Modeling and Control of Mechatronics Systems
Spring 2018-2019
HW I Solution

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Contents

1 Problem # 1

- **Section -a**

Using Lagrange's formula

$$\mathcal{T} = \frac{1}{2}m\dot{z}^2$$

$$\mathcal{V} = \frac{1}{2}k_1z^2 + \frac{1}{2}k_2(z-x)^2$$

$$\mathcal{D} = \frac{1}{2}b_1\dot{z}^2 + \frac{1}{2}b_2(\dot{y}-\dot{x})^2$$

$$\mathcal{L} = \mathcal{T} - \mathcal{V} = \frac{1}{2}[m\dot{z}^2 - k_1z^2 - k_2(z-x)^2]$$

where T,V and D stands for the kinetic, potential and dissipated energy of the system, respectively. There are 3 generalized coordinates of the system which are $q_1 = z$, $q_2 = x$ and $q_3 = y$.

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} + \frac{\partial \mathcal{D}}{\partial \dot{q}} = Q_{ext}$$

$$\frac{\partial \mathcal{L}}{\partial z} = -k_1z - k_2(z-x)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{z}} = m\dot{z} \quad and \quad \frac{\partial \mathcal{D}}{\partial \dot{z}} = b_1\dot{z}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{z}}\right) - \frac{\partial \mathcal{L}}{\partial z} + \frac{\partial \mathcal{D}}{\partial \dot{z}} = m\ddot{z} + (k_1+k_2)z - k_2x + b_1\dot{z} = 0$$

For the 2^{nd} coorrdinate x;

$$\frac{\partial \mathcal{L}}{\partial x} = k_2(z-x)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = 0 \quad and \quad \frac{\partial \mathcal{D}}{\partial \dot{x}} = -b_2(\dot{y}-\dot{x})$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} + \frac{\partial \mathcal{D}}{\partial \dot{x}} = k_2(x-z) - b_2(\dot{y}-\dot{x}) = 0$$

For the 3^{rd} coorrdinate y;

$$\frac{\partial \mathcal{L}}{\partial y} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \dot{y}} = 0 \quad and \quad \frac{\partial \mathcal{D}}{\partial \dot{y}} = b_2(\dot{y} - \dot{x})$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) - \frac{\partial \mathcal{L}}{\partial y} + \frac{\partial \mathcal{D}}{\partial \dot{y}} = b_2(\dot{y} - \dot{x}) = F_{y_{ext}}$$

Using these equations, transfer function can be derived as follows taking the Laplace Transforms(under the assumption of zero intiial conditions);

$$Z(s)[ms^2 + b_1 s + k_1 + k_2] = X(s)k_2$$

$$X(s)[sb_2 + k_2] = Z(s)k_2 + b_2 sY(s)$$

$$b_2 sY(s) = b_2 sX(s) + \boldsymbol{L}(F_{y_{ext}}) \implies b_2 sY(s) = b_2 sX(s) \quad (Assuming \quad Zero \quad force \quad exerted \quad to \quad the \quad system)$$

$$\frac{Z(s)}{Y(s)} = \frac{Z(s)}{X(s)}\frac{X(s)}{Y(s)} = \frac{k_2}{ms^2 + b_1 s + k_1 + k_2} * \frac{1}{1} = \frac{k_2}{ms^2 + b_1 s + k_1 + k_2}$$

for m=2, b1=b2=16, k1=k2=8

$$\frac{Z(s)}{Y(s)} = \frac{8}{2s^2 + 16s + 16} = \frac{4}{s^2 + 8s + 8}$$

I definitely have a mistake here as this system must be in the order of 3 as there is a damper at the free end. So I will continue with the right answer for the poles and zeros and step response of the system.

$$\frac{Z(s)}{Y(s)} = \frac{4s}{s^3 + 8.5s^2 + 12s + 2}$$

- **Section -b**

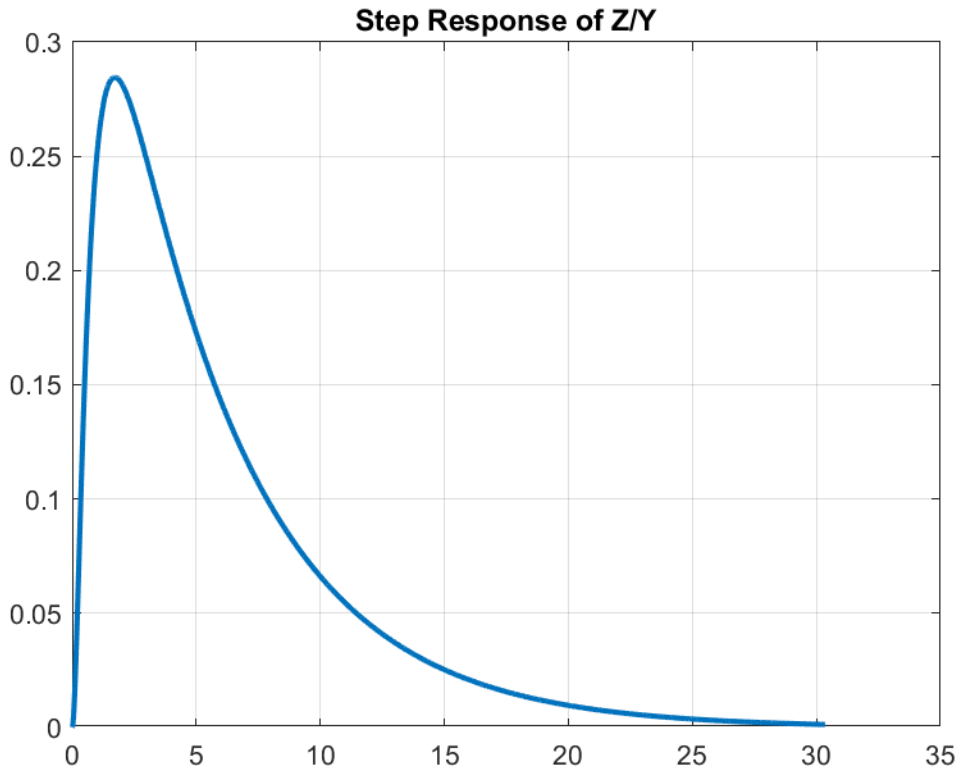


Figure 1: Step Response

- **Section -c**
  Pole-Zero table for the system is given below

Pole Zero Table For Problem 1		
	Poles	Zeros
$\frac{Z}{Y}$	-6.7715 -1.5363 -0.1923	0

