Final Review

Monday, April 24, 2023 14:41

25-final-review

QTM 385 Quantitative Finance

Lecture 25: Final review

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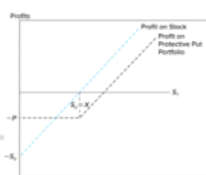
Question from Google form

- For the put-call parity relationship, we learned that the payoff, i.e. the profit or loss, are the same. From previous page we also learned to calculate the portfolio value in the situation that the stock price is 97 and 104. I'm just wondering is that how we calculate the payoff we discussed? If so, when we want to know the payoff, how do we do if we just have information about the call-plus-bill portfolio?
- *Answer: Call-plus-bill portfolio has a risk-free zero-coupon bond with face value X and hold a call with exercise price X*
- *If $S_T \le X$, then payoff is X; if $S_T > X$, then payoff is S_T*
- When $X = \$100$, *if* $S_T = \$97$, *then payoff is* $\$100$;

If $S_T = \$104$, *then payoff is* $\$104$. *These are just special cases*

	$S_T \le X$	$S_T > X$
Value of call option	0	$S_T - X$
Value of zero-coupon bond	X	X
Total	X	S_T



Question from Google form

- Since we have almost finished our semester, I'm really interested in the following discussion question: How can the development of ChatGPT and other language models be utilized in quantitative finance for tasks such as risk assessment, market prediction, and portfolio optimization?

- *Answer: Tasks in quantitative finance usually require sophisticated math models, which are different from large language models. But large language models can better analyze text (e.g., sentiments of news), which can be useful to improve the inputs of the math models.*

Announcement

- Solution for homework 3 and 4 are posted
- The final will be available from 04/30 12:00 am until 05/30 11:59pm at Quizzes on Canvas
- You can choose any 24h in between to finish it
- Once you decide to take it, you can open the quiz and the time starts to count
- Once you finish (within 24h), upload your solution (one pdf file) and click submit quiz on Canvas

Final exam

- Cover the material from Lectures 12-24
- Similar problems as those in Homework 3 and 4
- You need to finish it independently
- Open book, open notes
- You cannot talk to anyone about the exam until 05/03 11:59pm

CAPM
APT
options
etc.

Make sure you know how to solve homework

Xiong out of town & TH/B, Fri

There will be 4 questions in total
Points per question are higher
Won't be asked a lot of figures

CAPM

- Capital asset pricing model (CAPM): a prediction of the relationship between the risk of an asset and its expected return
- Security market line (SML): The security's risk premium is proportional to both beta and risk premium of the market portfolio, i.e.,

$$\underbrace{E[r_i] - r_f}_{\text{excess return}} = \beta\underbrace{[E(r_M) - r_f]}_{\text{market excess return}}$$

connects the two. Covariance between excess return & market excess return

β↑ then excess return ↑

- If an asset's return
  - on SML, fairly priced
  - above SML, underpriced
  - below SML, overpriced

(σ)
if risk ↑ excess return ↑
If β ↑ then volatility ↑ then excess return ↑

HW 3 Problem 1

Problem 1 (CAPM)

In CAPM, suppose the risk free rate is $r_f = 6\%$ and the expected return of market portfolio is $E(r_M) = 14\%$.

(a) What must be the β of a portfolio with $E(r_P) = 18\%$?

Suggested solution.

We can solve β from

$$E(r_P) = r_f + \beta(E(r_M) - r_f).$$

Therefore β is

$$\beta = \frac{E(r_P) - r_f}{E(r_M) - r_f} = \frac{18 - 6}{14 - 6} = 1.5$$

(b) What must be the return of a portfolio with $\beta = 0.5$?

Suggested solution.

The return of the portfolio with $\beta = 0.5$ is

$$E(r_P) = r_f + \beta(E(r_M) - r_f) = 6\% + 0.5(14\% - 6\%) = 10\%.$$

(c) If a portfolio has $\beta = 1$, $E(r_P) = 10\%$, and standard deviation of returns $\sigma_P = 10\%$. Is this portfolio underpriced, overpriced, or properly priced?

