Combining OR and Data Science

Summer Term 2022

6. Optimization under Uncertainty II: Risk Aversion and Chance-Constrained Programming

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Overview

Part I: Risk Orientation / Decision-Making as Shaping Distributions

- Decision Making under Uncertainty as Shaping Distributions
- Comparing Distributions: Stochastic Dominance
- Shaping Distributions by Incorporating Risk Attitude into Decision / Optimization Models

Part II: Feasibility Guarantees by Chance-Constrained Programming

- Feasibility Issues in Single-Stage Problems under Uncertainty
- Single Chance Constraints
- Joint Chance Constraints

Part I: Risk Orientation and Decision Making under Uncertainty as Shaping Distributions

Part I: Risk Orientation and Decision Making under Uncertainty as Shaping Distributions

In the first part, we will introduce the idea that:
Decision Making under Uncertainty means Shaping Distributions
In particular, we will answer the following questions:

- when it is appropriate to optimize expected values?
- what is an outcome distribution?
- what kind of useful information does the outcome distribution give us?
- how can we compare outcome distributions?
- how can we modify models to account for risk attitude?

Literature

Much of the literature dealing with Stochastic Programming is not very accessible for Business students. In my opinion, the following book is an exception:

- **Modeling with Stochastic Programming**, Authors: Alan King and Stein Wallace, Springer, 2012

Other Textbooks include:

- **Introduction to Stochastic Programming** (2nd edition), Authors: John Birge and Francois Louveaux, Springer, 2011
- **Stochastic Programming: Modeling Decision Problems Under Uncertainty**, Authors: Klein Haneveld et al. , Springer, 2020

Note that these Springer textbooks are available as full text pdfs from the university library!

This Meeting: The Outcome Distribution and Risk Orientation in Stochastic Programming

In this meeting, we will

- discuss how to obtain the outcome distribution
- see how manipulate the outcome distribution to account for risk aversion by:
    - penalizing shortfalls
    - mean-risk optimization with Conditional Value at Risk
- we will use the agricultural case study from video 2 for illustration and exercises

Review: A Stochastic Programming Formulation for our Case Study

Sets:

- S scenario set

Parameters:

- d_s: demand in units in scenario s
- p_s: probability of scenario s

Decision Variables:

- x: capacity installation decision (*first stage* / *here and now* decision)
- z_s: production decision for scenario s (*second stage* / *recourse* decision)

$$
\begin{array}{lll}
\max & -30x + \sum_{s \in S} p_s 40 z_s & \\
\text{s.t.} & z_s \leq x & \forall s \in S \\
& z_s \leq d_s & \forall s \in S \\
& x \geq 0 & \\
& z_s \geq 0 & \forall s \in S
\end{array}
$$

Review: Implementation in Python

In [8]:

```
# Create a new model
m = mip.Model("Capacity_Planning_Stochastic")

#sets
n_scenarios = n_samples
scenarios = np.arange(n_scenarios) #in stochastic programming, we call the samples scenarios

#probability of each scenario - in our case, each scenario has prob. 1/|S|
prob = np.full((n_scenarios), 1/n_scenarios)

#parameters
installation_cost = 30
contribution_margin = 40

#decision variables
capacity =  m.add_var(name="capacity", lb=0)

production = [m.add_var(name=f"production{s}", lb= 0) for s in scenarios]

m.objective =  maximize( -installation_cost*capacity + sum(prob[s] * contribution_margin * production[s] for s in scenarios))

for s in scenarios:
    m += production[s] <= demand_sample[s]
    m += production[s] <= capacity

m.optimize()

print(f'Capacity decision: {capacity.x:.02f}')

print(f'Expected Total Profit: {m.objective_value:.02f}' )
```

```
Capacity decision: 82.91
Expected Total Profit: 675.26
```

The Outcome Distribution in Stochastic Programming

Model Reformulation for Obtaining the Outcome Distribution

Remember: Decision making under uncertainty means shaping distributions

- Where is the outcome distribution in a stochastic program?

Idea: Introduce an outcome variable g_s per scenario (a second-stage variable) and reformulate the model as follows:

$$
\begin{aligned}
\max \quad & \sum_{s \in S} p_s g_s \\
\text{s.t.} \quad & g_s = -30x + 40z_s && \forall s \in S \\
& z_s \leq x && \forall s \in S \\
& z_s \leq d_s && \forall s \in S \\
& x \geq 0 \\
& z_s \geq 0 && \forall s \in S
\end{aligned}
$$

In a given solution:

- the vector $\mathbf{g} = [g_s]_{s \in S}$ is a sample approximation of the outcome (total profit) distribution given the optimal capacity decision x

