Coordination in Social Networks

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Abstract

I study the collective action problem in a discounted repeated coordination game, where players only know their neighbours' inclinations in participating a collective action, and can only monitor their neighbours' past actions. Given that the networks is fixed, finite, connected, commonly known, and undirected (FFCCU), if such networks are without circle, I show that there is a (week) sequential equilibrium in which the ex-post efficient level in static game can be coordinated in finite period when discounted factor is sufficient high. Given that the states of nature are discrete, this equilibrium is constructive and did not depend on public or private signals other than players' actions.

1 Introduction

This paper studies the collective action behaviour in a setting of repeated coordination games where information structure and monitoring structure is modelled as networks. Players face the uncertainty about the states of nature and can only observe their neighbours' actions. I then ask what kinds of networks can induce people to solve the uncertainty about underlying relevant information and coordinate to the ex-post efficient outcome. Though the main motivation lies in understanding the dynamic of social movement, a general interest is in the interaction between collective action and social structure.

Consider people's discontent in a rigid regime. A powerful discontent against this regime may exist, but these discontents are hard to put together due to the lack of complete information about how powerful it is and due to the communication barrier to know these discontents exist. In the era of East German, the voting system and mass media is controlled by the government and the eavesdrop by government impedes people to show their political discontent[1]. Before 1911 in China, several underground forces against Ching Dynasty scattered in the southern of China, but they have different opinions in when and how to make a revolution, and the ways in communication is

[1] e.g., [Lohmann, 2011]

dangerous. Though some social networks such as the networks of friends, the networks of organizations, served as routes in communication[2], but the communication is not free but costly in the sense that it is risky. While Berlin Wall become a history, we may then ask how a decisive collective action can be conducted within such information barrier. As social scientists has recognize it, the process of social movement can be traced back to periods ago, and an event can trigger another event[3]. In game theory, a well-known feature in the extensive form incomplete information game is that the information set a player faces is evolved with players' actions. When rebels know their actions can be used to transmit relevant information about how powerful this discontent is, they may be willing to take actions although taking actions is risky. I view such risky actions as parts of a equilibrium strategy and the entire movement as a learning process.

I model social network as a information structure. In this information structure, inspired by [Chwe, 2000], both of people's type and their actions can be observed perfectly and only by their neighbours. People's collective action is modelled as a k-*Threshold game*, with a threshold k. In this game, there are two types of players located in the network, one we called them *Rebel* and one we called them *Inert*. A Rebel has two actions, which are **revolt** or **stay**, while an Inert has only one action, which is **Inert**. The threshold k is relevant for Rebels' pay-off but is not relevant for Inerts' pay-off. The pay-off structure is modelled as the followings[4].

1. If a Rebel chooses **revolt**, and there are more than k people who **revolt**, then this Rebel will get a positive pay-off, simplified as 1;

2. If a Rebel chooses **revolt**, but there are less than k people who **revolt**, then this Rebel get negative pay-off, simplified as -1;

3. If a Rebel choose **stay**, then he will get a pay-off, simplified as 0 no matter how others play.

Rebels then play this game against all other players with the uncertainty about how many Rebels in this world. Rebels' pay-off structure captures the idea that **stay** is a safe arm while **revolt** is a risky arm. Given k and a prior π, those rebels play the k-Threshold game infinitely repeatedly with a common discount factor δ. The ways in communication is extremely restricted in the sense that cheap talks is not allowed, no outside mechanism serves as an information exchange device, and the static pay-off is unobservable[5].

[2] e.g, [Karl-Dieter and Christiane, 1993]

[3] e.g., [McAdam, Doung; Tarrow, Sidney;Tilly, 2001] [McAdam, 2003] [Lohmann, 2011]

[4] The pay-off structure can be relaxed later in Section 3.3. The things matter here are to keep there are two kinds of actions. One is interpreted as a safe action, such as **stay**. Another one is interpreted as a risky action, such as **revolt**.

[5] The assumption that the pay-off is hidden can be relaxed later in Section 3.3. One might think that what the Rebels fight for is actually to gather international attention. There is an autocrat can block and manipulate the international news, and therefore they can not (perfectly) observe the realized pay-off. The other explanation is to see the pay-off in static game is actually an expected pay-off when a collective action has been made. I admit that there is little room to justifies this assumption. Section 3.3 will discuss this issue.

The only way for Rebels to communicate is by their actions, which will incurs some risky actions, and these actions have to be parts of an equilibrium. With different k and different network structures, I am finding a sequential equilibrium which has the property of *approaching ex-post efficient* (APEX henceforth) to investigate the information sharing behaviour in a network to conduct coordination. We say a sequential equilibrium is approaching ex-post efficiency if and only if *the tails of actions in the equilibrium path repeats the ex-post efficient outcome in the underlying static game after a finite period.* [6]. This refinement mainly checks the in-path behaviour in which players learned the true state within finite period. In words, if there are at least k Rebels in this society, then *all* Rebels should revolt in the future; otherwise, *all* Rebels should stay in the future. In the hope that the uncertainty of "there are at least k Rebels in this society" can be solved, Rebels may then have incentive to cooperate in creating signals to reveal the state of nature if they care more about future pay-off. This incentive in creating signals are then affected by their positions in a network because both incomplete information structure and monitoring structure are aligned within network.

In order to get a quick intuition about Rebel's learning process in my framework, consider the case when $k = n$. Assume that the network is fixed, finite, connected, commonly known, and undirected (FFCCU henceforth), the a simple contagion argument will get the following result.

Result 1. *For n-person repeated k-Threshold game with parameter $k = n$ played in any network with FFCCU, then for any prior with full support there is a δ such that there is a sequential equilibrium which is approaching ex-post efficient.*

The argument to show the equilibrium path is to treat **stay** as the message of "there is an Inert out there"; and treat **revolt** as the message of "there could be no Inert out there ". If I (a Rebel) have an Inert neighbour, then I play stay for ever. If I have no Inert neighbours, then I play revolt in this period. If I have saw my neighbour played stay, then I play stay for ever. If I have saw my neighbour played revolt, then I continue to play revolt. I can learn the true state in finite period because the network is finite. One can then find a belief system and suitable off-path strategies to construct an equilibrium with approaching ex-post efficiency.

The non-trivial cases appear when $k < n$. The equilibrium construction in Result 1 is simple because these binary actions exactly separate the states into two parts, no Inerts or some Inerts, and that is a sufficient information Rebels need to know in making decision when $k = n$. It is similar to the full-rank condition in which actions can generate distinguishable distribution of signals about the true state[7]. However, when $1 < k < n$, as we will show later, more information (or more dimensions of information) needed to be carried by such binary actions, $\{\textbf{stay}, \textbf{revolt}\}$. Several sequence of actions has to be used to transmit Rebels' private information and to control Rebels'

[6] Note that the definition of approaching ex-post efficiency consider the tails of actions in the path, but did not consider players' expected pay-off in the path.

[7] e.g, [Fudenberg and Yamamoto, 2010] or [Fudenberg and Yamamoto, 2011] among others.

belief in order to explicitly write down the equilibrium strategies. In the equilibrium path, two kinds of binary sequences will be used to control them. The first one, called it *reporting messages*, is to report current information about the state of nature to other Rebels; the later one, called it *coordination messages*, is to report the information about whether the other Rebels has certain the successfulness of a coordination. The reporting messages mean to control individual's belief about *how may Rebels out there*, and the coordination messages mean to answer the question of *Have other Rebels known how many Rebels out there*. By exploiting the assumption on the network structure is FFCCU, the coordination message serve as a short-cut to bypass the tracking of individual's higher-order belief. [8].

However, the usage of messages is not free but costly in the sense that playing **revolt** is risky[9]. Due to there is a discounting, Rebels always seek the opportunity to manipulate their messages to save their cost in the time horizontal line. As the problem will be cleared later, the usage of combinations of those two kinds of messages will incur a free rider problem happened locally aligned with the network. The intuition behind this is to see the forthcoming coordination as a public good, which every Rebels can freely share that. This public good can be only made by Rebels' truthful reporting, which incurs some costs. [10]

In order to keep track the belief updating in the equilibrium path, a grim-trigger-like off-path belief will be used to enforce the strategies to follow a prescribed form of sequences. The criteria in judging a deviation need to be elaborated here if the grim trigger is adopted as a punishment. Some Rebels may fall outside a coordination with this grim trigger and so that the approaching ex-post efficiency can not be sustained. This is because the pay-off function in our k-threshold game is not strictly increasing with the total number of Rebels' revolts. A Rebel is more willing to share his information in the hope that at least k Rebels will join a coordination before he knows k Rebels will join, but he is less willing to share his information after he has known that, since more than k Rebels join an existing coordination did not change his pay-off and since there is always a cost to send message. Since the monitoring is limited, players' deviation may be only detected by some neighbours and thus Rebels always want to keep *only* k Rebels in a coordination[11].

[8] The major incentives to induce Rebels to reveal their information here is to give them some hope to coordinate to the ex-post efficient pay-off. It then require individual's growing confidence of the true number of Rebels out there, and individual's growing confidence of others' growing confidence of that, and so on. When the network getting complicated, it then seems intractable to keep track this giant belief profile.

[9] Indeed, allowing cheap talk or using limit-of-mean preference (e.g., [Renault and Tomala, 1998]) will let this coordination problem become trivial. As for the Folk theorem, a Folk theorem has not proposed to cover my setting as far as my knowledge.

[10] Consider the following situation, as Example 3.4. Suppose two nearby Rebels exchange information and the true state of nature can be revealed *only* by them, and suppose *every* Rebel can use a coordination message to trigger coordination *no matter* how his/her past reporting message looks like, then both players will shirk by not reporting anything since they can wait for other's reporting, but then no one can learn the true state of nature.

[11] A similar flavour, but in the opposite side, can be founded in the literature in the repeated prisoner dilemma game played in imperfect monitoring, e.g., [Ellison, 1994] or [Wolitzky, 2013]. Due to the imperfect monitoring, using grim

The main result in this paper will show that this coordination problem can be overcame in the network without circle given that the network is FFCCU. After I define *strong connectivity* as the property that there is always a path, which is consisting all Rebels, to connect any pairs of Rebels in a network, Result 2 shows the main result.

Result 2. ***Main Result*** *For n-person repeated k-Threshold game with parameter* $1 \leq k \leq n$ *played in FFCCU networks without cycle, if the state has strong connectivity and if* $\pi(\text{the states has strong connectivity}) = 1$ *with full support, then there is a* δ *such that there is a (weak) sequential equilibrium which is approaching ex-post efficient.*

This paper contribute to several fields of economics.

First, the future coordination can be viewed as a public good among all Rebels. This paper follows a long discussion among literatures in free rider problem, e.g., [OLSON, 1965][Granovetter, 1978]. A strand of this literatures, such as [Lohmann, 1994] is to view information as a public good while there is a cost in generating information[12]. This paper contribute in this literature by modelling the costly information generation in a repeated coordination game, while adding another respect, network monitoring structures, to investigate the collective action behaviour.

Second, this paper is also related to the literature in social learning[13] Several papers have considered social learnings in networks [14]. In those literature, when players are myopic, the information flows could be very complicated since players behaviour can be affected by the noise they sent[15]. The learning results is hard to converge in the same way. This paper then consider the social learning in networks as a learning-in-game procedure, where individual can put more weights on the future learning results. My results give a partial hint that the shape of network (without circle) did not matter too much if players are far-sighted.

Third, a growing literature consider the game played in networks where various games played in various definitions of networks[16]. Only few literatures among them discuss the repeated game.

trigger to punish some players who deviate will lose the opportunity to stay in a cooperation with others, and thus players may not follow the grim trigger strategies. Here, as Example 3.7 will show, by adopting the grim-trigger-like belief will let some Rebels be excluded from a coordination and thus the ex-post efficiency fails.

[12] For instance, [Lohmann, 1993][Lohmann, 1994] consider information generation by individuals' actions in a setting of finite dynamic game where individuals' actions are public signals. [Bolton and Harris, 1999] et al. consider team experiment in infinite time horizon by a group of decision makers where the outcomes of experiments are also public signals. [Bramoullé and Kranton, 2007] et al view information as a public good which is beneficial to neighbours, and consider a static game in public good provision in networks.

[13] [Bikhchandani et al., 1998] [Cao and Hirshleifer, 2001] gives the reviews.

[14] [Goyal, 2012] gives the reviews Recent papers such as [Acemoglu et al., 2011][Chatterjee and Dutta, 2011] can be documented

[15] For example, in [Gale and Kariv, 2003], even in a class of 3-person connected undirected network, the complete network and incomplete network will give different convergence results which highly depend on individuals' initial private signals and their allocation in a network. At least in [Golub and Jackson, 2010], instead of using Bayesian learning, they use a naive learning protocol to tackle the learning problem in a network.

[16] [Jackson, 2008][Goyal, 2012] gives the reviews.

In complete information game. [Laclau, 2012] prove a folk theorem where players play the game locally. [Wolitzky, 2013] [Wolitzky, 2014] consider network-like monitoring where a prisoner dilemma game played globally. This is the first paper to consider the incomplete information game repeatedly played in a network. The other related literatures is the folk theorems in repeated game with incomplete information. In those literatures, they consider a more general game than the game adopted here. However, as fas as my knowledge, a sufficient condition to show a folk theorem has not yet proposed to covered my setting[17]. I do not mean to prove a folk theorem here, my results however shows that the FFCCU networks without circle is sufficient to sustain the ex-post efficiency when discount factor is suffciently high..

A final comment here is that the free rider problem may become more severe when a network has a cycle. If a network has no cycle, a lemma in this paper will show that the potential free rider problem only happen between at most two connected Rebels. However, when a network has cycle, there is an example shows that this problem can happen among more Rebels, and these Rebels may not connected to each together. This problem will be discussed later.

The paper is organized as the followings. Section 2 introduce the model. Section 3 discuss the equilibrium construction and shows the main result. Some variations will be discussed in its subsection 3.3. Section 4 makes the conclusion. All the missing proofs can be found in Appendix.

2 Model

2.1 Model

Given a finite set A, denote $\#A$ as the cardinality of a set A, and denote ΔA as the set of probability distribution over A.

There are n players. Denote $N = \{1, 2, ..., n\}$ as the set of players. We say G is a network if G is a point-to-set function mapping from N to a subset of N containing i. Moreover, we denote $G_i = G(i)$ as i's neighbourhood and also denote $\bar{G}_i = G_i \backslash \{i\}$. We say G is fixed if G is given, and say G is undirected if for all i, j if $j \in G_i$ then $i \in G_j$. Throughout this paper, I assume G is finite, fixed, commonly known, and undirected. The set of states of nature is $\Theta = \{Rebel, Inert\}^n$. For convenience, denote $[Rebels](\theta) = \{j | \theta_j = Rebel\}$ be the set of Rebels given a state of nature. Given G, let $p_{G_i} : \Theta \to 2^\Theta$ be i's information partition function such that $p_{G_i}(\theta) = \times_{j \in G_i} \{\theta_j\} \times \{Rebel, Inert\}^{n - \#Gi}$. That is, whenever nature choose a state, i knows his own θ_i and his neighbour

[17] [Fudenberg and Yamamoto, 2010] [Fudenberg and Yamamoto, 2011] [Wiseman, 2012] considering n-person game; [Yamamoto, 2014] considering 2-person game, they relies on the assumption on distribution of signals to inform players publicly or privately, which is the lack in my setting. An interesting result by [Amitai] can be also documented. [Amitai] provide an example in n-person long cheap for incomplete information games. In that example, there is an incomplete information game such that if we enlarge the message space then the ex-post efficient outcome can be sustained, while limiting the message space will not sustain that outcome.

j's θ_j.

