Coordination in Social Networks

Communication by Actions

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MOTIVATION

- The relevant information in making joint decision is dispersed in the society. (Hayek 1945)
- If so, how people act collectively?

This paper shows

- In a long-term relationship, people can aggregate such information and coordinate their actions.

What this paper does?

- I model a repeated game with incomplete information and network-monitoring with discount factor.
  - Players can only observe own/neighbors' types and own/neighbors' actions.
- Look for an equilibrium in which the pay-off relevant information become commonly known in finite time.
- Such equilibrium can be constructed under some assumptions.

Time line

1. There is a fixed, finite, connected, undirected, and commonly known network.
2. Players of two types— S or B —chosen by nature according to a probability distribution.
    - S: Strategic type; B: Behavior type
3. Types are then fixed over time.
4. Players play a stage game— a collective action —infinitely repeatedly with common discount factor.

What player can/cannot observe

- Players can observe own/neighbors' types and actions, but not others'.
- Pay-off is hidden.
    - [Aumann and Maschiler 1990], [Miyahara and Sekiguchi 2013], [Wolitzky 2013], etc.

Model

- Stage game—*k*-threshold game: a protest ([Chwe 2000])
    - S-type's action set$= \{\mathbf{p}, \mathbf{n}\}$
    - B-type's action set$= \{\mathbf{n}\}$
    - Pay-offs for S-type:

$$
\begin{aligned}
u_{S_i}(a_{S_i}, a_{-\theta_i}) &= 1 \quad \text{if } a_{S_i} = \mathbf{p} \text{ and } \#\{j : a_{\theta_j} = \mathbf{p}\} \geq k \\
u_{S_i}(a_{S_i}, a_{-\theta_i}) &= -1 \quad \text{if } a_{S_i} = \mathbf{p} \text{ and } \#\{j : a_{\theta_j} = \mathbf{p}\} < k \\
u_{S_i}(a_{S_i}, a_{-\theta_i}) &= 0 \quad \text{if } a_{S_i} = \mathbf{n}
\end{aligned}
$$

Static ex-post Pareto efficient outcome

Type profile	Static ex-post efficient outcome
At least k S-types exist	All S-types play **p**
Otherwise	All S-types play **n**

Equilibrium concept

- WPBE (weak perfect Bayesian equilibrium)
- Sequential equilibrium

APEX (*approaching ex-post efficient*) equilibrium

DEFINITION (APEX STRATEGY)

An equilibrium is APEX $\Leftrightarrow$

$$\forall\theta, \text{ there is a finite time } T^{\theta}$$

such that the actions in the equilibrium path repeats the static ex-post efficient outcome after T^{θ}.

Theorem ($k = n$)

If $k = n$, then an APEX sequential equilibrium exists whenever discount factor is sufficiently high.

DEFINITION FOR APEX FOR $k < n$

DEFINITION

θ has **strong connectedness**$\Leftrightarrow$ for every pair of S-types, there is a path consisting of S-types to connect them.

DEFINITION

π has **full support on strong connectedness**$\Leftrightarrow$

$$\pi(\theta) > 0 \textbf{ if and only if } \theta \text{ has strong connectedness.}$$

Let k=2 and n=3

- A B-type will not reveal information.
- **Without** full support on strong connectedness, in general, an APEX equilibrium does not exist when pay-off is hidden.

Result 2: APEX for $k < n$

Theorem ($k < n$)

If $k < n$, then if network is a tree, if prior π has full support on strong connectedness, then an APEX WPBE exists whenever discount factor is sufficiently high.

Outline for Equilibrium Construction

1. APEX sequential equilibrium for $k = n$.
   - An example.
   - Sketch of proof.
2. APEX WPBE for $k < n$.
   1. Consider cheap talk.
   2. Consider “costly” talk.
   3. Sketch of proof.

