

Module 3, Part 1: Generalized Linear Models

BIOS 526

Concepts

- Link function.
- Logistic regression and odds ratio. Binary data
- Probit regression. Binary data
- Poisson regression. Count data

Readings

- Chapter 3, Wood, S. *Generalized Additive Models*, 2017. Has a nice, self-contained introduction to generalized linear models.

Linear Regression Model

Consider the following multiple linear regression model. For $i = 1, \dots, n$,

$$y_i = \beta_0 + \sum_{k=1}^p \beta_k x_{ik} + \epsilon_i \quad \epsilon_i \stackrel{iid}{\sim} N(0, \sigma^2),$$

where x_{ik} is the k th linear predictor for observation i .

The above model assumes

- $\beta_0, \beta_1, \dots, \beta_p$ are fixed unknown constants;
- only the residual error ϵ_i is **random**.

Therefore,

↳ y_i follows a normal distribution

$$\left(\begin{aligned} \text{2. } E[y_i | \tilde{x}_i] &= \beta_0 + \sum_{k=1}^p \beta_k x_{ik} \end{aligned} \right)$$

The **linear regression** part is used to model only the **mean function** of y_i .

Generalized Linear Regression Model

A generalized linear model (GLM) extends linear regression to other distributions, where the response variable is generated from a distribution in the exponential family.

A GLM involves three ingredients:

1. An exponential family of probability distributions.
2. A linear model $\mathbf{x}_i' \boldsymbol{\beta}$.
3. A link function $g()$ and its inverse $g^{-1}()$ relates the linear model to its expectation: ~~$\mathbf{x}_i' \boldsymbol{\beta}$~~

$$E[y_i | \mathbf{x}_i] = \mu_i = g^{-1}(\mathbf{x}_i' \boldsymbol{\beta})$$
$$Var[y_i | \mathbf{x}_i] = V(\mu_i) = V(g^{-1}(\mathbf{x}_i' \boldsymbol{\beta}))$$

Note: unlike ordinary least squares, the basic form of the GLM does not involve a noise variance (no σ^2).

Exponential Family

The basic form for an exponential family density is

$$f_{\theta}(y) = \exp [\{y\theta - b(\theta)\} / a(\phi) + c(y, \phi)],$$

where b , a , and c are known functions, and ϕ is a known scale parameter.

There is only one unknown parameter: θ .

In the GLM, θ will be a function of $x'_i\beta$.

(e.g. in normal linear model, known σ^2)

Examples of distributions in the exponential family include: normal distribution with **known** variance (link = identity), Bernoulli (logit or probit link), binomial (with fixed number of trials), gamma, exponential (link: negative inverse), and others. (binomial with n trials)

https://en.wikipedia.org/wiki/Generalized_linear_model

Rather than derive expressions for the general case, we will focus on the two most popular models:

1. Binary outcome: $y_i \stackrel{ind}{\sim}$ Bernoulli (p_i), where p_i is the probability of success.
2. Poisson outcome: $y_i \stackrel{ind}{\sim}$ Poisson (λ_i), λ_i is the rate parameter, equal to expected number of events.

GLMs

The mean and variance function of y_i can be expressed as a function of the distribution parameters (i.e. p_i for Bernoulli and λ_i for Poisson).

1. Binary outcome:

$$E[y_i] = \mu_i = p_i,$$

$$Var[y_i] = V[\mu_i] = V[p_i] = \underline{p_i(1 - p_i)}.$$

2. Poisson outcome:

$$E[y_i] = \mu_i = \lambda_i,$$

$$Var[y_i] = V[\mu_i] = V[\lambda_i] = \underline{\lambda_i}.$$

in Poisson, mean = variance

A natural approach is to model the mean as a function of linear predictors. A difficulty in modeling non-normal data is that the distributional parameters often have constraints.

1. Binary outcome has expected value $\underline{p_i} \in (0, 1)$
2. Poisson outcome has expected value $\underline{\lambda_i} > 0$.

$$\begin{aligned} Var(y_i) &= E[y_i - E[y_i]]^2 \\ &= \sum_{m=0}^1 [m - E[y_i]]^2 P(y_i = m) \\ &= (0 - p_i)^2 (1 - p_i) + (1 - p_i)^2 p_i \\ &= (1 - p_i) p_i^2 + (1 - p_i) p_i \\ &= (1 - p_i) p_i \end{aligned}$$

Workspace

Generalized Linear Regression Model

Our solution is to model the **transformed** mean function:

$$g(E[y_i | x_i]) = \beta_0 + \sum_{k=1}^p \beta_k x_{ik}$$

The function $g(\cdot)$ is known as the **link function**.