There is a game, k-threshold game, infinitely repeated played with common discounted factor δ in a fixed G. Time is discrete, infinite horizontal. At the beginning of this game, a state is realized and there is a common prior $\pi \in \Delta\Theta$ over Θ. After a state is realized, players simultaneously choose an action $a_{\theta_i} \in A_{\theta_i}$ afterwards. Note that the set of actions is dependent on their own θ_i. If $\theta_i = Rebel$, then $A_{\theta_i} = \{\textbf{revolt}, \textbf{stay}\}$. If $\theta_i = Inert$, then $A_{\theta_i} = \{\textbf{inert}\}$. Player i's static pay-off, denote as $u_{\theta_i} : A_{\theta_i} \to \mathbb{R}$, in this k-threshold game is defined as followings.

1. $u_{Rebel}(a_i, a_{-i}) = 1$ if $a_i =$ **revolt** and $\#\{j|a_j = \textbf{revolt}\} \geq k$
2. $u_{Rebel}(a_i, a_{-i}) = -1$ if $a_i =$ **revolt** and $\#\{j|a_j = \textbf{revolt}\} < k$
3. $u_{Rebel}(a_i, a_{-i}) = 0$ if $a_i =$ **stay**
4. $u_{Inert}(a_i, a_{-i}) = 1$ if $a_i =$ **inert**

The players can only observe the history of their neighbours' actions and the pay-off is hidden. Specifically, let $h^s_{G_i} \in H^s_{G_i} = \prod_{t=0}^{s} \prod_{j \in G_i} A^s_{\theta_j}$ be the set of histories player i can observe up to period s. Denote $H_{G_i} = \prod_{s=0}^{\infty} H^s_{G_i}$ be all the possible histories i can observe. i's pure behaviour strategy is a sequence $\tau_{\theta_i} = (\tau^0_{\theta_i}, ..., \tau^s_{\theta_i}, ...)$, where $\tau^s_{\theta_i} : p_{G_i}(\theta) \times H^s_{G_i} \to A_{\theta_i}$ is a measurable function with respect to $p_{G_i}(\theta)$ and $H^s_{G_i}$. For convenience, also denote $\tau = \{\tau_{\theta_i}\}_i$, $H^s = \prod_{m=0}^{s} \prod_{i=1}^{n} A_{\theta_i}$ and $H = \prod_{s=0}^{\infty} H^s$. The prior π, the network G, and strategies τ induce a joint distribution over $\Theta \times H$. Let $\beta^{\pi,\tau}_{G_i}(\theta|h^s_{G_i})$ be the conditional distribution over Θ given $h^s_{G_i} \in H^s_{G_i}$ at period s induced by τ for player i, and let $E^{\delta}_{G}(U_{\theta_i}(\tau)|\beta^{\pi,\tau}_{G_i}(\theta|h^s_{G_i}))$ be the conditional expected pay-off conditional on $h^s_{G_i}$ for player θ_i.

We say a prior π has full support if

Definition 2.1. *π has full support if and only if $\pi(\theta) > 0$ for all $\theta \in \Theta$.*

The equilibrium concept here is the (week) sequential equilibrium. Given a network G, I am finding a sequential equilibrium which is *approaching ex-post efficient*.

Definition 2.2. *A sequential equilibrium is approaching ex-post efficient (APEX) if there is a finite time T such that the tails of actions after T in equilibrium path repeats the ex-post efficient outcome.*

In words, in APEX, all the Rebels play revolt after some finite periods if there are more than k Rebels; otherwise, Rebels play stay after some finite periods. The definition of approaching ex-post efficient directly implies that Rebels must tell if there are more than k Rebels or not in the equilibrium path.

Lemma 2.1. *Given G, π, δ, k. If a sequential equilibrium τ^* is approaching ex-post efficient, then given a θ, there is a finite time T_i for a Rebel i such that*

1. $\sum_{\theta:\#[Rebels](\theta)\geq k} \beta_{G_i}^{\pi,\tau^*}(\theta|h_{G_i}^s) = 1$ *or*

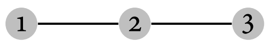
2. $\sum_{\theta:\#[Rebels](\theta)\geq k} \beta_{G_i}^{\pi,\tau^*}(\theta|h_{G_i}^s) = 0$

whenever $s \geq T_i$.

Proof. The proof is by contradiction. Suppose Rebels' strategies constitute an APEX. By definition of approaching ex-post efficient, there is a time T when actions start to repeat. Pick that time $T_i = T+1$ and suppose the consequence did not not holds so that $0 < \sum_{\theta:\#[Rebels](\theta)\geq k} \beta_{G_i}^{\pi,\tau^*}(\theta|h_{G_i}^s) < 1$ for some $s \geq T_i$. Then this Rebel put some positive weights on some $\theta \in \{\theta : \#[Rebels](\theta) < k\}$ and put some positive weights on $\theta \in \{\theta : \#[Rebels](\theta) \geq k\}$ at that time s. Note this Rebel i has already known θ_j if $j \in G_i$, and therefore Rebel i put some positive weights on $\theta \in \{\theta : \#[Rebels](\theta) < k, \theta_l = Rebel, l \notin G_i\}$ and $\theta \in \{\theta : \#[Rebels](\theta) < k, \theta_l = Inert, l \notin G_i\}$. Since actions start to repeat at T, all i's neighbours will play the same actions as the actions at time T, but then Rebel i can not update information from his neighbour by Bayesian rule. Suppose i's continuation strategy is to play **revolt** repeatedly, then this is not ex-post efficient if $\#[Rebels](\theta) < k$. Suppose i's continuation strategy is to play **stay** repeatedly, then this is not ex-post efficient if $\#[Rebels](\theta) \geq k$ $\square$

The following example shows an APEX can be founded if the δ is high enough.

Example 2.1. Suppose there are 3 players in a network. This network is set as $G_1 = \{1,2\}$, $G_2 = \{1,2,3\}$,and $G_3 = \{2,3\}$ as the following graph.



Let the k-threshold game with $k = 3$ infinitely repeated played. Note that after nature moves, player 2 can observe the true state of nature, while player 1 or 3 are not. We can construct an APEX as followings by letting Rebel 2 reveal the state.

- After nature moves, Rebel 2 (if he is) choose **revolt** if he observe $\theta = (Rebel, Rebel, Rebel)$, and play **revolt** in this period. Otherwise, he choose **stay** and keep playing **stay** afterwards.
- After nature moves, Rebel 1 and Rebel 3 (if they are) choose **stay**.
- If Rebel 2 choose **revolt** in the last period, then Rebel 1 (or Rebel 3) play **revolt** in this period; if Rebel 2 choose **stay** in the last period, then Rebel 1 (or Rebel 3) keep playing **stay** afterwards.

Given the prior has full support, the above strategies constitute an equilibrium if $\delta \geq \frac{1}{2}$. In the equilibrium, Rebel 1 and Rebel 3 believe that $\{\theta : \#[Rebel](\theta) \geq 3\}$ if they observe **revolt** played by Rebel 2 and believe $\{\theta : \#[Rebel](\theta) < 3\}$ if Rebel 2 played the other action.

In the following section, I begin to find an APEX in more general settings.

3 Equilibrium

3.1 The case: $k = n$

In the above special case, Example 2.1, the construction of an APEX relies on some important features. First, since $k = n$ in this special case, Rebel 2 will never play **revolt** if one of his neighbour is Inert. Thus, when Rebel 2 play **revolt**, it must be the case that all Rebel 2's neighbour are Rebels. Due to this feature, Rebel 2's actions separates the states into two events, $\{\theta : \#[Rebel](\theta) \geq k\}$ and $\{\theta : \#[Rebel](\theta) < k\}$. Second, Rebel 1 or Rebel 3 can force Rebel 2 to play **revolt** to reveal the true state in the first period since only Rebel 2 knows the true state and Rebel 2's action can separate the states. Third, since $k = n$, it is easy to punish a deviation by only one Rebel's shifting to play **stay** forever, and so that the group punishment is not necessary.[18] Note also that there is another APEX by letting Rebel 1 and Rebel 3 play **revolt** in the first period by using Rebel 2' punishment in shifting to play **stay** forever. I then generalize the $k = n$ case after one definition, the *connectivity* of a network.

Definition 3.1. *A path from i to j in a network G is a finite sequence $l^1, ..., l^q$ such that $l^1 = i, l^2 \in \bar{G}_{l^1}, l^3 \in \bar{G}_{l^2}, ..., l^q \in \bar{G}_j$. A network is connected if and only if for all i, j there is a path from i to j or j to i.*

Theorem 1. *For n-person repeated k-Threshold game with parameter $k = n$ played in any network with FFCCU, then for any prior with full support there is a δ such that there is a sequential equilibrium which is approaching ex-post efficient.*

Proof. The proof is constructive and is a generalization of Example. Let strategies τ^* as the followings. After nature moves, a Rebel play **revolt** if there is no Inert neighbour; a Rebel play **stay** forever if there is a Inert neighbour. After first period, if a Rebel has not detected a deviation and if such Rebel observed his Rebel neighbour play **revolt** continuously in the last periods, then keep playing **revolt** in current period; otherwise, play **stay** forever. If a Rebel deviate, then himself play **stay** forever.

According to τ^*, at period s, if a Rebel has not detected a deviation and if such Rebel observed his Rebel neighbour play **stay** once in the last periods, he form belief $\sum_{\theta:\#[Rebels](\theta)\geq k} \beta^{\pi,\tau^*}_{G_i}(\theta|h^s_{G_i}) = 0$ after period s and therefore play **stay** after period s is the best response. If a Rebel detects a deviation or himself deviate to play **stay**, play **stay** is the best response since at least one neighbour will play **stay**. [19]

[18] For instance, if a Rebel did not play **revolt** in the second period in the state $\theta = (Rebel, Rebel, Rebel)$, his neighbour can play **stay** forever. To give incentive to let such neighbour play **stay**, we can just let the deviating player also play **stay**.

[19] Note that we did not put additional assumptions on the off-path belief expect for the full support assumption on the prior.

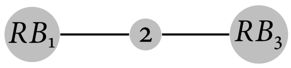
Since the network is FFCCU, there is a finite time t^s_θ such that all Rebel play **revolt** forever if $\{\theta : \#[Rebel](\theta) \geq k\}$; and there is a finite time t^f_θ such that all Rebel play **stay** forever if $\{\theta : \#[Rebel](\theta) < k\}$ in the equilibrium path. If a Rebel deviate, he at most get 0 after $\max\{t^s_\theta, t^f_\theta\}$, while he get $\max\{1, 0\}$ after $\max\{t^s_\theta, t^f_\theta\}$. Due to the full support assumption, he will not deviate if $\sum_{\theta:\#[Rebels](\theta)\geq k} \beta^{\pi,\tau^*}_{G_i}(\theta|h^s_{G_i}) > 0$ at some period s, otherwise he has a loss in expected continuation pay-off by $\delta^{t^s_\theta} \frac{\sum_{\theta:\#[Rebels](\theta)\geq k} \beta^{\pi,\tau^*}_{G_i}(\theta|h^s_{G_i})}{1-\delta}$ after t^s_θ. There is a δ_π such that he will not deviate.

To check if τ^* and $\{\beta^{\pi,\tau^*}_{G_i}(\theta|h^s_{G_i})\}_{i\in N}$ satisfy full consistency[20], take any $0 < \eta < 1$ such that Rebels play τ^* with probability $1 - \eta$, and play others with probability η. Clearly, when $\eta \to 0$, the belief system converge to $\{\beta^{\pi,\tau^*}_{G_i}(\theta|h^s_{G_i})\}_{i\in N}$. □

3.2 The case : $1 < k < n$

Different form the case of $k = n$, where playing **revolt** and **stay** separates the states into two events, $\{\theta : \#[Rebel](\theta) \geq k\}$ and $\{\theta : \#[Rebel](\theta) < k\}$, this property is generally fails if $1 < k < n$. We require more assumptions on the true state and on the priors to get APEX. This is because a Rebel still have incentive to play **revolt** if there is an Inert neighbour. Moreover, Inerts will never give information to Rebels since they have only one action. Rebels' learning will be limited if some Iners block the information flow. Consider the example.

Example 3.1. Let $k = 2$ and let the network as the following. Assume $\theta = (Rebel_1, Inert_2, Rebel_3)$.



Since $k = 2$, Rebel 1 has incentive to play **revolt** when $\pi(\{\theta : \theta_3 = Rebel\})$ is high enough given that Rebel 3 will play revolt. Moreover, Rebel 1 never learn the true θ_3 since Inert 2 block the information. We are now in the incomplete information game without communication. Clearly, the equilibrium which is APEX did not exist in this case.

According to Lemma 2.1, a Rebel's belief has to learn the state of nature in APEX. However, since Inerts' behaviour will not update such information, the state profiles and the prior to let APEX is possible have to be narrow down . Define *Strong connectivity* and *Full support on strong connectivity* as the following.

Definition 3.2. ***Strong Connectivity***: *Given G, a state θ has strong connectivity if and only if for every pair of Rebels, there is a path consisting all Rebels to connect them.*

Definition 3.3. ***Full support on strong connectivity***: *Given G, π has full support on strong connectivity if and only if $1 > \pi(\theta) > 0$ whenever θ has strong connectivity.*

[20] Krep and Wilson (1982)

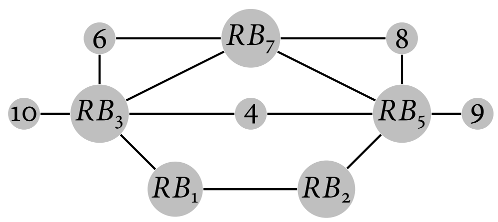
The goal of this paper is then to show that there is always a (weak) sequential APEX when $k < n$ in FFCCU networks without circle when δ is high enough.

Theorem 2. *For n-person repeated k-Threshold game with parameter $1 \leq k < n$ played in any FFCCU network without cycle, if the state θ has strong connectivity and $\pi(\{\theta : \theta \text{ has strong connectivity}\}) = 1$ with full support, then there is a δ such that there is a (weak) sequential equilibrium which is approaching ex-post efficient.*

The equilibrium in Theorem 3.1 is constructive as an automata and meant to overcome various potential problems in FFCCU networks when $k < n$. I begin illustrate the potential problems and describe the equilibrium construction. The automata and omitted proof are in Appendix.

Note that even if the state has strong connectivity, the coordination problem is still not trivial. Rebel may not play **stay** forever as in the case of $k = n$ if there are some Inert neighbours. With the full support assumption, they may loss the opportunity to coordinate other Rebels if they do so. Example 3.2 illustrate this situation.

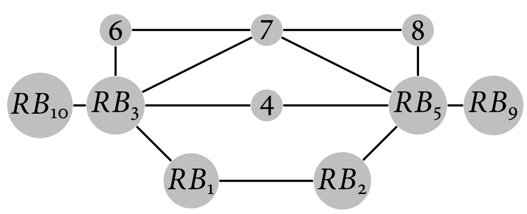
Example 3.2. Let $k = 5$ and let the network and the state θ as the following (the nodes other than RB are Inerts).



