# Lecture 19:  Term structures (Yield Curve) continued

Monday, April 3, 2023    2:34 PM

19-yield-curve

## QTM 385 Quantitative Finance

### Lecture 19: Term structures

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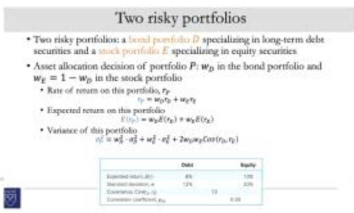
Suggested reading: Investments Ch 15

## Question from Google form

- Question: Is there a formula that exists to find the variance of a market portfolio with two different stocks in it? I think that I know what formula to use, but I am not seeing one in the slides. Thank you

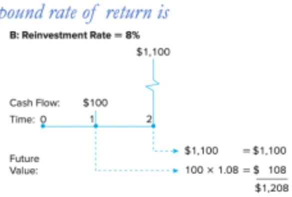
- *Answer: You can find it in slide 9 in lecture 9-diversification*




## Question from Google form

- Question: Can you specify more about the words on the PPT: "Coupons may not be reinvested to earn the bond's yield to maturity. Then realized compound return does not equal to YTM"?

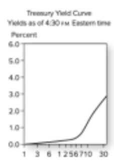
- *Answer: Suppose YTM is* $10\%$, *but the coupon payment after the first year can only be reinvested at rate* $8\%$. *Initial value of the investment is* $V_0 = 1000$. *Final value of the investment is* $V_2 = 1208$. *The realized compound rate of return is*
  - $r = \left(\frac{1208}{1000}\right)^{1/2} - 1 = 9.91\%$
  - *This is different from YTM* $(10\%)$




The takeaway here is that if the coupon payment cannot be reinvested at the same YTM rate, then we won't get as much out of it

## The yield curve and future interest rates

- The upward-sloping yield curve: short-term rates are going to be higher next year than they are now
- Spot rate: yield to maturity on zero-coupon bonds, meaning the rate that prevails today for a time period corresponding to the zero's maturity
- Short rate: refers to the interest rate for a given time interval (e.g., one year) available at different points in time



B. (December 2012)
Rising Yield Curve

Bonds are selling almost at par. They may have different time to maturity. Plot what is their corresponding yield  to maturity. Then we have different shapes of the YTM curves and we can see different expectations for the future

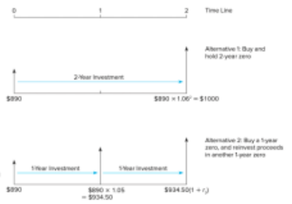
Spot rate can be used as building blocks to calculate's people expected interest rate for a given time period (I.e. short rate)

Currently: experiencing  high interest rate, expecting iinterest rate to fall
We will not see such a shape  today <<<-----

Instead, it will be a downward  shape because peoplees expectations of the future interest rate is lower

# Spot rate and short rate

- Consider two 2-year investment strategies with equal rate of returns
  - buying and holding a 2-year zero-coupon bond
  - buying a 1-year zero and rolling over the proceeds into a 1-year bond
- 2-year spot rate is an "average" of today's and next year's short rates
  - This year's short rate $r_1$. Next year's short rate $r_2$. Two year's spot rate $y_2$
  - $(1+y_2)^2 = (1+r_1)(1+r_2)$
  - Then $r_2 = \frac{(1+y_2)^2}{1+r_1} - 1 = 7.01\%$




For them to be indifferent between the two investors, they need to have the same initial value which is 890 and the same final value to be indifferent between investors (1000)

Short rate in year one is identical to the spot rate for year zero
But then the short rate for next year (interest rate between year 1 and year 2) needs to be calculated from spot rate of year 1 and year 0?????

# Spot rate and short rate

- Consider two 3-year investment strategies with equal rate of returns
  - buying and holding a 3-year zero-coupon bond
  - buying a 2-year zero and rolling over the proceeds into a 1-year bond
- Finding the short rate $r_3$ in year 3
  - $(1+y_3)^3 = (1+y_2)^2(1+r_3)$
  - Then $r_3 = \frac{(1+y_3)^3}{(1+y_2)^2} - 1 = \frac{1.07^3}{1.06^2} - 1 = 9.025\%$

