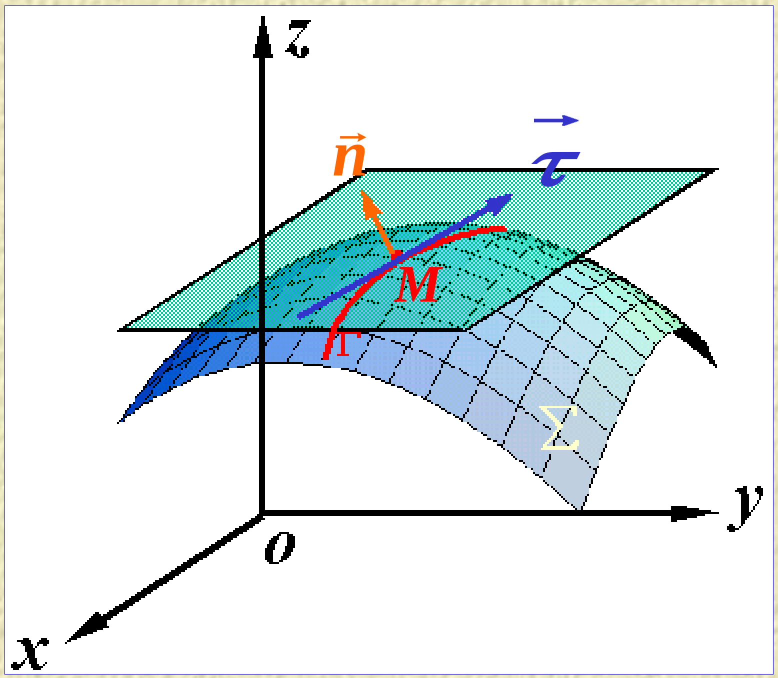
2. 设曲面方程

为$F(x,y,z)=0$,

在曲面上任取一条过点

$M(x_0,y_0,z_0)$的曲线

$$\Gamma:\begin{cases} x=x(t) \\ y=y(t), \\ z=z(t) \end{cases}$$



曲线在点M处的切向量为

$$\vec{\tau}=\left(x'(t_0),y'(t_0),z'(t_0)\right)$$

则 $F\left(x,y,z\right)=0,\begin{cases} x=x(t) \\ y=y(t), \\ z=z(t) \end{cases}$

$\Rightarrow F\left(x(t),y(t),z(t)\right)=0$

$\therefore t=t_0: F_x\cdot x'(t)+F_y\cdot y'(t)+F_z\cdot z'(t)=0$

$\vec{\tau}=\left(x'(t_0),y'(t_0),z'(t_0)\right)$

$\vec{n}=\left(F_x,F_y,F_z\right),F_z\left(x_0,y_0,z_0\right):=F_z$

$\because \vec{a}\perp\vec{b}\Leftrightarrow a_xb_x+a_yb_y+a_zb_z=0$

$\therefore \quad \vec{n}\perp\vec{\tau}$

令$\vec{n}=\left(F_x,F_y,F_z\right),F_z\left(x_0,y_0,z_0\right):=F_z$,则

$\vec{n}\perp\vec{\tau}$,由于曲线是曲面上过M点的任意一条曲线,它们在点M处的切线都与同一向量 $\vec{n}$正交,故曲面上过M点的任一曲线在点M处的切线都在同一平面上,该平面就是曲面在M处的切平面:

$$F_x\left(x_0,y_0,z_0\right)\left(x-x_0\right)+F_y\left(x_0,y_0,z_0\right)\left(y-y_0\right)$$
$$+F_z\left(x_0,y_0,z_0\right)\left(z-z_0\right)=0$$

切平面方程为

$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)$$
$$+F_z(x_0,y_0,z_0)(z-z_0)=0$$

过$M(x_0,y_0,z_0)$点而垂直于切平面的直线

——曲面在该点的法线,其方程为

$$\frac{x-x_0}{F_x(x_0,y_0,z_0)}=\frac{y-y_0}{F_y(x_0,y_0,z_0)}=\frac{z-z_0}{F_z(x_0,y_0,z_0)}$$

曲面在点$M(x_0,y_0,z_0)$处的法向量为

$$\vec{n}=\left(F_x,F_y,F_z\right)\Big|_{(x_0,y_0,z_0)}$$

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由隐函数定理知,由 $F(x,y,z)=0, F_z \neq 0,$

确定函数$z=f(x,y), \dfrac{\partial z}{\partial x}=-\dfrac{F_x}{F_z}=f_x(x,y),$

$$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=f_y(x,y),$$

由此可知,切平面的两种方程完全相同

$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)$$
$$+F_z(x_0,y_0,z_0)(z-z_0)=0,$$

$$f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)=z-z_0.$$

$F_z(x_0,y_0,z_0)$简记为F_z,余类推.

