Midterm Review

Wednesday, March 15, 2023 14:32

15-midterm-review

QTM 385 Quantitative Finance

Lecture 15: Midterm review

Instructor: Ruoxuan Xiong

EMORY

Question from Google form

- Question: Does the Silicon Valley Bank situation connect with anything we've learned?

- *Answer: Important reason of the collapse include the lack of diversification (high weight in long term bonds, bond value declines when interest rate increases) and bank run*

↳ classical bank run situation: No sufficient cash If investors want their $ from bank, not enough cash

- Question: Until what slides/class will the midterm cover?

- *Answer: Cover the material from Lectures 1-11 (up to Markowitz model)*

CAPM not covered

EMORY

Announcement

- Solution for homework 1 and 2 have been posted
- The midterm will be available from 03/17 12:00 am until 03/20 11:59pm at Quizzes on Canvas
- You can choose any 24h in between to finish it
- Once you decide to take it, you can open the quiz and the time starts to count
- Once you finish (within 24h), upload your solution (one pdf file) and click submit quiz on Canvas

NO CLASS ON MONDAY

can attach R file as supplementary

convert to pdf directly

Midterm

- Cover the material from Lectures 1-11 (up to Markowitz model)
- Similar problems as those in Homework 1 and 2
- You need to finish it independently
- Open book, open notes
- You cannot talk to anyone about the exam until 03/20 11:59 pm

Real assets vs financial assets

- Real assets: the land, buildings, machines, and knowledge that can be used to produce goods and services
  - Determine the productive capacity of its economy
  - Generate net income to the economy

→ main focus of this course

- Financial assets: claims to the income generated by real assets (or claims on income from the government)
  - E.g., stocks, bonds, and derivatives
  - Do not directly contribute to the productive capacity of the economy
  - Define the allocation of income or wealth among investors
  - Main focus of this course

HW 1 Problem 1

Problem 1

Lanni Products is a start-up computer software development firm. It currently owns computer equipment worth $30,000 and has cash on hand of $20,000 contributed by Lanni's owners. For each of the following transactions, identify the real and/or financial assets that trade hands. Are any financial assets created or destroyed in the transaction?

(a) Lanni takes out a bank loan. It receives $50,000 in cash and signs a note promising to pay back the loan over 3 years.

Suggested solution.

The bank loan $50,000 is the financial asset. The financial asset is created in this transaction.

(b) Lanni uses the cash from the bank plus $20,000 of its own funds to finance the development of new financial planning software.

Suggested solution.

Both the cash from the bank and $20,000 of its own funds are financial assets. The new financial planning software is the real asset. In this transaction, the financial asset is destroyed, while the real asset is created.

(c) Lanni sells the software product to Microsoft, which will market it to the public under the Microsoft name. Lanni accepts payment in the form of 1,250 shares of Microsoft stock.

Suggested solution.

The software is the real asset. The 1,250 shares of Microsoft stock are the financial assets. In this transaction, the real asset is destroyed, while the financial asset is created.

(d) Lanni sells the shares of stock for $100 per share and uses part of the proceeds to pay off the bank loan.

Suggested solution.

Both the shares of stock and bank loan are financial assets. In this transaction, financial asset is destroyed.

The bond market

- Bond market (as *fixed-income capital market*): longer term borrowing or debt instruments than those trade in the money market
- "treasury" US government as the issuer: Treasury bills, Treasury notes and Treasury bonds
- "municipal" State and local governments as the issuer: Municipal Bonds
  - *Equivalent taxable yield*: $r_{taxable}(1-t) = r_{muni}$
  - Attractive to high-tax-bracket investors
- Firms as the issuer: Corporate bonds
  - Secured bond, unsecured bond, callable bond

Exempt from fed tax & state tax, very appealing to high tax bracket investors bc $r_{muni} > r_{taxable}$

$\underbrace{\text{Secured}}$ → backed up by real assets

$\underbrace{\text{callable bond}}$ → firm can call back the bond at a rate higher than par value

Call back when interest rate is lower so you can refinance at lower rate But must pay a premium (coupon notes are higher than non-callable bonds)

HW 1 Problem 2

Problem 2

Find the equivalent taxable yield of a short-term municipal bond with a yield of 3% for tax brackets of

(a) zero

Suggested solution.

$$r_{taxable} = \frac{r_{muni}}{1-t} = \frac{0.03}{1-0} = 0.03$$

(b) 10%

Suggested solution.

$$r_{taxable} = \frac{r_{muni}}{1-t} = \frac{0.03}{1-0.1} = 0.033$$

HW 1 Problem 3

Problem 3

Looking at the Treasury bond maturing in May 2023 at the \$1,000 par value in Figure 1

LISTING OF TREASURY ISSUES					ASKED YIELD TO
MATURITY	COUPON	BID	ASKED	CHANGE	MATURITY
15-Feb-2019	2.750	100.0391	100.0547	0.0078	2.256
30-Apr-2021	2.250	99.7500	99.7656	0.2344	2.354
15-May-2023	1.750	97.4531	97.4688	0.4766	2.364
15-Aug-2029	6.125	132.7266	132.7891	1.1406	2.575
15-Feb-2036	4.500	125.4688	125.5313	1.5391	2.637
15-Aug-2048	3.000	101.8984	101.9297	1.5391	2.902

Figure 1: BID, ASKED and CHANGE columns are quoted as a percentage of par.