If a Rebel play stay forever, Rebels can not learn $\#[Rebels](\theta) \geq 5$.

Rebels then have to find a way to communicate with each other. Their actions will be a mixture of **revolt** and **stay** in transmitting information. The construction of APEX is not trivial now because the "dimension" of information needed to be generated is generally larger than the cardinality of their action space. Rebels then have to use the sequence of actions to transmit information, and thus we have to track the belief updating in the time horizontal line and check if such sequences constitute an equilibrium. To see that Rebel need to generate more dimensions of information, we may compare Example 3.2 and Example 3.3.

Example 3.3. Let $k = 6$ and let the network and the state θ_{example} as the following.



In Example, there are 6 Rebels, which is different from Example, where there are 5 Rebels. Suppose now we have a "talking strategies". Rebel 3 and Rebel 5 can talk about "how many" Rebels in their neighbourhood to Rebel 1 and Rebel 2. Rebel 1 and Rebel 2 then talk with each other about "how many" Rebels they have known conditional on Rebel 3 and Rebel 5's taking. In some ways, Rebel 1 and Rebel 2 can initiate the coordination to play revolt conditional on 'how many" Rebels they have known. The question is that Rebel 1 and Rebel 2 still don't know how many Rebels out there after Rebel 3 and Rebel 5's talking. In both Example Rebel 3 and Rebel 5 will talk the same number of Rebel neighbours to Rebel 1 and Rebel 2, and Rebel 1 and Rebel 2 can not distinguish if the true number of Rebels is 5 or 6. However, if Rebel 3 and Rebel 5 can also talk about "the locations" of nearby Rebels, then Rebel 1 and Rebel 2 can then tell. In order to get APEX, "talking about how many" Rebels nearby is enough, Rebels have to "talking about the locations" of Rebels.

I then prepare such "talking strategies" to let Rebels report both the number and locations of nearby Rebels. This talking strategies is, however, not enough to get APEX. Although Lemma 2.1 shows that there is a timing such that each Rebel have known the relevant information, but it did not say that all the Rebels have known others have known the relevant information. This higher-order information hierarchy has to be commonly known before that finite time T (in Definition 2.2) arrives when players play some actions repeatedly. Otherwise a tiny change in prior may let some Rebels has not arrive their T_i (in Lemma), and therefore other Rebels need to form a belief about such T_i, and then other Rebels need to form a belief about those Rebels' belief about those T_i,...,and so on. Since there is no learning after T, this common knowledge of T_i has to be formed previously.

This higher-order belief is an apparently giant object in the private imperfect monitoring setting here. By assuming the finite network, however, a preparation of *coordination messages* to let Rebels talk about their T_i might work. If we go back to the $k = n$ case, playing **stay** is as the role of coordination message to notify nearby Rebels the timing that coordination to **stay**, while playing **revolt** n times or less (recalled that there are n players) is as the coordination messages to the coordination to **revolt**. In the case of $k < n$, unfortunately, what kind of coordination message should be used in equilibrium is not obvious then. First, playing **stay** itself did not reveal $\#[Rebels](\theta) < k$ as case $k = n$ shows. Second, sending messages will incurs some expected costs (or expected benefits) and then the deviation in messages may need to be punished. It is hard to punish because the shifting to play **stay** forever may not be credible or is not enough to impede the deviation since a single Rebel's punishment may not be enough to change the continuation pay-off .

I construct the APEX with a weaker sequential consistent by assuming a off-path belief, which has a grim-trigger property. The equilibrium is constructed by three steps shown at next three subsections. In the first step, I define the *information hierarchy in G* which gives a characterization to specify those Rebels who are forced to report their information in equilibrium, and also gives the notations in constructing my APEX. In second step, I specify Rebels' strategies as several binary $\{\textbf{revolt}, \textbf{stay}\}$-sequences, and gives an automata where those binary sequences are used as inputs and outputs. Finally, I give a grim-trigger-like off-path belief in the third step and show that the sequences and automata defined in the first and second step constitute an APEX. I put the automata and the proofs in the Appendix for the convenience to read.

3.2.1 Step 1. Information hierarchy in G

The information hierarchy is defined on a network G after nature choose a state but before a game is played. I will use the term "node i" instead of "player i" in this step.

The definition is by iteration. We define information hierarchy by defining $\{N_i^{-1}, N_i^0, N_i^1 ...\}$ and $\{I_i^{-1}, I_i^0, I_i^1 ...\}$ for each i and each iteration in $(0, 1, 2, ...)$, and define $\{\leq^0, \leq^1, \leq^2\}$ and $\{R^0, R^1, R^2 ...\}$ for each iteration in $(0, 1, 2, ...)$. We also use the term "blocks" to represent the iterations.

Given θ, the information hierarchy is defined as the followings.

- **0-block** Denote

$$
\begin{aligned}
N_i^{-1} &\equiv i \\
I_i^{-1} &\equiv i
\end{aligned}
$$

Then define R^0 as

$$R^0 \equiv \{i : \theta_i \in [Rebels](\theta)\} \tag{1}$$

- **1-block** Denotes

$$
\begin{aligned}
N_i^0 &\equiv G_i \\
I_i^0 &\equiv G_i \cap R^0
\end{aligned}
$$

Define the set $\leq^0$ by defining

$$i \in \leq^0 \Leftrightarrow \exists j \in \bar{G}_i [I_i^0 \subseteq N_j^0 \cap R^0] \tag{2}$$

Then define R^1 as

$$R^1 \equiv \{i \in R^0 | i \notin \leq^0\} \tag{3}$$

- $t+1$-**block,** $t \geq 1$ Denote

$$
\begin{aligned}
N_i^t &\equiv \bigcup_{k \in I_i^{t-1}} G_k \\
I_i^t &\equiv \bigcup_{k \in G_i \cap R^t} I_k^{t-1}
\end{aligned}
$$

Define the set $\leq^t$ by defining

$$i \in \leq^t \Leftrightarrow \exists j \in \bar{G}_i [I_i^t \subseteq N_j^t \cap R^0] \tag{4}$$

Then define R^{t+1} as

$$R^{t+1} \equiv \{i \in R^t | i \notin \leq^t\} \tag{5}$$

Thus, the R^t nodes are those nodes who knows strictly more Rebels then any other their neighbours. Such nodes have the information about I^{t-1} at t-block, which contains the updating information about who are Rebels. If any reporting incurs some expected costs, I mainly control the incentives in next two steps by just letting R^t nodes to report their information, while non-R^t nodes will play safe action **stay**. Moreover, if the network has no circle, Theorem 3 shows that it is sufficient to just let R^t nodes to report information in the sense there is time t and a R^t node who will know the true state.

Theorem 3. *If the network is FFCCU without circle and if the state has strong connectivity, then*

$$R^0 \neq \emptyset \Rightarrow \exists t \geq 0[\exists i \in R^t[I_i^t = R^0]]$$

3.2.2 Step 2: Equilibrium strategies in the path

First, I assigned each player in a fixed network a distinguished prime number. Such indexation is starting from 3. I index players $(1, 2, ..., n)$ as $(3, 5, ..., x_n)$ where x_n is a prime number and the prime number assigned to i will be x_i. Since the multiplication of distinguish prime numbers can be uniquely de-factorized as those numbers, we then use this property to let Rebels simultaneous report the number and the locations of their Rebel-neighbours by report the multiplication of those prime numbers.

There are two classes of sequences, and we call them *reporting messages* and *coordination messages*. I denote $\langle\rangle$ as a form of sequence. The notations for the form of those sequences are shown in Table 1. Denote $\bar{N} \subset N$ be a non-empty subset of N.

Denote $|\langle\rangle|$ as the length of a form of sequence. The forms of sequences and the length of such forms will jointly determine the sequences I used in equilibrium. For example, if a sequence takes the form $\langle 1 \rangle$ and its length $|\langle 1 \rangle| = 3$, then this sequence is $\langle \mathbf{s}, \mathbf{s}, \mathbf{r} \rangle$. Thus, the length of a form is calculate from the end of such sequence.

In the equilibrium path, two kinds of periods, *reporting period* and *coordination period*, occur in turns in the following way,

$$\underbrace{< \text{coordination period} >}_{0-block}\underbrace{< \text{reporting period} >< \text{coordination period} >}_{1-block}\ldots$$

I.e. after nature chooses a state, all the Rebels start with 0-block, then enter to 1-block,...,and so on. 0-block has only one period, coordination period. The t-blocks, $t \geq 1$ has two periods, reporting period and coordination period, where reporting period occurs first and then coordination period follows. The length of each period in each block is finite but endogenous.

Table 1: Forms of messages

$X_{\bar{N}}$	$\equiv$	$\prod_{j\in\bar{N}} x_j$
$\mathbf{s}$	$\equiv$	$\mathbf{stay}$
$\mathbf{r}$	$\equiv$	$\mathbf{revolt}$
$\langle\mathbf{stay}\rangle$	$\equiv$	$\langle\mathbf{s},\dots,\mathbf{s}\rangle$
$\langle\mathbf{revolt}\rangle$	$\equiv$	$\langle\mathbf{r},\dots,\mathbf{r}\rangle$
$\langle\bar{N}\rangle$	$\equiv$	$\langle\mathbf{s},\dots,\mathbf{s}\,\underbrace{\mathbf{r},\mathbf{s},\dots,\mathbf{s}}_{X_{\bar{N}}}\rangle$
$\langle 1\rangle$	$\equiv$	$\langle\mathbf{s},\dots,\mathbf{s}\,\underbrace{\mathbf{r}}_{1}\rangle$
$\langle\mathbf{1}_i\rangle$	$\equiv$	$\langle\mathbf{s},\dots,\mathbf{s}\,\underbrace{\mathbf{r},\mathbf{s},\dots,\mathbf{s}}_{x_i}\rangle$

Table 2: Reporting messages

Reporting Messages
$\langle\mathbf{stay}\rangle$
$\langle I_i^{t-1}\rangle$
$\langle 1\rangle$

If a sequence of actions has been played in reporting period (or coordination period), we called it a *reporting messages* (or *coordination messages*). In reporting period in each t-block ($t \geq 1$), Rebels use the sequences defined in Table to report their I_i^{t-1} contingent on the histories they observed. In coordination period in each t-block ($t \geq 0$), Rebels check if there is one Rebel has initiate some coordination messages, which is defined in Table, and then start to coordination or continue to next reporting period in $t+1$-block. I start to give the details of these messages.

Reporting messages in reporting period

Denote $|\langle RP^t\rangle|$ be the total number of periods in t-block reporting period. The outcome of pure strategies in equilibrium path take the following forms of sequences in Table 2 with length $|\langle RP^t\rangle|$.

Any deviation from the forms defined in Table 2 will be considered as a deviation. In the equilibrium path, the beliefs a Rebel j will form after observing the reporting messages by neighbour i are shown in Table 3.

After reporting period, Rebel can tell who are R^t. Recall that R^t Rebels are those Rebels who have strictly more information than any of their neighbours, if a Rebel j have observed that none

Table 3: Belief updating after reporting period

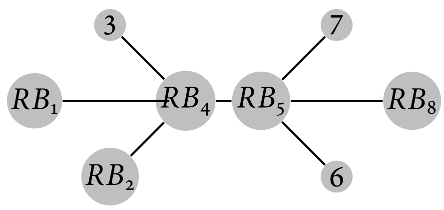
i plays	The events $j \in \bar{G}_i$ believe with probability one
$\langle$**stay**$\rangle$	$i \notin R^t$
$\langle I_i^{t-1} \rangle$	$i \in R^t$ and $\theta_l = Rebel$ if $l \in I_i^{t-1}$
$\langle 1 \rangle$	$i \in R^t$ and i has known $\#[Rebels](\theta) \geq k-1$

of his neighbours are in R^t, then he is sure that $\#[Rebels](\theta) < k$ if $\#I_j^{t-1} < k$.

An important feature here is the usage of $\langle 1 \rangle$. It serve as a signal to indicate i himself is a pivotal player and solve the free rider problem. I elaborate this issue here by providing some examples and give more details in the discussion in coordination period. As we have discussed in the Introduction, if we allow a coordination message such that every Rebels can use regardless how their reporting messages, then we have a free rider problem. Consider the Example 3.4.

Example 3.4. Free Rider Problem

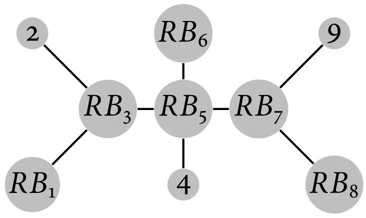
Let $k = 5$ and assume that there are message $\langle M_4 \rangle, \langle M_5 \rangle$ for Rebel 4, 5. To simply the analysis, let the game play starting from 1-block, i.e. by discarding the strategies in 0-block and staring from a reporting period. Further, assume Rebel will play **revolt** forever after observing $\langle M_4 \rangle$ or $\langle M_5 \rangle$ being played once in a period after reporting period; otherwise they will play **stay** forever. Let G is as the followings.



Note that Rebel 4 and Rebel 5 are R^1 members. Let $\langle\rangle_4$ and $\langle\rangle_5$ are the sequences of actions they may use to report the number of Rebel neighbours. These sequence will incur some expected costs. If Rebel 5 report truthfully, then Rebel 4 will not report truthfully by arranging the occurrence of **revolt**s. This is because he has a $\langle M_4 \rangle$ to initialize the coordination which all Rebels will follows by assumption. Same situation happens for Rebel 5, and then Rebel 4 and Rebel 5 will not report truthfully.

In the above example, two sources constitutes the free rider problem. One is that there is a coordination messages which can be used regardless the reporting messages. The other one is that Rebel 4 and Rebel 5 are pivotal in the sense that they are sure they will learn the true state given others' truthful reporting. To see the later source more clearly, consider the following Example 3.5.

Example 3.5. Pivotal player: Case 1 Let $k = 6$ and suppose that there are message $\langle M_3\rangle, \langle M_5\rangle, \langle M_7\rangle$ for Rebel 3,5,7. Let the game play starting from 1-block as Example. Further, suppose Rebel will play **revolt** forever after observing $\langle M_3\rangle$, $\langle M_5\rangle$, or $\langle M_7\rangle$ being played once in two periods[21] after reporting period; otherwise they will play **stay** forever. Let G is as the followings.



Note that Rebel 3, 5, 7 are R^1 members. Different from Example 3.4, although Rebel 3, 7 have coordination messages, they still have incentives to report the number of their Rebel neighbours truthfully. This is because there is a possibility that Rebel 5 misunderstood the true state if they did not report honestly, while they *can not* know the true state after reporting period. Since the coordination to **revolt** has to be initiated just after reporting period, they have incentives to report truthfully.

Rebel 5, however, has no incentive to report truthfully given others' truthful reporting. This is because he is sure that he will know the true state and he can initiate the coordination given that.

Combine the issue in Example 3.4 and Example 3.5 , a natural way to deal with the free rider problem is to identify who is the pivotal player in the reporting period while such pivotal player has to send the coordination messages. The question is how to identify the pivotal players. If the network is FFCCU without circle, Lemma shows that the free rider problem can be identified before the game enter to t-block and the pivotal player can be identified either. More precisely, first define the tree rooted in i node and it leaves spanning from $j \in \bar{G}_i$ as Tr_{ij}

Definition 3.4. $Tr_{ij} \equiv \{l \in N :$ *there is a unique path* $\{l, \ldots, j, i\}$ *from* l *to* i *through* $j\}$

and define the set

$$C^t = \{i \in R^t : \nexists j \in R^{t-1} \cap \bar{G}_i [\exists l, l' \in Tr_{ij}[l \in N_j^{t-1} \backslash I_i^{t-1} \text{ and } l' \in \bar{G}_l]]\}$$

be those R^t nodes so that there is no nodes connect with them more than "two walks". For instance, the nodes Rebel 4 and Rebel 5 in Example 3.4 are C^1 nodes and Rebel 5 in Example 3.5 is also a C^1 node. Then we can show the following lemmas. The proofs are in Appendix.

