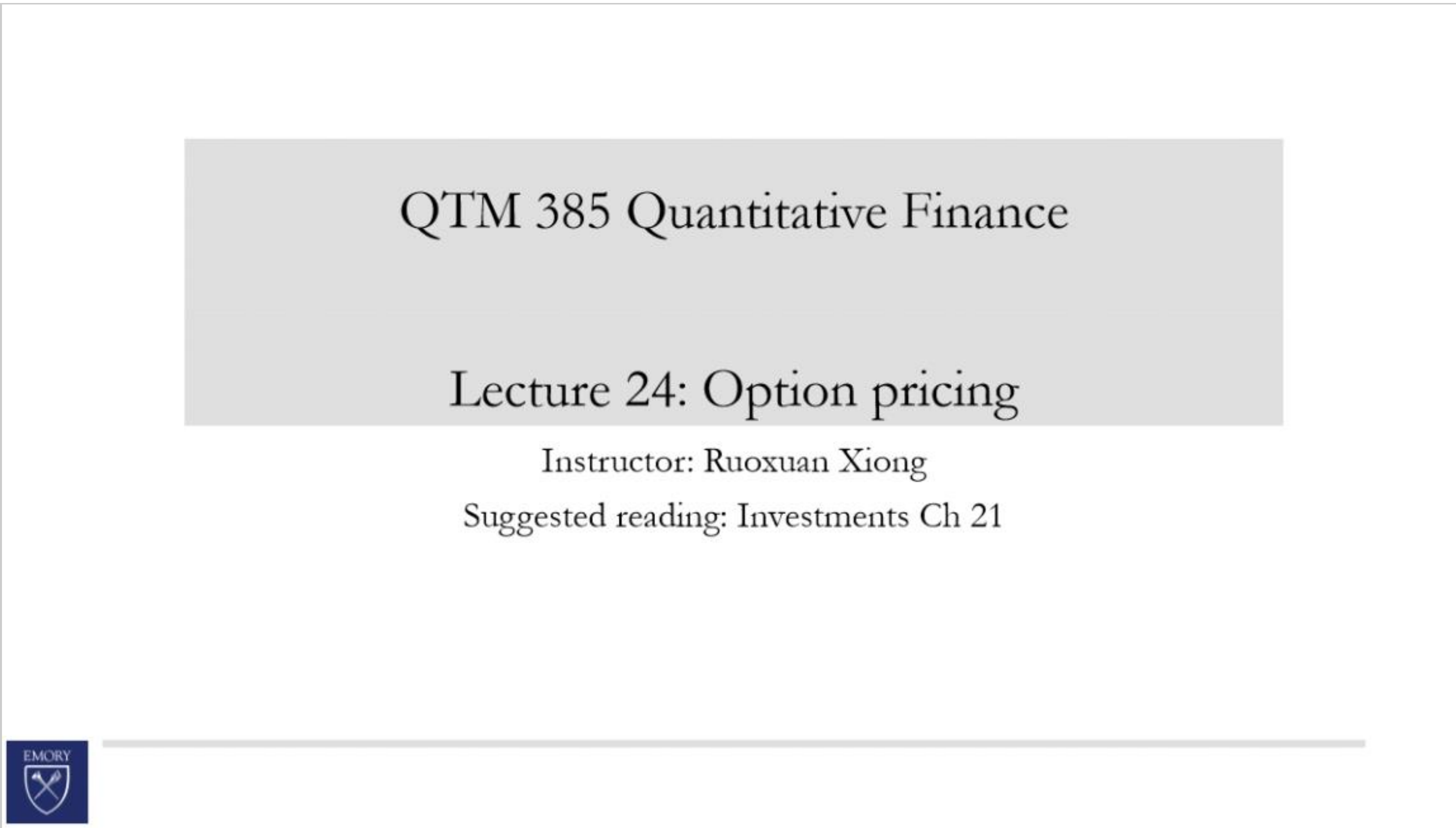
Lecture 24: Options Pricing

Monday, April 24, 2023 14:41

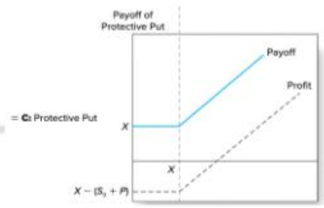
24-option-pricing



Question from Google form

- I'm curious about the trade-offs involved in using a protective put strategy. While a protective put option can limit the downside risk of investing in stocks, it also has costs. So how do we know if the cost of buying a put option over the long term will outweigh the potential return on the stock and reduce the profitability of the portfolio? How can investors judge when formulating investment policies? For example, some buy options and some buy cash equities?

