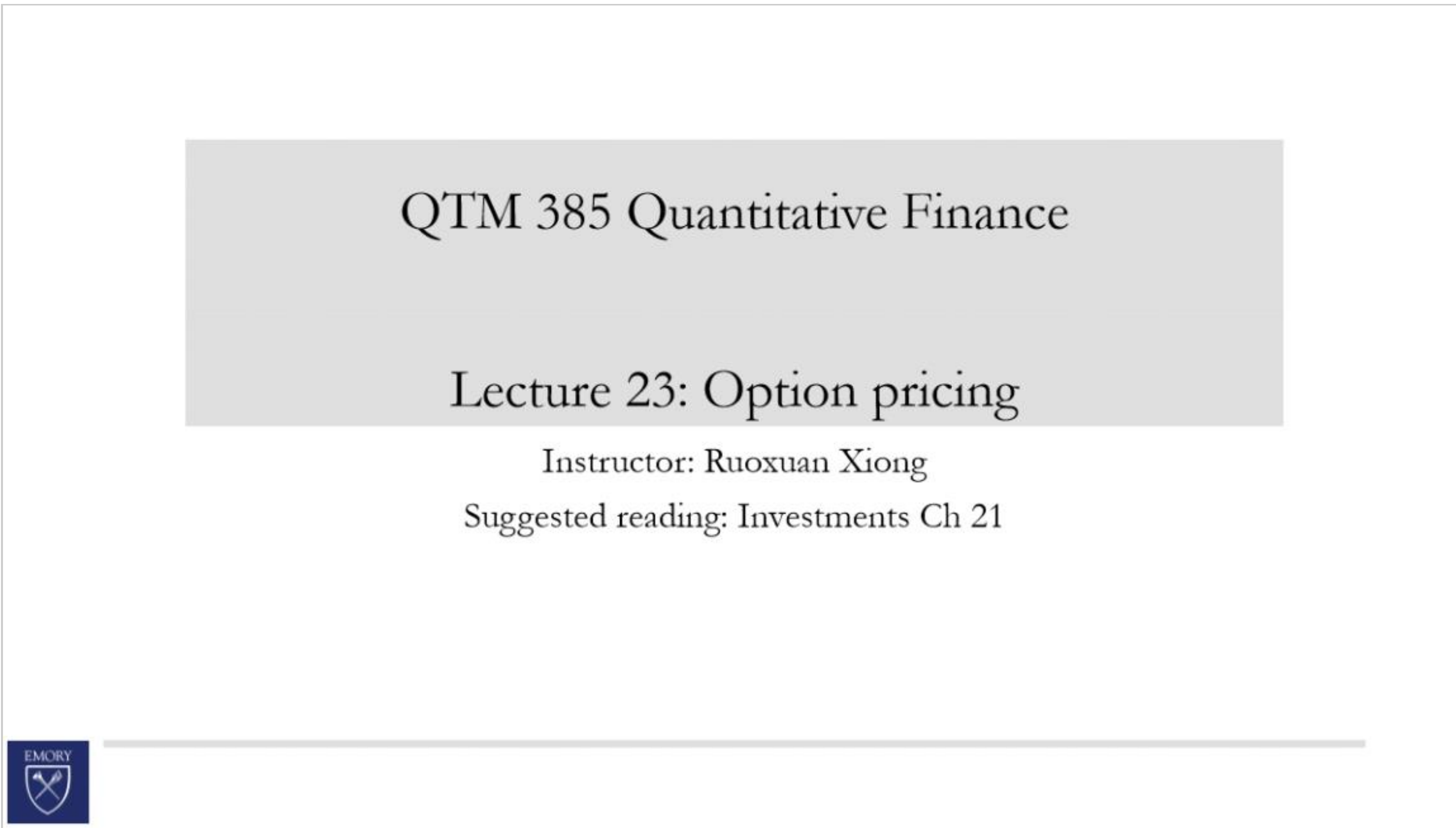
Lecture 23: Options Pricing

Wednesday, April 19, 2023 13:02

23-option-pricing



Feedback from Google form

- I thought the R coding are super efficient in calculations, but I was wondering if we get a chance to learn the same thing in Python, as it is a widely used language nowadays.

