Lecture 8: Capital allocation continued

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QTM 385 Quantitative Finance

Lecture 8: Capital allocation to risky assets

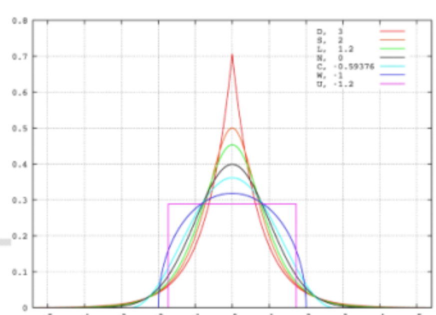
Instructor: Ruoxuan Xiong

Suggested reading: Investments Ch 6

Question from Google form

- Why would it make sense for things with high kurtosis to have both extreme outcomes and have a high density in the middle? Does that mean that either there is often no gain/loss in return, but when there is a gain/loss, it tends to be extreme?

- *Answer: Yes. It is possible that the density is high in the middle and at the tail. Then compared to normal distribution, it is more likely to have returns around zero and extreme gain/loss.*
    - *Recall* $Kurtosis = Average\left[\frac{(r-\bar{r})^4}{\hat{\sigma}^4}\right] - 3$



Question from Google form

- Is VaR used only for daily returns, or could you also use it for more long-term returns?

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- *Answer: VaR can also be used for long-term returns, e.g., monthly/quarterly returns*

Question from Google form

- When looking at a treasury bond information, how long of a given time period is the change in ask price in the change column?

- *Answer: Usually it is compared to yesterday's ask price.*

LISTING OF TREASURY ISSUES

MATURITY	COUPON	BID	ASKED	CHANGE	ASKED YIELD TO MATURITY
15-Feb-2019	2.750	100.0391	100.0547	0.0078	2.256
30-Apr-2021	2.250	99.7500	99.7656	0.2344	2.354
15-May-2023	1.750	97.4531	97.4688	0.4766	2.364
15-Aug-2029	6.125	132.7266	132.7891	1.1406	2.575
15-Feb-2036	4.500	125.4688	125.5313	1.5391	2.637
15-Aug-2048	3.000	101.8984	101.9297	1.5391	2.902

Question from Google form

- Doesn't the utility score formula contradict the Sharpe Ratio? Investors want to minimize risk while maximizing expected return. If an investor is a risk-lover, then the investor will favor the portfolio with the same level of return but a higher level of risk. It also doesn't follow the efficient frontier or intuitive sense. If an investor want to pursue above average return, why not pick stocks with higher beta instead?

- *Answer: Recall utility score is defined as* $U = E(r) - \frac{1}{2}A\sigma^2$ *and Sharpe ratio is defined as* $S = \frac{E(r)-r_f}{\sigma}$*. Given the same* σ*, the portfolio with a high* S *has a high* $E(r)$ *and high* U*. Therefore,* U *and* S *are consistent. Using efficient frontier, we can identify a set of portfolios with the highest* S*. We choose one among this set maximize investor's utility score.*

Lecture plan

- Risk aversion and utility values

- Portfolios of one risky asset and a risk-free asset

Risk aversion and utility values

- **Risk-averse** investor penalizes the expected rate of return by the risk involved
  - A portfolio is more attractive when its expected return is higher and its risk is lower
- **Utility score** to compare competing portfolios based on expected return and risk of those portfolios

$$U = E(r) - \frac{1}{2}A\sigma^2$$

  - U: utility value
  - A: index of the investor's risk aversion
    - Risk-averse investors: $A > 0$
    - Risk-neural investors: $A = 0$
    - Risk-lover investors: $A < 0$
  - Quantify the rate at which investors are willing to trade off return against risk

Evaluating investments by using utility scores

Portfolio	Risk Premium	Expected Return	Risk (SD)
L (low risk)	2%	7%	5%
M (medium risk)	4	9	10
H (high risk)	8	13	20

Table 6.1

Available risky portfolios (risk-free rate = 5%)

