BIOS 526 Course Summary

1. Ordinary least squares (M1):
$$\text{Cov}\,\mathbf{Y} = \sigma^2\mathbf{I}$$
$$\hat{\boldsymbol{\beta}}^{OLS} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$$

2. Generalized least squares (M4, slide 16) is a formulation for any given covariance matrix:
$$\text{Cov}\,\mathbf{Y} = \boldsymbol{\Sigma}$$
$$\hat{\boldsymbol{\beta}}^{GLS} = (\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y}$$
   - Need to know $\boldsymbol{\Sigma}$.
   - In practice, we impose structure on $\boldsymbol{\Sigma}$ and estimate a statistical model, e.g., mixed models with Gaussian assumptions, GEE with Gaussian assumptions.

3. Linear mixed models (M2)
   - Random intercept:
$$\begin{aligned} y_{ij} =& \boldsymbol{x}_i'\boldsymbol{\beta} + \theta_i + \epsilon_{ij} \\ \theta_i &\overset{iid}{\sim} N(0, \tau^2) \\ \epsilon_{ij} &\overset{iid}{\sim} N(0, \sigma^2) \\ \epsilon_{ij} &\perp\!\!\!\perp \theta_i \end{aligned}$$
   - Also see random slopes and hierarchical models

4. Generalized linear models (M3 part I):
$$y_i \overset{ind}{\sim} P(Y_i|\boldsymbol{x}_i'\boldsymbol{\beta})$$
$$g(E(Y_i)) = \boldsymbol{x}_i'\boldsymbol{\beta}$$
   - logistic regression for 0, 1 response:

  - Model:

$$y_i \overset{ind}{\sim} Bernoulli(E[Y_i])$$
$$\log\left\{\frac{E[Y_i]}{1-E[Y_i]}\right\} = \boldsymbol{x}_i'\boldsymbol{\beta}$$

  - $\text{Var}\,(Y_i) = E(Y_i)\,\{1 - E(Y_i)\}$.

- Poisson regression for count data.
  - Model:

$$y_i \overset{ind}{\sim} Poisson(E[Y_i]),$$
$$\log\{E[Y_i]\} = \boldsymbol{x}_i'\boldsymbol{\beta}.$$

  - $\text{Var}\,(Y_i) = E(Y_i)$.
  - Watch out for overdispersion, i.e., $Var(Y_i) > E(Y_i)$.

5. Generalized linear mixed models (M2, M3 part II):

Random intercept model:

$$y_{ij} \sim P(Y_{ij}|\boldsymbol{x}_{ij}'\boldsymbol{\beta} + \theta_i)$$
$$g\,\{E(y_{ij}|\theta_i)\} = \boldsymbol{x}_{ij}'\boldsymbol{\beta} + \theta_i$$
$$\theta_i \overset{iid}{\sim} N(0, \tau^2)$$

- Handle repeated measurements / longitudinal / clustered data.
- For Gaussian, interpretation of $\boldsymbol{\beta}$ not impacted by conditional versus marginal (the estimates of $\boldsymbol{\beta}$ from LMM and GEE are different but usually similar, in some cases GEE with exchangeable correlation structure and random intercept LMM have equivalent $\hat{\boldsymbol{\beta}}$).
- For logistic, interpretation of $\boldsymbol{\beta}$ in GLMM (conditional model) is different from the interpretation in a GEE (the estimates of $\boldsymbol{\beta}$ from GLMM and GEE are different).
- For Poisson, the interpretation of $\beta_1, \ldots, \beta_p$ in GLMM (conditional model) is the same as in the GEE due to a special property of the log link. The intercept changes, as the intercept in the marginal models includes $\tau^2/2$; see the R code.
- Use mixed models if interested in subject-specific predictions (shrinkage towards population effects).
- Can use if no overdispersion in logistic or Poisson, no heteroscedasticity in Gaussian.

