Lecture 10: Diversification continued

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10-diversification-cont

QTM 385 Quantitative Finance

Lecture 10: Efficient diversification

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Suggested reading: Investments Ch 7

EMORY

Two risky portfolios

- Two risky portfolios: a bond portfolio D specializing in long-term debt securities and a stock portfolio E specializing in equity securities
- Asset allocation decision of portfolio P: w_D in the bond portfolio and $w_E = 1 - w_D$ in the stock portfolio
  - Rate of return on this portfolio, r_P
    $$r_P = w_D r_D + w_E r_E$$ → recall this is linear
  - Expected return on this portfolio
    $$E(r_P) = w_D E(r_D) + w_E E(r_E)$$
  - Variance of this portfolio
    $$\sigma_P^2 = w_D^2 \cdot \sigma_D^2 + w_E^2 \cdot \sigma_E^2 + 2 w_D w_E Cov(r_D, r_E)$$

	Debt	Equity
Expected return, $E(r)$	8%	13%
Standard deviation, σ	12%	20%
Covariance, $Cov(r_D, r_E)$	72	
Correlation coefficient, ρ_{DE}		0.30

EMORY

Covariance between two risky portfolios

- Variance of portfolio P is affected by the covariance of two portfolios
  $$\sigma_P^2 = w_D^2 \cdot \sigma_D^2 + w_E^2 \cdot \sigma_E^2 + 2 w_D w_E Cov(r_D, r_E)$$
- Let ρ_{DE} be the correlation coefficient of two portfolio. Then
  $$Cov(r_D, r_E) = \rho_{DE} \sigma_D \sigma_E$$
  - Portfolio variance increases with ρ_{DE}

This is the worst case for diversification

correlation = 1 so highest possible correlation

  - Case 1: $\rho_{DE} = 1$, D and E are perfectly positively correlated and the standard deviation of portfolio P is the largest. Then plug in 1 for Cov(rD, rE)
    $$\sigma_P^2 = w_D^2 \cdot \sigma_D^2 + w_E^2 \cdot \sigma_E^2 + 2 w_D w_E \sigma_D \sigma_E = (w_D \sigma_D + w_E \sigma_E)^2$$
    $$\Rightarrow \sigma_P = w_D \sigma_D + w_E \sigma_E$$ std dev of portfolio = weighted std dev of E and D
  - Standard deviation σ_P is linear in w_E and σ_E (also linear in w_D and σ_D)

In this case There is NO DIVERSIFICATION regardless of w_D and w_E.

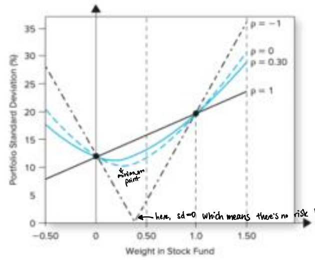
EMORY

Portfolio standard deviation as a function of w_E

- Variance of portfolio P is affected by the covariance of two portfolios

$$\sigma_P^2 = w_D^2 \cdot \sigma_D^2 + w_E^2 \cdot \sigma_E^2 + 2w_D w_E Cov(r_D, r_E)$$

Here we're assuming no arbitrage opportunity obviously



← here, sd=0 which means there's no risk but ⊕ return guarantee

How sd varies with the weights

$w_E \uparrow$ then $r_p \uparrow$ then $E(r_p) \uparrow$ (?)

Objective: How to solve for the minimum point? (Depends on ρ too)

Covariance between two risky portfolios

- Variance of portfolio P is affected by the covariance of two portfolios

$$\sigma_P^2 = w_D^2 \cdot \sigma_D^2 + w_E^2 \cdot \sigma_E^2 + 2\rho_{DE} w_D w_E \sigma_D \sigma_E$$

- Case 2: $-1 < \rho_{DE} < 1$

$$\sigma_P < w_D \sigma_D + w_E \sigma_E$$

- *Portfolios of less than perfectly correlated assets always offer some degree of diversification benefit*. The lower the correlation between the assets, the greater the gain in efficiency
- If $\rho_{DE} < 0$, then D is a hedge portfolio of E. In this case, diversification is particularly effective in reducing total risk (while not affecting total return)

→ If 1 portfolio has ⊖ return it's likely the other has ⊕ so we can hedge the negative return

