Coordination in Social Networks

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Abstract

This paper studies a collective action problem in a setting of discounted repeated coordination games in which players know their neighbors' inclination to participate as well as monitor their neighbors' past actions. I define *strong connectedness* to characterize those states in which, for every two players who incline to participate, there is a path consisting of players with the same inclination to connect them. Given that the networks are fixed, finite, connected, commonly known, undirected and without cycles, I show that if the priors have full support on the strong connectedness states, there is a (weak) sequential equilibrium in which the ex-post efficient outcome repeats after a finite time T in the path when discount factor is sufficiently high. This equilibrium is constructive and does not depend on public or private signals other than players' actions.

1 Introduction

This paper studies a collective action behavior in a setting of discounted repeated coordination games, where information and monitoring structures are modeled as networks. Players face the uncertainty about the states of nature and can observe their neighbors' actions. I ask what kinds of networks can induce people to solve the underlying uncertainty and to coordinate to the ex-post efficient outcome. Though the main motivation is to understand the dynamic of social movement, a general interest centers on the interaction between collective action behaviors and social structures.

Consider pro-democracy movement. Strong discontents against a regime may exist, but it is difficult to organized around these discontents because information about the existence of such discontents and the influence of they are not always transparent. In East Germany, the government had control over the electoral system and the mass media. Similarly, at the end of China's Qing Dynasty, the voting system has not been established and mass media was not popular, while several groups attempted to unit in Southern China and organize the overthrowing of the Qing Dynasty. In both instances, social networks served as ways to communicate although it was costly as it involved many risks. With the success of the Berlin Uprising, I ask how a decisive collective action might be conducted with the lack of transparency. As political scientists and sociologists explain that the process of social movement can be traced back to decades ago (e.g., [McAdam, Doung; Tarrow, Sidney;Tilly, 2001], [McAdam, 2003] in reviewing the historical background of social movements between 19th to 20th century), where an event may triggered another event (e.g., [Lohmann, 2011] in using informational cascade model to explain consecutive demonstrations in East Germany 1989-1991), in game theory a well-known feature in the extensive form game is that information sets a player faces are evolved with players' actions. When rebels are aware of the capacity to transmit relevant information about the level of collective discontent through their actions, they might be willing to act although it might be risky. I view such risky actions as a part of an equilibrium strategy and the entire movement as a learning process.

I model such dynamic collective action in the following way. Players repeatedly play a k-*Threshold game* with a parameter k in a network. There are two types of players located in the network, one we called them *Rebel* and one we called them *Inert*. Players' types and their actions can be observed only by their neighbors[1] . A Rebel has two actions, which are **revolt** or **stay**, while an Inert has only one action, which is **stay**. A Rebel will get pay-off as 1 if he chooses **revolt** and more than k players choose **revolt**; he will get pay-off as -1 if he chooses **revolt** and less than k players choose **revolt**; he will get pay-off as 0 if he chooses **stay**. An Inert will get pay-off as 1 if he chooses **stay**.

Since a Rebel may not know how many Rebels in this world, Rebels' pay-off structure captures the idea that **stay** is a safe arm and **revolt** is a risky arm. Given a common prior π over players' types, players play this k-Threshold game infinitely repeatedly with a common discount factor δ. Cheap talk is not allowed, no outside mechanism serves as an information exchange device.

Rebels then communicate with each other by playing actions. For different k and different network structures, I am looking for a sequential equilibrium which has the property of *approaching ex-post efficient* (APEX henceforth) to investigate the information sharing behavior in the networks. An equilibrium is APEX if and only if *the tails of actions in the equilibrium path repeats the static ex-post efficient outcome after a finite time T*. This refinement serves to check if players learned the relevant information in the equilibrium path. If there are at least k Rebels in this society, then *all* Rebels should **revolt** after T as if they have known that more than k Rebels exist; otherwise, *all* Rebels should **stay** after T. Rebels' incentives to communicate are affected by Rebels' positions in networks since networks are structuring the information and monitoring structure.

In order to get a quick intuition about Rebel's learning process in the proposed framework, consider the k-Threshold game with $k = n$ and assume pay-off is hidden. When $k = n$, a Rebel can get positive pay-off only if all the players are Rebels. Given that the networks are fixed, finite, connected, commonly known, and undirected (FFCCU henceforth), an APEX sequential equilibrium can be constructed by a contagion-like argument. This argument is to treat **stay** as the message of "there is an Inert out there"; and treat **revolt** as the message of "there could be no Inert out there

[1] As [Chwe, 2000].

". If a Rebel has an Inert neighbor, then he plays **stay** forever. If he has no Inert neighbors, then he plays **revolt** until he observes that some of his neighbors play **stay**, and then he shifts to play **stay** forever. Since the networks are FFCCU, within finite periods, a Rebel will learn that there is an Inert out there if some neighbors has played **stay** and learn that there is no an Inert out there otherwise.

The non-trivial cases appear when $k < n$. The $k = n$ case is easier because the underlying relevant information is to tell "Is there an Inert out there?". I can construct equilibrium when $k = n$ by using single-period binary actions, $\{$**stay**, **revolt**$\}$, to separate the states into two parts, "no Inerts" or "some Inerts". In other words, these single-period actions can generate distinguishable distribution of signals to inform players in telling the true states of nature[2]. However, when $k < n$, the relevant information is to tell "How many Inerts are there?", and thus these binary actions have to carry more information to reveal the states. As I will show later, several sequences of actions will be used to transmit Rebels' private informations and to control Rebels' beliefs in equilibrium. In the equilibrium path, two kinds of sequence will be used. The first kind, *reporting messages*, is to report their private information about the states of nature; the second one, *coordination messages*, is to inform Rebels about whether some other Rebels have known the relevant information. Specifically, in the equilibrium path, Rebels will play the coordination message to inform other Rebels whenever they have known the relevant information, and those other Rebels will play the same message again to inform other Rebels. The coordination message means to serve as a short-cut to track individuals' higher-order beliefs about "Have some Rebels known the relevant information?", "Have some Rebels known some Rebels have known the relevant information", etc.

Note that communication is not free but costly in the sense that playing **revolt** is risky. Due to being discounting, Rebels always seek the opportunity to manipulate their messages to save their costs in the time horizontal line[3]. A free rider problem may occur when reporting information incurs costs. I give an example here to illustrate this issue. Consider a situation where two nearby Rebels exchange information [4]. Suppose that these two Rebels can learn the true state after acquiring information from each other's truthful reporting. Further suppose that each of them can freely initiate the coordination after exchanging information. In this instance, truthful reporting is not a best response because a player can wait given that the other will report truthfully. The intuition behind the above scenario is to see the future coordination as a public good. This public good can only be made by Rebels' truthful reporting, which incurs some costs.

The main result will show that this coordination problem can be solved in the FFCCU networks *without cycles*. Here, I define a *path* in G is a sequence consisting of distinct nodes in which a node is a neighbor of some nodes there, and then I define an undirected network G without cycles

[2] e.g., [Fudenberg and Yamamoto, 2010] or [Fudenberg and Yamamoto, 2011].

[3] Indeed, allowing cheap talk or using limit-of-mean preference (e.g., [Renault and Tomala, 1998]) will solve this coordination problem.

[4] Example 3.4

by defining a network in which the path between different nodes is unique. After I define *strong connectedness* as the property that there is always a path consisting of Rebels to connect any pairs of Rebels, the main result shows:

Result 1. ***Main Result*** *In any FFCCU network without cycles, if π has full support on the strong connectedness states, then for n-person repeated k-Threshold game with parameter $1 \leq k \leq n$, there is a δ such that a (weak) sequential equilibrium which is APEX exists.*

Here, π has full support on strong connectedness states means that π assigns positive probability on same states if and only if those states has strong connectedness. This assumption is to make sure that the underlying game is not reduced to an incomplete information game without communication. To see this, recall that an Inert always plays **stay**. Rebels can not communicate with some Rebels by their actions if an Inert happens to separate them. For instance, in a wheel network, an incomplete game without communication is that the central player is an Inert while the peripheral players are all Rebels. It is impossible to find an APEX equilibrium in this instance unless $k = 1$.

The equilibrium construction is starting from building a communication protocol. By exploiting the assumption of finite and commonly known network, I assign each node a distinct prime number. Then I let reporting messages carry the information about the multiplication of nodes' prime numbers. Since the multiplication of prime numbers can be de-factorized uniquely, the reporting messages thus carry the information about those nodes' locations in a network. Next, I let two phases, reporting period and coordination period, occur in turns in the time horizon, where the reporting (resp. coordination) messages are used in the reporting (resp. coordination) period. In coordination period, whenever a Rebel can tell the relevant information, such Rebel inform his nearby Rebels by sending coordination messages. Those nearby Rebels then continue to inform their nearby Rebels by sending the same messages, etc. By exploiting the assumption of FFCCU network, after coordination period it is commonly known that all the Rebels can tell the relevant information if they have received coordination messages.

I control the inter-temporal incentives between reporting and coordination periods as follows. First, I let the coordination message that can initiate a coordination incurs no expected cost. Second, I let the reporting message before such coordination message incur some expected costs to "convince" Rebels that a coordination can be initiated. When a Rebel looks forward future coordination, he may have incentive to "burn money" to influence Rebels' beliefs forwardly. Next, in the equilibrium path, I make sure that Rebels will play ex-post efficient outcome repeatedly after a coordination period if some Rebels have initiated the coordination in that period. I will argue that only those Rebels who have been able to tell the relevant information after reporting period have incentive to initiate the coordination since they do not need further information to tell the states. This argument is to show that other Rebels besides them will not take advantage to send that free coordination message to initiate the coordination. The intuition behind is that players can not update their belief if all of their neighbors continue to play the same actions in the future. They will not initiate the coordination to impede their own learning process to achieve the ex-post efficient outcome when they have common interest.

The assumption of acyclic network is crucial to prove my main theorem. A lemma will show that the potential free rider problems during my equilibrium construction only happen between two nearby Rebels in only one reporting period if the network is acyclic. Moreover, these two nearby Rebels will both know that this free rider problem will happen after the end of previous coordination period but before that reporting period. A lemma also shows that these two Rebels will both know the true state given others' truthful reporting after that reporting period. Thus, in that reporting period in the equilibrium path, one of them is chosen to report truthfully and the other one is chosen to be "free rider", while such "free rider" has to decide if the coordination can be initiated right after that reporting period. I choose that Rebel with less prime-number index as the free rider. When a network has cycles, however, there is an example to illustrate that this free rider problem can happen among more than two Rebels who are not nearby each other. My equilibrium construction for acyclic networks can not directly extend to the cases of cyclic networks. I will discuss this issue later in Section 3.3.

In order to keep track the belief updating in the equilibrium path, I use an off-path belief to enforce Rebels' strategies to follow some specified forms. If the strategies did not follow those specified forms, they will be considered as deviations. This off-path belief serves as a grim trigger as follows. Whenever a Rebel detects some deviations, he believes that all other players outside his neighborhood are Inerts. Thus, if there are less than k Rebels in his neighborhood, he will play stay forever. The in-path beliefs and this grim-trigger off-path beliefs constitute a belief system that satisfies updating consistency([Perea, 2002]). Updating consistency requires that, for every player and for every information sets s^1, s^2 where s^2 follows s^1, if s^2 happens with positive probability given s^1 and players' strategies contingent on s^1, then the belief over s^2 should satisfy Bayesian updating by following the belief over s^1 and players' strategies contingent on s^1. In other words, the updating consistency require that players hold beliefs in every information sets and hold updated beliefs that follows previous beliefs. This requirement imposes restrictions on off-path beliefs that induce sequential rationality, although it is weaker than full consistency ([Kreps and Wilson, 1982] as [Perea, 2002] shows).

This paper contributes to several fields of economics.

First, the future coordination can be viewed as a public good among all Rebels. A strand of public good literature, such as [Lohmann, 1994], is to view information as a public good while generating information is costly[5]. This paper models costly information generation, while adding

[5]For instance, [Lohmann, 1993][Lohmann, 1994] consider that individuals generate information by their actions, where their actions are public signals. [Bolton and Harris, 1999] consider team experiment in infinite time horizon where the outcomes of experiments are public signals. [Bramoullé and Kranton, 2007] view information as a public good and consider public good provision in networks.

another aspect, network-monitoring, to investigate a collective action behavior.

Second, this paper is also related to the literature in social learning[6]. Several papers have considered social learning in networks [7]. In this literature, when players are myopic, the information flows could be very complicated because the information they sent can in turns affect their future behaviors. For instance, in [Gale and Kariv, 2003], even for 3-person connected undirected networks, the complete network and incomplete network will give different convergence results which highly depend on individuals' initial private signals and their allocations in a network. In [Golub and Jackson, 2010], instead of using Bayesian learning, they use a naive learning protocol to tackle with this social learning problem. I consider the social learning in networks as a learning-in-game procedure, where individual can put more weights on the future learning results. My result gives a hint that the shape of network (without cycle) did not matter too much if players are far-sighted.

Third, a growing literature consider the game played in networks where various games played in various networks with various definitions[8]. Only few papers in this literature discuss the repeated game. In complete information game. In [Laclau, 2012], she proved a folk theorem where players play the game locally. In [Wolitzky, 2013] [Wolitzky, 2014], he consider network-like monitoring where a prisoner dilemma game played globally. My paper is the first paper to consider the incomplete information game repeatedly played in a network.

My paper is also related to the literature in folk theorems in discounted repeated game with incomplete information. In this literature, they consider more general games than the games adopted here. [Fudenberg and Yamamoto, 2010] [Fudenberg and Yamamoto, 2011] [Wiseman, 2012] considering n-person game with public signals jointly generated by the states and actions; [Yamamoto, 2014] considering 2-person game with private signals jointly generated by the states and actions. There, the full-rank conditions are imposed to let single-period actions generate informative signals to separates the states. Here, I consider n-person game without signals and thus the single-period full-rank conditions are not imposed before solving the equilibrium. I do not mean to prove a folk theorem here, my results however shows that the FFCCU networks without cycles are sufficient to sustain the ex-post efficiency when discount factor is sufficiently high enough.

The paper is organized as the followings. Section 2 introduces the model. Section 3 discusses the equilibrium construction and shows the main result. Some variations of my model will be discussed in its subsection 3.3. Section 4 makes the conclusion. All the missing proofs can be found in Appendix.

[6] Reviews can be seen in [Bikhchandani et al., 1998] [Cao and Hirshleifer, 2001].

[7] [Goyal, 2012] gives the reviews. Recent papers, e.g., [Acemoglu et al., 2011][Chatterjee and Dutta, 2011], also discuss this topic

[8] [Jackson, 2008][Goyal, 2012] gives the reviews.

2 Model

2.1 Notations

Given a finite set A, denote $\#A$ as the cardinality of a set A, and denote ΔA as the set of probability distribution over A.

The square bracket $[]$ after a quantifier $\exists, \forall$ will be read as "*such that*". For instance, $\exists a \in A[c \in A, c = a]$ will be read as "exists a in A such that c in A and c is equal to a." .