The link function should have some desirable properties:

1. $g(\cdot)$ should have range $(-\infty, \infty)$
so that β s and x_{ik} s can have any value
2. $g(\cdot)$ should have domain correspond to possible values of μ_i (0,1) for Binary, (0, ∞) for Poisson
3. $g(\cdot)$ should be one-to-one.
$$\mu_i = g^{-1}\left(\beta_0 + \sum_{k=1}^p \beta_k x_{ik}\right)$$

• want a strictly \uparrow or \downarrow function

GLM for Binary Outcome

For $i = 1, \dots, n$, assume

$$y_i \stackrel{\text{ind}}{\sim} \text{Bernoulli}(\underline{p_i}),$$

where $y_i \in \{0, 1\}$, $y_i | p_i$ for $i = 1, \dots, n$ are independent, and p_i is the probability $y_i = 1$. The probability mass function is given by

$$f(y_i | p_i) = p_i^{y_i} (1 - p_i)^{1 - y_i}$$

We know that

$$\mu_i = p_i = \underline{P(Y_i = 1)}.$$

So we wish to model

$$g(\mu_i) = \beta_0 + \sum_{k=1}^p \beta_k x_{ik}$$

The two most commonly used link functions are the **logistic function** and the **probit function**.

Logistic Regression

The logistic regression is formulated as follows. For $i = 1, \dots, n$,

$$y_i \overset{\text{IND}}{\sim} \text{Bernoulli}(p_i)$$
$$\log\left(\frac{p_i}{1-p_i}\right) = \beta_0 + \sum_{k=1}^K \beta_k x_{ik}$$

.....

- $g(p_i)$ is the **log odds** of success probability.
- $\log\left(\frac{p_i}{1-p_i}\right) \rightarrow -\infty$ when $p_i \rightarrow 0$; and $\log\left(\frac{p_i}{1-p_i}\right) \rightarrow \infty$ when $p_i \rightarrow 1$.
- $\log\left(\frac{p_i}{1-p_i}\right)$ is strictly increasing.

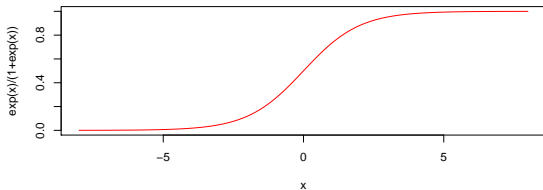
Likelihood function: logistic regression

$$\ell(\beta; \mathbf{y}, \mathbf{x}) =$$

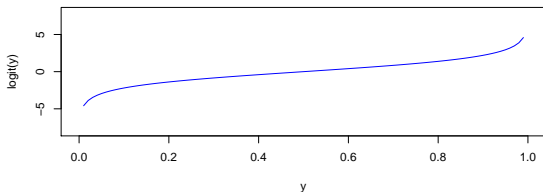
In practice, GLMs are estimated using iteratively reweighted least squares. We won't go into details, but see p. 107 in Wood for more info.

The **logistic** function:

$$p(g) = \exp(g)/(1 + \exp(g)) = 1/(1 + \exp(-g)), \quad (-\infty, \infty) \rightarrow (0, 1).$$



The **logit** function: $g(p) = \log(p/(1 - p))$, $(0, 1) \rightarrow (-\infty, \infty)$.



Logistic Regression: Interpretation

Consider a simple logistic model with only one predictor:

$$y_i \stackrel{\text{ind}}{\sim} \text{Bernoulli}(p_i)$$
$$\log\left(\frac{p_i}{1-p_i}\right) = \beta_0 + \beta_1 x_i.$$

When $x_i = 0$:

- $\log\left(\frac{p_i}{1-p_i}\right) = \beta_0.$
- β_0 is interpreted as the **baseline log odds**.

Function of the probability of success at baseline:

$$p_i = \frac{e^{\beta_0}}{1 + e^{\beta_0}}.$$

- Note that the above function satisfies
 - $p_i \in (0, 1)$ for $\beta_0 \in \mathbb{R}.$
 - p_i a strictly increasing function of β_0 .

Logistic Regression: Log odds and log odds ratio

Now we consider the effect of a unit change in x_i :

$$\log\left(\frac{p_i}{1-p_i}\right) = \beta_0 + \beta_1 x_i \quad \text{versus} \quad \log\left(\frac{p_i^*}{1-p_i^*}\right) = \beta_0 + \beta_1 (x_i + 1).$$

Then,

$$\begin{aligned} \log\left(\frac{p_i^*}{1-p_i^*}\right) - \log\left(\frac{p_i}{1-p_i}\right) &= \beta_0 + \beta_1 (x_i + 1) - (\beta_0 + \beta_1 x_i) \\ &= \beta_1. \end{aligned}$$

In words: β_1 is the change in log odds per unit change in x_i .

