Lecture 21: Bond Risk

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21-bond-risk

QTM 385 Quantitative Finance

Lecture 21: Bond duration and convexity

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Suggested reading: Investments Ch 16

There are only 4 ppl in class today

Interest rate sensitivity

- Bond *prices* and *yields* are *inversely related*: As yields increase, bond prices fall; as yields fall, bond prices rise
- When YTM is y, the bond price is $P = \sum_{t=1}^{T} \frac{Coupon}{(1+y)^t} + \frac{Par\ value}{(1+y)^T}$
- **Interest rate sensitivity**: The sensitivity of bond prices $\Delta P/P$ (percentage change in bond price) to changes in market interest rates and yields Δy

For zero-coupon, duration is TIME (to maturity?)

Measuring interest rate sensitivity by duration

- By taking the first-order derivative of P with respect to y, we can approximate $\Delta P/P$ by

$$\frac{\Delta P}{P} = -D \times \frac{\Delta(1+y)}{1+y} = -D^* \Delta y$$

Percentage change is linear?
If $\Delta y \uparrow$ bond price $\downarrow$

where $D^* = \frac{D}{1+y}$ is the modified duration, D is the Macaulay's duration defined as

Duration is always positive (duh)

$$D = \sum_{t=1}^{T} t \times w_t$$

and w_t is defined as (CF_t is the cash flow at time t and P is the current bond price)

$$w_t = \frac{CF_t/(1+y)^t}{P}$$

Why? derivative of bond price w/ respect to yield

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\overbrace{\sum_{t=1}^{T} \frac{CF_t}{(1+y)^t}}^{P} + \cancel{\frac{par}{(1+y)^t}}\right)$$

$$= \sum_{t=1}^{T}\left(-t \cdot \frac{CF_t}{(1+y)^{t+1}}\right)$$

Then duration is defined as derivative w/ respect to y.

Determinants of interest rate sensitivity

$$\sum_{t=1} \left(\quad \frac{}{(1+y)^{t+1}} \right)$$

Then duration is defined as derivative w/ respect to y.

$$\frac{\Delta P}{P} = \frac{1}{P}\left(\sum_{t=1}^{T} -t \frac{CF_t}{(1+y)^{t+1}}\right)\Delta y$$

then take out $-(1+y)$

$$= -\left(\sum_{t=1}^{T} \underbrace{\frac{CF_t/(1+y)^t}{P}}_{W_t} \cdot t\right) \cdot \frac{\Delta y}{1+y}$$

Determinants of interest rate sensitivity

$$\frac{\Delta P}{P} = -D^{*}\Delta y = \left(\sum_{t=1}^{T} t \times \frac{CF_t/(1+y)^t}{P}\right) \times \frac{\Delta(1+y)}{1+y}$$

- Determinants of interest rate sensitivity and duration include
  - Yield to maturity y
  - Time to maturity T
  - Coupon rate CF_t

So these can all affect % change

Calculating duration in excel

- Use the duration() and mduration() function to calculate Macaulay's duration and modified duration
  in excel
- Example: 2-year 8% semiannual coupon bond with 10% YTM

Inputs		Formula in column B
Settlement date	random/arbitrary 1/1/00	DATE(2000,1,1)
Maturity date	2 yrs after 1/1/02	DATE(2002,1,1)
Annual coupon date	0.08	= 8%
Yield to maturity	0.1	= 10%
Coupon per year	2	= Semi annual
Outputs		
Macaulay duration	1.8852	DURATION(B2,B3,B4,B5,B6)
Modified duration	1.7955	MDURATION(B2,B3,B4,B5,B6)

Par is $1000 by default

Alternative approach, calculates step by step

A	B	C	D	E	F	G
	Period	Time until payment (Years) (little t)	Cash Flow	PV of CF (Discount rate = 5% per period)	Weight* W_t	Column (C) times Column (F)
8% coupon bond	1	0.5	40	38.0952	0.0395	0.0197
	2	1	40	36.2812	0.0376	0.0376
	3	1.5	40	34.5535	0.0358	0.0537
	4	2	1040	855.6106	0.8871	1.7741
Sum				964.540	1.0000	1.8852

1000 × 0.08

coupon payments only

numerator in W_t equation

y = 0.05 bc YTM for whole year is 10%

bond price

Macaulay duration

Question

- Question: What is the duration and modified duration for a 2-year zero-coupon bond with 10% YTM? How does the duration change with the coupon rate?

No coupon rate

Duration is 2

modified 1.0948

If coupon rate ↑ then duration ↓ (but not at a constant rate)

Inputs =

1/1/00

1/1/02

0

0.1

2

Question

- Question: What is the duration and modified duration for a 4-year 8% semiannual coupon bond with 5% YTM? How does the duration change with the time to maturity?