2.2 Model

There are n players. Denote $N = \{1, 2, ..., n\}$ as the set of players. We say G is a network if G is a point-to-set function mapping from N to a subset of N containing $i \in N$. Moreover, we denote $G_i = G(i)$ as i's neighbors and also denote $\bar{G}_i = G_i \backslash \{i\}$ as i's neighborhood excluding i self. We say G is fixed if G is not random, and say G is undirected if for all i, j if $j \in G_i$ then $i \in G_j$. Throughout this paper, I assume G is finite, fixed, commonly known, and undirected. The set of states of nature is $\Theta = \{Rebel, Inert\}^n$ and let $\theta \in \Theta$ is a state of nature. For convenience, denote $[Rebels](\theta) = \{j : \theta_j = Rebel\}$ be the set of Rebels given $\theta \in \Theta$. Given G, let $p_{G_i} : \Theta \to 2^{\Theta}$ be i's information partition function such that $p_{G_i}(\theta) = \Pi_{j \in G_i}\{\theta_j\} \times \{Rebel, Inert\}^{n-\#G_i}$. That is, whenever nature chooses a state, i knows his own θ_i and his neighbor j's θ_j.

There is a game, k-threshold game, infinitely repeated played with common discounted factor δ in a fixed G. Time is discrete, infinite horizontal. At the beginning of this game, a state is realized and there is a common prior $\pi \in \Delta\Theta$ over Θ. After a state is realized, players simultaneously choose an action $a_{\theta_i} \in A_{\theta_i}$ in each period afterwards. If $\theta_i = Rebel$, then $A_{\theta_i} = \{\textbf{revolt}, \textbf{stay}\}$. If $\theta_i = Inert$, then $A_{\theta_i} = \{\textbf{stay}\}$. Let $a_{\theta_i} \in A_{\theta_i}$ be i's action if i's type is θ_i, and let $a_{-\theta_i} \in \Pi_{j \in N, j \neq i} A_{\theta_j}$ be the actions taken by players other than i. Player i's static pay-off function, denoted as $u_{\theta_i} : \Pi_{j \in N} A_{\theta_j} \to \mathbb{R}$, in this k-threshold game is defined as followings.

1. $u_{Rebel_i}(a_{Rebel_i}, a_{-\theta_i}) = 1$ if $a_{Rebel_i} = \textbf{revolt}$ and $\#\{j : a_{\theta_j} = \textbf{revolt}\} \geq k$
2. $u_{Rebel_i}(a_{Rebel_i}, a_{-\theta_i}) = -1$ if $a_{Rebel_i} = \textbf{revolt}$ and $\#\{j : a_{\theta_j} = \textbf{revolt}\} < k$
3. $u_{Rebel_i}(a_{Rebel_i}, a_{-\theta_i}) = 0$ if $a_{Rebel_i} = \textbf{stay}$
4. $u_{Inert_i}(a_{Inert_i}, a_{-\theta_i}) = 1$ if $a_{Inert_i} = \textbf{stay}$

The players can only observe their neighbor's types and the history of their neighbors' actions. For simplicity, I assume pay-off is hidden until Section 3.3. Specifically, let s be a period, and let $H^s_{G_i} = \prod_{t=0}^{s} \prod_{j \in G_i} A^s_{\theta_j}$ be the set of histories player i can observe up to period s. Denote $H_{G_i} = \prod_{s=0}^{\infty} H^s_{G_i}$ be all the possible histories i can observe. i's pure behavior strategy is a sequence $\tau_{\theta_i} = (\tau^0_{\theta_i}, ..., \tau^s_{\theta_i}, ...)$, where $\tau^s_{\theta_i} : p_{G_i}(\theta) \times H^s_{G_i} \to A_{\theta_i}$ is a measurable function with respect to $p_{G_i}(\theta) \times$

$H^s_{G_i}$. For convenience, given θ, also denote $\tau = \{\tau_{\theta_i}\}_i$, $H^s = \prod_{m=0}^{s}\prod_{i=1}^{n} A_{\theta_i}$ and $H = \prod_{s=0}^{\infty} H^s$. The prior π, the network G, and strategies τ induce a joint distribution over $\Theta \times H$.

Let $\beta^{\pi,\tau}_{G_i}(\theta|h^s_{G_i})$ be the conditional distribution over Θ conditional on $h^s_{G_i} \in H^s_{G_i}$ induced by τ for player i at period s, and let $E^{\delta}_{G}(u_{\theta_i}(\tau)|\beta^{\pi,\tau}_{G_i}(\theta|h^s_{G_i}))$ be the conditional expected pay-off conditional on $h^s_{G_i}$ for player i with θ_i.

We say a prior π over a finite set has full support if

Definition 2.1. *π has full support if and only if $\pi(\theta) > 0$ for all $\theta \in \Theta$.*

The equilibrium concept here is the (week) sequential equilibrium. Given a network G, I am looking for a sequential equilibrium which is APEX.

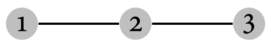
Definition 2.2. *A sequential equilibrium is APEX if and only if for all θ there is a finite time T^{θ} such that the tails of actions after T^{θ} in equilibrium path repeats the ex-post efficient outcome.*

In other words, in APEX, all the Rebels play **revolt** forever after some finite periods if there are more than k Rebels; otherwise, Rebels play **stay** forever after some finite periods. The definition of APEX directly implies that Rebels must tell if there are more than k Rebels or not in the equilibrium path as the following observation shows. This proof is in Appendix.

Lemma 2.1. *Given G, π, δ, k. If a sequential equilibrium τ^* is APEX, then given a θ, there is a finite time T^{θ}_i for a Rebel i such that $\sum_{\theta:\#[Rebels](\theta)\geq k} \beta^{\pi,\tau^*}_{G_i}(\theta|h^s_{G_i})$ either $= 1$ or $= 0$ whenever $s \geq T^{\theta}_i$.*

The following example shows that an APEX can be founded if the δ is high enough. In this example, some Rebels play the roles as "coordinators" to reveal the relevant information to others.

Example 2.1. Suppose there are 3 players in a network. This network is set as $G_1 = \{1,2\}$, $G_2 = \{1,2,3\}$,and $G_3 = \{2,3\}$ as the following graph.



They are playing the k-threshold game with $k = 3$ infinitely repeatedly. Note that after nature moves, player 2 can observe the true state of nature θ, while player 1 or 3 are not. Player 2 plays the role as a coordinator. We can construct an equilibrium which is APEX as the followings.

- After nature moves, Rebel 2 chooses **revolt** if he observes $\theta = (Rebel, Rebel, Rebel)$, and plays **revolt** in this period. Otherwise, he chooses **stay** and keeps playing **stay** afterwards.

- After nature moves, Rebel 1 and Rebel 3 play **stay**.

- If Rebel 2 chooses **revolt** in the last period, then Rebel 1 (or Rebel 3) plays **revolt** in this period; if Rebel 2 chooses **stay** in the last period, then Rebel 1 (or Rebel 3) keeps playing **stay** afterwards.

Given the prior has full support, the above strategies constitute an equilibrium if $\delta \geq \frac{1}{2}$. In the equilibrium, Rebel 1 and Rebel 3 believe that $\{\theta : \#[Rebels](\theta) \geq 3\}$ with probability one if they observe that Rebel 2 has played **revolt** and believe $\{\theta : \#[Rebels](\theta) < 3\}$ with probability one if Rebel 2 has played **stay**.

In the following section, I begin to find an APEX equilibrium in a more general setting.

3 Equilibrium

3.1 The case: $k = n$

In Example 2.1, the construction of an APEX equilibrium relies on some important features. First, since $k = n$, Rebel 2 will never play **revolt** if one of his neighbor is Inert. Thus, when Rebel 2 plays **revolt**, it must be the case that all Rebel 2's neighbor are Rebels. Second, Rebel 1 or Rebel 3 can force Rebel 2 to play **revolt** to reveal the true state in the first period since only Rebel 2 knows the true state and Rebel 2's actions can separate the states. Third, since $k = n$, one Rebel's shifting to play **stay** forever is enough to punish a deviation, and so that the group punishment is not necessary. For instance, if a Rebel did not play **revolt** in the first period at the state $\theta = (Rebel, Rebel, Rebel)$, his neighbor can punish him by playing **stay** forever. This punishment is credible by letting that player who deviated plays **stay** forever after he deviated. I then generalize the results for $k = n$ cases after one definition, *connected*.

Definition 3.1. *A path from i to j, $i \neq j$ in an undirected network G is a finite sequence $l_1, ..., l_q$ such that $l_1 = i, l_2 \in \bar{G}_{l_1}, l_3 \in \bar{G}_{l_2}, ..., l_q = j$ and $l_1, ..., l_q$ are all distinct. An undirected network is connected if and only if for all i, j, $i \neq j$ there is a path from i to j.*

Theorem 1. *In any FFCCU network, if the prior has full support on Θ, then for n-person repeated k-Threshold game with parameter $k = n$ played in such network, there is a δ such that a sequential equilibrium which is APEX exists.*

Proof. The proof is constructive and is a generalization of Example 2.1. Let strategies τ^* as the followings. After nature moves, a Rebel play **revolt** if there is no Inert neighbor; a Rebel play **stay** forever if there is a Inert neighbor. After first period, if a Rebel has not detected a deviation and if such Rebel observed his Rebel neighbor play **revolt** continuously in the last periods, then keep playing **revolt** in current period; otherwise, play **stay** forever. If a Rebel deviates, then he play **stay** forever.

According to τ^*, at period s, if a Rebel has not detected a deviation and if such Rebel observed his Rebel neighbors have played **stay** once in the last periods, he forms belief $\sum_{\theta:\#[Rebels](\theta)\geq k} \beta^{\pi,\tau^*}_{G_i}(\theta|h^s_{G_i}) = 0$ after period s and therefore plays **stay** after period s is the best response. If a Rebel detects a deviation or himself deviate to play **stay**, play **stay** is the best response since at least one neighbor will play **stay**.

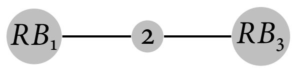
Since the network is FFCCU, there is a finite time t^s_θ such that all Rebel play **revolt** forever if $\{\theta : \#[Rebels](\theta) \geq k\}$; and there is a finite time t^f_θ such that all Rebel play **stay** forever if $\{\theta : \#[Rebels](\theta) < k\}$ in the equilibrium path. If a Rebel deviates, he at most get 0 after $\max\{t^s_\theta, t^f_\theta\}$, while he gets $\max\{1, 0\}$ after $\max\{t^s_\theta, t^f_\theta\}$. Due to the full support assumption, he will not deviate if $\sum_{\theta:\#[Rebels](\theta)\geq k} \beta^{\pi,\tau^*}_{G_i}(\theta|h^s_{G_i}) > 0$ at some period s, otherwise he has a loss in expected continuation pay-off by $\delta^{t^s_\theta} \frac{\sum_{\theta:\#[Rebels](\theta)\geq k} \beta^{\pi,\tau^*}_{G_i}(\theta|h^s_{G_i})}{1-\delta}$ after t^s_θ. There is a $0 < \delta < 1$ such that he will not deviate.

To check if τ^* and $\{\beta^{\pi,\tau^*}_{G_i}(\theta|h^s_{G_i})\}_{i\in N}$ satisfy full consistency[9], take any $0 < \eta < 1$ such that Rebels play τ^* with probability $1 - \eta$, and play others with probability η. Clearly, when $\eta \to 0$, the belief converges to $\{\beta^{\pi,\tau^*}_{G_i}(\theta|h^s_{G_i})\}_{i\in N}$. $\square$

3.2 The cases : $1 < k < n$

When $k < n$, the equilibrium constructed for the $k = n$ case will not work. First, a Rebel still has incentive to play **revolt** even if there is an Inert neighbor. Second, an Inert never transmit additional information about relevant information since they have only one action. We then require more assumptions on the states of natures and on the priors over the states of nature to get an APEX equilibrium. Example 3.1 shows why we need additional assumptions.

Example 3.1. Let $k = 2$ and let the network as the following. Assume $\theta = (Rebel_1, Inert_2, Rebel_3)$.



First, since $k = 2$, Rebel 1 has incentive to play **revolt** when $\pi(\{\theta : \theta_3 = Rebel\})$ is high enough given that Rebel 3 will play revolt . Second, Rebel 1 never learn θ_3 since Inert 2 can not reveal information about θ_3. We are now in the incomplete information game without communication. Clearly, an equilibrium which is APEX did not exist in this case.

In order to update Rebel's belief to get an equilibrium which is APEX (Lemma 2.1), I narrow down the states of natures and the priors on them to avoid similar cases as Example 3.1. Define *Strong connectedness* and *Full support on strong connectedness* as the following.

Definition 3.2. ***Strong connectedness***: *Given G, a state θ has strong connectedness if and only if for every pair of Rebels, there is a path consisting of Rebels to connect them.*

Definition 3.3. ***Full support on strong connectedness***: *Given G, π has full support on strong connectedness if and only if*

$$\pi(\theta) > 0 \Leftrightarrow \theta \text{ has strong connectedness}$$

[9] Krep and Wilson (1982)

The goal of this paper is to show that a (weak) sequential APEX equilibrium always exists when $k < n$ in FFCCU networks without cycle given that δ is sufficiently high. I define cycles in FFCCU networks in the following definition and state my main theorem in Theorem 3.1.

Definition 3.4. *A FFCCU network is without (with) cycles if and only if the path from i to j, for $i \neq j$, is (is not) unique.*

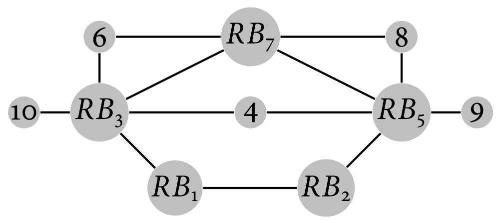
Theorem 2. *In any FFCCU network without cycles, if π has full support on strong connectedness, then for n-person repeated k-Threshold game with parameter $1 \leq k < n$ played in networks, there is a δ such that a (weak) sequential equilibrium which is APEX exists.*

The equilibrium in Theorem 3.1 is constructive. I begin with an overview of equilibrium construction, and then illustrate such construction. The whole equilibrium strategies and the omitted proofs are all in Appendix.

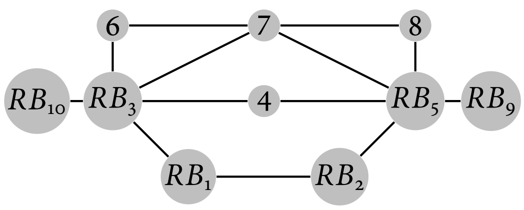
3.2.1 Overview

Given that the state has strong connectedness, Rebels have to find a way to communicate with each other. The construction of APEX is not trivial now because the "dimension" of information is generally larger than the cardinality of their own action space. Rebels then use several sequences of actions to transmit information, and thus we have to track the belief updating in the time horizontal line and to check if such sequences constitute an equilibrium. To see that Rebel need to communicate information with more dimensions, compare Example 3.2 and Example 3.3.

Example 3.2. Let $k = 5$ and let the network and the state θ as the following.



Example 3.3. Let $k = 6$ and let the network and the state θ as the following.