Equivalently, it is the log odds ratio per unit change in x_i :

Logistic Regression: Odds ratio

e^{β_1} is the **odds ratio**:

$$e^{\beta_1} = \frac{p_i^*/(1 - p_i^*)}{p_i/(1 - p_i)}.$$

Odds ratio interpretation helpful for indicator variables. Let $x_i = 1$ in exposed group, $x_i = 0$ in unexposed group. Then:

$$\begin{aligned} e^{\beta_1} &= \frac{\left\{ \frac{P(y_i=1|x_i=1)}{P(y_i=0|x_i=1)} \right\}}{\left\{ \frac{P(y_i=1|x_i=0)}{P(y_i=0|x_i=0)} \right\}} \\ &= \text{odds(Exposed)}/\text{odds(Unexposed)}. \end{aligned}$$

Logistic Regression: Background

- Logistic regression is commonly used because the slope coefficient corresponds to the log odds ratio (OR), a commonly used measure in epidemiology.
- OR is different from relative risk.
- Risk ratio (RR) is $P(y_i = 1|x_i = 1)/P(y_i = 1|x_i = 0)$
- OR is close to RR when the event $y_i = 1$ is rare, but in general, you need a different model to estimate RR.
- OR can be used in retrospective and observational studies.

Logistic Regression: Probability

The effect of a unit change in x_i depends on x_i on the probability scale.

E.g., $\beta_0 = 0$ and $\beta_1 = 2$:

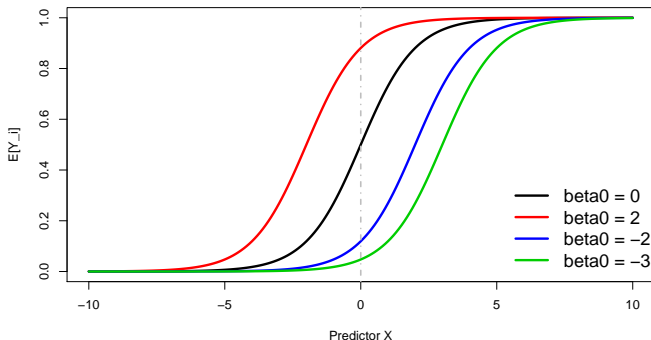
$$\frac{e^{2(0+1)}}{1 + e^{2(0+1)}} - \frac{e^{2(0)}}{1 + e^{2(0)}} \neq \frac{e^{2(1+1)}}{1 + e^{2(1+1)}} - \frac{e^{2(1)}}{1 + e^{2(1)}}$$

A change in x_i from 0 to 1 increases the probability by 0.38, but a change in x_i from 1 to 2 increases the probability by 0.10.

Intuitively, this has to be the case in order for the probability to max out at 1: $\frac{e^{2(100+1)}}{1+e^{2(100+1)}} - \frac{e^{2(100)}}{1+e^{2(100)}} \approx 1 - 1$.

Logistic Regression: Effects of Baseline Odds

$$\text{logit}(p_i) = \beta_0 + x_i.$$
$$\mu_i = P(Y_i = 1) = \frac{e^{\beta_0 + x_i}}{1 + e^{\beta_0 + x_i}}.$$



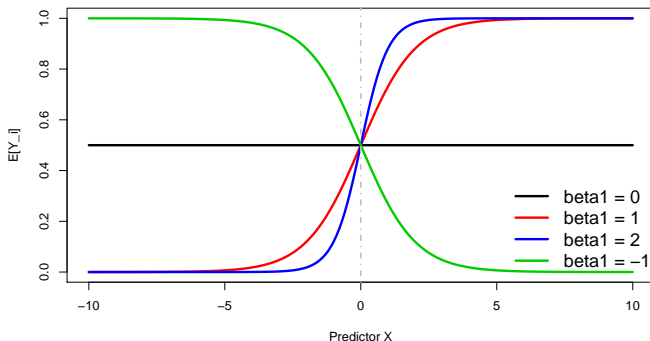
Note:

- The **shape** is maintained.
- The baseline (intercept) probability changes.

Logistic Regression: Effects of Slope

$$\text{logit}(\mu_i) = \beta_1 x_i.$$

$$\mu_i = P(Y_i = 1) = \frac{e^{\beta_1 x_i}}{1 + e^{\beta_1 x_i}}.$$



Note:

- How the steepness changes.
- How the direction of effect changes.

Logistic Regression: Interpretation

In multiple logistic regression, for $i = 1, \dots, n$,

- β_0 is the log odds when all covariate values equal zero.
- β_k is the log odds ratio associated with covariate k while controlling for other covariates.

