More on Sets 1

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2 Which of the following statements are true and which are false? Give proofs or counterexamples.

2.a For any sets A, B, C, we have:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

True.
$x \in A \cup (B \cap C) \Leftrightarrow x \in A$ or $x \in B \cap C \Leftrightarrow x \in A \cup B$ and $x \in A \cup C \Leftrightarrow x \in (A \cup B) \cap (A \cup C)$

2.b For any sets A, B, C, we have:

$$(A - B) - C = A - (B - C)$$

False.
If $A, B, C = \{1\}$, then $(A - B) - C = \emptyset$ and $A - (B - C) = \{1\}$.

2.c For any sets A, B, C, we have:

$$(A - B) \cup (B - C) \cup (C - A) = A \cup B \cup C$$

False.
If $A, B, C = \{1\}$, then $(A - B) \cup (B - C) \cup (C - A) = \emptyset$ and $A \cup B \cup C = \{1\}$.

11 Prove that if m and n are coprime positive integers, then $\phi(mn) = \phi(m)\phi(n)$

$m = p_1^{a_1}...p_k^{a_k}$
$n = q_1^{b_1}...q_i^{b_i}$
Since m and n are coprime, $mn = p_1^{a_1}...p_k^{a_k}q_1^{b_1}...q_i^{b_i}$.

Thus, $\phi(m) = m(1 - \frac{1}{p_1})...(1 - \frac{1}{p_k})$
$\phi(n) = n(1 - \frac{1}{q_1})...(1 - \frac{1}{q_i})$
$\phi(mn) = mn(1 - \frac{1}{p_1})...(1 - \frac{1}{p_k})(1 - \frac{1}{q_1})...(1 - \frac{1}{q_i})$

Therefore, $\phi(mn) = \phi(m)\phi(n)$

12 For a positive integer n, define

$$F(n) = \sum_{d|n} \phi(d)$$

where the sum is over the positive divisors d of n, including both 1 and n. (For example, the positive divisors of 15 are 1, 3, 5, and 15.)

12.a Calculate $F(15)$ and $F(100)$

$F(15) = \phi(1) + \phi(3) + \phi(5) + \phi(15) = 15$
$F(100) = 100$

12.b Calculate $F(p^r)$, where p is prime

$F(p^r) = \phi(1) + \phi(p) + \phi(p^2) + ... + \phi(p^r)$
$F(p^r) = 1 + p(1 - \frac{1}{p}) + p^2(1 - \frac{1}{p}) + ... + p^r(1 - \frac{1}{p})$
$F(p^r) = 1 + (p + ... + p^r)(1 - \frac{1}{p})$
$F(p^r) = 1 + (p + ... + p^r) - (1 + p + ...p^{r-1})$
$F(p^r) = p^r$

12.c Calculate $F(pq)$, where p, q are distinct primes.

$F(pq) = \phi(1) + \phi(p) + \phi(q) + \phi(pq)$
$F(pq) = \phi(1) + \phi(p) + \phi(q) + \phi(p)\phi(q)$
$F(pq) = 1 + (p - 1) + (q - 1) + (p - 1)(q - 1)$
$F(pq) = 1 + p - 1 + q - 1 + pq - p - q + 1$
$F(pq) = pq$

12.d Formulate a conjecture about $F(n)$ for an arbitrary positive integer n. Try to prove your conjecture.

Conjecture: $F(n) = n$.

For divisors d such that $d|n$, $\phi(\frac{n}{d})$ gives the number of integers x such that $gcd(x, n) = d$. Thus, since each number less than or equal to n has a gcd equal to a d, each integer less than or equal to n is counted once, accumulating to n.