Coordination in Social Networks

Chun-Ting Chen

December 31, 2014

Motivation

- An exogenous social network models **restricted information**

[Chwe] models	**incomplete information**
[Wolitzky] models	**network-monitoring**

MOTIVATION

- An exogenous social network models **restricted information** in repeated collective action

  incomplete information

  network-monitoring

- Will people solve the uncertainty and act collectively in networks eventually?

MOTIVATION

- An exogenous social network models **restricted information** in repeated collective action

  incomplete information

  network-monitoring

- Will people solve the uncertainty and act collectively in networks eventually?
- This paper provides a partial folk theorem with incomplete information and network-monitoring.

WHAT THIS PAPER DOES?

Model: repeated game of private provision of public good

- Players are allocated in a fixed and exogenous network.

What this paper does?

Model: repeated game of private provision of public good

- Players are allocated in a fixed and exogenous network.
- Time line
  - Nature choose a type distribution
  - Types are then fixed over time
  - Players play a “public good provision game” infinitely repeatedly with common discount factor.

Stage game: Features

- Players of two types: **Strategic type/Behavior type**

Strategic type	**provide/not to provide**
Behavior type	**not to provide**

Stage game: Features

- Players of two types: **Strategic type/Behavior type**

Strategic type	**provide/not to provide**
Behavior type	**not to provide**

- Strategic type's stage pay-off u:

$$u(\textit{own action}, \mathbb{1}(\textit{sufficient provision of public good}))$$

What this paper does?

Network-information-structure:

own/neighbors' types is perfectly observable

own/neighbors' actions is perfectly observable

Goal: looking for an equilibrium, in which the global type distribution becomes commonly known in finite time.

Result: such equilibrium can be constructed under some assumptions.

Model

- A fixed and finite network
  - n players; $N = \{1, ..., n\}$ is the set of players.
  - G_i is i's neighborhood; G_i is a subset of N such that $i \in G_i$.
  - $G = \{G_i\}_i$ is the network.
- Players of two types
  - Player i's type: $\theta_i \in \Theta_i = \{S, B\}$.
  - Type-contingent action set: $A_S = \{\mathbf{p}, \mathbf{np}\}$; $A_B = \{\mathbf{np}\}$
  - Type profile: $\theta \in \Theta = \times_{i \in N} \Theta_i$

Model

Stage game: *k*-threshold game

- Stage game payoff for S-type *i*: $u_{S_i}(a_{S_i}, a_{-\theta_i})$

$$
\begin{aligned}
u_{S_i}(a_{S_i}, a_{-\theta_i}) &= 1 && \text{if } a_{S_i} = \mathbf{p} \text{ and } \#\{j : a_{\theta_j} = \mathbf{p}\} \geq k \\
u_{S_i}(a_{S_i}, a_{-\theta_i}) &= \text{-1} && \text{if } a_{S_i} = \mathbf{p} \text{ and } \#\{j : a_{\theta_j} = \mathbf{p}\} < k \\
\\
u_{S_i}(a_{S_i}, a_{-\theta_i}) &= 0 && \text{if } a_{S_i} = \mathbf{np}
\end{aligned}
$$

Model

Stage game: *k*-threshold game

- Stage game payoff for S-type *i*: $u_{S_i}(a_{S_i}, a_{-\theta_i})$

$$
\begin{aligned}
u_{S_i}(a_{S_i}, a_{-\theta_i}) &= 1 \quad \text{if } a_{S_i} = \mathbf{p} \text{ and } \#\{j : a_{\theta_j} = \mathbf{p}\} \geq k \\
u_{S_i}(a_{S_i}, a_{-\theta_i}) &= \text{-1} \quad \text{if } a_{S_i} = \mathbf{p} \text{ and } \#\{j : a_{\theta_j} = \mathbf{p}\} < k \\
u_{S_i}(a_{S_i}, a_{-\theta_i}) &= 0 \quad \text{if } a_{S_i} = \mathbf{np}
\end{aligned}
$$

- Ex-post efficient outcome:

relevant information	ex-post efficient outcome
At least *k* S-types exist	All S-types play **p**
Otherwise	All S-types play **np**

Assumptions:

- Network G is commonly known, connected, and undirected.
- A common prior: $\pi \in \Delta\Theta$
- A common discount factor: $\delta \in (0, 1)$.
- Players perfectly observe their neighbors' types.
- Players perfectly observe their neighbors' actions.

