homework4_solution

Monday, April 24, 2023 14:44

homework
4_solution

Quantitative Finance: Homework 4 Solution

Ruoxuan Xiong

Problem 1 (Bond pricing, YTM and realized compound yield)

Consider an 8% coupon bond selling for \$950 with par value \$1,000 and three years until maturity making annual coupon payments. The interest rates in the next three years will be, with certainty, $r_1 = 8\%$, $r_2 = 10\%$, and $r_3 = 12\%$. Calculate the bond's

(a) yield to maturity

Suggested solution.

The yield to maturity is 10.01%.

```
knitr::include_graphics("problem1-a.png")
```

Settlement date	1/1/22
Maturity date	1/1/25
Annual coupon rate	0.08
Flat price (% of par)	95
Redemption value (% of face value)	100
Coupon payments per year	1
Yield to maturity (decimal)	0.1001

(b) realized compound yield

Suggested solution.

The investor gets \$80 at the end of year 1

As the reinvested rate is 10% in year 2 and 12% in year 3, the final value of the coupon paid at the end of year 1 is

$$80 \times (1 + 0.1) \times (1 + 0.12) = 98.56$$

the final value of the coupon paid at the end of year 2 is

$$80 \times (1 + 0.12) = 89.6$$

Therefore $V_0 = 950$ and

$$V_3 = 98.56 + 96 + 1080 = 1274.56$$

The realized compound yield is

$$r = \left(\frac{V_3}{V_0}\right)^{1/3} - 1 = 10.11\%$$

```r
coupon1.final <- 80 * 1.1 * 1.12
coupon1.final
[1] 98.56
coupon2.final <- 80 * 1.12
coupon2.final
[1] 89.6
V0 <- 950
V3 <- coupon1.final + coupon2.final + 1080
V3
[1] 1268.16
r <- (V3/V0)^{
    1/3
} - 1
r
[1] 0.1010748
```

(c) yield to maturity using bisection method

Suggested solution.

The yield to maturity solved by bisection method is also 10.01%.

```r
present_value <- function(par_value, coupon_rate, r, year) {
    value <- 0
    for (t in c(1:year)) {
        value <- value + par_value * coupon_rate/((1 + r)^t)
    }
    value <- value + par_value/((1 + r)^year)
    return(value)
}
bisection <- function(par_value, coupon_rate, current_price,
    year, thres = 1e-12, low = 1e-08, high = 1) {
    while (high - low > thres) {
        mid <- (low + high)/2
        mid_pv <- present_value(par_value, coupon_rate, mid,
            year) - current_price
        # print(c(mid, mid_pv))
        if (mid_pv > 0) {
            low <- mid
        } else {
            high <- mid
        }
    }
    r <- (low + high)/2
    return(r)
}
par_value <- 1000
coupon_rate <- 0.08
current_price <- 950
year <- 3
bisection(par_value, coupon_rate, current_price, year)
[1] 0.1001096
```

Suppose we have another 12% coupon bond with par value \$1,000 and three years until maturity making annual coupon payments. Suppose this bond has the same yield to maturity as the 8% coupon bond. Calculate the price of this 12% coupon bond.

Suggested solution.

The price of the bond is \$104.9480.

```
knitr::include_graphics("problem1-b.png")
```

Settlement date	1/1/22
Maturity date	1/1/25
Annual coupon rate	0.12
Yield to maturity	0.1001
Redemption value (% of face value)	100
Coupon payments per year	1
Flat price (% of par)	104.9480

Problem 2 (Yield to call) A 30-year maturity, 8% coupon bond paying coupons semiannually is callable in five years at a call price of \$1,100. The bond currently sells at a yield to maturity of 7% (3.5% per half-year).

(a) What is the yield to call?

Suggested solution.

The yield to call is 3.37% (semiannually).

```
knitr::include_graphics("problem2.png")
```

Settlement date	11/2/22	Settlement date	11/2/22	Settlement date	11/2/22
Maturity date	11/2/52	Maturity date	11/2/52	Maturity date	11/2/52
Annual coupon rate	0.08	Annual coupon rate	0.08	Annual coupon rate	0.08
Yield to maturity	0.07	Yield to maturity	0.07	Yield to maturity	0.07
Redemption value (% of face value)	100	Redemption value (% of face value)	100	Redemption value (% of face value)	100
Coupon payments per year	2	Coupon payments per year	2	Coupon payments per year	2
Flat price (% of par)	112.4724	Flat price (% of par)	112.4724	Flat price (% of par)	112.4724
	Yield to call		Yield to call		Yield to call
Settlement date	11/2/22	Settlement date	11/2/22	Settlement date	11/2/22
Maturity/Call date	11/2/27	Maturity/Call date	11/2/27	Maturity/Call date	11/2/24
Coupon rate	0.08	Coupon rate	0.08	Coupon rate	0.08
Price	112.4724	Price	112.4724	Price	112.4724
Redemption value (% of face value)	110	Redemption value (% of face value)	105	Redemption value (% of face value)	110
Coupon payments per year	2	Coupon payments per year	2	Coupon payments per year	2
Annual yield	0.067358094	Annual yield	0.059525174	Annual yield	0.060624897
Yield to call	0.033679047	Yield to call	0.029762587	Yield to call	0.030312449

(b) What is the yield to call if the call price is only \$1,050?

