0.1 Proof for equilibrium

Claim 0.1.0.1. *For $|Ex_{I_i^{m,t}} \cup I_i^{m,t}| \geq s$, where m is a period in reporting period. If i report $\langle 2 \rangle$ or $\langle 1 \rangle$, then i will know $|[H]| \geq s$ or $|[H]| < s$ after reporting period, and thus the coordination can be either initiated in t-block or be never initiated.*

Proof. By directly checking the equilibrium path, we have

1. if $\# I_i^{|RP^t|,t} \geq s$, then the coordination can be initiated by such i.
2. if $\# I_i^{|RP^t|,t} = s-1$, and if there is one more node who reported $\langle 1 \rangle$, then the coordination can be initiated by i.
3. if $\# I_i^{|RP^t|,t} = s-1$, and if there are no nodes who reported in current period, then $\# I_i^{|RP^t|,t} = \# I_i^t = s-1$. We now check the conditions guiding i to **POST-CHECK**.
   - If i is coming from the conditions in **MAIN**, it means that there is no further H-node outside I_i^{t-1}, and thus outside $\bigcup_{k \in I_i^{t-1}} N_k$.
   - If i is coming from the conditions in **CHECK.0**, it means that there is no further H-node outside $\bigcup_{k \in I_i^{t-1}} N_k \cap [H]$, and thus outside $\bigcup_{k \in I_i^{t-1}} N_k$.
   - If i is coming from the conditions in **CHECK.m**, it means that there is no further H-node outside $\bigcup_{k \in I_i^{t-1}} N_k \cap [H]$, and thus outside $\bigcup_{k \in I_i^{t-1}} N_k$.

   Then $\# I_i^t < k$, but $I_i^t = \bigcup_{k \in I_i^{t-1}} N_k \cap R^0$, and hence $\# R^0 < k$, and thus the coordination can never happen.

□

Lemma 0.1.1. *If the state has strong connectivity, then for all n-person repeated k-Threshold game with parameter $1 \leq k \leq n$ played in any finite connected undirected network without circle, the equilibrium path is approaching efficient.*

Proof. We want to show that when θ satisfying $\#[Rebels](\theta) \geq k$, all the Rebels play **revolt** eventually; when θ satisfying $\#[Rebels](\theta) < k$, all the Rebels play **stay** eventually.

1. If all the Rebels only play $\langle I^{t-1} \rangle$ or $\langle$**stay**$\rangle$ in reporting period for all $t \geq 1$ block, then by the equilibrium path, those nodes played $\langle I^{t-1} \rangle$ are R^t-node, and those nodes played $\langle$**stay**$\rangle$ are not-R^t nodes.

   If there are some Rebels play $\langle$**stay**$\rangle$ in the first division in t-block, then all the Rebels play **stay** eventually; If R^t Rebels play $\langle$**stay**$\rangle$ in the first sub-block in second division in t-block, then all the Rebels will play **stay** after third division in this block. Otherwise, all the Rebels go to the next reporting period.

   By Theorem **??**, there is a t^* such that there is a R^{t^*} node knows θ, and therefore he knows if θ satisfying $\#[Rebels](\theta) \geq k$ or $\#[Rebels](\theta) < k$. In equilibrium path, such node play $\langle$**stay**$\rangle$ either in the first sub-block in first division or in the first sub-block in second division in coordination period.Thus the equilibrium path is approaching efficient.

2. If there are some Rebels play $\langle 1 \rangle$ in reporting period for a $t \geq 1$ block, then by Claim 0.1.0.1, such nodes will knows if θ satisfying $\#[Rebels](\theta) \geq k$ or $\#[Rebels](\theta) < k$ after reporting period in this t-block. $\langle$**stay**$\rangle$ either in the first sub-block in first division or in the first sub-block in second division in coordination period. Thus the equilibrium path is approaching efficient.

□

0.1.1 Main claims in reporting period

We show the main claims here. The details of the other claims in equilibrium path will be in appendix.

Claim 0.1.1.1. *For $|Ex_{I_i^{m,t}} \cup I_i^{m,t}| \geq s$. Denote D be the set of H-neighbours who detect i's deviation. If $|I_i^{m,t}| < s$, and if $D \neq \emptyset$, then there is a $M < \infty$ and an event E such that i's expected continuation pay-off is less than that in equilibrium path by at least*

$$\delta^M \frac{\beta_i(E|h^m_{N_i})}{1-\delta}$$

, where $\beta_i(E|h^m_{N_i}) > 0$

Proof. Denote D be the set of neighbours who detect i's deviation. Let the events be

$$\begin{aligned} E_1 &= \{\theta : \#[Rebels](\theta) < k\} \\ E_2 &= \{\theta : k \leq \#[Rebels](\theta) < k + \#D\} \\ E_3 &= \{\theta : \#[Rebels](\theta) \geq k + \#D\} \end{aligned}$$

In equilibrium path, there are periods t^s (t^f) such that if θ satisfying $\#[\text{Rebels}](\theta) \geq k$ ($\#[\text{Rebels}](\theta) < k$) then Rebels play **revolt** (**stay**) forever. If i follows the equilibrium path, the expected static pay-off after $\max\{t^s, t^f\}$[1] is

$$\beta_i(E_2|h^m_{N_i}) + \beta_i(E_3|h^m_{N_i})$$

If i deviate, the expected static pay-off after $\max\{t^s, t^f\}$ is

$$\beta_i(E_3|h^m_{N_i})$$

Therefore there is a loss in expected static pay-off of

$$\beta_i(E_2|h^m_{N_i})$$

Thus, there is a loss in expected continuation pay-off contingent on E by

$$\delta^{\max\{t^s, t^f\}} \frac{\beta_i(E_2|h^m_{N_i})}{1-\delta}$$

□

[1] There is t^s or t^f for each θ. The maximum is among those possible θ.