Goal

Look for

- **An equilibrium, the ex-post efficient outcome repeats after some finite time T in the path** (APEX).

Goal

Look for

- **An equilibrium, the ex-post efficient outcome repeats after some finite time T in the path (APEX).**

$$\Downarrow$$

- **The relevant information must be commonly known after T in the path.**

Goal

Look for

- **An equilibrium, the ex-post efficient outcome repeats after some finite time T in the path** (APEX).

$\Downarrow \Uparrow$(with some additional assumptions)

- **The relevant information must be commonly known after T in the path.**

Notations:

- $\theta_{G_i} \in \Theta_{G_i}$: i's private information about the state.
- $h^m_{G_i} \in H^m_{G_i}$: the history of actions observed by i up to period m.
- $\Theta_{G_i} \times H^m_{G_i}$: i's observation up to time m.

Notations:

- $\theta_{G_i} \in \Theta_{G_i}$: i's private information about the state.
- $h^m_{G_i} \in H^m_{G_i}$: the history of actions observed by i up to period m.
- $\Theta_{G_i} \times H^m_{G_i}$: i's observation up to time m.
- h^m: a sequence of players' actions up to period m.
- h: an infinite sequence of players' actions.

Notations:

- $\tau_i : \Theta_{G_i} \times \bigcup_{m=0}^{\infty} H_{G_i}^m \to A_{\theta_i}$, i's strategy.
- $\tau = (\tau_1, ..., \tau_i, ..., \tau_n)$: a strategy profile.
- h_θ^τ : a history generated by τ given θ.
- τ-path: $\{h_\theta^\tau\}_{\theta \in \Theta}$

DEFINITION

The τ-path is **approaching ex-post efficient** (*APEX*) $\Leftrightarrow$

$$\forall \theta, \text{ there is a finite time } T^\theta$$

such that the actions after T^θ in h_θ^τ repeats the static ex-post efficient outcome.

Notations:

- $\beta_i^{\pi,\tau}(\theta|h_{G_i}^m)$: i's belief for a θ at period m given π, τ.
- $\phi_{G_i} : H^m \to H_{G_i}^m$: the projection mapping a h^m to $h_{G_i}^m$.

DEFINITION

$h_{G_i}^m$ is **reached** by τ iff there is a pair (θ, h^m) such that h^m is on the τ-path, and $h_{G_i}^m = \phi_{G_i}(h^m)$.

Definition (weak sequential equilibrium)

The pair (τ^*, β^*),

- τ^*: a strategy
- $\beta^* = \{\beta^{*,m}\}_m$: the belief system
  - $\beta_i^{*,m}$: $\Theta_{G_i} \times H_{G_i}^m \to \Delta(\Theta \times H^m)$

, is a weak sequential equilibrium iff

- $\beta_i^{*,m}(\theta | h_{G_i}^m) = \beta_i^{\pi,\tau^*}(\theta | h_{G_i}^m)$ whenever $h_{G_i}^m$ is reached by τ^* for all i.
- τ^* is sequential rational given β^*.

Definition (Sequential equilibrium)

A sequential equilibrium (τ^*, β^*) is a weak sequential equilibrium and β^* is fully consistent with τ^*[Krep and Wilson].

- Fully consistent: the β^* is **"very very similar with"** that belief system induced by a **"very very little perturbed"** strategies around τ^*.

APEX Equilibrium

- Finally, let the "(weak) APEX equilibrium" be the (weak) sequential equilibrium in which the equilibrium path is APEX.
- Does an APEX equilibrium exist?

Outline

- An example for APEX equilibrium
- Result 1: APEX equilibrium for $k = n$.
- Result 2: weak APEX equilibrium for $k < n$.
  - Idea in equilibrium construction: introducing a “mailing game”
  - Sketch of proof.
- Extension
- Further works

Leading example

- Let $G =$

- Let $k = n = 3$.