Investor Risk Aversion (*A*)	Utility Score of Portfolio *L* [$E(r) = 0.07$; $\sigma = 0.05$]	Utility Score of Portfolio *M* [$E(r) = 0.09$; $\sigma = 0.10$]	Utility Score of Portfolio *H* [$E(r) = 0.13$; $\sigma = 0.20$]
2.0	$0.07 - ½ \times 2 \times 0.05^2 = 0.0675$	$0.09 - ½ \times 2 \times 0.1^2 = 0.0800$	**$0.13 - ½ \times 2 \times 0.2^2 = 0.09$**
3.5	$0.07 - ½ \times 3.5 \times 0.05^2 = 0.0656$	**$0.09 - ½ \times 3.5 \times 0.1^2 = 0.0725$**	$0.13 - ½ \times 3.5 \times 0.2^2 = 0.06$
5.0	$0.07 - ½ \times 5 \times 0.05^2 = 0.0638$	**$0.09 - ½ \times 5 \times 0.1^2 = 0.0650$**	$0.13 - ½ \times 5 \times 0.2^2 = 0.03$

Interpretation of utility score

- The utility score of risky portfolios can be interpreted as a **certainty equivalent** rate of return

- Certainty equivalent is the rate that risk-free investment would need to offer to provide the same utility as the risky portfolio
  - A natural way to rank competing portfolios
  - A portfolio is desirable only if its certainty equivalent exceeds that of the risk-free equivalent

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- *Answer:*

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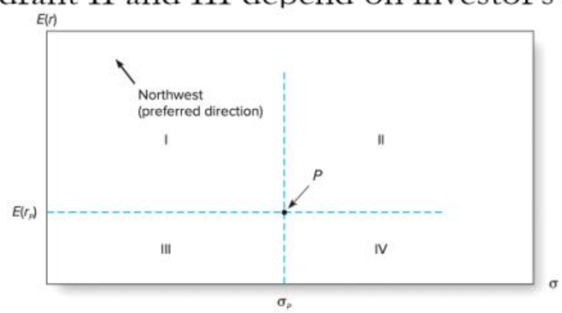
- *Answer:*
- *If* $A = 4$, *then* $U = E(r) - \frac{1}{2}A\sigma^2 = 0.2 - \frac{1}{2}\times 4 \times 0.3^2 = 2\% < 7\%$
- *If* $A = 2$, *then* $U = E(r) - \frac{1}{2}A\sigma^2 = 0.2 - \frac{1}{2}\times 2 \times 0.3^2 = 11\% > 7\%$

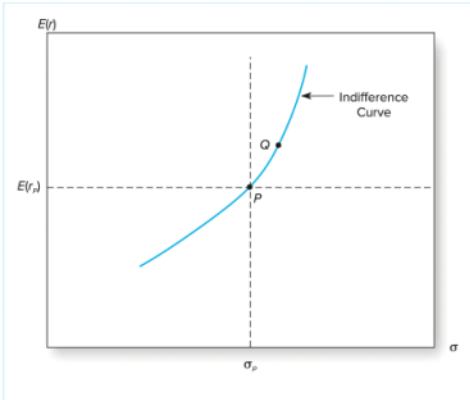
Risk and return tradeoff

- Portfolio P dominates any portfolio in quadrant IV (lower $E(r)$ and higher σ)
- Any portfolio in quadrant I (higher $E(r)$ and lower σ) dominates portfolio P
- Portfolios in quadrant II and III depend on investor's risk aversion



Indifference curve

- Indifference curve connects all portfolio points with the same utility value



Capital allocation across risky and risk-free portfolios

- **Asset allocation** of a complete portfolio: y in the risky assets and $1-y$ in the risk-free assets

- Example: total market value of a portfolio $\$300{,}000$
  - $\$90{,}000$ invested in risky assets
  - $\$210{,}000$ invested in risk-free assets
  - $y = \frac{210{,}000}{300{,}000} = .7$ (risky assets) and $1 - y = \frac{90{,}000}{300{,}000} = .3$ (risk-free assets)

- Treat all the risky assets as a portfolio, and all the risk-free assets a portfolio

Asset allocation decision

- The investor decides the proportion y of the risky portfolio P and risk-free asset in the complete portfolio C
- Suppose the risk-free rate is r_f and for the risky portfolio P
  - Rate of return r_P
  - Expected return $E[r_P]$
  - Standard deviation σ_P

