Lecture 12: CAPM

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12-capm

QTM 385 Quantitative Finance

Lecture 12: CAPM

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Suggested reading: Investments Ch 9

Question from Google form

- Just a quick clarification: $E(R_D) = E(r_D - r_f)$ which is different notation from $E(r_D)$?

- *Answer:*
  - r_D: *return of asset D*
  - $R_D = r_D - r_f$: *excess return of asset D*
  - $E(r_D)$: *expected return of asset D*
  - $E(R_D) = E(r_D - r_f)$: *risk premium (i.e., expected excess return) of asset D*

Question from Google form

- Personally, do you think it is worth investing in hedge funds?

- *Answer:*
- *It is difficult for individual investors to gain access to high quality hedge funds (may instead invest in stocks of a financial company that operates like a hedge fund).*
- *To invest in hedge fund as an individual, you must be an institutional investor (like a pension fund), or an accredited investor (with a net worth of $1M not including the value of primary residence or annual income of $200K ($300K for married coupon)).*

Investment strategies in hedge funds are quite complex and involve shorting and stuff
Usually for those strategies they need a protection period before releasing the portfolio to the investors so that when the market is down, the investors will not pull the money out

Most commonly, people will invest in ETFs or mutual funds
Hedge funds are not an easy option

Capital asset pricing model

- The capital asset pricing model (CAPM): a prediction of the relationship between the risk of an asset and its expected return

- Two set of assumptions
  1. Individual behavior (investors are alike)
  2. Market structure (markets are well-functioning)

- Central implication: Risk premium $E(r_i - r_f)$ is proportional to exposure to systematic risk and independent of firm-specific risk

See assumptions on next page
But this implication still generally holds in practice

Assumptions on individual investors

- Investors are rational, mean-variance optimizers
- Their common planning horizon is a single period

Different from dynamic asset pricing which extends to multiple periods

- Investors all use identical input lists (e.g., mean and covariance), an assumption often termed homogeneous expectations
  - homogeneous expectations are consistent with the assumption that all relevant information is publicly available

Investors are exposed to the same set of assets and use the same input list
And they solve for optimal asset allocation in portfolio -> this is called the market portfolio

In reality, we do not have the same expectations

Assumptions on market structure

- All assets are publicly held and trade on public exchanges
- Investors can borrow or lend at a common risk-free rate, and they can take short positions on traded securities
- No taxes

May need to add premium on top of risk free rate, but in this model we don't assume this

- No transaction costs

The market portfolio

- Under these assumptions, all investors draw the same efficient frontier and same tangent CAL
- All investors would choose the same P with the same set of weights for each risky asset
- This portfolio is the market portfolio, denoted as M, i.e., the value-weighted portfolio of all assets in the investable universe
- **Mutual fund theorem**: investing in a market-index portfolio is efficient (passive strategy)

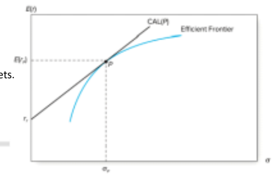
Investors are exposed to same assets, have the same expectations for mean and covariance. Each investor will make the mean-variance porfolio optimization. And the optimal solution will be the same.

For more risk averse investors they may choose a point closer to the bottom
Those that are less risk averse is choosing a point closer to the top

Market portfolio should be most efficient
Passive strategy means we invest in market portfolio and do not activiely increase/decrease asset allocation

In reality, there is some mispricing, and this is where hedge funds come in

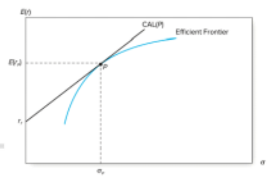
The top point is 100% allocation in risky assets?
Then the bottom point rf is 0% allocation in risky assets.



Capital market line

- Capital market line (CML): The capital allocation line (CAL) based on the market portfolio M
- Price adjustment process: If investors do not include the stock of some company in the portfolio, the price of this stock drops and then becomes attractive to be included in the portfolio

An example of when the price deviates from equilibrium and the price will go back to equilibrium
P is identical to the market portfolio



Risk premium of the market portfolio

- Individual investor with risk aversion A chooses a proportion y, allocated to the optimal portfolio M, to maximize utility

$$max_y U = E(r_C) - \frac{1}{2}A\sigma_C^2$$
$$= r_f + y[E(r_M) - r_f] - \frac{1}{2}Ay^2\sigma_M^2$$

  - Optimal $y^* = \frac{E(r_M)-r_f}{A\sigma_M^2} = \frac{E[R_M]}{A\sigma_M^2}$, where $R_M = r_M - r_f$

Solve y* from first order derivative set to zero

- In the simplified CAPM economy, net borrowing and lending across all investors must be zero, and the average position in M is 100%, or $\bar{y} = 1$
- Let a market representative investor be the one with $\bar{y} = 1$ and risk aversion $\bar{A}$. Then the risk premium of the market portfolio is

$$\frac{E[R_M]}{A\sigma_M^2} = \bar{y} = 1 \Rightarrow E(R_M) = \bar{A}\sigma_M^2$$