An APEX equilibrium can be constructed by

- At period 1
  - S-type 2: **p** in $\theta = (S, S, S)$;
  - S-type 2: **np** in $\theta \neq (S, S, S)$, and then **np** forever
  - S-type 1 (or S-type 3): **np**.
- After period 1
  - If S-type 2 chooses **p** in the last period, then S-type 1 (or S-type 3) chooses **p** forever;
  - If S-type 2 chooses **np** in the last period, then S-type 1 (or S-type 3) chooses **np** forever .
- Any deviation $\Rightarrow$ Choosing **np** forever .

LEADING EXAMPLE

- Let $G =$

- Let $k = n = 3$.

An APEX equilibrium can be constructed by

- At period 1
  - S-type 2: **p** in $\theta = (S, S, S)$;
  - S-type 2: **np** in $\theta \neq (S, S, S)$, and then **np** forever(the state is revealed)
  - S-type 1 (or S-type 3): **np**.
- After period 1
  - If S-type 2 chooses **p** in the last period, then S-type 1 (or S-type 3) chooses **p** forever;
  - If S-type 2 chooses **np** in the last period, then S-type 1 (or S-type 3) chooses **np** forever (undetectable deviation).
- Any deviation $\Rightarrow$ Choosing **np** forever(detectable deviation).

Main features in equilibrium construction

- Actions (in first period) serve as "messages" to reveal the relevant information.
- The "timing" (second period) to coordinate to ex-post efficient outcome is part of equilibrium strategy.
- Playing **np** forever serves as a "grim trigger".

RESULT 1: APEX FOR $k = n$

THEOREM ($k = n$)

In any network, if the prior has full support, then for repeated $k = n$ Threshold game, an APEX equilibrium exists whenever δ is sufficiently high.

Sketch of proof:

1. "messages" to reveal the relevant information.
   - Some B-types neighbors $\Rightarrow$ play **np** forever.
   - No B-type neighbor $\Rightarrow$ play **p** until **np** is observed, and then play **np** forever.

THEOREM ($k = n$)

In any network, if the prior has full support, then for repeated $k = n$ Threshold game, an APEX equilibrium exists whenever δ is sufficiently high.

Sketch of proof:

1. "messages" to reveal the relevant information.
    - Some B-types neighbors $\Rightarrow$ play **np** forever.
    - No B-type neighbor $\Rightarrow$ play **p** until **np** is observed, and then play **np** forever.
2. The "timing" to coordinate.
    - Finite network $\Rightarrow$ there is a finite time T^{θ} such that players coordinate to ex-post efficient outcome.

Theorem ($k = n$)

In any network, if the prior has full support, then for repeated $k = n$ Threshold game, an APEX equilibrium exists whenever δ is sufficiently high.

Sketch of proof:

1. "messages" to reveal the relevant information.
   - Some B-types neighbors $\Rightarrow$ play **np** forever.
   - No B-type neighbor $\Rightarrow$ play **p** until **np** is observed, and then play **np** forever.
2. The "timing" to coordinate.
   - Finite network $\Rightarrow$ there is a finite time T^{θ} such that players coordinate to ex-post efficient outcome.
3. Any deviation $\Rightarrow$ play **np** forever.

THEOREM ($k = n$)

In any network, if the prior has full support, then for repeated $k = n$ Threshold game, an APEX equilibrium exists whenever δ is sufficiently high.

Sketch of proof:

1. "messages" to reveal the relevant information.
   - Some B-types neighbors $\Rightarrow$ play **np** forever.
   - No B-type neighbor $\Rightarrow$ play **p** until **np** is observed, and then play **np** forever.
2. The "timing" to coordinate.
   - Finite network $\Rightarrow$ there is a finite time T^{θ} such that players coordinate to ex-post efficient outcome.
3. Any deviation $\Rightarrow$ play **np** forever.
4. A fully consistent belief system can be chosen.

Theorem ($k < n$)

In any acyclic network, if π has full support on strong connectedness, then for repeated $k < n$ Threshold game, a weak APEX equilibrium exists whenever δ is sufficiently high.

Main difficulties

- Using sequence of binary actions to reveal how many S-types out there.
- These sequences has to be incentive compatible.
- Explicitly calculating the timing to coordination may be intractable.
- Due to network-monitoring, group punishment is hard to be made.