An example for $k = n$

Let $k = n = 3$, when discount factor is high enough, an APEX sequential equilibrium can be constructed by

- Period 1
  - S-type 2: chooses **n** if $\theta \neq (S, S, S)$, and then choose **n** forever;
  - S-type 2: chooses **p** if $\theta = (S, S, S)$.
  - S-type 1, 3: chooses **p**.
- Period 2
  - If S-type 2 chooses **n** in the last period $\Rightarrow$ S-type 1 (or S-type 3) chooses **n** forever.
  - If S-type 2 chooses **p** in the last period $\Rightarrow$ S-type 1 (or S-type 3) chooses **p** forever;
- Any deviation $\Rightarrow$ Choosing **n** forever.

AN EXAMPLE FOR $k = n$

Main features in equilibrium construction in this example

- The 1st-period actions serve as “messages” to reveal the relevant information.
- The 2nd-period is a commonly known “timing” to coordinate (i.e. a part of equilibrium strategy).
- Playing **n** forever serves as a “grim trigger”.

EQUILIBRIUM CONSTRUCTION FOR $k = n$

Sketch of proof:

1. "messages" to reveal the relevant information.
    - Some B-types neighbors $\Rightarrow$ play **n** forever.
    - No B-type neighbor $\Rightarrow$ play **p** unless **n** is observed, and then play **n** forever.

Equilibrium construction for $k = n$

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   - Some B-types neighbors $\Rightarrow$ play **n** forever.
   - No B-type neighbor $\Rightarrow$ play **p** unless **n** is observed, and then play **n** forever.
2. "Timing" to coordinate.
   - Finite network $\Rightarrow$ there is a finite time $T(= n)$ such that players coordinate to the static ex-post efficient outcome.

Equilibrium construction for $k = n$

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2. "Timing" to coordinate.
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3. Any deviation $\Rightarrow$ play "**n** forever".

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Equilibrium construction for $k = n$

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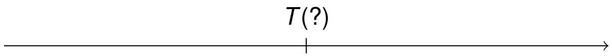
3. Any deviation $\Rightarrow$ play "**n** forever".
4. Let discount factor be sufficiently high to impede deviation.
5. A belief system for sequential equilibrium can be chosen.

Equilibrium construction for $k < n$



- Challenges:
  - Only two actions—$\{\mathbf{n}, \mathbf{p}\}$— used for transmit relevant information.
  - How to find that finite time "T" for every state?
  - Group punishment is hard to be made. (Network-monitoring)

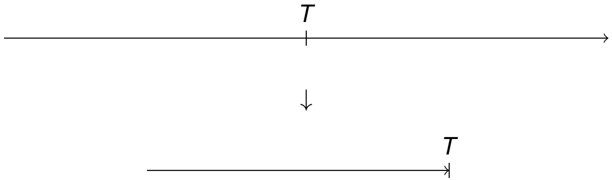
EQUILIBRIUM CONSTRUCTION FOR $k < n$

To solve the challenge:

1. We first consider fixed T.
2. Then allow indeterministic T.

EQUILIBRIUM CONSTRUCTION FOR $k < n$

Assume T is fixed, commonly known, and independent from states.



- By definition of APEX,
  - After T, actions are infinitely repeated and thus information can not be updated.
- Idea:
  - Suppose players can transmit information by “talking” within T rounds and then play a one-shot game.
    1. Consider an augmented T-round “cheap talk” phase.
    2. Consider an augmented T-round “costly talk” phase.

Time line

- Nature choose θ according to π.
- Types are then fixed over time.
- At the first T rounds, players play T-round cheap talk.
- At $T+1$ round, players play a one-shot k-Threshold game.
- Game ends.