**Table 15.1**

Prices and yields to maturity on zero-coupon bonds ($1,000 face value)

| Maturity (years) | Yield to Maturity (%) | Price |
|---|---|---|
| 1 | 5% | $952.38 = $1,000/1.05 |
| 2 | 6 | $890.00 = $1,000/1.06$^2$ |
| 3 | 7 | $816.30 = $1,000/1.07$^3$ |
| 4 | 8 | $735.03 = $1,000/1.08$^4$ |

We can consider two investent strategies
The first is we earn the spot rate of the 3-year zero-coupon bond, which is curretnly seeling at 816.30 with par to be 1000. **The YTM is 7%. This is also equall to the spot rate of y3.** We can use these terms interchangeably **because this is a zero-coupon bond.**

Spot rate y2 = 6%

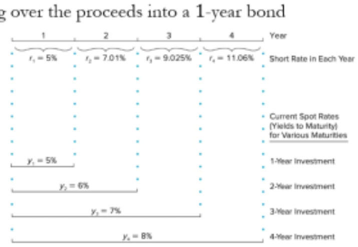
The interest rate from year 2 to year 3 is the short rate from year 2 to year 3. This is denoted as r3.

How can we use the spot rate to get the short rate??
<<--------

R3 is the interest rate we would earn from year 2 to year 3

# Forward rate

- Consider two $n$-year investment strategies with equal rate of returns
  - buying and holding an $n$ -year zero-coupon bond
  - buying an $(n-1)$-year zero and rolling over the proceeds into a 1-year bond
  - $(1+y_n)^n = (1+y_{n-1})^{n-1}(1+r_n)$
  - $r_n = \frac{(1+y_n)^n}{(1+y_{n-1})^{n-1}} - 1$
  - We also denote $r_n$ as $f_n$ to refer to **forward interest rate**




You can use the formula from the previous slide for any n year zero coupon bond.

We can also calculate for multiple years.
I.e. from year 1 to year 3

# Interest rate uncertainty and forward rates

- Under certainty, $(1+r_1)(1+r_2) = (1+y_2)^2$
  - $r_1$: short rate this year
  - $r_2$: short rate next year
  - $y_2$: spot rate/YTM for a two-year zero-coupon bond
- What if $r_2$ is uncertain?
- There are three cases

So the above formulas were only for CERTAIN rates
In a DETERMINISTIC WORLD

R2 is the random variable (interest rate, short rate from year1 to year2). It is uncerrtain.

But we can use 1 year zero and 2 year zero to provide us with some info.
Based on their price data, we can get investors expectations of thet interest rate from year 1 to year 2
We use the price data to calculate the number (can be higher or lower than investors expectations of the interest rate form year 1 to year 2)
Consider 1 year zero and 2 year zero. Calculate the YTM for each. These are DETERMINISTIC NUMBERS because they're based on the price data now. So y1 and y2 are deterministiic numbers.
Using these two numbers, we can use the f2 formula to calculate the interest rate from year 1 to year 2

Thhe forward rate is a deterministic number because y1 and y2 are deterministic
This forward rate intuitively should connect with randnom interest rate from year 1 to year 2
FRom now on, we will talk about what will be the connection between forward rate and interest rate from year one to year two

What if we're not reinvesting at the interest rate we expect?

In a special case where r2 iis certain, thhen f2 should be equal to r2.

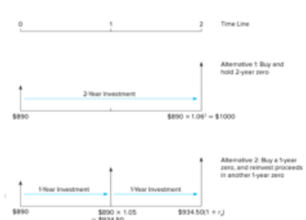
In the case where r2 is a random variaable, we would like to know the connectiono between f2 (what we calculated) and the expected interest rate (r2).
We assign the right hand side as f2. And we are interseted in how does the expected intnerest rate relate to what we calculate using such a formula?

# Case 1: Only expected short rate matters

- Suppose $r_1 = 5\%$ and $E[r_2] = 6\%$
- The yield to maturity on a 2-year zero is

$$(1+y_2)^2 = (1+r_1)(1+E[r_2]) = 1.05 \times 1.06$$

Go back to the example where we just have a two year zero investment horizon.



