Assignment 8: Complex Numbers 1

Declan Murphy Zink

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3.a Find the real and imaginary parts of $(\sqrt{3}-i)^{10}$ and $(\sqrt{3}-i)^{-7}$. For which values of n is $(\sqrt{3}-i)^n$ real?

$(\sqrt{3}-i)^{10} = (2e^{-\frac{\pi}{6}i})^{10} = 2^{10}e^{-\frac{10\pi}{6}i} = 2^9 \cdot 2e^{\frac{\pi}{3}i} = 2^9(1+\sqrt{3}i)$.

$(\sqrt{3}-i)^{-7} = 2^{-7}e^{\frac{7\pi}{6}i} = 2^{-8}(-\sqrt{3}-i)$.

for $(\sqrt{3}-i)^n$ to be real, $\frac{n\pi}{6} = k\pi, k \in \mathbb{Z}$, so n must be a multiple of 6.

3.b What is $\sqrt{i}$

$\sqrt{i} = (e^{\frac{\pi}{2}i})^{\frac{1}{2}} = e^{\frac{\pi}{4}i}$ or $e^{\frac{5\pi}{4}i}$

3.c Find all tenth roots of i. Which one is nearest to i in the Argand diagram?

$\sqrt[10]{i} = (e^{\frac{\pi}{2}i})^{\frac{1}{10}} = e^{\frac{\pi}{20}i}, e^{\frac{5\pi}{20}i}, e^{\frac{9\pi}{20}i}, e^{13\frac{\pi}{20}i}, e^{17\frac{\pi}{20}i}, e^{21\frac{\pi}{20}i}, e^{25\frac{\pi}{20}i}, e^{29\frac{\pi}{20}i}, e^{33\frac{\pi}{20}i}, e^{37\frac{\pi}{20}i}$.

$e^{\frac{9\pi}{20}i}$ is closest to i.

3.d Find the seven roots of the equation $z^7 - \sqrt{3} + i = 0$. Which one of these roots is closest to the imaginary axis?

$z^7 = \sqrt{3} - i$, so for $k \in \mathbb{Z}, 0 \leq k < 7, z = (2e^{(-\frac{\pi}{6}+2k\pi)i})^{\frac{1}{7}} = 2^{\frac{1}{7}}e^{(-\frac{\pi}{42}+\frac{12k\pi}{42})i}$

Closest to the imaginary axis $= 2^{\frac{1}{7}}e^{\frac{23\pi}{42}i}$.

4 Prove the "Triangle Inequality" for complex numbers: $|u+v| \leq |u|+|v|$ for all $u, v \in \mathbb{C}$.

For $a, b, c, d \in \mathbb{R}$:

$$|a+bi+c+di| \leq |a+bi| + |c+di|$$

$$\sqrt{(a+c)^2+(b+d)^2} \leq \sqrt{a^2+b^2} + \sqrt{c^2+d^2}$$

$$(a+c)^2+(b+d)^2 \leq a^2+b^2+c^2+d^2+2\sqrt{(a^2+b^2)(c^2+d^2)}$$

$$2ac + 2bd \leq 2\sqrt{(a^2+b^2)(c^2+d^2)}$$
$$ac + bd \leq \sqrt{a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2}$$
$$a^2c^2 + 2abcd + b^2d^2 \leq a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2$$
$$0 \leq a^2d^2 - 2abcd + b^2c^2$$
$$0 \leq (ad - bc)^2$$

This must be true since $a, b, c, d \in \mathbb{R}$.

5 Let z be a non-zero complex number. Prove that the three cube roots of z are the corners of an equilateral triangle in the Argand diagram.

Let $z = re^{(\theta+2k\pi)i}$ for $k \in \mathbb{Z}, 0 \leq k < 3$. Thus $\sqrt[3]{z} = r^{\frac{1}{3}}e^{(\frac{\theta}{3}+\frac{2k\pi}{3})i}$, which means that each cubic root is $\frac{2\pi}{3}$ radians from the other two roots. Since the modulus of each root is the same, $r^{\frac{1}{3}}$, this forms an equilateral triangle.