Suggested solution.

If the portfolio has $\beta = 1$, then the average return based on CAPM should be

$$r_f + \beta(E(r_M) - r_f) = 6\% + 1 \cdot (14\% - 6\%) = 14\%,$$

which is higher than $E(r_P) = 10\%$. Therefore this portfolio is overpriced (as this stock is plotted below the security market line).

Multifactor APT

more general than CAPM, less assumptions

- Consider a three-factor model for an asset i

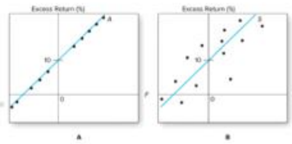
$$R_i = E(R_i) + \overbrace{\beta_{i1}F_1 + \beta_{i2}F_2 + \beta_{i3}F_3}^{\text{systematic}} + \overbrace{e_i}^{\text{idiosyncratic}}$$

  - R_i: excess return of asset i → E.g. market factors or interest factors
  - e_i: idiosyncratic risk of asset i
    ↳ risk specific to firm

- If well-diversified portfolio P has the same betas as i (i.e., $\beta_{P1} = \beta_{i1}$, $\beta_{P2} = \beta_{i2}$ and $\beta_{P3} = \beta_{i3}$), but diversifies the risk

$$R_P = E(R_P) + \beta_{P1}F_1 + \beta_{P2}F_2 + \beta_{P3}F_3$$

  - R_P: excess return of portfolio P
  - $E(R_P)$: risk premium of portfolio P
  - No idiosyncratic risk



E.g. suppose we have 1 factor and 2 portfolios

$R_{P_1} = E[R_{P_1}] + \beta F_1$

$R_{P_2} = E[R_{P_2}] + \beta F_1$

loan the first & short the second to get return of $E[R_{P_1}] - E[R_{P_2}]$??

Arbitrage opportunity ??

To push the two returns to be identical?

Using "no arbitrage" assumptions, same β → same risk premium??

Multifactor APT

- The risk premium of P is

$$E(R_P) = E(r_P) - r_f = \beta_{P1} \cdot \underbrace{(E(r_1) - r_f)}_{\text{risk premium of first factor}} + \beta_{P2} \cdot \underbrace{(E(r_2) - r_f)}_{\text{risk premium of second factor}} + \beta_{P3} \cdot \underbrace{(E(r_3) - r_f)}_{\text{risk premium of third factor}}$$

this is why it deviates from the mean

No arbitrage opportunity ??

- Fama-French three factor model is

$$E(r_P) - r_f = \beta_{P,M}(E(r_M) - r_f) + \beta_{P,SMB} \cdot (E(r_{SMB}) - r_f) + \beta_{P,HML} \cdot (E(r_{HML}) - r_f)$$

  - Market factor: market excess return
  - High-minus-low factor: the outperformance of high book/market versus low book/market companies
  - Small-minus-big factor: the outperformance of small versus big companies

HW 3 Problem 3

Problem 3 (APT factors)

The returns of two stocks are believed to satisfy the two-factor model

$$r_1 = E(r_1) + 2f_1 + 2f_2 + \varepsilon_1$$

$$r_2 = E(r_2) + 1f_1 + 4f_2 + \varepsilon_2$$

Assume that the factors are traded assets and the risk-free rate is also priced by the factor model. In addition, there is a risk-free asset with a rate of return of 4%. It is known that the expected return for asset 1 is $E[r_1] = 12\%$ and for asset 2 is $E[r_2] = 17\%$.

What are the values of r_f, risk premium of the first factor $E(f_1 - r_f)$, and risk premium of the second factor $E(f_2 - r_f)$ for this model?

Suggested solution.

The value of r_f is $r_f = 4\%$.

