Coordination in Social Networks

Chun-ting Chen

February 18, 2019

Motivation

- Consider a rigid regime, where "communication barrier" is imposed to impede people to show their discontents.
- Communication barrier
  1. Threatened by suppression, exile, eavesdropping, etc.
  2. No (fair) voting system, No (fair) mass media, No (uncensored) discussion forum, etc.
- *How do Rebels made decisive collective action?*

Motivation

History tells us:

- An event may trigger later events.
    - Benny Tai, a leader of Occupy Central, has said *"It (Umbrella Protest) is beyond what I imagined"*, while Occupy Central trigger the Umbrella Protest in Hong Kong.
- People communicate in their social network.
    - Ex., Gangster networks (1911 Revolution); Church networks (1989 Berlin Uprising, 2014 Umbrella Protest); Friend networks, etc.

Motivation

Question

- *If rational rebels know that a "tiny" event can trigger later events, how do they conduct a revolution under communication barrier?*

Objective

- What kinds of social networks can conduct such decisive collective action?

Model

1. Rebels communicate in network.
2. Communication is not free but costly.
3. Communication is through taking actions.

Motivation

Looking for

- An equilibrium, where the ex-post efficient outcome played repeatedly after a finite time T in the path when δ is high enough.

Related Literature

- Public good provision.
  - One strand: [Lohmann, 1993,1994], [Bolton and Harris, 1999], [Bramoullé and Kranton, 2007]
  - **This paper adds network-monitoring**
- Social learning.
  - One strand: [Goyal, 2012], [Acemoglu et al., 2011], [Chatterjee and Dutta, 2011].
  - **This paper considers farsighted-learning in the game**
- Repeated game.
  - One strand: [Laclau, 2012], [Wolitzky, 2013], [Wolitzky, 2014]
  - **This paper consider incomplete information and imperfect monitoring**
  - One strand: [Fudenberg and Yamamoto, 2010] [Fudenberg and Yamamoto, 2011] [Wiseman, 2012] [Yamamoto 2014]
  - **This paper consider n-person game without full-rank conditions on public or private signals generated by single-period actions.**

Model

Network

- Let $N = \{1, \ldots, n\}$ be the set of players.
- G_i is a subset of N, where $i \in G_i$
- G_i is i's neighborhood.
- $G = \{G_i\}_i$ is the network.

Definition

1. G is *fixed* if G is not random.
2. G is *finite* if N is finite.
3. G is *undirected* if $j \in G_i \Rightarrow i \in G_j$.
4. A *path* from i to j, $i \neq j$ in an undirected G is

$$(i, l_1, \ldots, l_q, j)$$

   such that $l_1 \in G_i, \ldots, l_q \in G_j$ and $i, l_1, \ldots, l_q, l$ are all distinct.
5. G is *connected*: An undirected G is connected $\Leftrightarrow \forall\ i, j, i \neq j$ there is a path from i to j.
6. G is *acyclic*: An undirected G is acyclic $\Leftrightarrow$ the path from i to j, for $i \neq j$, is unique.

Model

Static k-threshold game [Chwe 2000]

- i's type
- $\theta_i \in \Theta_i = \{Rebel, Inert\}$
- $\Theta = \times_{i \in N} \Theta_i$
- $\theta \in \Theta$
- $A_{Rebel_i} = \{\textbf{revolt}, \textbf{stay}\}$; $A_{Inert_i} = \{\textbf{inert}\}$
- $1 \leq k \leq n$
- Static game payoff for player i: $u_{\theta_i}(a_{\theta_i}, a_{-\theta_i})$

$$
\begin{aligned}
u_{Inert_i}(a_{Inert_i}, a_{-\theta_i}) &= 1 && \text{if } a_{Inert_i} = \textbf{inert} \\
\\
u_{Rebel_i}(a_{Rebel_i}, a_{-\theta_i}) &= 1 && \text{if } a_{Rebel_i} = \textbf{revolt} \text{ and } \#\{j : a_{\theta_j} = \textbf{revolt}\} \geq k \\
u_{Rebel_i}(a_{Rebel_i}, a_{-\theta_i}) &= \text{-1} && \text{if } a_{Rebel_i} = \textbf{revolt} \text{ and } \#\{j : a_{\theta_j} = \textbf{revolt}\} < k \\
u_{Rebel_i}(a_{Rebel_i}, a_{-\theta_i}) &= 0 && \text{if } a_{Rebel_i} = \textbf{stay}
\end{aligned}
$$