Claim 0.1.1.2. *For $|Ex_{I_i^{m,t}} \cup I_i^{m,t}| \geq s$. If $|I_i^{m,t}| < s$, and if i deviate by reporting $\bar{I}_i^{t-1} \supset I_i^{t-1}$ and such deviation is not detected by i's neighbour, then there is a loss compared to equilibrium in expected pay-off by*

1. *either $-\delta^{|RP^t|-|I_i^{t-1}|+1} + \delta^{|RP^t|-|\bar{I}_i^{t-1}|+1}$ in static pay-off*
2. *or, there is a $M < \infty$ and events E_1, E_2 such that i's expected continuation pay-off is less than that in equilibrium path by at least*

$$\delta^M \frac{\min\{\beta_i(E_1|h_{N_i}^m), \beta_i(E_2|h_{N_i}^m)\}}{1-\delta}$$

, where $\beta_i(E_1|h_{N_i}^m) > 0$ and $\beta_i(E_2|h_{N_i}^m) > 0$

Claim 0.1.1.3. *For $|Ex_{I_i^{m,t}} \cup I_i^{m,t}| \geq s$. If $|I_i^{m,t}| < s-1$, and if $i \notin C$, then if i deviate by reporting*

$$\langle \textbf{\textit{runs [POST-CHECK,1 or 2]}} \rangle$$

, then there is a $M < \infty$ and an event E such that i's expected continuation pay-off is less than that in equilibrium path by at least

$$\delta^M \frac{\beta_i(E|h_{N_i}^m)}{1-\delta}$$

or

$$\delta^M \frac{\min\{\beta_i(E|h_{N_i}^m), \beta_i(T \setminus E|h_{N_i}^m)\}}{1-\delta}$$

, where $\beta_i(E|h_{N_i}^m) > 0$ and $\beta_i(T \setminus E|h_{N_i}^m) > 0$

Claim 0.1.1.4. *For $|Ex_{I_i^{m,t}} \cup I_i^{m,t}| \geq s$,*

1. *If i can report $\langle 2 \rangle$, then $|I_i^{||RP^t-|2|+1|,t}| \geq s$, and thus $|[H]| \geq s$.*
2. *If i can report $\langle 1 \rangle$, then either $|I_i^{||RP^t-|1|+1|,t}| = s-1$ or there is one of i's neighbours has reported $\langle 2 \rangle$*

Proof. By checking condition **POST-CHECK** directly. □

Claim 0.1.1.5. *For $|Ex_{I_i^{m,t}} \cup I_i^{m,t}| \geq k$. If $\beta_i(\#[Rebels](\theta) \geq s|h_{G_i}^{|RP^t|}) > 0$, then if i can report $\langle 2 \rangle$, then i will not report $\langle 1 \rangle$ or $\langle l \rangle$ when δ is high enough.*

Proof. By Claim 0.1.1.4, We have $|I_i^{||RP^t-|2|+1|,t}| \geq s$.

Now let the event E be

$$E = [[H] = |I_i^{||RP^t-|2|+1|,t}|]$$

Note that, contingent on E, there is no more node can initiate the coordination. This is because for all $j \in O_i^{||RP^t-|2|+1|,t}$, j is with $|I_j^{t-1}| < s$, and there is no more H-node outside $I_i^{||RP^t-|2|+1|,t} =$

$\bigcup_{j\in O_i^{||RP^t-|2|+1|,t}} I_j^{t-1}$. Since i can not initiate the coordination, compared to equilibrium, there is a loss in expected continuation pay-off as

$$\delta^t \frac{\beta_i(E|h_{N_i}^{m'})}{1-\delta}, \text{when } m' > t$$

□

Claim 0.1.1.6. *For $|Ex_{I_i^{m,t}} \cup I_i^{m,t}| \geq s$. If $\beta_i(|[H]| \geq s|h_{N_i}^{||RP^t-|1|+1|}) > 0$, then if i can report $\langle 1\rangle$, then i will not report $\langle l\rangle$ when δ is high enough.*

Proof. By Claim 0.1.1.4, We have $|I_i^{||RP^t-|1|+1|,t}| = s-1$.

- Case 2: If there is no $j \in N_i \setminus i$ has reported $\langle 2\rangle$, since $\beta_i(|[H]| \geq s|h_{N_i}^{||RP^t-|1|+1|}) > 0$, the following event E_1 must have positive probability; otherwise, since no neighbours can report after current period, and thus $\beta_i(|[H]| \geq s|h_{N_i}^{||RP^t-|1|+1|}) = 0$.

