Lecture 7: Capital allocation

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7-capital-allocation

QTM 385 Quantitative Finance

Lecture 7: Capital allocation to risky assets

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Suggested reading: Investments Ch 6

Lecture plan

- Learning from historical data
- Risk aversion and utility values
- Portfolios of one risky asset and a risk-free asset

Expected returns and arithmetic average

- When we use historical data, we treat each observation as an equally likely "scenario". If there are n observations,

$$E(r) = \sum_s p(s)r(s) = \frac{1}{n}\sum_{i=1}^{n} r_i$$

  - Arithmetic average of historic rates of return
  - Example
    - $r_1 = \frac{110}{100} - 1 = 0.1$
    - $r_2 = \frac{132}{110} - 1 = 0.2$
    - $E(r) = \frac{(0.1+0.2)}{2} = 0.15$

Year 0	Year 1	Year 2
\$100	\$110	\$132

However, $100\times(1+0.15)^2 = 132.25 \neq 132$

N days
Suppose at the beginning of year 0 the stock sells at 100 and so on
Calculate rate of return from year 0 to year 1 and then from year 1 to year 2
Then average

Problem:
But suppose the mean rate of return is 15%. What is the end price at the end of year 2? If you work backwards, it's 132.25. But we finished at 132. So there's a discrepancy.

The geometric (time-weighted) average return

- *Geometric* or *compound* rate of return g: the fixed HPR that would compound to the same terminal value resulting from the sequence of actual returns in the time series

$$Initial\ value \times (1 + r_1) \times (1 + r_2) \times \ldots \times (1 + r_n) = Terminal\ value$$

So this is a much better way of calculating the average return

$$(1 + g)^n = \frac{Terminal\ value}{Initial\ value}$$

- *time-weighted* average return

Year 0 $100 — Year 1 $110 — Year 2 $132

- Example
  - $(1 + g)^n = 132/100$
  - $g = \left(\frac{132}{100}\right)^{1/2} - 1 = 14.89\%$

Estimating variance and standard deviation

- We can estimate the variance of the actual returns from historical data

$$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{s=1}^{n}(r(s) - \bar{r})^2$$

Use variance as an important measure of the risk

- $\bar{r} = E(r) = \frac{1}{n}\sum_{s=1}^{n} r(s)$

- The denominator $n - 1$ is to account for the degree of freedom

Since we're not dealing with a lot of df we can use n-1 or n

- Example
  - $\hat{\sigma}^2 = \frac{1}{2-1}[(0.2 - 0.15)^2 + (0.1 - 0.15)^2] = 0.005$
  - Standard deviation $\hat{\sigma} = 0.005^{1/2} = 0.071$

Year 0 $100 — Year 1 $110 — Year 2 $132

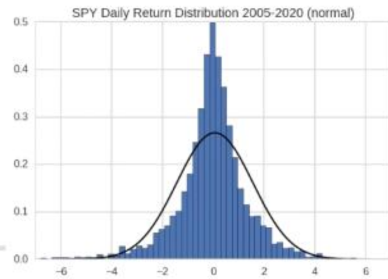
Normal approximation of distribution of returns

- Normal distribution appears naturally in many applications
  - E.g., the heights and weights of newborns, lifespans of many customers items
- Why does investment management use normal distribution?
  - Symmetricity: standard deviation is sufficient to capture the risk
  - Scenario analysis is simpler: only mean and variance are sufficient to estimate scenario probability
  - Easy to model statistical dependence of returns across assets: correlation is sufficient

Histogram of daily returns
Superimposed with normal distribution, you can see that it is a good approximation
Then we can use the SD to calculate risk and the symmetricity makes it easier for us

Deviation from normality

- First deviation: Asymmetry in the probability distribution of returns
- Second deviation: Likelihood of extreme values on either side of the mean



Much more likely to cluster around zero,

Or highly negative returns but few positive returns introduces skew, etc.

