Lecture 18: Term Structures

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18-yield-curve

QTM 385 Quantitative Finance

Lecture 18: Term structures

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Suggested reading: Investments Ch 15

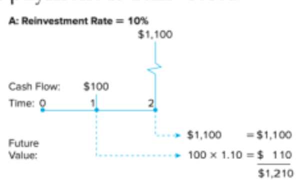
Realized compound return

- After an investment period T, we can calculate the realized compound return. Suppose the initial value is V_0, final value is V_T and realized compound return r is

$$V_0(1+r)^T = V_T$$

$$\Rightarrow r = \left(\frac{V_T}{V_0}\right)^{1/T} - 1$$

- Example: Consider, a 2-year bond selling at par value, paying a 10% coupon once a year and YTM is 10%. The coupon payment is reinvested at rate 10%
  - $r = \left(\frac{1210}{1000}\right)^{1/2} - 1 = 10\% = YTM$



Year 1: $100 coupon payment (10%)
Year 2: will get par value AND coupon payment which will be $1100 total

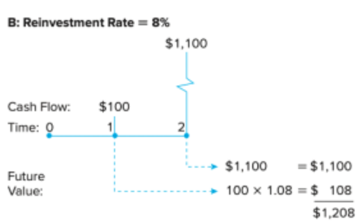
But what to do when we reinvest the coupon payment from year 1?
The final value turns out to be $1210 if you reinvest

Realized compound return vs YTM

- Coupons may not be reinvested to earn the bond's yield to maturity. Then realized compound return does not equal to YTM
- Initial value of the investment is $V_0 = 1000$. Final value of the investment is $V_2 = 1208$. The compound rate of return
  - $r = \left(\frac{1208}{1000}\right)^{1/2} - 1 = 9.91\%$

In many cases, we can't reinvest because market interest rate fluctuates

So we got the coupon payment after Year 1 but then the interest rate fell to 8%
So we can only reinvest at a lower percent
In that case, the 100$ can only be reinvested at 8%
So only get 108$ additional
For 1208 total



There is an issue in this process:
The issue is that when we calculate the realized compound rate, we need to use the reinvestment rate.
But how do we know this reinvestment rate? In practice, we don't know.

Calculation of realized compound return

- Realized compound return can be calculated either after the horizon or using a forecast of future reinvestment rates

Bond prices over time

- A bond sells at par value when its coupon rate equals the market interest rate
- When coupon rate is lower than the market interest rate, price should be lower than the par value; vice versa

But sometimes there are some bonds that only pay the par back? Zero-coupon bond (before maturity). They're very risky.

Why would you invest if it's so risky? It has a really high yield to maturity. So it will be sold much lower than the par. You are betting that the market interest rate will fall.

Or, bonds may pay coupon rate but not at market interest rate.

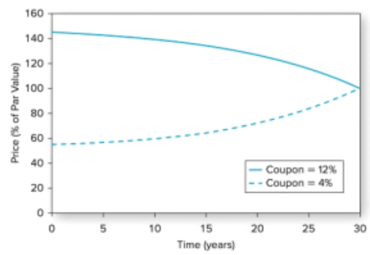


Figure 14.6 Price path of two 30-year maturity bonds, each selling at a yield to maturity of 8%. Bond price approaches par value as maturity date approaches.

If the coupon rate is the same rate as the market interest rate, it should be at 100% on the price y axis.
At left, it's higher than par.
At maturity: no coupon payments left. Last payment is par. So all we get is par and the last that we get is just the par.

Why does the top line shrink?
We have fewer coupon payments left, so the difference between the 8% market interest rate will shrink. (in beginning, lots of coupon payments left, each one is higher than market interest rate)

So every period, we get $120 when the market offers $80. The difference is the $40. This is why bond price is higher than par. But as time to maturity decreases, and we have received many coupon payments already, the 40$ will be fewer than at the beginning.

e.g. Year 1 we have 30 $40 left

e.g. Year 20 we have 10 $40 left

YTM versus holding-period return

- If YTM is unchanged over the holding period, the HPR on the bond equals YTM; Otherwise, a bond's HPR fluctuates with yields
  - If YTM rises, then the bond price falls and HPR falls
  - If YTM falls, then the bond price rises and HPR rises
- *Example*: Consider an 8% coupon, 30-year annual bond selling at par $1,000
  - If you hold the bond for a year and YTM remains 8%, then the price will remain at par and HPR is 8%
  - If YTM falls and the price rises to $1,050, then the HPR is greater than 8%

$$HPR = \frac{80 + (1050 - 1000)}{1000} = 13\%$$

HPR is holding period return

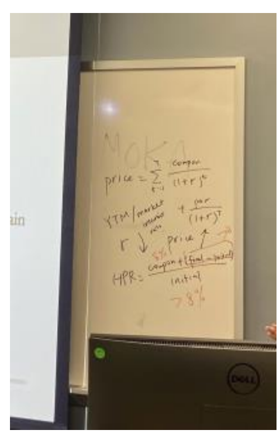
In a special case where the bond has YTM remain 8%
The price will auto remain at the par
So in the graph above the price stays $100 so it would be a straight line
So in this special case the HPR solely comes from coupon payment because the price stays constant