6. Generalized estimating equations (M4):

$$y_{ij} \sim P(Y_{ij}|\boldsymbol{x}'_{ij}\boldsymbol{\beta})$$

$$g(E[Y_{ij}]) = \boldsymbol{x}'_{ij}\boldsymbol{\beta}$$

$$\text{Cov}\,(\mathbf{Y}_i) = \mathbf{D}_i^{1/2}\mathbf{R}(\alpha)\mathbf{D}_i^{1/2}$$

   where $\mathbf{R}(\alpha)$ is the working correlation and $\mathbf{D}_i$ is a diagonal matrix with diagonal elements equal to the variance determined by the likelihood.

   - Handle repeated measurements / longitudinal / clustered data.
   - Use robust standard errors.
   - Use if heteroscedasticity and/or overdispersion (valid inference, unlike GLMM).
   - Marginal inference (no random effects).

7. Generalized additive models (M5):

$$g(E(Y_i)) = \beta_0 + s_1(x_{i1}) + \cdots + s_j(x_{ip})$$

   - Handle non-linear effects.
   - Can incorporate random effects for longitudinal / repeated measures / clustered data.
   - Can generalize interactions from linear models to bivariate splines, e.g., $s(x_{i1}, x_{i2})$, i.e., 2D surfaces.
   - Estimate $s(x_{ik})$ using either cross-validation or mixed model formulation of spline coefficients.

8. Bias-Variance Tradeoff (M5, part I, slides 33-43, M6, part II, slides 5-6)

   - $MSE(\hat{f}(x)) = Var(\hat{f}(x)) + Bias(\hat{f}(x))^2$
   - Fewer parameter: more bias, less variance
   - More parameters: less bias, more variance
   - Use cross-validation or generalized cross-validation to approximately minimize the MSE

9. Principal component analysis (M6 I): uses the singular value decomposition on standardized $N \times p$ data:

$$\mathbf{X}_{scaled} = \mathbf{UDV}'$$

   - Lower dimensional representation using first q left eigenvectors.
   - Principal component scores: $\mathbf{U}_{1:q}\mathbf{D}_{1:q}$,

- Can use in principal component regression when have issues with multicollinearity.

10. Ridge Regression (L2-norm regularization) (M5 part II, M6 part II):

$$\hat{\boldsymbol{\beta}}^{Ridge} = (\mathbf{X}'\mathbf{X} + \lambda\mathbf{I})^{-1}\mathbf{X}'\mathbf{Y}$$

- Regularization method with a nice closed form.
- Also extends to likelihoods (M6 part II):

$$\underset{\boldsymbol{\beta}}{\text{argmin}} \quad -\sum_{i=1}^{n} \ell(y_i; \boldsymbol{x}_i'\boldsymbol{\beta}) + \lambda||\boldsymbol{\beta}||_2^2.$$

- Can use when lots of covariates, $p > n$.
- Use for shrinking spline coefficients in GAMs (used in MGCV).

11. Lasso (L1-norm regularization) (M6 part II):

$$\hat{\boldsymbol{\beta}}^{Lasso} = \underset{\boldsymbol{\beta}}{\text{argmin}} \quad -\sum_{i=1}^{n} \ell(y_i; \boldsymbol{x}_i'\boldsymbol{\beta}) + \lambda||\boldsymbol{\beta}||_1$$

- Regularization that results in variable selection by setting many coefficients equal to 0.

12. Elastic net (L1-norm and L2-norm regularization) (M6 part II):

$$\hat{\boldsymbol{\beta}}^{ElNet} = \underset{\boldsymbol{\beta}}{\text{argmin}} \quad -\sum_{i=1}^{n} \ell(y_i; \boldsymbol{x}_i'\boldsymbol{\beta}) + \lambda\sum_{j=1}^{p}\left(\alpha|\beta_j| + \frac{(1-\alpha)}{2}\beta_j^2\right).$$

- A good choice when predictors are correlated.
- Use for variable selection.