In Example 3.3, there are 6 Rebels, while there are 5 Rebels in Example 3.2. Suppose now we have a "talking strategies". Rebel 3 and Rebel 5 can talk about "how many" Rebels in their neighborhood to Rebel 1 and Rebel 2. Rebel 1 and Rebel 2 then talk with each other about "how many" Rebels they have known conditional on Rebel 3 and Rebel 5's taking. In some ways, Rebel 1 and Rebel 2 can initiate the coordination to play revolt conditional on 'how many" Rebels they have known. The question is that Rebel 1 and Rebel 2 still don't know how many Rebels out there after Rebel 3 and Rebel 5's talking since Rebel 3 and Rebel 5 will reveal the same number in both cases. Thus, "talking about how many" nearby Rebels is not enough, Rebels have to "talking about the locations" of nearby Rebels to get APEX.

Only having "talking strategies" to reveal both number and locations is, however, not enough to get APEX. Although Lemma 2.1 shows that there is a timing such that each Rebel have known the relevant information, but it did not characterize how Rebels commonly known that. This higher-order information about "Have some Rebels known relevant information?" is an apparently giant object in the private monitoring setting. Here, Rebels use several "coordination sequences" to communicate this information by sending such sequences if they have known that some Rebels have known the relevant information.

I construct an APEX equilibrium with a weaker sequential consistent by assuming an off-path belief which has a grim-trigger property. More specifically, if a Rebel detect a deviation, he form the off-path belief as $\sum_{\theta\in\{\theta:\theta_j=Inert, j\notin G_i\}}\beta^{\pi,\tau}_{G_i}(\theta|h^{s'}_{G_i}) = 1$ for all $s' \geq s$. I.e., a Rebel will think all players outside his neighborhood are Inerts and stop update his belief. Thus, if a Rebel has less than k Rebel neighbors, he will play **stay** forever.

The equilibrium is constructed by three steps as the following three subsections shows. In the first step, I define the *information hierarchy in G* which gives a characterization to specify which Rebels have to report their information in equilibrium and gives the notations in constructing equilibrium. In second step, I show Rebels' strategies by showing the binary $\{\textbf{revolt}, \textbf{stay}\}$-sequences used in equilibrium path and give the belief updating in the path. Finally, I discuss such grim-trigger-like off-path belief in the third step and give the idea of proof to show Theorem 3.1.

3.2.2 Step 1. Information hierarchy in G

The information hierarchy is defined on a network G after nature chooses a state but before the game is played. I will use the term "node i" instead of "player i" in this step.

The definition is by iteration. I define information hierarchy by defining $\{N_i^{-1}, N_i^0, N_i^1 ...\}$ and $\{I_i^{-1}, I_i^0, I_i^1 ...\}$ for each $i \in N$, and then define $\{\preceq^0, \preceq^1, \preceq^2\}$ and $\{R^0, R^1, R^2 ...\}$ for each iteration in $(0, 1, 2, ...)$. I also use the term "blocks" to represent the "iterations".

Given θ, the information hierarchy is defined as the followings.

- **0-block** Denote

$$\begin{aligned} N_i^{-1} &\equiv i \\ I_i^{-1} &\equiv i \end{aligned}$$

Then define R^0 as

$$R^0 \equiv \{i : \theta_i \in [Rebels](\theta)\} \tag{1}$$

- **1-block** Denote

$$\begin{aligned} N_i^0 &\equiv G_i \\ I_i^0 &\equiv G_i \cap R^0 \end{aligned}$$

Define the set $\leq^0$ by defining

$$i \in \leq^0 \Leftrightarrow \exists j \in \bar{G}_i[I_i^0 \subseteq N_j^0 \cap R^0] \tag{2}$$

Then define R^1 as

$$R^1 \equiv \{i \in R^0 | i \notin \leq^0\} \tag{3}$$

- $t + 1$**-block,** $t \geq 1$ Denote

$$\begin{aligned} N_i^t &\equiv \bigcup_{k \in I_i^{t-1}} G_k \\ I_i^t &\equiv \bigcup_{k \in G_i \cap R^t} I_k^{t-1} \end{aligned}$$

Define the set $\leq^t$ by defining

$$i \in \leq^t \Leftrightarrow \exists j \in \bar{G}_i[I_i^t \subseteq N_j^t \cap R^0] \tag{4}$$

Then define R^{t+1} as

$$R^{t+1} \equiv \{i \in R^t | i \notin \leq^t\} \tag{5}$$

Thus, by examining the definition of $\leq^t$, the R^t nodes are those nodes who have known some other Rebels but none of their neighbors have known these Rebels. Such nodes have the information at t-block, I^{t-1}, which contains the updating information about θ. Since communication may incur expected costs, if a Rebel's private information can be fully reported by his neighbors and if there is no punishment, then he has no incentives to report it. Recall the Example 2.1, there Rebel 2 is a R^1 node and Rebel 1 or 3 are not.

I characterize Rebels' incentives in this step, and let only R^t nodes to report I^{t-1} in next two steps. Moreover, if the networks are without cycles, Theorem 3 shows that it is sufficient to just let R^t nodes to report information in the sense there is time t and a R^t node who will know the true state.

Theorem 3. *If the network is FFCCU without cycle and if the state has strong connectedness, then*

$$R^0 \neq \varnothing \Rightarrow \exists t \geq 0[\exists i \in R^t[I_i^t = R^0]]$$

Table 1: Notations

$X_{\bar{N}}$	$\equiv$	$\prod_{j\in\bar{N}} x_j$
$\mathbf{s}$	$\equiv$	**stay**
$\mathbf{r}$	$\equiv$	**revolt**
$\langle\mathbf{stay}\rangle$	$\equiv$	$\langle\mathbf{s},\dots,\mathbf{s}\rangle$
$\langle\mathbf{revolt}\rangle$	$\equiv$	$\langle\mathbf{r},\dots,\mathbf{r}\rangle$
$\langle\bar{N}\rangle$	$\equiv$	$\langle\mathbf{s},\dots,\mathbf{s},\underbrace{\mathbf{r},\mathbf{s},\dots,\mathbf{s}}_{X_{\bar{N}}}\rangle$
$\langle 1\rangle$	$\equiv$	$\langle\mathbf{s},\dots,\mathbf{s},\underbrace{\mathbf{r}}_{1}\rangle$
$\langle x_i\rangle$	$\equiv$	$\langle\mathbf{s},\dots,\mathbf{s},\underbrace{\mathbf{r},\mathbf{s},\dots,\mathbf{s}}_{x_i}\rangle$

3.2.3 Step 2: Equilibrium strategies in the path

First, I assigned each player in a fixed network a distinguished prime number. Such indexation is starting from 3. I index players $(1,2,\dots,n)$ as $(3,5,\dots,x_n)$ where x_n is a prime number and the prime number assigned to i will be x_i. Since the multiplication of distinguish prime numbers can be uniquely de-factorized as those numbers, I then use this property to let Rebels simultaneous report both the number and the locations of their Rebel-neighbors by reporting the multiplication of those prime numbers.

I denote $\langle\rangle$ as a form of sequence. Denote $\bar{N}\subset N$ as an non-empty subset of N. The notations for the forms of sequences are shown in Table 1.

Denote $|\langle\rangle|$ as the length of a form of finite sequence. The forms of sequences and the length of such forms will jointly determine the sequences I used in equilibrium. For example, if a sequence takes the form $\langle 1\rangle$ and its length $|\langle 1\rangle| = 3$, then this sequence is $\langle\mathbf{s},\mathbf{s},\mathbf{r}\rangle$. Note that the length of a form is calculate from the end of such sequence.

In the equilibrium path, two kinds of periods, *reporting period* and *coordination period*, occur in turns in the following way,

$$\underbrace{<\text{coordination period}>}_{0-block}\underbrace{<\text{reporting period}><\text{coordination period}>}_{1-block}\dots$$

I.e. after nature chooses a state, all the Rebels start with 0-block, then enter to 1-block,...,and so on. 0-block has only one period, coordination period. The t-blocks, $t\geq 1$ has two periods, reporting period and coordination period, where reporting period occurs first and then coordination period follows. The length of each period in each block is finite but endogenous.

If a sequence of actions has been played in reporting period (*resp.* coordination period), I called it a *reporting messages* (*resp. coordination messages*). In reporting period in each t-block ($t\geq 1$), Rebels use the sequences defined in Table 2 to report their I_i^{t-1} contingent on the histories they observed. In coordination period in each t-block ($t \geq 0$), Rebels use the sequences defined in Table 4 to check if there are some Rebels have known the relevant information. After the coordination period in each t-block ($t \geq 0$), they either start to coordination to some actions or enter to next reporting period in $t+1$-block. I start to give the details of these messages.

Reporting messages in reporting period

Denote $|\langle RP^t \rangle|$ be the total number of periods in t-block reporting period. I consider those pure strategies where the outcomes in equilibrium path take the following forms of sequences with length $|\langle RP^t \rangle|$ as Table 2 shows. Any sequence played differently from those forms in Table 2 will be considered as a deviation.

Table 2: Reporting messages

Reporting Messages
$\langle \mathbf{stay} \rangle$
$\langle I_i^{t-1} \rangle$
$\langle 1 \rangle$

In the equilibrium path, the beliefs a Rebel j will form after observing his neighbor i's reporting messages are shown in Table 3.

Table 3: Belief updating after reporting period

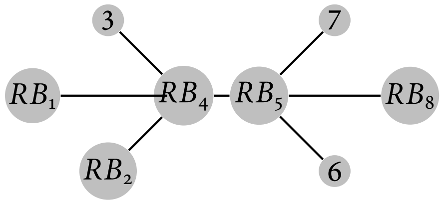
i plays	The events $j \in \bar{G}_i$ believe with probability one
$\langle \mathbf{stay} \rangle$	$i \notin R^t$
$\langle I_i^{t-1} \rangle$	$i \in R^t$ and $\theta_l = Rebel$ if $l \in I_i^{t-1}$
$\langle 1 \rangle$	$i \in R^t$ and i has known $\#[Rebels](\theta) \geq k-1$

Thus, Rebel can tell who are R^t after reporting period. Recall that R^t Rebels are those Rebels who have known some other Rebels any of their neighbors hasn't known, if a Rebel j have observed that all of his neighbors are not in R^t, then he is sure that $\#[Rebels](\theta) < k$ if $\#I_j^{t-1} < k$.

The important feature here is the usage of $\langle 1 \rangle$. It serve as a signal to indicate a pivotal player and to solve some potential free rider problems. I elaborate this issue here by providing some examples and give more details in the discussion for coordination period in next subsection. These problems happen when we allow a coordination message such that every Rebels can use regardless how their reporting messages. Consider the Example 3.4.

Example 3.4. Free Rider Problem

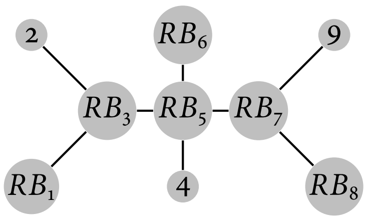
Let $k = 5$ and assume that there are message $\langle M_4\rangle, \langle M_5\rangle$ for Rebel 4, 5. To simply the analysis, let the game play starting from 1-block, i.e. by discarding the strategies in 0-block and staring the game from the reporting period. Further, assume Rebels will play **revolt** forever after observing $\langle M_4\rangle$ or $\langle M_5\rangle$ being played by Rebel 4 or 5 once immediately after reporting period; otherwise they will play **stay** forever. Let G be the followings.



Note that Rebel 4 and Rebel 5 are R^1 members. Let $\langle\rangle_4$ and $\langle\rangle_5$ are the sequences of actions they may use to report the number of Rebel neighbors. If Rebel 5 report truthfully, then Rebel 4 will not report truthfully by arranging the occurrence of **revolt**s. Since Rebel 4 has a $\langle M_4\rangle$ to initialize the coordination, such deviation is profitable. Same situation happens for Rebel 5, and then Rebel 4 and Rebel 5 will not report truthfully.

In the above example, two sources constitutes the free rider problem. One is that there is a coordination messages which can be used regardless the reporting messages. The other one is that Rebel 4 and Rebel 5 are *pivotal* since they are sure that they will learn the true state given others' truthful reporting immediately after reporting period. To see the later source more clearly, consider the following Example 3.5.

Example 3.5. Pivotal player: Case 1 Let $k = 6$ and suppose that there are message $\langle M_3\rangle, \langle M_5\rangle, \langle M_7\rangle$ for Rebel 3,5,7 to initiate a coordination. Let the game play starting from 1-block as Example 3.4. Further, suppose Rebel will play **revolt** forever after observing $\langle M_3\rangle$, $\langle M_5\rangle$, or $\langle M_7\rangle$ being played once in two periods immediately after this reporting period; otherwise they will play **stay** forever. Let G be the followings.



Note that Rebel 3, 5, 7 are R^1 members. Differently from Example 3.4, although Rebel 3, 7 have coordination messages, they still have incentives to report truthfully. This is because there is a possibility such that Rebel 5 will misunderstood the true state if they did not report truthfully to him, and they can not know the true state immediately after reporting period. Since the coordination to **revolt** has to be initiated immediately after this reporting period, they have incentives to report truthfully.

Rebel 5, however, has no incentive to report truthfully given others' truthful reporting since he is sure that he will know the true state immediately after reporting period.

Combine the discussions in Example 3.4 and Example 3.5 , a way to deal with the free rider problem is to identify who is the pivotal player in the reporting period. If we can identify them, we can let them report nothing but sending the coordination messages. If the networks are FFCCU without cycles, Lemma 3.1 shows that the free rider problem can be identified before the game enter to t-block and the pivotal player can be identified either. More precisely, first define Tr_{ij} as a tree rooted in i node while it leaves spanning from $j \in \bar{G}_i$.

Definition 3.5. $Tr_{ij} \equiv \{l \in N$: *there is a unique path* $\{l, ..., j, i\}$ *from* l *to* i *through* $j\}$

and define the set

$$C^t = \{i \in R^t : \nexists j \in R^{t-1} \cap \bar{G}_i[\exists l, l' \in Tr_{ij}[l \in N_j^{t-1} \backslash I_i^{t-1} \text{ and } l' \in \bar{G}_l]]\}$$

be those R^t nodes such that there are no possible Rebel nodes connect with them by a path which has more than "two walks". For instance, the nodes Rebel 4 and Rebel 5 in Example 3.4 are C^1 nodes and Rebel 5 in Example 3.5 is also a C^1 node. Then we can show the following lemmas.

Lemma 3.1. *If the network is FFCCU without cycle, and if the state has strong connectedness, then for each t-block*

1. $0 \leq |C^t| \leq 2$.
2. *Moreover, suppose there are two nodes in* C^t*, then they are each other's neighbor.*

Lemma 3.2. *If the network is FFCCU without cycle, and if the state has strong connectedness, then for each t-block*

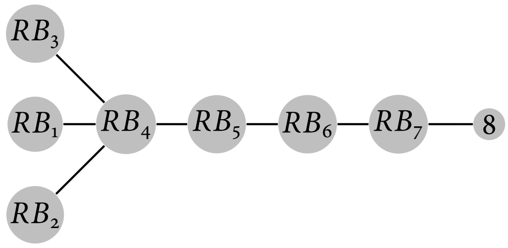
$$i \in C^t \Rightarrow \textit{there is no possible Rebel node outside of} \bigcup_{k \in N_i^{t-1}} G_k$$

Lemma 3.1 is crucial because a C^t node can identify with each other by just checking the definition of C^t. If there are more than one C^t nodes, we just pick a node who has smaller prime index to be a pivotal player[10]. Lemma 3.2 characterize the intuition behind Example 3.4 and 3.5.