Measuring asymmetry

- Skew measures asymmetry in the probability distribution of returns

$$\text{Skew} = \text{Average}\left[\frac{(r-\bar{r})^3}{\hat{\sigma}^3}\right]$$

  - E.g., large negative returns are more likely than large positive returns
  - Negative skew: extreme *bad* outcomes are more frequent than extreme positive ones (skew to the left, fatter left tail, underestimate risk)
  - Positive skew: opposite case (skew to the right)

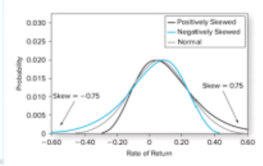


Figure 5.4 Normal and skewed distributions (mean = 6%, SD = 17%)

Gray line is the normal distribution
Consider two distributions that are asymmetric:
positively skewed (black) means more likely to have positive rate of returns
and negatively skewed (blue) means more likely to have negative rate of returns

Measuring extreme values

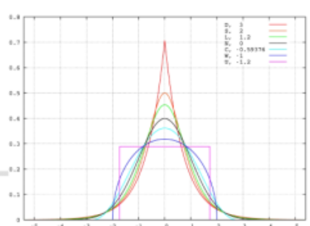
- Kurtosis measures likelihood of extreme values on either side of the mean

$$\text{Kurtosis} = \text{Average}\left[\frac{(r-\bar{r})^4}{\hat{\sigma}^4}\right] - 3$$

  - Deviations are raised to the fourth power so more sensitive to extreme outcomes
  - Subtract by 3 because the kurtosis for the normal distribution is 3

Use normal as a benchmark

Why is kurtosis a good measure?
If deviation is very large, and we take power, it will contribute a lot compared to if deviation is very small

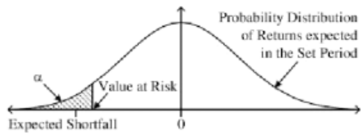


Black is the normal distribution with kertosis of 0
Then the purple has a uniform distribution so its kertosis is –1.2 because it's impossible to get a value larger than +-2 and also has zero probability of extreme events

Highest kertosis has the most density in the center and then drops quickly, and has relatively high density at the tail as compared to the uniform or the normal

Value at risk

- Practitioners are concerned about vulnerability to large losses
- One measure of downside risk: $q\%$ **value at risk** ($q\%$ VaR)
  - How much would I lose if my return was in the $q\%$ of the distribution
  - **VaR** differs from *Var*, the abbreviation for variance
  - Quantile of a distribution
  - Example: 1% VaR: 99% of returns will exceed the VaR and 1% of returns will be worse



If the return is in the q% of the distribution, how much would I lose?

The q% is the percentile of a distribution
e.g. there is q% of a change for the daily return to be lower than this value and a 100-q% chance for the daily return to be higher

Estimating VaR

- If portfolio/asset returns are normally distribution, then

$$VaR(1\%, normal) = Mean - 2.33\ SD$$

SD for standard deviations

  - -2.33 is the first percentile of the standard normal distribution (with mean = 0 and SD = 1)

e.g. mu +- 1.966 for 95% CI
So each would be 2.5 percentile and 97.5 percentile

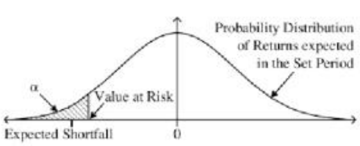
- Estimating VaR from the historical data
  - Sort the observations from high to low
  - VaR is the return at the $q\%$ percentile of the sample distribution

Expected shortfall

- Another measure of downside risk: $q\%$ **Expected shortfall (ES)** or **conditional tail expectation (CTE)**
  - Expected loss given that we find ourselves in one of the $q\%$ worst-case scenarios
  - As a comparison, $q\%$ VaR is the most optimistic measure of bad-case outcomes as it takes highest return (smallest loss) of all these cases

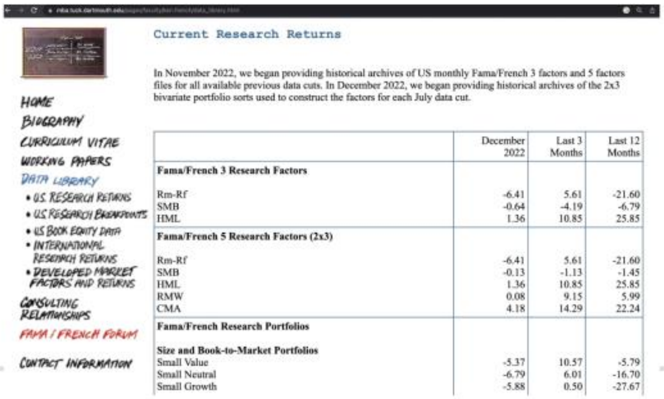
Expected shortfall accounts for ALL the worst case scenarios in the dashed region (in all the q%)
Then calculates the average loss

- Estimating ES from the historical data
  - 1% ES: identifying the worst 1% of all observations and taking their average



Example

- https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html



Current Research Returns

In November 2022, we began providing historical archives of US monthly Fama/French 3 factors and 5 factors files for all available previous data cuts. In December 2022, we began providing historical archives of the 2x3 bivariate portfolio sorts used to construct the factors for each July data cut.