  Let
  $$E_1 = [\exists j \in N_i \setminus i, j \notin O_i^{||RP^t-|2|+1|,t} \text{ such that } |I_j^{||RP^t-|1|+1|,t}| = s-1]$$
  [2]

  Now we can construct sub-events $E_1' \subset E_1$ as

  $$E_1' = [\text{there is only one } j \in N_i \setminus i, j \notin O_i^{||RP^t-|1|+1|,t} \text{ such that } |I_j^{||RP^t-|1|+1|,t}| = s-1]$$

  Now, dependent on such j, let

  $$E = [[H] = I_j^{||RP^t-|1|+1|,t} \cup I_i^{||RP^t-|1|+1|,t}]$$

  If i report $\langle l\rangle$, there are following consequences.

  - i will be consider as $\notin R^t$ by Claim **??**, and thus i can not initiate the coordination.
  - Such j will have $|I_j^{|RP^t|}| = |I_j^t| < s$. Since there is no more H-nodes outside $I_j^{||RP^t-|2|+1|,t} \cup I_i^{||RP^t-|2|+1|,t}$, contingent on E, such j will not initiate the coordination.
  - For other $k \in O_i^{||RP^t-|2|+1|,t}$, since k is with $|I_k^{t-1}| < s$, they will not initiate the coordination either

[2] More crucially, the following event F has zero probability,

$$F = [\exists j \in N_i \setminus i, j \notin O_i^{||RP^t-|2|+1|,t} \text{ such that } |I_j^{||RP^t-|1|+1|,t}| \geq s]$$

.

However, if i play $\langle 1 \rangle$, coordination can be initiated by himself in the following coordination period. Thus, there is a loss in expected continuation pay-off by

$$\delta^t \frac{\beta_i(E|h_{N_i}^{m'})}{1-\delta}, \text{when } m' > t$$

$\square$

Claim 0.1.1.7. *For $|Ex_{I_i^{m,t}} \cup I_i^{m,t}| \geq s$. If $i \notin R^t$, and if i did not observed $\langle 2 \rangle$, then i will not report $\langle 1 \rangle$.*

Proof. Otherwise, there is a loss in static pay-off and the continuation pay-off is the same for all states.

$\square$

0.1.2 Main claims in coordination period

We show the main claims here. The details of the other claims in equilibrium path will be in appendix.

Claim 0.1.2.1. *In **COORDINATION**. If there is a $j \in N_i \setminus i$ has played $\langle 2 \rangle$ or $\langle 1 \rangle$ in reporting period, and j did not play $\langle l \rangle$ in the **CHECK.f**, then either $|I_j^t| \geq s$ or $\exists k \in N_j \setminus j[|I_k^t| \geq s]$.*

Proof. Due to the equilibrium strategy in reporting period and in **CHECK.f**.

$\square$

Claim 0.1.2.2. *In **COORDINATION**. If there is no $j \in N_i$ has played $\langle 2 \rangle$ or $\langle 1 \rangle$, if $|I_i^t| < s$, and if there is no $j \in N_i$ has played play $\langle l \rangle$ in **CHECK.f**, then if i has not observed $\langle$coordination message$\rangle$ but i play those messages, then there is a $M < \infty$ and event E_1, E_2 such that i's expected continuation pay-off is less than that in equilibrium path by at least*

$$\delta^M \frac{\min\{\beta_i(E_1|h_{N_i}^m), \beta_i(E_2|h_{N_i}^m)\}}{1-\delta}$$

, where m is a period in pre- or post-coordination period, and $\beta_i(E_1|h_{N_i}^m) > 0, \beta_i(T \setminus E_2|h_{N_i}^m) > 0$

Claim 0.1.2.3. *In **COORDINATION**. If there is no $j \in N_i$ has played $\langle 2 \rangle$ or $\langle 1 \rangle$, if $|I_i^t| < s$, and if there is no $j \in N_i$ has played play $\langle l \rangle$ in **CHECK.f**, then if i deviate and such deviation is detected by some $j \in N_i \setminus i$, then there is a $M < \infty$ and event E such that i's expected continuation pay-off is less than that in equilibrium path by at least*

$$\delta^M \frac{\beta_i(E|h_{N_i}^m)}{1-\delta}$$

, where m is a period in pre- or post-coordination period, and $\beta_i(E_1|h_{N_i}^m) > 0$

A Proof

A.1 Proof for Lemma ??

Proof. The proof is by induction. We first show that the statement is true for $t = 1$.

Claim A.1.0.4. ***Base****:* $i \in R^1 \Leftrightarrow [i \in R^0] \wedge [\exists k_1, k_2 \in (R^0 \cap N_i \setminus i)]$.

Proof. $\Rightarrow$: Since $i \in R^1$, then $i \in R^0$ and $\forall j \in N_i \setminus i[I_i^0 \nsubseteq N_j^0]$ by definition. Since $I_i^0 = N_i \cap R^0$ and $i \in N_j^0$, then $\forall j \in N_i \setminus i[\exists k \in (R^0 \cap N_i \setminus i)[k \notin N_j^0]]$. Since the $j \in N_i \setminus i$ is arbitrary, we then have a pair of $k_1, k_2 \in (R^0 \cap N_i \setminus i)$ such that both $k_1 \notin N_{k_2}^0$ and $k_2 \notin N_{k_1}^0$.