若α,β,γ分别表示曲面在点M处的法向量与三根坐标轴ox, oy, oz轴正向的夹角,则点$M(x_0,y_0,z_0)$处的法向量的方向余弦为

$$\cos\alpha=\frac{F_x}{\pm\sqrt{F_x^{\ 2}+F_y^{\ 2}+F_z^{\ 2}}}$$

$$\cos\beta=\frac{F_y}{\pm\sqrt{F_x^{\ 2}+F_y^{\ 2}+F_z^{\ 2}}}$$

$$\cos\gamma=\frac{F_z}{\pm\sqrt{F_x^{\ 2}+F_y^{\ 2}+F_z^{\ 2}}}$$

单位法向量 $\overrightarrow{n^o}=(\cos\alpha,\cos\beta,\cos\gamma)$

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例2.试给出椭球面 $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$

上点$P_0\left(x_0,y_0,z_0\right)$处的切平面.

解　曲面$F\left(x,y,z\right)=0$的法向量$\vec{n}=\left(F_x,F_y,F_z\right)$,

$\therefore$ 椭球面上点P_0处的法向量$\vec{n}=\left(\dfrac{2x_0}{a^2},\dfrac{2y_0}{b^2},\dfrac{2z_0}{c^2}\right)$,

$\therefore$ 点P_0处切平面:

$$\frac{x_0}{a^2}\left(x-x_0\right)+\frac{y_0}{b^2}\left(y-y_0\right)+\frac{z_0}{c^2}\left(z-z_0\right)=0$$

即　$\dfrac{x_0x}{a^2}+\dfrac{y_0y}{b^2}+\dfrac{z_0z}{c^2}=1.$

例3.设$f(u)$可微,求证曲面$z = xf\left(\frac{y}{x}\right)$上任一点处的切平面都通过原点.

证明 $\frac{\partial z}{\partial x} = f\left(\frac{y}{x}\right) + xf'\left(\frac{y}{x}\right)\cdot\frac{-y}{x^2} = f(u) - \frac{y}{x}f'(u)$,

$\frac{\partial z}{\partial y} = xf'\left(\frac{y}{x}\right)\cdot\frac{1}{x} = f'(u)$. $\frac{y}{x} = u$

曲面上任取一点$P_0(x_0, y_0, z_0)$,记$u_0 = \frac{y_0}{x_0}$,

曲面的法向量 $\vec{n} = (z_x, z_y, -1)$,

$\therefore$曲面上P_0处的切平面方程为

$$z - z_0 = \left(f(u_0) - u_0 f'(u_0)\right)(x - x_0) + f'(u_0)(y - y_0),$$

由$u_0 = \frac{y_0}{x_0}$,$z = x_0 f(u_0)$知$O(0,0)$在该切平面上.

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Exercises :

1.证明:曲面$\sqrt{x}+\sqrt{y}+\sqrt{z}=\sqrt{a}$ $(a>0)$上任意一点处的切平面在各坐标轴上的截距之和为常数.

2.设函数$f(u,v)$有连续的偏导数,a,b为常数.证明:曲面$f(x-az,y-bz)=0$在任意一点处的切平面与一条定直线平行.

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Sec.18.4 条件极值

一.多元函数的无条件极值

二.条件极值拉格朗日乘数法

一.多元函数的无条件极值　回顾

Th.17.10.$\left(\text{必要条件}\right)$设函数$z = f\left(x, y\right)$在点$\left(x_0, y_0\right)$处有偏导数,且在该点处取得极值,则有$f_x\left(x_0, y_0\right) = 0,\ f_y\left(x_0, y_0\right) = 0.$

注意:　驻点 $\underset{\not\Longrightarrow}{\not\Longleftarrow}$ 极值点

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定理17.11.(充分条件)设函数$f(x,y)$在驻点$\left(x_0,y_0\right)$的某邻域内有连续的一阶和二阶偏导数.

