Lecture 6: Risk, return, derivatives continued

Monday, February 6, 2023 2:33 PM

6-risk-return-cont

QTM 385 Quantitative Finance

Lecture 6: Risk and return

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Suggested reading: Investments Ch 5

Question from Google form

- Question: Can we actually price the intrinsic value of derivatives correctly through simple models such as Black Scholes or Binomial? Or is the mechanism similar to stock price where profits come from mispriced derivatives and differences in investor expectations?

- *Answer: Intrinsic value is the value any given option would have if it were exercised today. If we know the distribution of stock price, then we can price the intrinsic value correctly*

Different people value the option on different models and different parameters

If we are clairvoyant and know the underlying distribution and know all the parameters then we can price the option correctly

Question from Google form

- Question: Are "return" and "yield" used synonymously?

- *Answer: Usually yield (to maturity in bonds) means the rate of return*

"Yield" is most frequently used in bonds. Also called "yield to maturity". We can think of yield to maturity as rate of return.

The return itself could be the raw return or the rate of return in capital. There is a subtle difference

Rate of return

- Rate of return: the percentage change in the value of an investment

$$r(T) = \frac{100}{P(T)} - 1$$

- Effective annual rate (EAR): the percentage increase in funds *per year*
  - Compare returns on investments with differing horizons
  - Accounts for compound return/interest

$$1 + EAR = \left(1 + r(T)\right)^{1/T}$$

Horizon, T	Price, $P(T)$	$r(T) = \frac{100}{P(T)} - 1$	EAR over Given Horizon
Half-year	\$97.36	100/97.36 − 1 = 0.0271 = 2.71%	$(1 + .0271)^2 - 1 = .0549$
1 year	\$95.52	100/95.52 − 1 = 0.0469 = 4.69%	$(1 + .0469) - 1 = 0.0469$
25 years	\$23.30	100/23.30 − 1 = 3.2918 = 329.18%	$(1 + 3.2918)^{1/25} - 1 = .060$

$$1 + EAR = \left(1 + r(0.5)\right)^{1/0.5}$$

$$1 + EAR = \left(1 + r(25)\right)^{1/25}$$

Usually the rate of return grows with the investment duration

TO make investment durations comparable, we talk about EAR

Why does the EAR relationship hold true?

EAR vs APR

- For short-term investments (with holding periods less than a year), rates of return are often annualized using simple interest that ignores compounding

- These are called annual percentage rates (APR)

- With n compounding periods per year, we can find EAR from APR by

$$1 + EAR = \left(1 + \frac{APR}{n}\right)^n$$

APR does not account for compound interest. Only accounts for simple interest. (Contrast with EAR which accounts for compound interest)

Continuous compounding

- As the number of compounding periods n gets larger, we effectively approach *continuous compounding (CC)*
- The continuous compounding rate r_{cc} is defined as the rate that satisfies

$$e^{r_{cc}} = \lim_{n\to\infty}\left(1 + \frac{APR}{n}\right)^n$$

where $e = 2.71828$ is the Euler's number

- Using the property $\lim_{x\to\infty}(1+x)^{1/x} = e$, we can solve r_{cc}

$$r_{cc} = \lim_{n\to\infty} \log\left(1 + \frac{APR}{n}\right)^n = \lim_{n\to\infty} \log\left(\left(1 + \frac{APR}{n}\right)^{n/APR}\right)^{APR} = \lim_{n\to\infty} \log e^{APR} = APR$$

Continuous compounding is a mathematical method used in finance to calculate the value of an investment over time, assuming that the interest rate is reinvested continuously. In continuous compounding, the interest earned in a given time period is immediately reinvested, leading to exponential growth of the investment over time. This is in contrast to simple or discrete compounding, where interest is calculated and reinvested at fixed intervals, such as annually or quarterly.

From <https://chat.openai.com/chat>

If the right hand side exists and is a finite number, we can always calculate the value of the RCC?

