Lecture 9: Diversification

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9-diversification

QTM 385 Quantitative Finance

Lecture 9: Efficient diversification

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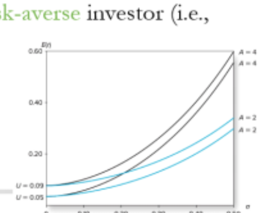
Suggested reading: Investments Ch 7

Lecture plan

- Indifference curve analysis
- Diversification and portfolio risk

Indifference curve analysis

- Indifference curve: all combinations of expected return $E(r)$ and volatility σ such that utility level is the same
$$U = E[r] - \frac{1}{2}\times A\times\sigma^2$$
- Given U, identify the required $E[r]$ for every σ through
$$E[r] = U + \frac{1}{2}\times A\times\sigma^2$$
- Given the same σ, the required $E[r]$ for the less risk-averse investor (i.e., smaller A) is lower to achieve the same U



Y axis is return, x axis is standard error
For each curve, each point on the curve is a combination of the return and std dev
For diff points on the curve, the utility is the same (e.g. U = 0.05)

Indifference curves have diff levels

Risk aversion A
If A larger, investor is more risk averse

Compare A = 4 to A = 2
For A = 4, the required return needs to be higher to achieve the same utility level

Comparative statics

- More risk-averse investors (i.e., larger A) have steeper indifference curves than less risk-averse investors
  - Steepness is measured by
$$\frac{dE[r]}{d\sigma} = \frac{d\left(U + \frac{1}{2}\times A\times\sigma^2\right)}{d\sigma} = A\sigma$$
  - Steepness increases with both A and σ, but does not vary with U
  - Steeper curves mean that investors require a great increase in expected returns to compensate for an increase in portfolio risk
- Higher indifference curves correspond to higher levels of utility

Gradient of the return
Derivative of utility plus adjusted variance of investment w respect to std dev
And slope is A * sigma

If investors are more risk averse, slope is larger
For 1 unit increase in std error, how mmuch increase we need in return so that utility stays the same

E.g. if std error is 0.4, we need a higher return for A = 4 in order for the utility to be the same

If utility increases while A stays constant, indifference curve stays the same shape but shifts up a little

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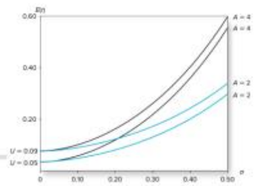

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E.g. if std error is 0.4, we need a higher return for A = 4 in order for the utility to be the same

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Diversification and portfolio risk

- Suppose your portfolio only contains one stock: Digital Computer Corp.
- Two sources of uncertainty:
  - Systematic risk: Risks from the condition of the general economy, e.g., business cycle, inflation, interest rates, exchange rates
  - Firm-specific risk: e.g., research and development and personnel changes

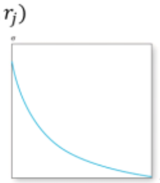
- A naïve **diversification** strategy: half your funds in ExxonMobil and half in Digital
  - When oil prices fall, hurting ExxonMobil, computer prices might rise, helping Digital; Vice versa
  - Diversification reduces portfolio risk: Two effects offset and stabilize portfolio return

We can still diversify if both have positive covariance

We can still reduce part of the risk but will not be as effective as when correlation is negative

Diversification and all risk is firm-specific

- If we diversify into many more securities, we spread out the exposure to firm-specific factors, and portfolio volatility falls
- When all risk is firm-specific, diversification can reduce risk to arbitrarily low levels. Firm-specific risk is also called nonsystematic risk, or diversifiable risk
- In portfolio P, weight of asset i is w_i. n assets in total.
  - Expected return $E[r_P] = \sum_{i=1}^{n} w_i E[r_i]$
  - Variance $\sigma_P^2 = Cov(\sum_{i=1}^{n} w_i r_i, \sum_{i=1}^{n} w_i r_i) = \sum_{i,j=1}^{n} w_i w_j Cov(r_i, r_j)$
  - All risk is firm specific: $Cov(r_i, r_j) = 0$ for $i \neq j$
  - Suppose $Cov(r_i, r_i) = \sigma^2$ and $w_i = 1/n$
  - Then $\sigma_P^2 = n \cdot \frac{1}{n^2} \cdot \sigma^2 = \frac{1}{n} \cdot \sigma^2 \to 0$



For now, let's assume weight is constant. Then we can take it out. And then variance is just the covariance?