令 $f_{xx}\left(x_0,y_0\right)=A, f_{xy}\left(x_0,y_0\right)=B, f_{yy}\left(x_0,y_0\right)=C.$

则函数$f(x,y)$在点$\left(x_0,y_0\right)$处取得极值的情况如下：

(1).$AC-B^2>0$ 时有极值：$A>0$ 时有极小值,

$A<0$ 时有极大值.

(2).$AC-B^2<0$ 时一定没有极值.

(3).$AC-B^2=0$ 时极值情况不确定.

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设$P_0(x_0, y_0)$为函数f的驻点,

$$H_{f(P_0)} = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}_{P_0} = \begin{pmatrix} A & B \\ B & C \end{pmatrix},$$

Hessian matrix

Hesse 矩阵

记$|H| = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = AC - B^2$.则

(1).当$|H| > 0$ 时,函数f在(x_0, y_0)处取极值,

且$A > 0$,函数取极小值,$A < 0$,函数取极大值.

(2).$|H| < 0$ 时,函数f在(x_0, y_0)处不取极值.

(3).$|H| = 0$ 时,函数f在(x_0, y_0)处的极值情况无法确定.

极值充分条件的定理要利用二元函数的泰勒公式来证明.

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二.条件极值,拉格朗日乘数法

极值问题

- 无条件极值:对自变量只有定义域限制.
- 条件极值:对自变量除了有定义域限制外还有其他限制条件.

例如,求函数 $f(x,y)=x^2+2y^2$ 在约束条件 $x^2+y^2=1$ 下的最大与最小值.

条件$\varphi(x,y)=0$下,求函数$z=f(x,y)$的极值,其直观的几何解释是:

曲面$z=f(x,y)$与以坐标面xoy上$\varphi(x,y)=0$为准线,母线平行于z轴的柱面相交的曲线

为$\begin{cases} z=f(x,y) \\ \varphi(x,y)=0 \end{cases}$,现在要求该空间曲线上点的竖坐标$z$的数值的极$\left(\text{大,小}\right)$值.

比如,求函数$f(x,y)=x^2+2y^2$在约束条件$x^2+y^2=1$下的最大与最小值.

就是确定椭圆抛物面$z=x^2+2y^2$被圆柱面$x^2+y^2=1$所截的截痕的最高点,最低点.

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条件极值的求法：

方法1.代入法

在条件$\varphi(x,y)=0$下求函数

$z=f(x,y)$的极值.

转化

从条件$\varphi(x,y)=0$

中解出$y=\psi(x)$

求函数$z=f\big(x,\psi(x)\big)$的无条件极值.

例如,求函数$f(x,y)=x^2+2y^2$在约束条件$x^2+y^2=1$下的最大,小值.

解 $x^2+y^2=1 \leftrightarrow y^2=1-x^2, x\in[-1,1]$

$$\therefore z=x^2+2y^2=2-x^2, x\in[-1,1],$$

所以问题就转变为求$z=2-x^2$在$[-1,1]$上的最大,小值了.

条件极值的求法：

方法2.*Lagrange*乘数法.

求在条件$\varphi(x,y)=0$下函数$z=f(x,y)$的极值：

引入辅助函数

$$L(x,y,\lambda)=f(x,y)+\lambda\varphi(x,y),$$

则极值点的坐标满足方程组

$$\begin{cases} f_x+\lambda\varphi_x=0 \\ f_y+\lambda\varphi_y=0 \\ \varphi(x,y)=0 \end{cases} \text{也就是} \begin{cases} L_x=0 \\ L_y=0 \\ L_\lambda=0 \end{cases}$$

解出x,y,λ,其中(x,y)就是可能的极值点.辅助函数L称为是*Lagrange*(*Lagrangian Multiplier*)乘子函数.

