homework2_solution

Wednesday, March 15, 2023 14:35

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Quantitative Finance: Homework 2 Solution

Ruoxuan Xiong

Problem 1

Suppose your expectations regarding the stock price are as follows:

State of the Market	Probability	HPR (including dividends)
Boom	0.35	30%
Normal growth	0.30	10%
Recession	0.35	−10%

Compute the mean and standard deviation of the HPR on stocks.

Suggested solution.

The mean of the HPR is

$$E(r) = 0.35 \cdot 0.3 + 0.3 \cdot 0.1 + 0.35 \cdot (-0.1) = 0.1$$

```
mu <- 0.35 * 0.3 + 0.3 * 0.1 + 0.35 * (-0.1)
mu
[1] 0.1
```

The standard deviation of the HPR is

$$Var(r) = 0.35 \cdot (0.3 - E(r))^2 + 0.3 \cdot (0.1 - E(r))^2 + 0.35 \cdot (-0.1 - E(r))^2 =$$

and

$$\sigma = Var(r)^{0.5} = 0.1673$$

```
sigma_sq = 0.35 * (0.3 - mu)**2 + 0.3 * (0.1 - mu)**2 + 0.35 * (-0.1 - mu)**2
sqrt(sigma_sq)
[1] 0.167332
```

Problem 2

Visit Professor Kenneth French's data library Web site: https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html and download the monthly returns of "Fama/French 3 Factors" from January 1927–December 2022. Split the sample in half and compute the average, SD, skew, kurtosis, 1% value at risk (VaR) and 1% expected shortfall (ES) for the market (Mkt-RF) factor, small minus big (SML) factor, and high minus low (HML) factor for the two halves. Do the three split-halves statistics suggest to you that returns come from the same distribution over the entire period?

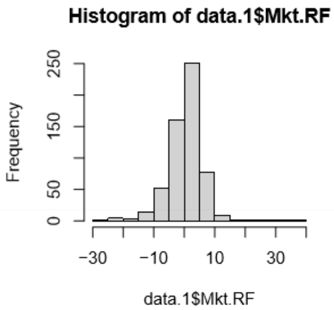
Suggested solution.

```
data <- read.csv("ff-3factors.csv")
data <- data[data$X >= 192701, ]
# split data in half
data.1 <- data[1: (nrow(data)/2),]
data.2 <- data[(nrow(data)/2+1):nrow(data),]
```

```r
# write a function to compute mean, sd, skew, kurtosis, 1\% value at risk (VaR) and 1\% expected shortf
summary_statistics <- function(ret) {
  ret.mu <- mean(ret)
  ret.sd <- sd(ret)
  ret.skew <- mean((ret - ret.mu)**3/ret.sd**3)
  ret.kurtosis <- mean((ret - ret.mu)**4/ret.sd**4) - 3
  ret.VaR <- quantile(ret, 0.01)
  ret.ES <- sum(ret * (ret < ret.VaR)) / sum(ret < ret.VaR)
  out <- c(ret.mu, ret.sd, ret.skew, ret.kurtosis, ret.VaR, ret.ES)
  out <- setNames(out, c("mean", "sd", "skew", "kurtosis", "VaR", "ES"))
  return(out)
}
```

First sample: the market factor

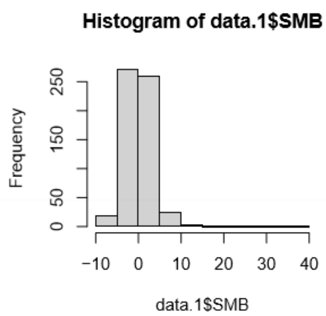
```r
hist(data.1$Mkt.RF)
```




```r
summary_statistics(data.1$Mkt.RF)
       mean          sd        skew    kurtosis         VaR          ES
  0.6217014   6.0834507   0.4781621   8.0265846 -16.6925000 -22.2483333
```

