

Assignment 7 for Statistical Computing and Empirical Methods: Sample and population

Henry Reeve

Introduction

This document describes your seventh assignment for Statistical Computing and Empirical Methods (Unit EMATM0061) on the MSc in Data Science. Before starting the assignment it is recommended that you first watch video lecture 7 entitled “Sample and population”.

It is recommended that you write your answers within an RMarkdown file using TEX <https://en.wikipedia.org/wiki/TeX> (<https://en.wikipedia.org/wiki/TeX>). However, this is optional.

1 Basic properties of random variables

Recall from the lectures that a random variable X with outcome space \mathcal{S} satisfies the following basic properties:

1. $\mathbb{P}(X \in \mathcal{S}) = 1$,
2. Given all well-behaved sets $A \subseteq \mathcal{S}$, we have $\mathbb{P}(X \in A) \in [0, 1]$,
3. Given sets $A_1, A_2, \dots, A_k \subseteq \mathcal{S}$ with $A_i \cap A_j = \emptyset$ for $i \neq j$ we have,

$$\mathbb{P}(X \in A_1 \cup \dots \cup A_k) = \mathbb{P}(X \in A_1) + \dots + \mathbb{P}(X \in A_k).$$

Recall that \emptyset denotes the empty set, the unique set which contains no points. Apply the basic properties of random variables to show that $\mathbb{P}(X \in \emptyset) = 0$.

Now, suppose that $A \subseteq \mathcal{S}$ and $B \subseteq \mathcal{S}$ are subsets of \mathcal{S} . What can we say about $\mathbb{P}(X \in A \cup B)$ in terms of $\mathbb{P}(X \in A)$ and $\mathbb{P}(X \in B)$?

2 Expectation and variance

Suppose that $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$. Suppose that X is a random variable with outcome space $\mathcal{S} = \{0, 1, 5\}$. Suppose that $\mathbb{P}(X = 1) = \alpha$ and $\mathbb{P}(X = 5) = \beta$.

What is the probability mass function $p : \mathcal{S} \rightarrow [0, 1]$ for X ?

What is the expectation of X ?

What is the variance of X ?

3 Transformations of random variables

Suppose that X is a discrete random variable with outcome space \mathcal{S}_X and probability mass function $p_X : \mathcal{S}_X \rightarrow [0, 1]$. Suppose that \mathcal{S}_Y is another space and $\varphi : \mathcal{S}_X \rightarrow \mathcal{S}_Y$ is a function. For each $A \subseteq \mathcal{S}_Y$ we define $\varphi^{-1}(A) \subseteq \mathcal{S}_X$ by $\varphi^{-1}(A) := \{x \in \mathcal{S}_X : \varphi(x) \in A\}$.

We define a new random variable $Y = \varphi(X)$ by $\mathbb{P}(\varphi(X) \in A) = \mathbb{P}(X \in \varphi^{-1}(A))$ for $A \subseteq \mathcal{S}_Y$.

Prove that $Y = \varphi(X)$ satisfies the three basic properties of a random variable with outcome space \mathcal{S}_Y .

Write the probability density function $p_Y : \mathcal{S}_X \rightarrow [0, 1]$ of $Y = \varphi(X)$ in terms of p_X .

4 Poisson random variables

Many discrete random variables have a finite outcome space. Poisson random variables are a family of discrete real-valued random variables with outcome space

$\mathbb{N}_0 := \{0, 1, 2, 3, \dots\}$. Take $\lambda > 0$. The Poisson random variable X with parameter λ has probability mass function $p_\lambda : \mathbb{N}_0 \rightarrow (0, \infty)$ defined by

$$p_\lambda(k) = \frac{\lambda^k e^{-\lambda}}{k!} \text{ for all } k \in \mathbb{N}_0.$$

Show that p_λ is a well-defined probability mass function.

Compute both the expectation and the variance of a Poisson random variable X with probability mass function p_λ .