(a) How much would you have to pay to purchase one of these bonds?

Ask price: price at which you can buy on the market (price the holders are willing to sell)

Suggested solution.

You have to pay

multiply percentage by par value (\$1000)

$$1,000 \times 97.4688\% = 974.688$$

HW 1 Problem 3

(b) How much would you get to sell one of these bonds?

bid price (price where bond holders are willing to buy)

Suggested solution.

You can get

$$1,000 \times 125.4688\% = 1254.688$$

wrong screenshot but it's correct on the key

to sell one of these bonds.

(c) What was its ask price the previous day?

subtract 97.4688 and 0.4766

Suggested solution.

The ask price on the previous day is

$$125.5313 - 1.5391 = 123.9922.$$

```
125.5313 - 1.5391
[1] 123.9922
```

(d) What is its coupon rate? If the bond makes semiannual coupon payments, then how much would you get in each coupon payment?

Suggested solution.

The coupon rate is 1.750%. Each coupon payment is

$$1000 \times 4.5\%/2 = 22.5.$$

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Common stock

- *Common stocks*, also known as *equity securities* or **equities**, represent ownership shares in a corporation
- Owner of common stock
  - Vote on any matters of corporate governance (can vote by proxy)
  - Share of the financial benefits of ownership
- **Residual claim**: Stockholders are the last to claim assets and income
- **Limited liability**: Not personally liable for the firm's obligations
- The dividend yield is only part of the return on a stock investment
- Another part is prospective capital gains (i.e., price increases) or losses

HW 1 Problem 4

WILL HAVE SIMILAR QUESTION ON MIDTERM

Problem 4

Looking at the listing for Hess Midstream Partners in Figure 2

NAME	SYMBOL	CLOSE	NET CHG	VOLUME	52 WK HIGH	52 WK LOW	DIV	YIELD	P/E	YTD %CHG
Herbalife Nutrition	HLF	57.94	−1.39	1,149,773	60.41	34.16	1.20	2.07	47.75	−1.71
Herc Holdings	HRI	26.86	−0.71	389,826	72.99	24.16	....	....	3.10	3.35
Heritage Insurance Holdings	HRTG	14.57	−0.38	81,929	19.15	12.85	0.24	1.65	22.01	−1.02
Hersha Hospitality Trust Cl A	HT	16.59	−0.16	732,879	24.16	16.50	1.12	6.75	...dd	−5.42
Hershey	HSY	106.24	0.80	1,145,889	114.63	89.10	2.89	2.72	22.00	−0.88
Hertz Global Holdings	HTZ	13.27	−0.77	2,965,201	25.14	13.01	....	....	2.24	−2.78
Hess Corp.	HES	42.39	0.15	5,969,511	74.81	35.59	1.00	2.36	...dd	4.67
Hess Midstream Partners	HESM	17.87	0.25	47,899	24.51	16.17	1.43	8.00	14.60	5.24
Hewlett Packard Enterprise	HPE	13.18	−0.28	11,756,695	19.48	12.09	0.45	3.41	11.46	−0.23

Figure 2: Listing of stocks traded on the New York Stock Exchange.

(a) How many shares could you buy for \$1,000?

Suggested solution.

The number of shares that we could buy for \$1,000 is

$$\frac{1000}{17.87} = 55.96$$

(It is ok to round it to 55 shares)

```
1000/17.87
[1] 55.95971
```

HW 1 Problem 4

(b) What would be your annual dividend income from those shares?

Suggested solution.

The annual dividend income from those shares is

$$\frac{1000}{17.87} \cdot 1.43 = 80.02$$

```
1000/17.87 * 1.43
[1] 80.02238
```

(c) What must be Hewlett Packard's earnings per share?

Suggested solution.

The earnings per share are

$$17.87/14.6 = 1.22$$

```
17.87 /14.6
[1] 1.223973
```

(d) What was the firm's closing price on the day before the listing?

Suggested solution.