No systematic risk if all risk is firm specific, meaning none of the firms are related at all

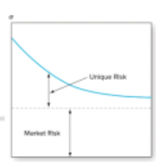
Also, assume stocks are homogeneous and return is identical and equally distributed
THEN WITH ALL THIS SIMPLIFICATION but also with the assumption that all risk is firm specific, we get the last line, since we are only looking at the Cov(ri, ri)

But the variance is inversely proportional to n so it will approach 0 as n gets larger
VARIANCE OF PORTFOLIO RETURN CONVERGES TO ZERO AS N INCREASES (variance of portfolio return will be lower than variance of individual stocks)

$$E[r_P] = n \frac{1}{n} E[r_1]$$

Diversification and some risk is systematic

- If all stocks are affected by the business cycle (or other macroeconomic factors), we cannot avoid exposure to business cycle risk no matter how many stocks we hold
- Systematic risk can not be eliminated even with extensive diversification, also called as market risk or nondiversifiable risk
- In portfolio P, weight of asset i is w_i. n assets in total.
  - Variance $\sigma_P^2 = Cov(\sum_{i=1}^{n} w_i r_i, \sum_{i=1}^{n} w_i r_i) = \sum_{i,j=1}^{n} w_i w_j Cov(r_i, r_j)$
  - Some risk is systematic: Suppose $Cov(r_i, r_j) = \rho\sigma^2$ for $i \neq j$
  - Suppose $Cov(r_i, r_i) = \sigma^2$ and $w_i = 1/n$
  - Then $\sigma_P^2 = n(n-1) \cdot \frac{1}{n^2} \cdot \rho\sigma^2 + n \cdot \frac{1}{n^2} \cdot \sigma^2$
    $= \frac{n-1}{n}\rho\sigma^2 + \frac{1}{n} \cdot \sigma^2 \to \rho\sigma^2$



So still assuming equal weights
Now because some risk is systematic, there is covariance between firms' stocks. For simplicity, It's roh sigma squared (roh being a number between 0 and 1)

There are n(n-1) terms where i != j
There are n terms where i =j
We have n^2 terms total

Power of diversification

- In our example, $\sigma_P^2 = \frac{n-1}{n}\rho\sigma^2 + \frac{1}{n}\cdot\sigma^2 \to \rho\sigma^2$
- $\rho = 0$: all risk is firm-specific, diversification is powerful
- $\rho = 0.4$: some risk is systematic, diversification is less powerful
- $\rho = 1$: all risk is systematic, diversification is useless

Universe Size n	Portfolio Weights $w = 1/n$ (%)	$\rho = 0$ Standard Deviation (%)	$\rho = 0$ Reduction in σ	$\rho = 0.40$ Standard Deviation (%)	$\rho = 0.40$ Reduction in σ
1	100	50.00	14.64	50.00	8.17
2	50	35.36		41.83	
5	20	22.36	1.95	36.06	0.70
6	16.67	20.41		35.36	
10	10	15.81	0.73	33.91	0.20
11	9.09	15.08		33.71	
20	5	11.18	0.27	32.79	0.06
21	4.76	10.91		32.73	
100	1	5.00	0.02	31.86	0.00
101	0.99	4.98		31.86	

Reduction level is (1-roh) *(sigma squared)

When roh is bigger than reduction, then some reduction but not as powerful

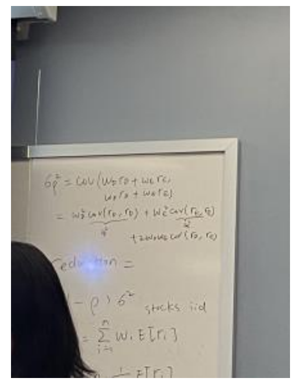
Two risky portfolios

- Two risky portfolios: a bond portfolio D specializing in long-term debt securities and a stock portfolio E specializing in equity securities
- Asset allocation decision of portfolio P: w_D in the bond portfolio and $w_E = 1 - w_D$ in the stock portfolio
  - Rate of return on this portfolio, r_P
    $$r_P = w_D r_D + w_E r_E$$
  - Expected return on this portfolio
    $$E(r_P) = w_D E(r_D) + w_E E(r_E)$$
  - Variance of this portfolio
    $$\sigma_P^2 = w_D^2 \cdot \sigma_D^2 + w_E^2 \cdot \sigma_E^2 + 2 w_D w_E Cov(r_D, r_E)$$

	Debt	Equity
Expected return, $E(r)$	8%	13%
Standard deviation, σ	12%	20%
Covariance, $Cov(r_D, r_E)$	72	
Correlation coefficient, ρ_{DE}	0.30	

In practice, expected returns of individual stocks, as well as their individual variances, will be different.
So the portfolio optimization problem becomes much harder.