Based on the returns of two stocks, we have

$$E(r_1) - r_f = 2E(f_1 - r_f) + 2E(f_2 - r_f) = 8\%$$

and

$$E(r_2) - r_f = 1E(f_1 - r_f) + 4E(f_2 - r_f) = 13\%.$$

Use ful to calculate risk premium

We can solve the risk premium of the two factors by solving the system of equations.

The risk premium of the first factor is

$$E(f_1 - r_f) = \frac{1}{3}\left[2 \times 8\% - 13\%\right] = 1\%$$

and the risk premium of the second factor is

$$E(f_2 - r_f) = \frac{1}{6}\left[2 \times 13\% - 8\%\right] = 3\%.$$

```
(2 * 13 - 8)/6
[1] 3
```

HW 3 Problem 4

Problem 4 (Single-Index Model)

Assume that security returns are generated by the single-index model,

$$R_i = \alpha_i + \beta_i R_M + e_i$$

where R_i is the excess return for security i and R_M is the market's excess return. The risk-free rate is 2%. Suppose also that there are three securities, A, B, and C, characterized by the following data:

Security	β_i	$E(R_i)$	$\sigma(e_i)$
A	0.8	10%	25%
B	1.0	10%	20%
C	1.2	14%	20%

(a) If $\sigma_M = 20\%$, calculate the variance of returns of securities A, B, and C.

Suggested solution.

The variance of returns of security A is

$$\sigma_A^2 = \overbrace{\beta_A^2\sigma_M^2}^{\text{systematic}} + \overbrace{\sigma^2(e_A)}^{\text{idiosyncratic}} = 0.8^2 \times 0.2^2 + 0.25^2 = 0.0881$$

The variance of returns of security B is

$$\sigma_B^2 = \beta_B^2\sigma_M^2 + \sigma^2(e_B) = 1.0^2 \times 0.2^2 + 0.2^2 = 0.08$$

The variance of returns of security C is

$$\sigma_C^2 = \beta_C^2\sigma_M^2 + \sigma^2(e_C) = 1.2^2 \times 0.2^2 + 0.2^2 = 0.0976$$

HW 3 Problem 4

(b) Now assume that there are an infinite number of assets with return characteristics identical to those of A, B, and C, respectively. If one forms a well-diversified portfolio of type A securities, what will be the mean and variance of the portfolio's excess returns? What about portfolios composed only of type B or C stocks?

Suggested solution.

The mean of the excess return of the portfolio of type A securities is

$$E(r_A) - r_f = 0.1 - 0.02 = 0.08$$

The variance of the portfolio is

$$\beta_A^2\sigma_M^2 = 0.0256$$

The mean of the excess return of the portfolio of type B securities is

$$E(r_B) - r_f = 0.1 - 0.02 = 0.08$$

The variance of the portfolio is

$$\beta_B^2\sigma_M^2 = 0.04$$

The mean of the excess return of the portfolio of type C securities is

$$E(r_C) - r_f = 0.14 - 0.02 = 0.12$$

The variance of the portfolio is

$$\beta_C^2\sigma_M^2 = 0.0576$$

HW 3 Problem 4

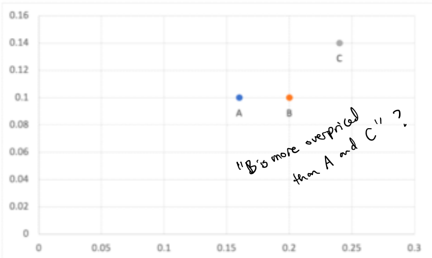
(c) Is there an arbitrage opportunity in this market? What is it? Analyze the opportunity graphically.

Suggested solution.

Note that the Sharpe ratio for the portfolio with type A's securities is $\frac{0.08}{0.16} = 0.5$. The Sharpe ratio for the portfolio with type B's securities is $\frac{0.08}{0.2} = 0.4$. The Sharpe ratio for the portfolio with type B's securities is $\frac{0.12}{0.24} = 0.5$.