The closing price on the day before listing is

$$17.87 - (0.25) = 17.62$$

```
17.87 - (0.25)
[1] 17.62
```

Stock market index

- Price-weighted average: E.g., Dow Jones Industrial Average
  - When a stock splits, we need to find a new divisor, *d*, that leaves the index unchanged
  - When splits happen, the relative weights of stocks in the price-weighted average change
- Market-value-weighted index: E.g., S&P 500, NASDAQ
- Equally weighted indexes: Equally weighted average of the returns of each stock in an index

HW 1 Problem 5

Problem 5

Consider the three stocks in the following table. P_t represents price at time t, and Q_t represents shares outstanding at time t. Stock Z splits two for one in the last period.

(a) Calculate the rate of return on a price-weighted index of the three stocks for the first period ($t = 0$ to $t = 1$).

	P_0	Q_0	P_1	Q_1	P_2	Q_2
X	100	200	95	200	95	200
Y	200	100	220	100	220	100
Z	180	200	200	200	100	400

Suggested solution.

The price-weighted index on $t = 0$ is

$$I_{pw,0} = \frac{100 + 200 + 180}{3}$$

The price-weighted index on $t = 1$ is

$$I_{pw,1} = \frac{95 + 220 + 200}{3}$$

The rate of return is

$$I_{pw,1}/I_{pw,0} - 1 = \frac{(95 + 220 + 200)/3}{(100 + 200 + 180)/3} - 1 = 7.29\%$$

```
(95+220+200)/(100+200+180) - 1
[1] 0.07291667
```

Initial value can be set as an arbitrary value (whatever value you want)

Price multiplied by quantity will be the market cap

HW 1 Problem 5

(d) A market-value-weighted index.

Suggested solution.

Suppose the market-value-weighted index on $t = 0$ is

$$I_{mvw,0} = 1$$

The market-value-weighted index on $t = 1$ is

$$I_{mvw,1} = \frac{95 \cdot 200 + 220 \cdot 100 + 200 \cdot 200}{100 \cdot 200 + 200 \cdot 100 + 180 \cdot 200} = 1.0658$$

The rate of return is

$$I_{mvw,1}/I_{mvw,0} - 1 = 6.58\%$$

(e) An equally weighted index.

Suggested solution.

Suppose we invest 1 in X, 1 in Y, and 1 in Z on $t = 0$. Then on $t = 0$

The equally weighted index on $t = 0$ is

$$I_{ew,0} = 3$$

The equally weighted index on $t = 1$ is

$$I_{ew,1} = \frac{95}{100} + \frac{220}{200} + \frac{200}{180} = 3.16$$

The rate of return is

$$I_{ew,1}/I_{ew,0} - 1 = 5.37\%$$

Derivatives

- Derivative contracts, e.g., futures and options, provide payoffs that depend on the values of other variables such as commodity prices, bond and stock prices, interest rates, or market index values
- Options (purchase price is a *premium*)
  - A call option gives its holder the right to *purchase* an asset for a specified price, called the exercise or strike price, on or before a specified expiration date
  - A put option gives its holder the right to *sell* an asset for a specified exercise price on or before a specified expiration date
  - For both call and put options, each option contract is for the purchase of 100 shares.
- Futures (obligation)
  - A futures contract calls for delivery of an asset at a specified maturity date for an agreed-upon price, called the futures price, to be paid at contract maturity

HW 1 Problem 6

Problem 6

Suppose you buy a February 2023 expiration Apple call option with exercise price $135.

(a) Suppose the stock price in September is $140. Will you exercise your call? What is the profit on your position?

Suggested solution.

I will exercise the call because the exercise price is lower than the market price. The profit per share is

$$140 - 135 = 5.$$

If the contract has 100 shares, then the total profit is 500.

(b) What if you had bought the February 2023 call with exercise price $145?

Suggested solution.

I will not exercise the call as the exercise price is higher than the market price. The profit is 0.

If the contract has 100 shares, then the total profit is 0.

(c) What if you had bought a February 2023 put with exercise price $145?

Suggested solution.

I will exercise the put as the exercise price is higher than the market price. The profit is

$$145 - 140 = 5.$$

If the contract has 100 shares, then the total profit is 500.