First sample: the SMB factor

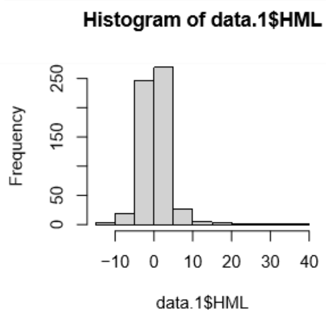
```r
hist(data.1$SMB)
```




```
summary_statistics(data.1$SMB)
      mean         sd       skew   kurtosis        VaR         ES
 0.1845139  3.3499747  2.8070595 25.9473287 -6.7325000 -7.9283333
```

First sample: the HML factor

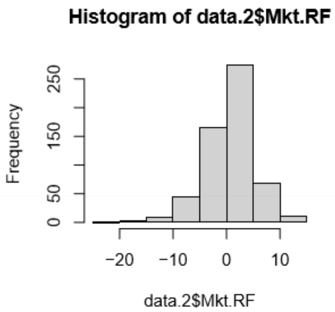
```
hist(data.1$HML)
```




```
summary_statistics(data.1$HML)
       mean          sd        skew    kurtosis         VaR          ES
  0.4441319   3.9980923   2.8724291  21.8765839  -8.6400000 -10.7666667
```

Second sample: the market factor

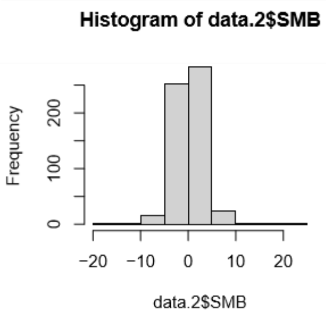
```
hist(data.2$Mkt.RF)
```




```
summary_statistics(data.2$Mkt.RF)
      mean         sd       skew   kurtosis         VaR          ES
 0.7070833  4.5387446 -0.5949319  1.8957459 -11.0175000 -15.7916667
```

Second sample: the SMB factor

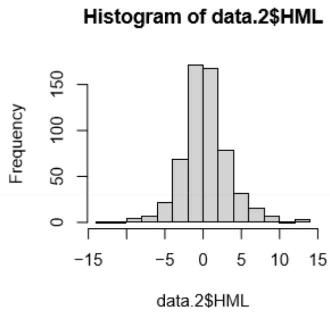
```
hist(data.2$SMB)
```




```
summary_statistics(data.2$SMB)
     mean        sd      skew  kurtosis        VaR         ES
0.2052431 2.9944645 0.4545983 6.3905119 -6.3325000 -9.2466667
```

Second sample: the HML factor

```
hist(data.2$HML)
```




```
summary_statistics(data.2$HML)
      mean         sd       skew   kurtosis        VaR          ES
 0.2781771  3.0824338  0.1931077  2.2689404 -7.9525000 -10.0033333
```

From the three split-halves statistics, the returns do not seeum to come from the same distribution over the entire period.

Problem 3

You manage a risky portfolio with an expected rate of return of 20% and a standard deviation of 30%. The T-bill rate is 5%.

(a) Your client chooses to invest 70% of a portfolio in your fund and 30% in an essentially risk-free money market fund. What are the expected value and standard deviation of the rate of return on his portfolio?

Suggested solution.

The expected rate of return is (with $y = 0.7$)

$$E(r_C) = r_f + y[E(r_P) - r_f] = 0.05 + 0.7(0.2 - 0.05) = 0.155$$

```
0.05 + 0.7 * (0.2 - 0.05)
[1] 0.155
```

The standard deviation of the rate of return is

$$\sigma_C = y\sigma_P = 0.7 \cdot 0.3 = 0.21$$

```
0.7 * 0.3
[1] 0.21
```

(b) Suppose that your risky portfolio includes the following investments in the given proportions:

Stock A	30%
Stock B	40%
Stock C	30%

What are the investment proportions of your client's overall portfolio, including the position in T-bills?

Suggested solution.

The investment proportion in stock A is $0.3 \cdot 0.7 = 0.21$. The investment proportion in stock B is $0.4 \cdot 0.7 = 0.28$. The investment proportion in stock C is $0.3 \cdot 0.7 = 0.21$. The investment proportion in T-bills is 0.3.