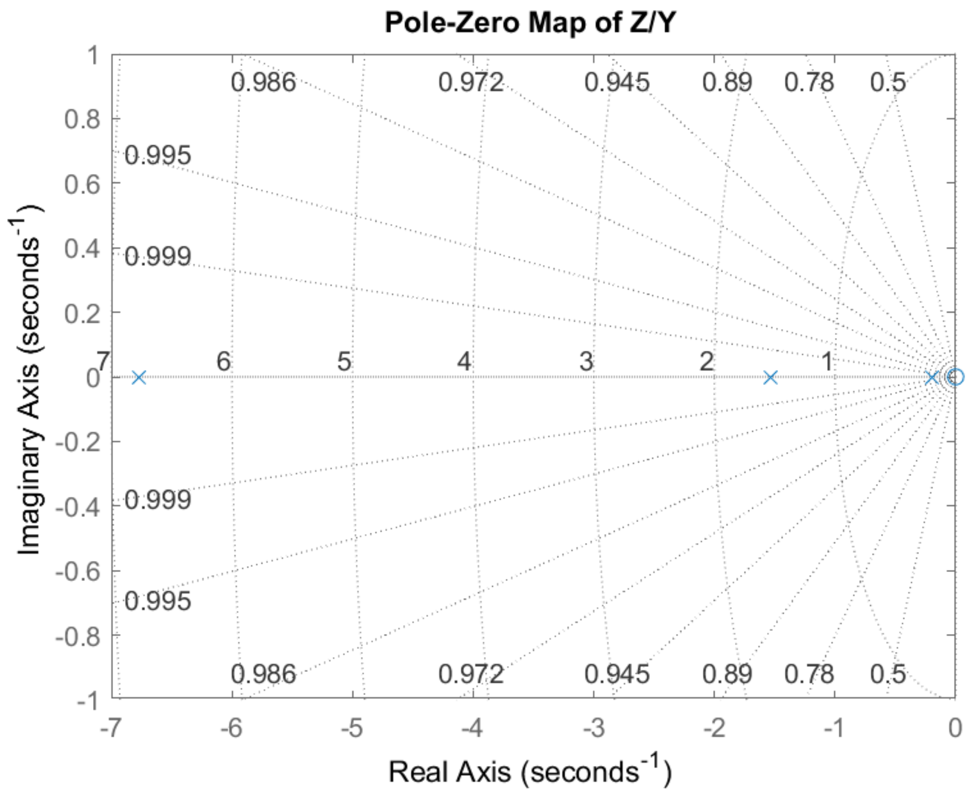


Figure 2: Pole-Zero Map of the system

All of the poles lie in the left hand side but there is a zero at the origin. system is stable but non-minimum phase.

2 Problem # 2

- **Section -a** Using Lagrange's formula

$$\mathcal{T} = \frac{1}{2}m_2\dot{y}^2 + \frac{1}{2}m_1\dot{y}^2$$

$$\mathcal{V} = \frac{1}{2}k_1(y_1 - y_0)^2 + \frac{1}{2}k_2(y_2 - y_1)^2$$

$$\mathcal{D} = \frac{1}{2}b(\dot{y}_2 - \dot{y}_1)^2$$

$$\mathcal{L} = \mathcal{T} - \mathcal{V} = \frac{1}{2}[m_2\dot{y_2}^2 + m_1\dot{y_1}^2 - k_1(y_1 - y_0)^2 - k_2(y_2 - y_1)^2]$$

where T,V and D stands for the kinetic, potential and dissipated energy of the system, respectively.

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} + \frac{\partial \mathcal{D}}{\partial \dot{q}} = Q_{ext}$$

Selecting q_1 as y_1 and q_2 as y_2 gives the following formulations,

$$\frac{\partial \mathcal{L}}{\partial q_1} = -k_1(y_1 - y_0) + k_2(y_2 - y_1)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q_1}} = m_1\dot{y_1} \quad and \quad \frac{\partial \mathcal{D}}{\partial \dot{q_1}} = -b(\dot{y_2} - \dot{y_1})$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q_1}}\right) - \frac{\partial \mathcal{L}}{\partial q_1} + \frac{\partial \mathcal{D}}{\partial \dot{q_1}} = m_1\ddot{y_1} + (k_1 + k_2)y_1 - k_2y_2 - k_1y_0 - b(\dot{y_2} - \dot{y_1}) = 0$$

and for the second generalized coordinate q_2

$$\frac{\partial \mathcal{L}}{\partial q_2} = -k_2(y_2 - y_1)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q_2}} = m_2\dot{y_2} \quad and \quad \frac{\partial \mathcal{D}}{\partial \dot{q_2}} = b(\dot{y_2} - \dot{y_1})$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q_2}}\right) - \frac{\partial \mathcal{L}}{\partial q_2} + \frac{\partial \mathcal{D}}{\partial \dot{q_2}} = m_2\ddot{y_2} + k_2(y_2 - y_1) + b(\dot{y_2} - \dot{y_1}) = 0$$

Selecting the states such that $x_1 = y_1 \quad x_2 = \dot{y_1} \quad x_3 = y_2 \quad x_4 = \dot{y_2}$ yields

$$\begin{aligned} &m_1\dot{x_2} + (k_1 + k_2)x_1 - k_2x_3 - k_1u_0 - b(x_4 - x_2) = 0 \\ &m_2\dot{x_4} + k_2(x_3 - x_1) + b(x_4 - x_2) = 0 \end{aligned}$$

In state space form

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & -\frac{b}{m_1} & \frac{k_2}{m_1} & \frac{b}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & \frac{b}{m_2} & -\frac{k_2}{m_2} & -\frac{b}{m_2} \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{k_1}{m_1} \\ 0 \\ 0 \end{bmatrix} u(t)$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u(t)$$

- **Section -b**

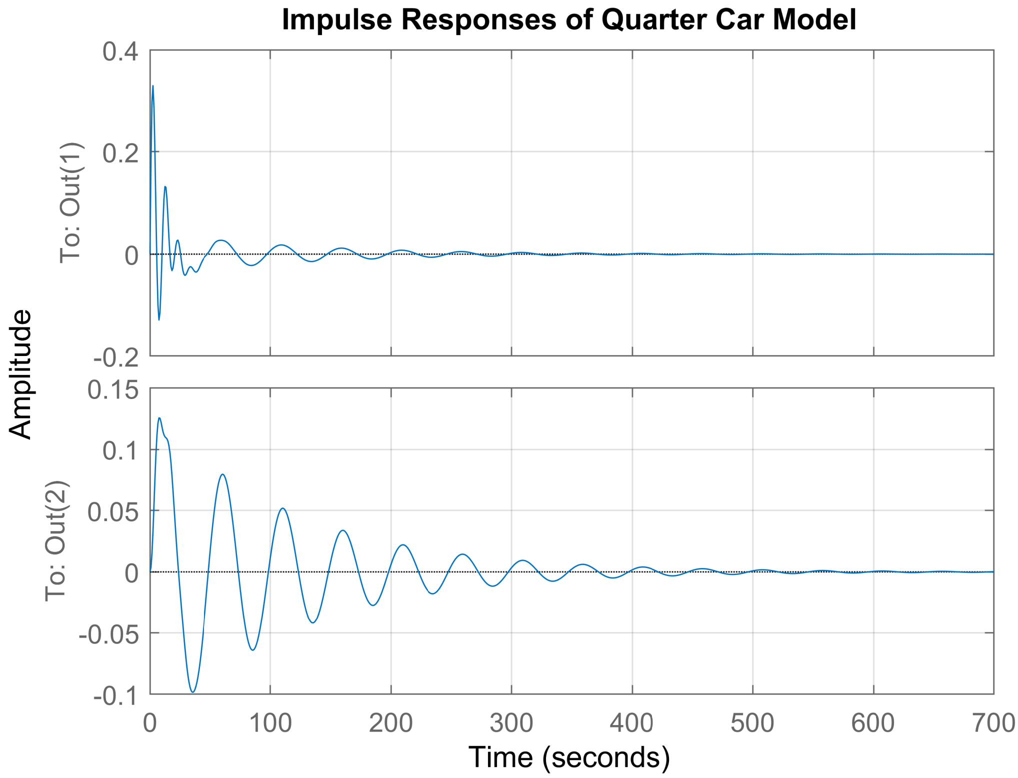


Figure 3: Impulse Responses (top-y_1 bottom-y_2)