Rate of return

- Rate of return
  - Effective annual rate (EAR): the percentage increase in funds *per year*
  - Annual percentage rates (APRs): Annualized using simple interest that ignores compounding
  - With n compounding periods per year, $1 + EAR = \left(1 + \frac{APR}{n}\right)^n$
  - As n gets larger, we approach *continuous compounding (CC)* $\left(1 + \frac{APR}{n}\right)^n \to e^{r_{cc}}$
    - $APR = r_{cc}$
  - HPR $= \dfrac{\text{Ending price of a share} - \text{Beginning price} + \text{Cash dividend}}{\text{Beginning price}}$

Initial price and end price

Evaluation of holding-period returns

- **Mean return**, $E(r)$: probability-weighted average of the rates of return in each scenario

$$E(r) = \sum_s p(s)r(s)$$

- **Variance**, σ^2: the expected value of the *squared* deviation from the mean

$$Var(r) = \sigma^2 = \sum_s p(s)[r(s) - E(r)]^2$$

- For an asset i
  - Holding-period return is denoted as r_i
  - Excess return is denoted as $R_i = r_i - r_f$
  - Risk premium is denoted as $E(R_i) = E(r_i) - r_f$

ri is a random variable

HW 2 Problem 1

Problem 1

Suppose your expectations regarding the stock price are as follows:

State of the Market	Probability	HPR (including dividends)
Boom	0.35	30%
Normal growth	0.30	10%
Recession	0.35	−10%

Compute the mean and standard deviation of the HPR on stocks.

Suggested solution.

The mean of the HPR is

$$E(r) = 0.35 \cdot 0.3 + 0.3 \cdot 0.1 + 0.35 \cdot (-0.1) = 0.1$$

```
mu <- 0.35 * 0.3 + 0.3 * 0.1 + 0.35 * (-0.1)
mu
[1] 0.1
```

The standard deviation of the HPR is

$$Var(r) = 0.35 \cdot (0.3 - E(r))^2 + 0.3 \cdot (0.1 - E(r))^2 + 0.35 \cdot (-0.1 - E(r))^2 =$$

and

$$\sigma = Var(r)^{0.5} = 0.1673$$

```
sigma_sq = 0.35 * (0.3 - mu)**2 + 0.3 * (0.1 - mu)**2 + 0.35 * (-0.1 - mu)**2
sqrt(sigma_sq)
[1] 0.167332
```

EMORY

Estimating evaluation metrics from historical data

- Suppose we have n observations. Return on day t is denoted as r_t
  - Arithmetic average return $\bar{r} = \frac{1}{n}\sum_{t=1}^{n} r_t$
  - Geometric average return $g = \left((1 + r_1) \times \ldots \times (1 + r_n)\right)^{1/n} - 1$
  - Variance $\hat{\sigma}^2 = \frac{1}{n-1}\sum_{s=1}^{n}[r_t - \bar{r}]^2$ (you can use $\frac{1}{n}$ when n is large)
  - Sharpe ratio $(\bar{r} - r_f)/\hat{\sigma}$
  - Skew $= \frac{1}{n}\sum_{t=1}^{n}\left[\frac{(r_t - \bar{r})^3}{\hat{\sigma}^3}\right]$
  - Kurtosis $= \frac{1}{n}\sum_{t=1}^{n}\left[\frac{(r_t - \bar{r})^4}{\hat{\sigma}^4}\right] - 3$

Skew is third moment, kurtosis is fourth moment

Normal distr is used as a benchmark for kurtosis
MAKE SURE TO SUBTRACT 3

EMORY

Estimating evaluation metrics from historical data

- Suppose we have n observations. On day t, return of security a and b are denoted as r_t^a and r_t^b
  - Let the arithmetic average return of a and b be $\bar{r}^a$ and $\bar{r}^b$
  - Let the standard error of return of a and b be $\hat{\sigma}^a$ and $\hat{\sigma}^b$
  - Covariance between a and b is $cov(r_t^a, r_t^b) = \frac{1}{n}\sum_{t=1}^{n}(r_t^a - \bar{r}^a)(r_t^b - \bar{r}^b)$
  - Correlation between a and b is $corr(r_t^a, r_t^b) = \frac{cov(r_t^a, r_t^b)}{\hat{\sigma}^a\hat{\sigma}^b}$

EMORY

Estimating evaluation metrics from historical data

- **$q\%$ value at risk** ($q\%$ VaR): $(100-q)\%$ of returns will exceed the VaR and $q\%$ of returns will be worse
  - Sort the observations from low to high
  - VaR is the return at the q-th percentile of the sample distribution
- $q\%$ expected shortfall ($q\%$ ES): expected loss given that the returns fall below $q\%$ VaR
  - Identify the worst $q\%$ of all observations and take their average

Can use quantile function to calculate VaR in R or Python

HW 2 Problem 2

Problem 2

Visit Professor Kenneth French's data library Web site: https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html and download the monthly returns of "Fama/French 3 Factors" from January 1927–December 2022. Split the sample in half and compute the average, SD, skew, kurtosis, 1% value at risk (VaR) and 1% expected shortfall (ES) for the market (Mkt-RF) factor, small minus big (SML) factor, and high minus low (HML) factor for the two halves. Do the three split-halves statistics suggest to you that returns come from the same distribution over the entire period?