(c) What is the reward-to-volatility (Sharpe) ratio (S) of your risky portfolio? What is the Sharpe ratio of your client's portfolio?

Suggested solution.

The Sharpe ratio of the risky portfolio is

$$S = \frac{E(r_P) - r_f}{\sigma_P} = \frac{0.2 - 0.05}{0.3} = 0.5.$$

The Sharpe ratio of client's portfolio is

$$S = \frac{E(r_P) - r_f}{\sigma_P} = \frac{0.155 - 0.05}{0.21} = 0.5.$$

(d) Suppose that your client decides to invest in your portfolio a proportion y of the total investment budget so that the overall portfolio will have an expected rate of return of 15%.

- What is the proportion y?
- What are your client's investment proportions in your three stocks and the T-bill fund? What is the standard deviation of the rate of return on your client's portfolio?

Suggested solution.

We solve y from the following equation

$$0.15 = 0.05 + y[0.2 - 0.05]$$

Then

$$y = \frac{0.15 - 0.05}{0.2 - 0.05} = \frac{2}{3}.$$

The investment proportion in stock A is $0.3 \cdot \frac{2}{3} = 0.2$. The investment proportion in stock B is $0.4 \cdot \frac{2}{3} = \frac{4}{15}$. The investment proportion in stock C is $0.3 \cdot \frac{2}{3} = 0.2$. The investment proportion in T-bills is $\frac{1}{3}$.

The standard deviation of the rate of return is

$$y\sigma_P = \frac{2}{3} \cdot 0.3 = 0.2.$$

(e) Suppose that your client prefers to invest in your fund a proportion y that maximizes the expected return on the complete portfolio subject to the constraint that the complete portfolio's standard deviation will not exceed 10%.

- What is the investment proportion, y?
- What is the expected rate of return on the complete portfolio?

Suggested solution.

Note that the return of the portfolio increases with y. We find the max y such that the following inequality holds

$$y\sigma_P = y \cdot 0.3 \leq 0.1.$$

Then

$$y = \frac{1}{3}.$$

The expected return is

$$E(r_C) = 0.05 + \frac{1}{3}[0.2 - 0.05] = 0.1.$$

(f) Your client's degree of risk aversion is $A = 3$.

What proportion, y, of the total investment should be invested in your fund?

- What proportion, y, of the total investment should be invested in your fund?
- What are the expected value and standard deviation of the rate of return on your client's optimized portfolio?
- What is your client's utility score of this optimized portfolio?
- Draw the indifference curve for your client in the expected return–standard deviation plane corresponding to the utility score of this optimized portfolio.

Suggested solution.

When the risk aversion is $A = 3$, the proportion y is

$$y = \frac{E(r_P) - r_f}{A\sigma_P^2} = \frac{0.2 - 0.05}{3 \cdot 0.03^2} = \frac{5}{9}.$$

The expected return is

$$E(r_C) = 0.05 + \frac{5}{9}[0.2 - 0.05] = 0.133.$$

The standard deviation of the return is

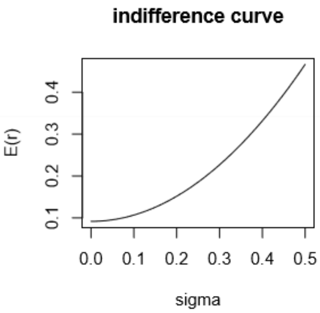
$$\sigma_C = y\sigma_P = \frac{5}{9} \cdot 0.3 = 0.167.$$

The utility score is

$$U = E(r_C) - 1/2 \cdot A \cdot \sigma_C^2 = 0.133 - 1/2 \cdot 3 \cdot 0.167^2 = 0.0917.$$

```
A = 3
mu_C = 0.05 + 5/9 * (0.2 - 0.05)
sigma_C = 5/9 * 0.3
U = mu_C - 1/2 * A * sigma_C^2

calc_ret <- function(U, A, sig) {
  ret <- U + 1/2 * A * sig^2
  return (ret)
}

sig.seq <- seq(0, 0.5, 0.001)
ret.seq <- c()
for (sig in sig.seq) {
  ret <- calc_ret(U, A, sig)
  ret.seq <- c(ret.seq, ret)
}

plot(sig.seq, ret.seq, type = 'l', xlab = 'sigma', ylab = 'E(r)', main = 'indifference curve')
```

	Expected Return	Standard Deviation
Stock fund (S)	20%	30%
Bond fund (B)	10%	10%



Problem 4

A pension fund manager is considering three mutual funds. The first is a stock fund, the second is a long-term bond fund, and the third is a money market fund that provides a safe return of 5%. The characteristics of the risky funds are as follows:

The correlation between the fund returns is .20.