However, Lemma 3.2 does not show only the node in C^t are pivotal. Some pivotal players thus can not be identified before the game enter t-block. Since that, we have to identify them during the game is played, and therefore we have to track their evolving information sets sequentially. This another source to let players become pivotal is that *a Rebel has already known too much*, and therefore he is sure that he will learn either $\#[Rebels](\theta) \geq k$ or $\#[Rebels](\theta) < k$ after current reporting period. Example 3.6 give a concrete example.

[10] This property is not generally hold if a network has cycle.

Example 3.6. Pivotal player: Case 2 Let $k = 6$. Again, assume that there are coordination message $\langle M\rangle$s for Rebels. Let the game play starting from 1-block as Example 3.4. Again, assume Rebel will play **revolt** forever after observing $\langle M\rangle$ being played once in four periods[11] immediately after reporting period; otherwise they will play **stay** forever. Let G be the followings.



In this case, no Rebels are in C^1, but Rebel 4 will deviate from reporting $\langle I_4^0\rangle$. Note that Rebel 4 has already known there are 5 Rebels in this world, therefore knowing one more Rebels is enough to initiate the coordination to **revolt**. Moreover, if there is no more Rebels, the only coordination is the coordination to **stay**. If node 6 is a Rebel, Rebel 5 will report that, and therefore he will know $\#[Rebels](\theta) \geq 6$; Otherwise, due to the state has strong connectedness, he will also know $\#[Rebels](\theta) < 6$ for sure immediately after reporting period. Since he can use the message $\langle M\rangle$ to initiate the coordination, this deviation is profitable.

After the discussion in the above examples, the message $\langle 1\rangle$ is introduced to specify the pivotal players. The pivotal players are those Rebels who have already known there are $k-1$ Rebels in the equilibrium path, or who are in C^t. The pivotal players will play $\langle 1\rangle$ in the equilibrium path, and therefore the beliefs after observing $\langle 1\rangle$ is that *a Rebel has known* $\#[Rebels](\theta) \geq k-1$ in the the path as Table 3 shows.

Coordination messages in coordination period

The ignorance of reporting messages after observed a coordination message $\langle M\rangle$ may incur untruthfully reporting as the above Example 3.4, 3.5, and 3.6 show, and the introducing of messages $\langle 1\rangle$ is meant to tackle with this issue. However, one may have observed that the combination of these two messages, $\langle 1\rangle\langle M\rangle$, themselves is another "coordination message". I.e., $\langle\langle \mathbf{s}, \mathbf{r}\rangle\langle M\rangle\rangle$ is another coordination message by truncating previous actions of $\langle 1\rangle$ and concatenate the remaining to $\langle M\rangle$. If the contingent behavior after observing this new coordination message is the same as seeing the original one, $\langle M\rangle$, the untruthfully reporting problem has not been solved. In this section, I still let $\langle 1\rangle$ as a reporting messages and call those $\langle M\rangle$ coordination messages, but I let the belief updating be contingent not only on the coordination messages but also on reporting messages.

[11]It requires four periods to let the coordination message transmit.

There are three divisions in coordination period and there are several sub-blocks in each division. In $t = 0$ block, the form is

$$\overbrace{\langle \underbrace{\langle\cdot\rangle}_{1\text{ sub-block}} \rangle}^{\text{1st division}} \overbrace{\langle \underbrace{\langle\cdot\rangle}_{1\text{ sub-blocks}} \rangle}^{\text{2nd division}} \overbrace{\langle \underbrace{\langle\cdot\rangle \cdots \langle\cdot\rangle}_{n\text{ sub-blocks}} \rangle}^{\text{3rd division}}$$

; in $t > 0$ blocks, the form is

$$\overbrace{\langle \underbrace{\langle\cdot\rangle \cdots \langle\cdot\rangle}_{n\text{ sub-blocks}} \rangle}^{\text{1st division}} \overbrace{\langle \underbrace{\langle\cdot\rangle \cdots \langle\cdot\rangle}_{t+1\text{ sub-blocks}} \rangle}^{\text{2nd division}} \overbrace{\langle \underbrace{\langle\cdot\rangle \cdots \langle\cdot\rangle}_{n\text{ sub-blocks}} \rangle}^{3rd division}$$

, where $n = \#N$.

Denote $CD^t_{m,q}$ be the m sub-block in q division, and denote $|\langle CD^t_{m,q}\rangle|$ be the total number of periods in $CD^t_{m,q}$. The outcome of pure strategies in equilibrium path takes the following forms of sequences with length $|\langle CD^t_{m,q}\rangle|$ as Table 4 shows.

Table 4: Coordination messages

Coordination messages
$\langle x_i \rangle$
$\langle \textbf{stay} \rangle$
r
s

In the following paragraphs, I will focus on the behaviors in the coordination period in $t > 0$ block, while Appendix shows the equilibrium path in coordination period in $t = 0$ block.

The belief a Rebel j form after observing i after $CD^t_{1,1}$ in equilibrium path is as Table 5 shows. After $CD^t_{1,1}$, Rebel j will be notified one more event: $\#[Rebels](\theta) < k$. Clearly, j will play **stay** forever if this event has been notified. In order to transmit this information about this event, Rebels will play $\langle x_i \rangle$ unless they observe someone play $\langle \textbf{stay} \rangle$ in the path in $CD^t_{m,1}$ where $m \geq 2$ as Table 6 and Table 7 shows. After $CD^t_{n,1}$, the information of $\#[Rebels](\theta) < k$ will be transmitted across all players.

Game enter to $CD^t_{1,2}$. In $CD^t_{1,2}$, Rebels start to check if the coordination to **revolt** can be initiated. The coordination message to initiate the coordination is $\langle \textbf{stay} \rangle$ as Table 8 shows. The important features here is that $\langle \textbf{stay} \rangle$ is a coordination message *only if* $\langle I^{t-1}_i \rangle$ or $\langle 1 \rangle$ has been played in reporting period. First, note that $\langle \textbf{stay} \rangle$ is also the message to coordinate to **stay** in $CD^t_{m,1}$, $m \geq 1$ and note that this is the only candidate to be a message to coordinate to **stay**. Rebels are not confused about the information which **stay** carries since the information is now contingent on the reporting

Table 5: Belief updating after $CD^t_{1,1}$, $t > 0$

In RP^t	In $CD^t_{1,1}$	
i plays	i plays	The events $j \in \bar{G}_i$ believe with probability one
$\langle$**stay**$\rangle$	$\langle x_i \rangle$	$i \notin R^t$
$\langle I_i^{t-1} \rangle$	$\langle$**stay**$\rangle$	$\#[Rebels](\theta) < k$
$\langle I_i^{t-1} \rangle$	$\langle x_i \rangle$	$i \in R^t$
$\langle 1 \rangle$	$\langle$**stay**$\rangle$	$\#[Rebels](\theta) < k$
$\langle 1 \rangle$	$\langle x_i \rangle$	$\#[Rebels](\theta) \geq k$

Table 6: In-path strategies in RP^t, $CD^t_{1,1}$, and $CD^t_{2,1}$, $t > 0$

In RP^t	In $CD^t_{1,1}$	In $CD^t_{2,1}$
i plays	i plays	$j \in \bar{G}_i$ plays
$\langle$**stay**$\rangle$	$\langle x_i \rangle$	$\langle x_i \rangle$
$\langle I_i^{t-1} \rangle$	$\langle$**stay**$\rangle$	$\langle$**stay**$\rangle$
$\langle I_i^{t-1} \rangle$	$\langle x_i \rangle$	$\langle x_i \rangle$
$\langle 1 \rangle$	$\langle$**stay**$\rangle$	$\langle$**stay**$\rangle$
$\langle 1 \rangle$	$\langle x_i \rangle$	$\langle x_i \rangle$

messages. Second, $\langle$**stay**$\rangle$ incurs no expected cost in equilibrium path. It is important because otherwise a free rider problem will happen again if such message incurs expected costs. Third, since this message is contingent on the reporting messages, initiating the coordination to **revolt** is "not free" since it requires Rebels to report something in reporting period to initiate that. This trade-off between reporting something and reporting nothing will force Rebels to report something in reporting period in order to take chances to initiate the coordination to **revolt**.

After the initiating in $CD^t_{1,2}$, Rebels start to transmit the information of $\#[Rebels](\theta) \geq k$ in $CD^t_{m,2}$ $m \geq 2$ as Table 9 and Table 10 shows. That is, they will play $\langle$**stay**$\rangle$ unless they observe someone play $\langle x_i \rangle$. After $CD^t_{t+1,1}$, the information of $\#[Rebels](\theta) \geq k$ will be transmitted across at least k Rebels.

Game finally enter to $CD^t_{1,3}$. In this period, those k Rebels who have known the information of $\#[Rebels](\theta) \geq k$ will start to play **revolt** forever, and this is the first period in the path in which a Rebel may get positive expected pay-off by playing **revolt**. After $CD^t_{m,3}$, $m \geq 2$, other Rebels start to transmit this information to all of the Rebels to coordinate to **revolt** as Table 12 shows.

Table 7: In-path strategies after $CD^t_{m,1}$, where $m \geq 2$, $t > 0$

In $CD^t_{m,1}$, $m \geq 2$	In $CD^t_{m+1,1}$, $m \geq 2$
i plays	$j \in \bar{G}_i$ plays
$\langle x_i \rangle$	$\langle x_i \rangle$
$\langle$**stay**$\rangle$	$\langle$**stay**$\rangle$

Table 8: Belief updating after $CD^t_{1,2}$, $t > 0$

In RP^t	In $CD^t_{1,1}$	In $CD^t_{1,2}$	
i plays	i plays	i plays	The events j believe with probability one
$\langle$**stay**$\rangle$	$\langle x_i \rangle$	$\langle$**stay**$\rangle$	$i \notin R^t$
$\langle I_i^{t-1} \rangle$	$\langle$**stay**$\rangle$	$\langle$**stay**$\rangle$	$\#[Rebels](\theta) < k$
$\langle I_i^{t-1} \rangle$	$\langle x_i \rangle$	$\langle$**stay**$\rangle$	$\#[Rebels](\theta) \geq k$
$\langle I_i^{t-1} \rangle$	$\langle x_i \rangle$	$\langle x_i \rangle$	$i \in R^t$
$\langle 1 \rangle$	$\langle$**stay**$\rangle$	$\langle$**stay**$\rangle$	$\#[Rebels](\theta) < k$
$\langle 1 \rangle$	$\langle x_i \rangle$	$\langle$**stay**$\rangle$	$\#[Rebels](\theta) \geq k$

3.2.4 Step 3: Off-path Belief

Whenever Rebel i detects a deviation, he forms the belief

$$\sum_{\theta \in \{\theta : \theta_j = Inert, j \notin G_i\}} \beta^{\pi,\tau}_{G_i}(\theta | h^{s'}_{G_i}) = 1 \tag{6}$$

for all $s' \geq s$. Thus, if $\#I_i^0 < k$, he will play **stay** forever and this is credible after t-block, $t > 0$. This off-path belief then serve as a grim trigger to impede Rebels' deviation.

As a discussion in Introduction, if the criteria in judging a deviation is too strict, an APEX equilibrium may not be sustained by using grim trigger. Due to the private monitoring structure, a deviation may be only detected by some Rebels while at least k Rebels cannot detect it. Since that, at least k Rebels may play **revolt** forever while the other Rebels may play **stay** forever if such judging is too strict. This phenomenon gives another reason why the message $\langle 1 \rangle$ is introduced. Example 3.7 (, a modification of Example 3.5) gives a concrete example to illustrate a case if $\langle 1 \rangle$ is not introduced, while this grim-trigger-like off-path belief is adopted.

Example 3.7. Let $k = 5$ and suppose that there are message $\langle M_3 \rangle, \langle M_5 \rangle, \langle M_7 \rangle$ for Rebel 3,5,7 to initiate a coordination. Let the game play starting from 1-block as Example 3.5. Suppose Rebel will play **revolt** forever after observing $\langle M_3 \rangle$, $\langle M_5 \rangle$, or $\langle M_7 \rangle$ *if no deviation be detected*; otherwise they

Table 9: In-path strategies in RP^t, $CD^t_{1,1}$, $CD^t_{1,2}$, and $CD^t_{2,2}$, $t > 0$

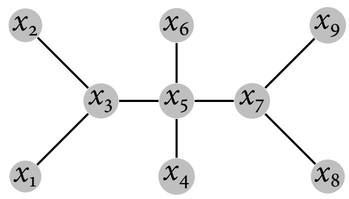
In RP^t	In $CD^t_{1,1}$	In $CD^t_{1,2}$	In $CD^t_{2,2}$
i plays	i plays	i plays	$j \in \bar{G}_i$ plays
$\langle$**stay**$\rangle$	$\langle x_i \rangle$	$\langle$**stay**$\rangle$	$\langle$**stay**$\rangle$
$\langle I_i^{t-1} \rangle$	$\langle$**stay**$\rangle$	$\langle$**stay**$\rangle$	$\langle$**stay**$\rangle$
$\langle I_i^{t-1} \rangle$	$\langle x_i \rangle$	$\langle$**stay**$\rangle$	$\langle x_i \rangle$
$\langle I_i^{t-1} \rangle$	$\langle x_i \rangle$	$\langle x_i \rangle$	$\langle$**stay**$\rangle$
$\langle 1 \rangle$	$\langle$**stay**$\rangle$	$\langle$**stay**$\rangle$	$\langle$**stay**$\rangle$
$\langle 1 \rangle$	$\langle x_i \rangle$	$\langle$**stay**$\rangle$	$\langle x_i \rangle$

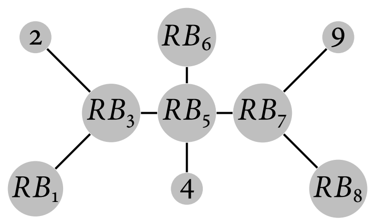
Table 10: In-path strategies after $CD^t_{m,2}$, where $m \geq 2$, $t > 0$

In $CD^t_{m,2}$, $m \geq 2$	In $CD^t_{m+1,2}$,$m \geq 2$
i plays	$j \in \bar{G}_i$ plays
$\langle x_i \rangle$	$\langle x_i \rangle$
$\langle$**stay**$\rangle$	$\langle$**stay**$\rangle$

will play **stay** forever. Moreover, assume that Rebels can only use the pure strategies in which the outcome satisfies the form of $\langle I_i^0 \rangle$ in reporting period; *otherwise, it will be consider as a deviation.*

Let the G and θ be the following, and let the players be indexed by distinct prime numbers.





Assume $X_{I_3^0} > X_{I_5^0}$ and $X_{I_7^0} > X_{I_5^0}$. Since that, Rebel 5 will get the information from Rebel 3,7 before he reports I_5^0. Since that, Rebel 5 has a profitable deviation by just reporting $\bar{I}_5^0 = x_3 \times x_5 \times x_7 < I_5^0$ to let Rebel 3, 7 think he is still in the path and to let the coordination to **revolt** be succeeded. Rebel 6 can detect such deviation since $\bar{I}_5^0$ did not include Rebel 6's own index x_6. However, if Rebel

Table 11: In-path strategies in $CD^t_{1,3}$, $t > 0$

In $CD^t_{m,2}$, $1 \le m \le t+1$	In $CD^t_{1,3}$
i has played	$j \in \bar{G}_i$ plays
$\langle x_i \rangle$	**r**
Otherwise	**s**

Table 12: In-path strategies after $CD^t_{m,3}$, where $m \ge 2$, $t > 0$

In $CD^t_{m,3}$, $m \ge 2$	In $CD^t_{m+1,3}$, $m \ge 2$
i plays	$j \in \bar{G}_i$ plays
r	**r**
s	**s**

6 form the off-path belief as Equation 6, he will then play **stay** forever although the coordination has been succeeded.