$\Leftarrow$: Pick $k \in \{k_1, k_2\} \subseteq N_i \cap R^0$, and pick an arbitrary $j \in (N_i \setminus \{i,k\})$. Note that $k \notin D_j^0$, otherwise there is a circle from i to i since $i \in N_j^0 \subseteq D_j^0$. Hence $[k \in N_i \cap R^0] \wedge [k \notin D_j^0]$, and therefore $[k \in I_i^0] \wedge [k \notin N_j^0]$. Then we have $I_i^0 \nsubseteq N_j^0$ for arbitrary $j \in N_i \setminus i$, and thus $i \in R^1$. □

Induction hypothesis: the statement is true up to t and $t \geq 1$.

Claim A.1.0.5. *If the hypothesis is true, then*

$$i \in R^{t+1} \Leftrightarrow [i \in R^t] \wedge [\exists k_1, k_2 \in R^t \cap N_i \setminus i]$$

Proof. $\Rightarrow$: since $i \in R^{t+1}$, then $i \in R^t$ and $\forall j \in N_i \setminus i[I_i^t \nsubseteq N_j^t]$ by definition. Recall Equation (**??**) and Equation (**??**), then for every $m \in I_i^{t-1}$, we can find a path connecting i to m (the existence of such path is by the induction hypothesis). If $j \in N_i \setminus i$, then we can find a path connecting j to m by connecting j to i, and then connecting i to m. Thus, if $m \in I_i^{t-1}$ then $m \in N_J^t$, and hence $I_i^{t-1} \subseteq N_j^t$ if $j \in N_i \setminus i$.

Further note that $I_i^t = \bigcup_{k \in N_i \cap R^t} I_k^{t-1}$, then we must have $\forall j \in N_i \setminus i[\exists k \in (R^t \cap N_i \setminus i)[I_k^{t-1} \nsubseteq N_j^t]]$, since $I_i^{t-1} \subseteq N_j^t$. Since the $j \in N_i \setminus i$ is arbitrary, we then have a pair of $k_1, k_2 \in (R^t \cap N_i \setminus i)$ such that both $k_1 \notin N_{k_2}^t$ and $k_2 \notin N_{k_1}^t$.

$\Leftarrow$: By the induction hypothesis, we have a chain $k_{1_0}, ..., k_1, i, k_2, ..., k_{2_0}$ with $k_{1_0} \in R^0$,..., $k_1 \in R^t$, $i \in R^t$, $k_2 \in R^t$,...,and $k_{1_0} \in R^0$. Note that $k_{1_0} \notin D_j^t$ whenever $j \in N_i \setminus i$, otherwise there is a circle from i to i since $\{i, k_2, ..., k_{2_0}\} \in N_j^t \subseteq D_j^t$. Hence $[k_{1_0} \in I_{k_1}^{t-1}] \wedge [k_{1_0} \notin D_j^t]$, and therefore $[I_{k_1}^{t-1} \in I_i^t] \wedge [I_{k_1}^{t-1} \notin N_j^t]$. Then we have $I_i^t = \bigcup_{k \in N_i \cap R^t} I_k^{t-1} \nsubseteq N_j^t$ for arbitrary $j \in N_i \setminus i$, and thus $i \in R^1$.

□

We can then conclude that the statement is true by induction.

□

A.2 Proof for Lemma ??

Proof. 1. The proof is by induction, and by Lemma **??**. Since the state has strong connectivity, all the R^0 nodes are connected, and thus we have a R^0-path connecting each pair of R^0 nodes. Since all pairs of R^0 nodes are connected by a R^0-path, then for all pairs of R^1 nodes must be in some of such paths by Lemma **??**, and then connected by a R^0-path. But then all the R^0-nodes in such path are all R^1 nodes by Lemma **??** again and by $R^t \subseteq R^{t-1}$. Thus, for all pairs of R^1 nodes has a R^1-path connecting them. The similar argument holds for $t > 1$, we then get the result.

2. The uniqueness is by the fact that the network is a tree, and therefore the path connecting all distinguish nodes is unique.

□

A.3 Proof for Lemma ??

Proof. We have to show that $R^{t-1} \supseteq \bigcup_{i \in R^t} N_i \cap [H]$ and $R^{t-1} \subseteq \bigcup_{i \in R^t} N_i \cap [H]$.

- $\supseteq$: Since R^t is not empty, we can pick a node $m \in \bigcup_{i \in R^t} N_i \cap [H]$. By Lemma **??**, $m \in R^t \cup R^{t-1} = R^{t-1}$, and therefore $m \in R^{t-1}$.
- $\subseteq$: Since both R^{t-1} and R^t are not empty, we can pick nodes $m_1 \in R^{t-1}$ and $m_2 \in R^t$. Since the state has strong connectivity, there is a R^{t-1} path connecting them by Lemma **??**. But then the nodes (expect for m_1, m_2) in this path are all R^t nodes by Lemma **??**, and then there is $m_1^{'} \in N_{m_1} \cap R^t$. Since the $m_1 \in R^{t-1}$ we picked is arbitrary, therefore it means for all $m \in R^{t-1}$ there is a $m^{'} \in N_m \cap R^t$, and hence $m \in N_{m^{'}} \cap [H]$ while $m^{'} \in R^t$. We then get the result.

□

A.4 Proof for Lemma ??