(a) What are the investment proportions in the minimum-variance portfolio of the two risky funds, and what are the expected value and standard deviation of its rate of return?

Suggested solution.

$$w_B = \frac{\sigma_S^2 - cov(r_S, r_B)}{\sigma_B^2 + \sigma_S^2 - 2cov(r_S, r_B)} = \frac{0.3^2 - 0.2 \cdot 0.3 \cdot 0.1}{0.1^2 + 0.3^2 - 2 \cdot 0.2 \cdot 0.3 \cdot 0.1} = 0.955$$

```
r_f <- 0.05
r_B <- 0.1
r_S <- 0.2
sigma_B <- 0.1
sigma_S <- 0.3
rho_BS <- 0.2
w_B <-  (sigma_S**2 - rho_BS * sigma_B * sigma_S)/(sigma_B**2 + sigma_S**2 - 2 * rho_BS * sigma_B * sig
w_S <-  1 - w_B
w_B
[1] 0.9545455
w_S
[1] 0.04545455
```

and

$$w_S = 1 - w_B = 0.045$$

The expected return is

$$E(r_P) = w_B E(r_B) + w_S E(r_S) = 0.955 \cdot 0.1 + 0.045 \cdot 0.2 = 0.105$$

```
ret_P <- w_B * r_B + w_S * r_S
ret_P
[1] 0.1045455
```

The variance of the portfolio is

$$\sigma_P^2 = w_B^2\sigma_B^2 + w_S^2\sigma_S^2 + 2w_Bw_Scov(r_B, r_S) = 0.955^2 \cdot 0.1^2 + 0.045^2 \cdot 0.3^2 + 2 \cdot 0.955 \cdot 0.045 \cdot 0.2 \cdot 0.3 \cdot 0.1 = 0.0098$$

The standard deviation is

$$\sigma_P = 0.099$$

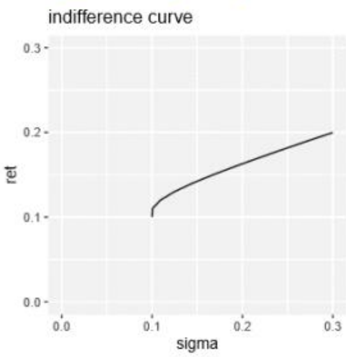
```
sigma_sq_P <- w_B ** 2 * (sigma_B ** 2) + w_S ** 2 * (sigma_S ** 2) + 2 * w_B * w_S * (rho_BS * sigma_B
sigma_sq_P
[1] 0.009818182
sigma_P <- sqrt(sigma_sq_P)
sigma_P
[1] 0.09908674
```

(b) Tabulate and draw the investment opportunity set of the two risky funds. Use investment proportions for the stock fund of 0% to 100% in increments of 10%.

