Functions 1

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1 For each of the following functions f, say whether f is 1-1 and whether f is onto:

1.a f:$\mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 2x$ for all $x \in \mathbb{R}$.

$f(x) = (x+1)^2 - 1 \Rightarrow$ Neither 1-1 or onto. Image only spans $[-1, \infty)$, and $f(-2) = f(0)$.

1.b f:$\mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x-2 & x > 1 \\ -x & -1 \leq x \leq 1 \\ x+2 & x < -1 \end{cases}$$

Onto, but not 1-1. Image spans $(-\infty, \infty) \Rightarrow \mathbb{R}$, but $f(0) = f(2)$.

1.c f:$\mathbb{Q} \to \mathbb{R}$ defined by $f(x) = (x + \sqrt{2})^2$.

1-1 because in order for f(a) = f(b) for two rational nubmers a and b, either $a = b$ or $a + b = -2\sqrt{2}$:

$$(a+\sqrt{2})^2 = (b+\sqrt{2})^2$$
$$a+\sqrt{2} = b+\sqrt{2} \; or \; a+\sqrt{2} = -b-\sqrt{2}$$
$$a = b \; or \; a+b = -2\sqrt{2}$$

$a + b \neq -2\sqrt{2}$ becuase they are rational, so it is 1-1.
Not onto because the image only spans $[0, \infty)$.

1.d $f : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $f(m,n,r) = 2^m 3^n 5^r$ for all $m,n,r \in \mathbb{N}$.

1-1 because 2,3, and 5 are prime factors. Not onto because Not all natural numbers can be created, such as 7.

1.e $f : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $f(m,n,r) = 2^m 3^n 6^r$ for all $m,n,r \in \mathbb{N}$.

Not 1-1 because $f(1,1,2) = f(2,2,1)$, $(2^m 3^n 6^r = 2^m 3^n 2^r 3^r)$. Not onto because Not all natural numbers can be created, such as 7.

1.f Let $\sim$ be the equivalence relation on $\mathbb{Z}$ defined by $a \sim b \Leftrightarrow a \equiv b$ mod 7, and let S be the set of equivalence classes of $\sim$. Define $f : S \to S$ by $f(cl(s)) = cl(s+1)$ for all $s \in \mathbb{Z}$.

$cl(0) \to cl(1) \to cl(2) \to cl(3) \to cl(4) \to cl(5) \to cl(6) \to cl(0)$, so both onto and 1-1.

3 Two functions $f, g : \mathbb{R} \to \mathbb{R}$ are such that for all $x \in \mathbb{R}$, $g(x) = x^2 + x + 3$, and $(g \circ f)(x) = x^2 - 3x + 5$. Find the possibilities for f.

$f(x) = y \Rightarrow y^2 + y + 3 = x^2 - 3x + 5 \Rightarrow (y)(y+1) = (x-1)(x-2)$. After graphing, $y = 1 - x$ or $x - 2$, so $f(x) \in \{1 - x, x - 2\}$.

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6.a Find an onto function from $\mathbb{N}$ to $\mathbb{Z}$.

$$f(x) = \begin{cases} -\frac{x-1}{2} & x \equiv 1 \; mod \; 2 \\ \frac{x}{2} & x \equiv 0 \; mod \; 2 \end{cases}$$

Or: $f(x) = (-1)^x (\frac{x-(x\%2)}{2})$, but I'm not sure if the modulus operator is allowed in pure math

6.b Find a 1-1 function from $\mathbb{Z}$ to $\mathbb{N}$.

$f(x) = x^2 + x + |x|$

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7.a Let $S = \{1, 2, 3\}$ and $T = \{1, 2, 3, 4, 5\}$. How many functions are there from S to T? How many of these are 1-1?

$5^3 = 125$ functions total.
$5 * 4 * 3 = 60$ 1-1 functions.

7.b Let $|S| = m, |T| = n$ with $m \leq n$. Show that the number of 1-1 functions from S to T is equal to $n(n-1)(n-2)...(n-m+1)$.

In a 1-1 function, each $s \in S$ must map to one $t \in T$. Thus, there are n choices for s_1, n-1 choices for s_2, and so on, continuing until the last s (s_m), which will have n-m+1 choices. Thus the total 1-1 functions is $n(n-1)(n-2)...(n-m+1)$.