Coordination in Social Networks

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  - [Wolitzky] models network-monitoring.
- This paper provides a partial folk theorem with incomplete information and network-monitoring.
  - Will people act collectively in networks **eventually**?

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- [Chwe]: one-shot collective action (in terms of revolution).
  - Players of two types (**Rebel**,**Inert**). They can **observe own/neighbor's type**.
  - Rebel's pay-off contingent on global type distribution.
- [Chwe]'s result: the ex-post efficient outcome "guaranteed" by complete network.

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  - Players can observe **own/neighbors' actions**.
- Goal: looking for an equilibrium, in which the global type distribution becomes commonly known in finite time.
- Result: such equilibrium can be constructed under some assumptions.

Related Literature

- Collective action.
  - One strand: [Chwe 2000], [Lohmann, 1993,1994], etc
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- Collective action.
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  - **This paper adds network-monitoring**
- Repeated game in networks.
  - One strand: [Wolitzky 2013]
  - **This paper adds incomplete information**

Network

- n players; $N = \{1, ..., n\}$ is the set of players.
- G_i is i's neighborhood; G_i is a subset of N such that $i \in G_i$.
- $G = \{G_i\}_i$ is the network.

ASSUMPTION

G is fixed (not random), finite, connected, commonly known, and undirected.

Static k-threshold game [Chwe 2000]

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- $\theta \in \Theta = \times_{i \in N} \Theta_i$: type profile
- $\pi \in \Delta\Theta$: the prior

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- $\pi \in \Delta\Theta$: the prior
- $A_{Rebel} = \{\textbf{revolt}, \textbf{stay}\}$; $A_{Inert} = \{\textbf{stay}\}$

Static k-threshold game [Chwe 2000]: ▸ Or

- Static game payoff for Rebel i: $u_{\text{Rebel}_i}(a_{\text{Rebel}_i}, a_{-\theta_i})$

$$
\begin{aligned}
u_{\text{Rebel}_i}(a_{\text{Rebel}_i}, a_{-\theta_i}) &= 1 \quad \text{if } a_{\text{Rebel}_i} = \textbf{revolt} \text{ and } \#\{j : a_{\theta_j} = \textbf{revolt}\} \geq k \\
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ASSUMPTION

Players perfectly observe their neighbors' types.

- *Remark*: **stay** is a safe arm; **revolt** is a risky arm.

Static *k*-threshold game [Chwe 2000]-example

- $n = 3$ and $k = 3$

- For some π, this network does not sustain ex-post efficient outcome.

MODEL

Repeated k-threshold game: time line

- Nature choose θ initially according to π.
- Types are then fixed over time.
- Players play the static k-threshold game infinitely repeatedly.

ASSUMPTION

- *Players perfectly observe their neighbors' types.*
- *Players perfectly observe their neighbors' actions.*
- π *has full support*
- *Common* δ.
- *Static pay-off could be observable, noisy or hidden.*

Look for

- An equilibrium, the ex-post efficient outcome repeats after some finite time T in the path.

Notations:

- $[\textit{Rebels}](\theta) = \{j : \theta_j = \textit{Rebel}\}$ for all $\theta \in \Theta$.
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- θ_{G_i}: i's private information about the state. $(\theta_{G_i} \in \Theta_{G_i} = \prod_{j \in G_i} \Theta_j)$
- $h^m_{G_i}$: the history observed by i up to period m. $(h^m_{G_i} \in H^m_{G_i} = \prod_{s=1}^{m} \prod_{j \in G_i} A_{\theta_j})$
- h: an infinite sequence of players' actions. $(h \in H = \prod_{s=1}^{\infty} \prod_{j \in N} A_{\theta_j})$

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- $\tau_i : \Theta_{G_i} \times \bigcup_0^\infty H^m_{G_i} \to A_{\theta_i}$, i's strategy.
- $\tau = (\tau_1, ..., \tau_i, ..., \tau_n)$: a strategy profile.

Model

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- $\beta_i^{\pi,\tau}(\theta | h^m_{G_i})$: i's belief for a θ at period m given τ.

Notations:

- h_θ^τ : a history generated by τ given θ.
- Call h_θ^τ a τ_θ-path.
- Call $\{h_\theta^\tau\}_{\theta\in\Theta}$ the τ-path

DEFINITION

The τ-path is **approaching ex-post efficient** (*APEX*) $\Leftrightarrow$

$$\forall\theta, \text{ there is a finite time } T^\theta$$

such that the actions after T^θ in τ_θ repeats the static ex-post efficient outcome.