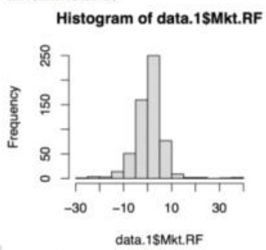
Suggested solution.

```
data <- read.csv("ff-3factors.csv")
data <- data[data$X >= 192701, ]
# split data in half
data.1 <- data[1: (nrow(data)/2),]
data.2 <- data[(nrow(data)/2+1):nrow(data),]
```

```
# write a function to compute mean, sd, skew, kurtosis
# 1\% value at risk (VaR) and 1\% expected shortfall (ES)
summary_statistics <- function(ret) {
  ret.mu <- mean(ret)
  ret.sd <- sd(ret)
  ret.skew <- mean((ret - ret.mu)**3/ret.sd**3)
  ret.kurtosis <- mean((ret - ret.mu)**4/ret.sd**4) - 3
  ret.VaR <- quantile(ret, 0.01)
  ret.ES <- sum(ret * (ret < ret.VaR)) / sum(ret < ret.VaR)
  out <- c(ret.mu, ret.sd, ret.skew, ret.kurtosis, ret.VaR, ret.ES)
  out <- setNames(out, c("mean", "sd", "skew", "kurtosis", "VaR", "ES"))
  return(out)
}
```

First sample: the market factor

```
hist(data.1$Mkt.RF)
```

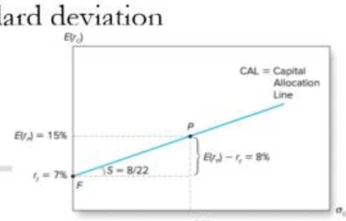



```
summary_statistics(data.1$Mkt.RF)
     mean        sd      skew  kurtosis         VaR          ES
0.6217014 6.0834507 0.4781621 8.0265846 -16.6925000 -22.2483333
```

Portfolios of one risky asset and a risk-free asset

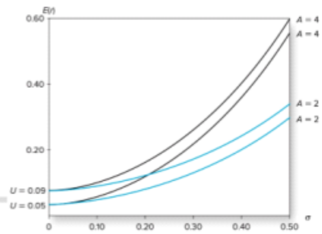
- **Asset allocation** of a complete portfolio: y in the risky assets and $1-y$ in the risk-free assets
- Rate of return: $r_C = y r_P + (1-y) r_f$
- Expected return: $E[r_C] = r_f + y[E(r_P) - r_f]$
- Standard error: $\sigma_C = y\sigma_P$
- Sharpe ratio: $S = \frac{E(r_P)-r_f}{\sigma_P}$ (does not depend on y)
- Capital allocation line (CAL): Expected return-standard deviation
  - $E[r_C] = r_f + S \cdot \sigma_C$

For whatever y we choose, sharpe ratio is the same



Risk tolerance and asset allocation

- Investors would like to choose y to maximize the utility given risk aversion A
  - $max_y U = E(r_C) - \frac{1}{2}A\sigma_C^2 = r_f + y[E(r_P) - r_f] - \frac{1}{2}Ay^2\sigma_P^2$
  - Optimal $y = \frac{E(r_P)-r_f}{A\sigma_P^2}$
- Indifference curve: $E(r)$ vs. σ given A such that utility level is the same
  $U = E[r] - \frac{1}{2}\times A\times\sigma^2$



Utility will always be greater than zero and will always be bounded by the risk free rate because we know we can always invest 0% in risky assets

HW 2 Problem 3

(f) Your client's degree of risk aversion is $A = 3$.

What proportion, y, of the total investment should be invested in your fund?

- What proportion, y, of the total investment should be invested in your fund?
- What are the expected value and standard deviation of the rate of return on your client's optimized portfolio?
- What is your client's utility score of this optimized portfolio?
- Draw the indifference curve for your client in the expected return–standard deviation plane corresponding to the utility score of this optimized portfolio.

Suggested solution.

When the risk aversion is $A = 3$, the proportion y is

$$y = \frac{E(r_P) - r_f}{A\sigma_P^2} = \frac{0.2 - 0.05}{3 \cdot 0.03^2} = \frac{5}{9}.$$

The expected return is

$$E(r_C) = 0.05 + \frac{5}{9}[0.2 - 0.05] = 0.133.$$

The standard deviation of the return is

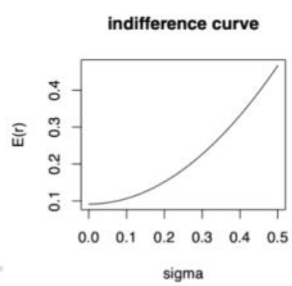
$$\sigma_C = y\sigma_P = \frac{5}{9} \cdot 0.3 = 0.167.$$