- T is a big number.
- A "letter-writing technology" for player i:
  - A set of sentences: $W = \{\mathbf{n}, \mathbf{p}\}^L$, where L is a big number.
  - A fixed grammar M for each round:

$$M_i^1 = \{f | f : \Theta_{G_i} \to W\} \cup \{\emptyset\}$$

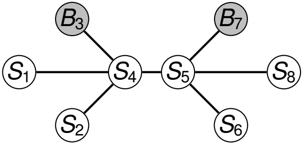
$$\text{for } 2 \leq t \leq T,\ M_i^t = \{f | f : \prod_{j \in G_i} M_j^{t-1} \to W\} \cup \{\emptyset\}$$

  - i's neighbors can observe what i write for each round.

k-THRESHOLD GAME AUGMENTED BY T-ROUND CHEAP TALK

Example of a WPBE construction:

- $k = 6$, $n = 8$ and $T = 2$.
- G and θ=



- Equilibrium path
  - At $t = 1$,

S-type 4	$(\overbrace{\mathbf{p}, \mathbf{p}, \mathbf{n}, \mathbf{p}, \mathbf{p}, \mathbf{n}, \mathbf{n}, \mathbf{n}}^{L=8})$
S-type 5	$(\overbrace{\mathbf{n}, \mathbf{n}, \mathbf{n}, \mathbf{p}, \mathbf{p}, \mathbf{p}, \mathbf{n}, \mathbf{p}}^{L=8})$
S-type 1,2,6,8	$\emptyset$

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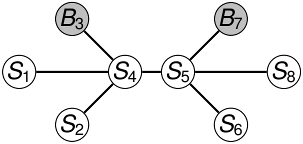

- Equilibrium path
    - At $t = T = 2$,

S-type 4	$(\overbrace{\mathbf{p}, \mathbf{p}, \mathbf{n}, \mathbf{p}, \mathbf{p}, \mathbf{p}, \mathbf{n}, \mathbf{p}}^{L=8})$
S-type 5	$(\overbrace{\mathbf{p}, \mathbf{p}, \mathbf{n}, \mathbf{p}, \mathbf{p}, \mathbf{p}, \mathbf{n}, \mathbf{p}}^{L=8})$
S-type 1,2,6,8	$\emptyset$

    - At $t = 3$, all S-types play **p**, then game ends.

- Off-path strategy
  - If a player make a detectable deviation (e.g. wrong sentence) $\Rightarrow$ the neighbor who detects that deviation plays **n** and then **n**.
- Off-path belief
  - If a player observes a detectable deviation $\Rightarrow$ he believes that all players outside neighborhood are B-types.

If there is a fixed cost ϵ to send the letter...

- Off-path strategy
  - If a player make a detectable deviation (e.g. wrong sentence, playing $\emptyset$) $\Rightarrow$ the neighbor who detects that deviation plays $\emptyset$ and then **n**.
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If there is a fixed cost ϵ to send the letter...

- Off-path strategy
  - If a player make a detectable deviation (e.g. wrong sentence, playing $\emptyset$) $\Rightarrow$ the neighbor who detects that deviation plays $\emptyset$ and then **n**.
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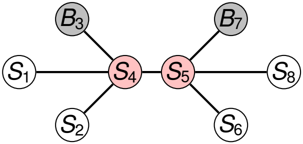
So, when ϵ is small enough and T is large enough, a WPBE can be constructed when ϵ is independent from messages.

However, if ϵ is not independent from messages, then a Free Rider Problem may occur.

- Suppose ϵ is decreasing when announcing more S-types in the round $t = 1$.
- $k = 6$, $n = 8$ and $T = 2$.
- G and θ=



1. S-type 4 and S-type 5 will deviate from truthfully announcement.
2. They will report more S-types to save costs in round $t = 1$ and wait for each others' truthfully announcement (Free Rider Problem).
3. Note that this deviation is not detectable.

Free rider problem is defined as

- There are multiple neighboring players who can assure that they can learn the relevant information given others' truthful announcement by knowing the network structure.