APEX

Definition (weak APEX equilibrium)

A weak sequential equilibrium (τ^*, β^*) is APEX $\Leftrightarrow$ τ^*-path is APEX, and β^* is the belief system consistent with τ^*.

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Definition (APEX equilibrium)

A sequential equilibrium (τ^*, β^*) is APEX $\Leftrightarrow$ (τ^*, β^*) is a weak APEX equilibrium and β^* is fully consistent with τ^*[Krep and Wilson 1982].

APEX for $k = n$

- $k = n$: For all networks, an APEX equilibrium can be found whenever δ is sufficiently high.

APEX-EXAMPLE FOR $k = n$

If pay-off is observable, for $k = n = 3$:

- All Rebels play **revolt** in the first period $\Rightarrow$ then state will be revealed.

APEX-EXAMPLE FOR $k = n$

If pay-off is hidden or noisy, for $k = n = 3$:

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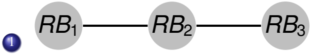

- Rebel 2 chooses **revolt** at the first period $\Rightarrow$ the state can be revealed.

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2. 

APEX-EXAMPLE FOR $k = n$

If pay-off is hidden or noisy, for $k = n = 3$:

1. 

- Rebel 2 chooses **revolt** at the first period $\Rightarrow$ the state can be revealed.

2. 

- Rebel 2 chooses **stay** at the first period $\Rightarrow$ the state can be revealed.

APEX FOR $k < n$

- $k < n$: with additional assumptions,
  - acyclic networks (tree networks): a weak APEX equilibrium can be found when δ is high enough.
  - cyclic networks: open question.

DEFINITION (PATH IN A NETWORK)

A **path** from node i to node j is a sequence of nodes

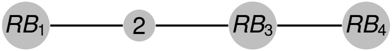
$$\{i, m_1, m_2, ..., m_n, j\} \text{ without repetition}$$

such that $i \in G_{m_1}, m_1 \in G_{m_2}, ..., m_n \in G_j$.

DEFINITION (ACYCLIC NETWORK (TREE))

A network is **acyclic** $\Leftrightarrow$ the path from node i to node j is unique for all nodes i, j.

If pay-off is observable, for $k = 3$ and $n = 4$:



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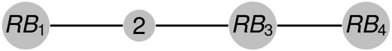

- An APEX equilibrium does not exist.

DEFINITION

θ has **Strong connectedness**$\Leftrightarrow$ for every pair of Rebels, there is a path consisting of Rebels to connect them.

DEFINITION

π has **full support on strong connectedness**$\Leftrightarrow$

$$\pi(\theta) > 0 \text{ if and only if } \theta \text{ has strong connectedness.}$$

- I.e. Commonly certainty of strong connectedness.

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ASSUMPTION

π *has **full support on strong connectedness**.*

APEX-EXAMPLE FOR $k < n$

If pay-off is hidden or noisy, for $k = 3$ and $n = 4$ with strong connectedness:

$$1 - RB_2 - RB_3 - RB_4$$

- An APEX equilibrium exists—same idea: Rebel 3 play a "*coordination message*"
  $\Rightarrow$ state can be revealed.

APEX-example for $k < n$

If pay-off is hidden or noisy, for $k = 3$ and $n = 4$ with strong connectedness:

$$1 \;\text{———}\; RB_2 \;\text{———}\; RB_3 \;\text{———}\; RB_4$$

- An APEX equilibrium exists—same idea: Rebel 3 play a "*coordination message*"

  $\Rightarrow$ state can be revealed.
- Later, I generalize the case of $k < n$ for acyclic networks.

Outline:

1. Communication by actions

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1. Communication by actions
2. Communication in the equilibrium
    1. Communication protocol
    2. In-the-path belief
    3. Off-path belief
    4. Sketch of proof

Communication by actions

Main idea

Ex. for $n = 5$ network:

- First step: index each node a distinguish prime number.

Ex. for $n = 5$ network:

$$2_1 - 3_2 - 5_3 - 7_4 - 11_5$$

- First step: index each node a distinguish prime number.
- This indexation is commonly known.

Communication by actions

Main idea-conti

Ex. for $k = 4$, $n = 5$ with strong connectedness:

- Second step: build a communication protocol.