Model Reformulation: Implementation in Python

In [9]:

```python
            # Create a new model
m = mip.Model("Capacity_Planning_Stochastic")

#decision variables
capacity =  m.add_var(name="capacity", lb=0)

production = [m.add_var(name=f"production{s}",lb=0) for s in scenarios]

## this is the new variable
profit = [m.add_var(name=f"profit{s}",  lb=-np.inf) for s in scenarios]

m.objective =  maximize( sum(prob[s] * profit[s] for s in scenarios))

for s in scenarios:
    m += profit[s] == -installation_cost*capacity + contribution_margin * production[s] # computes the profit for scenario s
    m += production[s] <= demand_sample[s]
    m += production[s] <= capacity

m.optimize()

print(f'Capacity decision: {capacity.x:.02f}')
print(f'Expected Total Profit: {m.objective_value:.02f}' )
```

```
Capacity decision: 82.91
Expected Total Profit: 675.26
```

Plotting the Outcome Distribution

As ab example, we can now use the vector $\mathbf{g} = [g_s]_{s \in S}$ from the optimal solution plot a histogram of the outcompe distribution:

In [10]:

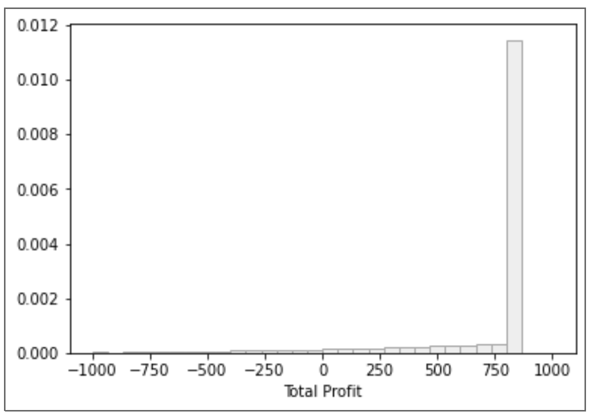
```
## collect the profit samples into an array
profit_samples = np.array( [ profit[s].x for s in scenarios ])

count, bins, ignored = plt.hist(profit_samples, bins=30, density=True, color='#EEEEEE', edgecolor="#AAAAAA", range =[-1000,1000])
plt.xlabel('Total Profit')

#plt.savefig('profit_distribution_stochastic_lp.pdf')
```

Out[10]:

```
Text(0.5, 0, 'Total Profit')
```



Risk Attitude and Shaping Distributions

Recall that **decision making under uncertainty means shaping outcome distributions**

- if the decision maker is risk averse, optimizing the expected value is not the best idea
- we may thus take the outcome distribution into account by introducing scenario-dependent outcome variables
- and we can change the objective function and add constraints to reflect the risk attitude of decision makers

We will consider the following approaches for **incorporating risk aversion into stochastic programs**:

- target values and shortfall penalties
- mean-risk objectives using CVaR as risk measure

Targets and Shortfall Penalties

Key Idea:

- set a target, e.g. g^t
- introduce a shortfall variable v_s for each scenario s
- in the objective, penalize expected shortfall with factor λ

Model:

$$
\begin{aligned}
\max \quad & \sum_{s \in S} p_s g_s - \lambda \sum_{s \in S} p_s v_s & \\
\text{s.t.} \quad & g_s = -30x + 40 z_s & \forall s \in S \\
& v_s \geq g^t - g_s & \forall s \in S \\
& z_s \leq d_s & \forall s \in S \\
& z_s \leq x & \forall s \in S \\
& x \geq 0 & \\
& v_s \geq 0, z_s \geq 0 & \forall s \in S
\end{aligned}
$$

Targets and Shortfall Penalties: Implementation in Python

In [ ]:

```
target_value = 500
lambd = 0.2

# Create a new model
m = mip.Model("Capacity_Planning_Stochastic_Shortfall")

#decision variables
capacity =  m.add_var(name="capacity", lb=0)
production = [m.add_var(name=f"production{s}",lb=0) for s in scenarios]

## this is the profit variable (scenario-dependent)
profit = [m.add_var(name=f"profit{s}",  lb=-np.inf) for s in scenarios]

## this is the shortfall variable (scenario-dependent)
shortfall = [m.add_var(name=f"shortfall{s}",  lb=0) for s in scenarios] # scenario-dependent!

# in the objective, we now add a variable for the shortfall
m.objective =  maximize( sum(prob[s] * profit[s] for s in scenarios)  - lambd*sum( prob[s]*shortfall[s] for s in scenarios ) )

for s in scenarios:
    m += profit[s]  == -installation_cost*capacity + contribution_margin * production[s] # computes the profit for scenario s
    m += production[s] <= demand_sample[s]
    m += production[s] <= capacity
    # computing shortfall
    m += shortfall[s] >= target_value - profit[s]

m.optimize()

print(f'Results for a target {target_value} and lambda = {lambd}')
print(f'Capacity decision: {capacity.x:.02f}')
print(f'Objective function value: {m.objective_value:.02f}' )

profit_samples = np.array( [ profit[s].x for s in scenarios ])
print(f' Expected Profit: {np.mean(profit_samples):.02f}')
```