Duration
↳ 3.5301
↳ 3.4490
Modified

time to maturity ↑ then duration ↑

but it increases at a decreasing rate (so not a constant rate)

Inputs:
1/1/00
1/1/04
0.08
0.05
2

Question

- Question: What is the duration and modified duration for a 2-year 8% semiannual coupon bond with 5% YTM? What if the YTM is 15%? How does the duration change with YTM?

Bond w/ lower YTM tends to be riskier

duration ↑ when YTM ↓

When you start with a higher YTM, the price Δ is smaller? So duration smaller So less risky

Inputs:

1/1/00	1/1/00
1/1/02	1/1/02
0.08	0.08
0.05	0.15
2	2
↓	↓
1.8910	1.8793
1.8449	1.7482

Comparing actual price change and duration formula

- Based on the duration formula $\Delta P/P = -D^{*}\Delta y$, the approximated price change is

$$\widetilde{\Delta P} = P\times(-D^{*}\Delta y) = 105.6430\times(-1.8449\times 0.02) = -3.8980 < -3.8064 = \Delta P$$

- The predicted change is more than the actual change

	2 year, 8% semiannual bond 5% YTM	
Settlement date	1/1/00	
Maturity date	1/1/02	
Annual coupon rate	0.08	
Yield to maturity	0.05	→ YTM = 5%
Redemption value (% of face value)	100	
Coupon payments per year	2	
Flat price (% of par)	105.6430	
	2% increase in YTM	
Settlement date	1/1/00	
Maturity date	1/1/02	
Annual coupon rate	0.08	
Yield to maturity	0.07	→ YTM = 7%
Redemption value (% of face value)	100	
Coupon payments per year	2	
Flat price (% of par)	101.8365	
Price difference	-3.8064	ΔP

Inputs		Formula in column B
Settlement date	1/1/00	DATE(2000,1,1)
Maturity date	1/1/02	DATE(2002,1,1)
Annual coupon rate	0.08	
Yield to maturity	0.05	
Coupon per year	2	
Outputs		
Macaulay duration	1.8910	DURATION(B2,B3,B4,B5,B6)
Modified duration	1.8449	MDURATION(B2,B3,B4,B5,B6)

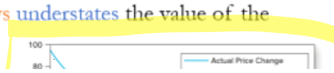
then $\Delta P = 105.6430 - 101.8365 = -3.8064$ $\quad -\frac{3.8}{105} \approx -1.89\cdot\frac{2\%}{1+5\%}$

The duration approximation underestimates the value

↳ & overestimates the risk

- The percentage change in the value of a bond *approximately equals* the product of modified duration times the change in the bond's yield
  - $\frac{\Delta P}{P} = -D^{*}\Delta y$
  - Approximates well for small changes
  - The duration approximation with slope $-D^{*}$ always understates the value of the bond

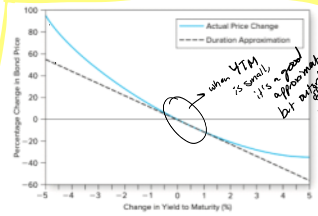
When using duration formula, ...



- Approximates well for small changes
- The duration approximation with slope $-D^*$ always understates the value of the bond

When using duration formula,
The predicted change is always smaller
than actual change

slope is − modified duration



when YTM is small, it's a good approximation but outside of y... not really

So it is a conservative approximation.

Convexity of a bond

- **Convexity** of a bond: the second derivative (the rate of change of the slope) of the price-yield curve divided by bond price →

$$\frac{\partial}{\partial y}\left(\frac{dP}{dy}\right) = \frac{d}{dy}\left(\sum_{t=1}^{T}\left(-t\,\frac{CF_t}{(1+y)^{t+1}}\right)\right)$$

$$= \sum_{t=1}^{T} -t\cdot\frac{CF_t}{(1+y)^{t+2}}\cdot(-(t+1))$$

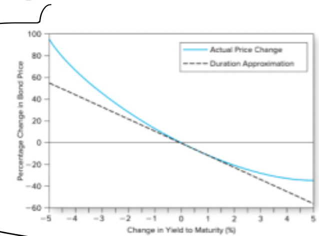
combine like terms
to get

$$= \sum_{t=1}^{T}\left(t(t+1)\cdot\frac{CF_t}{(1+y)^{t+2}}\right)$$

take out $(1+y)^2$
to get

$$Convexity = \frac{1}{P}\frac{\Delta^2 P}{\Delta y^2} = \frac{1}{P\times(1+y)^2}\sum_{t=1}^{T}\left[\frac{CF_t}{(1+y)^t}(t^2+t)\right]$$

assuming 1 in derivation to the right?