- *Some Python sample code (calculation of duration and convexity (in Duration_and_Convexity.ipynb), calculation of summary statistics (in Summary_Statistics.ipynb) has been provided (credited to our TA, Hao).*

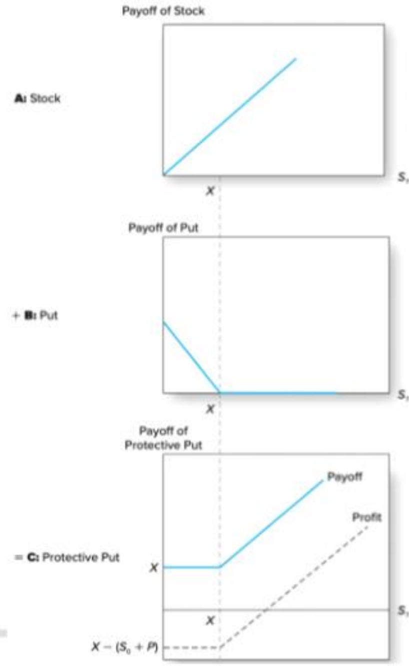
Options

- **American option:** gives its holder the right to **purchase/sell** an asset for a specified price, called the exercise or strike price, *on or before* some specified expiration date
  - Most traded options in the United States
- **European option:** allows for exercise *only on* the expiration date
- **Call option:** gives its holder the right to **purchase** an asset for a specified price, called the exercise or strike price, *on or before (or only on)* some specified expiration date
- **Put option:** gives its holder the right to **sell** an asset for a specified price, called the exercise or strike price, *on or before (or only on)* some specified expiration date

Protective put

- **Protective put**: *Invest* in stock (stock price at time 0: S_0) and *purchase a put* option on the stock (cost P)
- Suppose the strike of the put is $X = \$100$
- At expiration, if $S_T = \$97$, then the put is worth $X - S_T = \$3$ and portfolio value is $S_T + (X - S_T) = \$97 + \$3 = \$100$
- If stock is worth $S_T = \$104$, the put is worth $\$0$. the portfolio value is $S_T + 0 = \$104$

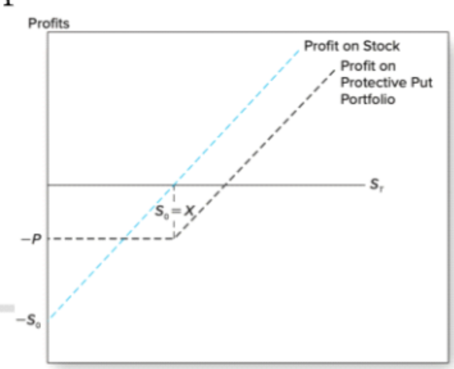
	$S_T \le X$	$S_T > X$
Stock	S_T	S_T
+ Put	$X - S_T$	0
Total	X	S_T



Call-plus-bills portfolio

- **Call-plus-bills portfolio**: purchase a risk-free zero-coupon bond with face value X and hold a call with exercise price X
- Suppose the strike of the call is $X = \$100$
- At expiration, if $S_T = \$97$, then the call is worth $\$0$ and portfolio value is $0 + X = \$100$
- If $S_T = \$104$, the call is worth $S_T - X = \$4$. the portfolio value is $X + (S_T - X) = \$100 + \$4 = \$104$

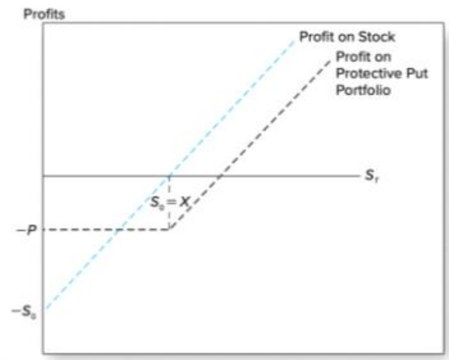
	$S_T \le X$	$S_T > X$
Value of call option	0	$S_T - X$
Value of zero-coupon bond	X	X
Total	X	S_T