Value at Risk and Conditional Value at Risk

Measuring Extremly Bad Outcomes

- Value at Risk (VaR_α): α -quantile of the outcome distribution, separating the α worst from the $1-\alpha$ best outcomes
- Conditional Value at Risk (CVaR_α) (Expected Shortfall): Average of the outcomes below VaR_α

A note on notation

In the literature, there are two ways of writing VaR and CVaR:

- using $1-\alpha$ as subscript: $\text{VaR}_{1-\alpha}$, e.g. $\text{VaR}_{0.95}$
- using only α as subscript: VaR_α, e.g. $\text{VaR}_{0.05}$
- we will use the latter version

VaR and CVaR in Optimization under Uncertainty

Ex-post-computation of VaR and CVaR

- we can use the distribution of outcomes, in case of sampling simply $\mathbf{g} = [g_s]_{s \in S}$
- and compute VaR and CVaR:

In [ ]:
```
VaR_sample = np.percentile(profit_samples, alpha*100,interpolation='lower')
profits_below_var = profit_samples[profit_samples <= VaR_sample]
CVaR_sample = profits_below_var.mean()

print(f'VaR from profit samples: {VaR_sample:.02f}')
print(f'CVaR from profit samples: {CVaR_sample:.02f}')
```

Mean-Risk-Optimization with VaR and CVaR

- surprisingly, optimizing VaR is (much) harder than optimizing CVaR
- in this part, we will see a model for optimizing CVaR
- we may discuss the techniques for optimizing VaR next week

Mean-Risk Optimization with CVaR: Optimization Model

New Parameters:

- α: probability parameter for VaR and CVaR
- λ: weight for CVaR

New Variables:

- v_s: shortfall below VaR
- VaR: Value at Risk
- $CVaR$: Conditional Value at Risk

$$
\begin{aligned}
& \max \sum_{s \in S} p_s g_s + \lambda CVaR \\
\text{s.t.} \quad & g_s = -30x + 40z_s && \forall s \in S \\
& z_s \leq d_s && \forall s \in S \\
& z_s \leq x && \forall s \in S \\
& v_s \geq VaR - g_s && \forall s \in S \\
& CVaR = VaR - \frac{1}{\alpha} \sum_{s \in S} p_s v_s \\
& x \geq 0 \\
& v_s \geq 0, z_s \geq 0 && \forall s \in S
\end{aligned}
$$

Mean-Risk Optimization with CVaR: Python Implemementation

In [ ]:

```
            # parameters for the mean-risk model
alpha = 0.05
lambd = 0.2

# Create a new model
m = mip.Model("Capacity_Planning_Stochastic_Mean_Risk")

#decision variables
capacity =  m.add_var(name="capacity", lb=0)
production = [m.add_var(name=f"production{s}",lb=0 ) for s in scenarios]

## this is the profit variable (scenario-dependent)
profit = [m.add_var(name=f"profit{s}",  lb=-np.inf) for s in scenarios]

shortfall = [m.add_var(name=f"shortfall{s}",  lb=0) for s in scenarios] # scenario-dependent!

VaR = m.add_var(name="VaR", lb=-np.inf)
CVaR = m.add_var(name="CVaR", lb=-np.inf)

# in the objective, we now add a variable for the shortfall
m.objective =  maximize( sum(prob[s] * profit[s] for s in scenarios)  + lambd*CVaR )

for s in scenarios:
    m += profit[s]  == -installation_cost*capacity + contribution_margin * production[s] # computes the profit for scenario s
    m += production[s] <= demand_sample[s]
    m += production[s] <= capacity

    # computing shortfall below VaR
    m += shortfall[s] >= VaR - profit[s]

# Relating CVar and VaR
m += CVaR == VaR - 1/alpha* sum((1/n_scenarios)*shortfall[s] for s in scenarios)
```

Mean-Risk Optimization with CVaR: Results

In [ ]:
```
m.optimize()

print(f'Results for a alpha = {alpha} and lambda = {lambd}')
print(f'Capacity decision: {capacity.x:.02f}')
print(f'Objective function value: {m.objective_value:.02f}' )

profit_samples = np.array( [ profit[s].x for s in scenarios ])
print(f' Expected Profit: {np.mean(profit_samples):.02f}')
print(f'CVaR variable: {CVaR.x:.02f}')
print(f'VaR variable: {VaR.x:.02f}')
```

Part II: Chance-Constrained Programming

Motivation

In our case studies discussed so far, we assumed that there are second-stage decisions:

- Capacity Planning: adapting production to demand
- Farmer's problem: buying and selling crops on the market

What if we assume that we cannot react?