	December 2022	Last 3 Months	Last 12 Months
Fama/French 3 Research Factors			
Rm-Rf	-6.41	5.61	-21.60
SMB	-0.64	-4.19	-6.79
HML	1.36	10.85	25.85
Fama/French 5 Research Factors (2x3)			
Rm-Rf	-6.41	5.61	-21.60
SMB	-0.13	-1.13	-1.45
HML	1.36	10.85	25.85
RMW	0.08	9.15	5.99
CMA	4.18	14.29	22.24
Fama/French Research Portfolios			
Size and Book-to-Market Portfolios			
Small Value	-5.37	10.57	-5.79
Small Neutral	-6.79	6.01	-16.70
Small Growth	-5.88	0.50	-27.67

In HW2, we will work with this dataset
We're going to download this dataset and calculate these metrics using the data in the dataset

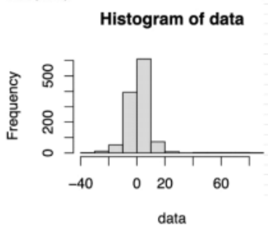
Example

- Portfolios formed on size: The portfolios are constructed at the end of each June using the June market equity and NYSE breakpoints

```
data <- read.csv("./Portfolios_Formed_on_ME.CSV")
data <- data[,"Lo.30"]

# write a function to compute mean, sd, skew, kurtosis
# 5% value at risk (VaR) and 5% expected shortfall (ES)
summary_statistics <- function(ret) {
  ret.mu <- mean(ret)
  ret.sd <- sd(ret)
  ret.skew <- mean((ret - ret.mu)**3/ret.sd**3)
  ret.kurtosis <- mean((ret - ret.mu)**4/ret.sd**4) - 3
  ret.VaR <- quantile(ret, 0.05)
  ret.ES <- sum(ret * (ret < ret.VaR)) / sum(ret < ret.VaR)
  out <- c(ret.mu, ret.sd, ret.skew, ret.kurtosis, ret.VaR, ret.ES)
  out <- setNames(out, c("mean", "sd", "skew", "kurtosis", "VaR", "ES"))
  return(out)
}
```

```
hist(data)
```



	<= 0	Lo 30	Med 40	Hi 30
192607	-99.99	0.14	1.54	3.42
192608	-99.99	3.19	2.73	2.91
192609	-99.99	-1.73	-0.88	0.8
192610	-99.99	-2.94	-3.26	-2.79
192611	-99.99	-0.38	3.73	2.74
192612	-99.99	4.15	1.66	3.04
192701	-99.99	1.21	1.57	-0.03
192702	-99.99	6.39	6.9	4.2
192703	-99.99	-1.97	-1.56	0.92
192704	-99.99	3.79	0.22	0.84
192705	-99.99	9.41	7.38	5.64
192706	-99.99	-2.34	-1.27	-2.19
192707	-99.99	4.74	5.3	7.87
192708	-99.99	-0.1	0.21	2.6
192709	-99.99	0.82	4.28	5.16
192710	-99.99	-4.29	-2.52	-4.14
192711	-99.99	12.8	7.72	6.64
192712	-99.99	2.29	3.1	2.31
192801	-99.99	5.31	1.85	-0.88
192802	-99.99	-2.73	-2.65	-1.12

So lo 30 is lowest 30 percentile
First column in the year and month eg. Year 1926 month 07 (july)

```
summary_statistics(data)
     mean        sd      skew  kurtosis        VaR         ES
 1.239309  8.260201  2.019101 20.774437 -10.684000 -16.542586
```

Expected shortfall – the avg loss we would experience if in that q%

Lecture plan

- Learning from historical data
- Risk aversion and utility values
- Portfolios of one risky asset and a risk-free asset

Risk aversion and utility values

- **Risk-averse** investor penalizes the expected rate of return by the risk involved
  - A portfolio is more attractive when its expected return is higher and its risk is lower
- **Utility score** to compare competing portfolios based on expected return and risk of those portfolios

$$U = E(r) - \frac{1}{2}A\sigma^2$$

  - U: utility value
  - A: index of the investor's risk aversion
    - Risk-averse investors: $A > 0$
    - Risk-neural investors: $A = 0$
    - Risk-lover investors: $A < 0$
  - Quantify the rate at which investors are willing to trade off return against risk

E(r) is expected return

Evaluating investments by using utility scores

Portfolio	Risk Premium	Expected Return	Risk (SD)
L (low risk)	2%	7%	5%
M (medium risk)	4	9	10
H (high risk)	8	13	20