Model

Repeated k-threshold game

- Time is infinite, discrete.
- Nature choose θ at 0 period.
- Players play the static k-threshold game infinitely repeatedly.

Assumption

- *Players know their neighbors' types.*
- *Players perfectly observe their neighbors' actions only.*
- *G is FFCCU (fixed, finite, connected, commonly known, undirected)*
- *Payoff is hidden.*
- *π has full support*
- *Common δ.*

Notations:

- $[Rebels](\theta) = \{j : \theta_j = Rebel\}$ for all $\theta \in \Theta$.
- τ: a strategy profile
- $h^m_{G_i}$: the history i can observe up to period m
- $\beta_i^{\pi,\tau}(\theta | h^m_{G_i})$: i's belief for a θ at period m.

APEX

Definition

A sequential equilibrium is *approaching efficient* (*APEX*) $\Leftrightarrow$

$$\forall \theta \text{ there is a finite time } T^{\theta}$$

such that ex-post efficient outcome repeats after T^{θ} in the path.

Lemma

If a sequential equilibrium τ^* is APEX $\Rightarrow$

$$\forall \theta \ \forall i, \textit{ there is a finite time } T_i^{\theta}$$

such that $\sum_{\theta:\#[Rebels](\theta)\geq k} \beta_{G_i}^{\pi,\tau^*}(\theta|h_{G_i}^s) = 1$ *or* $= 0$ *if* $s \geq T_i^{\theta}$.

Leading Example

An Apex Equilibrium for $k = n = 3$ in

- At 1st period
  - Rebel 2 chooses **revolt** if he observes $\theta = (Rebel, Rebel, Rebel)$; Otherwise, chooses **stay** forever.
  - Rebel 1 (or Rebel 3) choose **stay**.
- After 1st period
  - If Rebel 2 chooses **revolt** in the last period, then Rebel 1 (or Rebel 3) chooses **revolt** forever;
  - If Rebel 2 chooses **stay** in the last period, then Rebel 1 (or Rebel 3) chooses **stay** forever.
- Any deviation $\Rightarrow$
  - Choosing **stay** forever.

Goal
Can we generalize the result in Leading Example for all FFCCU networks?

Results

- $k = n$: we can.
- $k < n$: with additional assumption,
    - acyclic FFCCU: we can .
    - FFCCU: open question.

Result: $k = n$

Theorem

In *any FFCCU network,* ***if*** *the prior has full support,* ***then*** *for repeated $k = n$ Threshold game,* ***there is*** *a δ such that a sequential equilibrium which is APEX* ***exists****.*

Proof:

1. Some Inerts neighbors $\Rightarrow$ play **stay** forever.
2. No Inert neighbor $\Rightarrow$ play **revolt** until **stay** is observed, and then play **stay** forever.
3. Any deviation $\Rightarrow$ play **stay** forever.
4. Since networks are FFCCU, there is a finite time T^{θ} such that ex-post efficient outcome repeats afterwards.

Result: $k = n$

Comments:

1. **stay** $\Leftrightarrow$ some Inerts is observed.
2. single-period $\{$**revolt**, **stay**$\}$ $\Rightarrow$ reveals $\{$"no Inert", "some Inerts"$\}$.
3. Any deviation $\Rightarrow$ punished by shifting to **stay** forever by single player
   - Group punishment is not necessary.

Result and Conjecture: $k < n$

Definition

Strong connectedness$\Leftrightarrow$ for every pair of Rebels, there is a path consisting of Rebels to connect them.