Due to the pay-off function is not strictly increasing with the number of **revolts**, some Rebels will be excluded from coordination with this grim-trigger-like strategies. As Example 3.7 shows, some strategies other than I^t have to be considered as the strategies in the equilibrium path to avoid the situation where some Rebels are out of an existing coordination. The introducing of message $\langle 1 \rangle$ gives more equilibrium paths when grim trigger is adopted.

3.2.5 Sketch of the proof for Theorem 3.1

In previous subsections, I have listed the belief updating in equilibrium path by showing the the belief updating after observing various messages. Lemma 3.3 show that the equilibrium path is APEX where the consistent in-path belief updating is listed in Table 3, 5, and 8.

Lemma 3.3. *If the state has strong connectedness, then for all n-person repeated k-Threshold game with parameter $1 \le k \le n$ played in any FFCCU network without cycles, the equilibrium is APEX.*

For the histories outside of equilibrium path, they could be detectable or undetectable. The argument to prove Theorem 3.1 is as the following. First, I use off-path belief to prevent players from making detectable deviations, such as deviations from playing the specified forms of sequences listed in Table 2 and 4. Then I argue that any undetectable deviation made by a Rebel before he knows the relevant information, $\#[Rebels](\theta) \ge k$ or $\#[Rebels](\theta) < k$, will create noises in his own learning process and then reduce his own expected continuation pay-off. To see why making undetectable deviation will create noises to impede the learning process before he knows the relevant information, consider the case when a Rebel wants to mimic pivotal plays' behaviors by sending $\langle 1 \rangle$. According to Table 5 and Table 8, the continuation playing by his neighbors after observing $\langle 1 \rangle$ is to play **stay** forever or to play **revolt** forever. Since all his neighbors repeats the same action, he can not learn additional information about θ. When δ is high enough, since he can learn the relevant information by Lemma 2.1 if he stay in the path, he is better off by staying in the equilibrium path to achieve the maximum static pay-off as 1 when $\#[Rebels](\theta) \geq k$ and achieve the maximum static pay-off as 0 when $\#[Rebels](\theta) < k$. Claim 7 shows this argument.

3.3 Discussion

3.3.1 Variation: Pay-off as signals

The assumption on the observability of pay-off can be relaxed without change the result in Theorem 3.1. One may consider a situation such that the static pay-off depends not only on joint efforts but also depends on other random effects, says the weather.[12]. Specifically, consider there is a public signal $y \in \{y_1, y_2\}$ generated by Rebels' actions. Let Rebel i's pay-off function be $u_{Rebel}(a_{Rebel_i}, y)$, and let $u_{Rebel}(\mathbf{stay}, y_1) = u_{Rebel}(\mathbf{stay}, y_2) = u_0$. The distribution of y_1 and y_2 is

$$\begin{aligned} p_{1s} &= \Pr(y = y_1 | \#\mathbf{revolt} \geq k) \\ p_{1f} &= \Pr(y = y_1 | \#\mathbf{revolt} < k) \\ p_{2s} &= \Pr(y = y_2 | \#\mathbf{revolt} \geq k) \\ p_{2f} &= \Pr(y = y_2 | \#\mathbf{revolt} < k) \end{aligned}$$

with

$$p_{1s} u_{Rebel}(\mathbf{revolt}, y_1) + p_{2s} u_{Rebel}(\mathbf{revolt}, y_2) > u_0 > p_{1f} u_{Rebel}(\mathbf{revolt}, y_1) + p_{2f} u_{Rebel}(\mathbf{revolt}, y_2) \quad (7)$$

and

$$1 > p_{1s} > 0, 1 > p_{2s} > 0, p_{1f} = 1 - p_{1s}, p_{2f} = 1 - p_{2s} \quad (8)$$

Equation (6) is a generalization for the previous setting. Equation (7) is a full support assumption on signal y.

We can construct exactly the same equilibrium strategies by ignoring the noisy signal y when Equation (7) holds. By directly checking the equilibrium path constructed in previous sections (see also Table 2 and 4), *at most one* Rebel playing **revolt** in a period before some Rebels play $\langle 1 \rangle$, and thus the signal y is not relevant before some Rebels play $\langle 1 \rangle$. Moreover, playing $\langle 1 \rangle$ comes from a Rebel's observation in reporting period where Rebels' strategies are independent from y, and thus playing $\langle 1 \rangle$ is independent from y. We then just check if a Rebel want to deviate to play $\langle 1 \rangle$ to get additional information coming from y. According to equilibrium strategies in the path, however,

[12] e.g.,[SHADMEHR and BERNHARDT, 2011]

playing ⟨1⟩ will incur either coordination to **stay** or coordination to **revolt** after current block as Table 5 and Table 8 shows. Since signal y is noisy, and since Rebels' actions will repeat, he can not learn the relevant information. As the same argument in Claim 7, he is better off by staying in the equilibrium path.

If Equation (7) fails, and so that the signal y is not noisy, says $p_{1s} = p_{2f} = 1$, the equilibrium constructed in the previous section is no longer an equilibrium. However, another APEX equilibrium can be constructed by letting all Rebels play **revolt** in the first period, and then keep playing **revolt** or **stay** contingent on the signals $y = y_1$ or $y = y_2$ when δ is sufficiently high.

3.3.2 Variation: Rebels with different levels of efforts

Consider a model in which players have different levels of efforts to contribute to a collective action. Let the set of states of nature be $\hat{\Theta} = \Theta \times \Xi$, where $\Xi = \{1, 2, ..., k\}^n$. Let $\hat{\theta} = (\theta, e)$, where $\theta \in \Theta$ and $e \in \Xi$. $\hat{\theta}$ is interpreted as a state of nature in which a player i could be either a Rebel or Inert with e_i efforts, where $e_i \in \{1, 2, ..., k\}$. When a Rebel i plays **revolt**, he contributes e_i efforts to a collective action, while the success of such collective action requires k amount of efforts. If the collective action succeed, such Rebel i will get $b_i > 0$ as his reward. The pay-off structure is modified as the following.

1. $u_{Rebel_i}(a_{Rebel_i}, a_{-\theta_i}) = b_i$ if $a_{Rebel_i} =$ **revolt** and $\sum_{j:a_{\theta_j}=\textbf{revolt}} e_j \geq k$

2. $u_{Rebel_i}(a_{Rebel_i}, a_{-\theta_i}) = -e_i$ if $a_{Rebel_i} =$ **revolt** and $\sum_{j:a_{\theta_j}=\textbf{revolt}} e_j < k$

3. $u_{Rebel_i}(a_{Rebel_i}, a_{-\theta_i}) = 0$ if $a_{Rebel_i} =$ **stay**

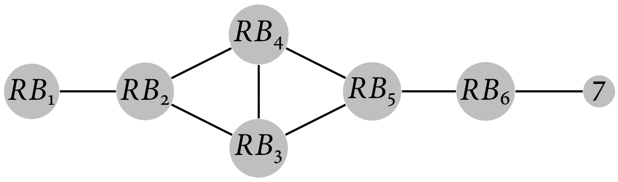
4. $u_{Inert_i}(a_{Inert_i}, a_{-\theta_i}) = 1$ if $a_{Inert_i} =$ **stay**

After the nature chooses a $\hat{\theta}$, players repeatedly play the above game in a network G. To see that the equilibrium constructed in previous section is still an equilibrium, transform the network and the state $(G, \hat{\theta})$ to $(G', \hat{\theta}')$ such that, in $(G', \hat{\theta}')$, there are e_i different players attached to a single player i for each $i \in N$, and all the players are with levels of efforts as 1. What matters here is that the states of nature are finite and discrete, and therefore we can use prime indexing to construct the equilibrium as previous section shows.

3.3.3 Variation: networks with cycles

The prime indexing can deal with a potential free problem when networks have cycles, although we may need to redefine the information hierarchy in order to redefine which Rebels are forced to report their private information. Consider the following Example 3.8.

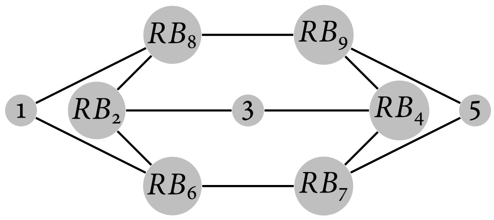
Example 3.8. Let $k = 6$. Rebel 3 and Rebel 4 have the same information $I_3^1 = I_4^1$. Since reporting is costly, if there is no punishment, Rebel 3 (or Rebel 4) may shirk and deviate from truthfully reporting if Rebel 4 (or Rebel 3) can reports truthfully. However, this kind of deviation can be detected by Rebel 5 (or Rebel 2) since I_3^1 should be equal to I_4^1.



Indeed, this monitoring technique will be less invalid if the network is not commonly known. In this example, if Rebels has asymmetric information about network structure, says Rebel 5 (or Rebel 2,3) does not certain about if there is a link between Rebel 4 and Rebel 2, then Rebel 4 can just pretend that he doesn't know Rebel 2[13]. The analysis in incomplete information about network structure is beyond the scope of this paper. Recent paper such as [Galeotti et al., 2010] deal with the issues when network is not commonly known.

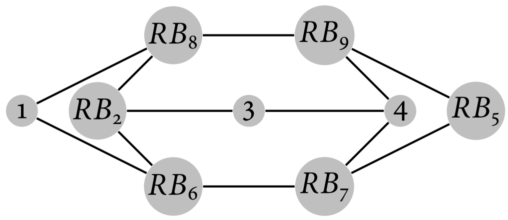
There is another free rider problem which is harder to deal with. Remind that the implicit technique in my equilibrium construction is that the pivotal Rebels can be identified before the game is played in each reporting period in each block. When the networks has cycles, the selection of pivotal Rebels needs more elaboration. Consider Example 3.9.

Example 3.9. Let $k = 6$. Suppose the network and θ is as the following.



Assume that one round of reporting is done. Rebel 2 has known $\{RB_2, RB_6, RB_8, RB_9, RB_7\}$, Rebel 4 has known $\{RB_4, RB_7, RB_9, RB_8, RB_6\}$, and so on. One more round of reporting will let Rebels 3,6,7,4,9,8 know the true state θ, and therefore Rebels 3,6,7,4,9,8 are all pivotal players. We may have a rule as Example 3.4 to pick up a pivotal Rebel, say we pick Rebel 4 before entering the next reporting period. However, this pivotal player selection is ex-post. The true state θ' could be as the following.

[13]However, if there is asymmetric information about network structure, then Example 3.8 is not exactly a free rider problem. Dependent on what asymmetric information is given, Rebel 3 and Rebel 4 then have different incentives in untruthful reporting. Untruthful reporting could be a dominant strategy for Rebel 4 but not necessary be a dominant strategy for Rebel 3.



Now node 4 is an Inert and so that he is not a pivotal Rebel. Some other rules are needed to be applied to select a pivotal Rebel (say, Rebel 5 in this case) during the game is played.

As Example 3.4 or 3.9 show, a free rider problem may occur if the selection of pivotal Rebels is not done before the game is played. When the networks has cycles, this problem seems more harsh and the selection rule may not be done before the game is played. Though it is possible to construct a selection rule, this rule is still infeasible in this paper.

I leave a conjecture here and end this section.

Conjecture 3.1. *For n-person repeated k-Threshold game with parameter $1 \leq k < n$ played in any FFCCU network, if the state θ has strong connectedness and $\pi(\{\theta : \theta$ has strong connectedness$\}) = 1$ with full support, then there is a δ such that there is a (weak) sequential equilibrium which is APEX.*

4 Conclusion

I model a coordination game and illustrate the learning processes generated by strategies in a sequential equilibrium and answer the question proposed in the beginning: what kind of networks can conduct coordination in a collective action game with information barrier. In the equilibrium, players transmit the relevant information by encoding such information by their actions in the time horizontal line. Since there is an expected cost in coding information, potential free rider problems may occur to impede the learning process. When the networks are FFCCU without cycle, players can always learn the underlying relevant information and conduct the coordination only by their actions in a equilibrium. However, what kinds of equilibrium strategies can constitute a learning process to learn the relevant information in the networks with cycles still remains to be answered.

Existing literatures in political science and sociology have recognized the importance of social network in influencing individual's behaviour in participating social movements, e.g., [Passy, 2003][McAdam, 2003][Siegel, 2009]. This paper views networks as routes for communication where rational individuals initially have local information, and they can influence nearby individuals by taking actions. Such influence may take long time to travel across individuals. A characterization in the speed of information transmission across a network is not answered here, although it is an important topic in order to give more attentions in the details of network structures. This question would remain for the future research.

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A Appendix

proof for Lemma 2.1

Proof. The proof is done by contradiction. Suppose Rebels' strategies constitute an APEX. By definition of APEX, there is a time T^{θ} when actions start to repeat at state θ. Let $T = \max_{\theta \in \Theta} T^{\theta}$. Pick that time $T_i = T+1$ and suppose the consequence did not not holds so that $0 < \sum_{\theta:\#[Rebels](\theta)\geq k} \beta^{\pi,\tau^*}_{G_i}(\theta|h^s_{G_i}) < 1$ for some $s \geq T_i$. Then this Rebel puts some positive weights on some $\theta \in \{\theta : \#[Rebels](\theta) < k\}$ and puts some positive weights on $\theta \in \{\theta : \#[Rebels](\theta) \geq k\}$ at that time s. Note this Rebel i has already known θ_j if $j \in G_i$, and therefore Rebel i put some positive weights on $\theta \in \{\theta : \#[Rebels](\theta) < k, \theta_l = Rebel, l \notin G_i\}$ and $\theta \in \{\theta : \#[Rebels](\theta) < k, \theta_l = Inert, l \notin G_i\}$. Since actions start to repeat at T, all i's neighbors will play the same actions as the actions at time T, but then Rebel i can not update information from his neighborhood by Bayesian rule. Suppose i's continuation strategy is to play **revolt** repeatedly, then this is not ex-post efficient if $\#[Rebels](\theta) < k$. Suppose i's continuation strategy is to play **stay** repeatedly, then this is not ex-post efficient if $\#[Rebels](\theta) \geq k$ □

proof for Theorem 3

This proof follows three useful claims, Claim 1, Claim 2 and Claim 3. First note that I^t_i and N^t_i, $t \geq 1$ can be expressed as

$$I^t_i = \bigcup_{k_0 \in G_i \cap R^t} \bigcup_{k_1 \in G_{k_0} \cap R^{t-1}} \cdots \bigcup_{k_{t-1} \in G_{k_{t-2}} \cap R^1} G_{k_{t-1}} \cap R^0 \tag{9}$$

, while H^t_i can be expressed as

$$N^t_i = \bigcup_{k_0 \in G_i \cap R^{t-1}} \bigcup_{k_1 \in G_{k_0} \cap R^{t-2}} \cdots \bigcup_{k_{t-2} \in G_{k_{t-3}} \cap R^0} G_{k_{t-2}} \tag{10}$$

Claim 1. *$I^t_i \subset N^t_i$ for $t \geq 1$*

Proof. $I^t_0 \subset N^t_0$ by definition. Since $R^t \subset R^{t-1}$ for $t \geq 1$, $I^t_i \subset N^t_i$ for $t \geq 1$ by comparing Equation 9 and Equation 10. □

Claim 2. *If the network is FFCCU without cycle, then for each $t \geq 1$ block, we have $i \in R^t \Leftrightarrow i \in R^{t-1}$ and $\exists k_1, k_2 \in R^{t-1} \cap \bar{G}_i$, where $k_1 \neq k_2$.*

Proof. The proof is by induction. We first show that the statement is true for $t = 1$.