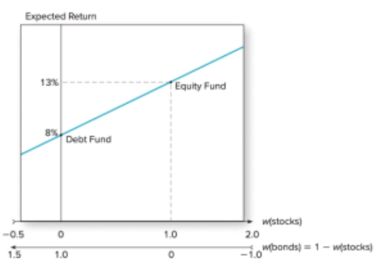
Let's look at 2 portfolios



0.3 is the SYSTEMATIC RISK

Portfolio expected return as a function of w_E

- Expected return is linear in w_E
$$E(r_P) = w_D E(r_D) + w_E E(r_E) = (1 - w_E)E(r_D) + w_E E(r_E)$$
$$= E(r_D) + w_E\big(E(r_E) - E(r_D)\big)$$



Replace wD with (1-wE)

X axis is the weight (wE)
When wE = 0, then the second term is 0, then we are left with only the first term, which in the table is 8%. So the intercept is 8%
When wE = 1 then the expected portfolio return is equal to 13% which is the equity

But we care most about the correlation and how that affects the variance

Covariance between two risky portfolios

- Variance of portfolio P is affected by the covariance of two portfolios
$$\sigma_P^2 = w_D^2 \cdot \sigma_D^2 + w_E^2 \cdot \sigma_E^2 + 2 w_D w_E Cov(r_D, r_E)$$
- Let ρ_{DE} be the correlation coefficient of two portfolio. Then
$$Cov(r_D, r_E) = \rho_{DE}\sigma_D\sigma_E$$
  - Portfolio variance increases with ρ_{DE}
  - Case 1: $\rho_{DE} = 1$, D and E are perfectly positively correlated and the standard deviation of portfolio P is the largest
    $$\sigma_P^2 = w_D^2 \cdot \sigma_D^2 + w_E^2 \cdot \sigma_E^2 + 2 w_D w_E \sigma_D \sigma_E = (w_D\sigma_D + w_E\sigma_E)^2$$
    $$\Rightarrow \sigma_P = w_D\sigma_D + w_E\sigma_E$$
  - Standard deviation σ_P is linear in w_E and σ_E (also linear in w_D and σ_D)

If rohDE is larger, then variance is larger

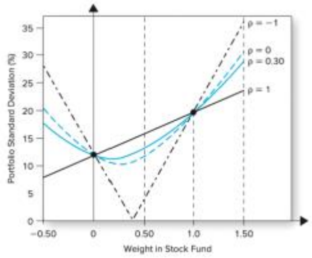
Case 1 is a special case
Expected return is completely proportional

Standard deviation of portfolio return is linear

Portfolio standard deviation as a function of w_E

- Variance of portfolio P is affected by the covariance of two portfolios

$$\sigma_P^2 = w_D^2 \cdot \sigma_D^2 + w_E^2 \cdot \sigma_E^2 + 2w_D w_E Cov(r_D, r_E)$$



You can see where it his the 12% and 20% , scroll up for the table on that

Weight in stock fund is wE

When roh = 1, no diversification, all return is coming from an increase in risk
But for all the other cases, for roh < 1, we indeed have diversification

So diversification is less useful when roh = 1 or sth

Covariance between two risky portfolios

- Variance of portfolio P is affected by the covariance of two portfolios

$$\sigma_P^2 = w_D^2 \cdot \sigma_D^2 + w_E^2 \cdot \sigma_E^2 + 2\rho_{DE} w_D w_E \sigma_D \sigma_E$$

- Case 2: $-1 < \rho_{DE} < 1$

$$\sigma_P < w_D \sigma_D + w_E \sigma_E$$

- *Portfolios of less than perfectly correlated assets always offer some degree of diversification benefit*. The lower the correlation between the assets, the greater the gain in efficiency
- If $\rho_{DE} < 0$, then D is a hedge portfolio of E. In this case, diversification is particularly effective in reducing total risk (while not affecting total return)

Covariance between two risky portfolios

- Variance of portfolio P is affected by the covariance of two portfolios

$$\sigma_P^2 = w_D^2 \cdot \sigma_D^2 + w_E^2 \cdot \sigma_E^2 + 2\rho_{DE} w_D w_E \sigma_D \sigma_E$$

- Case 3: $\rho_{DE} = -1$ D and E are perfectly negatively correlated and

$$\sigma_P^2 = (w_D \sigma_D - w_E \sigma_E)^2$$
$$\Rightarrow \sigma_P = |w_D \sigma_D - w_E \sigma_E|$$

- Standard deviation σ_P is the smallest

Suppose we have two scenarios:
Denote

Scen – 1 2
rD – 10% 6%
rE – 11% 15%
rP – 10.5% 10.5%

For scenario 1, this is good for the bond, since bond has higher return than expected value, but had for equity, because it has lower return than expected value. Then it's the other way for scenario 2
So these are perfectly negatively correlated

Perfectly hedging position

- A perfectly hedging position can be obtained by solving

$$w_D \sigma_D - w_E \sigma_E = 0$$

$$\Rightarrow w_D \sigma_D - (1 - w_D)\sigma_E = w_D(\sigma_D + \sigma_E) - \sigma_E = 0$$