结论解析：

求在条件$\varphi(x,y)=0$下函数$z=f(x,y)$的极值：如方法1所述，设$\varphi(x,y)=0$可确定函数$y=\psi(x)$，则问题就等价于求函数$z=f\left(x,\psi(x)\right)$的极值.若函数$f,\varphi$均有连续偏导数，则$z=f\left(x,\psi(x)\right)$在其取得极值的地方必有$\dfrac{dz}{dx}=0.$

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条件$\varphi(x,y)=0$下，求函数$z=f(x,y)$的极值

$\varphi(x,y)=0$确定隐函数$y=\psi(x)$,

则$z=f\left(x,\psi(x)\right)$在其取得极值的地方

有 $\dfrac{dz}{dx}=0$,由全导数公式得

$$\frac{dz}{dx}=f_x+f_y\frac{dy}{dx}=0\cdots\cdots(1)$$

$$\because\ \varphi(x,y)=0\Rightarrow\frac{dy}{dx}=-\frac{\varphi_x}{\varphi_y},$$

$$\therefore\ (1):\ f_x-f_y\frac{\varphi_x}{\varphi_y}=0\ \text{即}\ \frac{f_x}{\varphi_x}=\frac{f_y}{\varphi_y}$$

条件$\varphi(x,y)=0$下,函数$z=f(x,y)$的极值

点的坐标满足方程 $f_x - f_y\dfrac{\varphi_x}{\varphi_y}=0\cdots\cdots(1)$

记 $\dfrac{f_x}{\varphi_x}=\dfrac{f_y}{\varphi_y}=-\lambda$,则(1)改写为

$$\begin{cases} f_x+\lambda\varphi_x=0 \\ f_y+\lambda\varphi_y=0 \\ \varphi(x,y)=0 \end{cases}$$

引入辅助函数$L(x,y,\lambda)=f(x,y)+\lambda\varphi(x,y)$

则极值点的坐标满足方程组

$$\begin{cases} f_x+\lambda\varphi_x=0 \\ f_y+\lambda\varphi_y=0 \\ \varphi(x,y)=0 \end{cases}\text{也就是}\begin{cases} L_x=0 \\ L_y=0 \\ L_\lambda=0 \end{cases}$$

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利用拉格朗日乘子法求条件极值时,我们只得到了函数可能的极值点的坐标,至于这种点处函数是否取得极值,还需再作讨论,如把问题化为无条件极值问题,再用**定理17.11**

(充分条件) 利用二阶偏导数来进行判断.

不过在许多实际问题中,由客观意义知其必定存在最大值或最小值,又若只求得唯一的驻点,那么该驻点处函数就取得了最值.

推广:在条件$\varphi(x,y,z,t)=0,\psi(x,y,z,t)=0$下,

求函数$u=f(x,y,z,t)$ 的极值.

引入*Lagrange*乘子函数

$$L(x,y,z,t;\lambda,\mu)=f(x,y,z,t)+\lambda\varphi(x,y,z,t)+\mu\psi(x,y,z,t)$$

则函数的极值点必满足

$$\begin{cases} L_x=0 \\ L_y=0 \\ L_z=0 \\ L_t=0 \\ L_\lambda=0 \\ L_\mu=0 \end{cases} \quad \text{即} \quad \begin{cases} f_x+\lambda\varphi_x+\mu\psi_x=0 \\ f_y+\lambda\varphi_y+\mu\psi_y=0 \\ f_z+\lambda\varphi_z+\mu\psi_z=0 \\ f_t+\lambda\varphi_t+\mu\psi_t=0 \\ \varphi(x,y,z,t)=0 \\ \psi(x,y,z,t)=0 \end{cases}$$

例3.求$u = x - 2y + 2z$在条件

$x^2 + y^2 + z^2 = 1$下的最大,小值.

解 $\begin{cases} u = x - 2y + 2z \\ x^2 + y^2 + z^2 = 1 \end{cases}$,

$$L(x, y, z; \lambda) = x - 2y + 2z + \lambda\left(x^2 + y^2 + z^2 - 1\right)$$

求驻点$\begin{cases} \dfrac{\partial L}{\partial x} = 0, \dfrac{\partial L}{\partial y} = 0, \dfrac{\partial L}{\partial z} = 0 \\ \dfrac{\partial L}{\partial \lambda} = 0 \end{cases}$,

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得到驻点$\pm\left(\frac{1}{3},-\frac{2}{3},\frac{2}{3}\right)$, u在有界闭

集$x^2+y^2+z^2=1$上必有最大、小值,

$\therefore \max u=3, \min u=-3$

几何解释：等值平面$x-2y+2z=c$与

球面$x^2+y^2+z^2=1$相切时c相应取得

最大、小值.

例4.将数12分成三个正数x, y, z之和，使得$u = x^3 y^2 z$取得最大值.

解　我们可以将问题转化为求函数的无条件极值.

$$x + y + z = 12, x > 0, y > 0, z > 0.$$

$$\therefore u = x^3 y^2 z = x^3 y^2 (12 - x - y)$$

… …

例4.将数12分成三个正数x,y,z之和,

使得$u=x^3y^2z$取得最大值.