The put-call parity relationship

- **Protective put**: *invest* in stock and *purchase a put* with exercise price X
- **Call-plus-bills portfolio**: purchase a risk-free zero-coupon bond with face value X and hold a call with exercise price X
- Two portfolios have the same payoff: provide a guaranteed minimum payoff and unlimited upside potential
- Their cost should be the same

	$S_T \le X$	$S_T > X$
Value of call option	0	$S_T - X$
Value of zero-coupon bond	X	X
Total	X	S_T



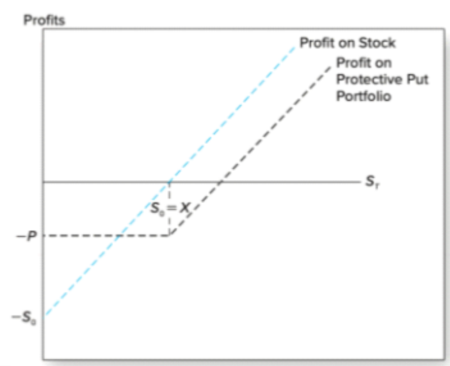
EMORY

The put-call parity relationship

- The cost of the protective put is $S_0 + P$
- The cost of the call-plus-bills is $C + X/(1+r_f)^T$
- Their cost should be the same, so the **put-call parity theorem** states

$$C + \frac{X}{(1+r_f)^T} = S_0 + P$$

	$S_T \le X$	$S_T > X$
Value of call option	0	$S_T - X$
Value of zero-coupon bond	X	X
Total	X	S_T



EMORY

Question

- Question: Suppose the current stock price is $S_0 = \$110$, 1-year expiration call with exercise price $X = \$105$ sells at $C = \$17$ and 1-year expiration put with exercise price $X = \$105$ sells at $P = \$5$. Suppose the risk-free interest rate is 5%. Is there an arbitrage opportunity?

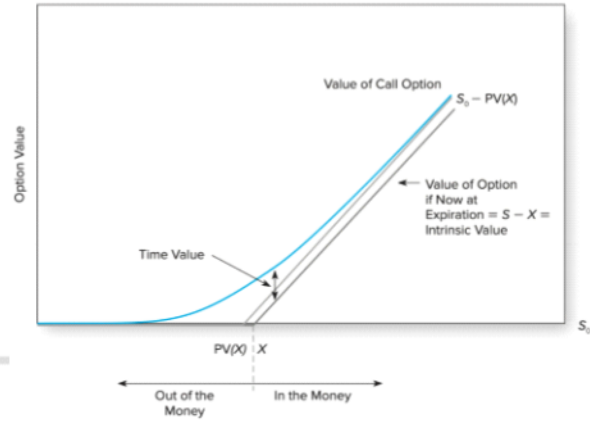
Question

- Question: Suppose the current stock price is $S_0 = \$110$, 1-year expiration call with exercise price $X = \$105$ sells at $C = \$17$ and 1-year expiration put with exercise price $X = \$105$ sells at $P = \$5$. Suppose the risk-free interest rate is 5%. Is there an arbitrage opportunity?

