Coordination in Social Networks: Communication by Actions

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Abstract

This paper studies a collective action problem in a setting of discounted repeated coordination games in which players know their neighbors' inclination to participate as well as monitor their neighbors' past actions. I define *strong connectedness* to characterize those states in which, for every two players who incline to participate, there is a path consisting of players with the same inclination to connect them. Given that the networks are fixed, finite, connected, commonly known, undirected and without cycles, I show that if the priors have full support on the strong connectedness states, there is a weak sequential equilibrium in which the ex-post efficient outcome repeats after a finite time T in the path when discount factor is sufficiently high. This equilibrium is constructive and does not depend on public or private signals other than players' actions.

1 Introduction

This paper studies repeated collective actions within social networks. Players hold local information but can perfectly observe their neighbors' actions. I explore how players share information through their past plays in order to achieve the ex-post efficient outcome in the

future. Though the motive of this study is to understand the dynamic of social movements, a general interest centers on the collective action behaviors within social structures.

Consider pro-democracy movements. Strong discontents overthrowing a regime may exist, but it is difficult to organize around these discontents because information about the existence of such discontents is not always transparent. For instance, in East Germany, the government had control over the electoral system and the mass media, and the eavesdropping of secret agents had impeded people from showing their discontents. As [Karl-Dieter and Christiane, 1993] or [Chwe, 2000] have suggested, such discontents may be revealed only to someone whom you trust or have an intimate relationship with, but they are hardly revealed publicly. This lack of common knowledge about the existence of strong discontent impedes people from conducting a one-shot uprising due to the fear of possible failure. However, an event may trigger a later event. When rebels are aware of the scale to share relevant information about the level of collective discontent through their actions, they might be willing to take actions although it is at risk of facing failure. I view the dynamic of their actions as an equilibrium so that the entire movement is as of a learning process.

I model such dynamic collective actions in the following way. Players infinitely play repeated *k-threshold games* within their social network ([Chwe, 2000]), where k is a commonly known parameter, with a common discount factor. There are two types of players, one we called them *Rebel* and the other *Inert*. The network constrains players' information structure so that players' types and their past play can be observed only by their neighbors. A Rebel has two options, which are **revolt** or **stay**, while an Inert has only one option, which is **stay**. If a Rebel chooses **revolt**, he will get the payoff of 1 if at least k players also choose **revolt**; he will get -1 otherwise. If a Rebel chooses **stay**, he will get the payoff of 0 no matter how many players choose **revolt**. Rebels have a common prior over type profiles but do not necessarily know how many Rebels exist in this society. Therefore, the parameter k represents the threshold for overthrowing a regime, and the payoff structure mimics that overthrowing a regime is risky. Cheap talk is not allowed, payoff is hidden, no outside mechanism serves as an communication device.

Rebels communicate by playing actions, while Inerts block the communication between

Rebels. For different k and different network structures, I am looking for a weak sequential equilibrium that has the property of *approaching ex-post efficient* or *APEX*.[1] An equilibrium is APEX if the tails of actions in the equilibrium path repeats the static ex-post efficient outcome after some finite period T. This notion serves to check whether players learn the relevant information on the equilibrium path in finite time: if there are at least k Rebels in this society, then all Rebels should **revolt** after T as if they were aware that at least k Rebels exist; otherwise, all Rebels should **stay** after T.

I focus my exploration on the networks that are finite, commonly known, fixed, undirected, and connected, or *networks* henceforth. On all *acyclic networks*, this paper's main result Theorem 2 affirms that Rebels can coordinate themselves to achieve the ex-post efficient outcome irrespective of the overthrowing threshold, given that they are sufficiently patient and under a *strong connectedness condition* on type profiles. This strong connectedness condition is meant to assure that no Inert can block the communication between Rebels. To be precise, the strong connectedness condition says that a type profile occurs if and only if it induces a path consisting of Rebels to connect every two Rebels. The "only-if" part in this condition is necessary in that if it fails, there must be some overthrowing threshold for which no APEX equilibrium exists.[2] This condition and the acyclic networks restriction can be dispensed together when the overthrowing threshold is the number of