- we will either have enforce a safe solution
- or tolerate a certain violation
- in many setting, we may see this as a **service level**
    - for example, on-time guarantees in logistics

Chance-Constrained Programming: Overview

In this part, we learn how to create optimization models under uncertainty providing probabilistic guarantees

In particular, we will discuss

- how to model cases in which a single constraints are affected by uncertainty using **chance constraints**
- how to model **joint chance constraints** providing a feasibility guarantee for problems in which multiple constraints are affected by uncertainty
- how to **improve joint chance constraint formulations** to make then solvable faster

Belt Manufacturing Case Study as an Illustration

Case Study: Manufacturing Belts

- A small company manufactures two types of belts: A and B. The contribution margin $2 for an A-belt and \$1.5 for a B-belt.
- It plans the production for a week, and it can sell its full production to small chain of shops.
- Producing a belt of type A takes twice as long as producing one of type B, and the total time available in that week would allow producing 1000 belts of type B if only B-belts were produced.
- Both types of belts require the same amount of leather, and there is enough leather to produce 800 belts.
- The total number that can be produced per type is limited by the number of available bucks: The company has 400 bucks for type A and 700 bucks for type B.

Create an LP model that determines the number of belts from each type to produce if the shop aims at maximizing the total contribution margin!

Case Study: Manufacturing Belts - Deterministic LP

Set

- $I = \{A, B\}$ belt types

Decision Variables

- x_i: number of belts to produce from type i

$$
\begin{aligned}
\max \ & 2x_A + 1.5x_B \\
\text{s.t. } & 2x_A + x_B \leq 1000 \\
& x_A + x_B \leq 800 \\
& 0 \leq x_A \leq 400 \\
& 0 \leq x_B \leq 700
\end{aligned}
$$

The optimal production plan is $x_A = 200, x_B = 600$ yielding a total profit of $1300.

Case Study: Manufacturing Belts - Deterministic LP in Python

In [78]:

```
belt_types = np.arange(2)
profit_contribution = [2, 1.5]
time_consumption = [2, 1]

time_available = 1000
leather_available = 800
bucks_available = [400, 700]


# Create a new model
m = mip.Model("Belt_Production_Deterministic")


#decision variables
production = [m.add_var(name= f"production{b}", lb=0, ub=bucks_available[b]) for b in belt_types]
m.objective = maximize(sum(profit_contribution[b] * production[b] for b in belt_types))

m += sum (time_consumption[b]*production[b] for b in belt_types) <= time_available
m += sum (production[b] for b in belt_types) <= leather_available

m.optimize()

production_decisions = [production[b].x for b in belt_types]
for b in belt_types:
    print(f'Production belt {b}: {production[b].x}')
print(f'Total Profit: {m.objective_value}' )
```

```
Production belt 0: 200.0
Production belt 1: 600.0
Total Profit: 1300.0
```

Uncertain Machine Time without Recourse

- let us now assume that due to random machine failures, the **available time is subject to uncertainty**
- however, due to contractual obligation, the company has to commit to a production plan **before** the machine time is known
  - the production decisions are thus **here and now decisions**
- in this part, we assume that there is **no recourse decision**
- given that the random variable describing the available time is B_T, the uncertain constraint is:

$$2x_A + x_B \leq B_T$$

$\rightarrow$ if our first-stage decisions are bad, the model becomes **infeasible for certain scenarios**

Uncertain Machine Time: The Data

We assume that the machine time follows a normal distribution:

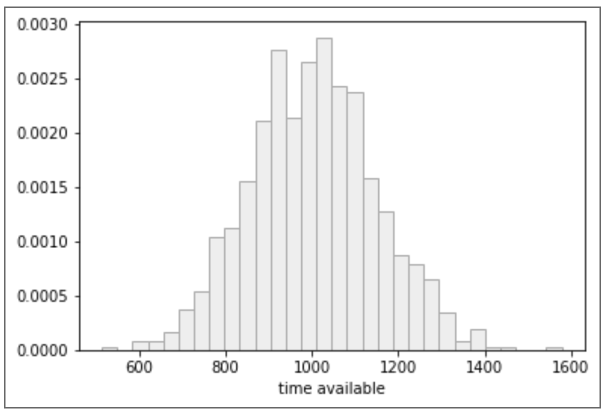
In [79]:

```
                # avaible_time
time_available_dist = stats.norm(1000,150)

n_samples = 1000

# we create a sample vector of demands (demand_dist was defined above), only using positive outcomes
time_available_samples = np.maximum(time_available_dist.rvs(n_samples, random_state=42),0)

count, bins, ignored = plt.hist(time_available_samples , bins=30, density=True, color='#EEEEEE', edgecolor="#AAAAAA")
plt.xlabel('time available')
plt.show()
```



Uncertain Machine Time: How likely is it that our Solution is Infeasible?

The following function can be used to estimate the feasibility of a production plan:

In [80]:

```
def probability_production_plan_is_feasible (production_decisions, time_available_samples):
    return np.mean([2 * production_decisions[0] + production_decisions[1] <= time_available for time_available in time_available_samples])
```

Let's apply it to our optimal deterministic decision:

In [81]:

```
probability_of_feasibility =  probability_production_plan_is_feasible (production_decisions, time_available_samples)

print (f'The production plan is feasible with probability {probability_of_feasibility:.02f}')
```

```
The production plan is feasible with probability 0.51
```

Enforcing Feasibility for each Scenario?