The point is to solve for the RCC
It turns out that the RCC IS EQUAL TO THE APR

When x converges to zero, it is equal to e
When we take the log of e^APR then it's equal to APR
So RCC = APR

Total return given continuous compounding

- Given a continuously compounded rate r_{cc}, the total return for any period T is

$$\exp(T \times r_{cc})$$

e^rcc = 1 + EAR

- Given a continuously compounded rate r_{cc}, the total return for any period T is

$$\exp(T \times r_{cc})$$

  - Simplify the calculation
  - For example, $APR = 18\%$ and the investment period is 2 years. Then

$$\lim_{n\to\infty}\left(1+\frac{APR}{n}\right)^{2n} = (\exp(r_{cc}))^2 = \exp(2r_{cc}) = \exp 0.36 = 1.4333$$

Rate of return is 43.33%

e^rcc = 1 + EAR

rcc = APR

EAR = (e^rcc) - 1

1 + r(T) = exp(T*rcc) = e^Trcc

r(T) for T = 2
1 + r(T) = exp(T*rcc) = e^Trcc
r(2) = e^(2*0.18) - 1 = (e^0.36) - 1
43.33%

T = 0.5
1 + r(0.5) = e^(0.5*0.18) -1 = (e^0.09) - 1

Question

- Question: A bank offers two alternative interest schedules for a savings account of $\$100{,}000$ locked in for 3 years: (a) a monthly rate of 1% and (b) an annually, continuously compounded rate, r_{cc} of 12%. Which alternative should you choose?

Answer:

For a, rate of return is 12.68%

(1+0.01)^12 = 1.1268

For b the rate of return is 12.75%

Exp(0.12) = 1.1275

So b is preferable

Lecture plan

- Risk and risk premiums
- Learning from historical data

Risk

- Any investment involves some uncertainty about future holding-period returns, and in many cases that uncertainty is considerable

- Sources of investment risk
  - macroeconomic fluctuations
  - changing fortunes of various industries
  - firm-specific unexpected developments

Holding-period returns

- The *holding-period return*, or HPR equals

$$\text{HPR} = \frac{\text{Ending price} - \text{Beginning price} + \text{Cash dividend}}{\text{Beginning price}}$$

$$= \underbrace{\frac{\text{Ending price} - \text{Beginning price}}{\text{Beginning price}}}_{\text{rate of capital gains}} + \underbrace{\frac{\text{Cash dividend}}{\text{Beginning price}}}_{\text{dividend yield}}$$

- Suppose the initial cost of investment is $\$100$. The end-of-year price is $\$110$ and cash dividends over the year amount to $\$4$. Then HPR of a year is

$$\text{HPR} = \frac{110 - 100 + 4}{100} = 0.14$$

HPR stands for "Holding Period Return," which is a measure of the return on an investment over a specific time period. It is calculated by taking the sum of all cash flows (such as dividends, coupon payments, and capital gains) received over the holding period and dividing it by the initial investment cost. The HPR is a way to calculate the total return on an investment and can be used to compare the performance of different investments over a similar time period.

From <https://chat.openai.com/chat>

For conventional firms, HPR mainly comes from second term. Mainly counts on dividend yield. Because the stock price is pretty stable.

Probability distribution of the HPR

- There is considerable uncertainty about share prices one year from now, so you cannot be sure about your eventual HPR

	A	B	C	D	E	F	G	H	I
1									
2									
3	Purchase Price =		$100			T-Bill Rate =	0.04		
4									
5							Squared		Squared
6	State of the		Year-End	Cash		Deviations	Deviations	Excess	Deviations
7	Economy	Probability	Price	Dividends	HPR	from Mean	from Mean	Returns	from Mean
8	Boom	0.25	126.50	4.50	0.3100	0.2124	0.0451	0.2700	0.0451
9	Normal growth	0.45	110.00	4.00	0.1400	0.0424	0.0018	0.1000	0.0018
10	Mild recession	0.25	89.75	3.50	−0.0675	−0.1651	0.0273	−0.1075	0.0273
11	Severe recession	0.05	46.00	2.00	−0.5200	−0.6176	0.3815	−0.5600	0.3815
12	Expected Value (mean)	SUMPRODUCT(B8:B11, E8:E11) =			0.0976				
13	Variance of HPR			SUMPRODUCT(B8:B11, G8:G11) =			0.0380		
14	Standard Deviation of HPR					SQRT(G13) =	0.1949		
15	Risk Premium					SUMPRODUCT(B8:B11, H8:H11) =		0.0576	
16	Standard Deviation of Excess Return						SQRT(SUMPRODUCT(B8:B11, I8:I11)) =		0.1949