- For the complete portfolio C, the rate of return is

$$r_C = yr_P + (1-y)r_f$$

and expected return is

$$E[r_C] = yE[r_P] + (1-y)r_f = r_f + y\left[E(r_P) - r_f\right]$$

Return and risk of complete portfolio

- For the complete portfolio C with proportion y in the risky portfolio,
  - The expected return is

$$E[r_C] = yE[r_P] + (1-y)r_f = r_f + y\left[E(r_P) - r_f\right]$$

    - The base rate of return for any portfolio is the risk-free rate
    - Expected return is proportional to y and the risk premium $E(r_P) - r_f$

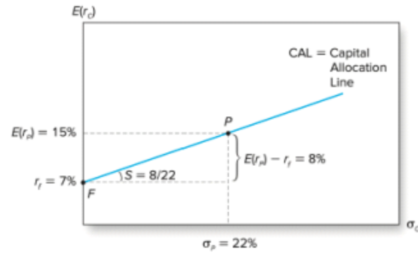
  - The standard deviation is

$$\sigma_C = y\sigma_P$$

    - Standard deviation is proportional to y and the standard deviation of the risky asset

Capital allocation line (CAL)

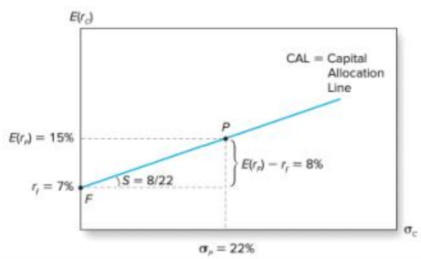
- Capital allocation line (CAL): A graph showing all feasible risk–return combinations of a risky and risk-free asset
  - y-axis: $E[r_C]$
    - Equals $r_f + y[E(r_P) - r_f]$
  - x-axis: σ_C
    - Equals $y\sigma_P$
  - CAL: $E[r_C] = r_f + S \cdot \sigma_C$
    - Intercept: r_f
    - Slope of the line (Sharpe ratio): $S = \frac{E(r_P)-r_f}{\sigma_P}$
    - return vs standard deviation of complete portfolio for every y
  - Example: $E[r_P] = 15\%, \sigma_P = 22\%, r_f = 7\%$
    - $S = \frac{15-7}{22} = .36$
    - CAL: $E[r_C] = .07 + .36 \cdot \sigma_C$



EMORY

Question

- Can the Sharpe ratio, $S = \frac{E(r_C)-r_f}{\sigma_C}$, of any combination of the risky asset and the risk-free asset be different from the ratio for the risky asset taken alone, $\frac{E(r_P)-r_f}{\sigma_P}$, which, in this case, is .36?



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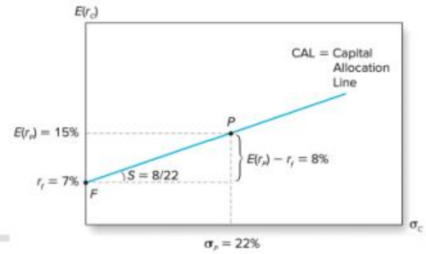
- *Answer: No. the Sharpe ratio of any complete portfolio must be* .36.

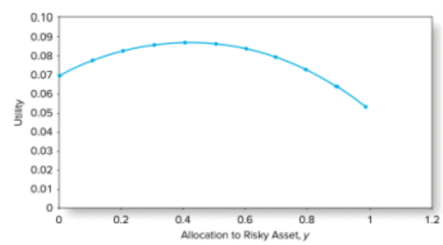
$$\frac{E(r_C)-r_f}{\sigma_C} = \frac{r_f + y[E(r_P)-r_f]-r_f}{y\sigma_P} = \frac{y[E(r_P)-r_f]}{y\sigma_P} = \frac{E(r_P)-r_f}{\sigma_P}$$