If market interest rate falls, YTM falls

YTM falls, price becomes higher bc theres a smaller denominator

In stock, dividend payment is equivalent to coupon payments in bonds

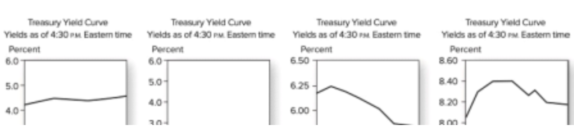
Stock is riskier because dividend payment fluctuates over time whereas coupon payment is fixed



The yield curve

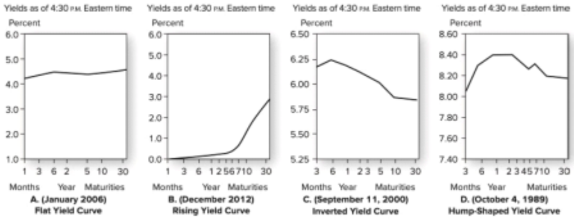
- **Yield curve**: Yield to maturity as a function of time to maturity
  - Central to bond valuation
  - Gauge their expectations for future interest rates against those of the market

on-the-run yield curve: the plot of yield as a function of



Y axis is YTM

on-the-run yield curve: the plot of yield as a function of maturity for recently issued coupon bonds selling at or near par value



Future interest rate increases

Future interest rate falls

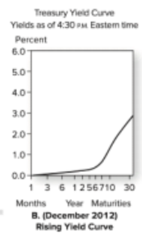
Y axis is YTM
X axis is time to maturity

Comparing bond with different time to maturities

B: people expecting future interest rate to grow
So investors don't care about very low interest rate rn because they'll get higher interest rate in the future
C: what we are experiencing at the moment
YTM for one year CD is 4% but for 5 year CD it's 3.5%
People expect that the future interest rate will fall
For longer term bonds, you don't need to offer a high coupon rate in the future to make it appealing for people
D appeared 2 years ago. Some peoples expectations were that interest rate would increase to 2023 and then decrease in a year or 2.

The yield curve and future interest rates

- The upward-sloping yield curve: short-term rates are going to be higher next year than they are now
- **Spot rate**: yield to maturity on zero-coupon bonds, meaning the rate that prevails today for a time period corresponding to the zero's maturity
- **Short rate**: refers to the interest rate for a given time interval (e.g., one year) available at different points in time



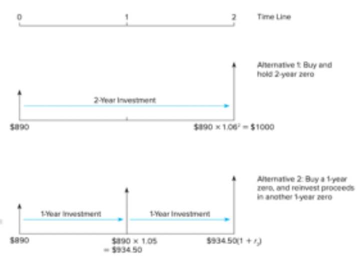
TO figure out what peoples expectations of future interest rates will be
This is the plot that can be calculated with bond price
BY looking at the plots. People can guess what the markets expectations will be

The formula from before has an important drawback: it assumes r stays constant
But r actually fluctuates

Short rate from today to a year etc.

Spot rate and short rate

- Consider two 2-year investment strategies with equal rate of returns
  - buying and holding a 2-year zero-coupon bond
  - buying a 1-year zero and rolling over the proceeds into a 1-year bond
- 2-year spot rate is an "average" of today's and next year's short rates
  - This year's short rate r_1. Next year's short rate r_2. Two year's spot rate y_2
  - $(1+y_2)^2 = (1+r_1)(1+r_2)$
  - Then $r_2 = \frac{(1+y_2)^2}{1+r_1} - 1 = 7.01\%$



Spot rate	1 yr zero	5% = y1
	2 y zero	6% = y2

Get shortrate from y1 --> y2

To make the 2 year appealing, we do y2 = sqrt(1000/890) -1 = 6%

At the end of the 1-year strategy one, it shouhld be the same value as the 2 year
Then we can get what the short rate shhould be

934.5 (1+ r2) = 1000
Solve for r2 to get 7.01%

Or you can write 1000 = [890 (1 + y2)^2] / [890 (1 + y1)]

Spot rate and short rate

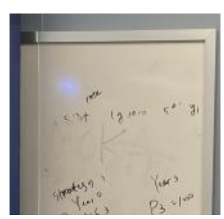
- Consider two 3-year investment strategies with equal rate of returns
  - buying and holding a 3-year zero-coupon bond
  - buying a 2-year zero and rolling over the proceeds into a 1-year bond
- Finding the short rate r_3 in year 3
  - $(1+y_3)^3 = (1+y_2)^2(1+r_3)$
  - Then $r_3 = \frac{(1+y_3)^3}{(1+y_2)^2} - 1 = \frac{1.07^3}{1.06^2} - 1 = 9.025\%$

Table 15.1 Prices and yields to maturity on zero-coupon bonds ($1,000 face value)

Maturity (years)	Yield to Maturity (%)	Price
1	5%	$952.38 = $1,000/1.05
2	6	$890.00 = $1,000/1.06^2
3	7	$816.30 = $1,000/1.07^3
4	8	$735.03 = $1,000/1.08^4

Something about RATIO?????