Dividends vary as well since it depends on the earnings of the firm

Mean of the HPR

- **Mean return**, $E(r)$: probability-weighted average of the rates of return in each scenario

$$E(r) = \sum_{s} p(s)r(s)$$

  - $p(s)$: the probability of scenario s
  - $r(s)$: the HPR in scenario s
  - $E(r) = (.25 \times .31) + (.45 \times .14) + [.25 \times (-.0675)] + [.05 \times (-.52)] = .0976$

	A	B	C	D	E	F	G	H	I
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Variance of the HPR

- **Variance**, σ^2: the expected value of the *squared* deviation from the mean

$$Var(r) = \sigma^2 = \sum_{s} p(s)[r(s) - E(r)]^2$$

  - $\sigma^2 = .25(.31 - .0976)^2 + .45(.14 - .0976)^2 + .25(-.0675 - .0976)^2 + .05(-.52 - .0976)^2 = .0380$

	A	B	C	D	E	F	G	H	I
1									
2									
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Excess return and risk premium

- Excess return: difference between HPR and risk-free rate r_f

$$r(s) - r_f$$

  - Excess return is random
  - Risk-free rate is nonrandom (rate of return of T-bills)
- Risk premium: difference between expected HPR and risk-free rate r_f

$$E(r) - r_f$$

  - Risk premium is nonrandom

In the context of Holding Period Return (HPR), excess return refers to the difference between the HPR of an investment and a benchmark rate of return, such as a benchmark index or a risk-free rate. Excess return measures the performance of an investment over and above the benchmark rate, and is a way to assess the risk-adjusted return of an investment. A positive excess return indicates that an investment has performed better than the benchmark, while a negative excess return suggests underperformance. The calculation of excess return helps investors to determine whether an investment is generating a return that compensates for the level of risk taken.

From <https://chat.openai.com/chat>

E(rf) = rf because no bias or variance

Risk premium is E(excess return)

Std error of excess return = standard error of HPR
This is because risk free rate rf has 0 variance

Sharpe ratio

- Higher risk premium is always associated with higher risk (measured by SD)

Why should we choose the risk free return if the high risk premium has low variance?

The Sharpe Ratio is a risk-adjusted performance measure

- Higher risk premium is always associated with higher risk (measured by SD)

- Sharpe ratio (reward-to-volatility ratio): risk premium divided by standard deviation

$$\text{Sharpe ratio} = \frac{\text{Risk premium}}{\text{SD of excess return}}$$
$$= \frac{E(r) - r_f}{\sigma}$$

Why should we choose the risk free return if the high risk premium has low variance?

The Sharpe Ratio is a risk-adjusted performance measure used in finance to assess the return of an investment relative to its risk. It is calculated by dividing the excess return of an investment over the risk-free rate by the standard deviation of returns, which represents the investment's volatility or risk.

The Sharpe Ratio is used to determine whether an investment's return is proportional to the amount of risk taken. A higher Sharpe Ratio indicates that an investment has a better risk-return trade-off compared to other investments with a lower Sharpe Ratio. The Sharpe Ratio is widely used by investors and financial professionals to evaluate the performance of various investment opportunities and to make informed investment decisions.