Suggested solution.

```
calc_ret_sd <- function(w_B, r_B, r_S, sigma_B, sigma_S) {
  w_S <- 1 - w_B
  ret_P <- w_B * r_B + w_S * r_S
  sigma_sq_P <- w_B ** 2 * (sigma_B ** 2) + w_S ** 2 * (sigma_S ** 2) + 2 * w_B * w_S * (rho_BS * sigma
  sigma_P <- sqrt(sigma_sq_P)
  return (c(ret_P, sigma_P))
}

w_B_seq <- seq(0, 1, 0.1)
ret_P_seq <- c()
sigma_P_seq <- c()
for (w_B in w_B_seq) {
  out <- calc_ret_sd(w_B, r_B, r_S, sigma_B, sigma_S)
  ret_P_seq <- c(ret_P_seq, out[1])
  sigma_P_seq <- c(sigma_P_seq, out[2])
}

df <- data.frame("wB" = w_B_seq, "ret" = ret_P_seq, "sigma" = sigma_P_seq)
df
   wB  ret     sigma
1  0.0 0.20 0.3000000
2  0.1 0.19 0.2721764
3  0.2 0.18 0.2447856
4  0.3 0.17 0.2179908
5  0.4 0.16 0.1920417
```

```
6  0.5 0.15 0.1673320
7  0.6 0.14 0.1444991
8  0.7 0.13 0.1245793
9  0.8 0.12 0.1091788
10 0.9 0.11 0.1003992
11 1.0 0.10 0.1000000

# plot(sigma_P_seq, ret_P_seq, type = 'ol', xlab = 'sigma', ylab = 'E(r)', main = 'indifference curve',

library(ggplot2)
ggplot(data = df, aes(x=sigma, y=ret)) + geom_path()+ xlim(0, 0.3) + ylim(0, 0.3) + ggtitle('indifferen
```



(c) Draw a tangent line from the risk-free rate to the opportunity set. What does your graph show for the expected return and standard deviation of the optimal portfolio?

Suggested solution.

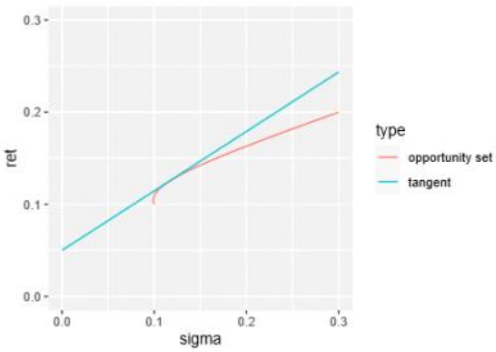
```
R_B <- r_B - r_f
R_S <- r_S - r_f

w_B <- (R_B * sigma_S**2 - R_S * rho_BS * sigma_S * sigma_B)/(R_B * sigma_S**2 + R_S * sigma_B**2 - (R_
# weight
w_B
[1] 0.75
w_S <- 1 - w_B
ret_P <- w_B  * r_B + w_S * r_S
# expected return
ret_P
[1] 0.125
sigma_sq_P <- w_B ** 2 * (sigma_B ** 2) + w_S ** 2 * (sigma_S ** 2) + 2 * w_B * w_S * (rho_BS * sigma_B
sigma_P <- sqrt(sigma_sq_P)
# standard deviation
sigma_P
```

10

Suggested solution.

```
R_B <- r_B - r_f
R_S <- r_S - r_f
w_B <- (R_B * sigma_S^2 - R_S * rho_BS * sigma_S * sigma_B)/(R_B *
    sigma_S^2 + R_S * sigma_B^2 - (R_S + R_B) * rho_BS * sigma_S *
    sigma_B)
# weight
w_B
[1] 0.75
w_S <- 1 - w_B
ret_P <- w_B * r_B + w_S * r_S
# expected return
ret_P
[1] 0.125
```

```
[1] 0.1161895
Sharpe <- (ret_P - r_f) / sigma_P
# Sharpe ratio
Sharpe
[1] 0.6454972

sigma_tangent_seq <- seq(0, 0.3, 0.01)
ret_tangent_seq <- c()
for (sig in sigma_tangent_seq) {
  ret <- r_f + sig * Sharpe
  ret_tangent_seq <- c(ret_tangent_seq, ret)
}


w_B_seq <- seq(0, 1, 0.001)
ret_P_seq <- c()
sigma_P_seq <- c()
for (w_B in w_B_seq) {
  out <- calc_ret_sd(w_B, r_B, r_S, sigma_B, sigma_S)
  ret_P_seq <- c(ret_P_seq, out[1])
  sigma_P_seq <- c(sigma_P_seq, out[2])
}

df <- data.frame("ret" = c(ret_tangent_seq, ret_P_seq), "sigma" = c(sigma_tangent_seq, sigma_P_seq), "t

library(ggplot2)
ggplot(data = df, aes(x=sigma, y=ret)) + geom_path(aes(colour=type)) + ylim(0, 0.3)
```



(d) Solve numerically for the proportions of each asset and for the expected return and standard deviation of the optimal risky portfolio.