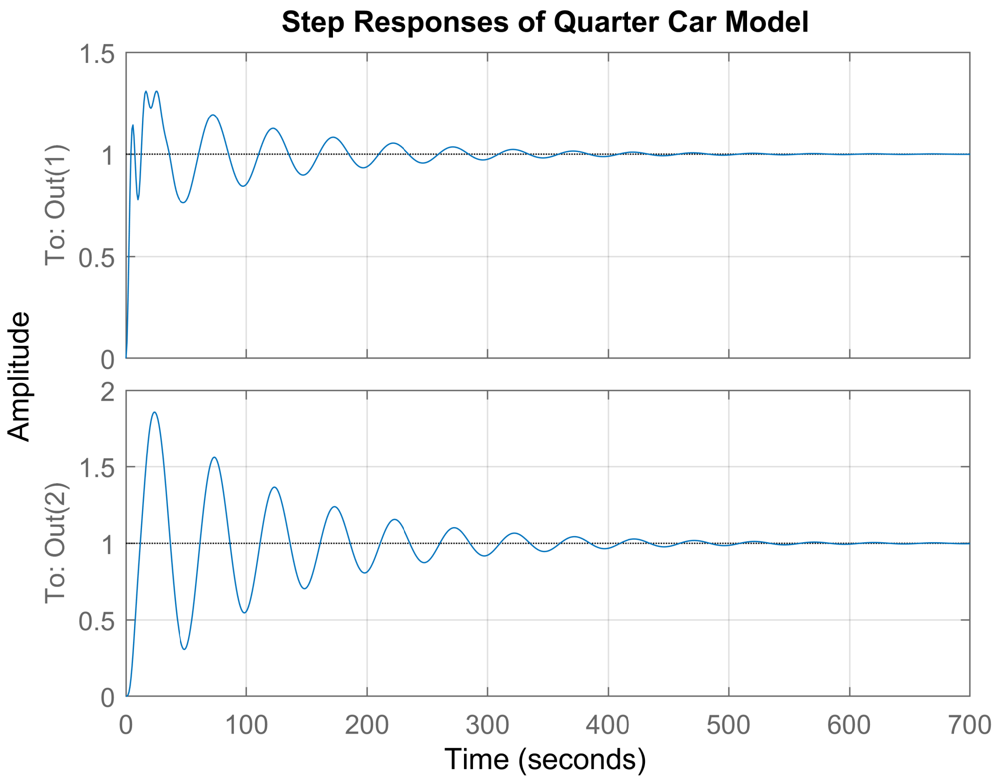


Figure 4: Step Responses (top-y_1 bottom-y_2)

- **Section -c** By taking the Laplace Transform of the system equations with zero initial conditions

$$m_1 s^2 Y_1(s) + (k_1 + k_2)Y_1(s) - k_2 Y_2(s) - k_1 U(s) - bs(Y_2(s) - Y_1(s)) = 0$$

$$m_2 s^2 Y_2(s) + k_2(Y_2(s) - Y_1(s)) + bs(Y_2(s) - Y_1(s)) = 0$$

Rewriting these equation

$$Y_2(s)[m_2 s^2 + bs + k_2] = Y_1(s)[bs + k_2]$$

$$Y_1(s)[m_1 s^2 + bs + k_1 + k_2] = Y_2(s)[bs + k_2] + k_1 U(s)$$

$$\frac{Y_2(s)}{Y_1(s)} = \frac{bs + k_2}{m_2 s^2 + bs + k_2}$$

$$\frac{Y_1(s)}{U(s)} = \frac{k_1(m_2 s^2 + bs + k_2)}{m_1 m_2 s^4 + b(m_1 + m_2)s^3 + (k_1 m_2 + k_2 m_2 + k_2 m_1)s^2 + bk_1 s + k_1 k_2}$$

$$\frac{Y_2(s)}{U(s)} = \frac{k_1(k_2 + bs)}{m_1 m_2 s^4 + b(m_1 + m_2)s^3 + (k_1 m_2 + k_2 m_1 + k_2 m_2)s^2 + bk_1 s + k_1 k_2}$$

And finally, substituting the values of the parameters into these equations transfer functions of the system can be obtained as follows

$$\frac{Y_1(s)}{U(s)} = \frac{12500s^2 + 500s + 300}{50000s^4 + 12000s^3 + 19700s^2 + 500s + 300}$$

$$\frac{Y_2(s)}{U(s)} = \frac{500s + 300}{50000s^4 + 12000s^3 + 19700s^2 + 500s + 300}$$

With these transfer functions it is possible to seperately recover the figures that obtained above through state space representation with Matlab