Market portfolio is market share weighted average of individual assets
To estimate the risk premium of each stock, we first need to estimate the risk premium of the market portfolio

Some facts

- Data from the last nine decades for the broad U.S. equity market
  - Excess return: 8.3%
  - Standard deviation: 20.1%

- Using $E(R_M) = \bar{A}\sigma_M^2$, the average risk aversion $\bar{A}$ is
  - $\bar{A} = \frac{E(R_M)}{\sigma_M^2} = \frac{8.3\%}{20.1\%^2} = 2.0544$

Risk premium of the market is about 8.3%, and the std dev is about 20.1% (see the little squared sign for the variance)

- Sharpe ratio
  - $S = \frac{E(R_M)}{\sigma_M} = 0.413$

Market portfolio of 0.4 so if u can identify another portfolio with better sharpe ratio then that's goood?

Expected returns on individual securities

- The risk premium on an asset is determined by its contribution to the risk of investors' overall portfolios (i.e., market portfolio)
- Take GE's stock as an example
  - GE's contribution to the risk premium of the market portfolio is $w_{GE}E(R_{GE})$
  - GE's contribution to the variance of the market portfolio is $w_{GE}Cov(R_M, R_{GE}) = w_{GE}Cov(\sum_{i=1}^n w_i R_i, R_{GE}) = w_{GE}\sum_{i=1}^n w_i Cov(R_i, R_{GE})$

Portfolio Weights	w_1	w_2	...	w_{GE}	...	w_n
w_1	$Cov(R_1, R_1)$	$Cov(R_1, R_2)$	...	$Cov(R_1, R_{GE})$	...	$Cov(R_1, R_n)$
w_2	$Cov(R_2, R_1)$	$Cov(R_2, R_2)$	...	$Cov(R_2, R_{GE})$	...	$Cov(R_2, R_n)$
⋮	⋮	⋮		⋮		⋮
w_{GE}	$Cov(R_{GE}, R_1)$	$Cov(R_{GE}, R_2)$	...	$Cov(R_{GE}, R_{GE})$	...	$Cov(R_{GE}, R_n)$
⋮	⋮	⋮		⋮		⋮
w_n	$Cov(R_n, R_1)$	$Cov(R_n, R_2)$	...	$Cov(R_n, R_{GE})$	...	$Cov(R_n, R_n)$

GE's risk premium should be proportional to exposure to market portfolio, so u can use this to calculate what its risk premium should be

$$E[R_M] = E[\sum_{i=1}^n w_i R_i] = E[W_{GE} \cdot R_{GE}]$$

$$Var(R_M) = cov(\sum_{i=1}^n w_i R_i, \sum_{i=1}^n w_i R_i)$$

$W_{GE} R_{GE}$

$$\Rightarrow W_{GE}\, cov(R_{GE}, \sum_{i=1}^n w_i R_i)$$

Risk to reward ratio of w_GE

Risk-to-reward ratios of GE and market portfolio

- The reward-to-risk ratio for investments in GE is

$$\frac{\text{GE's contribution to risk premium}}{\text{GE's contribution to variance}} = \frac{w_{GE}E(R_{GE})}{w_{GE}Cov(R_M, R_{GE})} = \frac{E(R_{GE})}{Cov(R_M, R_{GE})}$$

Risk to reward ratio of investing wGE's fraction in GE?

- The reward-to-risk ratio for market portfolio is (market price of risk)

$$\frac{\text{Market risk premium}}{\text{Market variance}} = \frac{E(R_M)}{\sigma_M^2}$$

- A basic principle of equilibrium: all investments should offer the same reward-to-risk ratio

$$\frac{E(R_{GE})}{Cov(R_M, R_{GE})} = \frac{E(R_M)}{\sigma_M^2}$$

For any stock, if this does not incur, then there is trading

For any stock I, the risk premium E[Ri] divided by the market portfolio cov(R_M, Ri) is equal to E[R_m]/sigma_M_^2

Risk premium of a stock

- The risk premium of GE is

Multiplied each side by cov(RM, RGE)

$$E(R_{GE}) = \frac{Cov(R_M, R_{GE})}{\sigma_M^2} E(R_M)$$

Use Beta to denote this ratio

$$\Rightarrow E(r_{GE}) = r_f + \beta_{GE}[E(r_M) - r_f] \text{ (expected return–beta relationship)}$$

  - $\beta_{GE} = Cov(R_M, R_{GE})/\sigma_M^2$: contribution of GE to the variance of market portfolio as a fraction of the total variance of the market portfolio

If beta is larger, the risk premium of GE is larger

- Expected return–beta relationship for any asset i

$$E(r_i) = r_f + \beta_i[E(r_M) - r_f]$$

Expected return-beta relationship

- Expected return-beta relationship for any asset i

$$E(r_i) = r_f + \beta_i[E(r_M) - r_f]$$

  - Sum of risk-free rate r_f (compensation for "waiting") and a risk premium $\beta_i[E(r_M) - r_f]$ (compensation for "worrying")
  - Firm-specific risk is not priced by the market

$$\beta_P = \sum_{i=1}^{n} w_i \beta_i$$

- Expected return-beta relationship for portfolio P with weight w_i in asset i

All the expected return is related to covariance between asset I and market portfolio

$$E(r_P) = w_1E(r_1) + w_2E(r_2) + \cdots + w_nE(r_n)$$
$$= \sum_{i}^{n} w_i(r_f + \beta_i[E(r_M) - r_f]) = r_f + \beta_P[E(r_M) - r_f]$$