Base: $i \in R^1 \Leftrightarrow [i \in R^0] \wedge [\exists k_1, k_2 \in (R^0 \cap \bar{G}_i)]$.

$\Rightarrow$: Since $i \in R^1$, then $i \in R^0$ and then $I^0_i \nsubseteq N^0_j$ for all $j \in \bar{G}_i$ by definition. Since $I^0_i = R^0 \cap G_i$, then $\forall j \in \bar{G}_i [\exists k \in (R^0 \cap \bar{G}_i)[k \notin N^0_j]]$. Since the $j \in \bar{G}_i$ is arbitrary, we then have a pair of $k_1, k_2 \in (R^0 \cap \bar{G}_i)$ such that $k_1 \notin N^0_{k_2}$ and $k_2 \notin N^0_{k_1}$.

$\Leftarrow$: Pick $k \in \{k_1, k_2\} \subseteq (R^0 \cap \bar{G}_i)$, and pick an arbitrary $j \in \bar{G}_i \backslash \{k\}$. Note that $k \notin N_j^0$, otherwise there is a cycle from i to i. Hence $[k \in (R^0 \cap \bar{G}_i)] \wedge [k \notin N_j^0]$ and therefore $[k \in I_i^0] \wedge [k \notin N_j^0]$. Then we have $I_i^0 \nsubseteq N_j^0$ for arbitrary $j \in \bar{G}_i$, and thus $i \in R^1$.

Induction hypothesis: the statement is true for $\{1, 2, .., t\}$ where $t \geq 1$.

If the hypothesis is true, then $i \in R^{t+1} \Leftrightarrow [i \in R^t] \wedge [\exists k_1, k_2 \in (R^t \cap \bar{G}_i)]$

$\Rightarrow$: since $i \in R^{t+1}$, then $i \in R^t$ and $I_i^t \nsubseteq N_j^t$ for all $j \in \bar{G}_i$ by definition. Recall that I_i^t can be expressed as Equation 9 and H_i^t can be expressed as Equation 10, then for every $l \in I_i^{t-1}$, we can find a path connecting i to l by the induction hypothesis. If $j \in \bar{G}_i$, then we can find a path connecting j to l by connecting j to i, and then connecting i to l. Thus, if $l \in I_i^{t-1}$ then $l \in N_j^t$, and hence $I_i^{t-1} \subseteq N_j^t$ for all $j \in \bar{G}_i$. Recall that $I_i^t = \bigcup_{k \in N_i \cap R^t} I_k^{t-1}$ and $i \in R^{t+1}$, then we must have $\forall j \in \bar{G}_i [\exists k \in (R^t \cap \bar{G}_i)[I_k^{t-1} \nsubseteq N_j^t]]$, since $I_i^{t-1} \subseteq N_j^t$. Note that such $j \in \bar{G}_i$ is arbitrary, we then have a pair of $k_1, k_2 \in (R^t \cap \bar{G}_i)$ such that $k_1 \notin N_{k_2}^t$ and $k_2 \notin N_{k_1}^t$.

$\Leftarrow$: By the induction hypothesis, we have a chain $k_{1_0}, \ldots, k_{1_t}, i, k_{2_t}, \ldots, k_{2_0}$ with $k_{1_0} \in R^0, \ldots, k_{1_t} \in R^t$, $i \in R^t$, $k_{2_t} \in R^t, \ldots, k_{1_0} \in R^0$, where $k_{1_t}, k_{2_t} \in (R^t \cap \bar{G}_i)$, $k_{1_0} \in I_{k_{1_t}}^{t-1}$ and $k_{2_0} \in I_{k_{2_t}}^{t-1}$. Note that $k_{1_0} \notin N_j^t$ whenever $j \in \bar{G}_i$, otherwise there is a cycle from i to i since $\{i, k_{2_t}, \ldots, k_{2_0}\} \in N_j^t$, and hence $[k_{1_0} \in I_{k_{1_t}}^{t-1}] \wedge [k_{1_0} \notin N_j^t]$ for all $j \in \bar{G}_i$. Therefore we have $[I_{k_{1_t}}^{t-1} \in I_i^t] \wedge [I_{k_{1_t}}^{t-1} \notin N_j^t]$ for all $j \in \bar{G}_i$ since $k_{1_t}, k_{2_t} \in (R^t \cap \bar{G}_i)$ and $[k_{1_0} \in I_{k_{1_t}}^{t-1}] \wedge [k_{1_0} \notin N_j^t]$ for all $j \in \bar{G}_i$. Then we have $I_i^t = \bigcup_{k \in N_i \cap R^t} I_k^{t-1} \nsubseteq N_j^t$ for arbitrary $j \in \bar{G}_i$, and thus $i \in R^{t+1}$.

We can then conclude that the statement is true by induction.

□

Claim 3. *If the network FFCCU without cycle and if the state has strong connectedness, then if there is a pair of R^t nodes then there exists a R^t-path connecting them.*

Proof. The proof is by induction and by Claim 2. Since the state has strong connectedness, we have a R^0-path connecting each pair of R^0 nodes. Since all pairs of R^0 nodes are connected by a R^0-path, then for all pairs of R^1 nodes must be in some of such paths by Claim 2, and then connected by a R^0-path. But then all the R^0-nodes in such path are all R^1 nodes by Claim 2 again and by $R^t \subseteq R^{t-1}$ for $t \geq 1$ by definition. Thus, for all pairs of R^1 nodes has a R^1-path connecting them. The similar argument holds for $t > 1$, we then get the result.

□

I begin to prove this Theorem 3. I first claim that if $R^t \neq \varnothing$ and if $R^{t+1} = \varnothing$, then $R^0 \subset I_i^t$ whenever $i \in R^t$. Then I claim that if $R^t \neq \varnothing$ then $\#R^{t+1} < \#R^t$. Finally, I iterate R^t with $t \geq 0$ to get the conclusion.

If $R^t \neq \varnothing$ but $R^{t+1} = \varnothing$, I claim that $R^0 \subset I_i^t$ for all $i \in R^t$. The proof is by contradiction. If $R^0 \not\subset I_i^t$, there is a $j \in R^0$ but $j \notin I_i^t$. Since I_i^t can be expressed as Equation 9, there is no such a path $\{i, k_0, k_1, \ldots, k_{t-1}, j\}$, where $k_0 \in G_i \cap R^t, k_1 \in G_{k_0} \cap R^{t-1}, \ldots, k_{t-1} \in N_{k_{t-2}} \cap R^1$. Since $R^{t+1} = \varnothing$ and therefore $R^{t'} = \varnothing$ if $t' \geq t+1$, and hence there is no such a path containing a node in $R^{t'} = \varnothing$, where

$t^{'} \geq t+1$ connecting i to j. But $i \in R^t$ and $i, j \in R^0$, if there is no such a path, then it violate either Claim 3 or Claim 2. Contradiction.

Next I claim that if $R^t \neq \varnothing$ then $\#R^{t+1} < \#R^t$. The proof is the followings. Given a node i in R^t, let $j \in R^t$ (could be i itself) be the node connected with i with the maximum shortest R^t path. This j can be found since $R^t \neq \varnothing$ and the network is finite. Then there is no R^t node in j's neighborhood other than the nodes in this path. Since the network is without cycle, there is at most one R^t node in j's neighborhood. But then $j \notin R^{t+1}$ since it violate Claim 2.

Starting from $R^0 \neq \varnothing$ and iterating R^t with $t \geq 0$, if $R^t \neq \varnothing$ but $R^{t+1} = \varnothing$, then there is some i with $R^0 \subset I_i^t$ as the above paragraph shows; if $R^t \neq \varnothing$ and $R^{t+1} \neq \varnothing$, then we starting from R^{t+1} and iterating R^{t+1} with $t \geq t+1$. Since $\#R^{t+1} < \#R^t$ as the above paragraph shows, there is a time t^* with $R^{t^*} = \varnothing$, then we get the conclusion.

proof for Lemma 3.1

Proof. Denote (i, j)-path as the set of paths from i to j. The proof is by contradiction. Suppose there are three or more R^t-nodes in C^t, then pick any three nodes of them, and denote them as i_1, i_2, i_3. Let's say i_2 is in a (i_1, i_3)-path by strong connectedness, and therefore $i_2 \in Tr_{i_1 i_2}$ and $i_3 \in Tr_{i_2 i_3}$. First we show that $i_1 \in G_{i_2}$ (or $i_3 \in G_{i_2}$). Suppose $i_1 \notin N_{i_2}$, since $i_1, i_2 \in R^t$, then the (i_1, i_2)-path is a R^t-path by Claim 2. Let this $(i_1 i_2)$-path be $\{i_1, j_1, \ldots, j_n, i_2\}$. Since $i_1, j_1, \ldots, j_n, i_2 \in R^t$, we then have $I_{i_1}^{t-1} \nsubseteq N_{j_1}^{t-1}, \ldots, I_{j_n}^{t-1} \nsubseteq N_{i_2}^{t-1}$ and $I_{j_1}^{t-1} \nsubseteq N_{i_1}^{t-1}, \ldots, I_{i_2}^{t-1} \nsubseteq N_{j_n}^{t-1}$. Since $I_{i_1}^{t-1} \subseteq N_{i_1}^{t-1}, \ldots, I_{i_2}^{t-1} \subseteq N_{i_2}^{t-1}$ by Claim 1, we further have $\exists k_1 \in R^0[k_1 \in N_{j_1}^{t-1} \backslash I_{i_1}^{t-1}]$,...,$\exists k_n \in R^0[k_n \in N_{j_n}^{t-1} \backslash I_{i_2}^{t-1}]$. Since the state has strong connectedness, there is a R^0 path connecting $k_1, \ldots, k_n$ by Claim 3. But then we have already found k_1, k_2 such that $k_1 \in N_{j_1}^{t-1} \backslash I_{i_1}^{t-1}$ and $k_2 \in \bar{G}_{k_1}$. It is a contradiction that $i_1 \in C$.

Now, i_1, i_2, i_3 will form a R^t-path as $\{i_1, i_2, i_3\}$. With the same argument as the above, we then have $\exists k_1 \in R^0[k_1 \in N_{i_2}^{t-1} \backslash I_{i_1}^{t-1}]$ and $\exists k_2 \in R^0[k_2 \in N_{i_3}^{t-1} \backslash I_{i_2}^{t-1}]$, and thus i_1 is not in C. □

proof for Lemma 3.2

Proof. The proof is done by contradiction. Since $i \in R^t$, there is a $j \in (R^{t-1} \cap \bar{G}_i)$ by Lemma 2. Note that $N_j^{t-1} \subseteq \bigcup_{k \in N_i^{t-1}} N_k$ since $N_j^{t-1} = \bigcup_{k \in I_j^{t-2}} N_k$, and $I_j^{t-2} \subseteq I_i^{t-1} \subseteq N_i^{t-1}$. If there is another node outside $\bigcup_{k \in N_i^{t-1}} N_k$ in Tr_{ij}, then there must be another node such that there is a path connected to some nodes in N_j^{t-1} since the network is connected. It is a contradiction that $i \in C$.

□

A.1 Equilibrium

A.1.1 Out-off-path belief

If Rebel i detects a deviation at m period, he form the belief as

$$\beta_i(\{\theta : \theta \in \times_{j \in G_i}\{\theta_j\} \times \{Inert\}^{n-\#G_i}\} | h_{G_i}^{m'}) = 1 \, , \, m' \geq m \tag{11}$$

A.1.2 Equilibrium Path: Notations

- $\langle\rangle_r$ be the set of finite sequences in which the action **r** occurs once and only once.
- $PF(\langle\rangle, m)$ be the m-periods prefix of a finite sequence $\langle\rangle$.
- (i, j)-path be the set of paths from i to j.

A.1.3 Equilibrium Path: reporting period

reporting period: notations

- m be a period in reporting period in t block.
- $|\langle RP^t\rangle|$ be the total periods in reporting period in t-block
- $O_i^{m,t}$ be the set of i's neighbors js who has played a sequence M such that $M = PF(\langle I_j^{t-1}\rangle, m)$ and $M \in \langle\rangle_r$ at period m.
- $I_i^{m,t} \equiv (\bigcup_{k\in O_i^{m,t}} I_k^{t-1}) \cup I_i^{t-1}$ be the updated relevant information gathered by i at period m. Note that $I_i^{0,t} = I_i^{t-1}$ and $I_i^{|\langle RP^t\rangle|,t} = I_i^t$.
- $N_i^{m,t} \equiv (\bigcup_{k\in O_i^{m,t}} N_k^{t-1}) \cup N_i^{t-1}$ be the updated neighborhood which contains $I_i^{m,t}$
- Let
$$Ex_{I_i^{m,t}} \equiv \{l \notin N_i^{m,t} | \exists l' \in I_i^{m,t} \text{ such that there exists a } (l, l')\text{-path}\}$$
be all the possible Rebel nodes outside of $N_i^{m,t}$ given $I_i^{m,t}$
- Let
$$Tr_{I_i^{m,t}j} \equiv Tr_{ij} \cap (Ex_{I_i^{m,t}} \cup I_i^{m,t})$$
be all the possible Rebel nodes in the Tr_{ij} given $I_i^{m,t}$.

reporting period: automata

$i \notin R^t$

- **WHILE LOOP**
  - At $m \geq 0$, if $\#Ex_{I_i^{m,t}} \cup I_i^{m,t} < k$, report $\langle$**stay**$\rangle$ and then play **stay** forever.
  - Otherwise, **runs POST-CHECK**

$i \in R^t$

- **WHILE LOOP**
  - At $m \geq 0$, if $\#Ex_{I_i^{m,t}} \cup I_i^{m,t} < k$, report $\langle$**stay**$\rangle$ and then play **stay** forever.
  - Otherwise, **runs MAIN**

- **MAIN**

  At $m \geq 0$,

  1. At $m = 0$ and if $\#I_i^{t-1} = \#I_i^{0,t} = k - 1$, then **runs POST-CHECK**
  2. At $m = 0$ and if $i \in R^t$ and

  $$\nexists j \in R^{t-1} \cap \bar{G}_i \text{ such that } \exists l_1, l_2 \in Tr_{ij}[[l_1 \in N_j^{t-1} \backslash I_i^{t-1}] \wedge [l_2 \in \bar{G}_{l_1}]]]$$

