homework3_solution

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homework
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Quantitative Finance: Homework 3 Solution

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Problem 1 (CAPM)

In CAPM, suppose the risk free rate is $r_f = 6\%$ and the expected return of market portfolio is $E(r_M) = 14\%$.

(a) What must be the β of a portfolio with $E(r_P) = 18\%$?

Suggested solution.

We can solve β from

$$E(r_P) = r_f + \beta(E(r_M) - r_f).$$

Therefore β is

$$\beta = \frac{E(r_P) - r_f}{E(r_M) - r_f} = \frac{18 - 6}{14 - 6} = 1.5$$

(b) What must be the return of a portfolio with $\beta = 0.5$?

Suggested solution.

The return of the portfolio with $\beta = 0.5$ is

$$E(r_P) = r_f + \beta(E(r_M) - r_f) = 6\% + 0.5(14\% - 6\%) = 10\%.$$

(c) If a portfolio has $\beta = 1$, $E(r_P) = 10\%$, and standard deviation of returns $\sigma_P = 10\%$. Is this portfolio underpriced, overpriced, or properly priced?

Suggested solution.

If the portfolio has $\beta = 1$, then the average return based on CAPM should be

$$r_f + \beta(E(r_M) - r_f) = 6\% + 1 \cdot (14\% - 6\%) = 14\%,$$

which is higher than $E(r_P) = 10\%$. Therefore this portfolio is overpriced (as this stock is plotted below the security market line).

Problem 2 (Simpleland)

In Simpleland there are only two risky stocks, A and B, whose details are listed in the table below.

Stock	Outstanding Shares	Price Per Share	Expected Rate of Return	Standard Deviation of Return
A	200	\$2.40	19%	30%
B	300	\$2.80	16%	20%

Furthermore the correlation ceofficient between the returns of stocks A and B is $\rho_{AB} = 0.30$. There is also a risk-free asset, and Simpleland satisfies the CAPM model exactly.

(a) What is the expected rate of return of the market portfolio?

Suggested solution.

The market share of stock A is

$$w_A = \frac{200 \times 2.4}{200 \times 2.4 + 300 \times 2.8} = 0.3636364$$

and the market share of stock B is

$$w_B = \frac{300 \times 2.8}{200 \times 2.4 + 300 \times 2.8} = 0.6363636$$

The expected return of the market portfolio is

$$r_M = w_A E(r_A) + w_B E(r_B) = 0.1709091$$

```
w_A <- (200 * 2.4)/(200 * 2.4 + 300 * 2.8)
w_B <- (300 * 2.8)/(200 * 2.4 + 300 * 2.8)
r_A <- 0.19
r_B <- 0.16
w_A
[1] 0.3636364
w_B
[1] 0.6363636
r_M <- w_A * r_A + w_B * r_B
r_M
[1] 0.1709091
```

(b) What is the standard deviation of the market portfolio?

Suggested solution.

The standard deviation of the market portfolio

$$\sigma_M = (w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_A w_B \rho_{AB}\sigma_A\sigma_B)^{0.5} = 0.1908658.$$

```
sigma_A <- 0.3
sigma_B <- 0.2
rho_AB <- 0.3
sigma_M_sq <- w_A**2 * sigma_A ** 2 + w_B**2 * sigma_B ** 2 + 2 * w_A * w_B * rho_AB * sigma_A * sigma_
sigma_M <- sigma_M_sq ** 0.5
sigma_M_sq
[1] 0.03642975
sigma_M
[1] 0.1908658
```

(c) What is the beta of stock A?

Suggested solution.

The beta of stock A is

$$\beta_A = \frac{cov(r_M, r_A)}{\sigma_M^2} = \frac{cov(w_A r_A + w_B r_B, r_A)}{\sigma_M^2} = \frac{w_A\sigma_A^2 + w_B\rho_{AB}\sigma_A\sigma_B}{\sigma_M^2} = 1.212795$$

```
beta_A <- (w_A * sigma_A ** 2 + w_B * rho_AB * sigma_A * sigma_B)/(sigma_M_sq)
beta_A
[1] 1.212795
```

(d) What is the risk-free rate in Simpleland?

Suggested solution.

Note that

$$E(r_A) = r_f + \beta_A(E(r_M) - r_f)$$

then the risk-free rate is

$$r_f = \frac{E(r_A) - \beta_A E(r_M)}{1 - \beta_A} = 0.08119403.$$

```
r_f <- (r_A - beta_A * r_M)/(1 - beta_A)
r_f
[1] 0.08119403
```

Problem 3 (APT factors)

The returns of two stocks are believed to satisfy the two-factor model

$$r_1 = E(r_1) + 2f_1 + 2f_2 + \varepsilon_1$$

$$r_2 = E(r_2) + 1f_1 + 4f_2 + \varepsilon_2$$

Assume that the factors are traded assets and the risk-free rate is also priced by the factor model. In addition, there is a risk-free asset with a rate of return of 4%. It is known that the expected return for asset 1 is $E[r_1] = 12\%$ and for asset 2 is $E[r_2] = 17\%$.

What are the values of r_f, risk premium of the first factor $E(f_1 - r_f)$, and risk premium of the second factor $E(f_2 - r_f)$ for this model?