Appears most when we talk about derivatives

Covariance between two risky portfolios

- Variance of portfolio P is affected by the covariance of two portfolios

$$\sigma_P^2 = w_D^2 \cdot \sigma_D^2 + w_E^2 \cdot \sigma_E^2 + 2\rho_{DE} w_D w_E \sigma_D \sigma_E$$

- Case 3: $\rho_{DE} = -1$ D and E are perfectly negatively correlated and

$$\sigma_P^2 = (w_D \sigma_D - w_E \sigma_E)^2$$
$$\Rightarrow \sigma_P = |w_D \sigma_D - w_E \sigma_E|$$

Plug $\rho_{DE} = -1$ into variance expression

Then write into quadratic form

- Standard deviation σ_P is the smallest

↓ You can see this is linear (aside from absolute value)

Perfectly hedging position (perfect negative correlation)

- A perfectly hedging position can be obtained by solving

$$w_D\sigma_D - w_E\sigma_E = 0$$

$$\Rightarrow w_D\sigma_D - (1-w_D)\sigma_E = w_D(\sigma_D+\sigma_E) - \sigma_E = 0$$

- The solution is

$$w_D = \frac{\sigma_E}{\sigma_D+\sigma_E}$$

$$w_E = \frac{\sigma_D}{\sigma_D+\sigma_E} = 1 - w_D$$

Two scenarios:

prob	50%	50%	
Scenario	1 rate of return	2 rate of return	expected return
D	6%	10%	8%
E	15%	11%	13%

$w_E = 0.5$
$w_D = 0.5$

Return is 10.5% — Return is 10.5% — 10.5% — $\sigma_p = 0$

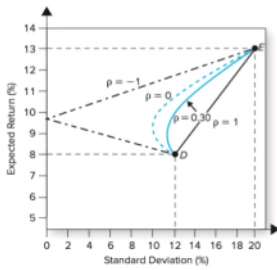
No variation. Perfect hedging position.

$\sigma_D^2 = \frac{1}{2}\left[(6\% - 8\%)^2 + (10\% - 8\%)^2\right]$
$= 2\%$
$\sigma_E^2 = \frac{1}{2}\left[(15\% - 13\%)^2 + (11\% - 13\%)^2\right]$
$= 2\%$

I'm so confused

Portfolio opportunity set

- Portfolio opportunity set: all combinations of portfolio expected return and standard deviation that can be constructed from the two available assets/portfolios
  Links σ_p with the RETURN
- The lower the correlation, the greater the potential benefit from diversification
- When $\rho_{DE} = -1$, there is a perfect hedging opportunity: Standard deviation can be driven all the way to zero



If we restrict the risk to a specific point, if the ρ is smaller, we can find w_E and w_D such that the return is higher

When $\rho \downarrow$ then portfolio return $\uparrow$

Minimum variance portfolio

How can we achieve the point with the lowest risk? Given ρ, what w_E and w_D we choose such that portfolio risk is the lowest

- Minimum-variance portfolio: Portfolio with w_D^* and w_E^* such that the corresponding σ_P^{*2} is the smallest among all possible choices of w_D and w_E

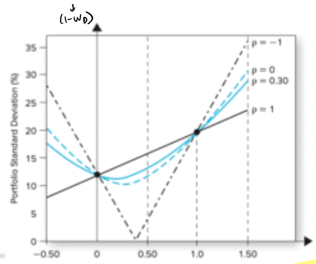
$$\sigma_P^2 = w_D^2\cdot\sigma_D^2 + w_E^2\cdot\sigma_E^2 + 2\rho_{DE}w_Dw_E\sigma_D\sigma_E$$

replace with $(1-w_D)^2$ — $(1-w_D)$

It's a quadratic fcn. Take derivative and find first order

- If $\rho_{DE} = -1$, then
  - $w_D^* = \frac{\sigma_E}{\sigma_D+\sigma_E} = \frac{20}{12+20} = .625$
  - $w_E^* = 1 - .625 = .375$

	Debt	Equity
Expected return, E(r)	8%	13%
Standard deviation, σ	12%	20%



w_E^* decreases with ρ or $Cov(r_D, r_E)$
w_D^* increases with " "

If covariance $\downarrow$, the denominator $\uparrow$ at a slower rate than numerator so w_D^* will increase?