Lemma 3.1. *If the network is FFCCU without circle, and if the state has strong connectivity, then for each t-block*

1. $0 \le |C^t| \le 2$.

[21] It requires two periods to let the coordination message transmit.

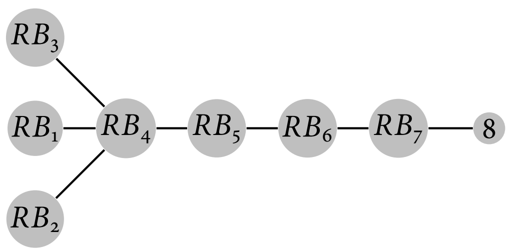
2. *Moreover, suppose there are two nodes in C^t, then they are the others' neighbour.*

Lemma 3.2. *If the network is FFCCU without circle, and if the state has strong connectivity, then for each t-block*

$$i \in C^t \Rightarrow \textit{there is no node outside of} \bigcup_{k \in N_i^{t-1}} N_k$$

Lemma 3.1 says there are at most 2 Rebels in each t-block and those Rebels are each others' neighbours. It is crucial because they can identify with each other by just checking the definition of C^t. Since they can identify with each other and since every nodes has been indexed a prime number, we just pick a node who has smaller index to be pivotal player[22]. Lemma 3.2 show that they can learn the true state after reporting period in t-block conditional on others' truthful reporting. Lemma 3.2 nevertheless show only the node in C^t is pivotal, and therefore there could be other nodes can learn the true state. It is troublesome because some pivotal players can not be identified before the game enter t-block and thus we have to identify them during the game is played by tracking the evolving information sets they face sequentially. This source to become a pivotal player is that *a Rebel has already known too much*, and therefore he is sure that he will learn *either* $\#[Rebels](\theta) \geq k$ *or* $\#[Rebels](\theta) < k$ after the reporting period. Example 3.6 give a more concrete example.

Example 3.6. Pivotal player: Case 2 Let $k = 6$. Again, assume that there are coordination message $\langle M \rangle$s for Rebels. Let the game play starting from 1-block as Example. Again, assume Rebel will play **revolt** forever after observing $\langle M \rangle$ being played once in four periods[23] after reporting period; otherwise they will play **stay** forever. Let G is as the followings.



In this case, no Rebels in C^1, but Rebel 4 will deviate from report $\langle I_4^0 \rangle$. Note that Rebel 4 has already known 5 Rebels in this world, so that one more Rebels is enough to initiate coordination to **revolt**. Moreover, if there is no more Rebels, the only coordination is to **stay**. If node 6 is a Rebel, Rebel 5 will report that and therefore he will know $\#[Rebels](\theta) \geq 6$. If not, due to the state has strong connectivity, he will also know $\#[Rebels](\theta) < 6$ for sure. Then he can use the message M to transmit the relevant information but not report $\langle I_4^0 \rangle$.

[22]This property is not generally hold if a network has circle.

[23]It requires four periods to let the coordination message transmit.

Table 4: Coordination messages

Coordination messages
$\langle \mathbf{1}_i \rangle$
$\langle \mathbf{stay} \rangle$
$\mathbf{r}$
$\mathbf{s}$

After the above examples, the message $\langle 1 \rangle$ has been used to specify the pivotal players. Playing this message gives the least expected costs to distinguish themselves from non R^t players. Overall speaking, the pivotal players is identified by those Rebels (1.) who have already known there are $k-1$ Rebels (sequentially) in the equilibrium path, or (2.) who are in C^t. The pivotal players i will play $\langle 1 \rangle$ in such cases and so that the beliefs after observing it is that *i has known* $\#[Rebels](\theta)$ in the equilibrium path, as Table 3 shows.

Coordination messages in coordination period

The ignorance of reporting messages after observed a coordination message $\langle M \rangle$ may incur the untruthfully reporting as the above examples show. The introducing of messages $\langle 1 \rangle$ is meant to tackle with that. However, one may have observed that these two messages $\langle 1 \rangle$, $\langle M \rangle$ themselves are another "coordination message", i.e. $\langle \langle \mathbf{s}, \mathbf{r} \rangle \langle M \rangle \rangle$ by truncating previous actions of $\langle 1 \rangle$ and concatenate them to $\langle M \rangle$. If the contingent behaviour after observing this new coordination message is the same as seeing the original one, the problem did not solve. In this section, I still call $\langle 1 \rangle$ as reporting messages and call those $\langle M \rangle$ as coordination messages, but I let the belief a Rebel will from contingent not only on the coordination message itself but also on reporting message.

There are three divisions in coordination period and there are several sub-blocks in each division. In $t = 0$ block, the form is

$$\langle \underbrace{\langle \cdot \rangle}_{1 \text{ sub-block}} \rangle \langle \underbrace{\langle \cdot \rangle}_{1 \text{ sub-blocks}} \rangle \langle \underbrace{\langle \cdot \rangle \cdots \langle \cdot \rangle}_{n \text{ sub-blocks}} \rangle$$

; in $t > 0$ block, the form is

$$\langle \underbrace{\langle \cdot \rangle \cdots \langle \cdot \rangle}_{n \text{ sub-blocks}} \rangle \langle \underbrace{\langle \cdot \rangle \cdots \langle \cdot \rangle}_{t+1 \text{ sub-blocks}} \rangle \langle \underbrace{\langle \cdot \rangle \cdots \langle \cdot \rangle}_{n \text{ sub-blocks}} \rangle$$

, where $n = \#N$.

Denote $CD^t_{m,q}$ be the m sub-block in q division, and denote $|\langle CD^t_{m,q} \rangle|$ be the total number of periods in $CD^t_{m,q}$. The outcome of pure strategies in equilibrium path takes the following forms of sequences with length $|\langle CD^t_{m,q} \rangle|$ as Table 4 shows.

Table 5: Belief updating after $CD^t_{1,1}$, $t > 0$

In RP^t	In $CD^t_{1,1}$	
i plays	i plays	The events $j \in \bar{G}_i$ believe with probability one
$\langle$**stay**$\rangle$	$\langle \mathbf{1}_i \rangle$	$i \notin R^t$
$\langle I_i^{t-1} \rangle$	$\langle$**stay**$\rangle$	$\#[Rebels](\theta) < k$
$\langle I_i^{t-1} \rangle$	$\langle \mathbf{1}_i \rangle$	$i \in R^t$
$\langle 1 \rangle$	$\langle$**stay**$\rangle$	$\#[Rebels](\theta) < k$
$\langle 1 \rangle$	$\langle \mathbf{1}_i \rangle$	$\#[Rebels](\theta) \geq k$

Table 6: In-path strategies in $CD^t_{2,1}$, $t > 0$

In RP^t	In $CD^t_{1,1}$	In $CD^t_{2,1}$
i plays	i plays	$j \in \bar{G}_i$ plays
$\langle$**stay**$\rangle$	$\langle \mathbf{1}_i \rangle$	$\langle \mathbf{1}_i \rangle$
$\langle I_i^{t-1} \rangle$	$\langle$**stay**$\rangle$	$\langle$**stay**$\rangle$
$\langle I_i^{t-1} \rangle$	$\langle \mathbf{1}_i \rangle$	$\langle \mathbf{1}_i \rangle$
$\langle 1 \rangle$	$\langle$**stay**$\rangle$	$\langle$**stay**$\rangle$
$\langle 1 \rangle$	$\langle \mathbf{1}_i \rangle$	$\langle \mathbf{1}_i \rangle$

In the following paragraphs, I will focus on the coordination period in $t > 0$ block, while Appendix shows the equilibrium path in coordination period in $t = 0$ block. The belief a Rebel j form after observing i after $CD^t_{1,1}$ in equilibrium path is as Table 5 shows. After $CD^t_{1,1}$, Rebel j will tell one more event: $\#[Rebels](\theta) < k$. Clearly, if this event has been observed, the j will play **stay** forever. In order to transmit this information, the strategies in $C^t_{m,1}$ where $m \geq 2$ is as Table 6 and Table 7 shows. That is, they will play $\langle \mathbf{1}_i \rangle$ unless they observe some one play $\langle$**stay**$\rangle$. After $CD^t_{n,1}$, the information of $\#[Rebels](\theta) < k$ will be transmitted across all players.

Game enter to $CD^t_{1,2}$. In $CD^t_{1,2}$, Rebels start to initiate the coordination to **revolt**. The coordination message to initiate the coordination is $\langle$**stay**$\rangle$. As Table 8 shows, the important features here is that $\langle$**stay**$\rangle$ is a coordination message *only if* $\langle I_i^{t-1} \rangle$ or $\langle 1 \rangle$ has been played in reporting period. This feature should be elaborated. First, note that $\langle$**stay**$\rangle$ is also the message to coordinate to **stay** in $CD^t_{m,1}$, $m \geq 1$, and this is the only candidate as a message to coordinate to **stay**. This is because whenever a Rebel believe $\#[Rebeles](\theta) < k$ with probability one, he has no incentives to play **revolt** again. Although the message to coordinate to **stay** or **revolt** is the same, Rebels are not confused since this message is contingent on the reporting messages now. Second, this message incurs no expected cost in equilibrium path. It is important because, otherwise, a free rider problem as

Table 7: In-path strategies after $CD^t_{m,1}$, where $m \geq 2$, $t > 0$

In $CD^t_{m,1}$, $m \geq 2$	In $CD^t_{m+1,1}$,$m \geq 2$
i plays	$j \in \bar{G}_i$ plays
$\langle \mathbf{1}_i \rangle$	$\langle \mathbf{1}_i \rangle$
$\langle$**stay**$\rangle$	$\langle$**stay**$\rangle$

Table 8: Belief updating after $CD^t_{1,2}$, $t > 0$

In RP^t	In $CD^t_{1,1}$	In $CD^t_{1,2}$	
i plays	i plays	i plays	The events j believe with probability one
$\langle$**stay**$\rangle$	$\langle \mathbf{1}_i \rangle$	$\langle$**stay**$\rangle$	$i \notin R^t$
$\langle I_i^{t-1} \rangle$	$\langle$**stay**$\rangle$	$\langle$**stay**$\rangle$	$\#[Rebels](\theta) < k$
$\langle I_i^{t-1} \rangle$	$\langle \mathbf{1}_i \rangle$	$\langle$**stay**$\rangle$	$\#[Rebels](\theta) \geq k$
$\langle I_i^{t-1} \rangle$	$\langle \mathbf{1}_i \rangle$	$\langle \mathbf{1}_i \rangle$	$i \in R^t$
$\langle 1 \rangle$	$\langle$**stay**$\rangle$	$\langle$**stay**$\rangle$	$\#[Rebels](\theta) < k$
$\langle 1 \rangle$	$\langle \mathbf{1}_i \rangle$	$\langle$**stay**$\rangle$	$\#[Rebels](\theta) \geq k$

Example 3.4 will happen again due to players can wait for the other's playing coordination messages. However, since this message incurs no expected cost, a Rebel who believe $\#[Rebeles](\theta) \geq k$ with probability one will initiate it. Third, since this message is contingent on the reporting messages, the initiating the coordination to **revolt** is not free. There is a trade-off between reporting something and reporting nothing in the reporting period which is the major force to let Rebels to report something in reporting period. In the proof for equilibrium in Appendix, I mainly argue that this trade-off will force Rebels to report something in reporting period and the Rebels will not take the advantage of this cost-free message to initiate the coordination to **revolt** if he has not believed $\#[Rebeles](\theta) \geq k$ with probability one.

After the initiating in $CD^t_{1,2}$, Rebels start to transmit the information of $\#[Rebels](\theta) \geq k$ in $CD^t_{m,2}$ $m \geq 2$ as Table 9 and Table 10 shows. That is, they will play $\langle$**stay**$\rangle$ unless they observe some one play $\langle \mathbf{1}_i \rangle$. After $CD^t_{t+1,1}$, the information of $\#[Rebels](\theta) \geq k$ will be transmitted across at least k Rebels. We need $t+1$ sub-blocks in this stage because such k Rebels are within a neighbourhood around that Rebel who initiate coordination at most $t+1$ sub-blocks.

Game finally enter to $CD^t_{1,3}$. In this period, those k Rebels who have known the information of $\#[Rebels](\theta) \geq k$ will start to play **revolt** forever. This is the first period to get positive expected pay-off by playing **revolt** if the coordination can be succeeded to **revolt**. After $CD^t_{m,3}$, $m \geq 2$, other Rebels start to transmit this information of coordination to **revolt** to all of the Rebels as Table 12

Table 9: In-path strategies in $CD^t_{2,2}$, $t > 0$

In RP^t	In $CD^t_{1,1}$	In $CD^t_{1,2}$	In $CD^t_{2,2}$
i plays	i plays	i plays	$j \in \bar{G}_i$ plays
$\langle$**stay**$\rangle$	$\langle \mathbf{1}_i \rangle$	$\langle$**stay**$\rangle$	$\langle$**stay**$\rangle$
$\langle I_i^{t-1} \rangle$	$\langle$**stay**$\rangle$	$\langle$**stay**$\rangle$	$\langle$**stay**$\rangle$
$\langle I_i^{t-1} \rangle$	$\langle \mathbf{1}_i \rangle$	$\langle$**stay**$\rangle$	$\langle \mathbf{1}_i \rangle$
$\langle I_i^{t-1} \rangle$	$\langle \mathbf{1}_i \rangle$	$\langle \mathbf{1}_i \rangle$	$\langle$**stay**$\rangle$
$\langle 1 \rangle$	$\langle$**stay**$\rangle$	$\langle$**stay**$\rangle$	$\langle$**stay**$\rangle$
$\langle 1 \rangle$	$\langle \mathbf{1}_i \rangle$	$\langle$**stay**$\rangle$	$\langle \mathbf{1}_i \rangle$

Table 10: In-path strategies after $CD^t_{m,2}$, where $m \geq 2$, $t > 0$

In $CD^t_{m,2}$, $m \geq 2$	In $CD^t_{m+1,2}$,$m \geq 2$
i plays	$j \in \bar{G}_i$ plays
$\langle \mathbf{1}_i \rangle$	$\langle \mathbf{1}_i \rangle$
$\langle$**stay**$\rangle$	$\langle$**stay**$\rangle$

shows.

3.2.3 Step 3: Off-path Belief and Incentives In-the-path

Whenever a deviation is detected by Rebel i at period s, he form the belief $\beta^{\pi,\tau}_{G_i}(\bar{\theta}|h^s_{G_i}) = 1$ for all $s' \geq s$, where $\bar{\theta} \in \times_{j \in G_i}\{\theta_j\} \times_{j \notin G_i} \{Inert\}$. Thus, if $\# I^0_i < k$, he will play **stay** forever and this is credible after 1-block. This off-path belief then serve as a grim trigger to impede Rebels' deviation. Then I check if this grim trigger can sustain my APEX. The details are given in the Appendix.