解二 设$L(x,y,z,\lambda)=x^3y^2z+\lambda\left(x+y+z-12\right)$

$$\begin{cases} L_x = 3x^2y^2z+\lambda=0 \\ L_y = 2x^3yz+\lambda=0 \\ L_z = x^3y^2+\lambda \quad =0 \\ x+y+z= \quad 12 \end{cases}$$

得唯一的驻点$(6,4,2)$.

由问题的意义知函数必存在最大值而不存在最小值,

$\therefore$函数的最大值为$u_{\max}=6^3\cdot4^2\cdot2=6912.$

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脑筋转个弯,我们把问题变换为

$$u=x^3y^2z=xxxyyz,\forall x,y,z>0$$

$$x+y+z=12\Leftrightarrow \frac{x}{3}+\frac{x}{3}+\frac{x}{3}+\frac{y}{2}+\frac{y}{2}+z=12$$

利用“几何平均—算术平均”不等式就有

$$u=xxxyyz=3^3\cdot 2^2\cdot\frac{x}{3}\cdot\frac{x}{3}\cdot\frac{x}{3}\cdot\frac{y}{2}\cdot\frac{y}{2}\cdot z$$

$$\le 3^3\cdot 2^2\cdot\left(\frac{\frac{x}{3}+\frac{x}{3}+\frac{x}{3}+\frac{y}{2}+\frac{y}{2}+z}{6}\right)^6$$

$$=3^3\cdot 2^2\cdot\left(\frac{12}{6}\right)^6=3^3\cdot 2^8$$ 岂不妙哉!

下面我们将介绍用条件极值的方法来证明:“几何平均—算术平均”不等式

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例4.(2).已知$a_1,a_2,\cdots,a_n$均非负,我们可以用求条件极值的方法证明:

$$\frac{a_1+a_2+\cdots+a_n}{n}\geq\sqrt[n]{a_1a_2\cdots a_n}$$

分析　我们可以设$a_1+a_2+\cdots+a_n=a$为常数,在此条件下可求得多元函数

$$u=\sqrt[n]{a_1a_2\cdots a_n}$$

的最大值$=\dfrac{a}{n}$的方法证明结论.

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例4.(2).已知$a_1,a_2,\cdots,a_n$均非负,我们可以用求条件极值的方法证明:$\dfrac{a_1+a_2+\cdots+a_n}{n}\geq\sqrt[n]{a_1a_2\cdots a_n}$.

分析　我们可以设$a_1+a_2+\cdots+a_n=a$为常数,在此条件下可用求得多元函数$u=\sqrt[n]{a_1a_2\cdots a_n}$的最大值$=\dfrac{a}{n}$的方法证明结论.

对偶的做法:设$\sqrt[n]{a_1a_2\cdots a_n}=a$为常数,在此条件下可用求得多元函数$u=\dfrac{a_1+a_2+\cdots+a_n}{n}$的最小值$=a$的方法证明结论.

解　如果某个$a_i=0$,则结论显然成立,

所以不妨假设$\forall a_i>0$. 把问题转化为

$$\begin{cases}\max u=\sqrt[n]{a_1a_2\cdots a_n}\\ a_1+a_2+\cdots+a_n=a\end{cases}\Leftrightarrow\begin{cases}\max\ln u=\dfrac{1}{n}\sum\limits_{i=1}^{n}\ln a_i\\ a_1+a_2+\cdots+a_n=a\end{cases}$$

$$L(a_1,a_2,\cdots,a_n;\lambda)=\frac{1}{n}\sum_{i=1}^{n}\ln a_i+\lambda(a_1+a_2+\cdots+a_n-a)$$

$$\begin{cases}\dfrac{\partial L}{\partial a_i}=0\\ \dfrac{\partial L}{\partial\lambda}=0\end{cases}\Rightarrow\begin{cases}\dfrac{1}{na_i}+\lambda=0,\forall i=1,2,\cdots,n\\ a_1+a_2+\cdots+a_n-a=0\end{cases}$$

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$\Rightarrow$ 得到驻点$a_1 = a_2 = \cdots = a_n = \dfrac{a}{n}$,

又根据实际问题的情况,该函数

有最大值没有最小值,

$\therefore$ 在$a_1 = a_2 = \cdots = a_n = \dfrac{a}{n}$时有$\max u = \dfrac{a}{n}$,

$$\therefore \sqrt[n]{a_1 a_2 \cdots a_n} \le \frac{a}{n} = \frac{a_1 + a_2 + \cdots + a_n}{n},$$

并且当$a_1 = a_2 = \cdots = a_n$时等号成立.