Market portfolio

- For the market portfolio,

$$E(r_M) = r_f + \beta_M[E(r_M) - r_f]$$

Beta_p = Cov(RM, Rp) / sigma squared M

But since Cov(RM, Rp) = sigma squared M then

Beta_p = 1

  - This implies that $\beta_M = 1$
  - The weighted average value of beta across all assets is 1

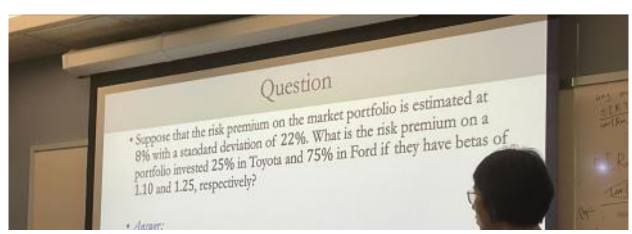
Question

- Suppose that the risk premium on the market portfolio is estimated at 8% with a standard deviation of 22%. What is the risk premium on a portfolio invested 25% in Toyota and 75% in Ford if they have betas of 1.10 and 1.25, respectively?

We didn't use SD because it is included in the beta

$$E(r_P) = w_1E(r_1) + w_2E(r_2) + \cdots + w_nE(r_n)$$
$$= \sum_{i}^{n} w_i(r_f + \beta_i[E(r_M) - r_f]) = r_f + \beta_P[E(r_M) - r_f]$$

9.7%

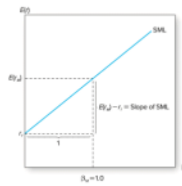


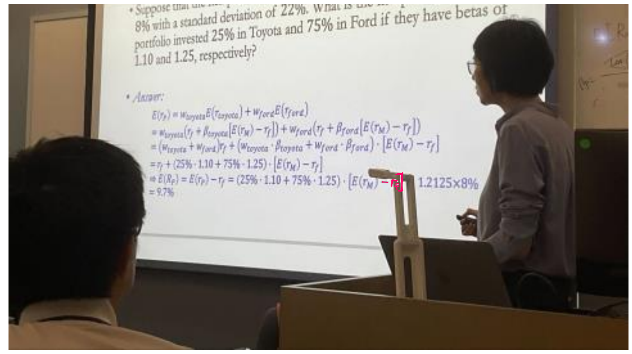
Security market line

- Security's risk premium is proportional to both beta and risk premium of the market portfolio, i.e.,

$$\beta[E(r_M) - r_f]$$

- Security market line (SML): expected return-beta relationship $E(r)$ vs β
  - Slope is $E(r_M) - r_f$
  - Graphs *individual asset* risk premiums as a function of asset risk (held as parts of well-diversified portfolios, i.e., contribution to portfolio variance)





Expected return of each individual asset is proportional to the beta
Per unit increase in beta results in an increase in market risk premium in stock's return
The SML graphs individual asset's risk premium as a function of asset risk

Beta only quantifies the risk as the part held in diversified portfolio?
Asset risk does not include firm specific risk?

Interpretation of β in security market line

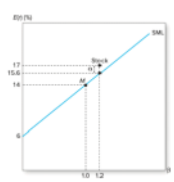
- In the security market line,

$$\beta[E(r_M) - r_f]$$

  - If $\beta = 0$, then $E(r_i) = r_f$ and the risk is diversifiable
  - If $\beta < 0$, then $E(r_i) < r_f$ and payoffs in bad times (insurance)
  - If $\beta > 0$, then $E(r_i) > r_f$ and the risk is compensated

Security market line

- If assets are "fairly priced", then they are exactly on the SML. All securities must lie on the SML in market equilibrium
- If a stock is underpriced, then it will plot above the SML
- The difference between the fair and actually expected rate of return on a stock is called the stock's **alpha,** denoted by α
- For example, $\alpha = 1.4\%$ for the stock in the figure



Question

- Stock XYZ has an expected return of 12% and risk of $\beta = 1$. Stock ABC has expected return of 13% and $\beta = 1.5$. The market's expected return is 11%, and $r_f = 5\%$

- According to the CAPM, which stock is a better buy?

The CAPM and single-index market model

- The single index model states

$$R_i = \alpha_i + \beta_i R_M + e_i$$

  - The realized excess return on any stock is the sum of three parts
    - Marketwide factors: $\beta_i R_M$
    - Nonmarket premium: α_i
    - Firm-specific outcomes: e_i
  - The risk premium has $E(R_i) = \alpha_i + \beta_i E(R_M) + e_i$

  - Based on CAPM, the equilibrium value of α_i is 0. Otherwise, if $\alpha_i > 0$, investors will buy the stocks, bid up the prices, and expected return will be lower