Assignment 10: Complex Numbers 2

Declan Murphy Zink

10/6/2020

6 Express $\frac{1+i}{\sqrt{3}+i}$ in the form $x+iy$, where $x, y \in \mathbb{R}$. By writing each of $1+i$ and $\sqrt{3}+i$ in polar from, deduce that

$$\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}},\ \sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

Part 1:

$$\frac{1+i}{\sqrt{3}+i} = \frac{\sqrt{2}e^{\frac{\pi}{4}i}}{2e^{\frac{\pi}{6}i}} = \frac{\sqrt{2}}{2}e^{\frac{\pi}{12}i} = \frac{\sqrt{2}}{2}(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12})$$

Part 2:

$$\frac{1+i}{\sqrt{3}+i} \cdot \frac{\sqrt{3}-i}{\sqrt{3}-i} = \frac{\sqrt{2}}{2}(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12})$$

$$\frac{\sqrt{3}+\sqrt{3}i-i+1}{3+1} \cdot \frac{2}{\sqrt{2}} = \cos\frac{\pi}{12} + i\sin\frac{\pi}{12}$$

$$\frac{\sqrt{3}+1}{2\sqrt{2}} + i\frac{\sqrt{3}-1}{2\sqrt{2}} = \cos\frac{\pi}{12} + i\sin\frac{\pi}{12}$$

$$\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}},\ \sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

7

7.a Show that $x^5 - 1 = (x-1)(x^4+x^3+x^2+x+1)$. Deduce that if $\omega = e^{\frac{2\pi i}{5}}$ then $\omega^4+\omega^3+\omega^2+\omega+1 = 0$.

Part 1:

$$(x-1)(x^4+x^3+x^2+x+1) = x^5+x^4+x^3+x^2+x-x^4-x^3-x^2-x-1 = x^5-1$$

Part 2:
$\omega^5 = 1$, so $\omega^5 - 1 = 0 = (\omega-1)(\omega^4+\omega^3+\omega^2+\omega+1)$. Thus, since $\omega \neq 1$, $\omega^4+\omega^3+\omega^2+\omega+1 = 0$.

7.b Let $\alpha = 2\cos\frac{2\pi}{5}$ and $\beta = 2\cos\frac{4\pi}{5}$. Show that $\alpha = \omega + \omega^4$ and $\beta = \omega^2 + \omega^3$. Find a quadratic equation with roots α, β. Hence show that

$$\cos\frac{2\pi}{5} = \frac{1}{4}(\sqrt{5} - 1)$$

$\omega^4 = e^{\frac{8\pi i}{5}} = e^{\frac{-2\pi i}{5}} = \bar{\omega}$ and $\omega + \bar{\omega} = 2\cos\frac{2\pi}{5}$, so $\omega + \omega^4 = \alpha$.
$\omega^3 = e^{\frac{6\pi i}{5}} = e^{\frac{-4\pi i}{5}} = (\bar{\omega^2})$ and $\omega^2 + (\bar{\omega^2}) = 2\cos\frac{4\pi}{5}$, so $\omega^2 + \omega^3 = \beta$.

Quadratic equation: $(x-\alpha)(x-\beta) = x^2 - x(\alpha+\beta) + \alpha\beta = 0$.

- $\alpha + \beta = \omega^4 + \omega^3 + \omega^2 + \omega = -1$
- $\alpha\beta = (\omega + \omega^4)(\omega^2 + \omega^3) = \omega^3 + \omega^4 + \omega^6 + \omega^7 = \omega^4 + \omega^3 + \omega^2 + \omega = -1$

So, α, β are the roots of $x^2 + x - 1 = 0$, which, with the quadratic equation, has roots $\frac{1}{2}(-1 \pm \sqrt{5})$. Since $\alpha = 2\cos\frac{2\pi}{5} > 0$, $\cos\frac{2\pi}{5} = \frac{1}{4}(-1 + \sqrt{5})$.

8 Find a formula for $\cos 4\theta$ in terms of $\cos\theta$. Hence write down a quartic equation (i.e., an equation of degree 4) that has $\cos\frac{\pi}{12}$ as a root. What are the other roots of your equation?

Part 1:

$$\cos 4\theta + i\sin 4\theta = (\cos\theta + i\sin\theta)^4$$

Let $c = \cos\theta, s = \sin\theta$

$$\cos 4\theta + i\sin 4\theta = c^4 - 6c^2s^2 + s^4 + i(4c^3s - 4cs^3)$$

$$\cos 4\theta = c^4 - 6c^2s^2 + s^4$$

$$\cos 4\theta = c^4 - 6c^2(1-c^2) + (1-c^2)^2$$

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$$

Part 2:

$$8(\cos\frac{\pi}{12})^4 - 8(\cos\frac{\pi}{12})^2 + 1 = \cos(4\cdot\frac{\pi}{12}) = \frac{1}{2}$$

$$16(\cos\frac{\pi}{12})^4 - 16(\cos\frac{\pi}{12})^2 + 2 = 1$$

So a quartic equation is $16x^4 - 16x^2 + 1 = 0$.

Part 3:
If $\cos(\alpha)$ is a root, then $\cos 4\alpha = \frac{1}{2}$, so $\alpha = \frac{\pi}{12},\ \frac{\pi}{12} + \frac{\pi}{2},\ \frac{\pi}{12} + \pi,\ \frac{\pi}{12} + \frac{3\pi}{2}$. So the roots of the quartic equation are $\cos\frac{\pi}{12},\ \cos(\frac{\pi}{12} + \frac{\pi}{2}),\ \cos(\frac{\pi}{12} + \pi),\ \cos(\frac{\pi}{12} + \frac{3\pi}{2})$. These can also be written as $\cos\frac{\pi}{12},\ -\cos\frac{\pi}{12},\ \sin\frac{\pi}{12},\ -\sin\frac{\pi}{12}$.