Suppose the single-index model holds. An arbitrage opportunity is to invest 50% in the portfolio with type A's securities, and another 50% in the portfolio with type C's securities, and short sell the portfolio with type B's securities, and we can earn 2% return in all scenarios.

```
knitr::include_graphics("Problem4-c.png")
```



"B's more overpriced than A and C" ?

3 dots NOT on the same line

If CAPM holds, 3 dots should be on same line

(we should be able to connect the dots w/ a straight line)

Bond price and yield to maturity

- Given *interest rate r, coupon rate, par value and time to maturity T*, present value of the bond equals to

$$\text{Bond value} = \sum_{t=1}^{T} \frac{\text{Coupon}}{(1+r)^t} + \frac{\text{Par value}}{(1+r)^T}$$

- Yield to maturity is the interest rate r that makes the present value of a bond's payments equal to its price. Given *bond price, coupon rate, par value and time to maturity T*, YTM is solved from

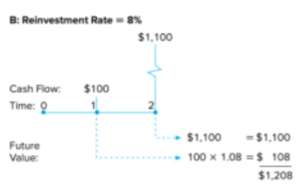
$$\text{Bond value} = \sum_{t=1}^{T} \frac{\text{Coupon}}{(1+r)^t} + \frac{\text{Par value}}{(1+r)^T}$$

Realized compound return

- Suppose the initial value of investment is V_0 and final value is V_T after T periods. The realized compound return is the r such that

$$V_0(1+r)^T = V_T$$

  - We first need to calculate V_T
  - Then we use V_0 and V_T to compute r
  - Coupon may be reinvested at a different rate than YTM



HW 4 Problem 1

Problem 1 (Bond pricing, YTM and realized compound yield)

Consider an 8% coupon bond selling for $950 with par value $1,000 and three years until maturity making annual coupon payments. The interest rates in the next three years will be, with certainty, $r_1 = 8\%$, $r_2 = 10\%$, and $r_3 = 12\%$. Calculate the bond's

(a) yield to maturity

Suggested solution.

The yield to maturity is 10.01%.

```
knitr::include_graphics("problem1-a.png")
```

Settlement date	1/1/22
Maturity date	1/1/25
Annual coupon rate	0.08
Flat price (% of par)	95
Redemption value (% of face value)	100
Coupon payments per year	1
Yield to maturity (decimal)	0.1001

(b) realized compound yield

Suggested solution.

The investor gets $80 at the end of year 1

As the reinvested rate is 10% in year 2 and 12% in year 3, the final value of the coupon paid at the end of year 1 is

$$80 \times (1 + 0.1) \times (1 + 0.12) = 98.56$$

the final value of the coupon paid at the end of year 2 is

$$80 \times (1 + 0.12) = 89.6$$

Therefore $V_0 = 950$ and

$$V_3 = 98.56 + 96 + 1080 = 1274.56$$

The realized compound yield is

$$r = \left(\frac{V_3}{V_0}\right)^{1/3} - 1 = 10.11\%$$

HW 4 Problem 1

(c) yield to maturity using bisection method

Suggested solution.

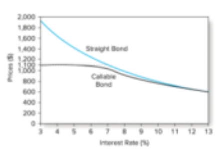
The yield to maturity solved by bisection method is also 10.01%.

```
present_value <- function(par_value, coupon_rate, r, year) {
    value <- 0
    for (t in c(1:year)) {
        value <- value + par_value * coupon_rate/((1 + r)^t)
    }
    value <- value + par_value/((1 + r)^year)
    return(value)
}
bisection <- function(par_value, coupon_rate, current_price,
    year, thres = 1e-12, low = 1e-08, high = 1) {
    while (high - low > thres) {
        mid <- (low + high)/2
        mid_pv <- present_value(par_value, coupon_rate, mid,
            year) - current_price
        # print(c(mid, mid_pv))
        if (mid_pv > 0) {
            low <- mid
        } else {
            high <- mid
        }
    }
    r <- (low + high)/2
    return(r)
}
par_value <- 1000
coupon_rate <- 0.08
current_price <- 950
year <- 3
bisection(par_value, coupon_rate, current_price, year)
[1] 0.1001098
```