Logistic Regression: Interpretation

The estimated (predicted) value is given by

$$\mu_i = p_i = \frac{e^{\beta_0 + \sum_{k=1}^p \beta_k x_{ik}}}{1 + e^{\beta_0 + \sum_{k=1}^p \beta_k x_{ik}}}$$

Again, the above function is non-linear in x_k , in contrast with normal regression

Inference: Likelihood Ratio Tests

To conduct inference, we appeal to **asymptotic results** that hold for large-ish n . There are two approaches:

1) Likelihood Ratio Tests (Difference in Deviance)

Let β_F be a vector of coefficients of interest. Then to test $H_0 : \beta_F = 0$, we create a full and reduced model. Let $\ell(\hat{\beta}_{\text{Full}})$ be the log-likelihood of the full model, and $\ell(\hat{\beta}_{\text{Reduced}})$ be the LL for the reduced model. Then we reject H_0 if

$$-2 \left\{ \ell(\hat{\beta}_{\text{Reduced}}) - \ell(\hat{\beta}_{\text{Full}}) \right\} > \chi^2_{\nu, 1-\alpha}$$

where ν is the difference in the number of parameters between the full and reduced, and $\chi^2_{\nu, 1-\alpha}$ is the critical value from a chi-squared distribution with ν degrees of freedom.

Inference: Wald Tests

2) Wald Tests. Under regularity conditions, asymptotically,

$$\begin{aligned}\hat{\beta} &\sim N(\beta, I(\beta)^{-1}), \\ I(\beta) &= E \left\{ \left(\frac{\partial \ell}{\partial \beta} \right) \left(\frac{\partial \ell}{\partial \beta} \right)' \right\} \\ &= -E \frac{\partial^2 \ell}{\partial \beta \partial \beta'},\end{aligned}$$

where the Hessian $I(\beta)$ is called the Fisher information matrix. See Wood p. 106 for details of the expected Hessian which is calculated during iteratively re-weighted least squares.

We can write $\hat{\beta} \sim N(\beta, (\mathbf{X}'\mathbf{W}\mathbf{X})^{-1}\phi)$ where \mathbf{W} contains the “Fisher weights” and $\phi = 1$ in the usual (not overdispersed) GLM.

In R, the default `summary(glmmodel)` is a Wald-type test: $\hat{\beta}_j/se(\hat{\beta}_j)$, where $se(\hat{\beta}_j)$ is extracted from the square root of the j th diagonal of the above covariance.

Binary Outcome Example

Dataset: a cohort of live births from Georgia born in the year 2001 ($N = 77,340$).

Variables:

- *ptb*: indicator for whether the baby from pregnancy i was born preterm (< 37 weeks).
- *age*: the mother's age at delivery (centered at age 25).
- *male*: indicator of the baby's sex (1 = male; 0=female).
- *tobacco*: indicator for mother's tobacco use during pregnancy (1 = yes; 0 = no)

The *glm* () Function

Fitting a GLM model in R is very similar to a linear regression model. We need to specify the distribution (**binomial**) and the link function (**logit**).

```
glm(formula = ptb ~ age + male + tobacco, family = binomial(link = "logit"),  
    data = dat)
```

Deviance Residuals:

Min	1Q	Median	3Q	Max
-0.5160	-0.4236	-0.4103	-0.4088	2.2500

Coefficients:

	Estimate	Std. Error	z value	Pr(> z)
(Intercept)	-2.4370033	0.0200791	-121.370	< 2e-16 ***
age	-0.0006295	0.0021596	-0.291	0.77068
maleM	0.0723659	0.0258672	2.798	0.00515 **
tobacco	0.4096495	0.0534627	7.662	1.83e-14 ***

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Birth Outcome Analysis

- Preterm delivery was significantly associated with male babies (p -value= 0.005) when controlling for age and mother's smoking status. The odds ratio of a preterm birth for a male baby versus a female baby was 1.07 (95% CI: 1.02, 1.13).

$$\text{OR} = e^{0.0723} = 1.07$$

$$\text{CI} = e^{(0.0723 \pm 1.96 * 0.0258)} = (1.02, 1.13)$$

- Note: **transform the intervals**. Do NOT transform standard errors (requires delta method).
- Preterm delivery was significantly associated with whether the mother smoked during pregnancy ($p < 0.001$) when controlling for age and the baby's sex. The odds ratio for mother's that smoked versus did not smoke was $e^{0.409} = 1.51$ (95% CI: 1.36, 1.67).

Birth outcome analysis, cont.

- The baseline proportion (female babies born to mother of age 25 who didn't smoke) of preterm delivery was

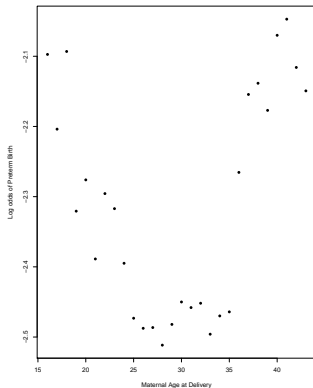
$$\frac{e^{-2.437}}{1 + e^{-2.437}} = 0.080.$$

- We didn't find an effect of mother's age.