Though the grim trigger strategy highly reduce players' incentives to deviate, a downside of using grim trigger strategy to sustain a APEX in this framework is due to the pay-off function is not continuous at Rebel's revolting. Recall that a APEX require all the players to play ex-post efficient outcome, not just k players. If the judgement for a deviation is too strict, such equilibrium may not be sustained by grim trigger since a deviation may be only detected by some Rebels, while there may be at least k Rebels can not detect it, and therefore there are k Rebels play **revolt** forever, while the others play **stay** for ever. This phenomenon gives another reason in explaining why the message $\langle 1 \rangle$ needed to be introduced. Consider the Example 3.7 (, a modification of Example 3.5).

Example 3.7. Let $k = 5$ and suppose that there are message $\langle M_3 \rangle$, $\langle M_5 \rangle$, $\langle M_7 \rangle$ for Rebel 3,5,7. Let the game play starting from 1-block as Example. Suppose Rebel will play **revolt** forever after observing

Table 11: In-path strategies in $CD^t_{1,3}$, $t > 0$

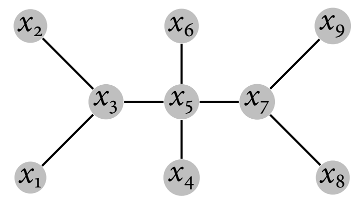
In $CD^t_{m,2}$, $1 \le m \le t+1$	In $CD^t_{1,3}$
i has played	$j \in \bar{G}_i$ plays
$\langle \mathbf{1}_i \rangle$	**r**
Otherwise	**s**

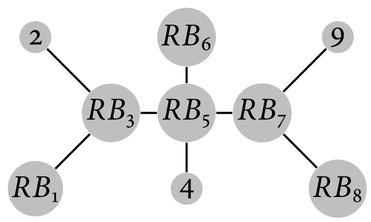
Table 12: In-path strategies after $CD^t_{m,3}$, where $m \ge 2$, $t > 0$

In $CD^t_{m,3}$, $m \ge 2$	In $CD^t_{m+1,3}$, $m \ge 2$
i plays	$j \in \bar{G}_i$ plays
r	**r**
s	**s**

$\langle M_3 \rangle$, $\langle M_5 \rangle$, or $\langle M_7 \rangle$ *if no deviation be detected*; otherwise they will play **stay** forever. Moreover, assume that Rebels can only use the pure strategies in which the outcome satisfies the form of $\langle I_i^\circ \rangle$ in reporting period; *otherwise, it will be consider as deviation.*

Let the G and θ is as the following, and let the players be indexed by prime numbers x_i in network G.





Assume $X_{I_3^\circ} > X_{I_5^\circ}$ and $X_{I_7^\circ} > X_{I_5^\circ}$. Rebel 5 will get the information from Rebel 3,7 before his reporting of I_5°. Rebel 5 will not report I_5°, although there is a punishment, since he can deviate to report $\bar{I}_5^\circ = x_3 \times x_5 \times x_7 < I_5^\circ$ by not letting Rebel 3, 7 detect, while the coordination can be succeeded. Rebel 6 can detect such deviation since the de-factorization of $X_{\bar{I}_5^\circ}$ did not include his own index x_6. However, Rebel 6 will then keep play **stay** forever although the coordination has been succeeded.

Due to the pay-off function is not continuous, a Rebel is more willing to report his information if he still don't know there are at least k Rebels, while he is less willing after he has known that.

More Rebels to join a coordination did not change his pay-off. The criteria in judging deviation should not be too strict as Example 3.7 shows. Otherwise, in the lack of paths to be considered as the equilibrium path, some Rebels will be excluded from coordination with this grim-trigger-like strategies. It gives another explanation to introduce the message $\langle 1 \rangle$ to give more paths considered as the equilibrium paths with grim trigger strategies.

3.2.4 In or Off the equilibrium path

In previous subsections, I have listed the belief updating in equilibrium path by showing the the belief after observing various messages. Lemma 3.3 show that the equilibrium constructed above is APEX. The proof is in Appendix.

Lemma 3.3. *If the state has strong connectivity, then for all n-person repeated k-Threshold game with parameter* $1 \leq k \leq n$ *played in any FFCCU network without circle, the equilibrium path is APEX.*

For the histories out of equilibrium path, the deviations could be detectable or undetectable. The argument is at first to use off-path belief to impede players from detectable deviating. Then I argue that any undetectable deviation (and therefore no punishment from other players) before a Rebel knows $\#[Rebels](\theta) \geq k$ or $\#[Rebels](\theta) < k$ will create noises in his own learning process, and will reduce his own expected continuation pay-off, and therefore such deviation will punish himself. Finally, by the equilibrium construction in coordination period, most profitable deviation in the coordination period can be detected, and therefore Rebels are forced to transmit coordination messages. To see that an undetectable deviation can impede the learning process, consider the case when a Rebel want to pretend himself as a pivotal player in order to send $\langle 1 \rangle$. According to Table 5 and Table 8, the continuation playing contingent on observing that is either playing **stay** forever or **revolt** forever after current block. This repeated playing will then block the information updating. By full support assumption and by the similar argument in the proof in Theorem 1, when δ is high enough, he is better off by staying in the equilibrium path since the states of nature can be learnt by Lemma 2.1. The proof in Appendix A.1 is to show this argument.

3.3 Variation

3.3.1 Variation: Pay-off as signals

The pay-off has been assumed to be hidden in the previous analysis, but it can be relaxed without change the result in Theorem. One may consider that the outcome of revolution may depend not only on their joint efforts but also on another effect, says the weather, and therefore the joint efforts did not guarantee the successfulness but it give higher expected pay-off[24]. Consider there is a public

[24] e.g.,[SHADMEHR and BERNHARDT, 2011]

signal $y \in \{y_1, y_2\}$ generated by Rebels' actions. Let Rebel i's pay-off function be $u_{Rebel}(a_i, y)$. Let $u_{Rebel}(\mathbf{stay}, y_1) = u_{Rebel}(\mathbf{stay}, y_2) = u_0$. The distribution of y_1 and y_2 is set as the followings.

$$
\begin{aligned}
p_{1s} &= \Pr(y_1|\#\mathbf{revolt} \geq k) \\
p_{1f} &= \Pr(y_1|\#\mathbf{revolt} < k) \\
p_{2s} &= \Pr(y_2|\#\mathbf{revolt} \geq k) \\
p_{2f} &= \Pr(y_2|\#\mathbf{revolt} < k)
\end{aligned}
$$

with

$$
p_{1s}u_{Rebel}(\mathbf{revolt}, y_1) + p_{2s}u_{Rebel}(\mathbf{revolt}, y_2) > u_0 > p_{1f}u_{Rebel}(\mathbf{revolt}, y_1) + p_{2f}u_{Rebel}(\mathbf{revolt}, y_2) \quad (6)
$$

$$
1 > p_{1s} > 0, 1 > p_{2s} > 0, p_{1f} = 1 - p_{1s}, p_{2f} = 1 - p_{2s} \quad (7)
$$

Equation (6) is a generalization of previous setting. Equation (7) is a full support assumption on signal y. With the full support assumption as Equation (7), we can construct the exactly same equilibrium strategies as previous analysis by ignore the noisy signal y. By directly checking the equilibrium strategy, there is at most one Rebel playing **revolt** in each period before some Rebels play $\langle 1 \rangle$, thus the signal y is not relevant before some Rebels play $\langle 1 \rangle$. However, a Rebel's playing $\langle 1 \rangle$ is due to his observation in reporting period, in which Rebels' strategies is independent from y, and thus the playing of $\langle 1 \rangle$ is independent from y. We then just check if Rebels can deviate to play $\langle 1 \rangle$ to get additional information coming from y. However, according to equilibrium strategies, if some Rebels play $\langle 1 \rangle$, it must be the case that either coordination to **stay** or coordination to **revolt** can be made after current block as Table 5 and Table 8 shows. Since the distribution of y has full support, and since Rebels' actions will repeat themselves after he play $\langle 1 \rangle$, he will not learn the true state of nature and thus there is a loss in expected continuation pay-off as Lemma 7 shows. It is then sequential rational for Rebels to stay in the path.

However, if Equation (7) fails and so that the pay-off can be perfectly observed, says $p_{1s} = p_{2f} = 1$, the equilibrium constructed above will be no more an equilibrium. A Rebel will deviate to send $\langle 1 \rangle$ and wait to see if the pay-off is getting to 1 by other players' playing in the equilibrium path. It is a profitable deviation since Rebels then keep playing **revolt** if they have observed the pay-off of 1 or keep playing **stay** if they did not observe it. In this case, however, it is easy to construct another APEX by letting all Rebels play **revolt** in the first period, and then keep playing **revolt** or **stay** contingent on the signals y.

3.3.2 Variation: Rebels with different levels of efforts

We can also consider a model in which players have different levels of efforts in contributing a collective action. Let the enlarged space of states of nature as $\hat{\Theta} = \Theta \times \Xi$, where $\Xi = \{1, 2, ..., k\}^n$. A $\hat{\theta} = (\theta, e)$, where $\theta \in \Theta$ and $e \in \Xi$, is interpreted as a state of nature in which a player i could be either a Rebel or Inert with levels of effort e_i, where $e_i \in \{1, 2, ..., k\}$. A Rebel playing **revolt** is interpreted as a contribution of e_1 efforts to a collective action, while such collective action require k amount of efforts. If such collective action succeed, a Rebel i will get an amount of $b_i > 0$ as his reward. Thus, we have a variation in pay-off structure as

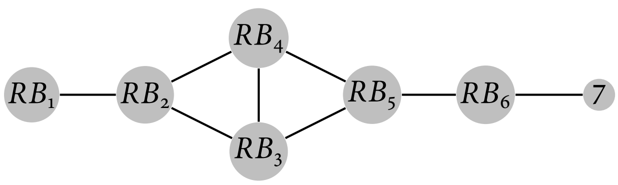
1. $u_{Rebel}(a_i, a_{-i}) = b_i$ if a_i = **revolt** and $\sum_{j:a_j=\textbf{revolt}} e_j \geq k$
2. $u_{Rebel}(a_i, a_{-i}) = -e_i$ if a_i = **revolt** and $\sum_{j:a_j=\textbf{revolt}} e_j < k$
3. $u_{Rebel}(a_i, a_{-i}) = 0$ if a_i = **stay**
4. $u_{Inert}(a_i, a_{-i}) = 1$ if a_i = **inert**

To see the equilibrium constructed in previous section still an equilibrium, just transform the network G to G' such that there is e_i different Rebels j who have $e_j = 1$ have attached a single Rebel i who has $e_i = 1$ after nature choose a state. What matters here is that the states of nature should be discrete, and therefore we can still use prime indexing to construct the equilibrium as previous result shows.

3.3.3 Variation: networks with circle

The prime indexing can deal with a potential free problem when networks have circle, although we may need to redefine the information hierarchy in order to redefine which Rebels are forced to report their private information. Consider the following Example 3.8.

Example 3.8. Let $k = 6$. Rebel 3 and Rebel 4 have the same information $I_3^1 = I_4^1$. Since reporting is costly, if there is no punishment, Rebel 3 (or Rebel 4) may shirk and deviate from truthfully reporting if Rebel 4 (or Rebel 3) can reports truthfully. However, this kind of deviation can be detected by Rebel 5 (or Rebel 2) since I_3^1 should be equal to I_4^1.



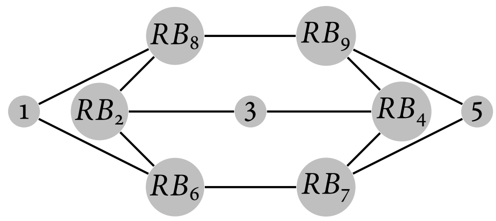
Indeed, this monitoring technique will be less invalid if the network is not commonly known. In Example 3.8, if Rebels has asymmetric information about network structure, says Rebel 5 (or Rebel 2,3) did not certain if there is a link between Rebel 4 and Rebel 2, then Rebel 4 can just pretend that he didn't know Rebel 2[25]. The analysis in incomplete information about network structure is

[25] However, if there is asymmetric information about network structure, then Example 3.8 is less than a free rider problem. Dependent on what asymmetric information is given, Rebel 3 and Rebel 4 then have different incentives in untruthful reporting. Untruthful reporting could be a dominant strategy for Rebel 4 but not necessary be a dominant strategy for Rebel 3.

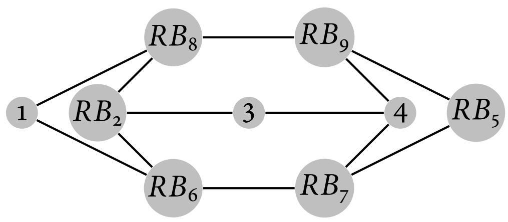
beyond the scope of this paper. Recent paper such as [Galeotti et al., 2010] deal with the issues when network is not commonly known.

There is another free rider problem which is harder to deal with. Remind that the implicit technique in my equilibrium construction is that the pivotal Rebels can be identified before the game is played in each reporting period in each block. When the network has circle, the selection of pivotal Rebels need more elaboration. Consider Example 3.9.

Example 3.9. Let $k = 6$. Suppose the network and θ is as the following.



Assume that one round of reporting is done. Rebel 2 has known $\{RB_2, RB_6, RB_8, RB_9, RB_7\}$, Rebel 4 has known $\{RB_4, RB_7, RB_9, RB_8, RB_6\}$, and so on. One more round of reporting will let Rebels 3,6,7,4,9,8 knows the true state θ, and therefore Rebels 3,6,7,4,9,8 are all pivotal Rebels conditional on others truthful reporting. We may have a rule as Example by picking up a pivotal Rebel dependent on its prime number index, say we pick Rebel 4. However, this selection of pivotal players is ex-post. The true state θ' could be as



Now node 4 is an Inert and so that he is not a pivotal Rebel. Some other rules are needed to selection a Rebel (say, Rebel 5 in this case) during the game is played.

As Example 3.9 shows, a free rider problem may occur if the selection of pivotal Rebels is not done before the game is played. When the network has circle, this problem seems more harsh and the selection rule may not be done before the game is played. Though it is possible to prove that if such pre-game selection fails, says an Inert has been selected (as the Inert 4 in Example 3.9), then the other Rebels can take over it (as the Rebel 5 in Example 3.9), the proof is still infeasible in this paper.

I leave a conjecture here and end this section.

Conjecture 3.1. *For n-person repeated k-Threshold game with parameter $1 \leq k < n$ played in any FFCCU network, if the state θ has strong connectivity and $\pi(\{\theta : \theta \text{ has strong connectivity}\}) = 1$ with full support, then there is a δ such that there is a (weak) sequential equilibrium which is approaching ex-post efficient.*

4 Conclusion

I model a coordination game and illustrate the learning processes generated by strategies in a sequential equilibrium and answer the question proposed in the beginning: what kind of networks can conduct coordination in a collective action game with information barrier. In the equilibrium strategies I constructed above, players transmit the relevant information by coding such information by their actions in the time horizontal line. Since there is an expected cost in coding information, potential free rider problem may occur to impede the learning process. When the networks is finite, fixed, connected, commonly known, and without circle, players can always learn the underlying relevant information and conduct the coordination only by their actions in a equilibrium. However, what kinds of equilibrium strategies can constitute a learning process in the networks with circle still remain to be answered.