Definition

Full support on strong connectedness$\Leftrightarrow$

$$\pi(\theta) > 0 \text{ if and only if } \theta \text{ has strong connectedness.}$$

Result and Conjecture: $k < n$

Theorem

In *any acyclic FFCCU network,* ***if*** *θ has strong connectedness and* ***if*** *π has full support on strong connectedness,* ***then*** *for repeated k < n Threshold game,* ***there is*** *a δ such that a weak sequential equilibrium which is APEX* ***exists****.*

Conjecture

In *any FFCCU network, ...[same as above]...*

Outline

1. The role of Strong Connectedness
2. Communication by actions
3. Communication in equilibrium
    1. Step 0: Build communication protocol
    2. Step 1: Characterize "information hierarchy" in communication.
    3. Step 2: Build reporting and coordination messages in the path, and characterize the in-path belief updating.
    4. Step 3: Set up off-path belief.

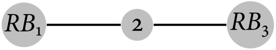
Equilibrium Construction: $k < n$

The role of Strong Connectedness:

- Otherwise, the game is reduced to incomplete information game without communication for some θ

Example,

1. Let $k = 2$. Assume $\theta = (Rebel_1, Inert_2, Rebel_3)$.
2. Let

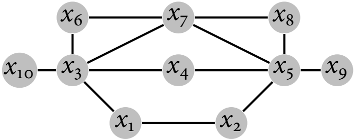


3. Then, Inert 2 block the information transmission.
4. This is an incomplete information game without communication.

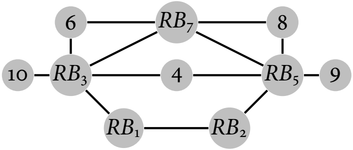
Equilibrium Construction: $k < n$

Communication by actions

1. Indexing each node i as a distinct prime number x_i. For instance,



2. Then, for instance,
    - If



    - Rebel 3 report $x_1 \times x_7 \times x_3$ to Rebel 1 by sending a finite sequence

$$\textbf{stay}, \ldots, \textbf{stay}, \underbrace{\textbf{revolt}, \textbf{stay}, \ldots, \textbf{stay}}_{x_1 \times x_7 \times x_3}$$

Equilibrium Construction: $k < n$

Communication in Equilibrium. Step 0

- Characterize the time horizontal line as

$$\underbrace{\langle \text{coordination period} \rangle}_{0-block} \underbrace{\langle \text{reporting period} \rangle \langle \text{coordination period} \rangle}_{1-block} \ldots$$

1. Reporting period: talking about θ
2. Coordination period: talking about "Have some Rebels known $\#[Rebels](\theta) \geq k$ or $\#[Rebels](\theta) < k$?"
3. Why do I need coordination period?

Equilibrium Construction: $k < n$

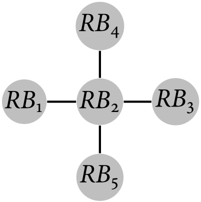
Communication in Equilibrium. Step 0.

- Q: Why do I need coordination period?
- A: Since higher-order belief is hard to track.
  - APEX: T^{θ} for all θ.
  - Calculating T^{θ} for all θ is tedious.
- I: If a Rebel knows $\#[Rebels](\theta) \geq k$ or $\#[Rebels](\theta) < k \Rightarrow$ sending messages to let others know.

Information Hierarchy

Why do I need "Information Hierarchy"?

- $\Rightarrow$ To ease the punishment scheme.
- Case 1:
- Let $k = 4$



1. Rebel 1 can only be monitored by Rebel 2.
2. Given some strategies, suppose Rebel 2,3,4,5 can coordinate at period T and play **revolt** forever.
3. If Rebel 1 deviate at period $T - 1$, Rebel 2 has no incentive to punish him.

Information Hierarchy

Why do I need "Information Hierarchy"?