In most cases, sharpe ratio is less than 1
If you get sharpe ratio higher than 2, you have bugs in your code

Lecture plan

- Risk and risk premiums

- Learning from historical data

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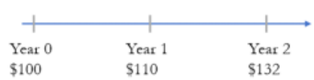
Expected returns and arithmetic average

- When we use historical data, we treat each observation as an equally likely "scenario". If there are n observations,

$$E(r) = \sum_s p(s)r(s) = \frac{1}{n}\sum_{i=1}^{n} r_i$$

  - Arithmetic average of historic rates of return

  - Example
    - $r_1 = \frac{110}{100} - 1 = 0.1$
    - $r_2 = \frac{132}{110} - 1 = 0.2$
    - $E(r) = \frac{(0.1+0.2)}{2} = 0.15$



However, $100 \times (1 + 0.15)^2 = 132.25 \neq 132$

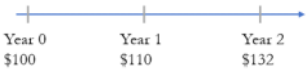
The geometric (time-weighted) average return

- *Geometric* or *compound* rate of return g: the fixed HPR that would compound to the same terminal value resulting from the sequence of actual returns in the time series

$$Initial\ value \times (1 + r_1) \times (1 + r_2) \times \ldots \times (1 + r_n) = Terminal\ value$$

$$(1 + g)^n = \frac{Terminal\ value}{Initial\ value}$$

  - *time-weighted* average return



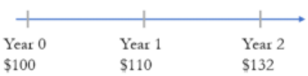
  - Example
    - $(1 + g)^n = 132/100$
    - $g = \left(\frac{132}{100}\right)^{1/2} - 1 = 14.89\%$

Estimating variance and standard deviation

- We can estimate the variance of the actual returns from historical data

$$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{s=1}^{n}(r(s) - \bar{r})^2$$

  - $\bar{r} = E(r) = \frac{1}{n}\sum_{s=1}^{n} r(s)$
  - The denominator $n - 1$ is to account for the degree of freedom
  - Example
    - $\hat{\sigma}^2 = \frac{1}{2-1}[(0.2 - 0.15)^2 + (0.1 - 0.15)^2] = 0.005$
    - Standard deviation $\hat{\sigma} = 0.005^{1/2} = 0.071$

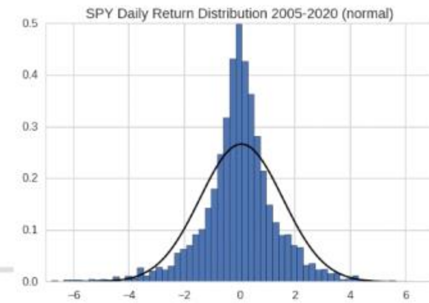


Normal approximation of distribution of returns

- Normal distribution appears naturally in many applications
  - E.g., the heights and weights of newborns, lifespans of many customers items
- Why does investment management use normal distribution?
  - Symmetricity: standard deviation is sufficient to capture the risk
  - Scenario analysis is simpler: only mean and variance are sufficient to estimate scenario probability
  - Easy to model statistical dependence of returns across assets: correlation is sufficient

Deviation from normality

- First deviation: Asymmetry in the probability distribution of returns
- Second deviation: Likelihood of extreme values on either side of the mean



Measuring asymmetry

- Skew measures asymmetry in the probability distribution of returns

$$\text{Skew} = \text{Average}\left[\frac{(r-\bar{r})^3}{\hat{\sigma}^3}\right]$$

  - E.g., large negative returns are more likely than large positive returns
  - Negative skew: extreme *bad* outcomes are more frequent than extreme positive ones (skew to the left, fatter left tail, underestimate risk)
  - Positive skew: opposite case (skew to the right)

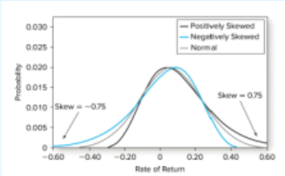


Figure 5.4 Normal and skewed distributions (mean = 6%, SD = 17%)

Measuring extreme values

- Kurtosis measures likelihood of extreme values on either side of the mean

$$\text{Kurtosis} = \text{Average}\left[\frac{(r-\bar{r})^4}{\hat{\sigma}^4}\right] - 3$$

  - Deviations are raised to the fourth power so more sensitive to extreme outcomes
  - Subtract by 3 because the kurtosis for the normal distribution is 3