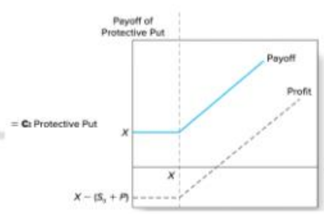
- *Answer: The present value of the expected payoff equals to the cost. An investor would choose the portfolio that maximize the utility (given the the portfolio's return and risk)*



Question from Google form

- Would a short position of put option be able to reduce risk for investment in a stock, just as the protective put?

- *Answer: A long position of the put (protective put) limits the downside risk. However, you need to pay a small premium for the put.*



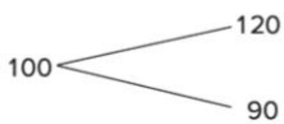
Question from Google form

- In the real world, how would hedge fund company leverage the strategy we learned in class to invest.

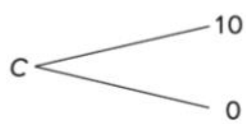
- *Answer: They use portfolio optimization and diversification to allocate the fund across assets/ strategies. They use asset pricing models a lot: Alpha seeking team constructs alpha strategies based on the term alpha in the pricing models with hundreds of factors. They use bond pricing and option pricing models to identify arbitrage opportunities. They also widely use derivatives e.g., options, to reduce downside risk or gain leverage.*

Binomial option pricing

- We start with two state option pricing
- The stock sells at $S_0 = 100$. The price will either increase to 120 or decrease to 90
- Consider two portfolios
  - Portfolio A: buy three calls
  - Portfolio B: buy a share of stock and borrow $\$81.82$ at interest rate 10%
  - Two portfolios have the same payoff



Stock price



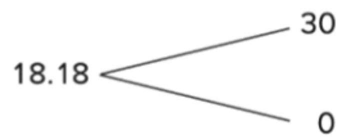
Call option value

Value of stock at year-end	\$90	\$120
– Repayment of loan with interest	–90	–90
Total	\$ 0	\$ 30

Price of the call

- Consider two portfolios
  - Portfolio A: buy three calls
  - Portfolio B: buy a share of stock and borrow $81.82 at interest rate 10%
  - Two portfolios have the same payoff

- The current cost is $\$18.18 = \$100 - \$81.82$ of portfolio B
- We have
  - $3C = \$18.18$
  - $C = \$6.06$



Hedge ratio

- A portfolio with one share of stock and three call options written is perfectly hedged. Its year-end value is independent of ultimate stock price

Stock price	$90	$120
– Obligations from 3 calls written	–0	–30
Portfolio value	$90	$ 90

- The hedge ratio is one share of stock to three calls, or one-third
  - Interpretation: range of call value is 10, range of stock price is 30, ratio of range is $10/30$

Hedge ratio

- More generally, the hedge ratio H for other two-state option problem is

$$H = \frac{C_u - C_d}{uS_0 - dS_0}$$

  - C_u or C_d refers to the call option's value when the stock goes up or down
  - uS_0 and dS_0 are the stock prices in two states

Option-pricing technique

- Step 1: Given possible end-of-year stock prices $uS_0 = \$120$ and $dS_0 = \$90$ and value of call option with exercise price $\$110$, $C_u = \$10$ or $C_d = 0$
- Step 2: Find the hedge ratio $H = \frac{C_u - C_d}{uS_0 - dS_0} = \frac{1}{3}$
- Step 3: find a portfolio made up of $\frac{1}{3}$ share of stock with one written call
  - This portfolio is a hedged portfolio with certainty payoff of $\$30$

Option-pricing technique

- Step 4: The present value of $30 is $\frac{\$30}{1.1} = \$27.27$

- Step 5: Set the present value of the hedge position to the present value of the certainty payoff