Yield to call

- For a callable bond, yield to call is the interest rate r that makes the present value of a callable bond's payments equal to its price if the bond is called on the call date. Given *bond price, coupon rate, par value and time to call* n, YTC is solved from

$$Bond\ value = \sum_{t=1}^{n} \frac{Coupon}{(1+r)^t} + \frac{Par\ value}{(1+r)^n}$$



HW 4 Problem 2

Problem 2 (Yield to call) A 30-year maturity, 8% coupon bond paying coupons semiannually is callable in five years at a call price of $1,100. The bond currently sells at a yield to maturity of 7% (3.5% per half-year).

(a) What is the yield to call?

Suggested solution.

YTC is lower than YTM and the bond will be called once it is callable

The yield to call is 3.37% (semiannually).

(b) What is the yield to call if the call price is only $1,050?

Suggested solution.

The yield to call is 2.98% (semiannually).

(c) What is the yield to call if the call price is $1,100 but the bond can be called in two years instead of five years?

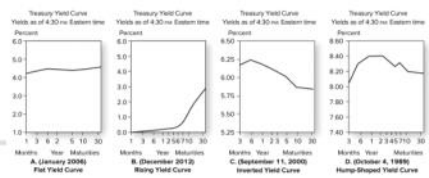
Suggested solution.

The yield to call is 3.03% (semiannually).

Settlement date	11/2/22	Settlement date	11/2/22	Settlement date	11/2/22
Maturity date	11/2/52	Maturity date	11/2/52	Maturity date	11/2/52
Annual coupon rate	0.08	Annual coupon rate	0.08	Annual coupon rate	0.08
Yield to maturity	0.07	Yield to maturity	0.07	Yield to maturity	0.07
Redemption value (% of face value)	100	Redemption value (% of face value)	100	Redemption value (% of face value)	100
Coupon payments per year	2	Coupon payments per year	2	Coupon payments per year	2
Flat price (% of par)	112.4724	Flat price (% of par)	112.4724	Flat price (% of par)	112.4724
	Yield to call		Yield to call		Yield to call
Settlement date	11/2/22	Settlement date	11/2/22	Settlement date	11/2/22
Maturity/Call date	11/2/27	Maturity/Call date	11/2/27	Maturity/Call date	11/2/24
Coupon rate	0.08	Coupon rate	0.08	Coupon rate	0.08
Price	112.4724	Price	112.4724	Price	112.4724
Redemption value (% of face value)	110	Redemption value (% of face value)	105	Redemption value (% of face value)	110
Coupon payments per year	2	Coupon payments per year	2	Coupon payments per year	2
Annual yield	0.067358094	Annual yield	0.059525174	Annual yield	0.060624897
Yield to call	0.033679047	Yield to call	0.029762587	Yield to call	0.030312449

Term structure

- *Term structure of interest rates*, or *yield curve*, refers to the market interest rates (i.e. spot rates) on bonds with different lengths of time to maturity, but with the same or similar risk (i.e. with the same credit rating)
  - **Spot rate**: yield to maturity on zero-coupon bonds
    - y_n: Spot rate/YTM for an n year zero coupon bond calculated from current value and —?
  - **Short rate**: refers to the interest rate for a given time interval → random variable
    - r_n: When interest rate is uncertain, r_n is a random variable and unknown today
  - **Forward rate**: future interest rate calculated from today's data pricing
    - $f_n = \frac{(1+y_n)^n}{(1+y_{n-1})^{n-1}} - 1$ YTM of zero-coupon bond
  - **Liquidity premium**: $\underset{\text{forward rate}}{f_2} - \underset{\text{expected}}{E}[r_2]$ short rate
    - If zero, follows expectation theory
    - If positive, market is dominated by short-term investors



HW 4 Problem 3

Problem 3 (Term structure) The yield to maturity on 1-year zero-coupon bonds is currently 7%; the YTM on 2-year zeros is 8%. The Treasury plans to issue a 2-year maturity coupon bond, paying coupons once per year with a coupon rate of 9%. The face value of the bond is $100.