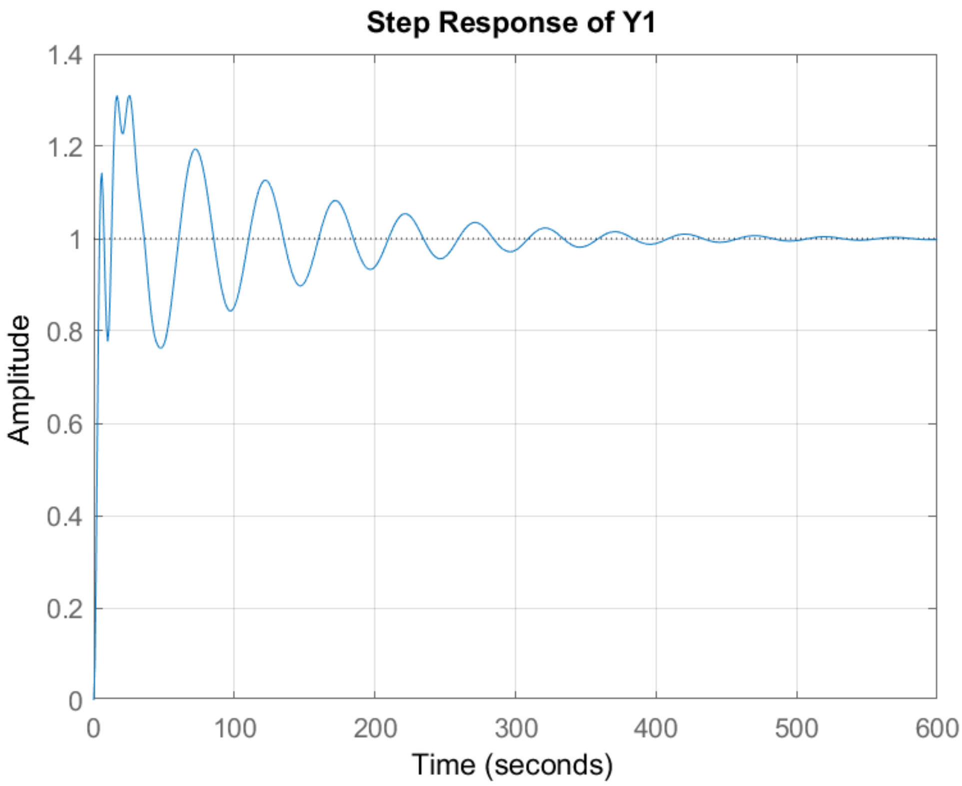


Figure 5: Step Response of the Output Y1

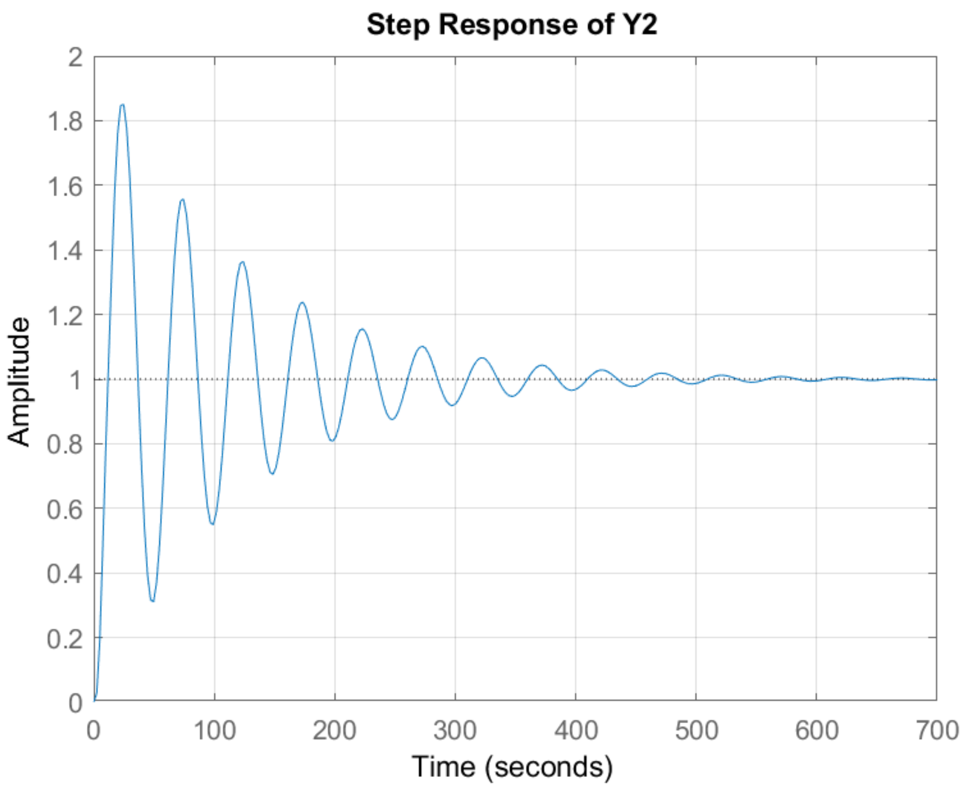
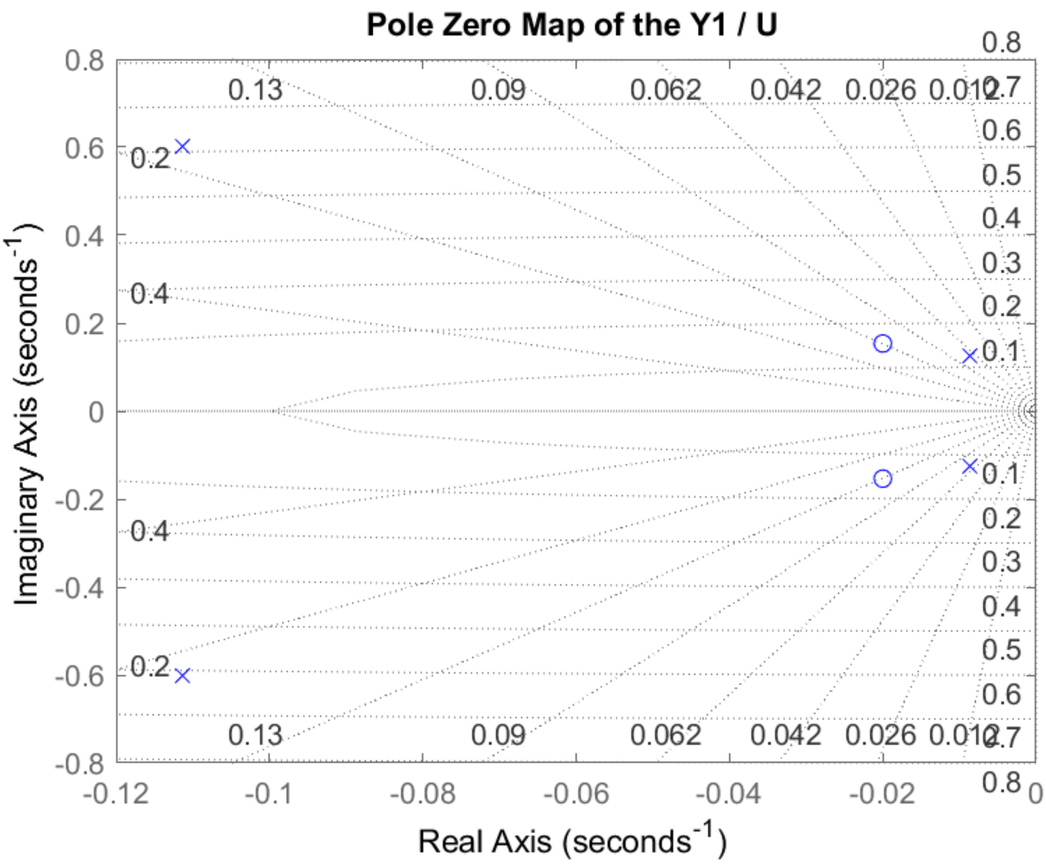
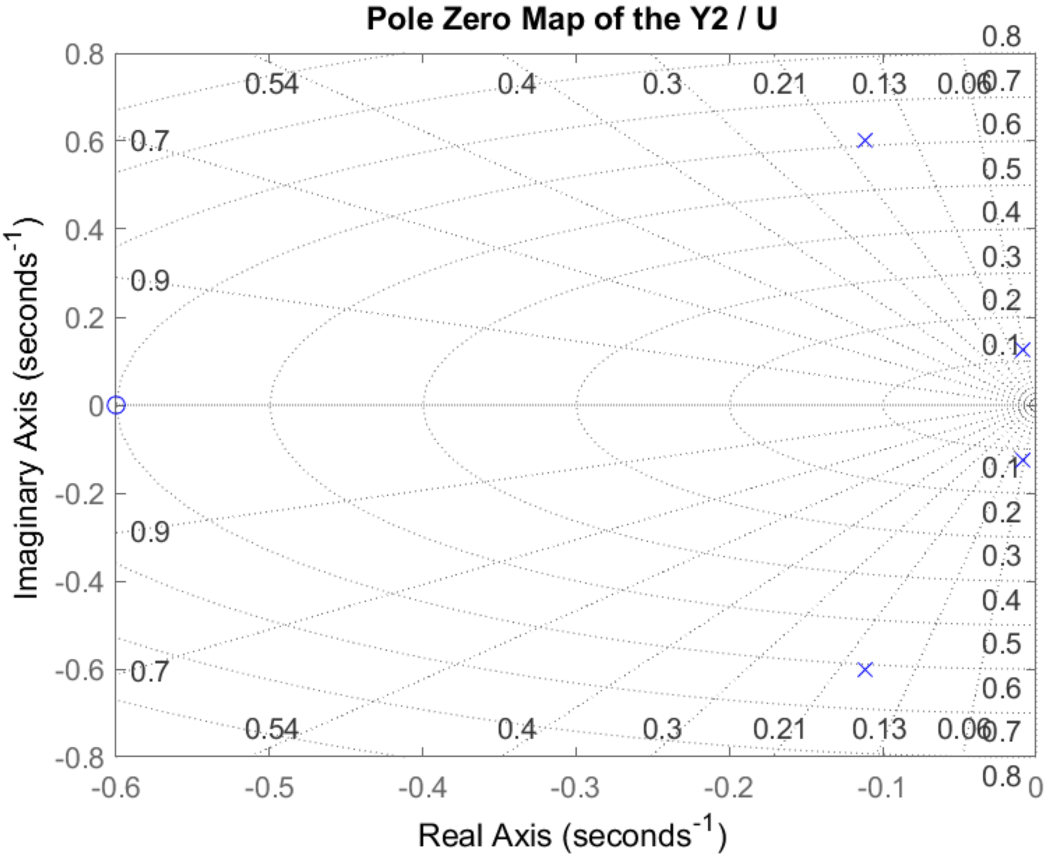


Figure 6: Step Response of the Output Y2

Pole-Zero table for the system is given below

Pole Zero Table For Quarter Car Model		
	Poles	Zeros
$\frac{Y_1}{U}$	-0.1114 + 0.6014i	-0.0200 + 0.1536i
	-0.1114 - 0.6014i	-0.0200 - 0.1536i
	-0.0086 + 0.1263i	
	-0.0086 - 0.1263i	
$\frac{Y_2}{U}$	-0.1114 + 0.6014i	-0.6
	-0.1114 - 0.6014i	
	-0.0086 + 0.1263i	
	-0.0086 - 0.1263i	





→ As all of the poles and zeros are in the Left Side of the Plane, system is stable and minimum phase.

