Lecture 11: Markowitz Model and CAPM

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# QTM 385 Quantitative Finance

## Lecture 11: Markowitz model and CAPM

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Suggested reading: Investments Ch 7 and 9

## Minimum-variance portfolio for general $\rho_{DE}$

- Based on the variance formula

$$\sigma_P^2 = w_D^2 \cdot \sigma_D^2 + w_E^2 \cdot \sigma_E^2 + 2w_D w_E Cov(r_D, r_E)$$

- Replace $w_E$ by $1 - w_D$ and take the derivative with respect to $w_D$. The optimal $w_D^*$ that minimizes $\sigma_P^2$ is

$$w_D^* = \frac{\sigma_E^2 - Cov(r_D, r_E)}{\sigma_D^2 + \sigma_E^2 - 2Cov(r_D, r_E)}$$

## Question

- The weight of in the minimum variance portfolio is

$$w_D^* = \frac{\sigma_E^2 - Cov(r_D, r_E)}{\sigma_D^2 + \sigma_E^2 - 2Cov(r_D, r_E)}$$

- Question: What is the weight of $D$ and $E$ when $\rho_{DE} = .3$? What if $\rho_{DE} = 0$?

| | Debt | Equity |
|---|---|---|
| Expected return, $E(r)$ | 8% | 13% |
| Standard deviation, $\sigma$ | 12% | 20% |

Recall

$$Cov(r_D, r_E) = \rho_{DE}\sigma_D\sigma_E$$

Plug in numbers into formula

(20^2 – 20*12*0.3) / (12^2 + 20^2 – 2*20*12*0.3 ) for wD and then you can get wE = 1 - wD

| D | E |
|---|---|
| 82% | 18% |
| 74% | 26% |

# Choosing a portfolio based on risk aversion

- For the utility $U = E(r) - \frac{1}{2}A\sigma^2$ with the risk aversion parameter $A$, the optimal investment proportions in the two funds are

$$w_D^* = \frac{E(r_D) - E(r_E) + A(\sigma_E^2 - Cov(r_D, r_E))}{A(\sigma_D^2 + \sigma_E^2 - 2Cov(r_D, r_E))}$$

$$w_E^* = 1 - w_D^*$$

E(rD) - E(rE) is the return difference
Investor should allocate more in the equity portfolio if E(rE) increases

- $w_D^*$ increases with $E(r_D)$
- $w_D^*$ decreases with $E(r_E)$
- $w_D^*$ increases with $A$
  - $w_D^* = \frac{(E(r_D)-E(r_E))/A+(\sigma_E^2-Cov(r_D,r_E))}{(\sigma_D^2+\sigma_E^2-2Cov(r_D,r_E))} = \frac{\frac{-a}{A}+b}{c}$, where $a = E(r_E) - E(r_E) > 0$
  - $\frac{dw_D^*}{dA} = \frac{a}{cA^2} > 0$

Replace E(rD) - E(rE) to be –a etc.

If the derivative is positive with respect to A, this means that the weight of D (bonds I think) increases with A. If negative, weight of D decreases with A

# Homework: Choosing a portfolio based on Sharpe ratio

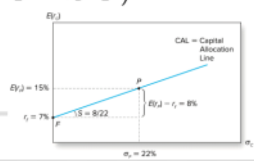
- We can find the weights $w_D$ and $w_E$ to maximize the Sharpe ratio. Then use this portfolio $P$ and a risk-free asset to construct the CAL

$$\max_{w_D, w_E} S_P = \frac{E(r_P) - r_f}{\sigma_P}$$

$$\text{s.t.}\ r_P = w_D r_D + w_E r_E$$

$$\sigma_P = \left(w_D^2 \cdot \sigma_D^2 + w_E^2 \cdot \sigma_E^2 + 2w_D w_E Cov(r_D, r_E)\right)^{0.5}$$

$$w_D + w_E = 1$$




# Optimal portfolio weights to maximize Sharpe ratio

- The optimal portfolio weights that maximize Sharpe ratio is

$$w_D^* = \frac{E(R_D)\sigma_E^2 - E(R_E)Cov(R_D,R_E)}{E(R_D)\sigma_E^2 + E(R_E)\sigma_D^2 - [E(R_D)+E(R_E)]Cov(R_D,R_E)}$$

$$w_E^* = 1 - w_D^*$$

- $R_D = r_D - r_f$ and $R_E = r_E - r_f$ are excess returns

MAJOR DIFFERENCE FROM THE PRIOR UTILITY MAXIMIZED WEIGHT IS THIS NEW NOTATION. CAPITAL RD AND RE ARE DIFFERENT FROM LOWERCASE