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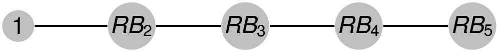

- Second step: build a communication protocol.
- If the incentive issue is ignored, ideally,

	Reporting period	*Coordination period*	...
	1,2,...,2310	2311,...,2421	2422,...
RB_2	$\mathbf{s}, \dots, \mathbf{s}, \overbrace{\mathbf{r}, \mathbf{s}, \dots, \mathbf{s}}^{2\times 3}$	$\neg$ send "coordination message"	play **revolt afterward**
RB_3	$\mathbf{s}, \dots, \mathbf{s}, \overbrace{\mathbf{r}, \mathbf{s}, \dots, \mathbf{s}}^{2\times 3\times 5}$	send "coordination message"	play **revolt afterward**

Communication Phases

Phases

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2. **CD** (Coordination period): coordinating the future actions.

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Phases

1. **RP** (Reporting period): revealing the information about θ.
2. **CD** (Coordination period): coordinating the future actions.
3. RP and CD alternate finitely.

$$\underbrace{\langle RP \rangle \langle CD \rangle}_{\textbf{block}} \ldots$$

4. Call a complete two phases, $\langle RP \rangle \langle CD \rangle$, a **block**.

In coordination period,

- “three” messages coordinate actions

Messages	Continuation actions
message to revolt	play **revolt** afterward
message to stay	play **stay** afterward
Other messages	continue to next block

- Communication either stops or continues after a CD.
  1. Stopping: If **Message to stay** or **Message to revolt** is sent $\Rightarrow$ all Rebels coordinate to play same actions.
  2. Continuing: Otherwise, go to the next block.

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Lemma

*Before a Rebel knows $\#[Rebels](\theta) < k$ or $\#[Rebels](\theta) \geq k$, he will not send **Message to stay** or **Message to revolt** if δ is high enough.*

- a "grim trigger".

Comment

- RP^t: the reporting period at t block
- $\langle RP^t \rangle$: the reporting message

Costly message	$\neg\langle \mathbf{stay} \rangle$	$\mathbf{s}, ..., \mathbf{s}, \mathbf{r}, \mathbf{s}, ..., \mathbf{s}$
Not costly message	$\langle \mathbf{stay} \rangle$	$\mathbf{s}, ..., \mathbf{s}, \mathbf{s}, \mathbf{s}, ..., \mathbf{s}$

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- Gives incentive to play costly message.
  1. Costly message+**message to revolt**: coordination to **revolt**
  2. Otherwise, no coordination to **revolt**
- How much cost should a Rebel take? — Characterization in the next slides.

Information Hierarchy

Information Hierarchy

- Characterizing Rebels' incentives in playing costly messages ▸ other reason

Ex:

$$0 - 1 - RB_2 - RB_3 - RB_4 - RB_5 - RB_6 - 7$$

Information Hierarchy

- Characterizing Rebels' incentives in playing costly messages ▸ other reason

Ex:

$0 - 1 - RB_2 - RB_3 - RB_4 - RB_5 - RB_6 - 7$

- **Rebel 2 has less incentive**: Rebel 2's information can be reported by Rebel 3 to Rebel 4.

Information Hierarchy

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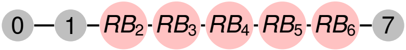

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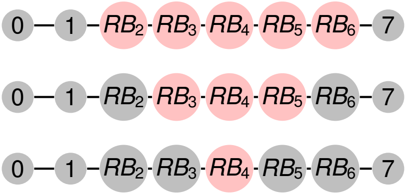

1. At 0-block, let $R^0 = \{2, 3, 4, 5, 6\}$
2. At 1-block, let $R^1 = \{\quad 3, 4, 5 \quad\}$
3. At 2-block, let $R^2 = \{\qquad 4 \qquad\}$

details

The Rebels known by i after t-block: I_i^t.

Theorem

Given θ, if

1. *the network is acyclic*
2. *the state has strong connectedness*

$\Rightarrow \exists t^\theta$ *and* $\exists i \in R^{t^\theta}$ *such that* $I_i^{t^\theta} \supset [Rebels](\theta)$.

Thus, ideally, APEX can be attained by

- At t block

R^t Rebels	play	$\langle I_i^{t-1} \rangle$	**s**, ..., **s**,	$\overbrace{\textbf{r}, \textbf{s}, \ldots, \textbf{s}}^{\text{Multiplication of } I_i^{t-1} \text{ Rebels' prime numbers}}$
non-R^t Rebels	play	$\langle$**stay**$\rangle$	**s**, ..., **s**, **s**, **s**, ..., **s**	

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- However, "Pivotal Rebels" will deviate.

Relevant information: $\#[Rebels](\theta) \geq k$ or $\#[Rebels](\theta) < k$.