3 Problem # 3

- **Section -a**

$$J = ml_3^2 \qquad \mathcal{T} = \frac{ml_3^2\dot{\theta}^2}{2} \qquad \mathcal{V} = \frac{1}{2}(x - l_1 sin(\theta))^2 + mgl_3 cos(\theta)$$

$$Assuming \quad small \quad displacements \quad sin(\theta) \implies \theta \quad and \quad cos(\theta) \implies 1$$

$$\mathcal{D} = \frac{1}{2}c(l_2 sin(\dot{\theta}))^2 \implies \mathcal{D} = \frac{1}{2}c(l_2\dot{\theta})^2 \qquad \mathcal{L} = \mathcal{T} - \mathcal{V}$$

$$\mathcal{L} = \frac{ml_3^2\dot{\theta}^2}{2} - \frac{1}{2}k(x - l_1\theta)^2 - mgl_3$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} + \frac{\partial \mathcal{D}}{\partial \dot{q}} = Q_{ext}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = k(x - \theta l_1) \qquad \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = ml_3^2\dot{\theta} \quad and \quad \frac{\partial \mathcal{D}}{\partial \dot{\theta}} = cl_2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} + \frac{\partial \mathcal{D}}{\partial \dot{\theta}} = 0$$

$$ml_3^2\ddot{\theta} - kl_1(x - \theta l_1) + cl_2^2\dot{\theta} = 0$$

$$ml_3^2\ddot{\theta} + kl_1^2\theta + cl_2^2\dot{\theta} = kl_1 x$$

State space form is obtained as follows

$$x_1 = \theta \quad x_2 = \dot{\theta} \implies \dot{x_1} = x_2 \quad \dot{x_2} = \ddot{\theta}$$

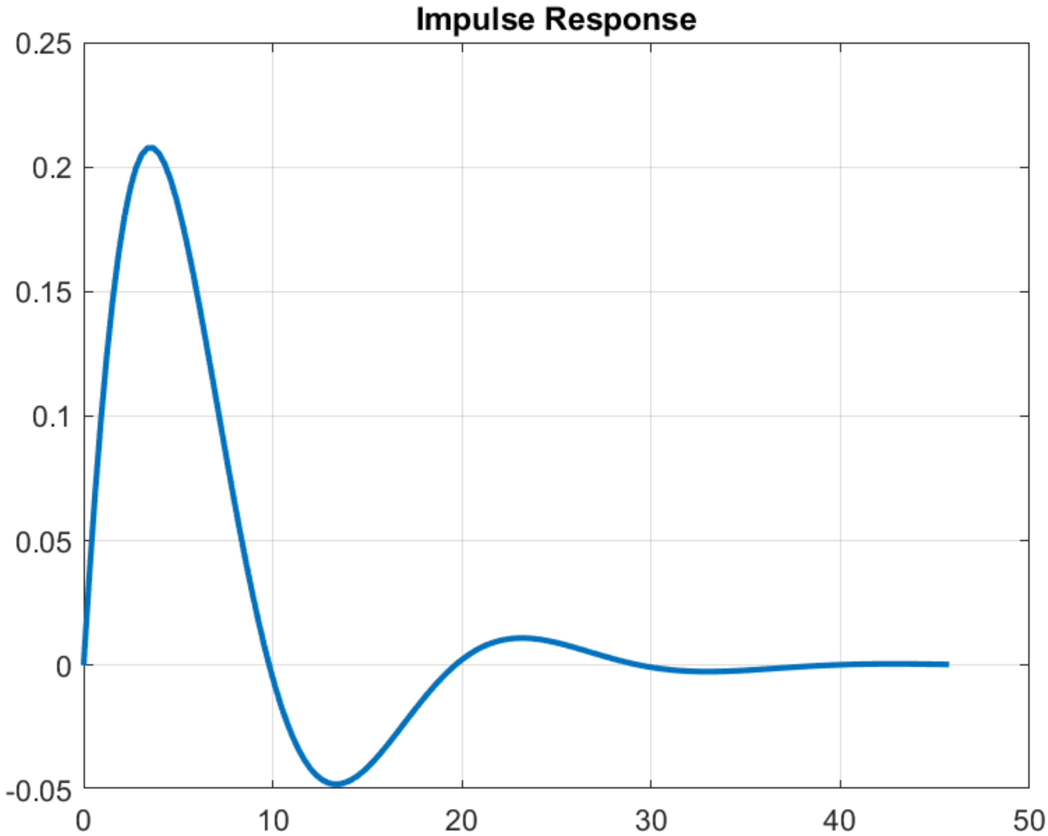
$$\ddot{\theta} = \dot{x_2} = -\frac{kl_1^2}{ml_3^2}x_1 - \frac{cl_2^2}{ml_3^2}x_2 + \frac{kl_1}{ml_3^2}u$$

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{kl_1^2}{ml_3^2} & -\frac{cl_2^2}{ml_3^2} \end{pmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{kl_1}{ml_3^2} \end{bmatrix} u$$

and if we are interested with only θ we can construct the output as follows

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]u$$

- **Section -b**



- **Section -c**

$$\theta(s)[s^2ml_3^2 + scl_2^2 + k1_1^2] = kl_1U(s)$$

$$\frac{\theta(s)}{U(s)} = \frac{kl_1}{s^2ml_3^2 + scl_2^2 + kl_1^2}$$

$$\frac{\dot{\theta}(s)}{U(s)} = \frac{kl_1s}{s^2ml_3^2 + scl_2^2 + kl_1^2}$$

Substituting the values into the transfer function gives

$$\frac{\theta(s)}{U(s)} = \frac{5}{40s^2 + 12s + 5}$$

$$Poles \quad are \quad p_1 = -0.1500 + 0.3202i, \quad p_2 = -0.1500 - 0.3202i$$
$$and \quad there \quad are \quad no \quad zeros.$$

All of the Poles are athe left hand side of the complex plane, so the system is stable and minimum phase.

4 Problem # 4

- **Section -a** Lagrange equations are derived as follows.
  For the first link of the elbow denoted by "1"

$$E_{k_1} = \frac{1}{2} m_1 l_{c_1}^2 \dot{q_1}^2 + \frac{1}{2} I_1 \dot{q_1}^2$$

$$E_{p_1} = m_1 g l_{c_1} sin(q_1)$$

Denoting the center coordinates of the second link with G_{x_2} and G_{y_2}

$$G_{x_2} = l_1 cos(q_1) + l_{c_2} cos(q_1 + q_2)$$

$$G_{y_2} = l_1 sin(q_1) + l_{c_2} sin(q_1 + q_2)$$

$$\dot{G_{x_2}} = -\dot{q_1} l_1 sin(q_1) - (\dot{q_1} + \dot{q_2}) l_{c_2} sin(q_1 + q_2) = A\dot{q_1} + B\dot{q_2}$$

$$\dot{G_{y_2}} = \dot{q_1} l_1 cos(q_1) + (\dot{q_1} + \dot{q_2}) l_{c_2} cos(q_1 + q_2) = C\dot{q_1} + D\dot{q_2}$$

$$A = -l_1 sin(q_1) - l_{c_2} sin(q_1 + q_2) \qquad B = -l_{c_2} sin(q_1 + q_2)$$

$$C = l_1 cos(q_1) + l_{c_2} cos(q_1 + q_2) \qquad D = l_{c_2} cos(q_1 + q_2)$$

$$\dot{G_{x_2}}^2 + \dot{G_{y_2}}^2 = (A^2 + C^2)\dot{q_1}^2 + (B^2 + D^2)\dot{q_2}^2 + 2(AB + CD)\dot{q_1}\dot{q_2}$$