Suggested solution.

The value of r_f is $r_f = 4\%$.

Based on the returns of two stocks, we have

$$E(r_1) - r_f = 2E(f_1 - r_f) + 2E(f_2 - r_f) = 8\%$$

and

$$E(r_2) - r_f = 1E(f_1 - r_f) + 4E(f_2 - r_f) = 13\%.$$

We can solve the risk premium of the two factors by solving the system of equations.

The risk premium of the first factor is

$$E(f_1 - r_f) = \frac{1}{3}\left[2 \times 8\% - 13\%\right] = 1\%$$

and the risk premium of the second factor is

Security	β_i	$E(R_i)$	$\sigma(e_i)$
A	0.8	10%	25%
B	1.0	10%	20%
C	1.2	14%	20%

$$E(f_2 - r_f) = \frac{1}{6}[2 \times 13\% - 8\%] = 3\%.$$

```
(2 * 13 - 8)/6
[1] 3
```

Problem 4 (Single-Index Model)

Assume that security returns are generated by the single-index model,

$$R_i = \alpha_i + \beta_i R_M + e_i$$

where R_i is the excess return for security i and R_M is the market's excess return. The risk-free rate is 2%. Suppose also that there are three securities, A, B, and C, characterized by the following data:

(a) If $\sigma_M = 20\%$, calculate the variance of returns of securities A, B, and C.

Suggested solution.

The variance of returns of security A is

$$\sigma_A^2 = \beta_A^2\sigma_M^2 + \sigma^2(e_A) = 0.8^2 \times 0.2^2 + 0.25^2 = 0.0881$$

The variance of returns of security B is

$$\sigma_B^2 = \beta_B^2\sigma_M^2 + \sigma^2(e_B) = 1.0^2 \times 0.2^2 + 0.2^2 = 0.08$$

The variance of returns of security C is

$$\sigma_C^2 = \beta_C^2\sigma_M^2 + \sigma^2(e_C) = 1.2^2 \times 0.2^2 + 0.2^2 = 0.0976$$

```
0.8**2 * 0.2**2 + 0.25**2
[1] 0.0881
1**2 * 0.2**2 + 0.2**2
[1] 0.08
1.2**2 * 0.2**2 + 0.2**2
[1] 0.0976
```

(b) Now assume that there are an infinite number of assets with return characteristics identical to those of A, B, and C, respectively. If one forms a well-diversified portfolio of type A securities, what will be the mean and variance of the portfolio's excess returns? What about portfolios composed only of type B or C stocks?

Suggested solution.

The mean of the excess return of the portfolio of type A securities is

$$E(r_A) - r_f = 0.1 - 0.02 = 0.08$$

The variance of the portfolio is

$$\beta_A^2 \sigma_M^2 = 0.0256$$

The mean of the excess return of the portfolio of type B securities is

$$E(r_B) - r_f = 0.1 - 0.02 = 0.08$$

The variance of the portfolio is

$$\beta_B^2 \sigma_M^2 = 0.04$$

The mean of the excess return of the portfolio of type C securities is

$$E(r_C) - r_f = 0.14 - 0.02 = 0.12$$

The variance of the portfolio is

$$\beta_C^2 \sigma_M^2 = 0.0576$$

```
0.8**2 * 0.2**2
[1] 0.0256
1**2 * 0.2**2
[1] 0.04
1.2**2 * 0.2**2
[1] 0.0576
```

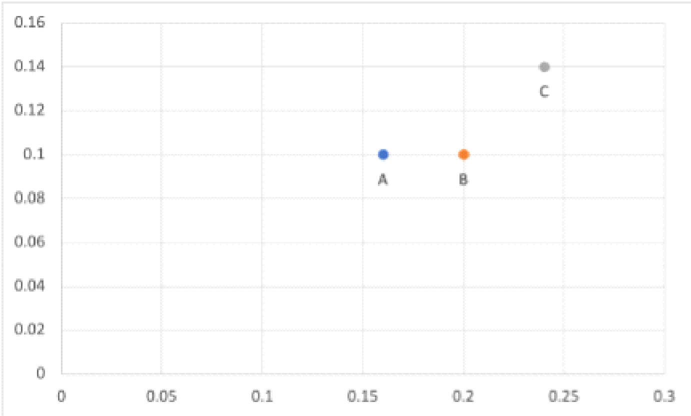
(c) Is there an arbitrage opportunity in this market? What is it? Analyze the opportunity graphically.

Suggested solution.

Note that the Sharpe ratio for the portfolio with type A's securities is $\frac{0.08}{0.16} = 0.5$. The Sharpe ratio for the portfolio with type B's securities is $\frac{0.08}{0.2} = 0.4$. The Sharpe ratio for the portfolio with type B's securities is $\frac{0.12}{0.24} = 0.5$.

Suppose the single-index model holds. An arbitrage opportunity is to invest 50% in the portfolio with type A's securities, and another 50% in the portfolio with type C's securities, and short sell the portfolio with type B's securities, and we can earn 2% return in all scenarios.

```
knitr::include_graphics("Problem5-c.png")
```


0.16
0.14
0.12
0.1
0.08
0.06
0.04
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0
0
0.05
0.1
0.15
0.2
0.25
0.3
A
B
C