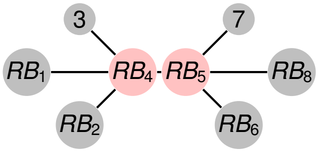
DEFINITION (PIVOTAL PLAYER IN RP^t)

i is **pivotal** in RP^t

$$\Leftrightarrow$$

$i \in R^t$ and i will learn the relevant info before I_i^{t-1} is reported given others' truthful reporting.

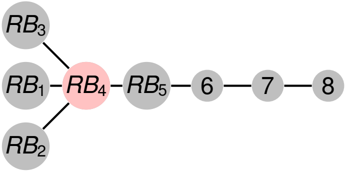
Ex. $k = 5$,



1. Rebel 4 and Rebel 5 are pivotal (**Free Rider problem**)
2. They can manipulate their reporting to save costs.

Go to discussion

Ex. $k = 6$,



1. Rebel 4 is pivotal (given Rebel 5's reporting)
2. He can manipulate his reporting to save costs.

Definition (Free rider in RP^t)

i is a **free rider** in RP^t $\Leftrightarrow$

1. i is pivotal in RP^t
2. i will learn $\#[Rebels](\theta)$ before I_i^{t-1} is reported.

Definition (Free Rider Problem in RP^t)

A **free rider problem** occurs in RP^t $\Leftrightarrow$ There are more than 2 free riders in RP^t.

LEMMA

If networks are acyclic, then

- *there is a unique PR^t where Free Rider Problem may occur.*
- *there are only two free riders i, j are involved. Moreover $i \in G_j$.*
- *Moreover, before PR^t and after CD^{t-1}, i, j both certain that they will be involved in free rider problem.*

Thus, before RP^t and after CD^{t-1}, pick one of them as a free rider.

Non-pivotal R^t Rebels	play	$\langle I_i^{t-1} \rangle$	$\mathbf{s}, ..., \mathbf{s}, \overbrace{\mathbf{r}, \mathbf{s}, ..., \mathbf{s}}^{\prod_{j \in I_i^{t-1}} x_j}$
Pivotal R^t Rebels	may play	$\langle 1 \rangle$	$\mathbf{s}, ..., \mathbf{s}, \mathbf{s}, \mathbf{s}, ..., \mathbf{r}$
non-R^t Rebels	play	$\langle \textbf{stay} \rangle$	$\mathbf{s}, ..., \mathbf{s}, \mathbf{s}, \mathbf{s}, ..., \mathbf{s}$

I.e. Add $\langle 1 \rangle$ into the equilibrium path.

In the equilibrium path,

LEMMA

If networks are acyclic,

$$i \text{ is pivotal but } i \text{ is not free rider in } RP^t$$

$\Rightarrow$

$$i \text{ has learned that } \#[Rebels](\theta) \geq k - 1 \text{ in } RP^t$$

LEMMA

If networks are acyclic,

$$i \text{ play } \langle 1 \rangle \text{ in } RP^t$$

$\Leftrightarrow$

$$i \text{ has learned that } \#[Rebels](\theta) \geq k - 1 \text{ in } RP^t$$

Consequently, if *i* play $\langle 1 \rangle$ in the path

In RP^t, i plays	is i a free rider?	In RP^t, $j \in G_i$ plays	After RP^t, i knows
$\langle 1 \rangle$	yes	$\langle \cdot \rangle$	$\#[Rebels](\theta) \geq k$

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$\langle 1 \rangle$	yes	$\langle \cdot \rangle$	$\#[Rebels](\theta) \geq k$
$\langle 1 \rangle$	no	$\langle 1 \rangle$	$\#[Rebels](\theta) \geq k$
$\langle 1 \rangle$	no	$\langle \textbf{stay} \rangle$	$\#[Rebels](\theta) < k$

$\Rightarrow$ i can tell the relevant info. after RP^t.

Consequently, pivotal i has to play **message to stay** or **message to revolt**

Table : Equilibrium path if i played $\langle 1 \rangle$

In RP^t	In $CD^t_{1,1}$	In $CD^t_{1,2}$	After CD^t
i plays	i plays	i plays	
$\langle 1 \rangle$	$\langle$**stay**$\rangle$	$\langle$**stay**$\rangle$	**stay**
$\langle 1 \rangle$	$\langle \mathbf{x}_i \rangle$	$\langle$**stay**$\rangle$	**revolt**