A way to solve Free Rider Problem:

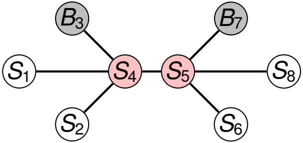
- Let some of them be free rider, while letting others report truthfully.

k-Threshold game augmented by T-round costly talk

Free Rider Problem

Example of solving Free Rider Problem (Suppose ϵ is decreasing when announcing more S-types in the 1^{st} round; After 1^{st} round, ϵ is fixed):

- $k = 6$, $n = 8$ and let $T = 3$.
- G and θ=



- Equilibrium path
  - At $t = 1$, let S-type 4 be free rider.

S-type 4 $\quad (\overbrace{\mathbf{p}, \mathbf{p}, \mathbf{p}, \mathbf{p}, \mathbf{p}, \mathbf{p}, \mathbf{p}, \mathbf{p}}^{L=8})$

S-type 5 $\quad (\overbrace{\mathbf{n}, \mathbf{n}, \mathbf{n}, \mathbf{p}, \mathbf{p}, \mathbf{p}, \mathbf{n}, \mathbf{p}}^{L=8})$

S-type 1,2,6,8 $\quad \emptyset$

Example of solving Free Rider Problem (Suppose ϵ is decreasing when announcing more S-types in the 1^{st} round; After 1^{st} round, ϵ is fixed):

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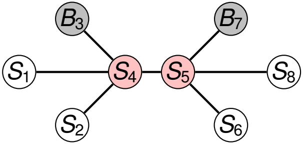
- G and θ=



- Equilibrium path
  - At $t = 2$,

S-type 4	$(\underbrace{\mathbf{p}, \mathbf{p}, \mathbf{p}, \mathbf{p}, \mathbf{p}, \mathbf{p}, \mathbf{p}, \mathbf{p}}^{L=8})$
S-type 5	$\emptyset$
S-type 1,2,6,8	$\emptyset$

Example of solving Free Rider Problem (Suppose ϵ is decreasing when announcing more S-types in the 1^{st} round; After 1^{st} round, ϵ is fixed):

- $k = 6$, $n = 8$ and let $T = 3$.
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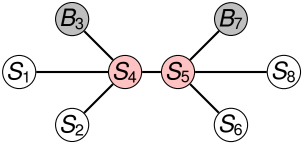

- Equilibrium path
  - At $t = T = 3$,

S-type 4	$\emptyset$
S-type 5	$(\overbrace{\mathbf{p}, \mathbf{p}, \mathbf{p}, \mathbf{p}, \mathbf{p}, \mathbf{p}, \mathbf{p}, \mathbf{p}}^{L=8})$
S-type 1,2,6,8	$\emptyset$

Example of solving Free Rider Problem (Suppose ϵ is decreasing when announcing more S-types in the 1^{st} round; After 1^{st} round, ϵ is fixed):

- $k = 6$, $n = 8$ and let $T = 3$.
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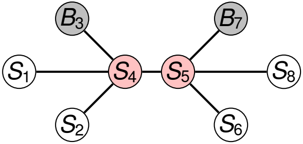


- Equilibrium path
  - At $t = T = 3$,

S-type 4	$\emptyset$
S-type 5	$(\overbrace{\mathbf{p}, \mathbf{p}, \mathbf{p}, \mathbf{p}, \mathbf{p}, \mathbf{p}, \mathbf{p}, \mathbf{p}}^{L=8})$
S-type 1,2,6,8	$\emptyset$

  - At $t = 4$, all S-types play **p**, then game ends.

LEMMA

If the network is a tree, then the Free Rider Problem only occurs between two players. Moreover, they are neighbors and will commonly know that they will be in this problem.

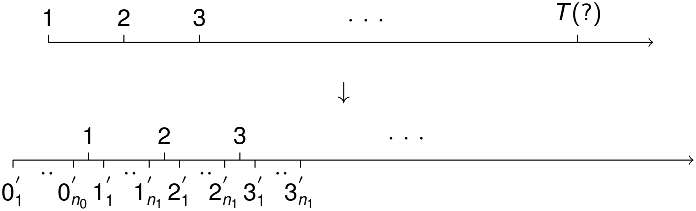
- Due to this, these two players will know that one of them has been picked as the free rider.