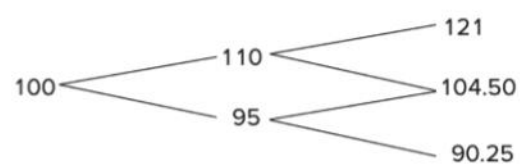
$$\frac{1}{3}S_0 - C_0 = \$27.27$$

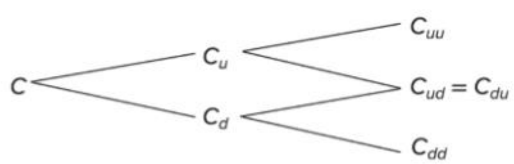
- Step 6: Solve the call's value

$$C_0 = \frac{1}{3}S_0 - \$27.27 = \$33.33 - \$27.27 = \$6.06$$

Generalization of the two-state approach

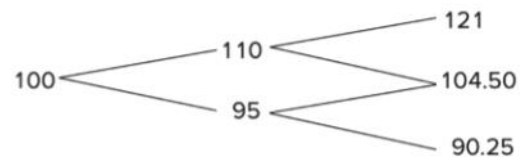
- We break up the year into two 6-month segments
- Over each half-year, the stock price could take on two values. Either increase 10% ($u = 1.10$) or decrease 5% ($d = .95$)
- Three possible end-of-year values for the stock and option

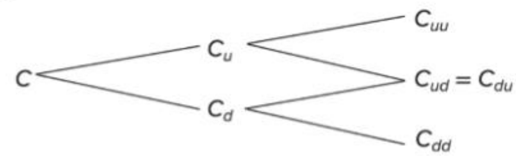




Two-period pricing

- We use a backward approach
- Suppose exercise price is $= \$110$
- We start by finding the value of C_u
- Step 1: From C_u, the call can rise to $C_{uu} = \$11$ or fall to $C_{ud} = \$0$
- Step 2: The hedge ratio at this point is
  - $H = \frac{C_{uu} - C_{ud}}{uuS_0 - udS_0} = \frac{\$11 - 0}{\$121 - 104.5} = \frac{2}{3}$
- Step 3: construct a hedged portfolio
  - 2 shares of stock and 3 written call: payoff is \$209

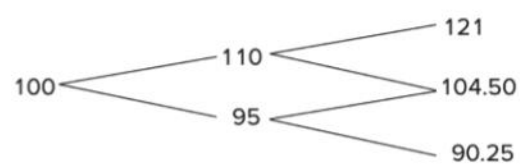


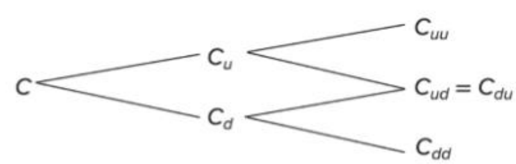


	$udS_0 = \$104.50$	$uuS_0 = \$121$
Buy 2 shares at price $uS_0 = \$110$	\$209	\$242
Write 3 calls at price C_u	0	−33
Total	\$209	\$209

Two-period pricing (continued)

- Step 4: The present value of the hedged portfolio is $\frac{\$209}{1.05} = \$199.047$
- Step 5: Set $2uS_0 - 3C_u = \$199.047$
- Step 6: $C_u = \frac{(2 \times \$110 - \$199.047)}{3} = \$6.984$



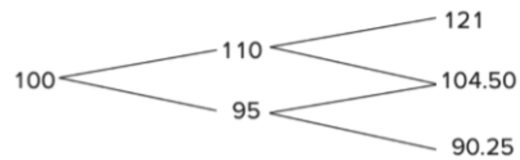


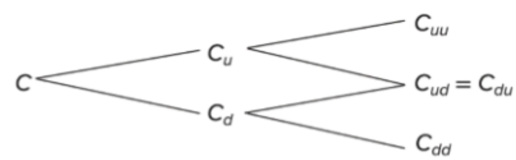
	$udS_0 = \$104.50$	$uuS_0 = \$121$
Buy 2 shares at price $uS_0 = \$110$	\$209	\$242
Write 3 calls at price C_u	0	−33
Total	\$209	\$209

Two-period pricing (continued)