Birth Outcome Analysis - Mother's Age

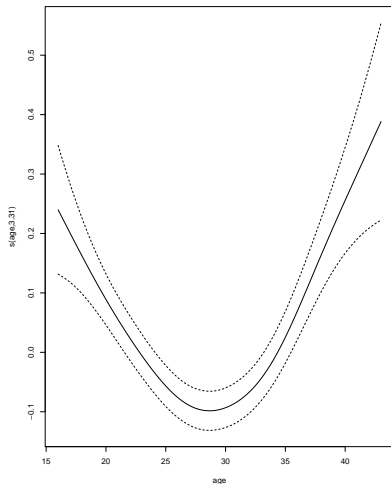
We assumed that the mother's age has a linear effect on log odds of preterm birth. Is this a reasonable assumption?

Explore this by calculating % preterm births for each age group.



Generalized Additive Model

In Module 5, we will model this non-linearly using splines:



P-values using LRTs

```
> library(car)
> # LRTs:
> Anova(fit)
Analysis of Deviance Table (Type II tests)
```

Response: ptb

	LR	Chisq	Df	Pr(>Chisq)
age	0.085	1		0.770669
male	7.834	1		0.005126 **
tobacco	53.648	1		2.399e-13 ***

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Probit Regression

Probit regression is an alternative approach to model binary data. It still assumes the Bernoulli model, but uses a different **link** function. For

$i = 1, \dots, n$,

$$y_i \stackrel{ind}{\sim} \text{Bernoulli}(p_i)$$

where Φ^{-1} is the **inverse cumulative distribution function** of a standard normal distribution. Recall $\Phi^{-1}(x)$ asks what Z-value gives a cumulative probability of x ?

- Example: $\Phi^{-1}(0.5) = 0$ and $\Phi^{-1}(0.975) = 1.96$.
- Note $\Phi^{-1}(p_i)$ is strictly increasing, has range $(-\infty, \infty)$ and domain $(0, 1)$.

Probit Regression

The probit link function results in

$$p_i = \Phi\left(\beta_0 + \sum_{k=1}^p \beta_k x_{ik}\right).$$

Therefore, $P(y_i = 1)$ is viewed as the probability of a standard normal variable being less than $\beta_0 + \sum_{k=1}^p \beta_k x_{ik}$.

Probit regression has a very attractive **latent variable** (i.e., unobserved) interpretation. Let Z_i denote a latent variable associated with each binary outcome.

$$Z_i \stackrel{ind}{\sim} N\left(\beta_0 + \sum_{k=1}^p \beta_k x_{ik}, 1\right).$$

Then

$$\begin{aligned} P(Z_i > 0) &= 1 - P(Z_i < 0) = 1 - P\left(\frac{0 - (\beta_0 + \sum_{k=1}^p \beta_k x_{ik})}{1} < 0\right) \\ &= 1 - \Phi\left(-(\beta_0 + \sum_{k=1}^p \beta_k x_{ik})\right) = \Phi\left(\beta_0 + \sum_{k=1}^p \beta_k x_{ik}\right). \end{aligned}$$

Probit Regression: Latent Variable Representation

We can rewrite

$$y_i \stackrel{ind}{\sim} \text{Bernoulli}(p_i) \quad \Phi^{-1}(p_i) = \beta_0 + \sum_{k=1}^p \beta_k x_{ik}$$

as a hierarchical model:

$$\begin{aligned} 1. \quad & Z_i \stackrel{ind}{\sim} N\left(\beta_0 + \sum_{k=1}^p \beta_k x_{ik}, 1\right) \\ 2. \quad & y_i = \begin{cases} 0 & \text{if } Z_i < 0 \\ 1 & \text{if } Z_i > 0 \end{cases} \end{aligned}$$

Therefore we assume the binary outcome $y_i = 1$ when its latent variable Z_i **passes the threshold 0**.

The probability of this occurring depends on the mean of the latent variable Z_i . Larger mean $(\beta_0 + \sum_{k=1}^p \beta_k x_{ik})$ increases the the probability of $y_i = 1$.

Probit Regression: Interpretation

Consider a simple probit model with only one predictor:

$$y_i \overset{ind}{\sim} \text{Bernoulli}(p_i)$$

$$p_i = \Phi(\beta_0 + \beta_1 x_i).$$

Interpretation of the regression coefficients is arguably more challenging.
Represents change in z-score.

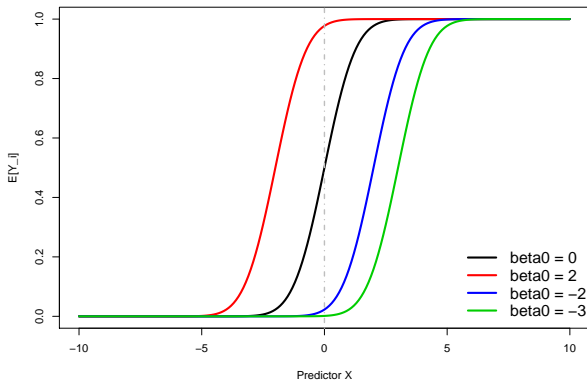
- The baseline probability is $\Phi(\beta_0)$.
- The effect of a unit increase in x_i on $P(y_i = 1)$ is

$$\Phi(\beta_0 + \beta_1 x_i + \beta_1) - \Phi(\beta_0 + \beta_1 x_i),$$

which again depends on the value of x_i .