$$A^2 + C^2 = l_1^2 + l_{c_2}^2 + 2 l_1 l_{c_2} cos(q_2) \qquad B^2 + D^2 = l_{c_2}^2$$

$$AB = l_1 l_{c_2} sin(q_1) sin(q_1 + q_2) + l_{c_2}^2 sin^2(q_1 + q_2)$$

$$CD = l_1 l_{c_2} cos(q_1) cos(q_1 + q_2) + l_{c_2}^2 cos^2(q_1 + q_2)$$

$$AB + CD = l_1 l_{c_2} cos(q2) + l_{c_2}^2 \qquad (Using \quad Trigonometric \quad identities)$$

$$\Longrightarrow \dot{G_{x_2}}^2 + \dot{G_{y_2}}^2 = \mathcal{V}_2^2 = \dot{q_1}^2(l_1^2 + l_{c_2}^2 + 2 l_1 l_{c_2} cos(q_2)) + \dot{q_2}^2 l_{c_2}^2 + 2\dot{q_1}\dot{q_2}(l_1 l_{c_2} cos(q2) + l_{c_2}^2)$$

And finally E_{k_2} and E_{p_2} can be derived as follows

$$E_{k_2} = \frac{1}{2} m_2 \mathcal{V}_2^2 + \frac{1}{2} I_2 (\dot{q_1} + \dot{q_2})^2$$

$$E_{p_2} = m_2 g [l_1 sin(q1) + l_{c_2} sin(q_1 + q_2)]$$

Total kinetic energy of the system is $E_{k_1} + E_{k_2}$ and total potential energy of the system is $E_{p_1} + E_{p_2}$ and $\mathcal{L} = E_{k_1} + E_{k_2} - (E_{p_1} + E_{p_2})$

$$\mathcal{L} = \frac{1}{2}m_1 l_{c_1}^2 \dot{q_1}^2 + \frac{1}{2}I_1\dot{q_1}^2 + \frac{1}{2}m_2[\dot{q_1}^2(l_1^2 + l_{c_2}^2 + 2l_1 l_{c_2} cos(q_2)) + \dot{q_2}^2 l_{c_2}^2 + 2\dot{q_1}\dot{q_2}(l_1 l_{c_2} cos(q_2) + l_{c_2}^2)] +$$

$$\frac{1}{2}I_2(\dot{q_1} + \dot{q_2})^2 - \left[m_1 g l_{c_1} sin(q_1) + m_2 g(l_1 sin(q1) + m_2 g l_{c_2} sin(q_1 + q_2))\right]$$

$$= \frac{1}{2}\dot{q_1}^2\left[m_1 l_{c_1}^2 + I_1 + I_2 + m_2 l_1^2 + m_2 l_{c_2}^2 + 2m_2 l_1 l_{c_2} cos(q_2)\right] + \frac{1}{2}\dot{q_2}^2\left[m_2 l_{c_2}^2 + I_2\right] +$$

$$\dot{q_1}\dot{q_2}\left[I_2 + m_2 l_1 l_{c_2} cos(q_2) + m_2 l_{c_2}^2\right] - g sin(q_1)[m_1 l_{c_1} + m_2 l_1] - sin(q_1 + q_2) m_2 g l_{c_2}$$

For q_1 and q_2 there will be 2 equations:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \dot{q_1}} =& \dot{q_1}\left[m_1 l_{c_1}^2 + I_1 + I_2 + m_2 l_1^2 + m_2 l_{c_2}^2 + 2m_2 l_1 l_{c_2} cos(q_2)\right] + \\ & \dot{q_2}\left[I_2 + m_2 l_1 l_{c_2} cos(q_2) + m_2 l_{c_2}^2\right]\end{aligned}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q_2}} = \dot{q_2}\left[m_2 l_{c_2}^2 + I_2\right] + \dot{q_1}\left[I_2 + m_2 l_1 l_{c_2} cos(q_2) + m_2 l_{c_2}^2\right]$$

$$\frac{\partial \mathcal{L}}{\partial q_1} = - g cos(q_1)[m_1 l_{c_1} + m_2 l_1] - cos(q_1 + q_2) m_2 g l_{c_2}$$

$$\frac{\partial \mathcal{L}}{\partial q_2} = - \dot{q_1}^2 m_2 l_1 l_{c_2} sin(q_2) - m_2 g l_{c_2} cos(q_1 + q_2) - \dot{q_1}\dot{q_2} m_2 l_1 l_{c_2} sin(q_2)$$

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q_1}}\right) =& \ddot{q_1}\left[m_1 l_{c_1}^2 + I_1 + I_2 + m_2 l_1^2 + m_2 l_{c_2}^2 + 2m_2 l_1 l_{c_2} cos(q_2)\right] - 2\dot{q_1}\dot{q_2}\left[m_2 l_1 l_{c_2} sin(q_2)\right] + \\ & \ddot{q_2}\left[I_2 + m_2 l_1 l_{c_2} cos(q_2) + m_2 l_{c_2}^2\right] - \dot{q_2}^2\left[m_2 l_1 l_{c_2} sin(q_2)\right]\end{aligned}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q_2}}\right) = \ddot{q_2}\left[m_2 l_{c_2}^2 + I_2\right] + \ddot{q_1}\left[I_2 + m_2 l_1 l_{c_2} cos(q_2) + m_2 l_{c_2}^2\right] - \dot{q_1}\dot{q_2}[m_2 l_1 l_{c_2} sin(q_2)]$$

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q_1}}\right) - \frac{\partial \mathcal{L}}{\partial q_1} =& \ddot{q_1}\left[m_1 l_{c_1}^2 + I_1 + I_2 + m_2 l_1^2 + m_2 l_{c_2}^2 + 2m_2 l_1 l_{c_2} cos(q_2)\right] - 2\dot{q_1}\dot{q_2}\left[m_2 l_1 l_{c_2} sin(q_2)\right] + \\ & \ddot{q_2}\left[I_2 + m_2 l_1 l_{c_2} cos(q_2) + m_2 l_{c_2}^2\right] - \dot{q_2}^2\left[m_2 l_1 l_{c_2} sin(q_2)\right] + g cos(q_1)[m_1 l_{c_1} + m_2 l_1] + \\ & cos(q_1 + q_2) m_2 g l_{c_2} = Q_1 = 0\end{aligned}$$