Suggested solution.

The yield to call is 2.98% (semiannually).

(c) What is the yield to call if the call price is \$1,100 but the bond can be called in two years instead of five years?

Suggested solution.

The yield to call is 3.03% (semiannually).

(d) Compare the yield to call from parts (a), (b), and (c). What do you find?

Suggested solution.

If the call price (repurchase price) is lower, then the yield to call is lower. This makes sense as the investor receives less when the bond is called.

If the call date is sooner, then the yield to call is lower. If the call price is lower than the present value, then the issuer will call the bond once it is callable. If the call date is sooner, then the rate of return (i.e., yield to call) is lower.

Problem 3 (Term structure) The yield to maturity on 1-year zero-coupon bonds is currently 7%; the YTM on 2-year zeros is 8%. The Treasury plans to issue a 2-year maturity coupon bond, paying coupons once per year with a coupon rate of 9%. The face value of the bond is $100.

(a) At what price will the bond sell?

Suggested solution.

The bond sells at $101.861.

$$P = \frac{9}{1+0.07} + \frac{109}{(1+0.08)^2} = 101.8611$$

```
P <- 9/1.07 + 109/(1.08^2)
```

```
knitr::include_graphics("problem3.png")
```

Settlement date	1/1/22
Maturity date	1/1/24
Annual coupon rate	0.09
Flat price (% of par)	101.8611
Redemption value (% of face value)	100
Coupon payments per year	1
Yield to maturity (decimal)	0.0796

(b) What will the yield to maturity on the bond be?

Suggested solution.

The yield to maturity is 7.96%.

```
# based on the solution from the spreadsheet
y <- 0.0796
```

(c) If the expectations theory of the yield curve is correct, what is the market expectation of the price for which the bond will sell next year?

Suggested solution.

The expected short rate in the second year is

$$E[r_2] = (1+y_2)^2/(1+y_1) - 1 = 9.01\%$$

The bond will be sold at price $99.99.

$$P_1 = 109/(1+E[r_2]) = 99.99$$

```
y1 <- 0.07
y2 <- 0.08
r2 <- (1 + y2)^2/(1 + y1) - 1
r2
```

```
[1] 0.09009346
P1 <- 109/(1 + r2)
P1
[1] 99.99143
```

(d) Recalculate your answer to part (c) if you believe in the liquidity preference theory and you believe that the liquidity premium is 1%. Calculate the expected holding period return and compare with the yield to maturity on 1-year zero. What do you find? Explain your finding.

Suggested solution.

If the liquidity premium is 1%, then the forward rate is 9.01%.

$$f_2 = (1 + y_2)^2/(1 + y_1) - 1 = 9.01\%$$

and the expected short rate is 8.01%

$$E[r_2] = f_2 - 1\% = 8.01\%$$

Then the price at the end of year 1 is \$100.9172

$$P_1 = 109/(1 + E[r_2]) = 100.9172$$

The expected holding period return is 7.91%

$$\text{HPR} = \frac{100.9172 + 9}{101.861} - 1 = 7.91\%$$

The holding period return is higher than the YTM for the one-year zero. This makes sense as the interest rate in the second year is uncertain and then the bond price by the end of the first year is also uncertain. The holding period return therefore needs to be higher to compensate the risk from the uncertainty in second year's interest rate.

```
f2 <- (1 + y2)^2/(1 + y1) - 1
Er2 <- f2 - 0.01
P1 <- 109/(1 + Er2)
P1
[1] 100.9172
hpr <- (P1 + 9)/P - 1
hpr
[1] 0.07908851
```

Problem 4 (Duration and convexity) A newly issued bond has a maturity of 10 years and pays a 7% coupon rate (with coupon payments coming once annually). The bond sells at par value.

(a) What are the duration and the convexity of the bond?

Suggested solution.

Without the loss of generality, suppose the par value is 100.