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例5.试计算椭球面的内接长方体体积的最大值.

与平面几何中相应问题做对比：

椭圆的内接矩形面积的最大值.

解　显然问题的结果与坐标系的选取无关.

故取直角坐标系中椭球面的标准方程：

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

再由几何对称性知：设椭球面内接长方体在第I卦限的顶点为(x,y,z),

$$\therefore V=8xyz.$$

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$$\begin{cases} \max V = 8xyz \\ \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1 \end{cases}, x>0, y>0, z>0,$$

直接用Lagrange乘子法计算,十分简单,

但计算略显繁琐:

$$L = 8xyz - \lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right),$$

由 $\dfrac{\partial L}{\partial x}=0, \dfrac{\partial L}{\partial y}=0, \dfrac{\partial L}{\partial z}=0, \dfrac{\partial L}{\partial \lambda}=0,$

求得唯一驻点,据问题的实际意义,知其

必有最大值,驻点 = 最值点.

按部就班地计算

按部就班地计算

按部就班地计算

繁！繁！繁！

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灵机一动 $\because \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1,$

$$V=8xyz=8abc\cdot\sqrt{\frac{x^2}{a^2}\cdot\frac{y^2}{b^2}\cdot\frac{z^2}{c^2}},$$

若记 $\dfrac{x^2}{a^2}=s,\dfrac{y^2}{b^2}=t,\dfrac{z^2}{c^2}=u.$

则相当于在 $s+t+u=1$ 的条件下求 $U=stu$ 的极值.

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灵活处理问题：

$$V=8xyz=8abc\cdot\sqrt{\frac{x^2}{a^2}\cdot\frac{y^2}{b^2}\cdot\frac{z^2}{c^2}},$$

我们先考虑计算：$U=stu$，为方便计，

计算：$\begin{cases}\max W=\ln s+\ln t+\ln u,\\ s+t+u=1, s>0, t>0, u>0,\end{cases}$

再用Lagrange乘子法，就简单得多：

$$L=\ln s+\ln t+\ln u-\lambda(s+t+u-1),$$

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计算$\begin{cases} \max W = \ln s + \ln t + \ln u, \\ s + t + u = 1, s > 0, t > 0, u > 0, \end{cases}$

$$L = \ln s + \ln t + \ln u - \lambda(s + t + u - 1),$$

由$\dfrac{\partial L}{\partial s} = 0, \dfrac{\partial L}{\partial t} = 0, \dfrac{\partial L}{\partial u} = 0, \dfrac{\partial L}{\partial \lambda} = 0,$

求得唯一驻点 $s = t = u = \dfrac{1}{3},$

据问题的实际意义,知其必有最大值,

$$\therefore \max U = \frac{1}{27}.$$

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$$s+t+u=1, s>0, t>0, u>0,$$

$$U=stu, \quad \max U=\frac{1}{27}.$$

$$\because \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

$$\therefore \text{当}\frac{x^2}{a^2}=\frac{y^2}{b^2}=\frac{z^2}{c^2}=\frac{1}{3}\text{时},$$

$$\max V=8xyz$$

$$=8abc\cdot\sqrt{\frac{x^2}{a^2}\cdot\frac{y^2}{b^2}\cdot\frac{z^2}{c^2}}=\frac{8abc}{3\sqrt{3}}.$$

众所周知,圆的内接长方形面积的最大值是在长方形为正方形时取得.

同理,球面的内接长方体体积的最大值是在长方体恰为正方体时取得,即在$s^2+t^2+u^2=1$上,内接长方体在第I卦限的顶点(s,t,u),当$s=t=u=\dfrac{1}{\sqrt{3}}$时长方体体积取得最大值.

那么,作仿射变换$\dfrac{x}{a}=s,\dfrac{y}{b}=t,\dfrac{z}{c}=u$,由此可知,椭球面$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$的内接长方体在第$I$卦限的顶点$(x,y,z)$当$\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}=\dfrac{1}{\sqrt{3}}$时体积取得最大值.