From T-round costly talk to indeterministic T-round costly talk

- Let's keep the assumption that ϵ is decreasing when announcing more S-types.
- Then we allow T to be indeterministic.

k-THRESHOLD GAME AUGMENTED BY INDETERMINISTIC T-ROUND

COSTLY TALK



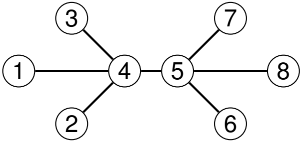
- There are countably infinite rounds. Rounds are linear ordered and labeled as

$$0_1^{'} < ... < 0_{l_0}^{'} < 1 < 1_1^{'} < ... < 1_{l_1}^{'} < 2 < ...$$

- This labeling is common known.
- $\Gamma = \{1, 2, 3, ...,\}$ is the set of rounds for announcing types.
- $\Gamma^{'} = \{0_1^{'}, ..., 0_{l_0}^{'}, 1_1^{'}, ..., 1_{l_1}^{'}, ...\}$ is the set of rounds to express the belief about whether or not this round is the terminal round T.

We will consider two examples given the following structure.

- $k = 6$, $n = 8$
- G=

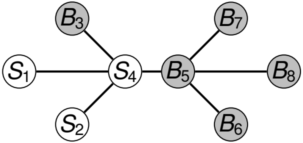


- The rounds be ordered and labeled as

$$0_1^{'} < 0_2^{'} < 1 < 1_1^{'} < 1_2^{'} < 2 < ...$$

- Also let us allow the lengths of letters could be different in different rounds.

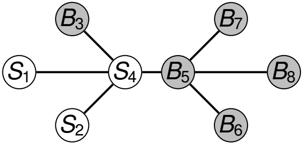
- Example 1
- $k = 6$, $n = 8$
- G and θ=



- Equilibrium path
  - At $t = 0'_1$, write $\emptyset$ if a player thinks [*a successful protest is impossible, and so that this round is the terminal round*]; write **n** if a player thinks [*a successful protest is still possible, and this round is not the terminal round*].

S-type 4	$\emptyset$
S-type 1,2	(**n**)

- Example 1
- $k = 6$, $n = 8$
- G and θ=

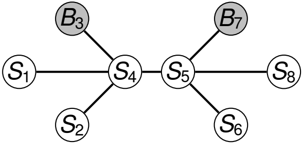


- Equilibrium path
  - At $t > 0'_1$, since all S-types have commonly known that [*a successful protest is impossible and* $0'_1$ *round is the terminal round*],

S-type 4 (**n**)

S-type 1,2 (**n**)

- Example 2
- $k = 6$, $n = 8$
- G and θ=



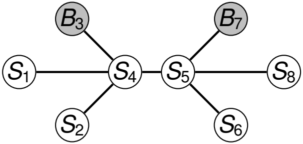
- Equilibrium path
  - At $t = 0'_1$, write (**n**) if a player thinks [*a successful protest is still possible and this round is not the terminal round*]

S-type 4	(**n**)
S-type 5	(**n**)
S-type 1,2,6,8	(**n**)

k-Threshold game augmented by indeterministic T-round

Costly talk

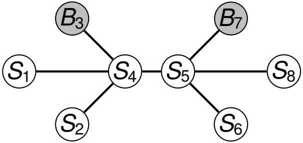
- Example 2
- $k = 6$, $n = 8$
- G and θ=



- Equilibrium path
  - At $t = 0'_2$, write (**n**) if a player thinks all his neighbors think [*a successful protest is still possible and* $0'_1$ *is not the terminal round*].