Existing literatures in political science and sociology have recognized the importance of social network in influencing individual's behaviour in participating social movements, e.g., [Passy, 2003][McAdam, 2003][Siegel, 2009]. This paper takes the view in considering networks as routes for communication where rational individuals have local information initially but they can influence nearby individuals by taking actions. Such local information may take long time to travel across individuals, although it will become globally in a strategical interaction. A characterization in the speed of information transmission across networks is not answered here, although it is an important topic in order to give more attentions in the details of network structures. This question would remain for the future research.

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A Appendix

proof for Theorem 3

Proof. This proof follow three useful claims, Claim 1, Claim 2 and Claim 3. To simplify notation, denote $\bar{G}_i = G_i \backslash \{i\}$. First note that I_i^t can be expressed as

$$I_i^t = \bigcup_{k_0 \in N_i \cap R^t} \bigcup_{k_1 \in N_{k_0} \cap R^{t-1}} \cdots \bigcup_{k_{t-1} \in N_{k_{t-2}} \cap R^1} N_{k_{t-1}} \cap R^0 \tag{8}$$

, while H_i^t can be expressed as

$$N_i^t = \bigcup_{k_0 \in N_i \cap R^{t-1}} \bigcup_{k_1 \in N_{k_0} \cap R^{t-2}} \cdots \bigcup_{k_{t-2} \in N_{k_{t-3}} \cap R^1} N_{k_{t-2}} \tag{9}$$

Claim 1. *$I_i^t \subset N_i^t$, $I_i^t = \bigcup_{k \in I_i^{t-1}} N_k \cap R^0$, and $N_i^t = \bigcup_{k \in N_i^{t-1}} N_k$ for all $t \geq 0$.*

Proof. $I_i^t \subset N_i^t$ is by definition. $I_i^t = \bigcup_{k \in I_i^{t-1}} N_k \cap R^0$ and $N_i^t = \bigcup_{k \in N_i^{t-1}} N_k$ is by comparing Equation 8 and Equation 9. □

Claim 2. *If the network is FFCCU without circle, then for each $t \geq 1$ block, we have $i \in R^t \Leftrightarrow i \in R^{t-1}$ and $\exists k_1, k_2 \in R^{t-1} \cap \bar{G}_i$, where $k_1 \neq k_2$.*

Proof. The proof is by induction. We first show that the statement is true for $t = 1$.

Base: $i \in R^1 \Leftrightarrow [i \in R^0] \wedge [\exists k_1, k_2 \in (R^0 \cap \bar{G}_i)]$.

$\Rightarrow$: Since $i \in R^1$, then $i \in R^0$ and then $I_i^0 \nsubseteq N_j^0$ for all $j \in \bar{G}_i$ by definition. Since $I_i^0 = R^0 \cap G_i$, then $\forall j \in \bar{G}_i [\exists k \in (R^0 \cap \bar{G}_i)[k \notin N_j^0]]$. Since the $j \in \bar{G}_i$ is arbitrary, we then have a pair of $k_1, k_2 \in (R^0 \cap \bar{G}_i)$ such that $k_1 \notin N_{k_2}^0$ and $k_2 \notin N_{k_1}^0$.

$\Leftarrow$: Pick $k \in \{k_1, k_2\} \subseteq (R^0 \cap \bar{G}_i)$, and pick an arbitrary $j \in \bar{G}_i \backslash \{k\}$. Note that $k \notin N_j^0$, otherwise there is a circle from i to i. Hence $[k \in (R^0 \cap \bar{G}_i)] \wedge [k \notin N_j^0]$ and therefore $[k \in I_i^0] \wedge [k \notin N_j^0]$. Then we have $I_i^0 \nsubseteq N_j^0$ for arbitrary $j \in \bar{G}_i$, and thus $i \in R^1$.

Induction hypothesis: the statement is true for $\{1, 2, .., t\}$ where $t \geq 1$.

If the hypothesis is true, then $i \in R^{t+1} \Leftrightarrow [i \in R^t] \wedge [\exists k_1, k_2 \in (R^t \cap \bar{G}_i)]$

$\Rightarrow$: since $i \in R^{t+1}$, then $i \in R^t$ and $I_i^t \nsubseteq N_j^t$ for all $j \in \bar{G}_i$ by definition. Recall that I_i^t can be expressed as $I_i^t = \bigcup_{k_0 \in N_i \cap R^t} \bigcup_{k_1 \in N_{k_0} \cap R^{t-1}} \dots \bigcup_{k_{t-1} \in N_{k_{t-2}} \cap R^1} N_{k_{t-1}} \cap R^0$ and H_i^t can be expressed as $N_i^t = \bigcup_{k_0 \in N_i \cap R^{t-1}} \bigcup_{k_1 \in N_{k_0} \cap R^{t-2}} \dots \bigcup_{k_{t-2} \in N_{k_{t-3}} \cap R^1} N_{k_{t-2}}$, then for every $l \in I_i^{t-1}$, we can find a path connecting i to l by the induction hypothesis. If $j \in \bar{G}_i$, then we can find a path connecting j to l by connecting j to i, and then connecting i to l. Thus, if $l \in I_i^{t-1}$ then $l \in N_j^t$, and hence $I_i^{t-1} \subseteq N_j^t$ for all $j \in \bar{G}_i$. Recall that $I_i^t = \bigcup_{k \in N_i \cap R^t} I_k^{t-1}$ and $i \in R^{t+1}$, then we must have $\forall j \in \bar{G}_i [\exists k \in (R^t \cap \bar{G}_i)[I_k^{t-1} \nsubseteq N_j^t]]$, since $I_i^{t-1} \subseteq N_j^t$. Note that such $j \in \bar{G}_i$ is arbitrary, we then have a pair of $k_1, k_2 \in (R^t \cap \bar{G}_i)$ such that $k_1 \notin N_{k_2}^t$ and $k_2 \notin N_{k_1}^t$.

$\Leftarrow$: By the induction hypothesis, we have a chain $k_{1_0}, \dots, k_{1_t}, i, k_{2_t}, \dots, k_{2_0}$ with $k_{1_0} \in R^0$,..., $k_{1_t} \in R^t$, $i \in R^t$, $k_{2_t} \in R^t$,...,$k_{1_0} \in R^0$, where $k_{1_t}, k_{2_t} \in (R^t \cap \bar{G}_i)$, $k_{1_0} \in I_{k_{1_t}}^{t-1}$ and $k_{2_0} \in I_{k_{2_t}}^{t-1}$. Note that $k_{1_0} \notin N_j^t$ whenever $j \in \bar{G}_i$, otherwise there is a circle from i to i since $\{i, k_{2_t}, \dots, k_{2_0}\} \in N_j^t$, and hence $[k_{1_0} \in I_{k_{1_t}}^{t-1}] \wedge [k_{1_0} \notin N_j^t]$ for all $j \in \bar{G}_i$. Therefore we have $[I_{k_{1_t}}^{t-1} \in I_i^t] \wedge [I_{k_{1_t}}^{t-1} \notin N_j^t]$ for all $j \in \bar{G}_i$ since $k_{1_t}, k_{2_t} \in (R^t \cap \bar{G}_i)$ and $[k_{1_0} \in I_{k_{1_t}}^{t-1}] \wedge [k_{1_0} \notin N_j^t]$ for all $j \in \bar{G}_i$. Then we have $I_i^t = \bigcup_{k \in N_i \cap R^t} I_k^{t-1} \nsubseteq N_j^t$ for arbitrary $j \in \bar{G}_i$, and thus $i \in R^{t+1}$.

We can then conclude that the statement is true by induction.

□

Claim 3. *If the network FFCCU without circle and if the state has strong connectivity, then if there is a pair of R^t nodes then there exists a R^t-path connecting them.*

Proof. The proof is by induction and by Claim 2. Since the state has strong connectivity, we have a R^0-path connecting each pair of R^0 nodes. Since all pairs of R^0 nodes are connected by a R^0-path, then for all pairs of R^1 nodes must be in some of such paths by Claim 2, and then connected by a R^0-path. But then all the R^0-nodes in such path are all R^1 nodes by Claim 2 again and by $R^t \subseteq R^{t-1}$ for $t \geq 1$ by definition. Thus, for all pairs of R^1 nodes has a R^1-path connecting them. The similar argument holds for $t > 1$, we then get the result.

□

I begin to prove this Theorem 3. I first claim that if $R^t \neq \emptyset$ and if $R^{t+1} = \emptyset$, then $R^0 \subset I_i^t$ whenever $i \in R^t$. Then I claim that if $R^t \neq \emptyset$ then $\#R^{t+1} < \#R^t$. Finally, I iterate R^t with $t \geq 0$ to get the conclusion.

If $R^t \neq \emptyset$ but $R^{t+1} = \emptyset$, I claim that $R^0 \subset I_i^t$ for all $i \in R^t$. The proof is by contradiction. If $R^0 \not\subset I_i^t$, there is a $j \in R^0$ but $j \notin I_i^t$. Since $I_i^t = \bigcup_{k_0 \in N_i \cap R^t} \bigcup_{k_1 \in N_{k_0} \cap R^{t-1}} \dots \bigcup_{k_{t-1} \in N_{k_{t-2}} \cap R^1} N_{k_{t-1}} \cap R^0$, then there is no such a path $\{i, k_0, k_1, \dots, k_{t-1}, j\}$, where $k_0 \in G_i \cap R^t, k_1 \in G_{k_0} \cap R^{t-1}, \dots, k_{t-1} \in N_{k_{t-2}} \cap R^1$. Since $R^{t+1} = \emptyset$ and therefore $R^{t'} = \emptyset$ if $t' \geq t+1$, and hence there is no such a path containing a node in $R^{t'} = \emptyset$, where $t' \geq t+1$ connecting i to j. But $i \in R^t$ and $i, j \in R^0$, if there is no such a path, then it violate either Claim 3 or Claim 2. Contradiction.

Next I claim that if $R^t \neq \emptyset$ then $\#R^{t+1} < \#R^t$. The proof is the followings. Given a node i in R^t, let $j \in R^t$ (could be i itself) be the node connected with i with the maximum shortest R^t path. This j can be found since $R^t \neq \emptyset$ and the network is finite. Then there is no R^t node in j's neighbourhood other than the nodes in this path. Since the network is without circle, there is at most one R^t node in j's neighbourhood. But then $j \notin R^{t+1}$ since it violate Claim 2.

Starting from $R^0 \neq \emptyset$ and iterating R^t with $t \geq 0$, if $R^t \neq \emptyset$ but $R^{t+1} = \emptyset$, then there is some i with $R^0 \subset I_i^t$ as the above paragraph shows; if $R^t \neq \emptyset$ and $R^{t+1} \neq \emptyset$, then we starting from R^{t+1} and iterating R^{t+1} with $t \geq t+1$. Since $\#R^{t+1} < \#R^t$ as the above paragraph shows, there is a time t^* with $R^{t^*} = \emptyset$, then we get the conclusion.

□

proof for Lemma 3.1

Proof. Denote (i, j)-path as the set of paths from i to j. The proof is by contradiction. Suppose there are three or more R^t-nodes in C^t, then pick any three nodes of them, and denote them as i_1, i_2, i_3. Let's say i_2 is in a (i_1, i_3)-path by strong connectivity, and therefore $i_2 \in Tr_{i_1 i_2}$ and $i_3 \in Tr_{i_2 i_3}$. First we show that $i_1 \in G_{i_2}$ (or $i_3 \in G_{i_2}$). Suppose $i_1 \notin N_{i_2}$, since $i_1, i_2 \in R^t$, then the (i_1, i_2)-path is a R^t-path by Claim 2. Let this $(i_1 i_2)$-path be $\{i_1, j_1, \dots, j_n, i_2\}$. Since $i_1, j_1, \dots, j_n, i_2 \in R^t$, we then have $I_{i_1}^{t-1} \not\subseteq N_{j_1}^{t-1}, \dots, I_{j_n}^{t-1} \not\subseteq N_{i_2}^{t-1}$ and $I_{j_1}^{t-1} \not\subseteq N_{i_1}^{t-1}, \dots, I_{i_2}^{t-1} \not\subseteq N_{j_n}^{t-1}$. Since $I_{i_1}^{t-1} \subseteq N_{i_1}^{t-1}, \dots, I_{i_2}^{t-1} \subseteq N_{i_2}^{t-1}$ by Lemma 1, we further have $\exists k_1 \in R^0[k_1 \in N_{j_1}^{t-1} \backslash I_{i_1}^{t-1}]$,...,$\exists k_n \in R^0[k_n \in N_{j_n}^{t-1} \backslash I_{i_2}^{t-1}]$. Since the state has strong connectivity, there is a R^0 path connecting $k_1, \dots, k_n$ by Claim 3. But then we have already found k_1, k_2 such that $k_1 \in N_{j_1}^{t-1} \backslash I_{i_1}^{t-1}$ and $k_2 \in \bar{G}_{k_1}$. It is a contradiction that $i_1 \in C$.

Now, i_1, i_2, i_3 will form a R^t-path as $\{i_1, i_2, i_3\}$. With the same argument as the above, we then have $\exists k_1 \in R^0[k_1 \in N_{i_2}^{t-1} \backslash I_{i_1}^{t-1}]$ and $\exists k_2 \in R^0[k_2 \in N_{i_3}^{t-1} \backslash I_{i_2}^{t-1}]$, and thus i_1 is not in C. □

proof for Lemma 3.2

Proof. The proof is by contradiction. Since $i \in R^t$, there is a $j \in (R^{t-1} \cap \bar{G}_i)$ by Lemma 2. Note that $N_j^{t-1} \subseteq \bigcup_{k \in N_i^{t-1}} N_k$ since $N_j^{t-1} = \bigcup_{k \in I_j^{t-2}} N_k$, and $I_j^{t-2} \subseteq I_i^{t-1} \subseteq N_i^{t-1}$. If there is another node outside $\bigcup_{k \in N_i^{t-1}} N_k$ in Tr_{ij}, then there must be another node such that there is a path connected to some nodes in N_j^{t-1} since the network is connected. It is a contradiction that $i \in C$.

□

A.1 Equilibrium

A.1.1 Out-off-path belief

If Rebel i detects a deviation at m period, he form the belief as

$$\beta_i(\{\theta : \theta \in \times_{j \in G_i}\{\theta_j\} \times \{Inert\}^{n-\#G_i}\} | h_{G_i}^{m'}) = 1 \, , m' \geq m \tag{10}$$

A.1.2 Equilibrium Path: Notations

- $\langle\rangle_r$ be the set of finite sequence of actions in which the action **r** occurs once and only once.
- $PF(\langle\rangle, m)$ be the m-periods prefix of $\langle\rangle$.
- (i, j)-path be the set of paths from i to j.