- Next we find the value of C_d
  - The value C_d must be 0
  - The stock price at expiration is either $104.50 and $90.25
  - $C_{ud} = C_{dd} = 0$

- Finally we solve the values for C





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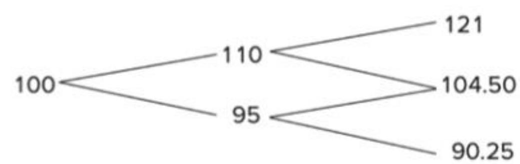
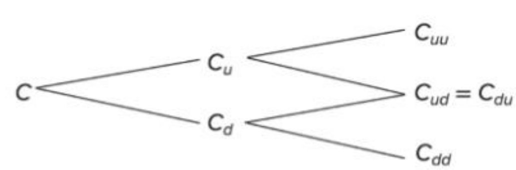
Two-period pricing (continued)

- Finally we solve the values for C
- Step 1: $C_u = \$6.984$ or fall to $C_d = \$0$ with stock price $uS_0 = \$110$ or $dS_0 = \$95$
- Step 2: The hedge ratio at this point is
  - $H = \frac{C_u - C_d}{uS_0 - dS_0} = \frac{\$6.984 - 0}{\$110 - 95} = 0.4656$
- Step 3: construct a hedged portfolio
  - 0.4656 share of stock with 1 written call: payoff is $44.232



	A	B	C	D	E
1			dS0	uS0	
2			95	110	
3	stock share	0.4656	44.232	51.216	
4	written call	1	0	-6.984	
5			44.232	44.232	
6					
7			present value	42.12571429	=D6/1.05
8					
9			C0	4.434285714	=B4*100-D8



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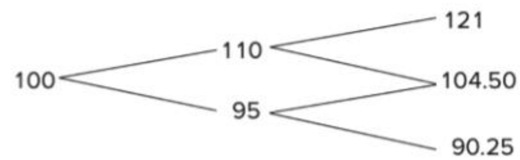
Two-period pricing (continued)

- Step 4: The present value of the hedged portfolio is $\frac{\$44.232}{1.05} = \$42.1257$
- Step 5: Set $0.4656S_0 - C_0 = \$42.1257$
- Step 6: $C_0 = 0.4656 \times \$100 - 42.1257 = \$4.434$



	A	B	C	D	E
1			dS0	uS0	
2			95	110	
3	stock share	0.4656	44.232	51.216	
4	written call	1	0	-6.984	
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7			present value	42.12571429	=D6/1.05
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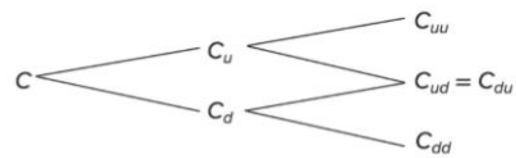
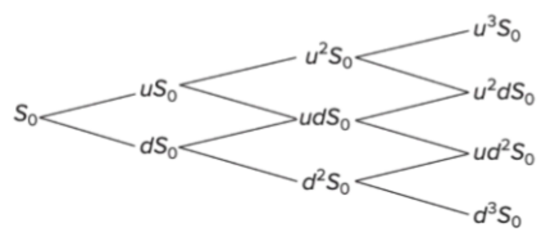


Making the valuation model practical

- We can consider more periods
- The multiperiod approach to option pricing is called the binomial model
- We can use the backward approach, but the computation is very tedious
- We use an option-pricing formula



Black-Scholes option valuation

- Two more assumptions
  - risk-free interest rate
  - stock price volatility are constant over the life of the option

- Black-Scholes formula to price European call options

$$C_0 = S_0 N(d_1) - X e^{-rT} N(d_2)$$

  - $d_1 = \frac{\log\left(\frac{S_0}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$
  - $d_2 = d_1 - \sigma\sqrt{T} = \frac{\log\left(\frac{S_0}{X}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$

Black-Scholes option valuation

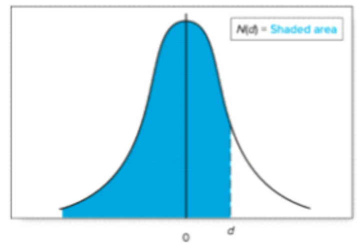
$$C_0 = S_0 N(d_1) - X e^{-rT} N(d_2)$$

- C_0: Current call option value
- S_0: Current stock price
- X: Exercise price
- r: Risk-free interest rate
- e^{-rT}: discount factor for continuous compounding rate