What if we want to establish feasibility for each scenario?

- the time constraint becomes: $2x_A + x_B \leq b_{T,s} \qquad \forall s \in S$
- which is equivalent to the single constraint: $2x_A + x_B \leq \min_{s \in S} b_{T,s}$
- note that $\min_{s \in S} b_{T,s}$ can be computed **before** solving the problem

Note that this approach, sometimes called **fat solution**,

- is extremely conservative / pessimistic
- may lead completely infeasible model instances in more complex cases
- depends a lot on the samples that are drawn:
    - different samples can result in very different solutions
    - more samples tend to result in more pessimistic decisions

Ensuring Feasibility for each Scenario

If we want the model to be feasible for each scenario, we just use the minimum of the available time over all sample values.

In [82]:

```python
                # Create a new model
m = mip.Model("Belt_Production_Pessimistic")

min_time_avaible = np.min(time_available_samples)

#decision variables
production = [m.add_var(name= f"production{b}", lb=0, ub=bucks_available[b]) for b in belt_types]

m.objective = maximize(sum(profit_contribution[b] * production[b] for b in belt_types))

# we change this variable
m += sum (time_consumption[b]*production[b] for b in belt_types) <= min_time_avaible
m += sum (production[b] for b in belt_types) <= leather_available

m.optimize()

production_decisions = [production[b].x for b in belt_types]
for b in belt_types:
    print(f'Production belt {b}: {production[b].x}')
print(f'Total Profit: {m.objective_value}' )
```

```
Production belt 0: 0.0
Production belt 1: 513.8098989896391
Total Profit: 770.7148484844587
```

Ensuring Feasibility for each Scenario: Evaluating the Solution

We may evaluate the solution, and of course it should be feasible

In [83]:
```
probability_of_feasibility =  probability_production_plan_is_feasible (production_decisions, time_available_samples)
print (f'The production plan is feasible with probability {probability_of_feasibility:.02f}')
```

```
The production plan is feasible with probability 1.00
```

We may also evaluate it using a new sample, resulting in a so-called *out-of-sample* evaluation

In [84]:
```
probability_of_feasibility =  probability_production_plan_is_feasible (production_decisions, time_available_dist.rvs(100000, random_state=42))
print (f'The production plan is feasible with probability {probability_of_feasibility}')
```

```
The production plan is feasible with probability 0.99943
```

Exercise:
Experiment with different samples of the same size and samples of different sizes and see the effects on the decisions and on the profit!

Chance Constraints: Key Idea

Key Idea:

- 100% feasibility is often very expensive
- instead ensure that the probability of feasibility is $\geq \alpha$

Formally expressed, given a constraint C, a **chance constraint** can be written as:

$$P(C \text{ holds}) \geq \alpha$$

In our example, given B_T as the random variable for available time:

$$P(2x_A + x_B \leq B_T) \geq \alpha$$

- if only a single constraint is affected by uncertainty, a single chance constraint ensures that the solution is feasible with a probability $\geq \alpha$
- if more constraints are uncertain, we have **joint chance constraints**

How to Model a Chance Constraint?

Given that B_T is a random variable and α is the desired probability of feasibility, how can we model the chance constraint:

$$P(2x_A + x_B \leq B_T) \geq \alpha$$

Key Idea:
Assume that we have a set S of scenarios. Then we can approximate the chance constraint by ensuring that the constraint is violated in at most $(1-\alpha)|S|$ scenarios

- we can achieve this by sorting the scenarios from the most to the least restricting and
- taking the value b_T^α below which $(1-\alpha)|S|$ values fall and using it in the right hand side of the constraint:

$$2x_A + x_B \leq b_T^{1-\alpha}$$

- this way, there are at most $\alpha|S|$ scenarios violating the constraint
- for example, in case of $\alpha = 0.95$ and $|S|$ = 1000, $b_{T,s}^{1-\alpha}$ is the 51th-smallest value from all scenarios

Chance Constraints Using Quantiles

- observe that the value $b_T^{1-\alpha}$ corresponds to the $(1-\alpha)$ quantile of the sample distribution
- thus, if we have a random distribution for which we can compute the $(1-\alpha)$ quantile directly (without sampling), we can use it without resorting to sampling
- for our example, the full model looks as follows:

$$\begin{aligned} \max\ & 2x_A + 1.5x_B \\ \text{s.t. } & 2x_A + x_B \leq b_T^{1-\alpha} \\ & x_A + x_B \leq 800 \\ & 0 \leq x_A \leq 400 \\ & 0 \leq x_B \leq 700 \end{aligned}$$

- observe that the model does *not* involve scenarios but has the same size as the deterministic model - we only replace b_T with $b_T^{1-\alpha}$.

Individual Chance Constraints

Given that we have a single chance constraint, we can ensure feasibility with probability α by replacing the uncertain right hand side parameter with the $1-\alpha$-quantile of the distribution.