(a) At what price will the bond sell?

Suggested solution.

The bond sells at $101.861.

$$P = \frac{9}{1+0.07} + \frac{109}{(1+0.08)^2} = 101.8611$$

```
P <- 9/1.07 + 109/(1.08^2)
```

```
knitr::include_graphics("problem3.png")
```

Settlement date	1/1/22
Maturity date	1/1/24
Annual coupon rate	0.09
Flat price (% of par)	101.8611
Redemption value (% of face value)	100
Coupon payments per year	1
Yield to maturity (decimal)	0.0796

HW 4 Problem 3

(b) What will the yield to maturity on the bond be?

Suggested solution.

The yield to maturity is 7.96%.

```r
# based on the solution from the spreadsheet
y <- 0.0796
```

(c) If the expectations theory of the yield curve is correct, what is the market expectation of the price for which the bond will sell next year?

Suggested solution.

The expected short rate in the second year is

$$E[r_2] = (1+y_2)^2/(1+y_1) - 1 = 9.01\%$$

The bond will be sold at price \$99.99.

$$P_1 = 109/(1+E[r_2]) = 99.99$$

```r
y1 <- 0.07
y2 <- 0.08
r2 <- (1 + y2)^2/(1 + y1) - 1
r2
```

HW 4 Problem 3

(d) Recalculate your answer to part (c) if you believe in the liquidity preference theory and you believe that the liquidity premium is 1%. Calculate the expected holding period return and compare with the yield to maturity on 1-year zero. What do you find? Explain your finding.

Suggested solution.

If the liquidity premium is 1%, then the forward rate is 9.01%.

$$f_2 = (1+y_2)^2/(1+y_1) - 1 = 9.01\%$$

and the expected short rate is 8.01%

$$E[r_2] = f_2 - 1\% = 8.01\%$$

Then the price at the end of year 1 is \$100.9172

$$P_1 = 109/(1+E[r_2]) = 100.9172$$

The expected holding period return is 7.91%

$$\text{HPR} = \frac{100.9172 + 9}{101.861} - 1 = 7.91\%$$

The holding period return is higher than the YTM for the one-year zero. This makes sense as the interest rate in the second year is uncertain and then the bond price by the end of the first year is also uncertain. The holding period return therefore needs to be higher to compensate the risk from the uncertainty in second year's interest rate.

Duration

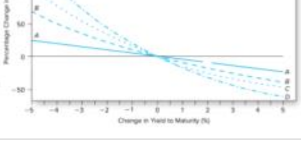
- *Interest rate sensitivity*: The sensitivity of bond prices to changes in market interest rates
- **Macaulay's duration**: the weighted average of the times to each coupon or principal payment

$$w_t = \frac{CF_t/(1+y)^t}{Bond\ price}$$

  - $\sum_{t=1}^{T} w_t = 1$
- **Macaulay's duration**: $D = \sum_{t=1}^{T} t \times w_t$
- **Modified duration**: $D^* = \frac{D}{1+y}$
- Modified duration rule

$$\frac{\Delta P}{P} = -D^* \Delta y$$

- Sensitivity increases with duration



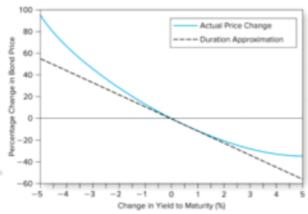
Convexity

- **Convexity**: the second derivative (the rate of change of the slope) of the price-yield curve divided by bond price

$$Convexity = \frac{1}{P}\frac{\Delta^2 P}{\Delta y^2} = \frac{1}{P \times (1+y)^2}\sum_{t=1}^{T}\left[\frac{CF_t}{(1+y)^t}(t^2+t)\right]$$

