Assignment 4: Rational Numbers

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1.a Prove that $\sqrt{3}$ is irrational.

By contradiction. If $\sqrt{3}$ is rational, then for relatively prime integers p, q, $\sqrt{3} = \frac{p}{q}$. This can be rewritten as $\sqrt{3}q = p$, and after squaring both sides we get $3q^2 = p^2$. Thus, p^2 is a multiple of 3 and therefore p is a multiple of 3, so for some integer k, $p^2 = 9k$. Thus $3q^2 = 9k \Rightarrow q^2 = 3k$, but this is a contradiction because p and q can't both be mutiples of 3 if they are relatively prime.

1.b Prove that there are no rationals r, s such that $\sqrt{3} = r + s\sqrt{2}$.

By contradiction. If $\sqrt{3} = r + s\sqrt{2}$, then after squaring we get $3 = r^2 + 2rs\sqrt{2} + 2q^2$. This can be rewritten as $\frac{3-r^2-2q^2}{2rs} = \sqrt{2}$. However this is a contradiction because $\frac{3-r^2-2q^2}{2rs}$ is rational but $\sqrt{2}$ is irrational.

2 Which of the following numbers are rational and which are irrational?

2.a $\sqrt{2} + \sqrt{\frac{3}{2}}$.

Irrational. Proof by contradiction: Assume $\sqrt{2} + \sqrt{\frac{3}{2}}$ is rational, and therefore $\sqrt{2} + \sqrt{\frac{3}{2}} = p$ for $p \in \mathbb{Q}$. After squaring this is: $2 + \frac{3}{2} + 2\sqrt{3} = p^2$, which can be rewritten as: $\sqrt{2} = \frac{p^2}{2} - \frac{7}{4}$. This is a contradiction because $\sqrt{3}$ is irrational but $\frac{p^2}{2} - \frac{7}{4}$ is rational by closure.

2.b $1 + \sqrt{2} + \sqrt{\frac{3}{2}}$.

Irrational. Because rational + irrational = irrational.

2.c $2\sqrt{18} - 3\sqrt{8} + \sqrt{4}$.

Rational. Because $2\sqrt{18} = \sqrt{72} = 3\sqrt{8}$, $2\sqrt{18} - 3\sqrt{8} + \sqrt{4} = \sqrt{4} = 2$, which is rational.

2.d $\sqrt{2}+\sqrt{3}+\sqrt{5}$.

Irrational. Proof by contradiction: Assume $\sqrt{2}+\sqrt{3}+\sqrt{5}$ is rational, and therefore for $p \in \mathbb{Q}$:

$$\begin{aligned}
\sqrt{2}+\sqrt{3}+\sqrt{5} &= p \\
\sqrt{2}+\sqrt{3} &= p-\sqrt{5} \\
5+2\sqrt{6} &= p^2-2p\sqrt{5}+5 \\
2\sqrt{6}+2p\sqrt{5} &= p^2 \\
24+8p\sqrt{30}+20p^2 &= p^4 \\
\sqrt{30} &= \frac{p^4-20p^2-24}{8p}
\end{aligned}$$

This is a contradiction because $\frac{p^4-20p^2-24}{8p}$ is rational by closure, but $\sqrt{30}$ is irrational.

2.e $\sqrt{2}+\sqrt{3}-\sqrt{5+2\sqrt{6}}$.

Rational. Because $\sqrt{5+2\sqrt{6}}^2 = 5+2\sqrt{6} = (\sqrt{2}+\sqrt{3})^2$, therefore $\sqrt{5+2\sqrt{6}} = \sqrt{2}+\sqrt{3}$ (since both must be positive numbers). Thus, $\sqrt{2}+\sqrt{3}-\sqrt{5+2\sqrt{6}} = \sqrt{2}+\sqrt{3}-\sqrt{2}-\sqrt{3} = 0$, which is rational.

3 For each of the following statements, either prove it is true or give a counterexample to show it is false.

3.a The product of two rational numbers is always rational.

True. If $x=\frac{a}{b}$ and $y=\frac{c}{d}$ for $a,b,c,d \in \mathbb{Z}$ and $x,y \in \mathbb{Q}$, then $xy = \frac{ac}{bd}$, which is rational because ac and bd are integers by closure.

3.b The product of two irrational numbers is always irrational.

False. Counterexample: $\sqrt{2} * 2\sqrt{2} = 4$, which is rational.

3.c The product of two irrational numbers is always rational.

False. Counterexample: $\sqrt{2} * \sqrt{3} = \sqrt{6}$, which is irrational.

3.d The product of a non-zero rational and an irrational is always irrational.

True. Proof by contradiction: If $a = bc$ where a, b are rational and c is irrational, then $\frac{a}{b} = c$. This is a contradiction because $\frac{a}{b}$ is rational by closure but c is irrational.