- $\Rightarrow$ To characterize Rebels' incentives in communication.
- Case 2:
- Let $k = 4$

$$RB_1 — RB_2 — RB_3 — RB_4 — RB_5$$

1. Rebel 2 has more incentive than Rebel 1 in sending messages.
2. Compare Rebel 3 and Rebel 2, etc.

Information Hierarchy

At 0-block, let

- $$R^0 = [Rebels](\theta)$$

Information Hierarchy

At 1-block, let

$$\begin{aligned} N_i^0 &\equiv G_i \\ I_i^0 &\equiv G_i \cap R^0 \end{aligned}$$

Define $\leq^0$ by

$$i \in \leq^0 \Leftrightarrow \exists j \in \bar{G}_i (I_i^0 \subseteq N_j^0 \cap R^0)$$

Let

$$R^1 \equiv \{ i \in R^0 | i \notin \leq^0 \}$$

Information Hierarchy

Ex., Rebel 1 is a non-R^1 node. Rebel 2 is a R^1 node.

$$RB_1 - RB_2 - RB_3 - RB_4 - RB_5$$

Calculation:

$$\begin{aligned} I_1^0 &= \{1, 2\} & N_1^0 \cap R^0 &= \{1, 2\} \\ I_2^0 &= \{1, 2, 3\} & N_2^0 \cap R^0 &= \{1, 2, 3\} \\ I_3^0 &= \{2, 3, 4\} & N_3^0 \cap R^0 &= \{2, 3, 4\} \end{aligned}$$

Main idea:

- Rebel 2 is more “important” than Rebel 1.
- Rebel 3 and Rebel 2 are equally “important”, etc.

Information Hierarchy

In $t+1$-block, denote

$$\begin{aligned} N_i^t &\equiv \bigcup_{k \in I_i^{t-1}} G_k \\ I_i^t &\equiv \bigcup_{k \in G_i \cap R^t} I_k^{t-1} \end{aligned}$$

- N_i^t is i's *extended* neighborhood given i's information I_i^{t-1}
- I_i^t is i's *extended* Rebel neighbors given j's information I_j^{t-1}, where j is a R^t Rebel.

Define $\preceq^t$ by

$$i \in \preceq^t \Leftrightarrow \exists j \in \bar{G}_i (I_i^t \subseteq N_j^t \cap R^0)$$

Let

$$R^{t+1} \equiv \{ i \in R^t | i \notin \preceq^t \}$$

Information Hierarchy

Ex.,

- Rebel 1 is a non-R^1 node. Rebel 2 is a R^1 node. Rebel 3 is a R^1 node.
- Rebel 1 is a non-R^2 node. Rebel 2 is a non-R^2 node. Rebel 3 is a R^2 node.

$$RB_1 — RB_2 — RB_3 — RB_4 — RB_5$$

Calculation:

at 1-block

$I_1^0 = \{1, 2\}$	$N_1^0 \cap R^0 = \{1, 2\}$
$I_2^0 = \{1, 2, 3\}$	$N_2^0 \cap R^0 = \{1, 2, 3\}$
$I_3^0 = \{2, 3, 4\}$	$N_3^0 \cap R^0 = \{2, 3, 4\}$

at 2-block

$I_1^1 = \{1, 2, 3\}$	$N_1^1 \cap R^0 = \{1, 2, 3\}$
$I_2^1 = \{1, 2, 3, 4\}$	$N_2^1 \cap R^0 = \{1, 2, 3, 4\}$
$I_3^1 = \{1, 2, 3, 4, 5\}$	$N_3^1 \cap R^0 = \{1, 2, 3, 4, 5\}$

Information Hierarchy

Theorem

Given θ, if the network is FFCCU and acyclic and if the state has strong connectedness $\Rightarrow \exists t^{\theta}$ *such that some* $R^{t^{\theta}}$ *Rebels whose* $I^{t^{\theta}} \supset [Rebels](\theta)$.