- Modified duration-with-convexity rule

$$\frac{\Delta P}{P} = -D^*\Delta y + \frac{1}{2} \times Convexity \times (\Delta y)^2$$



HW 4 Problem 4

Problem 4 (Duration and convexity) A newly issued bond has a maturity of 10 years and pays a 7% coupon rate (with coupon payments coming once annually). The bond sells at par value.

(a) What are the duration and the convexity of the bond?

Suggested solution.

Without the loss of generality, suppose the par value is 100.

The duration of the bond is 7.515232.

$$\text{duration} = \sum_{t=1}^{T}[t \times w_t] = \sum_{t=1}^{T}\left[t \times \frac{CF_t}{P \cdot (1+y)^t}\right] = 7.515232$$

```
P <- 100
rate <- 0.07
y <- rate
year <- 10
CF <- rep(P * rate, year)
CF[year] <- CF[year] + P
duration <- 0
for (t in c(1:year)) {
    duration <- duration + t * CF[t]/((1 + y)^t)/P
}
duration
[1] 7.515232
```

The convexity of the bond is 64.93296.

$$\text{convexity} = \frac{1}{P \times (1+y)^2}\sum_{t=1}^{T}[t \times w_t] = \frac{1}{P \times (1+y)^2}\sum_{t=1}^{T}\left[(t^2+t) \times \frac{CF_t}{(1+y)^t}\right] = 64.93296$$

```
convexity <- 0
for (t in c(1:year)) {
    convexity <- convexity + (t^2 + t) * CF[t]/((1 + y)^t)
}
convexity <- convexity/(P * (1 + y)^2)
convexity
[1] 64.93296
```

HW 4 Problem 4

(b) Find the actual price of the bond assuming that its yield to maturity immediately increases from 7% to 8% (with maturity still 10 years).

Suggested solution.

The actual price is 93.2899.

```
P.actual <- 93.2899
```

```
knitr::include_graphics("problem4.png")
```

Settlement date	1/1/22
Maturity date	1/1/32
Annual coupon rate	0.07
Yield to maturity	0.08
Redemption value (% of face value)	100
Coupon payments per year	1
Flat price (% of par)	93.2899

(c) What price would be predicted by the modified duration rule? What is the percentage error of that rule?

Suggested solution.

$$\frac{\Delta P}{P} = -\frac{\text{duration}}{1+y} \cdot \Delta y = -0.070$$

The price predicted by the modified duration rule is \$92.976.

$$P' = P \cdot \left(1 + \frac{\Delta P}{P}\right) = 92.976$$

The percentage error of this rule is 0.336%.

$$(P' - P_{actual})/P_{actual} = 0.336\%$$

```
delta.y <- 0.01
price.change <- -duration/(1 + y) * delta.y
price.change
[1] -0.07023582
P.prime <- P * (1 + price.change)
P.prime
[1] 92.97642
abs(P.prime/P.actual - 1)
[1] 0.003360295
```

(d) What price would be predicted by the modified duration-with-convexity rule? What is the percentage error of that rule?

Suggested solution.

$$\frac{\Delta P}{P} = -\frac{\text{duration}}{1+y} \cdot \Delta y + 1/2 \times \text{convexity} \times (\Delta y)^2 = -0.067$$

The price predicted by the modified duration-with-convexity rule is \$93.301.

$$P' = P \cdot \left(1 + \frac{\Delta P}{P}\right) = 93.301$$

The percentage error of this rule is 0.01%.

```
price.change <- -duration/(1 + y) * delta.y + 1/2 * convexity *
    delta.y^2
price.change
[1] -0.06698917
P.prime <- P * (1 + price.change)
P.prime
[1] 93.30108
abs(P.prime/P.actual - 1)
[1] 0.0001198764
```

(e) Compare your solution from parts (c) and (d). What do you find? Explain your finding.