- The yield to maturity on a 2-year zero is

$$(1 + y_2)^2 = (1 + r_1)(1 + E[r_2]) = 1.05 \times 1.06$$


Investors only care about expected return E[r2] = 7.01%
If the expected rate is 7.01% then the two returns are equivalent to one another

$$r_2 = \frac{(1+y_2)^2}{1+r_1} - 1$$

First case: f2 = E[r2]

## Case 2: Short-term investor

- Consider a short-term investor who wishes to invest only for one year
- Two possible strategies
  - Strategy 1: Purchase the 1-year zero for $\$1{,}000/1.05$
  - Strategy 2: Purchase the 2-year zero and sell it at the end of year $1$. The expected rate of return in year $1$ is $5\%$. The expected rate of return in year $2$ is $6\%$ with price $\$943.40$
- However, the rate of return on the $2$-year bond is risky. The price can higher or lower than $\$943.40$

Second case: f2 > E[r2]

## Case 2: Short-term investor

- To compensate the risk in the end price of year $1$, the expected short rate $E[r_2]$ should be **less than** the forward rate $f_2$
- *Why?*
  - The short-term investors would prefer strategy 1, if the $2$-year zero sells at $\frac{\$943.40}{1.05} = \$898.47$
  - Suppose short-term investors are indifferent only if $2$-year zero sells at $\$881.47$
  - At this price, the HPR for the $2$-year zero is $\frac{\$943.40}{\$881.47} - 1 = 1.07 - 1 = 7\% > 5\%$
  - The YTM $y_2$ for the $2$-year zero satisfies $(1 + y_2)^2 = \frac{\$1000}{\$881.47} = 1.1344$
  - The forward rate $f_2$ satisfies $1 + f_2 = \frac{(1+y_2)^2}{1+y_1} = \frac{1.1344}{1.05} = 1.08$
  - The expected short rate $E[r_2] = \frac{\$1000}{\$943.40} - 1 = 6\% < f_2$

Only if the 2 y zero is curretnly selling at a price lower than 890 will the investors choose the 2 year zero? Or something

881.47 < 890 or something

## Liquidity premium

- The liquidity premium is $f_2 - E[r_2]$: compensates short-term investors for the uncertainty about the price at which they will be able to sell their long-term bonds at the end of the year

In this case, f2 needs to be higher than E[r2] for the short term investors to be indifferent

The liquidity premium shows how much the investor prefers the short term over the long term investment

## Case 3: Long-term investor

- Consider a long-term investor who wishes to invest for a full $2$-year period
- Two strategies:
  - Strategy 1: Purchase the 2-year zero for $\$890$ and lock in a guaranteed yield to maturity of $6\%$
  - Strategy 2: Roll over two 1-year investments. In this case, an investment of $\$890$ will grow in two years to $890 \times 1.05 \times (1 + r_2)$
- However, the rate of return in year 2 is uncertain

Third case: f2 < E[r2]

maturity of 6%

- Strategy 2: Roll over two 1-year investments. In this case, an investment of $890 will grow in two years to $890 \times 1.05 \times (1 + r_2)$

- However, the rate of return in year 2 is uncertain

We need to find where the two strategies are indifferent. We will only choose strategy 2 if E[r2] > f2 in this case

- To compensate the risk of interest rate in year 2, $E[r_2]$ exceeds $f_2$

## $E[r_2]$ vs $f_2$

- The relationship between $E[r_2]$ and $f_2$ depends on
  - Investors' readiness to bear interest rate risk
  - Investors' willingness to hold bonds that do not correspond to their investment horizons

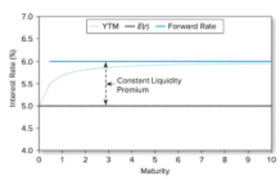
## Theories of the term structure

- **Theory 1**: The expectations hypothesis
  - The forward rate equals the market consensus expectation of the future short interest rate
  - $E[r_2] = f_2$ and liquidity premium is 0
  - An upward-sloping yield curve: investors anticipate increases in interest rates

## Theories of the term structure

- **Theory 2**: Liquidity preference theory
  - Short-term investors dominate the market so that the forward rate will generally exceed the expected short rate
  - $f_2$ exceeds $E[r_2]$ and liquidity premium is positive

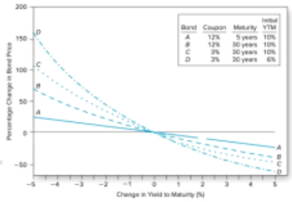
A



# Interest rate sensitivity

- **Interest rate sensitivity**: The sensitivity of bond prices to changes in market interest rates is obviously of great concern to investors
- **Property 1**: Bond *prices* and *yields* are *inversely related*: As yields increase, bond prices fall; as yields fall, bond prices rise
- **Property 2**: An *increase* in a bond's *yield* to maturity results in a *smaller* price change than a *decrease* in *yield* of equal magnitude