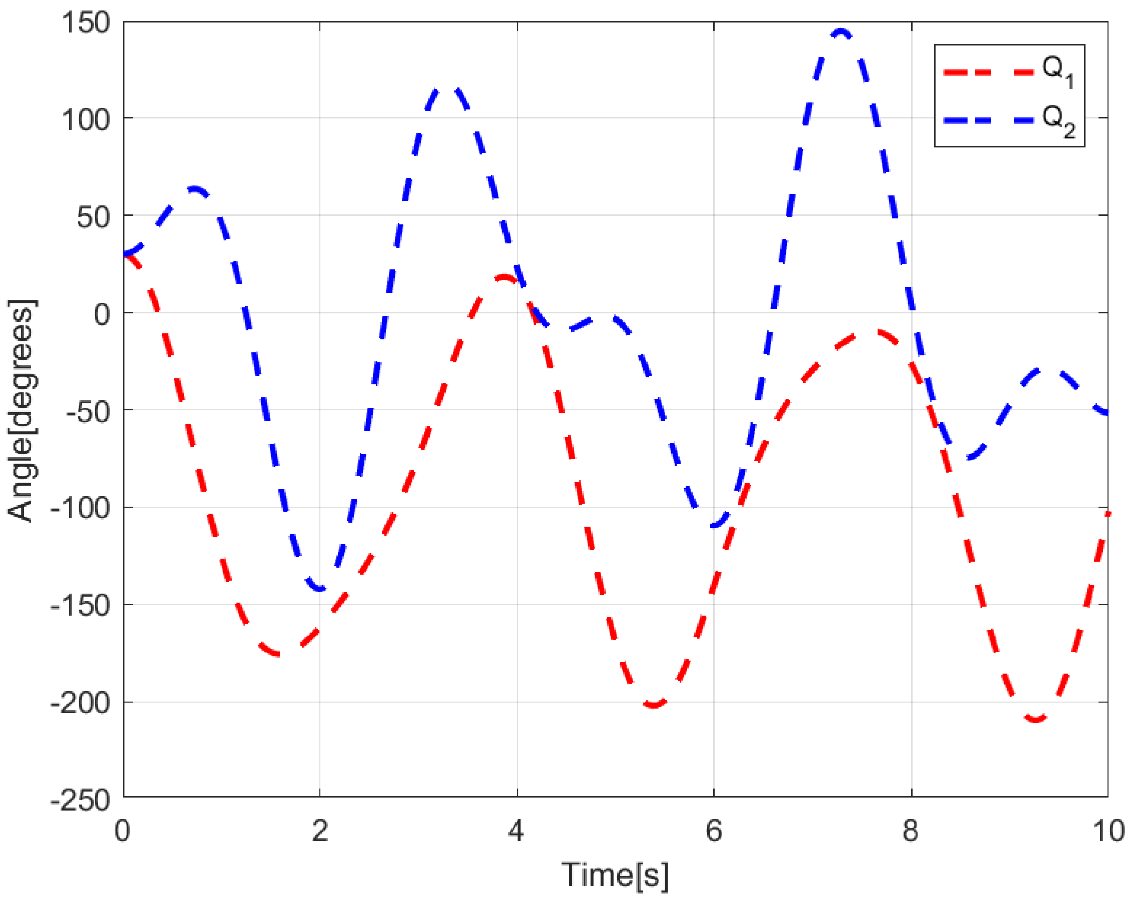
$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q_2}}\right) - \frac{\partial \mathcal{L}}{\partial q_2} =& \ddot{q_2}\left[m_2 l_{c_2}^2 + I_2\right] + \ddot{q_1}\left[I_2 + m_2 l_1 l_{c_2} cos(q_2) + m_2 l_{c_2}^2\right] + \dot{q_1}^2 m_2 l_1 l_{c_2} sin(q_2) + \\ & m_2 g l_{c_2} cos(q_1 + q_2) = Q_2 = 0\end{aligned}$$

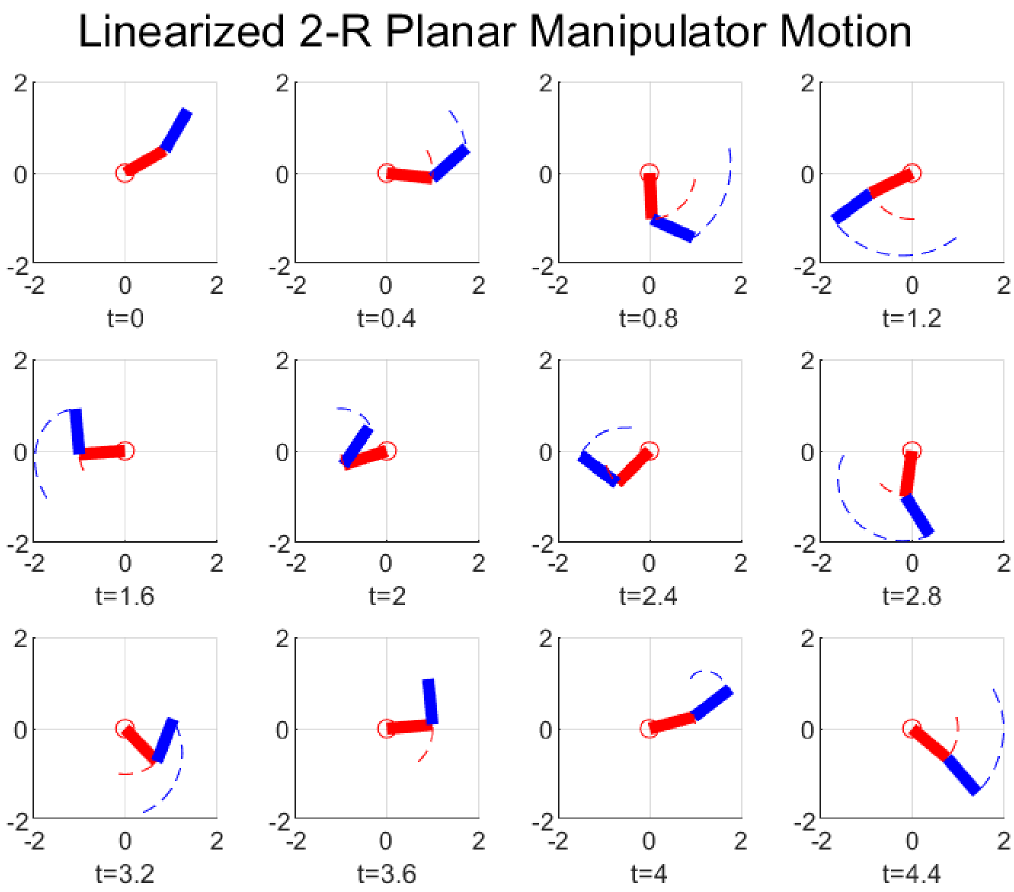
Assuming unit values and linearization around $q_1 = \pi/2$ *and* $q_2 = 0$ where the system is at the downward position and hence stable. Jacobians are derived in Matlab symbolic toolbox and A and B matrices are calculated assuming $\dot{q}_1 = 0$ and $\dot{q}_2 = 0$ to be able to obtain a linear model.

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4.5510 & 1.8204 & 0 & 0 \\ 4.2476 & -7.5851 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.2165 & -0.4021 \\ -0.4021 & 1.1753 \end{bmatrix} \begin{bmatrix} C_1(t) \\ C_2(t) \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad D = [0]$$

- **Section -b**





- **Section -c**

$$\frac{\theta_1(s)}{C_1(s)} = \frac{0.2165s^2 + 0.9102}{s^4 + 12.14s^2 + 26.79}$$

$$\frac{\theta_2(s)}{C_1(s)} = \frac{-0.4021s^2 - 0.9102}{s^4 + 12.14s^2 + 26.79}$$

$$\frac{\theta_1(s)}{C_2(s)} = \frac{-0.4021s^2 - 0.9102}{s^4 + 12.14s^2 + 26.79}$$

$$\frac{\theta_2(s)}{C_2(s)} = \frac{1.175s^2 + 3.641}{s^4 + 12.14s^2 + 26.79}$$

Poles and zeros are not simply the poles and zeros of the transfer functions anymore. They are calculated as in the course notes[1] And the poles of the system are

$$p_1 = 3.0390i \quad p_2 = -3.0390i \quad p_3 = 1.7031i \quad p_4 = -1.7031i$$

[1] Feedback Control Systems `https://ocw.mit.edu/courses/aeronautics-and-astronautics/16-30-feedback-control-systems-fall-2010/lecture-notes/MIT16_30F10_lec08.pdf`.

Or simply eigenvalues of the A matrix gives the poles. There are no zeros even it may seem like there are zeros by looking at the transfer functions.