Suggested solution.

The price predicted by the modified duration-with-convexity rule is much more accurate than the price predicted by the modified duration rule. This is because the modified duration-with-convexity rule uses both the first and second order information, while the duration rule only uses the first order information.

Option

- *Call option* gives its holder the right to **purchase** an asset for a specified price

$$payoff\ to\ call\ holder = \begin{cases} S_T - X & if\ S_T > X \\ 0 & if\ S_T \leq X \end{cases}$$

- *Put option* gives its holder the right to **sell** an asset for a specified price

$$payoff\ to\ put\ holder = \begin{cases} 0 & if\ S_T > X \\ X - S_T & if\ S_T \leq X \end{cases}$$

- The put-call parity relationship

$$C + \frac{X}{(1 + r_f)^T} = S_0 + P$$

Option pricing

- Six-step procedure (call option and one period as an example)
  - Step 1: Given possible end-of-year stock prices $uS_0 = \$120$ and $dS_0 = \$90$ and value of call option with exercise price $\$110$, $C_u = \$10$ or $C_d = 0$
  - Step 2: Find the hedge ratio $H = \frac{C_u - C_d}{uS_0 - dS_0} = \frac{1}{3}$
  - Step 3: find a portfolio made up of $\frac{1}{3}$ share of stock with one written call
  - Step 4: The present value of $\$30$ is $\frac{\$30}{1.1} = \$27.27$
  - Step 5: Set the present value of the hedge position to the present value of the certainty payoff
    $$\frac{1}{3}S_0 - C_0 = \$27.27$$
  - Step 6: Solve the call's value
    $$C_0 = \frac{1}{3}S_0 - \$27.27 = \$33.33 - \$27.27 = \$6.06$$

HW 4 Problem 5

Problem 5 (Option) We will derive a two-state put option value in this problem. Data: $S_0 = 100$; $X = 110$; $1 + r = 1.10$. The two possibilities for S_T are 130 and 80.

(a) Show that the range of S is 50, whereas that of P is 30 across the two states. What is the hedge ratio of the put?

Suggested solution.

The range of S is \$50 because

$$uS_0 - dS_0 = \$130 - \$80 = \$50.$$

In the state of $uS_0 = \$130$, the put has zero value and then $P_u = 0$. In the state of $dS_0 = \$80$, the put holder will exercise the put and then the value is $P_d = \$110 - \$80 = \$30$.

The range of P is \$30 because

$$P_d - P_u = \$30.$$

The hedge ratio of the put is

$$\text{hedge ratio} = \frac{P_u - P_d}{uS_0 - dS_0} = -\frac{30}{50} = -\frac{3}{5}.$$

HW 4 Problem 5

(b) Form a portfolio of three shares of stock and five puts. What is the (nonrandom) payoff to this portfolio?

Suggested solution.

The payoff when the stock is up is

$$3 \times uS_0 + 5 \times P_u = \$390$$

The payoff when the stock is down is

$$3 \times dS_0 + 5 \times P_d = 3 \times \$80 + 5 \times \$30 = \$390$$

(c) What is the present value of the portfolio?

Suggested solution.

The present value of the portfolio is \$354.5455

$$PV = \frac{\$390}{1+r} = \frac{\$390}{1.10} = \$354.5455$$

```
PV <- 390/1.1
PV
[1] 354.5455
```

(d) Given that the stock currently is selling at 100, solve for the value of the put.

Suggested solution.

The value of the put can be solved from

$$3 \times S_0 + 5 \times P_0 = \$354.5455$$

and then the value of the put is \$10.90909

$$P_0 = \frac{\$354.5455 - \$300}{5} = \$10.90909$$

```
(PV - 300)/5
[1] 10.90909
```

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