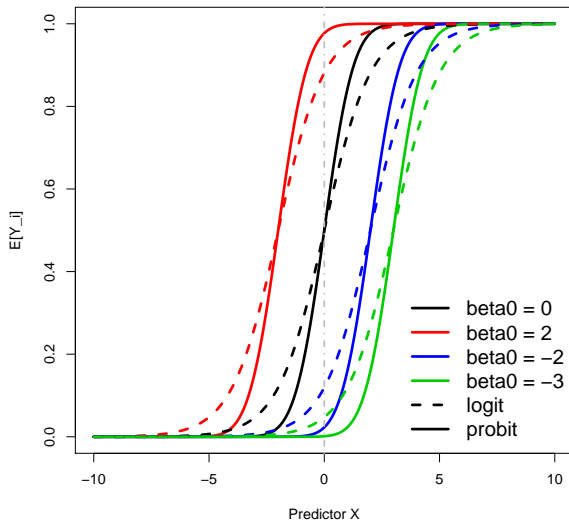
Probit Regression: Effects of Intercept

$$p_i = \mu_i = \Phi(\beta_0 + \beta_1 x_i).$$



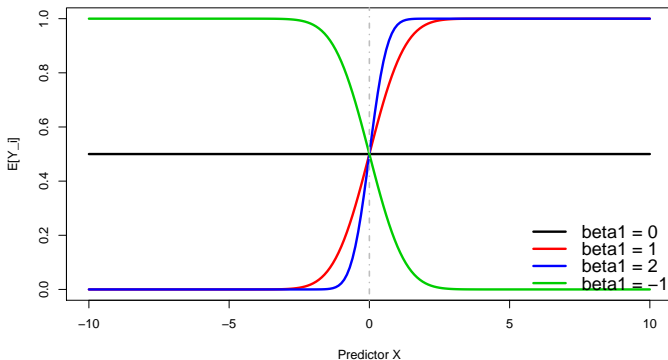
Very similar behaviors as logistic regression. Slightly different tail behaviors compared to a logit link function.

Logit and Probit Regression: Effects of Intercept



Probit Regression: Effects of Slope

$$p_i = \mu_i = \Phi(\beta_1 x_i).$$



The *glm* () Function with probit

```
glm(formula = ptb ~ age + male + tobacco, family = binomial(link = "probit"),  
    data = dat)
```

Deviance Residuals:

Min	1Q	Median	3Q	Max
-0.5158	-0.4237	-0.4104	-0.4087	2.2509

Coefficients:

	Estimate	Std. Error	z value	Pr(> z)	
(Intercept)	-1.4024285	0.0099738	-140.612	< 2e-16	***
age	-0.0003746	0.0010793	-0.347	0.7285	
maleM	0.0363566	0.0129215	2.814	0.0049	**
tobacco	0.2102264	0.0281156	7.477	7.59e-14	***

Comparing the logistic and probit regression model, we note the regression coefficients are **qualitatively** similar but the magnitude differs.

Poisson Regression

A Poisson regression is specified as follows. For $i = 1, \dots, n$,

For a Poisson distributed random variable,

Poisson regression is often used to model **count data**. Examples include daily mortality in a city, number of HIV infected individuals in a neighborhood, and number of medical errors at a hospital.

- Here the link function is $\log(\cdot)$.
- $\log(\cdot)$ has domain $(0, \infty)$ and range $(-\infty, \infty)$, and is strictly increasing.

Poisson Regression Interpretation

Consider a simple Poisson regression model with only one covariate:

$$y_i \overset{ind}{\sim} \text{Poisson}(\lambda_i)$$

$$\log(\lambda_i) = \beta_0 + \beta_1 x_i.$$

When $x_i = 0$, $\log(\lambda_i) = \beta_0$.

- $e^{\beta_0} = \lambda_i$ is the **baseline expected counts**.

Poisson Regression Interpretation

Now consider a unit change in x

$$\log(\lambda_i) = \beta_0 + \beta_1 x_i \quad \log(\lambda_i^*) = \beta_0 + \beta_1 (x_i + 1).$$

Note that

$$\beta_1 = \log(\lambda_i^*) - \log(\lambda_i)$$

- $e^{\beta_1} = \lambda_i^* / \lambda_i$ is the **relative change in count** per unit change in x . Also called the relative rate. Also called the incident rate ratio.
- For continuous variables with $\beta_1 > 0$, the rate increases by $100 * (e^{\beta_1} - 1)\%$ for every unit increase in x .
- For factors with $\beta_j > 0$, the rate increases by $100 * (e^{\beta_j} - 1)\%$ for level j relative to baseline.