5 Problem # 5

- **Section -a**

$$\begin{aligned} x_a &= -\alpha a \qquad x_b = -\alpha a - l sin(\theta) \qquad y_b = a + l cos(\theta) \\ \dot{x_a} &= -\dot{\alpha} a \qquad \dot{x_b} = -\dot{\alpha} a - l cos(\theta)\dot{\theta} \qquad \dot{y_b} = -l sin(\theta)\dot{\theta} \\ \dot{x_b}^2 + \dot{y_b}^2 &= \dot{\alpha}^2 a^2 + l^2\dot{\theta}^2 + 2la\dot{\alpha}\dot{\theta} \end{aligned}$$

Using Lagrange's formula

$$\begin{aligned} \mathcal{T} &= \frac{1}{2}m[\dot{\alpha}^2 a^2 + l^2\dot{\theta}^2 + 2la\dot{\alpha}\dot{\theta}cos(\theta)] + \frac{1}{2}M\dot{\alpha}^2 a^2 + \frac{1}{2}J_1\dot{\alpha}^2 + \frac{1}{2}J_2\dot{\theta}^2 \\ &= \frac{1}{2}[(M+m)a^2 + J_1]\dot{\alpha}^2 + \frac{1}{2}[ml^2 + J_2]\dot{\theta}^2 + mla\dot{\alpha}\dot{\theta}cos(\theta) \\ \mathcal{V} &= mglcos(\theta) \\ \\ \mathcal{D} &= 0 \\ \\ \mathcal{L} &= \mathcal{T} - \mathcal{V} = \frac{1}{2}[(M+m)a^2 + J_1]\dot{\alpha}^2 + \frac{1}{2}[ml^2 + J_2]\dot{\theta}^2 + mla\dot{\alpha}\dot{\theta}cos(\theta) - mglcos(\theta) \end{aligned}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = [ml^2 + J_2]\dot{\theta} + mla\dot{\alpha}cos(\theta) \qquad \frac{\partial \mathcal{L}}{\partial \theta} = -mla\dot{\alpha}\dot{\theta}sin(\theta) + mglsin(\theta)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\alpha}} = [(M+m)a^2 + J1]\dot{\alpha} + mla\dot{\theta}cos(\theta) \qquad \frac{\partial \mathcal{L}}{\partial \alpha} = 0$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\alpha}}\right) - \frac{\partial \mathcal{L}}{\partial \alpha} = [(M+m)a^2 + J1]\ddot{\alpha} + mla[\ddot{\theta}cos(\theta) - \dot{\theta}^2 sin(\theta)] = u(t)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = [ml^2 + J_2]\ddot{\theta} + mla[\ddot{\alpha}cos(\theta)] - mglsin(\theta) = -u(t)$$

$$\begin{aligned} k_1\ddot{\alpha} + k_3(\ddot{\theta}cos(\theta) - \dot{\theta}^2 sin(\theta)) &= u(t) \\ k_2\ddot{\theta} + k_3\ddot{\alpha}cos(\theta) - k_4 sin(\theta) &= -u(t) \end{aligned}$$

Solving these 2 equations for $\ddot{\alpha}$ and $\ddot{\theta}$ gives

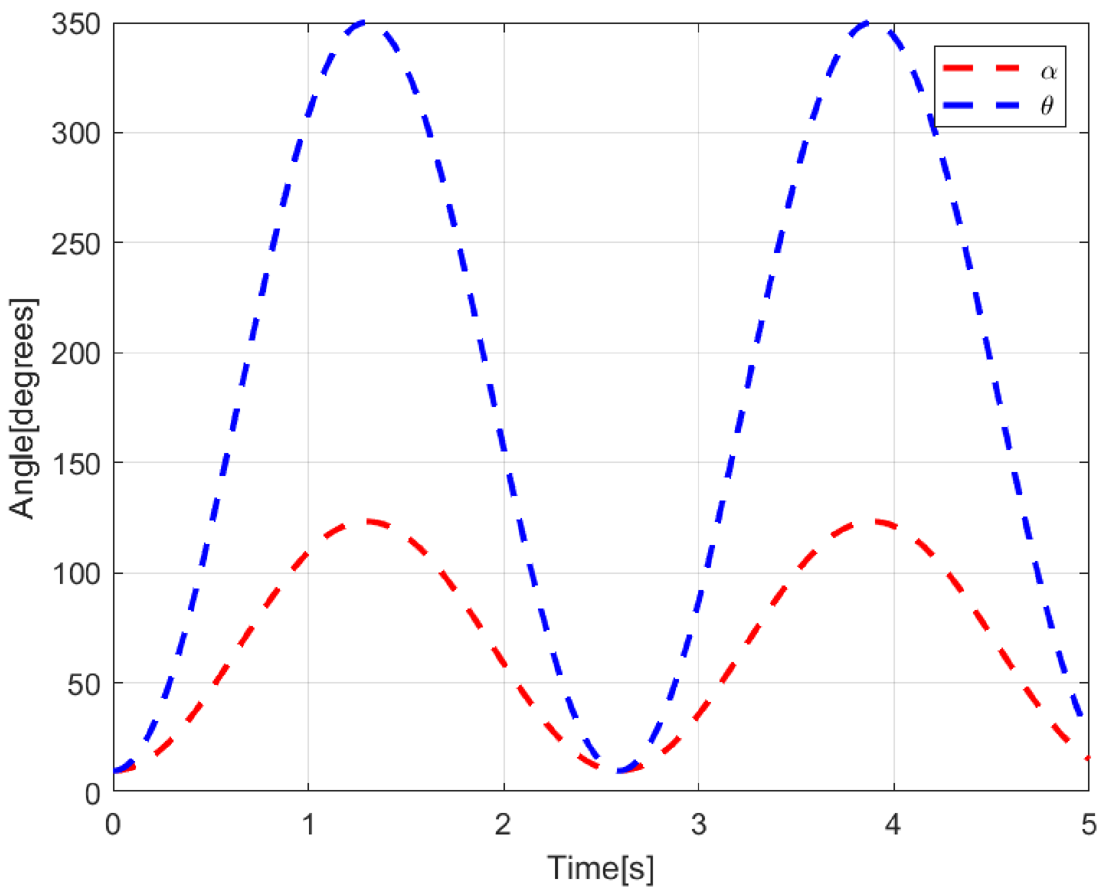
$$\ddot{\alpha} = \frac{k_2 k_3 sin(\theta)}{k_1 k_2 - k_3^2 cos^2(\theta)}\dot{\theta}^2 - \frac{k_3 k_4 sin(\theta)cos(\theta)}{k_1 k_2 - k_3^2 cos^2(\theta)} + \frac{k_2 + k_3 cos(\theta)}{k_1 k_2 - k_3^2 cos^2(\theta)}u(t)$$

$$\ddot{\theta} = -\frac{k_3^2 sin(\theta)cos(\theta)}{k_1 k_2 - k_3^2 cos^2(\theta)}\dot{\theta}^2 + \frac{k_1 k_4 sin(\theta)}{k_1 k_2 - k_3^2 cos^2(\theta)} - \frac{k_1 + k_3 cos(\theta)}{k_1 k_2 - k_3^2 cos^2(\theta)}u(t)$$

- **Section -b** Linearized around $\theta = \pi$

$$\begin{bmatrix} \dot{\alpha} \\ \dot{\theta} \\ \ddot{\alpha} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1.962 & 0 & 0 \\ 0 & -5.886 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \theta \\ \dot{\alpha} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.2 \\ -0.4 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad D = [0]$$



- **Section -c**

$$\frac{\alpha(s)}{U(s)} = \frac{0.2s^2 + 1.962}{s^2(s^2 + 5.886)}$$

$$\frac{\theta(s)}{U(s)} = \frac{-0.4}{s^2 + 5.886}$$

System poles are at $p_1 = 0 \qquad p_2 = 0 \qquad p_3 = 0.4261i \qquad p_4 = -0.4261i$

And there are no zeros in the system.

Poles are on the imaginary axis, which explains the oscillatory response of the system.