南京農業大學

计算机视觉与图形图像处理 HW1

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题目 1

题目描述：

The continuous convolution of two functions $f(x)$ and $g(x)$ is given by

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(y)\, g(x-y)\, \mathrm{d}y. \tag{1}$$

The Gaussian function at scale s is defined as

$$G_s(x) = \frac{1}{\sqrt{2\pi s}} \exp\left(-\frac{x^2}{2s}\right), \tag{2}$$

and has the property that

$$\int_{-\infty}^{+\infty} G_s(x)\, \mathrm{d}x = 1. \tag{3}$$

Prove that this class of functions satisfies the *semigroup property*: the convolution of one Gaussian with another produces a third Gaussian with scale equal to their sum, or

$$(G_{s_1} * G_{s_2})(x) = G_{s_1+s_2}(x). \tag{4}$$

题解：

首先，代入高斯函数的定义：

$$G_{s_1}(y) = \frac{1}{\sqrt{2\pi s_1}} \exp\left(-\frac{y^2}{2s_1}\right),$$
$$G_{s_2}(x-y) = \frac{1}{\sqrt{2\pi s_2}} \exp\left(-\frac{(x-y)^2}{2s_2}\right).$$

现在，计算卷积 $(G_{s_1} * G_{s_2})(x)$：

$$(G_{s_1} * G_{s_2})(x) = \int_{-\infty}^{+\infty} G_{s_1}(y) G_{s_2}(x-y) \mathrm{d}y.$$

代入高斯函数的表达式：

$$(G_{s_1} * G_{s_2})(x) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi s_1}} \exp(-\frac{y^2}{2s_1}) \cdot \frac{1}{\sqrt{2\pi s_2}} \exp(-\frac{(x-y)^2}{2s_2}) dy$$

整理可得：

$$\frac{1}{\sqrt{2\pi s_1}\sqrt{2\pi s_2}} \int_{-\infty}^{+\infty} \exp(-(\frac{s_2+s_1}{2s_1 s_2})y^2 + \frac{x}{s_2}y - \frac{x^2}{2s_2}) dy.$$

设 $A = \frac{s_1+s_2}{2s_1 s_2}$, $B = \frac{x}{s_2}$，因此，积分可以表示为：

$$\frac{1}{\sqrt{2\pi s_1}\sqrt{2\pi s_2}} \int_{-\infty}^{+\infty} \exp\left(-Ay^2 + By - \frac{x^2}{2s_2}\right) dy.$$

我们可以将常数项从积分中提取出来：

$$= \frac{1}{\sqrt{2\pi s_1}\sqrt{2\pi s_2}} \exp\left(-\frac{x^2}{2s_2}\right) \int_{-\infty}^{+\infty} \exp\left(-Ay^2 + By\right)\, dy.$$

对积分项进行变换可得：

$$= \frac{1}{\sqrt{2\pi s_1}\sqrt{2\pi s_2}} \exp\left(-\frac{x^2}{2s_2}\right) \int_{-\infty}^{+\infty} \exp\left(A\left(y - \frac{B}{2A}\right)^2 - \frac{B^2}{4A}\right)\, dy.$$

由高斯函数的性质和高斯函数积分的标准结果可知：

$$\int_{-\infty}^{+\infty} \exp\left(-a(y-b)^2\right)\, dy = \sqrt{\frac{\pi}{a}}.$$

所以我们的积分结果为：

$$= \frac{1}{\sqrt{2\pi s_1}\sqrt{2\pi s_2}} \cdot \exp\left(-\frac{x^2}{2s_2}\right) \cdot \exp\left(\frac{B^2}{4A}\right) \sqrt{\frac{\pi}{A}}.$$

将 A 和 B 代入可得：

$$= \frac{1}{\sqrt{2\pi(s_1+s_2)}} \exp(-\frac{x^2}{2(s_1+s_2)}) = G_{s_1+s_2}(x).$$

所以：

$$\left(G_{s_1} * G_{s_2}\right)(x) = G_{s_1+s_2}(x).$$

这表明两个高斯函数的卷积仍然是一个高斯函数，其尺度为 $s_1 + s_2$，从而证明了高斯函数满足半群性质。

题目 2

题目描述：

In class we derived a finite-difference approximation to the derivative of the univariate function $f(x)$ by considering the Taylor polynomial approximations of $f(x+h)$ and $f(x-h)$. We showed that

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2),$$

so that the derivative can be approximated by convolving a discrete version of $f(x)$—a vector of values (..., $f(x_0 - \Delta)$, $f(x_0)$, $f(x_0 + \Delta)$,...)—with kernel ($\frac{1}{2}$, 0, -$\frac{1}{2}$). This is termed a because *central difference* because its interval is symmetric about a sample point.

1. Derive a higher order central-difference approximation to $f'(x)$ such that the truncation error tends to zero as h^4 instead of h^2. Hint: consider Taylor polynomial approximations of $f(x \pm 2h)$ in addition to $f(x \pm h)$.

2. What is the corresponding convolution (not correlation!) kernel?

题解：

要推导一个更高阶的中心差分近似以计算 $f'(x)$，我们可以考虑 $f(x \pm h)$ 和 $f(x \pm 2h)$ 的泰勒展开。对于 $f(x+h)$ 和 $f(x-h)$ 我们有：

$$
\begin{aligned}
f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f^{(4)}(x) + O(h^5), \\
f(x-h) &= f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f^{(4)}(x) + O(h^5).
\end{aligned}
$$

将这两个展开结合，得到：

$$
f(x+h) - f(x-h) = 2hf'(x) + \frac{h^3}{3}f'''(x) + O(h^5).
$$

对于 $f(x+2h)$ 和 $f(x-2h)$ 我们有：

$$
\begin{aligned}
f(x+2h) &= f(x) + 2hf'(x) + 2h^2f''(x) + \frac{8h^3}{6}f'''(x) + \frac{16h^4}{24}f^{(4)}(x) + O(h^5), \\
f(x-2h) &= f(x) - 2hf'(x) + 2h^2f''(x) - \frac{8h^3}{6}f'''(x) + \frac{16h^4}{24}f^{(4)}(x) + O(h^5).
\end{aligned}
$$

将这两个展开结合，得到：

$$
f(x+2h) - f(x-2h) = 4hf'(x) + \frac{8h^3}{3}f'''(x) + O(h^5).
$$

设要找的公式形式为：

$$
f'(x) = \frac{a[f(x+h) - f(x-h)] + b[f(x+2h) - f(x-2h)]}{2h} + O(h^4)
$$

将泰勒展开式代入，整理得：

$$
f'(x)(a+2b) + \frac{h^2}{6}f'''(x)(a+8b) + O(h^4) = f'(x)
$$

要使误差为 $O(h^4)$, 需要：

$$
\begin{aligned}
a + 2b &= 1(f'(x)\text{项系数}) \\
a + 8b &= 0(f'''(x)\text{项系数})
\end{aligned}
$$

解这个方程组得：$a = \frac{4}{3}, b = -\frac{1}{6}$.
所以我们可以得到：

1. 四阶精度的中心差分公式为：

$$
\begin{aligned}
f'(x) &= \frac{4[f(x+h) - f(x-h)] - [f(x+2h) - f(x-2h)]}{6h} + O(h^4) \\
&= \frac{f(x-2h) - 4f(x-h) + 0f(x) + 4f(x+h) - f(x+2h)}{6h} + O(h^4)
\end{aligned}
$$

2. 相应的卷积核为：

$$
(\frac{1}{6}, -\frac{2}{3}, 0, \frac{2}{3}, -\frac{1}{6})
$$