Equilibrium path

At t-block, looking for messages (strategies) such that

- The length of players' messages is the same as the length of corresponding period.
- RP^t: the reporting period

$$\overbrace{\langle \cdots \rangle}^{\text{reporting period}}$$

- CD^t: the coordination period

$$\overbrace{\underbrace{\langle \langle_{\text{sub-block}} \rangle \cdots \langle \cdot \rangle \rangle}_{\text{division}} \underbrace{\langle \langle \cdot \rangle \cdots \langle \cdot \rangle \rangle}_{\text{division}} \underbrace{\langle \langle \cdot \rangle \cdots \langle \cdot \rangle \rangle}_{\text{division}}}^{\text{coordination period}}$$

- $\langle RP^t \rangle$: the reporting messages
- $\langle CD^t \rangle$: the coordination messages

Equilibrium path

Denote

- $\langle I_i^{t-1} \rangle$=$\mathbf{s}, \dots, \mathbf{s}, \underbrace{\mathbf{r}, \mathbf{s}, \dots, \mathbf{s}}_{X}$
    - $X = \times_{j \in I_i^{t-1}}$ j's prime index
- $\langle \mathbf{stay} \rangle$=$\mathbf{s}, \dots, \mathbf{s}$

Ideally, (by information hierarchy theory),

- R^t: report $\langle I^{t-1} \rangle$.
- Non-R^t: report $\langle \mathbf{stay} \rangle$.
- $\Rightarrow$ some Rebels knows the state.

Equilibrium path

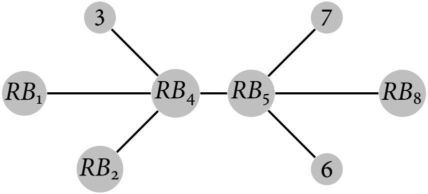
However, not so obvious.

- Not cheap talks.
- Consider next 3 problems, where we suppose
  - An action-irrelevant message $\langle M \rangle$.
  - Starting with a *RP* and then a *CD* follows.
  - Observing $\langle M \rangle$ in *CD* $\Rightarrow$ play **revolt** forever; Otherwise $\Rightarrow$ play **stay** forever.

Equilibrium path

Pivotal player case 1: Free Rider Problem. (Rebel 4 and Rebel 5)

- $k = 5$

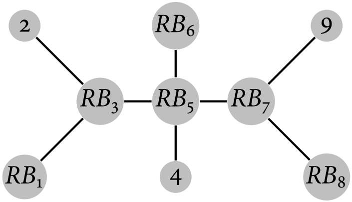


- **Problem**: Both Rebel 4 and Rebel 5 are pivotal ⇒ they will shift to play ⟨**stay**⟩ if others report truthfully.

Equilibrium path

Pivotal Player Case 2 (Rebel 5)

- $k = 6$

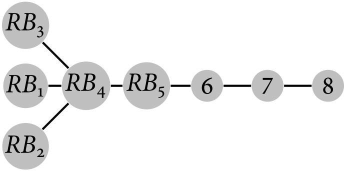


- **Problem**: Rebel 5 is pivotal $\Rightarrow$ he shifts to play $\langle$**stay**$\rangle$ given others' truthful reporting.

Equilibrium path

Pivotal Player Case 3 (Rebel 4)

- $k = 6$



- **Problem**: Rebel 4 is pivotal ⇒ he shifts to play ⟨**stay**⟩ given others' truthful reporting.

Equilibrium path

Problem

- Rebels may deviate $\langle I^{t-1} \rangle$ to $\langle \textbf{stay} \rangle$.

Remedy

- $\langle 1 \rangle = \mathbf{s}, \ldots, \mathbf{s}, \mathbf{r}$, as the message used by pivotal player.
- Continuation behavior contingent on both *RP* and *CD*.

Good news.

