Proof of Theorem 1

Proof. As the computation of MEU proceeds bottom up through the SPMN, we prove the theorem by induction bottom up. We assume that max and sum nodes have two children each. The proof easily generalizes to nodes with more children.

Base case In the base case we consider a max node connected to two utility nodes that are leaves. Let the maximization be over decision variable D and the utility nodes represent values of utility variable U. MEU computation on this SPMN yields $S() = \max_D U(D)$, which satisfies the Sum-Max-Sum rule.

Inductive hypothesis Let n_0 be an internal node with two children n_1 and n_2. Let the scope of n_0 be set V_0 (which includes decision variables as well), state of V_0 be x_0 and $S_0(e)$ be the MEU from the SPMN rooted at n_0. Analogously, for the SPMNs rooted at n_1 and n_2. Let $\phi_1(x_1)$ be the (unnormalized) network polynomial that gives the joint probability distribution embedded in the SPMN rooted at n_1. Then, the inductive hypothesis is $S_1(e) = \sum_{X_1|_e} \phi_1(x_1)\, U(x_1, D^*)$ where D^* is the set of maximizing decisions taken on the max nodes contained in the SPMN rooted at n_1; these decisions appear in the scope of n_1; U is the utility variable associated with the utility nodes; and $X_1|_e$ is the set of all values of X_1 that are consistent with evidence e. Similarly, assume that $S_2(e) = \sum_{X_2|_e} \phi_2(x_1)\, U(x_2, D^*)$.

Inductive proof If n_0 is a sum node, and w_{01} and w_{02} are the weights on its two edges that connect to the children. Then,

$$\begin{aligned} S_0(e) &= w_{01} S_1(e) + w_{02} S_2(e) \\ &= w_{01} \sum_{X_1} \phi_1(x_1)\, U(x_1, D^*) + w_{02} \sum_{X_2} \phi_2(x_2)\, U(x_2, D^*) \end{aligned}$$

If the sum node is not sum-complete $\implies V_1 \neq V_2$. Then, $V_0 = V_1 \cup V_2$ and we lose the one-one correspondence between a state in X_0 and exactly one monomial in ϕ_0 that is non zero. This implies that $\phi_0(x_0)$ is not a network polynomial and the correct MEU is not guaranteed. Therefore, $V_0 = V_1 = V_2$ $\implies$

$$\begin{aligned} S_0(e) &= w_{01} \sum_{X_0|_e} \phi_1(x_0)\, U(x_0, D^*) + w_{02} \sum_{X_0} \phi_2(x_0)\, U(x_0, D^*) \\ &= \sum_{X_0|_e} w_{01}\phi_1(x_0)\, U(x_0, D^*) + w_{02}\phi_2(x_0)\, U(x_0, D^*) \\ &= \sum_{X_0|_e} [w_{01}\phi_1(x_0) + w_{02}\phi_2(x_0)] U(x_0, D^*) \qquad (1) \end{aligned}$$

This is an instance of the Sum-Max-Sum rule.

If n_0 is a product node then,

$$S_0(e) = \left(\sum_{X_1|_e} \phi_1(x_1)\, U(x_1, D_1^*) \right) \left(\sum_{X_2|e} \phi_2(x_1)\, U(x_2, D_2^*) \right)$$

If $V_1 \cap V_2 = \emptyset$, it is immediately obvious that the above satisfies the Sum-Max-Sum rule yielding the MEU.

If n_0 is a max node associated with decision variable D_0, then $S_0(e) = \max_{D_0}\{S_1(e), S_2(e)\}$. From the induction hypothesis, $S_1(e)$, and $S_2(e)$ return the MEU computed for the SPMN rooted at nodes n_1 and n_2, respectively. Then it follows trivially that maximization over these returns the MEU at the max node.

Let D_0 be the decision variable also associated with a max node in a path in the SPMN rooted at n_1. Then, w.l.o.g. let d_0 be the maximizing decision, which is added to the scope of n_1. If n_0 is also a max node, and because $V_1 \subset V_0$, then d_0 is already present in V_0. Thus, a second max node associated with the same variable D_0 violates the Sum-Max-Sum rule. $\square$