Minimum-variance portfolio for general ρ_{DE}

- Based on the variance formula

$$\sigma_P^2 = w_D^2\cdot\sigma_D^2 + w_E^2\cdot\sigma_E^2 + 2w_Dw_E Cov(r_D, r_E)$$

- Replace w_E by $1 - w_D$ and take the derivative with respect to w_D. The optimal w_D^* that minimizes σ_P^2 is

$$w_D^* = \frac{\sigma_E^2 - Cov(r_D,r_E)}{\sigma_D^2 + \sigma_E^2 - 2Cov(r_D,r_E)}$$

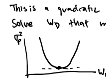
Proof: Solve minimum variance portfolio

$\sigma_p^2 = w_D^2\sigma_D^2 + w_E^2\sigma_E^2 + 2w_Dw_E Cov(r_D, r_E)$

replace $w_E = 1 - w_D$

$\sigma_p^2 = w_D^2\sigma_D^2 + (1-w_D)^2\sigma_E^2 + \overbrace{2w_D(1-w_D)}^{(2w_D - 2w_D^2)} Cov(r_D, r_E)$

This is a quadratic fcn of w_D
Solve w_D that minimizes σ_P^2



at which pt the derivative = 0
so

$0 = \cancel{2}w_D\sigma_D^2 + \cancel{2}(w_D - 1)\sigma_E^2 + (\cancel{2} - \cancel{4}w_D)\, Cov(r_D, r_E)$

$0 = w_D(\sigma_D^2 + \sigma_E^2 - 2Cov(r_D, r_E)) - \sigma_E^2 + Cov(r_D, r_E)$

Then solve for w_D

$\frac{\sigma_E^2 - Cov(r_D, r_E)}{\sigma_D^2 + \sigma_E^2 - 2Cov(r_D, r_E)} = w_D^*$ and then $w_E^* = 1 - w_D^*$

What does this expression mean?

when $\rho = -1$ then $Cov(r_D, r_E) = \rho\sigma_D\sigma_E = -\sigma_D\sigma_E$

then $w_D^* = \frac{\sigma_E^2 + \sigma_D\sigma_E}{\ }= \frac{\sigma_E(\cancel{\sigma_E + \sigma_D})}{\ } = \frac{\sigma_E}{\ } = 50\%$

What does this expression mean?

when $\rho = -1$ then $cov(r_D, r_E) = \rho\sigma_D\sigma_E = -\sigma_D\sigma_E$

then $W_D^* = \frac{\sigma_E^2 + \sigma_D\sigma_E}{\sigma_D^2 + \sigma_E^2 + 2\sigma_D\sigma_E} = \frac{\sigma_E \cancel{(\sigma_E + \sigma_D)}}{(\sigma_D + \sigma_E)^{\cancel{2}}} = \frac{\sigma_E}{\sigma_D + \sigma_E} = 50\%$

Question

- The weight of in the minimum variance portfolio is

$$w_D^* = \frac{\sigma_E^2 - Cov(r_D, r_E)}{\sigma_D^2 + \sigma_E^2 - 2Cov(r_D, r_E)}$$

- Question: What is the weight of D and E when $\rho_{DE} = .3$? What if $\rho_{DE} = 0$?

	Debt	Equity
Expected return, $E(r)$	8%	13%
Standard deviation, σ	12%	20%

Choosing a portfolio based on risk aversion

- For the utility $U = E(r) - \frac{1}{2}A\sigma^2$ with the risk aversion parameter A, the optimal investment proportions in the two funds are

The investor may care most about utility

$$w_D^* = \frac{E(r_D) - E(r_E) + A(\sigma_E^2 - Cov(r_D, r_E))}{A(\sigma_D^2 + \sigma_E^2 - 2Cov(r_D, r_E))}$$

$$w_E^* = 1 - w_D^*$$

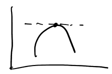
What's the optimal portfolio to maximize utility?

Solution will be
we plug expression of
Expected return & variances
take derivative
solve w_D first order

Proof: Maximizing utility

risk aversion

$U = E(r_p) - \frac{1}{2}A\sigma_p^2 = w_D E(r_D) + w_E E(r_E) - \frac{1}{2}A(w_D^2\sigma_D^2 + w_E^2\sigma_E^2 + 2w_D w_E Cov(r_D, r_E))$



still a quadratic function

Then procedure is exactly the same

Replace $w_E = 1 - w_D$

$U = w_D E(r_D) + (1-w_D)E(r_E) - \frac{1}{2}A(w_D^2\sigma_D^2 + (1-w_D)^2\sigma_E^2 + 2w_D(1-w_D)Cov(r_D, r_E))$

