Assignment 18: Prime Factorization

Declan Murphy Zink

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2.a Which positive integers have exactly three positive divisors?

Squared prime numbers have exactly three positive divisors. If the prime is a, then these divisors are 1, a, and a^2.

2.b Which positive integers have exactly four positive divisors?

Cubed prime numbers have exactly four positive divisors. If the prime is a, then these divisors are 1, a, a^2, and a^3.

Also products of two distinct prime numbers have exactly four positive divisors. If the primes are a and b, then these divisors are 1, a, b, and ab.

3 Suppose $n \geq 2$ is an integer with the property that whenever a prime p divides n, p^2 also divides n (i.e., all primes in the prime factorization of n appear at least to the power 2). Prove that n can be written as the product of a square and a cube.

Let $n = p_1^{s_1}...p_k^{s_k}, s_i \geq 2$. We can write s_i as $2u_i + 3v_i$ for integers $u_i, v_i \geq 0$ since $2u_i + 3v_i$ can produce all integers ≥ 2. (This can be proven because all even numbers ≥ 2 can be created with $2u_i$, all odd numbers ≥ 2 that are multiples of 3 can be created with $3v_i$, and all odd numbers ≥ 2 that are not multiples of 3 can be created with $2u_i + 3(1)$.) Thus $n = p_1^{2u_1+3v_1}...p_k^{2u_k+3v_k} = (p_1^{2u_1}...p_k^{2u_k})(p_1^{3v_1}...p_k^{3v_k}) = (p_1^{u_1}...p_k^{u_k})^2(p_1^{v_1}...p_k^{v_k})^3$, which is the product of a square and a cube.

4 Prove that $lcm(a,b) = \frac{ab}{hcf(a,b)}$ for any positive integers a, b without using prime factorization.

$\frac{ab}{hcf(a,b)}$ is a multiple of a and b because $\frac{ab}{hcf(a,b)} = a(\frac{b}{hcf(a,b)}) = b(\frac{a}{hcf(a,b)})$ and $\frac{b}{hcf(a,b)}$ and $\frac{a}{hcf(a,b)}$ are both integers.

Let u be a multiple of a and b such that $u = \frac{ab}{s}, s \in \mathbb{Z}$. Thus, $\frac{u}{a} = \frac{b}{s} \Rightarrow s|b$ since $\frac{u}{a}$ is an integer. Likewise $\frac{u}{b} = \frac{a}{s} \Rightarrow s|a$. Because $s|a$ and $s|b$, it must be true that $s|hcf(a,b)$.

Thus, the highest value possible for $s = hcf(a, b)$, so the lowest possible value for $u = \frac{ab}{hcf(a,b)}$. Hence, $lcm(a, b) = \frac{ab}{hcf(a,b)}$.