% change in bond price

Approximating price change by convexity

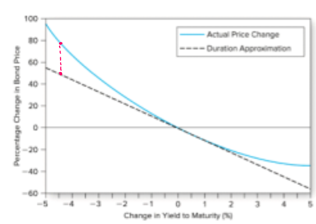
- Approximate the change in bond price by both duration and convexity

$$\frac{\Delta P}{P} = -D^*\Delta y + \underbrace{\frac{1}{2}\times Convexity\times(\Delta y)^2}$$

also non-negative

convexity is always non-negative?

- Similar to Taylor expansion up to 2nd order derivative
- The first term is the same as the duration rule
- The second term is the modification for convexity



Example

- Consider a 30-year, 8% annual bond selling at par (initial YTM is 8%)
- The modified duration is 11.26

In excel

Inputs		Formula in column B
Settlement date	1/1/00	DATE(2000,1,1)
Maturity date	1/1/30	DATE(2002,1,1)
Annual coupon rate	0.08	
Yield to maturity	0.08	
Coupon per year	1	
Outputs		
Macaulay duration	12.1584	DURATION(B2,B3,B4,B5,B6)
Modified duration	11.2578	MDURATION(B2,B3,B4,B5,B6)

In excel, no formula for convexity → use R

The duration is

$$D = \sum_{t=1}^{T} [t \times w_t] = \sum_{t=1}^{T} \left[t \times \frac{CF_t}{P \cdot (1+y)^t} \right] = 12.15841$$

The modified duration is

$$D^* = D/(1+y) = 11.25778$$

In R

She posted an R markdown notebook

```
P <- 1000
rate <- 0.08
y <- rate
year <- 30
CF <- rep(P * rate, year)
CF[year] <- CF[year] + P
duration <- 0
for (t in c(1:year)) {
    duration <- duration + t * CF[t]/((1 + y)^t)/P
}
duration
[1] 12.15841
duration
[1] 12.15841
modified.duration <- duration/(1 + y)
modified.duration
[1] 11.25778
```

Convexity

- Consider a 30-year, 8% annual bond selling at par (initial YTM is 8%)
- The modified duration is 11.26
- The convexity is 212.4

The convexity is

$$\text{convexity} = \frac{1}{P \times (1+y)^2} \sum_{t=1}^{T} \left[(t^2 + t) \times \frac{CF_t}{(1+y)^t} \right] = 212.4325$$

```
convexity <- 0
for (t in c(1:year)) {
    convexity <- convexity + (t^2 + t) * CF[t]/((1 + y)^t)
}
convexity <- convexity/(P * (1 + y)^2)
convexity
[1] 212.4325
```

Yield change and price change

- Consider a 30-year, 8% annual bond selling at par (initial YTM is 8%)
- The modified duration is 11.26
- The convexity is 212.4
- Suppose the yield is increased to 10%
- The price falls to $811.46, a decline of 18.85%

Settlement date	1/1/00
Maturity date	1/1/30
Annual coupon rate	0.08
Yield to maturity	0.1
Redemption value (% of	100
Coupon payments per yea	1
Flat price (% of par)	81.1462

Duration approximation

- Consider a 30-year, 8% annual bond selling at par (initial YTM is 8%)
- The modified duration is 11.26
- Suppose the yield is increased to 10%
- The price falls to \$811.46, a decline of 18.85%
- The duration rule would predict a price decline of 22.52%
  - $\frac{\Delta P}{P} = -D^{*}\Delta y = -11.26 \times 0.02 = -.2252$

→ confirms duration formula

overestimates risk & underestimates price

Duration with convexity approximation

- Consider a 30-year, 8% annual bond selling at par (initial YTM is 8%)
- The modified duration is 11.26
- The convexity is 212.4
- Suppose the yield is increased to 10%
- The price falls to \$811.46, a decline of 18.85%
- The duration-with-convexity rule would predict a price decline of 18.27%
  - $\frac{\Delta P}{P} = -D^{*}\Delta y + \frac{1}{2} \times convexity \times \Delta y^2 = -11.26 \times 0.02 + \frac{1}{2} \times 212.4 \times 0.02^2 = -.1827$
  - More accurate than the duration rule

much closer

→ we may underestimate the risk w/ convexity (overestimate price) in contrast to just the duration formula

Question

- Question: What if the yield only increases by 0.1%? How much does the bond price fall? What is the decline predicted by the duration rule? What about the duration-with-convexity rule?

- *Answer: The modified duration is* 11.26 *and the convexity is* 212.4
- *The duration rule would predict a price decline of* 1.1126%
  - $\frac{\Delta P}{P} = -D^{*}\Delta y = -11.26 \times 0.001 = -0.01126$
- *The duration-with-convexity rule would predict a price decline of* 1.115%
  - $\frac{\Delta P}{P} = -D^{*}\Delta y + \frac{1}{2} \times convexity \times \Delta y^2 = -11.26 \times 0.001 + \frac{1}{2} \times 212.4 \times 0.001^2 = -.01115$