  , then runs **CHECK.**0. Otherwise, recall **MAIN**

  3. At $0 \leq m \leq |RP^t| - |\langle I_i^{t-1} \rangle|$, play

  stay

  4. At $m = |RP^t| - |\langle I_i^{t-1} \rangle| + 1$, then
     (a) if $O_i^{m,t} = \varnothing$, then report

     $$\langle I_i^{t-1} \rangle$$

     (b) if $O_i^{m,t} \neq \varnothing$, then **runs CHECK.k**

- **CHECK.**0

  At $m = 0$, if $i \in C^t$, i.e. if $i \in R^t$ and

  $$\nexists j \in R^{t-1} \cap \bar{G}_i \text{ such that } [\exists l_1, l_2 \in Tr_{ij}[[l_1 \in N_j^{t-1} \setminus I_i^{t-1}] \wedge [l_2 \in \bar{G}_{l_1}]]]$$

  , then

  1. If $\#C^t = 1$, then **runs POST-CHECK**
  2. If $\#C^t = 2$, then denote $i_1, i_2 \in C$ such that $I_{i_1}^{t-2} < I_{i_2}^{t-2}$, and then
     - if $i = i_1$, then **runs POST-CHECK**
     - if $i = i_2$, then report

     $$\langle I_i^{t-1} \rangle$$

- **CHECK.**m

  At $m > 0$, if $O_i^{m,t} \neq \varnothing$, then there are two cases,

1. Case 1: If $i \in R^t$ and

$$\exists j \in O_i^m \text{ such that } \exists l_1, l_2 \in Tr_{I_i^{m,t} j}[[l_1 \in I_j^{t-1} \backslash I_i^{t-1}] \wedge [l_2 \in \bar{G}_{l_1}]]]$$

, then report

$$\langle I_i^{t-1} \rangle$$

2. Case 2: If $i \in R^t$ and

$$\nexists\, j \in O_i^m \text{ such that } \exists l_1, l_2 \in Tr_{I_i^{m,t} j}[[l_1 \in I_j^{t-1} \backslash I_i^{t-1}] \wedge [l_2 \in \bar{G}_{l_1}]]]$$

   (a) Case 2.1: If $i \in R^t$ and

$$\nexists\, j \in R^{t-1} \cap (G_i \backslash O_i^{m,t}) \text{ such that } [\exists l_1, l_2 \in Tr_{I_i^{m,t} j}[[l_1 \in N_j^{t-1} \backslash I_i^{t-1}] \wedge [l_2 \in \bar{G}_{l_2}]]]$$

Note: this case is the case when $i \in C$, thus recall Check.o

   (b) Case 2.2: If $i \in R^t$ and

$$\exists j \in R^{t-1} \cap (G_i \backslash O_i^{m,t}) \text{ such that } [\exists l_1, l_2 \in Tr_{I_i^{m,t} j}[[l_1 \in N_j^{t-1} \backslash I_i^{t-1}] \wedge [l_2 \in \bar{G}_{l_2}]]]$$

   – if $\# I_i^{m,t} = k - 1$, then **runs POST-CHECK**
   – if $\# I_i^{m,t} < k - 1$, then report

$$\langle I_i^{t-1} \rangle$$

- **CHECK.k**

  At $m \geq 1$,

1. $O_i^{m,t} \neq \varnothing$, and

$$\# I_i^{m,t} \geq k$$

, then **runs POST-CHECK**

2. $O_i^{m,t} \neq \varnothing$, and

$$\# I_i^{m,t} < k$$

, then **runs CHECK.m**

- **POST-CHECK**

1. At $m = |RP^t|$, then
   (a) If $i \in R^t$ and if $|I_i^{m,t}| \geq k - 1$, then play **revolt**
   (b) if $i \notin R^t$, then play **stay**

A.1.4 Equilibrium path: coordination period

coordination period: notations

- m be a sub-block in coordination period.
- Let
$$Ex_{I_i^t} \equiv \{l \notin I_i^t | \exists l' \in I_i^t \setminus I^{t-1} \text{ such that there exists a } (l, l')\text{-path}\}$$
be all the possible Rebel nodes outside of N_i^t given I_i^t.
- Let
$$Tr_{I_i^t j} \equiv Tr_{ij} \cap (Ex_{I_i^t} \cup I_i^t)$$
be the set of possible Rebel nodes in the Tr_{ij} given I_i^t.

coordination period: automata

- **1st Division**

  In 1st division, for $t = 0$ block,

  - If $\#Ex_{I_i^t} \cup I_i^t < k$, then play **stay** forever.
  - If $\#Ex_{I_i^t} \cup I_i^t \geq k$, and if $i \notin R^1$, then play
$$\langle \textbf{stay} \rangle$$
  - If $\#Ex_{I_i^t} \cup I_i^t \geq k$, and if $i \in R^1$, then play
$$\langle x_i \rangle$$

  In 1st division, for $t > 0$ block and for $1 \leq m \leq n$ sub-block,

  - If i has played $\langle 1 \rangle$, then play
$$\langle x_i \rangle$$
  - If $\#Ex_{I_i^t} \cup I_i^t < k$, then play **stay** forever.
  - If $\#Ex_{I_i^t} \cup I_i^t \geq k$, and there are some $j \in \bar{G}_i$ have played $\langle \textbf{stay} \rangle$, then play **stay** forever.
  - If $\#Ex_{I_i^t} \cup I_i^t \geq k$, and there is no $j \in \bar{G}_i$ has played $\langle \textbf{stay} \rangle$, then play
$$\langle x_i \rangle$$

- **2nd Division**

  In $t = 0$ block

- If $i \notin R^1$, play
$$\langle \textbf{stay} \rangle$$
.
- If $i \in R^1$, and if $\#I_i^0 \geq k$, play
$$\langle \textbf{stay} \rangle$$
.
- If $i \in R^1$, if $\#I_i^0 < k$, if $\#Ex_{I_i^t} \cup I_i^t \geq k$ and if some $j \in \bar{G}_i$ have played play $\mathbf{1}_j$ in the 1st division, then play
$$\langle \textbf{stay} \rangle$$
.
- If $i \in R^1$, if $\#I_i^0 < k$, if $\#Ex_{I_i^t} \cup I_i^t \geq k$ and if no $j \in \bar{G}_i$ has played play $\mathbf{1}_j$ in the 1st division, then play **stay** forever.

In $t > 0$ block, if there is no $j \in G_i$ such that j has played $\langle \textbf{stay} \rangle$ in the **1st Division** , then run the following automata. Otherwise, play **stay** forever.

- $i \notin R^t$
  * In the 1-sub-block: play
  $$\langle \textbf{stay} \rangle$$
  * In the $2 \leq m \leq t+1$ sub-blocks:
    1. If $i \in R^{t'}$ for some $t' \geq 0$ and if there is a $j \in R^{t'+1} \cap \bar{G}_i$ has played
       (a) $\langle \textbf{stay} \rangle$ in $m = 1$ sub-block
       (b) or $\langle \mathbf{1}_j \rangle$ in $m \geq 2$ sub-blocks

       , then play
       $$\langle x_i \rangle$$
       in $m+1$ sub-block.
    2. Otherwise, play
       $$\langle \textbf{stay} \rangle$$
       in current sub-block
- $i \in R^t$
  * In the 1-sub-block:
    1. If i has played $\langle 1 \rangle$, then play
       $$\langle \textbf{stay} \rangle$$

2. If i has not played $\langle 1\rangle$ and if there is a $j \in \bar{G}_i$ has played $\langle 1\rangle$, then play

$$\langle \textbf{stay} \rangle$$

3. If i has not played $\langle 1\rangle$ and if there is no $j \in \bar{G}_i$ has played $\langle 1\rangle$, then
   · If $\#I_i^{|RP^t|,t} \geq k$, then play
   $$\langle \textbf{stay} \rangle$$
   · If $\#I_i^{|RP^t|,t} < k$, then play
   $$\langle \mathbf{1}_i \rangle$$

* In the $m \geq 2$-sub-block:
  1. If $i \in R^{t'}$ for some $t' \geq 0$ and if there is a $j \in R^{t'} \cap \bar{G}_i$ has played
     (a) $\langle \textbf{stay} \rangle$ in $m = 1$ sub-block, or
     (b) $\langle \mathbf{1}_j \rangle$ in $m \geq 2$ sub-blocks
     , then play
     $$\langle x_i \rangle$$
     in $m + 1$ sub-block.
  2. Otherwise, play
     $$\langle \textbf{stay} \rangle$$
     in current sub-block.

- **3rd Division**

  1. **INITIATING**
     If i has observed $j \in \bar{G}_i$ has played
     (a) $\langle \textbf{stay} \rangle$ in 1-sub-block in **2nd Division** or
     (b) $\langle \mathbf{1}_j \rangle$ in $m \geq 2$ sub-blocks **2nd Division** or
     (c) **s** in the **3rd Division**
     , then play **revolt** forever
  2. **NOT INITIATING**
     Otherwise, play **stay** in current period.

A.1.5 Proof for Theorem 3.1

The proof is organized as the following. In Claim 4 and Lemma 3.3, we show that a Rebel will learn $\#[Rebels](\theta) \geq k$ or $\#[Rebels](\theta) < k$ in the equilibrium path. Lemma 3.3 also show that the equilibrium path is ex-post efficient. Since that, there is a time T such that a Rebel's static pay-off after T is 1 if $\#[Rebels](\theta) \geq k$ or 0 if $\#[Rebels](\theta) \geq k$. Such pay-offs after time T is the maximum static pay-off contingent on θ. In Claim 5, I show that if a Rebel makes detectable deviation, then there is a positive probability event E (by the full support assumption) contingent on this deviation such that his expected continuation static pay-off is strictly lower than that in equilibrium path after T. Finally, in Claim 6, Claim 7, Claim 8, and Claim 9, I show that if a Rebel makes undetectable deviation, then there is a positive event E (by the full support assumption) contingent on this deviation such that his expected continuation static pay-off is also strictly lower than that in equilibrium path after T. Since the static pay-off after T is maximum for all θ in equilibrium path, there is a δ such that a Rebel will not deviate. I then conclude this theorem.

To simplify the notations, if $P(\theta)$ is a property of θ, then I abuse the notations by letting $\beta^{\pi,\tau^*}_{G_i}(P(\theta)|h^m_{G_i}) \equiv \sum_{\theta:P(\theta)} \beta^{\pi,\tau^*}_{G_i}(\theta|h^m_{G_i})$. I also say "$i$ knows $P(\theta)$" to mean $\beta^{\pi,\tau^*}_{G_i}(P(\theta)|h^m_{G_i}) = 1$.

Claim 4. *In the equilibrium path and for $\#Ex_{I^{m,t}_i} \cup I^{m,t}_i \geq k$, where m is a period in reporting period. If i report $\langle 1 \rangle$, then either Rebels coordinate to **revolt** after t-block or $\#R^\circ < k$.*

Proof. By directly checking the equilibrium path, we have

1. if $\#I_i^{|RP^t|,t} \geq k$, then the coordination can be initiated by such i.
2. if $\#I_i^{|RP^t|,t} = k-1$, and if there is one more node who reported $\langle 1 \rangle$, then the coordination can be initiated by i.
3. if $\#I_i^{|RP^t|,t} = k-1$, and if there are no nodes who reported in current reporting period, then $\#I_i^{|RP^t|,t} = \#I_i^t = k-1$. We now check the conditions guiding i to **POST-CHECK**.
   - If i is coming from the conditions in **MAIN**, it means that there are no further Rebels outside I_i^{t-1}, thus outside $\bigcup_{k\in I_i^{t-1}} G_k$.
   - If i is coming from the conditions in **CHECK.o**, it means that there are no further Rebels outside $\bigcup_{k\in I_i^{t-1}} G_k \cap R^\circ$, and thus outside $\bigcup_{k\in I_i^{t-1}} G_k$.
   - If i is coming from the conditions in **CHECK.m**, it means that there are no further Rebels outside $\bigcup_{k\in I_i^{t-1}} G_k \cap R^\circ$, and thus outside $\bigcup_{k\in I_i^{t-1}} G_k$.

   Since $I_i^t = \bigcup_{k\in I_i^{t-1}} G_k \cap R^\circ \subset \bigcup_{k\in I_i^{t-1}} G_k$ and $\#I_i^t < k$, and hence $\#R^\circ < k$.

□

proof for Lemma 3.3

Proof. We want to show that when θ satisfying $\#[Rebels](\theta) \geq k$, all the Rebels play **revolt** eventually; when θ satisfying $\#[Rebels](\theta) < k$, all the Rebels play **stay** eventually.

1. If all the Rebels only play $\langle I^{t-1} \rangle$ or $\langle$**stay**$\rangle$ in reporting period for all $t \geq 1$ block, then by the equilibrium path, those nodes played $\langle I^{t-1} \rangle$ are R^t-node, and those nodes played $\langle$**stay**$\rangle$ are non-R^t nodes.

   If there are some Rebels play $\langle$**stay**$\rangle$ in $CD^t_{1,1}$, then all the Rebels play **stay** eventually; If R^t Rebels play $\langle$**stay**$\rangle$ in $CD^t_{1,2}$, then all the Rebels will play **revolt** after third division in coordination period in this block. Otherwise, all the Rebels go to the next reporting period.

   By Theorem 3, there is a t^* such that there is a R^{t^*} node knows θ, and therefore he knows if θ satisfying $\#[Rebels](\theta) \geq k$ or $\#[Rebels](\theta) < k$. In equilibrium path, such node play $\langle$**stay**$\rangle$ either in $CD^{t^*}_{1,1}$ or in $CD^{t^*}_{1,2}$. Thus the equilibrium path is APEX.

2. If there are some Rebels play $\langle 1 \rangle$ in reporting period for a $t \geq 1$ block, then by Claim 4, such nodes will knows if θ satisfying $\#[Rebels](\theta) \geq k$ or $\#[Rebels](\theta) < k$ after reporting period in this t-block. Then $\langle$**stay**$\rangle$ is either played in the first sub-block in first division or played in the first sub-block in second division in coordination period. Thus the equilibrium path is APEX.

□

Next, I prepare the claims to show that a Rebel will not deviate. I start with Claim 5 in which the deviation is detectable.

Claim 5. *For $\#Ex_{I_i^{m,t}} \cup I_i^{m,t} \geq k$, where m is a period. Denote D be the set of Rebels who detect i's deviation. If $\#I_i^{m,t} < k$ and if $D \neq \emptyset$, then there is a δ such that i will not deviate.*

Proof. Denote D be the set of neighbors who detect i's deviation. Let the events be

$$\begin{aligned} E_1 &= \{\theta : \#[Rebels](\theta) < k\} \\ E_2 &= \{\theta : k \leq \#[Rebels](\theta) < k + \#D\} \\ E_3 &= \{\theta : \#[Rebels](\theta) \geq k + \#D\} \end{aligned}$$

In equilibrium path, there are periods t^s (t^f) such that if θ satisfying $\#[\text{Rebels}](\theta) \geq k$ ($\#[\text{Rebels}](\theta) < k$) then Rebels play **revolt** (**stay**) forever. If i follows the equilibrium path, the expected static pay-off after $\max\{t^s, t^f\}$[14] is

$$\beta_i(E_2|h^m_{N_i}) + \beta_i(E_3|h^m_{N_i})$$

[14]There is t^s or t^f for each θ. The maximum is among those possible θ.