Proof. 1. If $1 \leq |R^t| \leq 2$, then by Lemma **??** and by Lemma **??**, we have a spanning tree consisting the nodes in R^{t-1},...,R^0. Since the state has strong connectivity, all the H-nodes are in this tree. By Lemma **??**, we have

$$R^0 = \bigcup_{k_1 \in R^1} N_{k_1} \cap [H] = \bigcup_{k_1 \in N_{k_2} \cap R^1} \bigcup_{k_2 \in N_{k_3} \cap R^2} \cdots \bigcup_{k_{t-1} \in N_{k_t} \cap R^t} N_{k_t} \cap [H]$$

Then by Equation (14), if $i \in R^t$ we have

$$I_i^t = \bigcup_{k_0 \in N_i \cap R^t} \bigcup_{k_1 \in N_{k_0} \cap R^{t-1}} \cdots \bigcup_{k_{t-1} \in N_{k_{t-2}} \cap R^1} N_{k_{t-1}} \cap R^0$$

Now note that $R^0 = [H]$, and compare the above two equations, we got $[H] = I_i^t$ if $i \in R^t$.

2. For a given t-block, in the case when $R^t \neq \emptyset$ and $R^{t+1} \neq \emptyset$, the proof is a direct application of Lemma **??**, and we continue taking the union of nodes' information set. Since the network is finite, the $[H]$ will be a subset of some nodes' information set eventually.

   We then only consider the case when $R^t \neq \emptyset$ and $R^{t+1} = \emptyset$. But in such case, it means that there is no R^t node which has more than two distinguish R^t neighbours by Lemma **??**, and then $1 \leq |R^t| \leq 2$ since all pairs of R^t nodes are connected by R^t-path by Lemma **??**. The first part of this Lemma **??** then shows the result.

□

A.5 Proof for Lemma ??

Proof. Suppose there are three or more R^t-nodes in C, then pick any three nodes of them, and denote them as i_1, i_2, i_3. Let's say i_2 is in the $(i_1 i_3)$-path, and therefore $i_2 \in Tr_{i_1 i_2}$ and $i_3 \in Tr_{i_2 i_3}$. First we show that $i_1 \in N_{i_2}$ (or $i_3 \in N_{i_2}$). Suppose $i_1 \notin N_{i_2}$, since $i_1, i_2 \in R^t$, then the $(i_1 i_2)$-path is a R^t-path by Lemma **??**. Let this $(i_1 i_2)$-path be $(i_1, j_1, ..., j_n, i_2)$. Since $i_1, j_1, ..., j_n, i_2 \in R^t$, we then have $I^{t-1}_{i_1} \nsubseteq N^{t-1}_{j_1}, ..., I^{t-1}_{j_n} \nsubseteq N^{t-1}_{i_2}$ and $I^{t-1}_{j_1} \nsubseteq N^{t-1}_{i_1}, ..., I^{t-1}_{i_2} \nsubseteq N^{t-1}_{j_n}$. Since $I^{t-1}_{i_1} \subseteq N^{t-1}_{i_1}, ..., I^{t-1}_{i_2} \subseteq N^{t-1}_{i_2}$ by Lemma **??**, we further have $\exists k_1 \in [H][k_1 \in N^{t-1}_{j_1} \setminus I^{t-1}_{i_1}]$,...,$\exists k_n \in [H][k_n \in N^{t-1}_{j_n} \setminus I^{t-1}_{i_2}]$. Since the state has Strong Connectivity, such $k_1, ..., k_n$ are connected. But then we have already found k_1, k_2 such that $k_1 \in N^{t-1}_{j_1} \setminus I^{t-1}_{i_1}$ and $k_2 \in N_{k_1} \setminus k_1$. It is a contradiction that $i_1 \in C$.

Now, i_1, i_2, i_3 will form a R^t-path as (i_1, i_2, i_3). With the same argument as the above, we then have $\exists k_1 \in [H][k_1 \in N^{t-1}_{i_2} \setminus I^{t-1}_{i_1}]$ and $\exists k_2 \in [H][k_2 \in N^{t-1}_{i_3} \setminus I^{t-1}_{i_2}]$, and thus i_1 is not in C. □

A.6 Proof for Lemma ??

Proof. Since $i \in R^t$, there is a $j \in R^{t-1}$ and $j \in N_i \setminus i$ by Lemma **??**. Given any $j \in R^{t-1} \cap (N_i \setminus i)$, first note that $N^{t-1}_j \subseteq \bigcup_{k \in N^{t-1}_i} N_k$ by $N^{t-1}_j \equiv \bigcup_{k \in I^{t-2}_j} N_k$, and $I^{t-2}_j \subseteq I^{t-1}_i \subseteq N^{t-1}_i$. If there is another node outside $\bigcup_{k \in N^{t-1}_i} N_k$ in Tr_{ij}, then there must be another node connected to N^{t-1}_j since the network is connected. It is a contradiction that $i \in C$.

□

A.7 Proof for equilibrium

A.7.1 Proof for Claim 0.1.0.1

A.8 Proof for reporting period

A.8.1 Proof for Claim 0.1.1.1

Proof. Assume $\bar{I}^{t-1}_i \neq I^{t-1}_i$. Since a detection of deviation has not occur, it must be the case that there is a non-empty set $F = \{j \in \bar{I}^{t-1}_i : \theta_j = Inerts\}$[3].