Let's say $r_f = 5\%$ → is a constant

$E(R_D]$ risk premium of D

$cov(R_D, R_E) = Cov(r_D, r_E) = Cov(r_D - r_f, r_E - r_f)$

since $r_f$ is just a constant

# Example

- The optimal portfolio weights that maximize Sharpe ratio is

$$w_D^* = \frac{E(R_D)\sigma_E^2 - E(R_E)Cov(R_D,R_E)}{E(R_D)\sigma_E^2 + E(R_E)\sigma_D^2 - [E(R_D)+E(R_E)]Cov(R_D,R_E)}$$

$$= \frac{(8-5)\times 400 - (13-5)\times 72}{(8-5)\times 400 + (13-5)\times 144 - (8-5+13-5)\times 72} = .40$$

$$w_D^* = 1 - .40 = .60$$

$$E(r_P) = (.4\times 8) + (.6\times 13) = 11\%$$

$$\sigma_P = [(.4^2\times 144) + (.6^2\times 400) + (2\times .4\times .6\times 72)]^{0.5} = 14.2\%$$

$$S_P = \frac{11-5}{14.2} = .42$$

| | Debt | | Equity |
|---|---|---|---|
| Expected return, $E(r)$ | 8% | | 13% |
| Standard deviation, $\sigma$ | 12% | | 20% |
| Covariance, $Cov(r_D, r_E)$ | | 72 | |
| Correlation coefficient, $\rho_{DE}$ | | 0.30 | |

If everyone jumps on a stock, it's likely that the price increases and the return decreases (but the first person to buy it will have much bigger return but last buyers will not. First movers have the most profit)

# Hint for homework

- Write $S_P^2$ in terms of $w_D$ only (using $w_D = 1 - w_E$)
- Take the derivative of $S_P^2$ with respect to $w_D$ and set it to zero

Hints: First take the square of the sharpe ratio

$R_P = r_P - r_f = w_D R_D + w_E R_E$

# Hint for homework

- Write $S_P^2$ in terms of $w_D$ only (using $w_D = 1 - w_E$)
- Take the derivative of $S_P^2$ with respect to $w_D$ and set it to zero

$$\frac{dS_P^2}{dw_D} = 0$$

- Solve $w_D$ from the first order condition and then we can show it takes the form in Problem 5

Hints: First take the square of the sharpe ratio

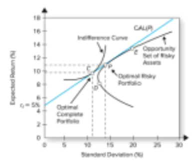
$R_P = r_P - r_f = w_D R_D + w_E R_E$

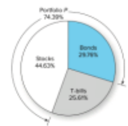
$S_P^2 = \frac{E[R_P]^2}{\sigma_P^2}$ quadratic in $w_D$ and $w_E$ / quadratic in $w_D$ and $w_E$ $\Rightarrow$ $S_P^2$ is a ratio of two quadratic functions of $w_D$ (sub in $w_E = 1 - w_D$) $\rightarrow$ take the derivative w/ respect to $w_D$ and set equal to 0, solve for $w_D$

$S_P < 1 \rightarrow E[R_P] < \sigma_P \Rightarrow E[R_P] > 0$
then $r_P > r_f$

# Optimal complete portfolio

1. Specify the return characteristics of all securities: expected returns, variances, covariances
2. Asset allocation decision to construct optimal risky portfolio $P$ (same for all investors)
   - Solve the weight of each asset to maximize the Sharpe ratio of $P$
3. Capital allocation between risky portfolio $P$ and risk-free assets (e.g., T-bills) to maximize an investor's utility (vary with investors)
   - Solve the weight $y$ in $P$ and weight $1 - y$ in risk-free assets to maximize utility score ($U = E(r) - \frac{1}{2}A\sigma^2$)




# Markowitz model

- Markowitz model: portfolio optimization model, also called mean-variance model
  - A mathematical framework for assembling a portfolio of assets such that the expected return is maximized for a given level of risk
- Asset allocation among $n$ assets: weight of asset $i$ is $w_i$

$$E(r_P) = \sum_{i=1}^{n} w_i E(r_i)$$

$$\sigma_P^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j Cov(r_i, r_j)$$

  - Portfolio manager has an estimate of $E(r_i)$ for all $i$ and an estimate of $Cov(r_i, r_j)$ for all $i$ and $j$

# Minimum variance frontier

- Minimum variance frontier: A graph of the lowest possible variance that can be attained for a given portfolio expected return

Basically a careful choice of wi

- All the individual assets lie to the right inside the frontier (i.e., return is lower or risk is higher)
  - Short sales are allowed in the construction of risky portfolios
  - Portfolio with a single asset is inefficient
- Diversification can increase expected returns and lower standard deviations


THE BLUE CURVE is the minimum variance frontier
Y axis is expected return
X axis is std error

Given target return, what is the lowest possible variance?
The point on the blue curve

A vertical line through the frontier intersects with the frontier twice
You might notice that for these two intersections, the top is preferable over the lower because the top has higher return but the same variance. So ignore everything on the bottom half of the curve.

# Efficient frontier

- Efficient frontier is the portion of the minimum-variance frontier that lies above the global minimum-variance portfolio
- The bottom part of the minimum-variance frontier is inefficient
  - There is always a portfolio with the same standard deviation and a greater expected return positioned directly above it


So we ignore everything on the bottom. And everything on top is the efficient frontier.