Main idea

- Consider a simple version of equilibrium construction in a "mailing game".

DEFINITION (PATH IN A NETWORK)

A **path** from node i to node j is a sequence of nodes

$$\{i, m_1, m_2, ..., m_n, j\} \text{ without repetition}$$

such that $i \in G_{m_1}, m_1 \in G_{m_2}, ..., m_n \in G_j$.

DEFINITION (ACYCLIC NETWORK (TREE))

A network is **acyclic** $\Leftrightarrow$ the path from node i to node j is unique for all nodes i, j.

Definition

θ has **strong connectedness** $\Leftrightarrow$ for every pair of S-types, there is a path consisting of S-types to connect them.

Definition

π has **full support on strong connectedness** $\Leftrightarrow$

$\pi(\theta) > 0$ **if and only if** θ has strong connectedness.

Strong connectedness

Definition

θ has **strong connectedness** $\Leftrightarrow$ for every pair of S-types, there is a path consisting of S-types to connect them.

Definition

π has **full support on strong connectedness** $\Leftrightarrow$

$\pi(\theta) > 0$ **if and only if** θ has strong connectedness.

- An B-type will not reveal information.
- Without **full support on strong connectedness**, in general, an Apex equilibrium does not exist when pay-off (as a signal) is hidden or noisy.
- Ex. $k = 2$. G and $\theta =$

T-PERIOD MAILING GAME

- A fixed and commonly known number T, where T is big enough.
- A fixed and finite network
  - n players; $N = \{1, ..., n\}$ is the set of players.
  - G_i is i's neighborhood; G_i is a subset of N such that $i \in G_i$.
  - $G = \{G_i\}_i$ is the network.
- Players of two types
  - Player i's type: $\theta_i \in \Theta_i = \{S, B\}$.
  - Type profile: $\theta \in \Theta = \times_{i \in N} \Theta_i$.
  - A common prior over Θ: π

T-PERIOD MAILING GAME

- A "letter-writing technology" for player i:
  - A set of sentences: $W = \{n, p\}^L$, where L is a big number.
  - $M_i^1 = \{f | f : \Theta_{G_i} \to W\}$; $M_i^{t+1} = \{f | f$ is a selection from $\prod_{j \in G_i} M_j^t\}$ for $T \geq t \geq 1$.
- Type-contingent action set for player i:
  - $A_{S_i}^t = \{\textbf{send}, \textbf{hold}\} \times M_i^t$; $A_{B_i}^t = \{\textbf{hold}\}$, for $t \geq 0$.
- A "letter-sending technology" for player i:
  - If $(\textbf{send}, m_i^t)$ is chosen, a fixed cost of ϵ incurs, where ϵ is small enough.
  - If $(\textbf{send}, m_i^t)$ is chosen, m_i^t is observable by G_i.
  - If $(\textbf{hold}, m_i^t)$ is chosen, no cost incurs and m_i^t is not observable by G_i.

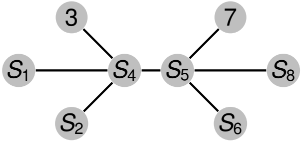
Time line

- Nature choose θ according to π.
- Types are then fixed over time.
- At the first T periods, players play T-period Mailing game.
- At $T+1$ period, players play a one-shot k-Threshold game.
- Game ends.

k-Threshold game augmented by T-period Mailing game

Example of a weak equilibrium construction:

- Let $k = 5$, $T = 2$.
- Suppose G and θ=

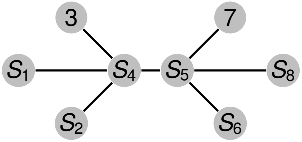


- Equilibrium path
  - At $t = 1$,

S-type 4	(**send**	$(\overbrace{n, n, \ldots, \underbrace{p, n, \ldots, n}_{4}}^{L})$)
S-type 5	(**send**	$(\overbrace{n, n, \ldots, \underbrace{p, n, \ldots, n}_{4}}^{L})$)
S-type 1,2,6,8	(**not send**	$(\overbrace{n, n, \ldots, n, n, \ldots, n}^{L})$)

Example of a weak equilibrium construction:

