Assignment 16: Integers

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1 For each of the following pairs a, b of integers, find the highest common factor $d = hef(a, b)$, and find integers s, t such that $d = sa + tb$:

1.a $a = 17, b = 29$

$$\begin{aligned}
29 &= (17) + 12 \\
17 &= 1(12) + 5 \\
12 &= 2(5) + 2 \\
5 &= 2(2) + 1 \\
2 &= 2(1) + 0
\end{aligned}$$

$hef(17, 29) = 1$

$$\begin{aligned}
1 &= 5 - 2(2) \\
1 &= 5 - 2(12 - 2(5)) \\
1 &= -2(12) + 5(5) \\
1 &= -2(12) + 5(17 - 12) \\
1 &= 5(17) - 7(12) \\
1 &= 5(17) - 7(29 - 17) \\
1 &= 12(17) - 7(29)
\end{aligned}$$

$s = 12, t = -7$

1.b $a = 552, b = 713$

$$\begin{aligned}
713 &= 1(552) + 161 \\
552 &= 3(161) + 69 \\
161 &= 2(69) + 23 \\
69 &= 3(23) + 0
\end{aligned}$$

$hef(552, 713) = 23$

$$
\begin{aligned}
23 &= 161 - 2(69) \\
23 &= 161 - 2(552 - 3(161)) \\
23 &= -2(552) + 7(161) \\
23 &= -2(552) + 7(713 - 552 \\
23 &= -9(552) + 7(713)
\end{aligned}
$$

$s = -9, t = 7$

1.c $a = 345, b = 299$

$$
\begin{aligned}
345 &= 1(299) + 46 \\
299 &= 6(46) + 23 \\
46 &= 2(23) + 0
\end{aligned}
$$

$hef(345, 288) = 23$

$$
\begin{aligned}
23 &= 299 - 6(46) \\
23 &= 299 - 6(345 - 299) \\
23 &= -6(345) + 7(299)
\end{aligned}
$$

$s = -6, t = 7$

4

4.a Show that for all positive integers n, $hcf(6n + 8, 4n + 5) = 1$.

$$
\begin{aligned}
6n + 8 &= 1(4n + 5) + (2n + 3) \\
4n + 5 &= 1(2n + 3) + (2n + 2) \\
2n + 3 &= 1(2n + 2) + 1
\end{aligned}
$$

Thus, $hcf(6n + 8, 4n + 5) = 1$

4.b Suppose a, b are integers such that $a|b$ and $b|a$. Prove that $a = \pm b$.

If $a|b$ then $b = ca, c \in \mathbb{Z}$. If $b|a$ then $a = db, d \in \mathbb{Z}$. Thus, $b = cdb$ so $cd = 1$. Since $c, d \in \mathbb{Z}$, either $c = 1, d = 1$ or $c = -1, d = -1$, so $a = \pm b$.

4.c Suppose s, t, a, b are integers such that $sa + tb = 1$. Show that $hcf(a, b) = 1$.

Any common factor of a and b would also be a factor of $sa + tb$, thus any common factor of a and b is also a factor of 1. The only positive factor of 1 is 1 itself, so $hcf(a, b) = 1$.

5

5.a Let m, n be coprime integers, and suppose a is an integer which is divisible by both m and n. Prove that mn divides a.

Since $m|a$ and $n|a$, $a = bm$ and $a = cn$ for some $b, c \in \mathbb{Z}$. Since m and n are coprime, $sm + tn = 1$ for some $s, t \in \mathbb{Z}$.
Multiplying $sm + tn = 1$ by b:

$$\begin{aligned} bsm + btn &= b \\ as + btn &= b \\ cns + btn &= b \\ (cs + bt)n &= b \end{aligned}$$

Therefore $n|b$, so $b = dn$ for some $d \in \mathbb{Z}$. Thus, $a = dnm$, which means $mn|a$.

5.b Show that the conclusion of part (a) is false if m and n are not coprime (i.e., show that if m and n are not coprime, there exists an integer a such that $m|a$ and $n|a$, but mn does not divide a).

Counterexample: If $m = 8, n = 4, a = 16$, $m|a$ because $8|16$ and $n|a$ because $4|16$, however mn does not divide a because $4 \cdot 8 = 32$ does not divide 16.

Generalized: If m and n are not coprime, then, if $d = hcf(m, n)$, $d > 1$. m and n can be rewritten as $m = dj$ and $n = dk$ for some $j, k \in \mathbb{Z}$. If $a = djk$, then $m|a$ and $n|a$, but mn will not divide a because $\frac{djk}{(dj)(dk)} = \frac{1}{d}$, which is not an integer since $d > 1$.

5.c Show that if $hcf(x, m) = 1$ and $hcf(y, m) = 1$, then $hcf(xy, m) = 1$.

If $hcf(x, m) = 1$ and $hcf(y, m) = 1$, then $sx + tm = 1$ and $ey + bm = 1$ for some $s, t, e, b \in \mathbb{Z}$.
Multiplying these equations:

$$\begin{aligned} sexy + sbxm + temy + tbm^2 &= 1 \\ sexy + (sbx + tey + tbm)m &= 1 \end{aligned}$$

Let $c = xy$ and $d = sbx + tey + tbm$, then $c(xy) + dm = 1$. Since c and d are both integers, this shows that $hcf(xy, m) = 1$.