Let the set

$$E_1 = \{\bar{\theta} : \bar{\theta}_j = Rebel \text{ if } j \in F \text{ and } \bar{\theta}_j = \theta_j \text{ if } j \notin F\}$$

[3]Otherwise, there is a detection of deviation. Recall the definition in information hierarchy: $I^{-1}_i \subset I^0_i \subset ... \subset I^{t-1}_i$

be the set of pseudo events by changing θ_j where $j \in F$. And let

$$E_2 = \{\theta : \theta_j = Inert \text{ if } j \in F \text{ and } \bar{\theta}_j = \theta_j \text{ if } j \notin F\}$$

be the set of true event.

Then consider the event

$$\begin{aligned} E &= \{\bar{\theta} \in E_1 : \#[Rebels](\bar{\theta}) \geq k\} \\ &= \{\theta \in E_2 : \#[Rebels](\theta) \geq k - \#F\} \end{aligned}$$

Partition E as sub events

$$\begin{aligned} E_3 &= \{\theta \in E_2 : \#[Rebels](\theta) \geq k\} \\ E_4 &= \{\theta \in E_2 : k > \#[Rebels](\theta) \geq k - \#F\} \end{aligned}$$

By Lemma and following the strategies in equilibrium path (since i have not been detected), there is a block $\bar{t}^s$ with respect to $\bar{\theta}$ such that if $\bar{\theta} \in E$ then there some $R^{\bar{t}^s}$ Rebel js, says J, will initiate the coordination, and then Rebels play **revolt** forever after $\bar{t}^s$-block.

If $i \in J$, his own initiation will only depends on $\#I_i^{\bar{t}^s}$ by Claim, which is the same as he has reported $\langle I_i^{t-1} \rangle$. It is strictly better by not deviating since playing $\langle \bar{I}_i^{t-1} \rangle$ is more costly than $\langle \bar{I}_i^{t-1} \rangle$ ($X_{\bar{I}_i^{t-1}} > X_{I_i^{t-1}}$).

If there is another j who $\bar{I}_i^{t-1} \not\subset I_j^{\bar{t}^s}$, then which is the same as he has reported $\langle I_i^{t-1} \rangle$. It is strictly better by not deviating since playing $\langle \bar{I}_i^{t-1} \rangle$ is more costly than $\langle \bar{I}_i^{t-1} \rangle$.

If there is another j who $\bar{I}_i^{t-1} \subset \bar{I}_j^{\bar{t}^s}$ and $\#I_i^{\bar{t}^s} \geq k$, take the event $E = \{\theta : \theta_j = Rebel\}$. In this event, no one will initiate. Note that this event is independent of $\bar{I}_i^{t-1}$.

If the event $\{\bar{\theta} : \#\bar{I}_j^{\bar{t}^s} \geq k\}$ E_2 and E_3 has positive probability. If each one has zero probability. same for not deviating.

After $\bar{t}^s$, i's expected static pay-off at most

$$\max\{\beta_i(E_3|h_{N_i}^m) \times 1 + \beta_i(E_4|h_{N_i}^m) \times (-1), 0\}$$

If i deviate, the expected static pay-off after $\max\{t^s, t^f\}$ is

$$\max\{\beta_i(E_3|h_{N_i}^{m'}), 0\}, \text{when } m' > \max\{t^s, t^f\}$$

$$\max\{\beta_i(E_1|h_{N_i}^{m'})(1), 0\}, \text{when } m' > \max\{t^s, t^f\} \tag{1}$$

Then there is a loss compared with equilibrium path in expected static pay-off contingent on E by

$$\min\{\beta_i(E_1|h_{N_i}^{m'}), \beta_i(E_2|h_{N_i}^{m'}), \text{when } m' > \max\{t^s, t^f\} \tag{2}$$

Thus, there is a loss in expected continuation pay-off contingent on E by

$$\delta^{\max\{t^s, t^f\}} \frac{\min\{\beta_i(E_1|h_{N_i}^{m'}), \beta_i(E_2|h_{N_i}^{m'})}{1 - \delta}$$

$\square$

A.8.2 Proof for Claim 0.1.1.2

Proof. Since $\bar{I}_i^{t-1} \subset I_i^{t-1}$, and such deviation is not detected by i's neighbour, then i must report some L-nodes; otherwise, due to $I_i^{t-1} = \bigcup_{j \in N_i \cap R^{t-1}} I_j^{t-2}$, some neighbours has detected it.

Denote L_D be the set of L-nodes i reported. Let the events be

$$\begin{aligned} E_1 &= [|[H]| < s - |L_D|] \\ E_2 &= [s - |L_D| \leq |[H]| < s] \\ E_3 &= [|[H]| \geq s] \end{aligned}$$

Now contingent on the event $E_2 \cup E_3$, and consider the following cases.