A.1.3 Equilibrium Path: reporting period

reporting period: notations

- m be a period in reporting period.
- $|\langle RP^t \rangle|$ be the total periods in reporting period in t-block
- $O_i^{m,t}$ be the set of i's neighbours js who has played a sequence M such that $M = PF(\langle I_j^{t-1} \rangle, m)$ and $M \in \langle\rangle_r$ at m period in the reporting period of t-block
- $I_i^{m,t} \equiv (\bigcup_{k \in O_i^{m,t}} I_k^{t-1}) \cup I_i^{t-1}$ be the updated relevant information gathered by i at period m in the reporting period of t-block. Note that $I_i^{0,t} = I_i^{t-1}$ and $I_i^{|\langle RP^t | ,t \rangle} = I_i^t$.
- $N_i^{m,t} \equiv (\bigcup_{k \in O_i^{m,t}} N_k^{t-1}) \cup N_i^{t-1}$ be the updated neighbourhood which contains $I_i^{m,t}$

- Let
$$Ex_{I_i^{m,t}} \equiv \{l \notin N_i^{m,t} | \exists l' \in I_i^{m,t} \text{ such that there exists a } (l, l')\text{-path}\}$$
be all the possible Rebel nodes outside of $N_i^{m,t}$ given $I_i^{m,t}$

- Let
$$Tr_{I_i^{m,t} j} \equiv Tr_{ij} \cap (Ex_{I_i^{m,t}} \cup I_i^{m,t})$$
be all the possible Rebel nodes in the Tr_{ij} given $I_i^{m,t}$.

reporting period: automata

$i \notin R^t$

- **WHILE LOOP**
  - At $m \geq 0$, if $\#Ex_{I_i^{m,t}} \cup I_i^{m,t} < k$, report $\langle$**stay**$\rangle$ and then play **stay** forever.
  - Otherwise, **runs POST-CHECK**

$i \in R^t$

- **WHILE LOOP**
  - At $m \geq 0$, if $\#Ex_{I_i^{m,t}} \cup I_i^{m,t} < k$, report $\langle$**stay**$\rangle$ and then play **stay** forever.
  - Otherwise, **runs MAIN**
- **MAIN**

  At $m \geq 0$,

  1. At $m = 0$ and if $\#I_i^{t-1} = \#I_i^{0,t} = k - 1$, then **runs POST-CHECK**
  2. At $m = 0$ and if $i \in R^t$ and
  $$\nexists j \in R^{t-1}\bar{G}_i \text{ such that } \exists l_1, l_2 \in Tr_{ij}[[l_1 \in N_j^{t-1} \backslash I_i^{t-1}] \wedge [l_2 \in \bar{G}_{l_1}]]]$$
  , then runs **CHECK.**0. Otherwise, recall **MAIN**
  3. At $0 \leq m \leq |RP^t| - |\langle I_i^{t-1} \rangle|$, play
  $$\textbf{stay}$$
  4. At $m = |RP^t| - |\langle I_i^{t-1} \rangle| + 1$, then
     (a) if $O_i^{m,t} = \varnothing$, then report
     $$\langle I_i^{t-1} \rangle$$

(b) if $O_i^{m,t} \neq \varnothing$, then **runs CHECK.k**

- **CHECK.0**

  At $m = 0$, if $i \in C$, i.e. if $i \in R^t$ and

  $$\nexists j \in R^{t-1} \cap \bar{G}_i \text{ such that } [\exists l_1, l_2 \in Tr_{ij}[[l_1 \in N_j^{t-1} \setminus I_i^{t-1}] \wedge [l_2 \in \bar{G}_{l_1}]]]$$

  , then

  1. If $\#C = 1$, then **runs POST-CHECK**
  2. If $\#C = 2$, then denote $i_1, i_2 \in C$ such that $I_{i_1}^{t-2} < I_{i_2}^{t-2}$, and then
     - if $i = i_1$, then **runs POST-CHECK**
     - if $i = i_2$, then report

       $$\langle I_i^{t-1} \rangle$$

- **CHECK.m**

  At $m > 0$, if $O_i^{m,t} \neq \varnothing$, then there are two cases,

  1. Case 1: If $i \in R^t$ and

     $$\exists j \in O_i^m \text{ such that } \exists l_1, l_2 \in Tr_{I_i^{m,t} j}[[l_1 \in I_j^{t-1} \backslash I_i^{t-1}] \wedge [l_2 \in \bar{G}_{l_1}]]]$$

     , then report

     $$\langle I_i^{t-1} \rangle$$

  2. Case 2: If $i \in R^t$ and

     $$\nexists j \in O_i^m \text{ such that } \exists l_1, l_2 \in Tr_{I_i^{m,t} j}[[l_1 \in I_j^{t-1} \backslash I_i^{t-1}] \wedge [l_2 \in \bar{G}_{l_1}]]]$$

     (a) Case 2.1: If $i \in R^t$ and

     $$\nexists j \in R^{t-1} \cap (G_i \backslash O_i^{m,t}) \text{ such that } [\exists l_1, l_2 \in Tr_{I_i^{m,t} j}[[l_1 \in N_j^{t-1} \backslash I_i^{t-1}] \wedge [l_2 \in \bar{G}_{l_2}]]]$$

     Note: this case is the case when $i \in C$, thus recall Check.0

     (b) Case 2.2: If

     $$\exists j \in R^{t-1} \cap (G_i \backslash O_i^{m,t}) \text{ such that } [\exists l_1, l_2 \in Tr_{I_i^{m,t} j}[[l_1 \in N_j^{t-1} \backslash I_i^{t-1}] \wedge [l_2 \in \bar{G}_{l_2}]]]$$

     - if $\#I_i^{m,t} = k - 1$, then **runs POST-CHECK**
     - if $\#I_i^{m,t} < k - 1$, then report

       $$\langle I_i^{t-1} \rangle$$

- **CHECK.k**

  At $m \geq 1$,

  1. $O_i^{m,t} \neq \varnothing$, and
  $$\# I_i^{m,t} \geq k$$
  , then **runs POST-CHECK**
  2. $O_i^{m,t} \neq \varnothing$, and
  $$\# I_i^{m,t} < k$$
  , then **runs CHECK.m**

- **POST-CHECK**

  1. At $m = |RP^t|$, then
     (a) If $i \in R^t$ and if $|I_i^{m,t}| \geq k - 1$, then play **revolt**
     (b) if $i \notin R^t$, then play **stay**

A.1.4 Equilibrium path: coordination period

coordination period: notations

- m be a sub-block in coordination period.
- Let
  $$Ex_{I_i^t} \equiv \{l \notin I_i^t | \exists l' \in I_i^t \setminus I^{t-1} \text{ such that there exists a } (l, l')\text{-path}\}$$
  be all the possible Rebel nodes outside of N_i^t given I_i^t.
- Let
  $$Tr_{I_i^t j} \equiv Tr_{ij} \cap (Ex_{I_i^t} \cup I_i^t)$$
  be the set of possible Rebel nodes in the Tr_{ij} given I_i^t.

coordination period: automata

- **1st Division**

  In 1st division, for $t = 0$ block,

  – If $\# Ex_{I_i^t} \cup I_i^t < k$, then play **stay** forever.
  – If $\# Ex_{I_i^t} \cup I_i^t \geq k$, and if $i \notin R^1$, then play
  $$\langle \textbf{stay} \rangle$$

- If $\#Ex_{I_i^t} \cup I_i^t \geq k$, and if $i \in R^1$, then play

$$\langle \mathbf{1}_i \rangle$$

In 1st division, for $t > 0$ block and for $1 \leq m \leq n$ sub-block,

- If i has played $\langle 1 \rangle$, then play
$$\langle \mathbf{1}_i \rangle$$
- If $\#Ex_{I_i^t} \cup I_i^t < k$, then play **stay** forever.
- If $\#Ex_{I_i^t} \cup I_i^t \geq k$, and there are some $j \in \bar{G}_i$ have played $\langle \textbf{stay} \rangle$, then play **stay** forever.
- If $\#Ex_{I_i^t} \cup I_i^t \geq k$, and there is no $j \in \bar{G}_i$ has played $\langle \textbf{stay} \rangle$, then play

$$\langle \mathbf{1}_i \rangle$$

- **2nd Division**

  In $t = 0$ block

  - If $i \notin R^1$, play
  $$\langle \textbf{stay} \rangle$$
  .
  - If $i \in R^1$, and if $\#I_i^0 \geq k$, play
  $$\langle \textbf{stay} \rangle$$
  .
  - If $i \in R^1$, if $\#I_i^0 < k$, if $\#Ex_{I_i^t} \cup I_i^t \geq k$ and if some $j \in \bar{G}_i$ have played play $\mathbf{1}_j$ in the 1st division, then play
  $$\langle \textbf{stay} \rangle$$
  .
  - If $i \in R^1$, if $\#I_i^0 < k$, if $\#Ex_{I_i^t} \cup I_i^t \geq k$ and if no $j \in \bar{G}_i$ has played play $\mathbf{1}_j$ in the 1st division, then play **stay** forever.

  In $t > 0$ block, if there is no $j \in G_i$ such that j has played $\langle \textbf{stay} \rangle$ in the **1st Division** , then run the following automata. Otherwise, play **stay** forever.

  - $i \notin R^t$
    * In the 1-sub-block: play
    $$\langle \textbf{stay} \rangle$$
    * In the $2 \leq m \leq t + 1$ sub-blocks:

1. If $i \in R^{t'}$ for some $t' \geq 0$ and if there is a $j \in R^{t'+1} \cap \bar{G}_i$ has played
   (a) $\langle \mathbf{stay} \rangle$ in $m = 1$ sub-block
   (b) or $\langle \mathbf{1}_j \rangle$ in $m \geq 2$ sub-blocks

   , then play

$$\langle \mathbf{1}_i \rangle$$

   in $m+1$ sub-block.
2. Otherwise, play

$$\langle \mathbf{stay} \rangle$$

   in current sub-block

– $i \in R^t$

  * In the 1-sub-block:
    1. If i has played $\langle 1 \rangle$, then play

$$\langle \mathbf{stay} \rangle$$

    2. If i has not played $\langle 1 \rangle$ and if there is a $j \in \bar{G}_i$ has played $\langle 1 \rangle$, then play

$$\langle \mathbf{stay} \rangle$$

    3. If i has not played $\langle 1 \rangle$ and if there is no $j \in \bar{G}_i$ has played $\langle 1 \rangle$, then
       · If $\# I_i^{|RP^t|,t} \geq k$, then play

$$\langle \mathbf{stay} \rangle$$

       · If $\# I_i^{|RP^t|,t} < k$, then play

$$\langle \mathbf{1}_i \rangle$$

  * In the $m \geq 2$-sub-block:
    1. If $i \in R^{t'}$ for some $t' \geq 0$ and if there is a $j \in R^{t'} \cap \bar{G}_i$ has played
       (a) $\langle \mathbf{stay} \rangle$ in $m = 1$ sub-block, or
       (b) $\langle \mathbf{1}_j \rangle$ in $m \geq 2$ sub-blocks

       , then play

$$\langle \mathbf{1}_i \rangle$$

       in $m+1$ sub-block.
    2. Otherwise, play

$$\langle \mathbf{stay} \rangle$$

       in current sub-block.

- **3rd Division**

1. **INITIATING**
   If i has observed $j \in \bar{G}_i$ has played
   (a) $\langle$**stay**$\rangle$ in 1-sub-block in **2nd Division** or
   (b) $\langle \mathbf{1}_j \rangle$ in $m \geq 2$ sub-blocks **2nd Division** or
   (c) **s** in the **3rd Division**
   , then play **revolt** forever
2. **NOT INITIATING**
   Otherwise, play **stay** in current period.

A.1.5 Proof for Theorem 3.1

Proof. The proof is organized as the following. In Claim 4 and Lemma 3.3, we show that a Rebel will learn $\#[Rebel](\theta) \geq k$ or $\#[Rebel](\theta) < k$ in the equilibrium path. Lemma 3.3 also show that the equilibrium path is ex-post efficient. Since that, there is a time T such that a Rebel's static pay-off after T is 1 if $\#[Rebel](\theta) \geq k$ or 0 if $\#[Rebel](\theta) \geq k$, which is the maximum static pay-off contingent on all events. Then in the proof in Claim 5, I show that if a Rebel makes detectable deviation, then there is an event E contingent on this deviation such that his expected continuation static pay-off is strictly lower than that in equilibrium path after T. Finally, in Claim 6, Claim 7, Claim 8, and Claim 9, I show that if a Rebel makes undetectable deviation, then there is an event E contingent on this deviation such that his expected continuation static pay-off is also strictly lower than that in equilibrium path after T. By the full support assumption, such events can be constructed such that those events has positive probability. Since such event contingent on a deviation can always be found, and since the static pay-off after T is maximum for all events, there is a δ such that a Rebel will not deviate. I then conclude this theorem.

To simplify the notations, if $P(\theta)$ is a property of θ, then I abuse the notations by letting $\beta^{\pi,\tau^*}_{G_i}(P(\theta)|h^m_{G_i}) \equiv \sum_{\theta:P(\theta)} \beta^{\pi,\tau^*}_{G_i}(\theta|h^m_{G_i})$. I also say "$i$ knows $P(\theta)$" to mean $\beta^{\pi,\tau^*}_{G_i}(P(\theta)|h^m_{G_i}) = 1$. Also that that $\bar{G}_i = G_i \backslash \{i\}$.

Claim 4. *In the equilibrium path and for $\#Ex_{I^{m,t}_i} \cup I^{m,t}_i \geq k$, where m is a period in reporting period. If i report $\langle 1 \rangle$, then either coordinate to* **revolt** *after t-block or $\#R^\circ < k$.*

Proof. By directly checking the equilibrium path, we have

1. if $\#I^{|RP^t|,t}_i \geq s$, then the coordination can be initiated by such i.
2. if $\#I^{|RP^t|,t}_i = k-1$, and if there is one more node who reported $\langle 1 \rangle$, then the coordination can be initiated by i.
3. if $\#I^{|RP^t|,t}_i = k-1$, and if there are no nodes who reported in current period, then $\#I^{|RP^t|,t}_i = \#I^t_i = k-1$. We now check the conditions guiding i to **POST-CHECK**.

- If i is coming from the conditions in **MAIN**, it means that there is no further H-node outside I_i^{t-1}, and thus outside $\bigcup_{k\in I_i^{t-1}} G_k$.
- If i is coming from the conditions in **CHECK.o**, it means that there is no further H-node outside $\bigcup_{k\in I_i^{t-1}} G_k \cap R^o$, and thus outside $\bigcup_{k\in I_i^{t-1}} G_k$.
- If i is coming from the conditions in **CHECK.m**, it means that there is no further H-node outside $\bigcup_{k\in I_i^{t-1}} G_k \cap R^o$, and thus outside $\bigcup_{k\in I_i^{t-1}} G_k$.

Then $\#I_i^t < k$, but $I_i^t = \bigcup_{k\in I_i^{t-1}} N_k \cap R^o$, and hence $\#R^o < k$.

□

proof for Lemma 3.3

Proof. We want to show that when θ satisfying $\#[Rebels](\theta) \geq k$, all the Rebels play **revolt** eventually; when θ satisfying $\#[Rebels](\theta) < k$, all the Rebels play **stay** eventually.

1. If all the Rebels only play $\langle I^{t-1}\rangle$ or $\langle$**stay**$\rangle$ in reporting period for all $t \geq 1$ block, then by the equilibrium path, those nodes played $\langle I^{t-1}\rangle$ are R^t-node, and those nodes played $\langle$**stay**$\rangle$ are not-R^t nodes.

   If there are some Rebels play $\langle$**stay**$\rangle$ in the first division in t-block, then all the Rebels play **stay** eventually; If R^t Rebels play $\langle$**stay**$\rangle$ in the first sub-block in second division in t-block, then all the Rebels will play **stay** after third division in this block. Otherwise, all the Rebels go to the next reporting period.

   By Theorem 3, there is a t^* such that there is a R^{t^*} node knows θ, and therefore he knows if θ satisfying $\#[Rebels](\theta) \geq k$ or $\#[Rebels](\theta) < k$. In equilibrium path, such node play $\langle$**stay**$\rangle$ either in the first sub-block in first division or in the first sub-block in second division in coordination period.Thus the equilibrium path is approaching efficient.