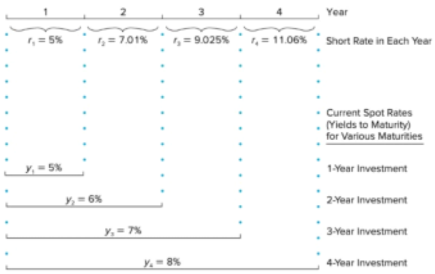
(1+y3)^3 is like the growth factor
To make the right hand side equal to the left, the value at the end of year 3 needs to be 1000, so then consider what would be the present value of the 2nd strategy which should be identical to the price of the left
To make different, change peooples expectations about what they can make in 2 years??

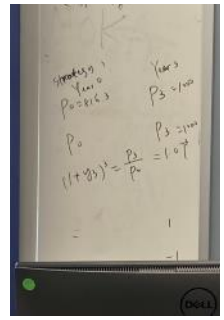


4	8	$735.03 = $1,000/1.08^4

Forward rate

- Consider two n-year investment strategies with equal rate of returns
  - buying and holding an n -year zero-coupon bond
  - buying an $(n-1)$-year zero and rolling over the proceeds into a 1-year bond
  - $(1+y_n)^n = (1+y_{n-1})^{n-1}(1+r_n)$
  - $r_n = \frac{(1+y_n)^n}{(1+y_{n-1})^{n-1}} - 1$
  - We also denote r_n as f_n to refer to **forward interest rate**





Interest rate uncertainty and forward rates

- Under certainty, $(1+r_1)(1+r_2) = (1+y_2)^2$
  - r_1: short rate this year
  - r_2: short rate next year
  - y_2: spot rate/YTM for a two-year zero-coupon bond
- What if r_2 is uncertain?
- There are three cases

Case 1: Only expected short rate matters

- Suppose $r_1 = 5\%$ and $E[r_2] = 6\%$
- The yield to maturity on a 2-year zero is

$$(1+y_2)^2 = (1+r_1)(1+E[r_2]) = 1.05\times1.06$$

Case 2: Short-term investor

- Consider a short-term investor who wishes to invest only for one year

- Two possible strategies
  - Strategy 1: Purchase the 1-year zero for $1,000/1.05

  - Strategy 2: Purchase the 2-year zero and sell it at the end of year 1. The expected rate of return in year 1 is 5%. The expected rate of return in year 2 is 6% with price $943.40

- However, the rate of return on the 2-year bond is risky. The price can higher or lower than $943.40

Case 2: Short-term investor

- To compensate the risk in the end price of year 1, the expected short rate $E[r_2]$ should be **less than** the forward rate f_2
- *Why?*
  - The short-term investors would prefer strategy 1, if the 2-year zero sells at $\frac{\$943.40}{1.05} = \$898.47$
  - Suppose short-term investors are indifferent only if 2-year zero sells at $881.47
  - At this price, the HPR for the 2-year zero is $\frac{\$943.40}{\$881.47} - 1 = 1.07 - 1 = 7\% > 5\%$
  - The YTM y_2 for the 2-year zero satisfies $(1 + y_2)^2 = \frac{\$1000}{\$881.47} = 1.1344$
  - The forward rate f_2 satisfies $1 + f_2 = \frac{(1+y_2)^2}{1+y_1} = \frac{1.1344}{1.05} = 1.08$
  - The expected short rate $E[r_2] = \frac{\$1000}{\$943.40} - 1 = 6\% < f_2$

Liquidity premium

- The liquidity premium is $f_2 - E[r_2]$: compensates short-term investors for the uncertainty about the price at which they will be able to sell their long-term bonds at the end of the year

Case 3: Long-term investor

- Consider a long-term investor who wishes to invest for a full 2-year period

- Two strategies:
  - Strategy 1: Purchase the 2-year zero for $890 and lock in a guaranteed yield to maturity of 6%
  - Strategy 2: Roll over two 1-year investments. In this case, an investment of $890 will grow in two years to $890 \times 1.05 \times (1 + r_2)$

- However, the rate of return in year 2 is uncertain

- To compensate the risk of interest rate in year 2, $E[r_2]$ exceeds f_2

$E[r_2]$ vs f_2

- The relationship between $E[r_2]$ and f_2 depends on

  - Investors' readiness to bear interest rate risk

  - Investors' willingness to hold bonds that do not correspond to their investment horizons