The duration of the bond is 7.515232.

$$\text{duration} = \sum_{t=1}^{T} [t \times w_t] = \sum_{t=1}^{T} \left[t \times \frac{CF_t}{P \cdot (1 + y)^t} \right] = 7.515232$$

```r
P <- 100
rate <- 0.07
y <- rate
year <- 10
CF <- rep(P * rate, year)
CF[year] <- CF[year] + P
duration <- 0
for (t in c(1:year)) {
    duration <- duration + t * CF[t]/((1 + y)^t)/P
}
duration
[1] 7.515232
```

The convexity of the bond is 64.93296.

$$\text{convexity} = \frac{1}{P \times (1+y)^2} \sum_{t=1}^{T} [t \times w_t] = \frac{1}{P \times (1+y)^2} \sum_{t=1}^{T} \left[(t^2 + t) \times \frac{CF_t}{(1+y)^t} \right] = 64.93296$$

```r
convexity <- 0
for (t in c(1:year)) {
    convexity <- convexity + (t^2 + t) * CF[t]/((1 + y)^t)
}
convexity <- convexity/(P * (1 + y)^2)
convexity
[1] 64.93296
```

(b) Find the actual price of the bond assuming that its yield to maturity immediately increases from 7% to 8% (with maturity still 10 years).

Suggested solution.

The actual price is 93.2899.

```r
P.actual <- 93.2899
```

```r
knitr::include_graphics("problem4.png")
```

Settlement date	1/1/22
Maturity date	1/1/32
Annual coupon rate	0.07
Yield to maturity	0.08
Redemption value (% of face value)	100
Coupon payments per year	1
Flat price (% of par)	93.2899

(c) What price would be predicted by the modified duration rule? What is the percentage error of that rule?

Suggested solution.

$$\frac{\Delta P}{P} = -\frac{\text{duration}}{1+y} \cdot \Delta y = -0.070$$

The price predicted by the modified duration rule is \$92.976.

$$P' = P \cdot \left(1 + \frac{\Delta P}{P}\right) = 92.976$$

The percentage error of this rule is 0.336%.

$$(P' - P_{actual})/P_{actual} = 0.336\%$$

```
delta.y <- 0.01
price.change <- -duration/(1 + y) * delta.y
price.change
[1] -0.07023582
P.prime <- P * (1 + price.change)
P.prime
[1] 92.97642
abs(P.prime/P.actual - 1)
[1] 0.003360295
```

(d) What price would be predicted by the modified duration-with-convexity rule? What is the percentage error of that rule?

Suggested solution.

$$\frac{\Delta P}{P} = -\frac{\text{duration}}{1+y} \cdot \Delta y + 1/2 \times \text{convexity} \times (\Delta y)^2 = -0.067$$

The price predicted by the modified duration-with-convexity rule is \$93.301.

$$P' = P \cdot \left(1 + \frac{\Delta P}{P}\right) = 93.301$$

The percentage error of this rule is 0.01%.

$$(P' - P_{actual})/P_{actual} = 0.01\%$$

```
price.change <- -duration/(1 + y) * delta.y + 1/2 * convexity *
    delta.y^2
price.change
[1] -0.06698917
P.prime <- P * (1 + price.change)
P.prime
[1] 93.30108
abs(P.prime/P.actual - 1)
[1] 0.0001198764
```

(e) Compare your solution from parts (c) and (d). What do you find? Explain your finding.

Suggested solution.

The price predicted by the modified duration-with-convexity rule is much more accurate than the price predicted by the modified duration rule. This is because the modified duration-with-convexity rule uses both the first and second order information, while the duration rule only uses the first order information.

Problem 5 (Option) We will derive a two-state put option value in this problem. Data: $S_0 = 100$; $X = 110$; $1 + r = 1.10$. The two possibilities for S_T are 130 and 80.

(a) Show that the range of S is 50, whereas that of P is 30 across the two states. What is the hedge ratio of the put?

Suggested solution.

The range of S is \$50 because

$$uS_0 - dS_0 = \$130 - \$80 = \$50.$$

In the state of $uS_0 = \$130$, the put has zero value and then $P_u = 0$. In the state of $dS_0 = \$80$, the put holder will exercise the put and then the value is $P_d = \$110 - \$80 = \$30$.

The range of P is \$30 because

$$P_d - P_u = \$30.$$

The hedge ratio of the put is

$$\text{hedge ratio} = \frac{P_u - P_d}{uS_0 - dS_0} = -\frac{30}{50} = -\frac{3}{5}.$$

(b) Form a portfolio of three shares of stock and five puts. What is the (nonrandom) payoff to this portfolio?

Suggested solution.

The payoff when the stock is up is

$$3 \times uS_0 + 5 \times P_u = \$390$$

The payoff when the stock is down is

$$3 \times dS_0 + 5 \times P_d = 3 \times \$80 + 5 \times \$30 = \$390$$

(c) What is the present value of the portfolio?

Suggested solution.

The present value of the portfolio is \$354.5455

$$PV = \frac{\$390}{1+r} = \frac{\$390}{1.10} = \$354.5455$$

```
PV <- 390/1.1
PV
[1] 354.5455
```

(d) Given that the stock currently is selling at 100, solve for the value of the put.

Suggested solution.

The value of the put can be solved from

$$3 \times S_0 + 5 \times P_0 = \$354.5455$$

and then the value of the put is \$10.90909

$$P_0 = \frac{\$354.5455 - \$300}{5} = \$10.90909$$

```
(PV - 300)/5
[1] 10.90909
```