- If there is no positive probability such that there is a node who will initiate the coordination contingent on $E_2 \cup E_3$, then there is a static pay-off loss as $-\delta^{|RP^t|-|I_i^{t-1}|+1} + \delta^{|RP^t|-|\bar{I}_i^{t-1}|+1}$.
- If there is a positive probability such that i will initiate the coordination contingent on $E_2 \cup E_3$, then there is a static pay-off loss as $-\delta^{|RP^t|-|I_i^{t-1}|+1} + \delta^{|RP^t|-|\bar{I}_i^{t-1}|+1}$.
- If there is a positive probability on E such that $E =$ [there is another node other than i who will initiate the coordination contingent on $E_2 \cup E_3$, but i can not send the coordination messages.], then, by similar argument as Claim 0.1.2.3, there is loss compared to equilibrium in expected continuation pay-off. Denote t^s be the block when such coordination happen. The loss will be

$$\delta^{t^s} \frac{\min\{\beta_i(E \cap E_3 | h_{N_i}^m), \beta_i(E \cap E_2 | h_{N_i}^m)\}}{1-\delta}$$

□

A.8.3 Proof for Claim 0.1.1.3

Proof. We first prepare some facts. Since $|Ex_{I_i^{m,t}} \cup I_i^{m,t}| \geq s$ and $|I_i^{m,t}| < s-1$, we have $|Ex_{I_i^{m,t}}| \geq 2$. Since $i \notin C$, then

$$\exists j \in R^{t-1} \cap (N_i \setminus i)[\exists k_1, k_2 \in Tr_{ij}[[k_1 \in N_j^{t-1} \setminus I_i^{t-1}] \wedge [k_2 \in N_{k_1} \setminus k_1]]]$$

no matter such j has reported or not.

Now we argue that i will not deviate.

1. Case $j \in O_i^{m,t}$:
   - We first argue that if i deviated by reporting $\langle 1 \rangle$, then such deviation has been detected by j. Then we can follow the Claim 0.1.1.1 to construct the event E. This is because i can report $\langle 1 \rangle$ only if $|I^{m,t}| = s-1$ with the condition **CHECK.m.Case.2** or $|I_i^{t-1}| = s-1$ with the condition **MAIN.1**. Clearly, i did not satisfy condition **CHECK.m.Case.2**, since j has reported and there is a node outside N_j^{t-1}. If i satisfies condition **MAIN.1**, then since j has reported, i should already have $|I_i^{m,t}| \geq s$, and then i should have reported $\langle 2 \rangle$.

- Next we argue i will not deviate by reporting $\langle 2 \rangle$. Since there is a node outside N_j^{t-1}, such node is outside I_j^{t-1}, and therefore outside $I_i^{m,t}$ (since j has reported). Then there is an event E which has positive probability,

  $$E = [|I_i^{RP^t - |2|,t}| = s-1]$$

  Contingent on E, there is a sub-event $E^{'} \subset E$ which has positive probability,

  $$E^{'} = [|I_i^{RP^t,t}| = s-1]$$

  Therefore, contingent on $E^{'}$, i will be with $|I_i^{RP^t,t}| = |I_i^t| = s-1 < s$. Moreover, the belief of i after $|RP^t|$ is with $\beta_i(|[H]| \geq s | h_{N_i}^{|RP^t|}) > 0$ since $|Ex_{I_i^{m,t}}| \geq 1$ by the face that there is one more node outside N_j^{t-1}.
  Now if i report $\langle 2 \rangle$, in the equilibrium path, then i should play $\mathbf{1}_i$ in **CHECK.f** and initiate the coordination in the coordination period in the current block, and then i will initiate the coordination before observing coordination message with $|I_i^t| = s-1 < s$ and $\beta_i(|[H]| \geq s | h_{N_i}^{|RP^t|}) > 0$. By Claim 0.1.2.3, we can then find an event $E^{''} \subseteq E^{'}$ and $M < \infty$ such that there is a loss compared to equilibrium path in expected continuation pay-off by

  $$\delta^M \frac{\min\{\beta_i(E^{''}|h_{N_i}^m), \beta_i(T \setminus E^{''}|h_{N_i}^m)\}}{1-\delta}$$

2. Case $j \notin O_i^{m,t}$: Since $|Ex_{I_i^{m,t}} \cup I_i^{m,t}| \geq s$, $|I_i^{m,t}| < s-1$ and there is one more node outside N_j^{t-1}, then the following event E has positive probability,

   $$E = [|I_i^{|RP^t|-|2|,t}| = s-1]$$

   Contingent on E, there is an event $E^{'} \subseteq E$,

   $$E^{'} = E \cap [j \text{ will report at } m^{'}, \text{ where } m < m^{'} < |RP^t| - |2|]$$

   . Note that $E^{'}$ has positive probability. This is because $|I_i^{t-1}| \leq |I_i^{m,t}| < s-1$ and $|I_j^{t-1} \cap I_j^{t-1}| \geq 2$, therefore the event $[|I_j^{t-1}| < s-1]^4$ has positive probability, and therefore j will report later in the equilibrium.

   Then, contingent on $E^{'}$, such j will be in $O_i^{m^{'},t}$ at $m < m^{'} < |RP^t| - |2|$. We apply the argument in Case $j \in O_i^{m,t}$ again by replacing m by $m^{'}$. We now conclude that the statement in this claim holds.