Black-Scholes option valuation

$$C_0 = S_0 N(d_1) - Xe^{-rT} N(d_2)$$

- C_0: Current call option value
- S_0: Current stock price
- $N(d)$: The probability that a random draw from a standard normal is less than d
- $d_1 = \frac{\log\left(\frac{S_0}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$
- $d_2 = d_1 - \sigma\sqrt{T} = \frac{\log\left(\frac{S_0}{X}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$



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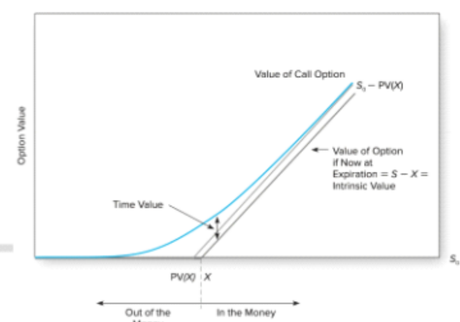
Black-Scholes option valuation

$$C_0 = S_0 N(d_1) - Xe^{-rT} N(d_2)$$

- $d_1 = \frac{\log\left(\frac{S_0}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$
- For a very large d_1 (e.g., $S_0 \gg X$), both $N(d_1)$ and $N(d_2)$ are close to 1. Then $C_0 = S_0 - Xe^{-rT} = S_0 - PV(X)$
- For a very small d_1 (e.g., $S_0 \ll X$), both $N(d_1)$ and $N(d_2)$ are close to 0. Then $C_0 = 0$



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Black-Scholes option valuation

$$C_0 = S_0 N(d_1) - Xe^{-rT} N(d_2)$$

- $d_1 = \frac{\log\left(\frac{S_0}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$
- $\log\left(\frac{S_0}{X}\right)$: roughly the percentage amount by which the option is in or out of the money
  - E.g., $S_0 = 105$ and $X = 100$, then $\log\left(\frac{S_0}{X}\right) = \log\frac{105}{100} = 0.049$
  - E.g., $S_0 = 95$ and $X = 100$, then $\log\left(\frac{S_0}{X}\right) = \log\frac{95}{100} = -0.051$
- $\sigma\sqrt{T}$: volatility of the stock price over the remaining life of the option

Example

$$C_0 = S_0 N(d_1) - Xe^{-rT} N(d_2)$$

- $S_0 = 100, X = 95, r = 10\%$ (per year), $T = .25$ (3 months), $\sigma = .50$ (per year)
- $d_1 = \frac{\log\left(\frac{S_0}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = \frac{\log\left(\frac{100}{95}\right) + \left(.10 + \frac{0.5^2}{2}\right).25}{0.5\sqrt{.25}} = .439$
- $d_2 = d_1 - \sigma\sqrt{T} = .43 - 0.5\sqrt{.25} = .189$
- $N(d_1) = N(.439) = .6697, N(d_2) = N(.189) = .5750$
- $C_0 = S_0 N(d_1) - Xe^{-rT} N(d_2) = 100\times .6697 - 95e^{-.10\times.25}\times.5750 = 66.97 - 53.28 = \$13.69$

Put option valuation

- We can use put-call parity theorem $C_0 + PV(X) = S_0 + P_0$
- Then $P_0 = C_0 + Xe^{-rT} - S_0$
- Using $C_0 = S_0N(d_1) - Xe^{-rT}N(d_2)$
- We have $P_0 = Xe^{-rT}[1 - N(d_2)] - S_0[1 - N(d_1)]$

Example

$$P_0 = Xe^{-rT}[1 - N(d_2)] - S_0[1 - N(d_1)]$$

- $S_0 = 100, X = 95, r = 10\%$ (per year), $T = .25$ (3 months), $\sigma = .50$ (per year)
- $P_0 = Xe^{-rT}[1 - N(d_2)] - S_0[1 - N(d_1)] = 95e^{-.10\times.25}\times(1 - .5750) - 100\times(1 - .6697) = \$6.35$