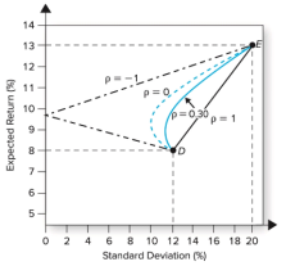
- The solution is

$$w_D = \frac{\sigma_E}{\sigma_D + \sigma_E}$$

$$w_E = \frac{\sigma_D}{\sigma_D + \sigma_E} = 1 - w_D$$

Portfolio opportunity set

- Portfolio opportunity set: all combinations of portfolio expected return and standard deviation that can be constructed from the two available assets/portfolios
- The lower the correlation, the greater the potential benefit from diversification
- When $\rho_{DE} = -1$, there is a perfect hedging opportunity: Standard deviation can be driven all the way to zero



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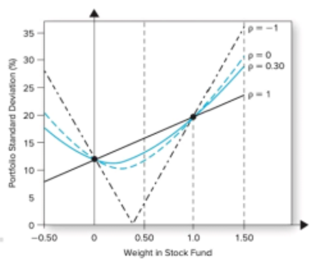
Minimum variance portfolio

- Minimum-variance portfolio: Portfolio with w_D^* and w_E^* such that the corresponding σ_P^{*2} is the smallest among all possible choices of w_D and w_E

$$\sigma_P^2 = w_D^2 \cdot \sigma_D^2 + w_E^2 \cdot \sigma_E^2 + 2\rho_{DE} w_D w_E \sigma_D \sigma_E$$

- If $\rho_{DE} = -1$, then
  - $w_D^* = \frac{\sigma_E}{\sigma_D + \sigma_E} = \frac{20}{12+20} = .625$
  - $w_E^* = 1 - .625 = .375$

	Debt	Equity
Expected return, *E*(*r*)	8%	13%
Standard deviation, σ	12%	20%



Minimum-variance portfolio for general ρ_{DE}

- Based on the variance formula

$$\sigma_P^2 = w_D^2 \cdot \sigma_D^2 + w_E^2 \cdot \sigma_E^2 + 2 w_D w_E Cov(r_D, r_E)$$

- Replace w_E by $1 - w_D$ and take the derivative with respect to w_D. The optimal w_D^* that minimizes σ_P^2 is

$$w_D^* = \frac{\sigma_E^2 - Cov(r_D, r_E)}{\sigma_D^2 + \sigma_E^2 - 2Cov(r_D, r_E)}$$

Question

- The weight of in the minimum variance portfolio is

$$w_D^* = \frac{\sigma_E^2 - Cov(r_D, r_E)}{\sigma_D^2 + \sigma_E^2 - 2Cov(r_D, r_E)}$$

- Question: What is the weight of D and E when $\rho_{DE} = .3$? What if $\rho_{DE} = 0$?

	Debt	Equity
Expected return, *E*(*r*)	8%	13%
Standard deviation, σ	12%	20%

Choosing a portfolio based on risk aversion

- For the utility $U = E(r) - \frac{1}{2}A\sigma^2$ with the risk aversion parameter A, the optimal investment proportions in the two funds are

$$w_D^* = \frac{E(r_D) - E(r_E) + A(\sigma_E^2 - Cov(r_D, r_E))}{A(\sigma_D^2 + \sigma_E^2 - 2Cov(r_D, r_E))}$$

$$w_E^* = 1 - w_D^*$$

https://www.mathworks.com/help/finance/portfolio.estimatemaxsharperatio.html

Algorithms

The maximization of the Sharpe ratio is accomplished by either using the `'direct'` or `'iterative'` method. For the `'direct'` method, consider the following scenario. To maximize the Sharpe ratio is to:

$$\text{Maximize} \frac{\mu^T x - r_f}{\sqrt{x^T C x}}, s.t. \sum x_i = 1,\ 0 \le x_i \le 1,$$

where μ and C are the mean and covariance matrix, and r_f is the risk-free rate.

If $\mu^T x - r_f \le 0$ for all x the portfolio that maximizes the Sharpe ratio is the one with maximum return.

If $\mu^T x - r_f > 0$, let $t = \frac{1}{\mu^T x - r_f}$

and $y = tx$ (Cornuejols [1] section 8.2). Then after some substitutions, you can transform the original problem into the following form,

$$\text{Minimize } y^T C y,\ s.t. \sum y_i = t,\ t > 0,\ 0 \le y_i \le t,\ \mu^T y - r_f t = 1.$$

Only one optimization needs to be solved, hence the name "direct". The portfolio weights can be recovered by $x^* = y^* / t^*$.

For the `'iterative'` method, the idea is to iteratively explore the portfolios at different return levels on the efficient frontier and locate the one with maximum Sharpe ratio. Therefore, multiple optimization problems are solved during the process, instead of only one in the `'direct'` method. Consequently, the `'iterative'` method is slow compared to `'direct'` method.