- Pivotal problems: only above three cases.
- The free rider problem: only the above case.
  - Two nearby Rebels. (only for acyclic G)

Equilibrium path

Good news

- With suitable coordination messages and continuation behavior
  1. Pivotal players will not deviate from playing $\langle 1 \rangle$.
  2. Only pivotal players will play $\langle 1 \rangle$

Equilibrium path

Good news

- By adding a $\langle \mathbf{x}_i \rangle = \mathbf{s}, \ldots, \mathbf{s}, \underbrace{\mathbf{r}, \mathbf{s}, \ldots, \mathbf{s}}_{x_i}.$
  - To create more equilibrium paths in coordination period.
- The belief updating after CD^t, $t > 0$ in the equilibrium path will be

Table: Belief updating after CD^t, $t > 0$

In RP^t	In $CD^t_{1,1}$	In $CD^t_{1,2}$	
i plays	i plays	i plays	The events j believe with probability one
$\langle \mathbf{stay} \rangle$	$\langle \mathbf{x}_i \rangle$	$\langle \mathbf{stay} \rangle$	$i \notin R^t$
$\langle I_i^{t-1} \rangle$	$\langle \mathbf{stay} \rangle$	$\langle \mathbf{stay} \rangle$	$\#[Rebels](\theta) < k$
$\langle I_i^{t-1} \rangle$	$\langle \mathbf{x}_i \rangle$	$\langle \mathbf{stay} \rangle$	$\#[Rebels](\theta) \geq k$
$\langle I_i^{t-1} \rangle$	$\langle \mathbf{x}_i \rangle$	$\langle \mathbf{x}_i \rangle$	$i \in R^t$
$\langle 1 \rangle$	$\langle \mathbf{stay} \rangle$	$\langle \mathbf{stay} \rangle$	$\#[Rebels](\theta) < k$
$\langle 1 \rangle$	$\langle \mathbf{x}_i \rangle$	$\langle \mathbf{stay} \rangle$	$\#[Rebels](\theta) \geq k$

Off-path Belief

i detects a deviation at s period, he forms off-path belief

$$\sum_{\theta \in \{\theta : \theta_j = Inert, j \notin G_i\}} \beta_{G_i}^{\pi,\tau}(\theta | h_{G_i}^{s'}) = 1 \tag{1}$$

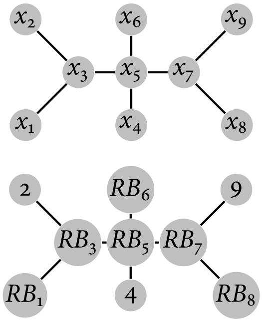
for all $s' \geq s$.

1. If $\# I_i^o < k$, he will play **stay** forever.
2. This off-path belief then serve as a grim trigger.

Off-path Belief

Without ⟨1⟩, using this grim-trigger-like belief may not sustain APEX

- $k = 5$



- **Problem**: Without ⟨1⟩ being considered as an in-path strategies;
- Rebel 4 is pivotal; He shifts to report $x_3 \times x_5 \times x_7$ instead of $x_3 \times x_5 \times x_7 \times x_6$.
- Coordination can be made, but Rebel 6 is out of coordination since he detects a deviation.

Result: $k < n$

Comments:

1. **stay** $\not\Leftrightarrow$ some Inerts be observed.
2. single-period $\{\textbf{revolt}, \textbf{stay}\} \not\Rightarrow$ reveals $\{\#[Rebels](\theta)\}$
3. Any deviation $\Rightarrow$ punished by shifting to **stay** forever by some players.

Discussion

1. From the above steps, an APEX equilibrium is constructed.
2. We can relaxed the assumption that payoff is hidden.
   - payoff is perfectly observed: easy to construct an APEX equilibrium.
   - payoff is noisy: with full support assumption, the existing equilibrium is APEX
3. This proof is still open for FFCCU network with cycles.
4. Off-path belief did not satisfy full consistency property for FFCCU network without cycles.
5. Prime number indexing also works for other discreet and finite state space.

Conclusion

1. I show that, without cheap talk, in this repeated k-threshold game played in FFCCU networks without cycles, coordination still can happen.
   - Using sequence of actions to communicate.
2. The equilibrium is constructive and does not rely on public or private signals other than actions.
3. We can use prime number to index the states given that states are discrete and finite.
4. For the network with circle, it is still remaining to tackle with.