$\frac{dU}{dw_D} = E(r_D) + E(r_E) - \frac{1}{\cancel{2}}A\cancel{2}[w_D\sigma_D^2 + (w_D - 1)\sigma_E^2 + (1-2w_D)Cov(r_D, r_E))$

Combine like terms & set to 0

$0 = -Aw_D(\sigma_D^2 + \sigma_E^2 - 2Cov(r_D, r_E)) + E[r_D] - E[r_E] + A(\sigma_E^2 - Cov(r_D, r_E))$

Solve for w_D

$w_D^* = \frac{E[r_D] - E[r_E] + A(\sigma_E^2 - Cov(r_D, r_E))}{A(\sigma_D^2 + \sigma_E^2 - 2Cov(r_D, r_E)}$

$w_E^* = 1 - w_D^*$

$w_D^* \uparrow$ with risk aversion

If investor is more risk averse, more weight is in bonds

$w_D^* \uparrow$ with $E(r_D)$
$\downarrow$ with $E(r_E)$

Homework: Choosing a portfolio based on Sharpe ratio

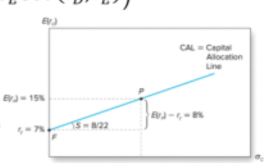
- We can find the weights w_D and w_E to maximize the Sharpe ratio. Then use this portfolio P and a risk-free asset to construct the CAL

$$\max_{w_D, w_E} S_P = \frac{E(r_P) - r_f}{\sigma_P}$$

$$\text{s.t.}\ r_P = w_D r_D + w_E r_E$$

$$\sigma_P = \left(w_D^2 \cdot \sigma_D^2 + w_E^2 \cdot \sigma_E^2 + 2w_D w_E Cov(r_D, r_E)\right)^{0.5}$$

$$w_D + w_E = 1$$



Optimal portfolio weights to maximize Sharpe ratio

- The optimal portfolio weights that maximize Sharpe ratio is

$$w_D^* = \frac{E(R_D)\sigma_E^2 - E(R_E)Cov(R_D,R_E)}{E(R_D)\sigma_E^2 + E(R_E)\sigma_D^2 - [E(R_D)+E(R_E)]Cov(R_D,R_E)}$$

$$w_E^* = 1 - w_D^*$$

- $R_D = r_D - r_f$ and $R_E = r_E - r_f$ are excess returns

Example

- The optimal portfolio weights that maximize Sharpe ratio is

$$w_D^* = \frac{E(R_D)\sigma_E^2 - E(R_E)Cov(R_D,R_E)}{E(R_D)\sigma_E^2 + E(R_E)\sigma_D^2 - [E(R_D)+E(R_E)]Cov(R_D,R_E)}$$

$$= \frac{(8-5)\times 400 - (13-5)\times 72}{(8-5)\times 400 + (13-5)\times 144 - (8-5+13-5)\times 72} = .40$$

$$w_D^* = 1 - .40 = .60$$

$$E(r_P) = (.4\times 8) + (.6\times 13) = 11\%$$

$$\sigma_P = [(.4^2\times 144) + (.6^2\times 400) + (2\times .4\times .6\times 72)]^{0.5} = 14.2\%$$

$$S_P = \frac{11-5}{14.2} = .42$$

	Debt		Equity
Expected return, $E(r)$	8%		13%
Standard deviation, σ	12%		20%
Covariance, $Cov(r_D, r_E)$		72	
Correlation coefficient, ρ_{DE}		0.30	

Hint for homework

- Write S_P^2 in terms of w_D only (using $w_D = 1 - w_E$)
- Take the derivative of S_P^2 with respect to w_D and set it to zero

$$\frac{dS_P^2}{dw_D} = 0$$

- Solve w_D from the first order condition and then we can show it takes the form in Problem 5

Optimal complete portfolio

1. Specify the return characteristics of all securities: expected returns, variances, covariances
2. Asset allocation decision to construct optimal risky portfolio P (same for all investors)
   - Solve the weight of each asset to maximize the Sharpe ratio of P
3. Capital allocation between risky portfolio P and risk-free assets (e.g., T-bills) to maximize an investor's utility (vary with investors)
   - Solve the weight y in P and weight $1 - y$ in risk-free assets to maximize utility score ($U = E(r) - \frac{1}{2}A\sigma^2$)