Covariate impacts are **multiplicative** rather than additive (applies to log models in general) on the count scale:

Example: bacteria counts

Dataset: antibiotic resistance in a mutation of *E. coli*.

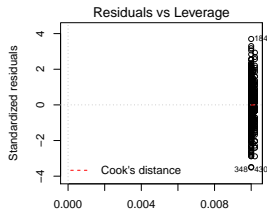
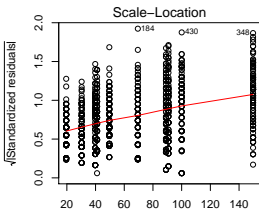
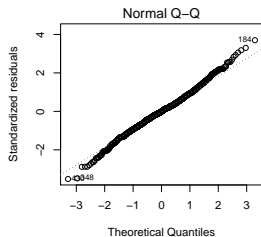
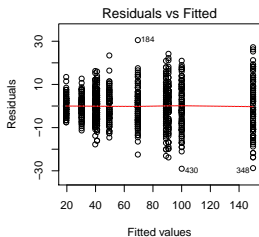
Variables:

- *Colony*: the number of ampicillin-resistant mutant colonies
- *Conc*: the concentration of novobiocin ($\mu\text{g}/\text{ml}$)
- *Media*: the type of media used for bacterial growth.

The experiment involved two media preparations (LB and M9), 5 concentrations of novobiocin, and 100 replicates for each media-concentration combination. TNTC (too numerous to count) were recorded when the number of colonies exceeded 300.

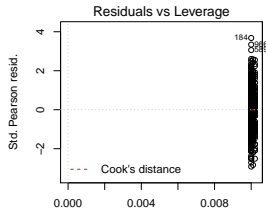
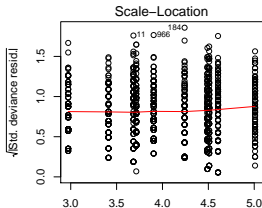
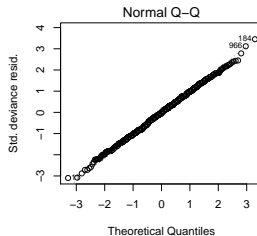
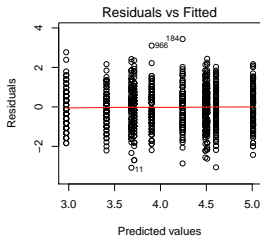
Modeling count data

```
lm_colony = lm(Colony_numeric~factor(Conc)*Media,data=colonydata)
```



Modeling count data: Poisson

```
glm_colony = glm(Colony_numeric~factor(Conc)*Media,data=colonydata,family = "poisson")
```



Residuals in glms

The usual plot of residual versus fitted is not useful in GLMs because of the relationship between the mean and variance.

The **deviance residuals** have an approximate normal distribution.

The deviance residual is

$$\hat{\epsilon}_i^d = \text{sign}(y_i - \hat{\mu}_i) \sqrt{d_i}.$$

where d_i is the i th term in the calculation of the deviance. For details, see p.113 in Wood 2017. (This plot is not useful for 0/1 data as common in logistic regression.)

Goodness of fit tests, quasipoisson

In glms, the **deviance** performs a role similar to the sum of squared errors in OLS:

where $\ell(\hat{\beta}_{\max}; \mathbf{y})$ is the “saturated model,” equal to likelihood evaluated at $\hat{\mu}_i = y_i$.

Asymptotically, $D(\hat{\beta}) \sim \chi^2_{n-p}$

One can perform a deviance test to examine goodness of fit. The null hypothesis is that the model fits the data. $p < 0.05$ indicates a problem (i.e., lack of fit).

```
with(glm_colony, cbind(res.deviance = deviance, df = df.residual,  
p = pchisq(deviance, df.residual, lower.tail=FALSE)))
```

```
res.deviance  df      p  
[1,]      973.8841 987 0.6108368
```

Here, $p > 0.05$, from which we conclude that the model provides an adequate fit.

Overdispersion

In Poisson, the assumption that $Ey_i = Vy_i$ is often violated.

One can add an additional **overdispersion parameter**, also called a **scale parameter**.

One can adjust the parameter variances by the scale parameter:

$$\hat{\beta} \sim N(0, I(\beta)^{-1} \phi)$$

We will see this again in GEEs.

Section 3.1.5 in Wood describes three different estimators of ϕ .