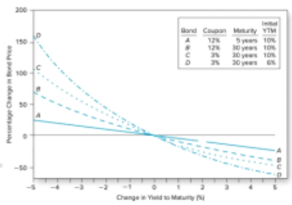
| Bond A | 12% coupon, semi-annual bond | Bond A | changes in YTM | percentage changes in bond pri |
|---|---|---|---|---|
| | 5 year maturity bond | | | |
| | 10% YTM | | | |
| Settlement date | 11/15/20 | | -0.05 | 0.212682137 |
| Maturity date | 11/15/25 | | -0.045 | 0.188991689 |
| Annual coupon date | 0.12 | | -0.04 | 0.165879928 |
| Yield to maturity | 0.1 | | -0.035 | 0.143330884 |
| Redemption value (% of face value) | 100 | | -0.03 | 0.121329074 |
| Coupon payments per year | 2 | | -0.025 | 0.09985948 |
| | | | -0.02 | 0.078907536 |
| = PRICE(settlement date, maturity date, annual coupon rate, yield tc | | | -0.015 | 0.058459115 |
| Flat price (% of par) | 107.7217 | | -0.01 | 0.038500516 |
| | | | -0.005 | 0.019018443 |
| | | | 0 | 0 |
| | | | 0.005 | -0.018567326 |
| | | | 0.01 | -0.036695677 |
| | | | 0.015 | -0.054396836 |
| | | | 0.02 | -0.071682237 |




# Interest rate sensitivity

- **Property 3**: Prices of *long-term* bonds tend to be *more sensitive* to interest rate changes than prices of *short-term* bonds
  - Bond B (longer maturity) exhibits greater sensitivity to interest rate changes than bond A (shorter maturity)
  - Intuition: If rate increases, the bond price decreases; the impact is higher for more distant cash flows

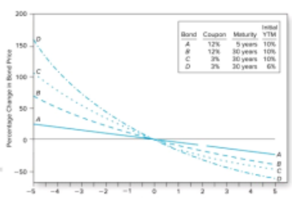
| Bond A | changes in YTM | percentage changes in bond pri | Bond B | changes in YTM | percentage changes in bond pri |
|---|---|---|---|---|---|
| | -0.05 | 0.212682137 | | -0.05 | 0.750454396 |
| | -0.045 | 0.188991689 | | -0.045 | 0.639408192 |
| | -0.04 | 0.165879928 | | -0.04 | 0.538953875 |
| | -0.035 | 0.143330884 | | -0.035 | 0.447898623 |
| | -0.03 | 0.121329074 | | -0.03 | 0.365196378 |
| | -0.025 | 0.09985948 | | -0.025 | 0.289928759 |
| | -0.02 | 0.078907536 | | -0.02 | 0.221288553 |
| | -0.015 | 0.058459115 | | -0.015 | 0.15856543 |
| | -0.01 | 0.038500516 | | -0.01 | 0.101133569 |
| | -0.005 | 0.019018443 | | -0.005 | 0.04844094 |
| | 0 | 0 | | 0 | 0 |
| | 0.005 | -0.018567326 | | 0.005 | -0.04462038 |
| | 0.01 | -0.036695677 | | 0.01 | -0.085801929 |
| | 0.015 | -0.054396836 | | 0.015 | -0.123883071 |
| | 0.02 | -0.071682237 | | 0.02 | -0.159164236 |




# Interest rate sensitivity

- **Property 4**: The sensitivity of bond prices to changes in yields increases at a decreasing rate as maturity increases. In other words, interest rate risk is *less than proportional* to bond maturity
  - Bond B has six times the maturity of bond A
  - Bond B has less than six times the interest rate sensitivity

