Assignment 2: Logic

Declan Murphy Zink

9/14/2020

6 Write down careful proofs of the following statements:

6.a $\sqrt{6} - \sqrt{2} > 1$.

$\sqrt{6} - \sqrt{2} \leq 1 \Rightarrow (\sqrt{6} - \sqrt{2})^2 \leq 1 \Rightarrow 10 - 2\sqrt{12} \leq 1 \Rightarrow 9 \leq 2\sqrt{12} \Rightarrow 81 \leq 48$
Contradiction

6.b If n is an integer such that n^2 is even, then n is even.

If n is odd, $n = 2a + 1$ for $a \in \mathbb{Z}$. Therefore $n^2 = 4a^2 + 4a + 1$, which is always odd. Thus if n^2 is even, n must be even.

6.c If $n = m^3 - m$ for some integer m, then n, is a multiple of 6.

For all integers m, $(m^2 - m)(m + 1)$ will produce a multiple of 3:

- If m is a multiple of 3 such that $m = 3a, a \in \mathbb{Z}$:

$$((3a)^2 - 3a)(3a + 1) = 27a^3 - 3a$$

- If m is one more than a multiple of 3 such that $m = 3a + 1, a \in \mathbb{Z}$:

$$((3a + 1)^2 - (3a + 1))((3a + 1) + 1) = (9a^2 + 3a)(3a + 2) = 27a^3 + 27a^2 + 6a$$

- If m is two more than a multiple of 3 such that $m = 3a + 2, a \in \mathbb{Z}$:

$$((3a + 2)^2 - (3a + 2))((3a + 2) + 1) = (9a^2 + 9a + 2)(3a + 3) = 27a^3 + 54a^2 + 33a + 6$$

For all integers m, $m(m - 1)$ will produce a multiple of 2:

- If m is a multiple of 2 such that $m = 2a, a \in \mathbb{Z}$:

$$2a(2a - 1) = 4a^2 - 2a$$

- If m is one more than a multiple of 2 such that $m = 2a + 1, a \in \mathbb{Z}$:

$$(2a + 1)(2a + 1 - 1) = 4a^2 + 2a$$

Becuase $m^3 - m = (m^2 - m)(m + 1)$, n must be a multiple of 3. Becacuse $(m^2 - m) = m(m - 1)$, n must be a multiple of 2. Because n is a multiple of 2 and 3, n must be a multiple of 6.

7 Disprove the following statements:

7.a If n and k are positive integers, then $n^k - n$ is always divisible by k.

This is false when $n = 3$ and $k = 4$, because $3^4 - 3 = 78$, which is not divisible by 4.

7.b Every positive integer is the sum of three squares.

The integer 7 cannot be created, so this is false.

8 Given that the number 8881 is not a prime number, prove that it has a prime factor that is at most 89.

The smallest prime factor of 8881 must be less than or equal to its square root because, otherwise, the prime factors would always multiply to be greater than 8881. $\sqrt{8881} \approx 94$, therefore, since the nearest prime number below 94 is 89, 8881 must have a prime factor that is at most 89.