The utility score is

$$U = E(r_C) - 1/2 \cdot A \cdot \sigma_C^2 = 0.133 - 1/2 \cdot 3 \cdot 0.167^2 = 0.0917.$$

HW 2 Problem 3

```
A = 3
mu_C = 0.05 + 5/9 * (0.2 - 0.05)
sigma_C = 5/9 * 0.3
U = mu_C - 1/2 * A * sigma_C^2
calc_ret <- function(U, A, sig) {
    ret <- U + 1/2 * A * sig^2
    return(ret)
}
sig.seq <- seq(0, 0.5, 0.001)
ret.seq <- c()
for (sig in sig.seq) {
    ret <- calc_ret(U, A, sig)
    ret.seq <- c(ret.seq, ret)
}
plot(sig.seq, ret.seq, type = "l", xlab = "sigma", ylab = "E(r)",
    main = "indifference curve")
```



Portfolios of two risky assets

- Allocation decision of two-asset risky portfolios to construct a risky portfolio P: a bond portfolio D and a stock portfolio E
  - Can be any two stock portfolios, or two bond portfolios
  - w_D in the bond portfolio, $w_E = 1 - w_D$ in the stock portfolio
  - Return $r_P = w_D r_D + w_E r_E$
  - Expected return $E(r_P) = w_D E(r_D) + w_E E(r_E)$
  - Variance $\sigma_P^2 = w_D^2 \cdot \sigma_D^2 + w_E^2 \cdot \sigma_E^2 + 2 w_D w_E Cov(r_D, r_E)$

  - Minimum-variance portfolio: $w_D^* = \frac{\sigma_E^2 - Cov(r_D, r_E)}{\sigma_D^2 + \sigma_E^2 - 2Cov(r_D, r_E)}$

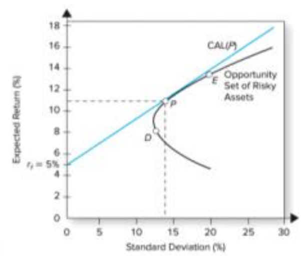
Portfolios of two risky assets

- Utility maximization portfolio (given A)
  - $w_D^* = \frac{E(r_D) - E(r_E) + A(\sigma_E^2 - Cov(r_D, r_E))}{A(\sigma_D^2 + \sigma_E^2 - 2Cov(r_D, r_E))}$

- Sharpe ratio maximization portfolio (R_D and R_E are excess return)
  - $w_D^* = \frac{E(R_D)\sigma_E^2 - E(R_E)Cov(R_D, R_E)}{E(R_D)\sigma_E^2 + E(R_E)\sigma_D^2 - [E(R_D) + E(R_E)]Cov(R_D, R_E)}$

- Investment opportunity set
  - Combinations of r_P and σ_P for varying w_D

- CAL with highest Sharpe ratio
  - Tangent with the investment opportunity set



HW 2 Problem 4

(c) Draw a tangent line from the risk-free rate to the opportunity set. What does your graph show for the expected return and standard deviation of the optimal portfolio?

(d) Solve numerically for the proportions of each asset and for the expected return and standard deviation of the optimal risky portfolio.

Suggested solution.

The optimal weight is

$$w_B^* = \frac{E(R_B)\sigma_S^2 - E(R_S)Cov(R_B, R_S)}{E(R_B)\sigma_S^2 + E(R_S)\sigma_B^2 - [E(R_S) + E(R_B)]Cov(R_B, R_S)} = 0.75$$

$$w_S^* = 1 - w_B^* = 0.25$$

The expected return is

$$E(r_P) = w_B E(r_B) + w_S E(r_S) = 0.75 \cdot 0.1 + 0.25 \cdot 0.2 = 0.125$$

The standard deviation is

$$\sigma_P = (w_B^2\sigma_B^2 + w_S^2\sigma_S^2 + 2w_B w_S cov(r_B, r_S))^{1/2} = 0.116.$$

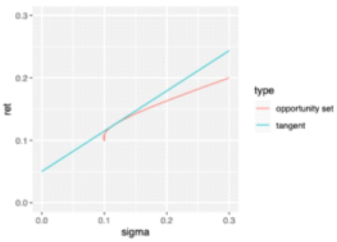
```
r_f <- 0.05
r_B <- 0.1
r_S <- 0.2
sigma_B <- 0.1
sigma_S <- 0.3
rho_BS <- 0.2
```

Suggested solution.

```
R_B <- r_B - r_f
R_S <- r_S - r_f
w_B <- (R_B * sigma_S^2 - R_S * rho_BS * sigma_S * sigma_B)/(R_B *
    sigma_S^2 + R_S * sigma_B^2 - (R_S + R_B) * rho_BS * sigma_S *
    sigma_B)
# weight
w_B
[1] 0.75
w_S <- 1 - w_B
ret_P <- w_B * r_B + w_S * r_S
# expected return
ret_P
[1] 0.125
```