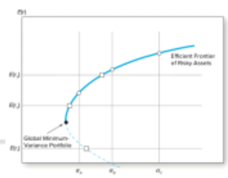
# Two equivalent approaches to construct efficient frontier

- Approach 1: Minimize variance for any target expected return $\mu$ (e.g., points marked by squares)

$$\min_{w} \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j Cov(r_i, r_j)$$
$$s.t.\ \sum_{i=1}^{n} w_i E(r_i) = \mu$$
$$\sum_{i=1}^{n} w_i = 1$$

- Approach 2: Maximize expected return for any target risk level $\sigma^2$ (e.g., points marked by circles)

$$\max_{w} \sum_{i=1}^{n} w_i E(r_i)$$
$$s.t. \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j Cov(r_i, r_j) = \sigma^2$$
$$\sum_{i=1}^{n} w_i = 1$$




This is called convex optimization

Convex means you take second order derivative
Constraints are linear

So this is a quadratic objective function with linear constraints

Cvxpy in python and cvx in matlab – package for optimization

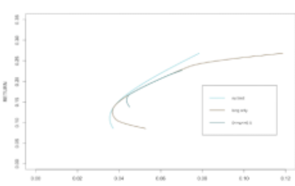
Approach 1: Drawing a horizontal line through any return
Approach 2: Drawing a vertical line through any sigma squared

# Efficient frontier with constraints

- In the optimization problem

$$\min_{w} \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j Cov(r_i, r_j)$$
$$s.t.\ \sum_{i=1}^{n} w_i E(r_i) = \mu$$
$$\sum_{i=1}^{n} w_i = 1$$

add additional constraints, e.g.,
  - Short-sell restrictions (non-negative weights)
    $w_i \geq 0$
  - Weight restrictions
    - $w_i \leq \bar{w}$
- As we add more constraints, the frontier naturally moves more "to the right", i.e., towards higher risk and lower return

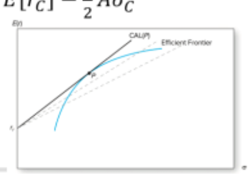


Long only – cannot short sell

Adding more constraints pushes the frontier to the right
Having less choices makes variance higher/return lower

# Markowitz model with risk-free assets

- Identify the point with the highest Sharpe ratio on the efficient frontier. This portfolio is denoted as $P$ (shared across all investors)
- Construct the CAL that connects the risk-free asset with $P$
- The CAL is tangent to the efficient frontier
- For individual investors, choose the optimal complete portfolio with $y$ in $P$ and $1 - y$ in the risk-free asset to maximize $U = E[r_C] - \frac{1}{2} A\sigma_C^2$



# Capital asset pricing model

- The capital asset pricing model (CAPM): a prediction of the relationship between the risk of an asset and its expected return
- Two set of assumptions
  1. Individual behavior
  2. Market structure

# Assumptions on individual investors

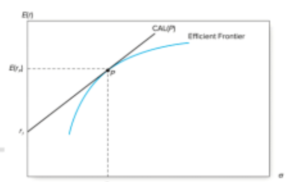
- Investors are rational, mean-variance optimizers
- Their common planning horizon is a single period
- Investors all use identical input lists (e.g., mean and covariance), an assumption often termed homogeneous expectations
  - homogeneous expectations are consistent with the assumption that all relevant information is publicly available

# Assumptions on market structure

- All assets are publicly held and trade on public exchanges
- Investors can borrow or lend at a common risk-free rate, and they can take short positions on traded securities
- No taxes
- No transaction costs

# The market portfolio

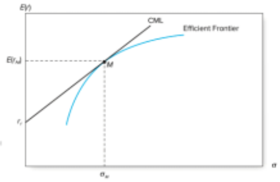
- Under these assumptions, same efficient frontier and same tangent CAL for all investors
- All investors would choose the same *P* with the same set of weights for each risky asset
- This portfolio is the market portfolio, denoted as *M*, i.e., the value-weighted portfolio of all assets in the investable universe
- **Mutual fund theorem**: investing in a market-index portfolio is efficient (passive strategy)



# Capital market line

- Capital market line (CML): The capital allocation line (CAL) based on the market portfolio $M$

- Price adjustment process: If investors do not include the stock of some company in the portfolio, the price of this stock drops and then becomes attractive to be included in the portfolio




# Risk premium of the market portfolio

- Individual investor with risk aversion $A$ chooses a proportion $y$, allocated to the optimal portfolio $M$, to maximize utility

$$max_y U = E(r_C) - \frac{1}{2}A\sigma_C^2$$
$$= r_f + y[E(r_M) - r_f] - \frac{1}{2}Ay^2\sigma_M^2$$

  - Optimal $y^* = \frac{E(r_M)-r_f}{A\sigma_M^2} = \frac{E[R_M]}{A\sigma_M^2}$, where $R_M = r_M - r_f$
- In the simplified CAPM economy, net borrowing and lending across all investors must be zero, and the average position in $M$ is 100%, or $\bar{y} = 1$
- Let a market representative investor be the one with $\bar{y} = 1$ and risk aversion $\bar{A}$. Then the risk premium of the market portfolio is

$$\frac{E[R_M]}{A\sigma_M^2} = \bar{y} = 1 \Rightarrow E(R_M) = \bar{A}\sigma_M^2$$