□

[4]More specifically, let $|I_j^{t-1}| = |I_j^{t-1} \cap I_j^{t-1}| + 1$. Since $|I_i^{t-1}| = |I_j^{t-1} \cap I_j^{t-1}| + k < s-1$ for some $k \geq 0$, and since $i \in R^t$, therefore $k \geq 1$. Then when $k = 1$, we have $|I_j^{t-1}| < s-1$

A.8.4 Proof for Claim 0.1.1.5

A.8.5 Proof for Claim 0.1.1.6

A.9 Proof for coordination period

A.9.1 Proof for Claim 0.1.2.2

Proof. If i has not observed $\langle$coordination message$\rangle$ but i play those messages: Since $|I_i^t| < s$ and since i has not observed $\langle$coordination message$\rangle$, and due to the equilibrium strategies playing by neighbours, we have

$$0 < \beta_i(|[H]| \geq s | h_{N_i}^m) < 1$$

Since i play coordination message, all i's neighbour who did not detect the deviation will form their belief as Equation **??** and Equation **??**, and all his neighbour who detected this deviation will form the belief as Equation **??** and Equation **??**. But i's neighbour will play h or l forever in all following reporting period and coordination period, and then

$$h_{N_i}^{m'} = h_{N_i}^m \text{ whenever } m' > m$$

, hence

$$\beta_i(|[H]| \geq s | h_{N_i}^{m'}) = \beta_i(|[H]| \geq s | h_{N_i}^m) \text{ whenever } m' > m$$

Let $E_1 = [|[H]| \geq s]$, and then $E_2 = [|[H]| < s]$.

Let t^s be the block when coordination success, and let t^f be the block when coordination fail. Since i can not change his neighbours' beliefs after-ward by Equation **??** and Equation **??**, the expected static pay-off for i after $\max\{t^s, t^f\}$ contingent on E is at most

$$\beta_i(E|h_{N_i}^{m'})(1) + \beta_i(T \setminus E|h_{N_i}^{m'})(-1), 0\}, \text{when } m' > \max\{t^s, t^f\} \tag{3}$$

, where $\beta_i(E_1|h_{N_i}^{m'})(1) + \beta_i(E_2|h_{N_i}^{m'})(-1)$ is the expected pay-off by playing h, and 0 is the expected pay-off by playing l.

However, if i follow the equilibrium, i's expected static pay-off contingent on E is

$$\max\{\beta_i(E_1|h_{N_i}^{m'})(1), 0\}, \text{when } m' > \max\{t^s, t^f\} \tag{4}$$

Then there is a loss compared with equilibrium path in expected static pay-off contingent on E by

$$\min\{\beta_i(E_1|h_{N_i}^{m'}), \beta_i(E_2|h_{N_i}^{m'}), \text{when } m' > \max\{t^s, t^f\} \tag{5}$$

Thus, there is a loss in expected continuation pay-off contingent on E by

$$\delta^{\max\{t^s, t^f\}} \frac{\min\{\beta_i(E_1|h_{N_i}^{m'}), \beta_i(E_2|h_{N_i}^{m'})}{1 - \delta}$$

□

A.9.2 Proof for Claim 0.1.2.3

Proof. We will discuss two cases. Case 1 is that there is a neighbour has sent coordination messages, but i did not follow to send messages. Case 2 is that no neighbours has sent coordination messages, but i deviate and detected by some neighbours. Denote D be the set of neighbours who detect this deviation.

1. Case 1: Let $j \in N_i \setminus i$ be the neighbour who has sent the messages. When $t = 0$, since there is only one sub-block in pre-coordination period, i has no incentive to play l in the following post-coordination period since the coordination has happened. When $t \geq 1$, note that $j \in D$. Now let the event E be

$$E = [\text{there is no more } H\text{-node in } Tr_{ik} \text{ outside } I_i^t, \text{ where } k \notin D\,]$$

   Those $j \in D$ will play l forever since $|I_j^0| < s$; otherwise, j has already initiated the coordination in $t = 0$. Moreover, the nodes in Tr_{ij} will play l forever eventually since j will play $\langle l \rangle$ in **CHECK.f**. Contingent on E, those $k \notin D$ has not played $\langle 2 \rangle$ or $\langle 1 \rangle$ by assumption, therefore the event of $[|I_k^t| < s]$ has positive probability, and thus E has positive probability.

2. Case 2: If what i deviate is to send $\langle$coordination messages$\rangle$, then the argument is by Claim 0.1.2.2. If not, we first denote the events

$$\begin{aligned} E_1 &= [|[H]| < s] \\ E_2 &= [s \leq |[H]| < s + |D|] \\ E_3 &= [|[H]| \geq s + |D|] \end{aligned}$$

   , and let the event E be

$$E = E_2$$

In equilibrium path, let t^s be the period when coordination success, and let t^f be the period when coordination fail. Since i deviate, contingent on E, the most expected static pay-off i can get after t^s is

$$0, \text{when } m' > t^s$$

If i follows the equilibrium path, contingent on E, the expected static pay-off after t^s is

$$\beta_i(E|h_{N_i}^{m'})(1), \text{when } m' > t^s$$

For the other event in $T \setminus E$, after $\max\{t^s, t^f\}$, the expected static pay-off is at most the same as following equilibrium. Thus, there is a loss in expected continuation pay-off contingent on E by

$$\delta^{t^s} \frac{\beta_i(E|h_{N_i}^{m'})}{1-\delta}, \text{when } m' > t^s$$

□