HW Problem 4

```
sigma_sq_P <- w_B^2 * (sigma_B^2) + w_S^2 * (sigma_S^2) + 2 *
    w_B * w_S * (rho_BS * sigma_B * sigma_S)
sigma_P <- sqrt(sigma_sq_P)
# standard deviation
sigma_P
[1] 0.1161895
Sharpe <- (ret_P - r_f)/sigma_P
# Sharpe ratio
Sharpe
[1] 0.6454972
sigma_tangent_seq <- seq(0, 0.3, 0.01)
ret_tangent_seq <- c()
for (sig in sigma_tangent_seq) {
    ret <- r_f + sig * Sharpe
    ret_tangent_seq <- c(ret_tangent_seq, ret)
}
w_B_seq <- seq(0, 1, 0.001)
ret_P_seq <- c()
sigma_P_seq <- c()
for (w_B in w_B_seq) {
    out <- calc_ret_sd(w_B, r_B, r_S, sigma_B, sigma_S)
    ret_P_seq <- c(ret_P_seq, out[1])
    sigma_P_seq <- c(sigma_P_seq, out[2])
}
df <- data.frame(ret = c(ret_tangent_seq, ret_P_seq), sigma = c(sigma_tangent_seq,
    sigma_P_seq), type = c(rep("tangent", length(ret_tangent_seq)),
    rep("opportunity set", length(ret_P_seq))))
library(ggplot2)
ggplot(data = df, aes(x = sigma, y = ret)) + geom_path(aes(colour = type)) +
    ylim(0, 0.3)
```



HW 2 Problem 4

(f) You require that your portfolio yield an expected return of 12%, and that it be efficient, that is, on the steepest feasible CAL.
- What is the standard deviation of your portfolio?
- What is the proportion invested in the money market fund and each of the two risky funds?

Suggested solution.

The standard deviation is

$$\sigma_P = \frac{E(r_P) - r_f}{S} = \frac{0.12 - 0.05}{0.645} = 0.108$$

```
(0.12 - 0.05)/Sharpe
[1] 0.1084435
```

We first solve y from the following equation

$$0.12 = 0.05 + y[0.125 - 0.05]$$

Then

$$y = \frac{0.12 - 0.05}{0.125 - 0.05} = 0.933$$

The proportion invested in the money market fund is $1 - y = 0.067$.

The proportion invested in the bond fund is $y \cdot w_B^* = 0.933 \cdot 0.75 = 0.7$.

The proportion invested in the stock fund is $y \cdot w_S^* = 0.933 \cdot 0.25 = 0.233$.

EMORY

HW 2 Problem 4

(g) If you were to use only the two risky funds and still require an expected return of 12%, what would be the investment proportions of your portfolio? Compare its standard deviation to that of the optimized portfolio in (f). What do you conclude?

Suggested solution.

We solve w_B from

$$E(r_P) = w_B E(r_B) + (1 - w_B)E(r_S) = w_B(E(r_B) - E(r_S)) + E(r_S) = 0.12$$

i.e.,

$$w_B = \frac{0.12 - 0.2}{0.1 - 0.2} = 0.8$$

and

$$w_S = 1 - w_B = 0.2.$$

The standard deviation is

$$\sigma_P^2 = w_B^2\sigma_B^2 + w_S^2\sigma_S^2 + 2w_B w_S cov(r_B, r_S) = 0.8^2 \cdot 0.1^2 + 0.2^2 \cdot 0.3^2 + 2 \cdot 0.8 \cdot 0.2 \cdot 0.2 \cdot 0.3 \cdot 0.1 = 0.109.$$

```
calc_ret_sd(0.8, r_B, r_S, sigma_B, sigma_S)
[1] 0.1200000 0.1091788
```

The standard deviation is higher than that of the optimized portfolio in (f). Therefore, it is better to use the money market fund to construct an investment portfolio.

EMORY

HW 2 Problem 5

Problem 5

Let R_B be the rate of excess return on the bond fund and R_S be the rate of return on the stock fund. Let the variance of R_B be σ_B^2, the variance of R_S be σ_S^2, and the covariance between R_B and R_S be $Cov(R_B, R_S)$.

Suppose a portfolio has w_B proportion invested in the bond fund and the remainder $w_S = 1 - w_B$ in the stock fund. Show that the weight w_B that maximizes the Sharpe ratio equals

$$w_B^* = \frac{E(R_B)\sigma_S^2 - E(R_S)Cov(R_B, R_S)}{E(R_B)\sigma_S^2 + E(R_S)\sigma_B^2 - [E(R_S) + E(R_B)]Cov(R_B, R_S)}.$$

Suggested solution.