In [85]:

```
# Create a new model
m = mip.Model("Belt_Production_Single_Chance_Constraint")


alpha = 0.95

#in scipy stats, we get the quantile using the function ppf (inverse cumulative distribution function)
# we may also get the quantile from the set of samples!
time_available_quantile = time_available_dist.ppf(1-alpha)

#decision variables
production = [m.add_var(name= f"production{b}", lb=0, ub=bucks_available[b]) for b in belt_types]

m.objective = maximize(sum(profit_contribution[b] * production[b] for b in belt_types))

# we change this variable
m += sum (time_consumption[b]*production[b] for b in belt_types) <= time_available_quantile
m += sum (production[b] for b in belt_types) <= leather_available
```

Individual Chance Constraints: Solving and Evaluating the Model

In [86]:

```
             m.optimize()

production_decisions = [production[b].x for b in belt_types]

for b in belt_types:
    print(f'Production belt {b}: {production[b].x}')
print(f'Total Profit: {m.objective_value}' )
```

```
Production belt 0: 26.635977978639573
Production belt 1: 700.0
Total Profit: 1103.2719559572793
```

Let us use Monte-Carlo simulation to verify:

In [88]:

```
# let's evaluate feasibility using sampling!
probability_of_feasibility_from_sampling =  probability_production_plan_is_feasible (production_decisions, time_available_dist.rvs(50000, random_state=42))

print (f'Using Monte Carlo, the production plan is feasible with probability {probability_of_feasibility_from_sampling :.02f}')
```

```
Using Monte Carlo, the production plan is feasible with probability 0.95
```

Joint Chance Contraints

But: What Happens in Case of Multiple Uncertain Constraints?

Let us now assume that both resource constraints are uncertain:

- no only time, but also leather capacity is affected by uncertainty, e.g. due to quality issues, represented by the random variable B_L
- as in the time case, there is no second-stage / recourse decision is available

Question: How can we ensure that our plan is feasible with a probability of α?

- more formally, how can we ensure the so-called *joint chance constraint*

$$P\begin{pmatrix} 2x_A + x_B \leq B_T \\ x_A + x_B \leq B_L \end{pmatrix} \geq \alpha$$

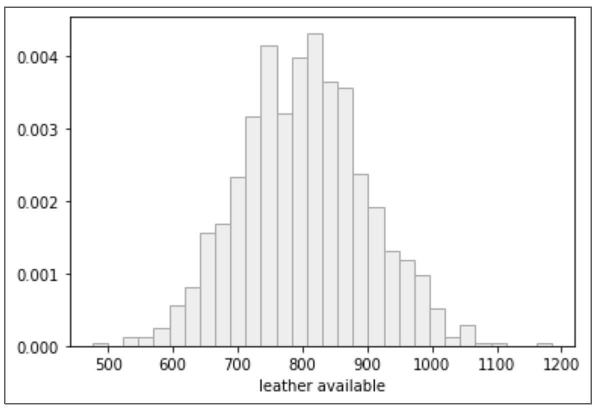
Uncertain Leather Availability: Probability Distribution

We assume that leather availability is normally distributed:

In [89]:
```
# available leather
leather_available_dist = stats.norm(800,100)


# we create a sample vector of demands (demand_dist was defined above), only using positive outcomes
leather_available_samples = np.maximum(leather_available_dist.rvs(n_samples, random_state=42),0)

count, bins, ignored = plt.hist(leather_available_samples , bins=30, density=True, color='#EEEEEE', edgecolor="#AAAAAA")
plt.xlabel('leather available')
plt.show()
```



What about using two single chance constraints to model the joint chance constraint?

We may enforce that
$P(2x_A + x_B \leq B_T) \geq \alpha$ **and** $P(x_A + x_B \leq B_L) \geq \alpha$
by the following constraints:

$$2x_A + x_B \leq b_T^{1-\alpha}$$
$$x_A + x_B \leq b_L^{1-\alpha}$$

Does this imply that the problem is feasible with probability α?

- Let's check in Python

Quantile-Based Model for Two Chance Constraints in Python

In [90]:

```python
               # Create a new model
m = mip.Model("Belt_Production_Single_Chance_Constraint")
m.verbose=1
belt_types = np.arange(2)
profit_contribution = [2, 1.5]
time_consumption = [2, 1]
bucks_available = [400, 700]

alpha = 0.95

#in scipy stats, we get the quantile using the function ppf (inverse cumulative distribution function)
# we may also get the quantile from the set of samples!
time_available_quantile = time_available_dist.ppf(1-alpha)
leather_available_quantile = leather_available_dist.ppf(1-alpha)


#decision variables
production = [m.add_var(name= f"production{b}", lb=0, ub=bucks_available[b]) for b in belt_types]

m.objective = maximize(sum(profit_contribution[b] * production[b] for b in belt_types))

# we change this variable
m += sum (time_consumption[b]*production[b] for b in belt_types) <= time_available_quantile
m += sum (production[b] for b in belt_types) <= leather_available_quantile

m.optimize()
production_decisions = [production[b].x for b in belt_types]

for b in belt_types:
    print(f'Production belt {b}: {production[b].x}')
print(f'Total Profit: {m.objective_value}' )
```

```
Production belt 0: 117.7573186524263
Production belt 1: 517.7573186524265
Total Profit: 1012.1506152834925
```

Quantile-Based Model for Two Chance Constraints: Evaluation in Python

- in our model, we built the two constraints with $\alpha = 0.95$
- how high is the probability that the solution is feasible with respect to both constraints?