Quasipoisson in GLM

```
> glm_colony_quasi = glm(Colony_numeric~factor(Conc)*Media,data=colonydata,family = "quasi")
> summary(glm_colony_quasi)
```

Call:

```
glm(formula = Colony_numeric ~ factor(Conc) * Media, family = "quasipoisson",
    data = colonydata)
```

Deviance Residuals:

Min	1Q	Median	3Q	Max
-3.0815	-0.7615	-0.0047	0.6573	3.4344

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	3.683308	0.015807	233.023	< 2e-16 ***
factor(Conc)100	0.557587	0.019786	28.181	< 2e-16 ***
factor(Conc)200	0.806339	0.018981	42.481	< 2e-16 ***
factor(Conc)250	1.325726	0.017764	74.631	< 2e-16 ***
factor(Conc)300	0.922162	0.018660	49.418	< 2e-16 ***
MediaM9	-0.710845	0.027448	-25.898	< 2e-16 ***
factor(Conc)100:MediaM9	-0.118573	0.034923	-3.395	0.000713 ***
factor(Conc)200:MediaM9	-0.064499	0.033221	-1.942	0.052480 .
factor(Conc)250:MediaM9	0.210650	0.030490	6.909	8.75e-12 ***
factor(Conc)300:MediaM9	0.009164	0.032406	0.283	0.777404

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for quasipoisson family taken to be 0.983908)

Back to original model

When the data are slightly underdispersed, i.e., dispersion parameter < 1 , and there is no evidence of lack of fit, I suggest using the original model:

```
> summary(glm_colony)
```

Call:

```
glm(formula = Colony_numeric ~ factor(Conc) * Media, family = "poisson",  
    data = colonydata)
```

Deviance Residuals:

Min	1Q	Median	3Q	Max
-3.0815	-0.7615	-0.0047	0.6573	3.4344

Coefficients:

	Estimate	Std. Error	z value	Pr(> z)
(Intercept)	3.683308	0.015935	231.140	< 2e-16 ***
factor(Conc)100	0.557587	0.019947	27.953	< 2e-16 ***
factor(Conc)200	0.806339	0.019136	42.138	< 2e-16 ***
factor(Conc)250	1.325726	0.017908	74.028	< 2e-16 ***
factor(Conc)300	0.922162	0.018812	49.019	< 2e-16 ***
MediaM9	-0.710845	0.027671	-25.689	< 2e-16 ***
factor(Conc)100:MediaM9	-0.118573	0.035208	-3.368	0.000758 ***
factor(Conc)200:MediaM9	-0.064499	0.033491	-1.926	0.054126 .
factor(Conc)250:MediaM9	0.210650	0.030738	6.853	7.23e-12 ***
factor(Conc)300:MediaM9	0.009164	0.032670	0.280	0.779097

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Interpretation

- The rate here is colonies per petri dish (for some fixed amount of time).
- Intercept: the log of the expected count is 3.68 in LB media with no novobiocin. Equivalently, the log rate is 3.68 in LB media with no novobiocin.
- The estimated number of colonies and 95% CI for this baseline is $e^{3.68}$, $(e^{3.68-1.96*0.016}, e^{3.68+1.96(0.016)}) = 39.6$ (38.4, 41).
- The multiplicative effect of the interaction between M9 and 300 is $e^{0.009}$, i.e., the rate increases by 0.9% relative to no interaction, which is not significant ($p > 0.05$).
- More on M9 with 300: ratio of counts in M9 300 to counts in M9 with no novobiocin:
$$e^{3.68+0.92-0.71+0.009}/e^{3.68-0.71} = e^{0.92+0.009} = 2.5.$$

Example with overdispersion

For educational purposes, consider this poor fitting model:

```
> glm_nomedia_quasi = glm(Colony_numeric~factor(Conc),data=colonydata,  
family = "quasipoisson")  
> summary(glm_nomedia_quasi)
```

Call:

```
glm(formula = Colony_numeric ~ factor(Conc), family = "quasipoisson",  
data = colonydata)
```

Deviance Residuals:

	Min	1Q	Median	3Q	Max
	-5.6832	-2.8772	-0.2983	2.5557	6.2330

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	3.38805	0.03727	90.90	<2e-16 ***
factor(Conc)100	0.52177	0.04701	11.10	<2e-16 ***
factor(Conc)200	0.78726	0.04493	17.52	<2e-16 ***
factor(Conc)250	1.40432	0.04162	33.74	<2e-16 ***
factor(Conc)300	0.92691	0.04400	21.06	<2e-16 ***

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for quasipoisson family taken to be 8.185913)

Null deviance: 21484.0 on 996 degrees of freedom
Residual deviance: 8249.4 on 992 degrees of freedom
AIC: NA

GOF test

```
> glm_nomedia = glm(Colony_numeric~factor(Conc),data=colonydata,family = "poisson")
> summary(glm_nomedia)
```

Call:

```
glm(formula = Colony_numeric ~ factor(Conc), family = "poisson",
     data = colonydata)
```

...

(Dispersion parameter for poisson family taken to be 1)

```
Null deviance: 21484.0 on 996 degrees of freedom
Residual deviance: 8249.4 on 992 degrees of freedom
AIC: 14127
```

```
> # versus:
```

```
> with(glm_nomedia, cbind(res.deviance = deviance, df = df.residual, p = pchisq(deviance, df.residual)
  res.deviance df p
[1,]      8249.368 992 0
```