Suggested solution.

The optimal weight is

$$w_B^* = \frac{E(R_B)\sigma_S^2 - E(R_S)Cov(R_B, R_S)}{E(R_B)\sigma_S^2 + E(R_S)\sigma_B^2 - [E(R_S) + E(R_B)]Cov(R_B, R_S)} = 0.75$$

$$w_S^* = 1 - w_B^* = 0.25$$

The expected return is

$$E(r_P) = w_B E(r_B) + w_S E(r_S) = 0.75 \cdot 0.1 + 0.25 \cdot 0.2 = 0.125$$

The standard deviation is

$$\sigma_P = (w_B^2\sigma_B^2 + w_S^2\sigma_S^2 + 2w_B w_S cov(r_B, r_S))^{1/2} = 0.116.$$

(e) What is the Sharpe ratio of the best feasible CAL?

Suggested solution.

The Sharpe ratio is

$$S = \frac{E(r_P) - r_f}{\sigma_P} = 0.645$$

(f) You require that your portfolio yield an expected return of 12%, and that it be efficient, that is, on the steepest feasible CAL.

- What is the standard deviation of your portfolio?
- What is the proportion invested in the money market fund and each of the two risky funds?

Suggested solution.

The standard deviation is

$$\sigma_P = \frac{E(r_P) - r_f}{S} = \frac{0.12 - 0.05}{0.645} = 0.108$$

```
(0.12 - 0.05)/Sharpe
[1] 0.1084435
```

We first solve y from the following equation

$$0.12 = 0.05 + y[0.125 - 0.05]$$

Then

$$y = \frac{0.12 - 0.05}{0.125 - 0.05} = 0.933$$

The proportion invested in the money market fund is $1 - y = 0.067$.

The proportion invested in the bond fund is $y \cdot w_B^* = 0.933 \cdot 0.75 = 0.7$.

The proportion invested in the stock fund is $y \cdot w_S^* = 0.933 \cdot 0.25 = 0.233$.

(g) If you were to use only the two risky funds and still require an expected return of 12%, what would be the investment proportions of your portfolio? Compare its standard deviation to that of the optimized portfolio in (f). What do you conclude?

Suggested solution.

We solve w_B from

$$E(r_P) = w_B E(r_B) + (1 - w_B)E(r_S) = w_B(E(r_B) - E(r_S)) + E(r_S) = 0.12$$

i.e.,

$$w_B = \frac{0.12 - 0.2}{0.1 - 0.2} = 0.8$$

and

$$w_S = 1 - w_B = 0.2.$$

The standard deviation is

$$\sigma_P^2 = w_B^2\sigma_B^2 + w_S^2\sigma_S^2 + 2w_B w_S cov(r_B, r_S) = 0.8^2 \cdot 0.1^2 + 0.2^2 \cdot 0.3^2 + 2 \cdot 0.8 \cdot 0.2 \cdot 0.2 \cdot 0.3 \cdot 0.1 = 0.109.$$

```
calc_ret_sd(0.8, r_B, r_S, sigma_B, sigma_S)
[1] 0.1200000 0.1091788
```

The standard deviation is higher than that of the optimized portfolio in (f). Therefore, it is better to use the money market fund to construct an investment portfolio.

Problem 5

Let R_B be the rate of excess return on the bond fund and R_S be the rate of return on the stock fund. Let the variance of R_B be σ_B^2, the variance of R_S be σ_S^2, and the covariance between R_B and R_S be $Cov(R_B, R_S)$.

Suppose a portfolio has w_B proportion invested in the bond fund and the remainder $w_S = 1 - w_B$ in the stock fund. Show that the weight w_B that maximizes the Sharpe ratio equals

$$w_B^* = \frac{E(R_B)\sigma_S^2 - E(R_S)Cov(R_B, R_S)}{E(R_B)\sigma_S^2 + E(R_S)\sigma_B^2 - [E(R_S) + E(R_B)]Cov(R_B, R_S)}.$$

Suggested solution.