In [91]:

```
print(f'We want to ensure feasibility with probability {alpha}')


def probability_production_plan_is_feasible_time_and_leather  (production_decisions, time_available_samples, leather_available_samples):
    n_samples = time_available_samples.size
    feasible_samples = np.array([2 * production_decisions[0] + production_decisions[1] <= time_available_samples[s]
                                 and production_decisions[0] + production_decisions[1] <= leather_available_samples[s] for s in range(n_samples)])
    return np.mean(feasible_samples)

n_samples_for_evaluation = 5000
samples_time = time_available_dist.rvs(n_samples_for_evaluation)
samples_leather = leather_available_dist.rvs(n_samples_for_evaluation)

probability_of_feasibility_from_sampling = probability_production_plan_is_feasible_time_and_leather (production_decisions, samples_time, samples_leather )


print (f'Based on a Monte-Carlo approximation the production plan is feasible with probability {probability_of_feasibility_from_sampling :.02f}')
```

```
We want to ensure feasibility with probability 0.95
Based on a Monte-Carlo approximation the production plan is feasible with probability 0.9
0
```

Using Multiple Individual Chance Constraints?

As we learned in our example, in general, given a set of n constraints affected by uncertainty and a desired feasibility probability α, the *joint chance constraint*

$$P(C_j \text{ holds} \quad \forall j = 1 \ldots n) \geq \alpha$$

is **not** enforced by using a set of single chance constraints of the form

$$P(C_j \text{ holds}) \geq \alpha \quad \forall j = 1 \ldots n$$

Question: How can we enforce a such joint chance constraint?

- note that in case of multiple uncertain constraints, the scenarios can no longer be ordered according to strictness of constraints
- for example, given two scenarios in the belt production case study, one may have a smaller b_T and the other a smaller b_L

Modeling Chance Constraints using Scenarios and Binary Variables

Key Idea: Let the solver decide which scenarios should hold

- introduce a binary variable y_s for each scenario $s \in S$:
  - $y_s = 0$ means that in scenario s, all uncertain constraints hold
  - $y_s = 1$ means that in scenario s, at least one constraint may be violated
- introduce a constraint that ensures that the total probability of all scenarios for which $y_s = 1$ is $\leq 1 - \alpha$:

$$\sum_{s \in S} p_s y_s \leq 1 - \alpha$$

- introduce a "scenario realization" for each uncertain constraint and each scenario s and link that constraint to y_s to switch it on or off

$$\begin{aligned} 2x_A + x_B &\leq b_{T,s} + M_T y_s \qquad && \forall s \in S \\ x_A + x_B &\leq b_{L,s} + M_L y_s \qquad && \forall s \in S \end{aligned}$$

The Model with Joint Chance Constraints

Sets

- S: scenarios

Decision Variables:

- x_i: number of belts to produce from type i
- y_s: binary indicator variable for infeasibility in scenario s

Parameters:

- p_s: probability of scenarios s
- $b_{r,s}$: availability from resource r (time, leather) in scenario s
- M_r: big-M value for resource r used to switch off constraint

$$
\begin{aligned}
\max\ & 2x_A + 1.5x_B & \\
\text{s.t. } & 2x_A + x_B \leq b_{T,s} + M_T y_s & \forall s \in S \\
& x_A + x_B \leq b_{L,s} + M_L y_s & \forall s \in S \\
& \sum_{s \in S} p_s y_s \leq 1 - \alpha & \\
& 0 \leq x_A \leq 400 & \\
& 0 \leq x_B \leq 700 & \\
& y_s \in \{0, 1\} & \forall s \in S
\end{aligned}
$$

The Model with Joint Chance Constraints in Python

In [92]:

```
alpha = 0.95
n_scenarios = 500
scenarios = np.arange(n_scenarios)
time_available = time_available_dist.rvs(n_scenarios, random_state=5)
leather_available = leather_available_dist.rvs(n_scenarios,  random_state=7)
```