If i deviate, the expected static pay-off after $\max\{t^s, t^f\}$ is

$$\beta_i(E_3|h^m_{N_i})$$

Therefore there is a loss in expected static pay-off of

$$\beta_i(E_2|h^m_{N_i})$$

Thus, there is a loss in expected continuation pay-off contingent on E_2 by

$$\delta^{\max\{t^s,t^f\}}\frac{\beta_i(E_2|h^m_{N_i})}{1-\delta}$$

Note that $\beta_i(E_2|h^m_{N_i}) > 0$, since $\#Ex_{I^{m,t}_i} \cup I^{m,t}_i \geq k$ and therefore $\beta_i(\#[Rebels](\theta) \geq k|h^m_{N_i}) > 0$ by full support assumption. □

Next, I prepare the claims to show that a Rebel will not deviate if such deviation is undetectable.

Claim 6. *In reporting period for $\#Ex_{I^{m,t}_i} \cup I^{m,t}_i \geq k$, if $\#I^{m,t}_i < k$, then there is a δ such that i will not deviate by reporting $\bar{I}^{t-1}_i \neq I^{t-1}_i$ if such deviation is not detected by i's neighbor.*

Proof. Assume $\bar{I}^{t-1}_i \neq I^{t-1}_i$. Since a detection of deviation has not occur, it must be the case that there is a non-empty set $F = \{j \in \bar{I}^{t-1}_i : \theta_j = Inerts\}$[15].

Let the set

$$E_1 = \{\bar{\theta} : \bar{\theta}_j = Rebel \text{ if } j \in F \text{ and } \bar{\theta}_j = \theta_j \text{ if } j \notin F\}$$

be the set of pseudo events by changing θ_j where $j \in F$. And let

$$E_2 = \{\theta : \theta_j = Inert \text{ if } j \in F \text{ and } \bar{\theta}_j = \theta_j \text{ if } j \notin F\}$$

be the set of true event.

Then consider the event

$$\begin{aligned} E &= \{\bar{\theta} \in E_1 : \#[Rebels](\bar{\theta}) \geq k\} \\ &= \{\theta \in E_2 : \#[Rebels](\theta) \geq k - \#F\} \end{aligned}$$

Partition E as sub events

$$\begin{aligned} E_3 &= \{\theta \in E_2 : \#[Rebels](\theta) \geq k\} \\ E_4 &= \{\theta \in E_2 : k > \#[Rebels](\theta) \geq k - \#F\} \end{aligned}$$

By Lemma 3 and following the strategies in equilibrium path (since i have not been detected), there is a block $\bar{t}^s$ with respect to $\bar{\theta}$ such that if $\bar{\theta} \in E$ then there some $R^{\bar{t}^s}$ Rebel js, says J, will initiate the coordination, and then Rebels play **revolt** forever after $\bar{t}^s$-block. Note that such j is with $\#I^{\bar{t}^s}_i \geq k$ by checking the equilibrium path.

We have several cases:

[15] Otherwise, there is a detection of deviation. Recall the definition in information hierarchy: $I^{-1}_i \subset I^0_i \subset \ldots \subset I^{t-1}_i$ for all $i \in R^0$

1. Case 1: If $i \in J$, his own initiation will only depends on $\#I_i^{\bar{t}^s}$ by Claim 7 and Claim 8, which is the same as he has reported $\langle I_i^{t-1}\rangle$. He is strictly better off by not deviating since playing $\langle \bar{I}_i^{t-1}\rangle$ is more costly than $\langle \bar{I}_i^{t-1}\rangle$ (since $X_{\bar{I}_i^{t-1}} > X_{I_i^{t-1}}$).

2. Case 2: If there is another j such that $\bar{I}_i^{t-1} \notin I_j^{\bar{t}^s}$, then since such j's initiation of coordination dependent on his own information $I_j^{\bar{t}^s}$ by Claim 7 and Claim 8, and i's deviation did not change j's information. It is strictly better by not deviating since playing $\langle \bar{I}_i^{t-1}\rangle$ is more costly than $\langle \bar{I}_i^{t-1}\rangle$ (since $X_{\bar{I}_i^{t-1}} > X_{I_i^{t-1}}$).

3. Case 3: Suppose there is another j such that $\bar{I}_i^{t-1} \subset I_j^{\bar{t}^s}$ and $\#I_i^{\bar{t}^s} \geq k$, then such j will initiate the coordination to **revolt**. If i did not follow j's initiation of coordination, then there is a detection of deviation by checking the equilibrium path. i will not deviate as Claim 5 shows. If i follows, and $\#I_i^{\bar{t}^s} \geq s$, we are in the Case 1. If i follows, but $\#I_i^{\bar{t}^s} < s$, then i's expected static pay-off after $\bar{t}^s$ is at most

   $$\max\{\beta_i(E_3|h_{N_i}^m) \times 1 + \beta_i(E_4|h_{N_i}^m) \times (-1), 0\}$$

   However, if i follow the equilibrium path, there is are t^s, t^f such that the expected static pay-off after $\max\{t^s, t^f\}$ is

   $$\max\{\beta_i(E_3|h_{N_i}^{m'}), 0\}$$

   Thus, there is a loss in expected continuation pay-off contingent on E by

   $$\delta^{\max\{t^s,t^f\}} \frac{\min\{\beta_i(E_3|h_{G_i}^m), \beta_i(E_4|h_{G_i}^m)\}}{1-\delta}$$

Note that $\beta_i(E_3|h_{N_i}^m) > 0$ and $\beta_i(E_3|h_{N_i}^m) > 0$, since $\#Ex_{I_i^{m,t}} \cup I_i^{m,t} \geq k$ and $\#I_i^{m,t} < k$, and therefore $1 > \beta_i(\#[Rebels](\theta) \geq k|h_{N_i}^m) > 0$ by full support assumption. □

Claim 7. *In reporting period for $\#Ex_{I_i^{m,t}} \cup I_i^{m,t} \geq k$, if $\#I_i^{m,t} \leq k-1$ and if $i \notin C^t$ or i did not satisfy the condition to play $\langle 1\rangle$ in equilibrium path, then i will not play $\langle 1\rangle$.*

Proof. Let

$$E' = \{\theta : \#I_i^{RP^t,t} \leq k-1\}$$

. Note that such event is not empty by checking the timing where i deviated:

1. If i has a neighbor $j \in C^t$, then $j \notin O_i^{RP^t,t}$, and therefore we can construct E' by assuming that all other neighbors (other than i, j and other than $l \in O_i^{m,t}$) are non-R^t.

2. If

   $$\exists j \in R^{t-1} \cap \bar{G}_i \text{ such that } \exists k_1, k_2 \in Tr_{ij}[[k_1 \in N_j^{t-1} \backslash I_i^{t-1}] \wedge [k_2 \in \bar{G}_{k_2}]]$$

   , then just let $E' = \{\theta : N_i^t \cap R^0 \leq k-1\} = \{\theta : I_i^t \leq k-1\}$[16].

[16] note that $I_i^t = I_i^{RP^t,t}$

Next, let

$$\begin{aligned} E_1 &= \{\theta : \#[Reble](\theta) < k\} \cap E' \\ E_2 &= \{\theta : \#[Reble](\theta) \geq k\} \cap E' \end{aligned}$$

Note that E_1 and E_2 are not empty. According to equilibrium path, if i did not follow the conditions to play $\langle 1 \rangle$, it must be the case that there are some nodes outside I_i^t but there is a path consisting of Rebels to connect them. By strong connectedness, E_1 and E_2 are not empty.

Since i deviate to play $\langle 1 \rangle$, his behavior after $CD_{1,1}^t$ will decide the following three cases:

1. If i play $\langle$**stay**$\rangle$ in $CD_{1,1}^t$, then the coordination to **stay** starts after $CD_{1,1}^t$.
2. If i play $\langle x_i \rangle$ in $CD_{1,1}^t$, then the coordination to **revolt** will be initiate after $CD_{1,2}^t$ if he mimic the behavior of a pivotal player (i.e., by mimicking those players who played $\langle 1 \rangle$ in equilibrium path).
3. If i play $\langle x_i \rangle$ in $CD_{1,1}^t$, but he did not mimic the behavior of pivotal player, then such deviation will be detected.

Thus, i's expected static pay-off after the coordination period in this t-block is at most

$$\max\{\beta_i(E_2|h_{N_i}^m) \times 1 + \beta_i(E_1|h_{N_i}^m) \times (-1), 0\}$$

However, if he stay in the equilibrium, there is a t^s (t^f) such that Rebels play **revolt** (**stay**) contingent on E_2 (E_1), and thus after $t^* = \max\{t^s, t^f\}$ he get the expected pay-off as

$$\max\{\beta_i(E_2|h_{N_i}^m) \times 1, 0\}$$

After some calculation, after t^*, there is a loss of

$$\delta^{t^*} \frac{\min\{\beta_i(E_2|h_{G_i}^m), \beta_i(E_1|h_{G_i}^m)\}}{1-\delta}$$

Note that $\beta_i(E_1|h_{N_i}^m) > 0$ and $\beta_i(E_2|h_{N_i}^m) > 0$, by E_1 and E_2 are not empty and by full support assumption.

□

Claim 8. *In reporting period for $\beta_i(\#[Rebels](\theta) \geq k|h_{G_i}^{RP^t-1}) > 0$, then if i can report $\langle 1 \rangle$, then i will not report $\langle l \rangle$ when δ is high enough.*

Proof. There are two cases when i can play $\langle 1 \rangle$.

- Case 1: If $\#I_i^{|RP^t|-1,t} \geq k$, let the event E be

$$E = \{\theta : \#[Rebels](\theta) = \#I_i^{|RP^t|,t}\}$$

That is, the event that no more Rebels outside i's information about Rebels. Contingent on E, there is no more Rebel can initiate the coordination. This is because for all $j \in O_i^{|RP^t|-1,t}$, j is with $\#I_j^{t-1} < k-1$, and for all $j \in \bar{G}_i$ who have not yet reported, $j \notin R^t$ since all the Rebels are in $I_i^{|RP^t|-1,t}$. Since only i can initiate the coordination, if i deviated, compared to equilibrium, there is a loss in expected continuation pay-off as

$$\delta^q \frac{\beta_i(E|h_{N_i}^m)}{1-\delta}$$

, where q is a period after t-block.

- Case 2: If $\#I_i^{|RP^t|-1,t} = k-1$, since $\beta_i(\#[Rebels](\theta) \geq k|h_{G_i}^{|RP^t|}) > 0$, the following event E_1 must have positive probability; otherwise, since no neighbors can report after current period, and thus $\beta_i(\#[Rebels](\theta) \geq k|h_{G_i}^{|RP^t|}) = 0$.

  Let

$$E_1 = \{\theta : \exists j \in \bar{G}_i, j \notin O_i^{|RP^t|-1,t}[\#I_j^{|RP^t|-1,t} \geq k-1]\}$$

  Let sub-events $E_1' \subset E_1$ as

$$E_1' = \{\theta : \text{ exist a unique} j \in \bar{G}_i, j \notin O_i^{|RP^t|-1,t}[\#I_j^{|RP^t|-1,t} \geq k-1]\}$$

  Note that this E_1' can be constructed since the network is tree. If there is θ admits 2 or more js in the definition E_1, these js are not each others' neighbor. Suppose there are two js, says j, j', there must be at least one node in $I_j^{|RP^t|-1,t}$ but outside of $I_{j'}^{|RP^t|-1,t}$. We then pick a j, and suppose those nodes outside $I_j^{|RP^t|-1,t}$ are all Inerts.

  Now, dependent on such j, let

$$E = \{\theta : \#[Rebels](\theta) = \#I_j^{|RP^t|-1,t} \cup I_i^{|RP^t|-1,t}\}$$

  If i report $\langle l \rangle$, there are following consequences.

  - i will be consider as $\notin R^t$ by j, and thus i can not initiate the coordination.
  - Such j has $\#I_j^{|RP^t|,t} = \#I_j^t < k$. Since there is no more Rebel outside $I_j^{|RP^t|-1,t} \cup I_i^{|RP^t|-1,t}$ contingent on E, such j will then play stay forever after t-block.
  - Without such extra Rebels in $I_j^{|RP^t|,t}$, only $\#I_i^{|RP^t|-1,t} = k-1$ Rebels may play **revolt**, and therefore there is no coordination to **revolt**

However, if i play $\langle 1 \rangle$, coordination can be initiated by himself in the following coordination period. Thus, there is a loss in expected continuation pay-off by

$$\delta^q \frac{\beta_i(E|h^m_{N_i})}{1-\delta}$$

, where q is a period after t-block.

$\square$

Claim 9. *Given a m period in coordination period in t-block, suppose there is no $j \in G_i$ has played $\langle 1 \rangle$ in reporting period in t-block, suppose $\#I^t_i < k$, and suppose $\#Ex_{I^t_i} \cup I^t_i \geq k$, then there is δ such that*

- *if i has not observed $\langle \textbf{stay} \rangle$ played by $j \in G_i$ in $CD^t_{1,2}$, or*
- *if i has not observed $\langle \mathbf{1}_j \rangle$ played by $j \in G_i$ in $CD^t_{q,2}$, $g \geq 2$*

, then i will not play

- *$\langle \textbf{stay} \rangle$ in $CD^t_{1,2}$ and*
- *$\langle \mathbf{1}_j \rangle$ in $CD^t_{q+1,2}$, $g \geq 2$*

Proof. If i deviate, all i's neighbor who did not detect the deviation will play **revolt** after coordination period in this block; if i's deviation is detected by some neighbors, we are in the case of Claim 5 and so that i will not deviate. We then check if i deviate but no neighbor detect it. Let

$$E' = \{\theta : \#I^t_i \leq k-1\}$$

and let

$$\begin{aligned} E_1 &= \{\theta : \#[Reble](\theta) < k\} \cap E' \\ E_2 &= \{\theta : \#[Reble](\theta) \geq k\} \cap E' \end{aligned}$$

Since $\#I^t_i < k$ and $\#Ex_{I^t_i} \cup I^t_i \geq k$, due to the full support assumption and the equilibrium strategies played by i's neighbors, we have

$$0 < \beta_i(\#[Rebels](\theta)| \geq k|h^m_{G_i}) < 1$$

, and thus E_1 and E_2 have positive probability. Since after i's deviation, all the Rebels will play **revolt** after this block, i's expected static pay-off after the coordination period in this t-block is at most

$$\max\{\beta_i(E_2|h^m_{N_i}) \times 1 + \beta_i(E_1|h^m_{N_i}) \times (-1), 0\}$$

However, if he stay in the equilibrium, there is a t^s (t^f) such that Rebels play **revolt** (**stay**) contingent on E_2 (E_1), and thus after $t^* = \max\{t^s, t^f\}$ he get the expected pay-off as

$$\max\{\beta_i(E_2|h^m_{N_i}) \times 1, 0\}$$

After some calculation, after t^*, there is a loss of

$$\delta^{t^*} \frac{\min\{\beta_i(E_2|h^m_{G_i}), \beta_i(E_1|h^m_{G_i})\}}{1-\delta}$$

$\square$

After the above claims, we can take a sufficiently high δ to let all the above claims hold. Since a deviation is either detectable or non-detectable, and a deviation happens either in reporting period or coordination period, I conclude that this theorem holds by above claims.