- Let $k = 5$, $T = 2$.
- Suppose G and θ=



- Equilibrium path
  - At $t = 2$,

S-type 4 (**send** $(\overbrace{n, n, \dots, \underbrace{p, n, \dots, n}_{4}}^{L}), (\overbrace{n, n, \dots, \underbrace{p, n, \dots, n}_{4}}^{L})$)

S-type 5 (**send** $(\overbrace{n, n, \dots, \underbrace{p, n, \dots, n}_{4}}^{L}), (\overbrace{n, n, \dots, \underbrace{p, n, \dots, n}_{4}}^{L})$)

S-type 1,2,6,8 (**not send** $(\overbrace{n, n, \dots, n, n, \dots, n}^{L}), (\overbrace{n, n, \dots, n, n, \dots, n}^{L})$)

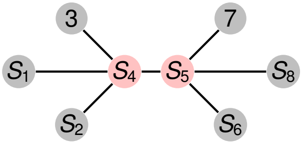
- Equilibrium path (conti.)
  - At $t = 3$, all S-types play **p**
- Off-path strategy
  - If S-type 4 (or 5) make detectable deviation $\Rightarrow$ others play **hold** and then **np**.
  - If S-type 4 (or 5) make undetectable deviation $\Rightarrow$ he is facing a possibility of failure to coordinate to **p**.
- Off-path belief
  - Detectable deviation $\Rightarrow$ believing that all players outside neighborhood are B-types.

- Equilibrium path (conti.)
  - At $t = 3$, all S-types play **p**
- Off-path strategy
  - If S-type 4 (or 5) make detectable deviation $\Rightarrow$ others play **hold** and then **np**.
  - If S-type 4 (or 5) make undetectable deviation $\Rightarrow$ he is facing a possibility of failure to coordinate to **p**.
- Off-path belief
  - Detectable deviation $\Rightarrow$ believing that all players outside neighborhood are B-types.

So, when ϵ is small enough and T is large enough, an weak equilibrium can be constructed when ϵ is independent from messages.

However, if ϵ is not independent from messages, then a Free Rider Problem may occur.

- Suppose $\epsilon \downarrow$ when announce more S-types in the 1^{st} period.
- Let $k = 5$, $T = 2$.
- Suppose G and θ=



1. S-type 4 and S-type 5 will deviate from truthfully announce(**Free Rider Problem**).
2. Why? They will report more S-types to save costs.

THEOREM ($k < n$)

In any acyclic network, if π has full support on strong connectedness, then for repeated $k < n$ Threshold game, a weak APEX equilibrium exists whenever δ is sufficiently high.

Sketch of proof for Result 2:

1. The Free Rider Problem can be solved in acyclic networks.
2. An Apex equilibrium path can be constructed.
3. APEX outcome gives maximum ex-post continuation pay-off after some T.
4. Detectable deviation $\Rightarrow$ playing **np** forever (by off-path belief).
5. Undetectable deviation $\Rightarrow$ facing a possibility of coordination failure.
6. Any deviation will let APEX fail with positive probability.
7. Sufficiently high δ will impede deviation.

1. payoff is perfectly observed
   - Play **p** in the first period, then the relevant information revealed.
2. payoff is noisy
   - With full support assumption, the existing equilibrium is APEX.
   - Ex:
     - $u_{s_i}(a, y)$ is dependent on a random variable $y \in \{y_1, y_2\}$
     - full support:

$$\begin{aligned} p_{1,\geq k} &= \Pr(y = y_1 | \#\mathbf{p} \geq k) > 0 \\ p_{1,<k} &= \Pr(y = y_1 | \#\mathbf{p} < k) > 0 \\ p_{2,\geq k} &= \Pr(y = y_2 | \#\mathbf{p} \geq k) > 0 \\ p_{2,<k} &= \Pr(y = y_2 | \#\mathbf{p} < k) > 0 \end{aligned}$$

     - S-type: <u>the expected payoff on $\#\mathbf{p} \geq k$</u> strictly larger than <u>the expected payoff on $\#\mathbf{p} < k$</u>

Further works

1. Cyclic networks.
2. A general model in which players can communicate only by their actions to learn the relevant information in finite time when $\delta < 1$, while the communication protocol itself is an equilibrium.
3. Equilibrium selection.