例6.我们知道平面外一点$P(x_0,y_0,z_0)$到平面$\Pi: Ax+By+Cz+D=0\left(A^2+B^2+C^2\neq 0\right)$的距离为

$$d=\frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}.$$

显然,点P到平面Π的距离就是点P与平面Π上的点之间距离的最小值.试用条件极值的方法推导该距离公式.

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解 考虑 $\min d=\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}$,

$$s.t.\ \ Ax+By+Cz+D=0.$$

设 $L(x,y,z;\lambda)=\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}$

$$+\lambda(Ax+By+Cz+D),$$

发现求导求驻点比较麻烦.

故考虑 $\min d^2=(x-x_0)^2+(y-y_0)^2+(z-z_0)^2$,

$$s.t.\ \ Ax+By+Cz+D=0.$$

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考虑 $\min d^2 = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2$,

$$s.t.\ \ Ax + By + Cz + D = 0.$$

设 $L(x,y,z;\lambda) = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2$
$$+\lambda(Ax + By + Cz + D),$$

求驻点以及下面的计算仍稍嫌麻烦,为避开

那个引起麻烦的 $\frac{1}{2}$,我们可以设:

$$L(x,y,z;\lambda) = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2$$
$$-2\lambda(Ax + By + Cz + D).$$

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设$L(x,y,z;\lambda)=(x-x_0)^2+(y-y_0)^2+(z-z_0)^2$

$$-2\lambda(Ax+By+Cz+D),$$

$$\begin{cases} L_x=0 \\ L_y=0 \\ L_z=0 \\ L_\lambda=0 \end{cases} \Rightarrow \begin{cases} 2(x-x_0)-2\lambda A=0 \\ 2(y-y_0)-2\lambda B=0 \\ 2(z-z_0)-2\lambda C=0 \\ Ax+By+Cz+D=0 \end{cases} \Rightarrow$$

$$x-x_0=\lambda A, y-y_0=\lambda B, z-z_0=\lambda C,$$

$$Ax+By+Cz+D=0 \Leftrightarrow$$

$$A(x-x_0)+B(y-y_0)+C(z-z_0)$$

$$+(Ax_0+By_0+Cz_0+D)=0$$

$$\Rightarrow (A^2+B^2+C^2)\lambda=-(Ax_0+By_0+Cz_0+D)$$

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$$x - x_0 = \lambda A, y - y_0 = \lambda B, z - z_0 = \lambda C,$$

$$\left(A^2 + B^2 + C^2\right)\lambda = -\left(Ax_0 + By_0 + Cz_0 + D\right)$$

得唯一驻点.根据问题的实际情况,该函数有最小值没有最大值.

$\therefore$ 在$x - x_0 = \lambda A, y - y_0 = \lambda B, z - z_0 = \lambda C,$

$\left(A^2 + B^2 + C^2\right)\lambda = -\left(Ax_0 + By_0 + Cz_0 + D\right)$时

有$\min d^2 = \dfrac{\left(Ax_0 + By_0 + Cz_0 + D\right)^2}{A^2 + B^2 + C^2}$,

$$\therefore \min d = \frac{\left|Ax_0 + By_0 + Cz_0 + D\right|}{\sqrt{A^2 + B^2 + C^2}}.$$

例7.在第一卦限内作椭球面 $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ 的切平面,使切平面与三个坐标平面所围成的四面体体积最小.求切点坐标.

解 设$P(x_0,y_0,z_0)$是椭球面上的点,

令$F(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$,

则$F_x\big|_P=\frac{2x_0}{a^2},F_y\big|_P=\frac{2y_0}{b^2},F_z\big|_P=\frac{2z_0}{c^2}$,

过$P(x_0,y_0,z_0)$的切平面方程为

$$\frac{x_0}{a^2}(x-x_0)+\frac{y_0}{b^2}(y-y_0)+\frac{z_0}{c^2}(z-z_0)=0$$

化简为 $\dfrac{x_0 x}{a^2}+\dfrac{y_0 y}{b^2}+\dfrac{z_0 z}{c^2}=1.$

该切平面在三坐标轴上的截距为

$$x=\frac{a^2}{x_0}, y=\frac{b^2}{y_0}, z=\frac{c^2}{z_0},$$

$\therefore$ 四面体体积为 $V=\dfrac{1}{6}xyz=\dfrac{a^2b^2c^2}{6x_0y_0z_0}.$

在条件$\dfrac{x_0^2}{a^2}+\dfrac{y_0^2}{b^2}+\dfrac{z_0^2}{c^2}=1$下求$V$的最小值.