Table 6.1
Available risky portfolios (risk-free rate = 5%)

Investor Risk Aversion (A)	Utility Score of Portfolio L [$E(r) = 0.07; \sigma = 0.05$]	Utility Score of Portfolio M [$E(r) = 0.09; \sigma = 0.10$]	Utility Score of Portfolio H [$E(r) = 0.13; \sigma = 0.20$]
2.0	$0.07 - \frac{1}{2} \times 2 \times 0.05^2 = 0.0675$	$0.09 - \frac{1}{2} \times 2 \times 0.1^2 = 0.0800$	$\mathbf{0.13 - \frac{1}{2} \times 2 \times 0.2^2 = 0.09}$
3.5	$0.07 - \frac{1}{2} \times 3.5 \times 0.05^2 = 0.0656$	$\mathbf{0.09 - \frac{1}{2} \times 3.5 \times 0.1^2 = 0.0725}$	$0.13 - \frac{1}{2} \times 3.5 \times 0.2^2 = 0.06$
5.0	$0.07 - \frac{1}{2} \times 5 \times 0.05^2 = 0.0638$	$\mathbf{0.09 - \frac{1}{2} \times 5 \times 0.1^2 = 0.0650}$	$0.13 - \frac{1}{2} \times 5 \times 0.2^2 = 0.03$

Risk premium is the difference between expected return and risk free rate
So 2 % is the difference between 7% and 5%
4% is the difference between 9% and 5%

Interpretation of utility score

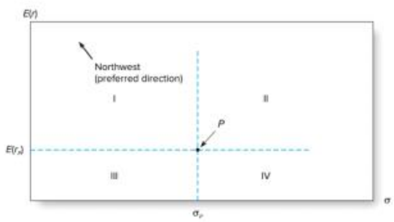
- The utility score of risky portfolios can be interpreted as a **certainty equivalent** rate of return
- Certainty equivalent is the rate that risk-free investment would need to offer to provide the same utility as the risky portfolio
  - A natural way to rank competing portfolios
  - A portfolio is desirable only if its certainty equivalent exceeds that of the risk-free equivalent

Question

- Question: A portfolio has an expected rate of return of 20% and standard deviation of 30%. T-bills offer a safe rate of return of 7%. Would an investor with risk-aversion parameter $A = 4$ prefer to invest in T-bills or the risky portfolio? What if $A = 2$?

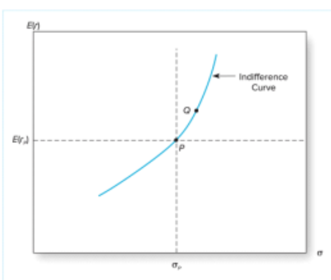
Risk and return tradeoff

- Portfolio P dominates any portfolio in quadrant IV
- Any portfolio in quadrant I dominates portfolio P
- Portfolios in quadrant II and III depend on investor's risk aversion



Indifference curve

- Indifference curve connects all portfolio points with the same utility value



Capital allocation across risky and risk-free portfolios

- **Asset allocation** of a complete portfolio: y in the risky assets and $1-y$ in the risk-free assets

- Example: total market value of a portfolio $\$300{,}000$
  - $\$90{,}000$ invested in risky assets
  - $\$210{,}000$ invested in risk-free assets
  - $y = \frac{210{,}000}{300{,}000} = .7$ (risky assets) and $1 - y = \frac{90{,}000}{300{,}000} = .3$ (risk-free assets)

- Treat all the risky assets as a portfolio, and all the risk-free assets a portfolio
- Next week we learn how to construct the risky portfolio

Asset allocation decision

- The investor decides the proportion y of the risky portfolio P and risk-free asset in the complete portfolio C
- Suppose the risk-free rate is r_f and for the risky portfolio P
  - Rate of return r_P
  - Expected return $E[r_P]$
  - Standard deviation σ_P

- For the complete portfolio C, the rate of return is

$$r_C = yr_P + (1-y)r_f$$

and expected return is

$$E[r_C] = yE[r_P] + (1-y)r_f = r_f + y[E(r_P) - r_f]$$

Return and risk of complete portfolio

- For the complete portfolio C with proportion y in the risky portfolio,
  - The expected return is

$$E[r_C] = yE[r_P] + (1-y)r_f = r_f + y[E(r_P) - r_f]$$

    - The base rate of return for any portfolio is the risk-free rate
    - Expected return is proportional to y and the risk premium $E(r_P) - r_f$

  - The standard deviation is

$$\sigma_C = y\sigma_P$$

    - Standard deviation is proportional to y and the standard deviation of the risky asset