TABLE : Belief updating after CD^t, $t > 0$

In RP^t	In $CD^t_{1,1}$	In $CD^t_{1,2}$	
i plays	i plays	i plays	The events $j \in G_i$ believes with probability one
$\langle I_i^{t-1} \rangle$	$\langle$**stay**$\rangle$	$\langle$**stay**$\rangle$	$\#[Rebels](\theta) < k$
$\langle I_i^{t-1} \rangle$	$\langle \mathbf{x}_i \rangle$	$\langle$**stay**$\rangle$	$\#[Rebels](\theta) \geq k$
$\langle 1 \rangle$	$\langle$**stay**$\rangle$	$\langle$**stay**$\rangle$	$\#[Rebels](\theta) < k$
$\langle 1 \rangle$	$\langle \mathbf{x}_i \rangle$	$\langle$**stay**$\rangle$	$\#[Rebels](\theta) \geq k$

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In RP^t	In $CD^t_{1,1}$	In $CD^t_{1,2}$	
i plays	i plays	i plays	The events $j \in G_i$ believes with probability one
$\langle \textbf{stay} \rangle$	$\langle \mathbf{x}_i \rangle$	$\langle \textbf{stay} \rangle$	$i \notin R^t$
$\langle I_i^{t-1} \rangle$	$\langle \mathbf{x}_i \rangle$	$\langle \mathbf{x}_i \rangle$	$i \in R^t$

Belief updating in equilibrium path

Table : Belief updating after CD^t, $t > 0$

In RP^t	In $CD^t_{1,1}$	In $CD^t_{1,2}$	
i plays	i plays	i plays	The events $j \in G_i$ believes with probability one
$\langle \textbf{stay} \rangle$	$\langle \mathbf{x}_i \rangle$	$\langle \textbf{stay} \rangle$	$i \notin R^t$
$\langle I_i^{t-1} \rangle$	$\langle \textbf{stay} \rangle$	$\langle \textbf{stay} \rangle$	$\#[Rebels](\theta) < k$
$\langle I_i^{t-1} \rangle$	$\langle \mathbf{x}_i \rangle$	$\langle \textbf{stay} \rangle$	$\#[Rebels](\theta) \geq k$
$\langle I_i^{t-1} \rangle$	$\langle \mathbf{x}_i \rangle$	$\langle \mathbf{x}_i \rangle$	$i \in R^t$
$\langle 1 \rangle$	$\langle \textbf{stay} \rangle$	$\langle \textbf{stay} \rangle$	$\#[Rebels](\theta) < k$
$\langle 1 \rangle$	$\langle \mathbf{x}_i \rangle$	$\langle \textbf{stay} \rangle$	$\#[Rebels](\theta) \geq k$

Off-path Belief

Whenever i detects a deviation, he believes that

$$\text{for all } j \notin G_i,\ \theta_j \neq \mathit{Rebel}$$

1. If he has less than k Rebel-neighbors, he will play **stay** forever.

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1. If he has less than k Rebel-neighbors, he will play **stay** forever.
2. This off-path belief then also serve as another “grim trigger” (belief-grim-trigger).

Sketch of proof

1. The equilibrium path is APEX.
2. APEX outcome gives maximum ex-post continuation pay-off after some T.
3. Detectable deviation $\Rightarrow$ belief-grim-trigger. ▸ belief-grim-trigger
4. Undetectable deviation $\Rightarrow$ protocol-grim-trigger. ▸ protocol-grim-trigger
5. Any deviation will let APEX fail in a positive probability.
6. Sufficiently high δ will impede deviation.

1. From the above steps, an APEX equilibrium for **acyclic** networks is constructed.
   - At most **2** free riders will occur. ▸ example
2. Solving Pivotal-player problem for **cyclic** networks need more elaboration.
   - More than **3** free riders will occur. ▸ example

1. payoff is perfectly observed
    - Play **revolt** in the first period, then the relevant information revealed.
2. payoff is noisy
    - With full support assumption, the existing equilibrium is APEX.
    - Ex.

$$
\begin{aligned}
p_{1s} &= \Pr(y = y_1 | \#\textbf{revolt} \geq k) \\
p_{1f} &= \Pr(y = y_1 | \#\textbf{revolt} < k) \\
p_{2s} &= \Pr(y = y_2 | \#\textbf{revolt} \geq k) \\
p_{2f} &= \Pr(y = y_2 | \#\textbf{revolt} < k)
\end{aligned}
$$

$$1 > p_{1s} > 0, 1 > p_{2s} > 0, p_{1f} = 1 - p_{1s}, p_{2f} = 1 - p_{2s} \tag{1}$$

Further works

1. Cyclic networks.
2. A general model in which players can communicate only by their actions to learn the relevant information in finite time when $\delta < 1$, while the communication protocol itself is an equilibrium.
3. Equilibrium selection.