令 $u=\ln x_0+\ln y_0+\ln z_0$,

$$L(x_0,y_0,z_0)=\ln x_0+\ln y_0+\ln z_0$$

$$+\lambda\left(\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}+\frac{z_0^2}{c^2}-1\right)$$

$$\begin{cases} L_{x_0}=0,\ L_{y_0}=0,\ L_{z_0}=0 \\ \dfrac{x_0^2}{a^2}+\dfrac{y_0^2}{b^2}+\dfrac{y_0^2}{c^2}-1=0 \end{cases},$$

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即 $\begin{cases} \dfrac{1}{x_0}+\dfrac{2\lambda x_0}{a^2}=0 \\ \dfrac{1}{y_0}+\dfrac{2\lambda y_0}{b^2}=0 \\ \dfrac{1}{z_0}+\dfrac{2\lambda z_0}{c^2}=0 \\ \dfrac{x_0^2}{a^2}+\dfrac{y_0^2}{b^2}+\dfrac{z_0^2}{c^2}-1=0 \end{cases}$, $\begin{cases} x_0=\dfrac{a}{\sqrt{3}} \\ y_0=\dfrac{b}{\sqrt{3}} \\ z_0=\dfrac{c}{\sqrt{3}} \end{cases}$

$\therefore$ 当切点坐标为 $\left(\dfrac{a}{\sqrt{3}},\dfrac{b}{\sqrt{3}},\dfrac{c}{\sqrt{3}}\right)$ 时,

四面体的最小体积为 $V_{\min}=\dfrac{\sqrt{3}}{2}abc.$

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又及 求四面体体积$V=\dfrac{a^2b^2c^2}{6x_0y_0z_0}$在条件$\dfrac{x_0^2}{a^2}+\dfrac{y_0^2}{b^2}+\dfrac{z_0^2}{c^2}=1$ 下的最小值.

$$V=\frac{a^2b^2c^2}{6x_0y_0z_0}=\frac{abc}{6\left(\dfrac{x_0}{a}\right)\left(\dfrac{y_0}{b}\right)\left(\dfrac{z_0}{c}\right)}$$

$$\because\left(\frac{x_0}{a}\right)^2+\left(\frac{y_0}{b}\right)^2+\left(\frac{z_0}{c}\right)^2=1$$

$$\text{又}\sqrt[3]{\left(\frac{x_0}{a}\right)^2\left(\frac{y_0}{b}\right)^2\left(\frac{z_0}{c}\right)^2}\le\frac{\left(\dfrac{x_0}{a}\right)^2+\left(\dfrac{y_0}{b}\right)^2+\left(\dfrac{z_0}{c}\right)^2}{3}$$

…　…

思考练习

1.设实数满足$x+y+z=1$,求证:$x^2+y^2+z^2\geq\frac{1}{3}$.

2. 求证二次型$f\left(x,y,z\right)=Ax^2+By^2+Cz^2+2Dyz+2Ezx+2Fxy$ 在单位球面$x^2+y^2+z^2=1$上的最大,小值恰好是二次型的矩阵的最大,小特征值.

1.设实数满足$x+y+z=1$,求证：$x^2+y^2+z^2\geq\frac{1}{3}$.

证明 Lagrange乘子法,$L=x^2+y^2+z^2-\lambda(x+y+z-1)$

有唯一驻点,且显然函数 $H=x^2+y^2+z^2$ 有最小值无最大值,

故驻点处函数H取得最小值,得证.

法二 说明函数$G=x^2+y^2+(1-x-y)^2$取得最小值$(\geq)\frac{1}{3}$.

法三 利用“几何平均–算术平均不等式”,

$x+y+z=1,\Rightarrow x^2+y^2+z^2+2xy+2yz+2zx=1,$

而 $2xy+2yz+2zx\leq x^2+y^2+y^2+z^2+z^2+x^2=2(x^2+y^2+z^2)$,

$\therefore 3(x^2+y^2+z^2)\geq 1.$

法四 利用几何意义,平面$x+y+z=1$ 与三坐标平面围成一个

四面体$O-ABC$,O点到平面$ABC: x+y+z=1$ 的距离为$\frac{1}{\sqrt{3}}\cdots$