- *Answer: Based on put-call parity theorem,*
  - $C + \frac{X}{(1+r_f)^T} = 17 + \frac{105}{1.05} = 117$
  - $S_0 + P = 110 + 5 = 115$
  - *We can buy a share of stock and put, and sell the call and risk-free one-year zero*

Option valuation

- **Intrinsic value**: $S_0 - X$ for the in-the-money call options
  - Intrinsic value is zero for out-of-the-money or at-the-money options
- Option's **time value**: difference between option price and intrinsic value
  - A type of volatility value
  - The stock price may rise above the exercise price of the call by expiration date
- **Adjusted intrinsic value**: $S_0 - PV(X)$
  - If you are virtually certain to exercise the option (at the expiration date), then present value of the option is $PV(X)$



Determinants of option values

- **Determinant 1**: stock price S_0
  - The value of the call increases with the stock price
  - Expected payoff increases with $S_0 - X$
- **Determinant 2**: exercise price X
  - The value of the call decreases with the exercise price X
- **Determinant 3**: volatility of the stock price
  - The value increases with the volatility

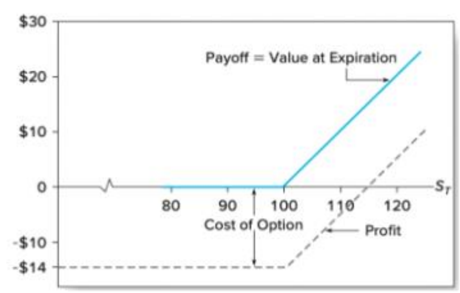
High-Volatility Scenario					
Stock price	\$10	\$20	\$30	\$40	\$50
Option payoff	0	0	0	10	20
Low-Volatility Scenario					
Stock price	\$20	\$25	\$30	\$35	\$40
Option payoff	0	0	0	5	10

Determinants of option values

- **Determinant 4**: expiration date T
  - Option value increases with time to expiration
  - More time for unexpected events, and present value of exercise price X is lower
- **Determinant 5**: interest rate r_f
  - Option value increases with interest rate
  - Present value of X is lower
- **Determinant 6**: dividend payouts
  - Option value decreases with the dividend payouts
  - The stock price S_T decreases with dividend payouts

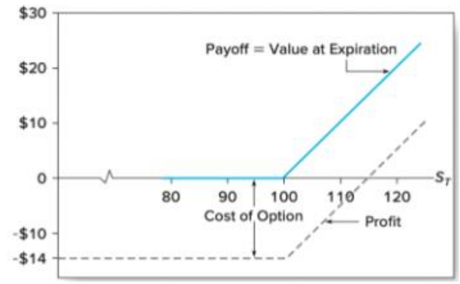
Question

If this variable increases	Value of American call
Stock price S	Increases
Exercise price X	Decreases
Volatility σ	
Time to expiration T	
Interest rate r_f	
Dividend payouts	



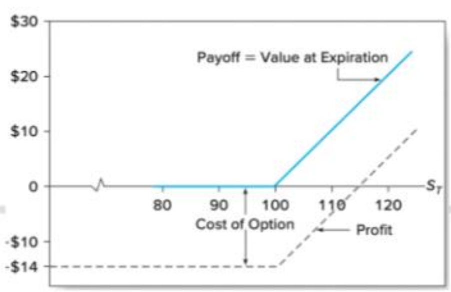
Question

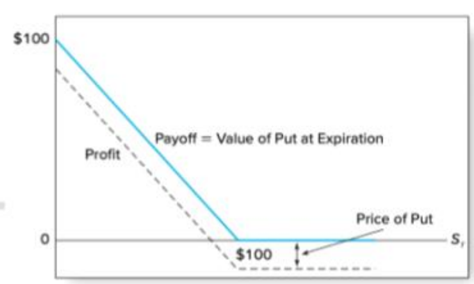
If this variable increases	Value of American call
Stock price S	Increases
Exercise price X	Decreases
Volatility σ	Increases
Time to expiration T	Increases
Interest rate r_f	Increases
Dividend payouts	Decreases



Question

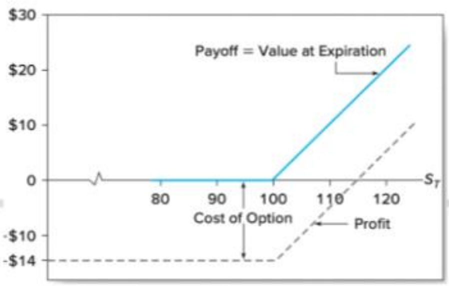
If this variable increases	Value of American call	Value of American put
Stock price S	Increases	
Exercise price X	Decreases	
Volatility σ	Increases	
Time to expiration T	Increases	
Interest rate r_f	Increases	
Dividend payouts	Decreases	