Note that the w_B that maximizes the Sharpe ratio S_P also maximizes the squared Sharpe ratio S_P^2.

We solve the w_B from the first order condition

$$\frac{dS_P^2}{dw_B} = 0.$$

Recall the definition of Sharpe ratio, we have S_P^2 equal to

$$S_P^2 = \frac{E(R_P)^2}{\sigma_P^2} = \frac{(w_B E(R_B) + (1 - w_B)E(R_S))^2}{w_B^2\sigma_B^2 + (1 - w_B)^2\sigma_S^2 + 2w_B(1 - w_B)Cov(R_B, R_S)}.$$

Then the derivative of S_P^2 with respect to w_B is

$$\begin{aligned}
\frac{dS_P^2}{dw_B} =& \frac{1}{\sigma_P^4}\left[\frac{dE(R_P)^2}{dw_B}\cdot\sigma_P^2 - E(R_P)^2\frac{d\sigma_P^2}{dw_B}\right] \\
=& \frac{1}{\sigma_P^4}\left[2[E(R_S) + w_B(E(R_B) - E(R_S))]\cdot(E(R_B) - E(R_S))\cdot\sigma_P^2\right. \\
& \left.-(E(R_S) + w_B(E(R_B) - E(R_S)))^2\left[2w_B\sigma_B^2 + 2(w_B - 1)\sigma_S^2 + 2(1 - 2w_B)Cov(R_B, R_S)\right]\right].
\end{aligned}$$

w_B satisfies $\frac{dS_P^2}{dw_B} = 0$, then w_B also satisfies the following equation

$$\begin{aligned}
0 =& (E(R_B) - E(R_S))\cdot\left[w_B^2\sigma_B^2 + (1 - w_B)^2\sigma_S^2 + 2w_B(1 - w_B)Cov(R_B, R_S)\right] \\
& - (E(R_S) + w_B(E(R_B) - E(R_S)))\left[w_B\sigma_B^2 + (w_B - 1)\sigma_S^2 + (1 - 2w_B)Cov(R_B, R_S)\right].
\end{aligned}$$

/e rearrange terms in the above equal and have

$$\begin{aligned}
0 =& (E(R_B) - E(R_S))\cdot\left\{\left[w_B^2\sigma_B^2 + (1 - w_B)^2\sigma_S^2 + 2w_B(1 - w_B)Cov(R_B, R_S)\right]\right. \\
& \left.- \left[w_B^2\sigma_B^2 + w_B(w_B - 1)\sigma_S^2 + w_B(1 - 2w_B)Cov(R_B, R_S)\right]\right\} \\
& - E(R_S)\left[w_B\sigma_B^2 + (w_B - 1)\sigma_S^2 + (1 - 2w_B)Cov(R_B, R_S)\right] \\
=& (E(R_B) - E(R_S))\cdot\left[(1 - w_B)\sigma_S^2 + w_B Cov(R_B, R_S)\right] \\
& - E(R_S)\left[w_B\sigma_B^2 + (w_B - 1)\sigma_S^2 + (1 - 2w_B)Cov(R_B, R_S)\right] \\
=& E(R_B)(1 - w_B)\sigma_S^2 + (E(R_B) + E(R_S))w_B Cov(R_B, R_S) - E(R_S)Cov(R_B, R_S) - E(R_S)w_B\sigma_B^2
\end{aligned}$$

/e move the terms that contain w_B to the left hand size, and then we have

$$w_B\left[E(R_B)\sigma_S^2 + E(R_S)\sigma_B^2 - (E(R_B) + E(R_S))w_B Cov(R_B, R_S)\right] = E(R_B)\sigma_S^2 - E(R_S)Cov(R_B, R_S).$$

'herefore the optimal w_B is

$$w_B^* = \frac{E(R_B)\sigma_S^2 - E(R_S)Cov(R_B, R_S)}{E(R_B)\sigma_S^2 + E(R_S)\sigma_B^2 - [E(R_S) + E(R_B)]Cov(R_B, R_S)}.$$

Markowitz model

- Markowitz model: portfolio optimization model, also called mean-variance model (asset allocation among n assets)
- Minimum variance frontier: A graph of the lowest possible variance that can be attained for a given portfolio expected return
- Efficient frontier is the portion of the minimum-variance frontier that lies above the global minimum-variance portfolio
- Two equivalent approaches to construct efficient frontier
  - Approach 1: Minimize variance for any target expected return μ
  - Approach 2: Maximize expected return for any target risk level σ^2

WILL HAVE CONCEPTUAL QUESTIONS ABOUT THIS PART

EMORY