S-type 4	(**n**)
S-type 5	(**n**)
S-type 1,2,6,8	(**n**)

- Example 2
- $k = 6$, $n = 8$
- G and θ=

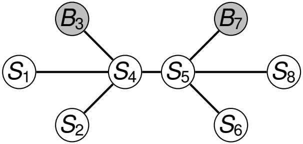


- Equilibrium path
  - At $t = 1$,

S-type 4	$(\overbrace{\mathbf{p}, \mathbf{p}, \mathbf{p}, \mathbf{p}, \mathbf{p}, \mathbf{p}, \mathbf{p}, \mathbf{p}}^{L=8})$
S-type 5	$(\overbrace{\mathbf{n}, \mathbf{n}, \mathbf{n}, \mathbf{p}, \mathbf{p}, \mathbf{p}, \mathbf{n}, \mathbf{p}}^{L=8})$
S-type 1,2,6,8	$\emptyset$

k-THRESHOLD GAME AUGMENTED BY INDETERMINISTIC T-ROUND COSTLY TALK

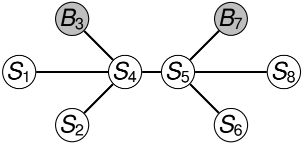
- Example 2
- $k = 6$, $n = 8$
- G and θ=



- Equilibrium path
  - At $t = 1'_1$, write (**p**) if a player thinks [*a successful protest can be made and this round is the terminal round*]; write (**n**) if a player thinks [*a successful protest is still possible, and this round is not the terminal round*].

S-type 4	(**p**)
S-type 5	(**n**)
S-type 1,2,6,8	(**n**)

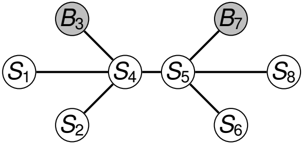
- Example 2
- $k = 6$, $n = 8$
- G and θ=



- Equilibrium path
  - At $t = 1_1'$, write (**p**) if a player thinks all his neighbors think [*a successful protest can be made and round* $1_1'$ *is the terminal round*]. write (**n**) if a player thinks [*a successful protest is still possible, and this round is not the terminal round*].

S-type 4	(**p**)
S-type 5	(**p**)
S-type 1,2,6,8	(**n**)

- Example 2
- $k = 6$, $n = 8$
- G and θ=



- Equilibrium path
  - At $t > 1'_2$, since all S-types commonly known that [*a successful protest can be made and* $1'_1$ *round is the terminal round*],

S-type 4	(**p**)
S-type 5	(**p**)
S-type 1,2,6,8	(**p**)

Finally, we can take our repeated game as an analogue of indeterministic T-round costly talk in the following sense.

augmented indeterministic T-round costly talk	repeated game
a round	a range of periods
the length of a sentence in a round	the length of a range of periods
a chosen digit in a sentence	a chosen action
the cost of writing a sentence	the expected payoff in a range of perio
the fixed grammar	the equilibrium path

More relevant details:

the sentence	the sequence of actions
$\emptyset$	$< \mathbf{n}, ..., \mathbf{n} >$
the sentence in announcing $i, ..., j$ are S-types	$< \mathbf{n}, ..., \underbrace{\mathbf{p}, ..., \mathbf{n}}_{z_i \times \ldots \times z_j} >$, $z_i, ..., z_j$ are distinct prime numbers
the least costly sentence used for solving free rider problem	$< \mathbf{n}, ..., \mathbf{n}, \mathbf{p} >$

THEOREM ($k < n$)

If $k < n$, then if network is a tree, if prior π has full support on strong connectedness, then an APEX WPBE exists whenever discount factor is sufficiently high.

Sketch of proof:

1. The Free Rider Problem may exist in tree networks, but it can be solved.
2. Detectable deviation $\Rightarrow$ playing **n** forever.
   - To rationalize it, the player who detects deviations will believe all players outside his neighborhood are B-types.
3. Undetectable deviation $\Rightarrow$ facing a possibility of coordination failure.
4. Any deviation will let APEX fail with positive probability.
5. APEX outcome gives maximum ex-post continuation pay-off after T.
6. Sufficiently high discount factor will impede deviation.

Further works

1. Look for a general model such that a finite-time communication protocol exists and this protocol can be extended to an equilibrium.