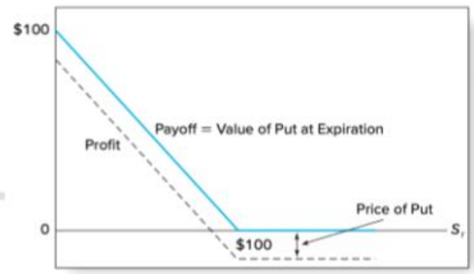




Question

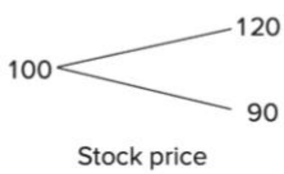
If this variable increases	Value of American call	Value of American put
Stock price S	Increases	Decreases
Exercise price X	Decreases	Increases
Volatility σ	Increases	Increases
Time to expiration T	Increases	Increases
Interest rate r_f	Increases	Decreases
Dividend payouts	Decreases	Increases

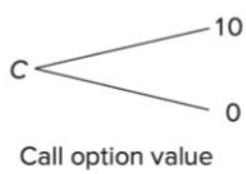




Binomial option pricing

- We start with two state option pricing
- The stock sells at $S_0 = 100$. The price will either increase to 120 or decrease to 90
- Consider two portfolios
  - Portfolio A: buy three calls
  - Portfolio B: buy a share of stock and borrow $\$81.82$ at interest rate 10%
  - Two portfolios have the same payoff



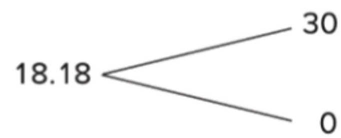


Value of stock at year-end	$90	$120
– Repayment of loan with interest	−90	−90
Total	$ 0	$ 30

Price of the call

- Consider two portfolios
  - Portfolio A: buy three calls
  - Portfolio B: buy a share of stock and borrow $81.82 at interest rate 10%
  - Two portfolios have the same payoff

- The current cost is $\$18.18 = \$100 - \$81.82$ of portfolio B
- We have
  - $3C = \$18.18$
  - $C = \$6.06$



Hedge ratio

- A portfolio with one share of stock and three call options written is perfectly hedged. Its year-end value is independent of ultimate stock price

Stock price	$90	$120
– Obligations from 3 calls written	–0	–30
Portfolio value	$90	$ 90

- The hedge ratio is one share of stock to three calls, or one-third
  - Interpretation: range of call value is 10, range of stock price is 30, ratio of range is $10/30$

Hedge ratio

- More generally, the hedge ratio H for other two-state option problem is

$$H = \frac{C_u - C_d}{uS_0 - dS_0}$$

  - C_u or C_d refers to the call option's value when the stock goes up or down
  - uS_0 and dS_0 are the stock prices in two states

Option-pricing technique

- Step 1: Given possible end-of-year stock prices $uS_0 = \$120$ and $dS_0 = \$90$ and value of call option with exercise price $\$110$, $C_u = \$10$ or $C_d = 0$
- Step 2: Find the hedge ratio $H = \frac{C_u - C_d}{uS_0 - dS_0} = \frac{1}{3}$
- Step 3: find a portfolio made up of $\frac{1}{3}$ share of stock with one written call
  - This portfolio is a hedged portfolio with certainty payoff of $\$30$

Option-pricing technique

- Step 4: The present value of $30 is $\frac{\$30}{1.1} = \$27.27$

- Step 5: Set the present value of the hedge position to the present value of the certainty payoff

$$\frac{1}{3}S_0 - C_0 = \$27.27$$

- Step 6: Solve the call's value

$$C_0 = \frac{1}{3}S_0 - \$27.27 = \$33.33 - \$27.27 = \$6.06$$

EMORY