2. If there are some Rebels play $\langle 1\rangle$ in reporting period for a $t \geq 1$ block, then by Claim 4, such nodes will knows if θ satisfying $\#[Rebels](\theta) \geq k$ or $\#[Rebels](\theta) < k$ after reporting period in this t-block. Then $\langle$**stay**$\rangle$ is either played in the first sub-block in first division or played in the first sub-block in second division in coordination period. Thus the equilibrium path is approaching efficient.

□

Next, I prepare the claims to show that a Rebel will not deviate. I start with Claim 5 in which the deviation is detectable.

Claim 5. *For $\#Ex_{I_i^{m,t}} \cup I_i^{m,t} \geq s$, where m is a period. Denote D be the set of Rebels who detect i's deviation. If $\#I_i^{m,t} < k$. If $D \neq \varnothing$, then there is a δ such that i will not deviate.*

Proof. Denote D be the set of neighbours who detect i's deviation. Let the events be

$$\begin{aligned} E_1 &= \{\theta : \#[Rebels](\theta) < k\} \\ E_2 &= \{\theta : k \leq \#[Rebels](\theta) < k + \#D\} \\ E_3 &= \{\theta : \#[Rebels](\theta) \geq k + \#D\} \end{aligned}$$

In equilibrium path, there are periods t^s (t^f) such that if θ satisfying $\#[\text{Rebels}](\theta) \geq k$ ($\#[\text{Rebels}](\theta) < k$) then Rebels play **revolt** (**stay**) forever. If i follows the equilibrium path, the expected static pay-off after $\max\{t^s, t^f\}$[26] is

$$\beta_i(E_2|h^m_{N_i}) + \beta_i(E_3|h^m_{N_i})$$

If i deviate, the expected static pay-off after $\max\{t^s, t^f\}$ is

$$\beta_i(E_3|h^m_{N_i})$$

Therefore there is a loss in expected static pay-off of

$$\beta_i(E_2|h^m_{N_i})$$

Thus, there is a loss in expected continuation pay-off contingent on E_2 by

$$\delta^{\max\{t^s,t^f\}} \frac{\beta_i(E_2|h^m_{N_i})}{1-\delta}$$

□

Next, I prepare the claims to show that a Rebel will not deviate if such deviation is undetectable.

Claim 6. *In reporting period for $\#Ex_{I^{m,t}_i} \cup I^{m,t}_i \geq k$, if $\#I^{m,t}_i < k$, then there is a δ such that i will not deviate by reporting $\bar{I}^{t-1}_i \neq I^{t-1}_i$ if such deviation is not detected by i's neighbour.*

Proof. Assume $\bar{I}^{t-1}_i \neq I^{t-1}_i$. Since a detection of deviation has not occur, it must be the case that there is a non-empty set $F = \{j \in \bar{I}^{t-1}_i : \theta_j = Inerts\}$[27].

Let the set

$$E_1 = \{\bar{\theta} : \bar{\theta}_j = Rebel \text{ if } j \in F \text{ and } \bar{\theta}_j = \theta_j \text{ if } j \notin F\}$$

be the set of pseudo events by changing θ_j where $j \in F$. And let

$$E_2 = \{\theta : \theta_j = Inert \text{ if } j \in F \text{ and } \bar{\theta}_j = \theta_j \text{ if } j \notin F\}$$

be the set of true event.

[26] There is t^s or t^f for each θ. The maximum is among those possible θ.

[27] Otherwise, there is a detection of deviation. Recall the definition in information hierarchy: $I^{-1}_i \subset I^0_i \subset \ldots \subset I^{t-1}_i$ for all $i \in R^0$

Then consider the event

$$\begin{aligned} E &= \{\bar{\theta} \in E_1 : \#[Rebels](\bar{\theta}) \geq k\} \\ &= \{\theta \in E_2 : \#[Rebels](\theta) \geq k - \#F\} \end{aligned}$$

Partition E as sub events

$$\begin{aligned} E_3 &= \{\theta \in E_2 : \#[Rebels](\theta) \geq k\} \\ E_4 &= \{\theta \in E_2 : k > \#[Rebels](\theta) \geq k - \#F\} \end{aligned}$$

By Lemma 3 and following the strategies in equilibrium path (since i have not been detected), there is a block $\bar{t}^s$ with respect to $\bar{\theta}$ such that if $\bar{\theta} \in E$ then there some $R^{\bar{t}^s}$ Rebel js, says J, will initiate the coordination, and then Rebels play **revolt** forever after $\bar{t}^s$-block. Note that such j is with $\#I_i^{\bar{t}^s} \geq k$ by Claim.

We have several cases:

1. Case 1: If $i \in J$, his own initiation will only depends on $\#I_i^{\bar{t}^s}$ by Claim, which is the same as he has reported $\langle I_i^{t-1} \rangle$. It is strictly better by not deviating since playing $\langle \bar{I}_i^{t-1} \rangle$ is more costly than $\langle \bar{I}_i^{t-1} \rangle$ ($X_{\bar{I}_i^{t-1}} > X_{I_i^{t-1}}$).

2. Case 2: If there is another j who $\bar{I}_i^{t-1} \not\subset I_j^{\bar{t}^s}$, then such j's initiation of coordination dependent of his own information about θ, $\subset I_j^{\bar{t}^s}$, by Claim and i's deviation did not change j's information. It is strictly better by not deviating since playing $\langle \bar{I}_i^{t-1} \rangle$ is more costly than $\langle \bar{I}_i^{t-1} \rangle$.

3. Case 3: If there is another j who $\bar{I}_i^{t-1} \subset I_j^{\bar{t}^s}$ such that $\#I_i^{\bar{t}^s} \geq k$. If i did not follow j's initiation of coordination, then there is a detection of deviation by checking the equilibrium path. Such detection will let i's continuation expected pay-off down to zero, and therefore i should follow this initiation as Claim shows. If i follows, and $\#I_i^{\bar{t}^s} \geq s$, we are in the Case 1. If i follows, but $\#I_i^{\bar{t}^s} < s$, then i's expected static pay-off after $\bar{t}^s$ is at most

$$\max\{\beta_i(E_3 | h_{N_i}^m) \times 1 + \beta_i(E_4 | h_{N_i}^m) \times (-1), 0\}$$

   However, if i follow the equilibrium path, there is are t^s, t^f such that the expected static pay-off after $\max\{t^s, t^f\}$ is

$$\max\{\beta_i(E_3 | h_{N_i}^{m'}), 0\}$$

   Thus, there is a loss in expected continuation pay-off contingent on E by

$$\delta^{\max\{t^s,t^f\}} \frac{\min\{\beta_i(E_3 | h_{G_i}^m), \beta_i(E_4 | h_{G_i}^m)\}}{1-\delta}$$

□

Claim 7. *In reporting period for* $\# Ex_{I_i^{m,t}} \cup I_i^{m,t} \geq k$*, if* $\# I_i^{m,t} \leq s-1$ *and if* $i \notin C^t$ *or* i *did not satisfy the condition to play* $\langle 1 \rangle$ *in equilibrium path, then* i *will not play* $\langle 1 \rangle$.

Proof. Let

$$E' = \{\theta : \# I_i^{RP^t,t} \leq k-1\}$$

The event is not empty by checking the timing where i deviated. We have two case:

1. If i has a neighbour $j \in C$, then $j \notin O_i^{RP^t,t}$, and then suppose all other neighbour are not in R^t.
2. If
$$\exists j \in R^{t-1} \cap \bar{G}_i \text{ such that } \exists k_1, k_2 \in Tr_{ij}[[k_1 \in N_j^{t-1} \backslash I_i^{t-1}] \wedge [k_2 \in \bar{G}_{k_2}]]$$
, then just let $E = \{\theta : N_i^t \cap R^0 \leq k-1\} = \{\theta : I_i^t \leq k-1\} = E'$[28].

Next, let

$$\begin{aligned} E_1 &= \{\theta : \#[Reble](\theta) < k\} \cap E' \\ E_2 &= \{\theta : \#[Reble](\theta) \geq k\} \cap E' \end{aligned}$$

be the event contingent on i's information $I_i^{RP^t,t}$. Since i deviate to play $\langle 1 \rangle$ and note that this deviation can not be detected, his behaviour, $\langle$**stay**$\rangle$ and $\langle \mathbf{1}_i \rangle$, in the first sub-block at first division in coordination period will decide his neighbours' belief as if his neighbours think he is still on the path. In that sub-block, we have two case:

1. If i play $\langle$**stay**$\rangle$, then the coordination to **stay** starts.
2. If i play $\langle \mathbf{1}_i \rangle$, then the coordination to **revolt** starts.

But due to E_1 and E_2 still have positive probability (due to his own prior and others' strategies), i's expected static pay-off after the coordination period in this t-block is at most

$$\max\{\beta_i(E_2|h_{N_i}^m) \times 1 + \beta_i(E_1|h_{N_i}^m) \times (-1), 0\}$$

However, if he stay in the equilibrium, there is a t^s (t^f) such that Rebels play **revolt** (**stay**) contingent on E_2 (E_1), and thus after $t^* = \max\{t^s, t^f\}$ he get the expected pay-off as

$$\max\{\beta_i(E_2|h_{N_i}^m) \times 1, 0\}$$

After some calculation, after t^*, there is a loss of

$$\delta^{t^*} \frac{\min\{\beta_i(E_2|h_{G_i}^m), \beta_i(E_1|h_{G_i}^m)\}}{1-\delta}$$

□

[28] note that $I_i^t = I_i^{RP^t,t}$

Claim 8. *In reporting period for* $\#Ex_{I_i^{m,t}} \cup I_i^{m,t} \geq k$*, if* $\beta_i(\#[Rebels](\theta) \geq k | h_{G_i}^{|RP^t|}) > 0$ *then if* i *can report* $\langle 1 \rangle$*, then* i *will not report* $\langle l \rangle$ *when* δ *is high enough.*

Proof. There are two cases when i can play $\langle 1 \rangle$.

- Case 1: If $\#I_i^{|RP^t|,t} \geq k$, let the event E be

  $$E = \{\theta : \#[Rebel](\theta) = \#I_i^{|RP^t|,t}\}$$

  That is, the event that no more Rebels outside i's information about Rebels. Contingent on E, there is no more Rebel can initiate the coordination. This is because for all $j \in O_i^{|RP^t|,t}$, j is with $\#I_j^{t-1} < k-1$, and for all $j \in \bar{G}_i$ who have not yet reported, $j \notin R^t$ since all the Rebels are in $|I_i^{|RP^t|,t}|$. Since only i can initiate the coordination, if i deviated, compared to equilibrium, there is a loss in expected continuation pay-off as

  $$\delta^t \frac{\beta_i(E|h_{N_i}^m)}{1-\delta}$$

- Case 2: If $\#I_i^{|RP^t|,t} = k-1$, since $\beta_i(\#[Rebels](\theta) \geq k | h_{G_i}^{|RP^t|}) > 0$, the following event E_1 must have positive probability; otherwise, since no neighbours can report after current period, and thus $\beta_i(\#[Rebels](\theta) \geq k | h_{G_i}^{|RP^t|}) = 0$.

  Let

  $$E_1 = \{\theta : \exists j \in \bar{G}_i, j \notin O_i^{|RP^t|,t}[\#I_j^{|RP^t|,t} \geq k-1]\}$$

  Let sub-events $E_1' \subset E_1$ as

  $$E_1' = \{\theta : \text{ exist a unique} j \in \bar{G}_i, j \notin O_i^{|RP^t|,t}[\#I_j^{|RP^t|,t} \geq k-1]\}$$

  Note that this E_1' can be constructed since the network is tree. If there is θ admits 2 or more js in the definition E_1, these js must be not each others' neighbour. Suppose there are two js, says j, j', there must be at least one node in $I_j^{|RP^t|,t}$ but outside of $I_{j'}^{|RP^t|,t}$. We then pick a j, and suppose those nodes outside of $I_j^{|RP^t|,t}$ are Inert.

  Now, dependent on such j, let

  $$E = \{\theta : \#[Rebel](\theta) = \#I_j^{|RP^t|,t} \cup I_i^{|RP^t|,t}\}$$

  If i report $\langle l \rangle$, there are following consequences.

  - i will be consider as $\notin R^t$ by j, and thus i can not initiate the coordination.

- Such j has $\#I_j^{|RP^t|} = \#I_j^t < k$. Since there is no more Rebel outside $I_j^{|RP^t|,t} \cup I_i^{|RP^t|,t}$ contingent on E, such j will then play stay forever after t-block.
- Without the extra Rebels in $I_j^{|RP^t|}$, only $\#I_i^{|RP^t|,t} = k - 1$ Rebels may play **revolt**, and therefore there is no coordination to **revolt**

However, if i play $\langle 1 \rangle$, coordination can be initiated by himself in the following coordination period. Thus, there is a loss in expected continuation pay-off by

$$\delta^{|t|} \frac{\beta_i(E|h_{N_i}^m)}{1-\delta}$$

□

Claim 9. *In coordination period, suppose there is no $j \in G_i$ has played $\langle 1 \rangle$ in reporting period, suppose $\#I_i^t < k$, suppose $\beta_i(\#[Rebel](\theta) \geq k|h_{G_i}^m) > 0$, then there is δ such that*

- *if i has not observed $\langle$**stay**$\rangle$ played by $j \in G_i$ in the first sub-block at second division, or*
- *if i has not observed $\langle \mathbf{1}_j \rangle$ played by $j \in G_i$ after first sub-block at second division*

, then i will not play

- *$\langle$**stay**$\rangle$ in the first sub-block at second division and*
- *$\langle \mathbf{1}_j \rangle$ after first sub-block at second division*

Proof. Since $\#I_i^t| < k$ and due to the equilibrium strategies played by i's neighbours, we have

$$0 < \beta_i(\#[Rebel](\theta)| \geq k|h_{G_i}^m) < 1$$

If i deviate, all i's neighbour who did not detect the deviation will play **revolt** after coordination period in this block; if i's deviation is detected by some neighbours, we are in the case of Claim and so that i will not deviate. We then check if i deviate but no neighbour detect it. Let

$$E' = \{\theta : \#I_i^t \leq k-1\}$$

and let

$$\begin{aligned} E_1 &= \{\theta : \#[Reble](\theta) < k\} \cap E' \\ E_2 &= \{\theta : \#[Reble](\theta) \geq k\} \cap E' \end{aligned}$$

E_1 and E_2 have positive probability (due to his own prior and others' strategies). Since after i deviated, all the Rebels will play **revolt** after this block, i's expected static pay-off after the coordination period in this t-block is at most

$$\max\{\beta_i(E_2|h^m_{N_i}) \times 1 + \beta_i(E_1|h^m_{N_i}) \times (-1), 0\}$$

However, if he stay in the equilibrium, there is a t^s (t^f) such that Rebels play **revolt** (**stay**) contingent on E_2 (E_1), and thus after $t^* = \max\{t^s, t^f\}$ he get the expected pay-off as

$$\max\{\beta_i(E_2|h^m_{N_i}) \times 1, 0\}$$

After some calculation, after t^*, there is a loss of

$$\delta^{t^*} \frac{\min\{\beta_i(E_2|h^m_{G_i}), \beta_i(E_1|h^m_{G_i})\}}{1-\delta}$$

□

After the above claims, I conclude that this theorem holds.

□