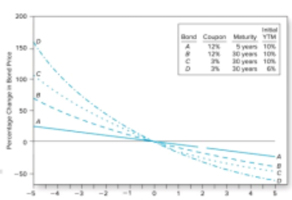
| Bond A | changes in YTM | percentage changes in bond pri | Bond B | changes in YTM | percentage changes in bond pri |
|---|---|---|---|---|---|
| | -0.05 | 0.212682137 | | -0.05 | 0.750454396 |
| | -0.045 | 0.188991689 | | -0.045 | 0.639408192 |
| | -0.04 | 0.165879928 | | -0.04 | 0.538953875 |
| | -0.035 | 0.143330884 | | -0.035 | 0.447898623 |
| | -0.03 | 0.121329074 | | -0.03 | 0.365196378 |
| | -0.025 | 0.09985948 | | -0.025 | 0.289928759 |
| | -0.02 | 0.078907536 | | -0.02 | 0.221288553 |
| | -0.015 | 0.058459115 | | -0.015 | 0.15856543 |
| | -0.01 | 0.038500516 | | -0.01 | 0.101133569 |
| | -0.005 | 0.019018443 | | -0.005 | 0.04844094 |
| | 0 | 0 | | 0 | 0 |
| | 0.005 | -0.018567326 | | 0.005 | -0.04462038 |
| | 0.01 | -0.036695677 | | 0.01 | -0.085801929 |
| | 0.015 | -0.054396836 | | 0.015 | -0.123883071 |
| | 0.02 | -0.071682237 | | 0.02 | -0.159164236 |




# Interest rate sensitivity

- **Property 5**: Interest rate risk is inversely related to the bond's coupon rate. Prices of *low-coupon* bonds are *more sensitive* to changes in interest rates than prices of high-coupon bonds
  - Bond B has a higher coupon rate than bond C
  - Bond C is more sensitive to changes in interest rate
  - Intuition: price of low coupon bond depends more on distant cash flows (that are more sensitive to interest rate changes)

| Bond B | changes in YTM | percentage changes in bond pri | Bond C | changes in YTM | percentage changes in bond pri |
|---|---|---|---|---|---|
| | -0.05 | 0.750454396 | | -0.05 | 1.04730338 |
| | -0.045 | 0.639408192 | | -0.045 | 0.880782208 |
| | -0.04 | 0.538953875 | | -0.04 | 0.733066871 |
| | -0.035 | 0.447898623 | | -0.035 | 0.601781555 |
| | -0.03 | 0.365196378 | | -0.03 | 0.484867074 |
| | -0.025 | 0.289928759 | | -0.025 | 0.380538041 |
| | -0.02 | 0.221288553 | | -0.02 | 0.287244751 |
| | -0.015 | 0.15856543 | | -0.015 | 0.203640966 |
| | -0.01 | 0.101133569 | | -0.01 | 0.128556158 |
| | -0.005 | 0.04844094 | | -0.005 | 0.060971633 |
| | 0 | 0 | | 0 | 0 |
| | 0.005 | -0.04462038 | | 0.005 | -0.055132508 |
| | 0.01 | -0.085801929 | | 0.01 | -0.105101258 |
| | 0.015 | -0.123883071 | | 0.015 | -0.150496341 |
| | 0.02 | -0.159164236 | | 0.02 | -0.191833881 |



# Interest rate sensitivity

- **Property 6**: The sensitivity of a bond's price to a change in its yield is *inversely related* to the *yield to maturity* at which the bond currently is selling
  - Bond C has a higher yield to maturity than bond D
  - Bond C is less sensitive to changes in yields
  - Intuition: Higher yield reduces the present value of all bond's payments, more so for more-distant payments. At a higher yield, a higher proportion of the bond's value is due to its earlier payments

| Bond C | changes in YTM | percentage changes in bond pri | Bond D | changes in YTM | percentage changes in bond pri |
|---|---|---|---|---|---|
| | | | | | |
| | | | | | |
| | -0.05 | 1.04730338 | | -0.05 | 1.594191137 |
| | -0.045 | 0.880782208 | | -0.045 | 1.32754004 |
| | -0.04 | 0.733066971 | | -0.04 | 1.094110533 |
| | -0.035 | 0.601781555 | | -0.035 | 0.889467758 |
| | -0.03 | 0.484867074 | | -0.03 | 0.709791761 |
| | -0.025 | 0.380538041 | | -0.025 | 0.551789966 |
| | -0.02 | 0.287244751 | | -0.02 | 0.412622373 |
| | -0.015 | 0.203640966 | | -0.015 | 0.289837588 |
| | -0.01 | 0.128556158 | | -0.01 | 0.181318099 |
| | -0.005 | 0.060971633 | | -0.005 | 0.085233427 |
| | 0 | 0 | | 0 | 0 |
| | 0.005 | -0.055132508 | | 0.005 | -0.075753227 |
| | 0.01 | -0.105101258 | | 0.01 | -0.143214256 |
| | 0.015 | -0.150496341 | | 0.015 | -0.203413334 |
| | 0.02 | -0.191833881 | | 0.02 | -0.257244658 |