Assignment 17: Integers 2

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6 **Let $a, b, c \in \mathbb{Z}$. Define the highest common factor $hcf(a, b, c)$ to be the largest positive integer that divides a, b and c. Prove that there are integers s, t, u such that**

$$hcf(a, b, c) = sa + tb + uc.$$

Find such integers s, t, u when $a = 91, b = 903, c = 1792$.

$hcf(a, b, c)$ can be rewritten as $hcf(hcf(a, b), c)$. We know $hcf(a, b) = xa + yb$ for some integers x and y, so $hcf(a, b, c) = hcf(xa + yb, c) = w(xa + yb) + uc = wxa + wyb + uc$ for some integers w and u. We can define $s = wx$ and $t = wy$. Thus $hcf(a, b, c) = sa+tb+uc$.

$hcf(91, 903, 1792) = hcf(hcf(91, 903), 1792)$.
Find $hcf(91, 903)$:

$$\begin{aligned} 903 &= 9(91) + 84 \\ 91 &= 1(84) + 7 \\ 84 &= 12(7) + 0 \end{aligned}$$

$hcf(91, 903) = 7$
Find $hcf(7, 1792)$:

$$1792 = 256(7) + 0$$

$hcf(7, 1792) = 7$
Now find x and y for $hcf(91, 903) = 91x + 903y$:

$$\begin{aligned} 7 &= 91 - 84 \\ 7 &= 91 - (903 - 9(91)) \\ 7 &= 10(91) - 903 \end{aligned}$$

So $hcf(91, 903) = 10(91) - 903y$.
Since $hcf(91, 903) = hcf(91, 903, 1792)$, $hcf(91, 903, 1792) = 10(91) - 1(903) + 0(1792)$.
Thus $s = 10, t = -1, u = 0$.

9 Let a, b be coprime positive integers. Prove that for any integer n there exist integers s, t with $s > 0$ such that $sa + tb = n$.

If a and b are coprime, there exist integers c and d such that $ca + db = 1$. Mutliplying by n, this means $nca + ndb = n$. Define $s = nc$ and $t = nd$. Thus $sa + tb = n$.