Risk tolerance and asset allocation

- Investors confronting CAL need to choose one optimal complete portfolio C (or one y) from the set of feasible choices
  - Investors maximize utility $U = E(r) - \frac{1}{2}A\sigma^2$
  - The expected return is $E[r_C] = r_f + y[E(r_P) - r_f]$
  - The variance is $\sigma_C^2 = y^2\sigma_P^2$
  - Plug $E[r_C]$ and σ_C^2 into U
  - The decision problem is to choose y that maximizes

$$U = r_f + y[E(r_P) - r_f] - \frac{1}{2}Ay^2\sigma_P^2$$



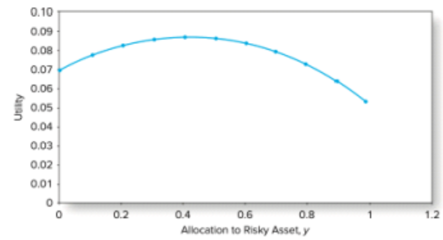
Solve the optimal portfolio

- Solve the y that maximizes

$$U = r_f + y[E(r_P) - r_f] - \frac{1}{2}Ay^2\sigma_P^2$$

- Take the derivative of U with respect to y and set the derivative to 0
  - $[E(r_P) - r_f] - Ay\sigma_P^2 = 0$

- Optimal y
  - $y^* = \frac{E(r_P) - r_f}{A\sigma_P^2}$

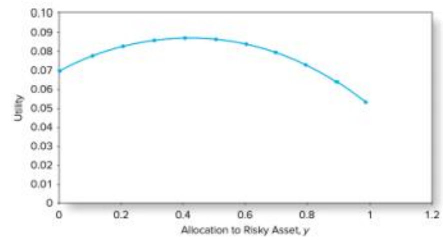


Example

- Suppose $r_f = 7\%, E(r_P) = 15\%, \sigma_P = 22\%$
- For an investor with risk aversion $A = 4$

$$y^* = \frac{E(r_P) - r_f}{A\sigma_P^2} = \frac{.15 - .07}{4 \times .22^2} = .41$$

- 41% in the risky asset and 59% in the risk-free asset
- $E[r_C] = 7 + [.41 \times (15 - 7)] = 10.28\%$
- $\sigma_C = .41 \times 22 = 9.02\%$
- $U = E[r_C] - \frac{1}{2}A\sigma_C^2 = 10.28\% - \frac{1}{2} \cdot 4 \cdot 9.02\%^2$
  $= 0.08653$
- Sharpe ratio is $\frac{E[r_C] - r_f}{\sigma_C} = \frac{3.28}{9.02} = .36$



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Indifference curve analysis

- Indifference curve: all combinations of expected return $E(r)$ and volatility σ such that utility level is the same

$$U = E[r] - \frac{1}{2} \times A \times \sigma^2$$

- For example, an investor with risk aversion $A = 4$
  - Portfolio 1: Risk-free portfolio with $r_f = 5\%$
    - $U_1 = 0.05 - \frac{1}{2} \times 4 \times 0 = 0.05$
  - Portfolio 2: Risky portfolio with return $E[r] = .0502$ and $\sigma = 1\%$
    - $U_2 = 0.0502 - \frac{1}{2} \times 4 \times 1\% = 0.05$

➢Two portfolios are on the same indifference curve with utility $U = .05$

Indifference curve analysis

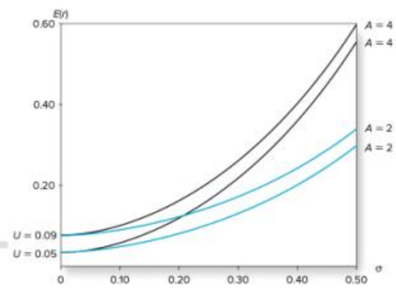
- Indifference curve: all combinations of expected return $E(r)$ and volatility σ such that utility level is the same

$$U = E[r] - \frac{1}{2} \times A \times \sigma^2$$

- Given U, identify the required $E[r]$ for every σ through

$$E[r] = U + \frac{1}{2} \times A \times \sigma^2$$

- Given the same σ, the required $E[r]$ for the less risk-averse investor (i.e., smaller A) is lower to achieve the same U



Comparative statics

- More risk-averse investors (i.e., larger A) have steeper indifference curves than less risk-averse investors
  - Steepness is measured by
$$\frac{dE[r]}{d\sigma} = \frac{d\left(U + \frac{1}{2}\times A\times\sigma^2\right)}{d\sigma} = A\sigma$$
  - Steepness increases with both A and σ, but does not vary with U
  - Steeper curves mean that investors require a great increase in expected returns to compensate for an increase in portfolio risk
- Higher indifference curves correspond to higher levels of utility