In [93]:

```
m = mip.Model("Belt_Production_Joint_Chance_Constraints")

#probability of each scenario
prob = np.full((n_scenarios), 1.0/n_scenarios)


#bigM is used for switching off constraints by adding it to the resource capacity in a scenario
#if it is chosen as difference between the max and min capacity, it is definitely large enough to switch off the constraint
bigM = [max(time_available)-min(time_available),max(leather_available)-min(leather_available)]

#decision variables
production = [m.add_var(name= f"production{b}", lb=0, ub=bucks_available[b]) for b in belt_types]

#y-variable
is_scenario_violated = [m.add_var(name=f"is_feasible{s}", var_type=mip.BINARY ) for s in scenarios]

m.objective = maximize(sum(profit_contribution[b] * production[b] for b in belt_types))

# one set of constraints per scenario
for s in scenarios:
    m += sum (time_consumption[b]*production[b] for b in belt_types) <= time_available[s] + bigM[0]*(is_scenario_violated[s])
    m += sum (production[b] for b in belt_types) <= leather_available[s] + bigM[1]*(is_scenario_violated[s])

# ensure that the right number of scenarios hold
m += sum(prob[s]*is_scenario_violated[s] for s in scenarios) <= 1-alpha - 2.0 / n_scenarios
```

The Model with Joint Chance Constraints In Python

In [94]:
```
from time import time

start_time = time()
m.optimize()
print(f"Solution time: {time()-start_time} seconds")

production_decisions = [production[b].x for b in belt_types]
for b in belt_types:
    print(f'Production belt {b}: {production[b].x}')
print(f'Total Profit: {m.objective_value}' )
```

```
Solution time: 19.707329273223877 seconds
Production belt 0: 103.71945455868126
Production belt 1: 510.89995829054806
Total Profit: 973.7888465531846
```

Let us now evaluate the probability of having an infeasible model.

- first, using the same 500 samples we used in the model

In [95]:
```
print(f'We want to ensure feasibility with probability {alpha}')
probability_of_feasibility_from_sampling = probability_production_plan_is_feasible_time_and_leather (production_decisions, time_available, leather_available)

print (f'Using Monte Carlo for the samples used in the optimization, the production plan is feasible with probability {probability_of_feasibility_from_sampling :.02f}')
```

```
We want to ensure feasibility with probability 0.95
Using Monte Carlo for the samples used in the optimization, the production plan is feasib
le with probability 0.95
```

- and then, using 5000 new sample (out-of-sample evaluation)

In [96]:
```
n_samples_for_evaluation = 10000
probability_of_feasibility_from_sampling = probability_production_plan_is_feasible_time_and_leather (production_decisions, time_available_dist.rvs(n_samples_for_evaluation), leather_available_dist.rvs(n_samples_for_evaluation)

print (f'Using Monte Carlo with different samples, the production plan is feasible with probability {probability_of_feasibility_from_sampling :.03f}')
```

```
Using Monte Carlo with different samples, the production plan is feasible with probabilit
y 0.936
```

Exercise: Capacity Planning Case Study - Reaching A Certain Profit with a Desired Probability

Exercise: Capacity Planning Case Study

- Here you once again find the implementation of the capacity planning case study, this time with the modification from last week for obtaining an approximation of a profit distribution

In [ ]:

```
demand_dist = stats.norm(100,25)
n_samples = 500

# we create a sample vector of demands (demand_dist was defined above), only using positive outcomes
demand_sample = np.maximum(demand_dist.rvs(n_samples),0)
prob = np.full((n_scenarios), 1/n_scenarios)

m = mip.Model("Capacity_Planning_Stochastic")

#in stochastic programming, we call the samples scenarios
n_scenarios = n_samples
scenarios = np.arange(n_scenarios) # set of scenarios

#parameters
installation_cost = 30
contribution_margin = 40

#decision variables
capacity =  m.add_var(name="capacity", lb=0)

production = [m.add_var(name=f"production{s}", lb= 0) for s in scenarios]

## this is the new variable
profit = [m.add_var(name=f"profit{s}",  lb=-np.inf) for s in scenarios]

m.objective =  maximize( sum(prob[s] * profit[s] for s in scenarios))

for s in scenarios:
    m += profit[s] == -installation_cost*capacity + contribution_margin * production[s] # computes the profit for scenario s
    m += production[s] <= demand_sample[s]
    m += production[s] <= capacity

m.optimize()

print(f'Capacity decision: {capacity.x:.02f}')
print(f'Expected Total Profit: {m.objective_value:.02f}' )

profit_samples = np.array( [ profit[s].x for s in scenarios ])
```

Exercise: Capacity Planning Case Study: Tasks

- what is the probability that the profit exceeds 500?

```
In [ ]:
```

- modify the model in a way that we obtain a profit exceeding 500 with a probability of 90%
- hint: use the basic idea of using binary variables for modeling (joint) chance constraints!

Summary

In the two parts of this meeting, we dealt with approaches for avoiding undesired outcomes:

PART I: RISK ORIENTATION / DECISION-MAKING AS SHAPING DISTRIBUTIONS

- Decision Making under Uncertainty as Shaping Distributions
- Comparing Distributions: Stochastic Dominance
- Shaping Distributions by Incorporating Risk Attitude into Decision / Optimization Models

PART II: FEASIBILITY GUARANTEES BY CHANCE-CONSTRAINED PROGRAMMING

- Feasibility Issues in Single-Stage Problems under Uncertainty
- Single Chance Constraints
- Joint Chance Constraints