Note that the w_B that maximizes the Sharpe ratio S_P also maximizes the squared Sharpe ratio S_P^2.

We solve the w_B from the first order condition

$$\frac{dS_P^2}{dw_B} = 0.$$

Recall the definition of Sharpe ratio, we have S_P^2 equal to

$$S_P^2 = \frac{E(R_P)^2}{\sigma_P^2} = \frac{(w_B E(R_B) + (1 - w_B)E(R_S))^2}{w_B^2\sigma_B^2 + (1 - w_B)^2\sigma_S^2 + 2w_B(1 - w_B)Cov(R_B, R_S)}.$$

Then the derivative of S_P^2 with respect to w_B is

$$\begin{aligned}
\frac{dS_P^2}{dw_B} =& \frac{1}{\sigma_P^4}\left[\frac{dE(R_P)^2}{dw_B} \cdot \sigma_P^2 - E(R_P)^2\frac{d\sigma_P^2}{dw_B}\right] \\
=& \frac{1}{\sigma_P^4}\left[2[E(R_S) + w_B(E(R_B) - E(R_S))] \cdot (E(R_B) - E(R_S)) \cdot \sigma_P^2\right. \\
& \left.-(E(R_S) + w_B(E(R_B) - E(R_S)))^2\left[2w_B\sigma_B^2 + 2(w_B - 1)\sigma_S^2 + 2(1 - 2w_B)Cov(R_B, R_S)\right]\right].
\end{aligned}$$

If w_B satisfies $\frac{dS_P^2}{dw_B} = 0$, then w_B also satisfies the following equation

$$\begin{aligned}
0 =& (E(R_B) - E(R_S)) \cdot \left[w_B^2\sigma_B^2 + (1 - w_B)^2\sigma_S^2 + 2w_B(1 - w_B)Cov(R_B, R_S)\right] \\
&- (E(R_S) + w_B(E(R_B) - E(R_S))) \left[w_B\sigma_B^2 + (w_B - 1)\sigma_S^2 + (1 - 2w_B)Cov(R_B, R_S)\right].
\end{aligned}$$

We rearrange terms in the above equal and have

$$\begin{aligned}
0 =& (E(R_B) - E(R_S)) \cdot \left\{\left[w_B^2\sigma_B^2 + (1 - w_B)^2\sigma_S^2 + 2w_B(1 - w_B)Cov(R_B, R_S)\right]\right. \\
&\left. - \left[w_B^2\sigma_B^2 + w_B(w_B - 1)\sigma_S^2 + w_B(1 - 2w_B)Cov(R_B, R_S)\right]\right\} \\
&- E(R_S)\left[w_B\sigma_B^2 + (w_B - 1)\sigma_S^2 + (1 - 2w_B)Cov(R_B, R_S)\right] \\
=& (E(R_B) - E(R_S)) \cdot \left[(1 - w_B)\sigma_S^2 + w_BCov(R_B, R_S)\right] \\
&- E(R_S)\left[w_B\sigma_B^2 + (w_B - 1)\sigma_S^2 + (1 - 2w_B)Cov(R_B, R_S)\right] \\
=& E(R_B)(1 - w_B)\sigma_S^2 + (E(R_B) + E(R_S))w_BCov(R_B, R_S) - E(R_S)Cov(R_B, R_S) - E(R_S)w_B\sigma_B^2
\end{aligned}$$

We move the terms that contain w_B to the left hand size, and then we have

$$w_B\left[E(R_B)\sigma_S^2 + E(R_S)\sigma_B^2 - (E(R_B) + E(R_S))w_BCov(R_B, R_S)\right] = E(R_B)\sigma_S^2 - E(R_S)Cov(R_B, R_S).$$

Therefore the optimal w_B is

$$w_B^* = \frac{E(R_B)\sigma_S^2 - E(R_S)Cov(R_B, R_S)}{E(R_B)\sigma_S^2 + E(R_S)\sigma_B^2 - [E(R_S) + E(R_B)]Cov(R_B, R_S)}.$$