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[1]The APEX equilibrium I constructed would not satisfy full consistency ([Kreps and Wilson, 1982]) for some k, but does satisfy updating consistency ([Perea, 2002]). Updating consistency requires that, for (1) every player, (2) every player's strategies, and (3) two information sets s^1, s^2, where s^2 follows s^1, if s^2 happens with positive probability given s^1 and given players' strategies contingent on s^1, then the belief over s^2 should satisfy Bayesian updating conditional on the belief over s^1 and players' strategies contingent on s^1. In other words, the updating consistency requires that players hold belief in every information set and hold updated belief that follows the previous belief. This requirement imposes restrictions on off-path beliefs that induce sequential rationality, although it is weaker than full consistency in that full consistency implies updating consistency.

[2]The falsification of "only-if" part is equivalent to saying that the network induced by Rebels is not connected. It implies that some Rebels never learn each other's type. When the overthrowing threshold is high enough, it then violates Lemma 2.1 below, which states that in every APEX equilibrium under some overthrowing threshold, every Rebel must learn whether the number of Rebels exceeds this threshold. The "if" part is a technical assumption for the proof easiness.

players. My Theorem 1 shows that, if the overthrowing threshold is the number of players, an APEX equilibrium will exist in any network if Rebels are sufficiently patient. In the aspect of characterizing the network structure that sustains the ex-post efficient outcome, Theorem 1 and Theorem 2 are in sharp contrast to the sufficient network result of [Chwe, 2000]: a network and a prior always exist so that the ex-post efficient outcome cannot be an equilibrium in the one-shot k-threshold game when $k \geq 3$, but that can be achieved in the long term in an equilibrium if this game is played repeatedly and infinitely.

The existence of APEX equilibrium is proved by construction. I let two phases, the *reporting phase* and the *coordination phase*, consecutively alternate on the equilibrium path, each of which endures a specified length of periods. Ignoring incentive compatibility, this construction idea is straightforward.[3] Rebels truthfully inform their neighbors *which* Rebels they have been aware of by varying their actions in the reporting phases. They can do so because the network is commonly known: the players are not anonymous, and their locations are equivalent to their indexes. Rebels, moreover, can observe their neighbors' past play. What follows is that Rebels are aware of each other iteratively as reporting phases go by. In the coordination phase, several sequences of actions are specified to represent three kinds of knowledge: (1) "I am certain that the number of Rebels exceeds k." (2) "I am certain that the number of Rebels cannot exceed k." (3) "I am still uncertain." Rebels then play these specified sequences to disseminate their knowledge until a consensus is reached. Whenever a consensus is reached, they either play the ex-post efficient outcome together or enter the next reporting phase together.

When the incentive compatibility is taken into consideration, this construction, however, constitutes a free-rider problem. This problem stems from the fact that a Rebel concerns short-term incentive and always seeks the opportunity to manipulate his sequence of actions. Consider a scenario where two nearby Rebels vary actions to share information in a reporting phase.[4] Suppose that these two Rebels can learn the true state from each other's truthful information sharing. Further suppose that each of them has a continuation strategy on the

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[3]Indeed, if cheap talk is allowed or if there is no discounting, Theorem 2 immediately holds since Rebels have no incentive to lie by that the k-threshold game is a coordination game without conflict of interest.

[4]Example 5 below exams this scenario in details.

equilibrium path to initiate the ex-post efficient outcome. Then sharing truthful information is not the best response since either Rebel can wait for the other's information sharing. This is to say the coordination of ex-post outcome is as a public good. This public good can only be made by information sharing, which incurs costs since playing **revolt** is risky. To solve the free-rider problem implied by this simple construction idea, the assumption of acyclic network becomes crucial. If the networks are acyclic, I will show that, there are at most two Rebels that subject to the free-rider problem. Moreover, these two Rebels are neighbors and commonly know that they are subject to the free-rider problem. The equilibrium then prescribes that one of them is the free-rider, who chooses his most profitable sequence of actions in the reporting phase, and the other one shares truthful information. However, this common knowledge property does not hold in cyclic networks. I leave a concrete example and a discussion for this issue in Section 5.2.

The paper is organized as follows. I will introduce the model in Section 2, and illustrate my equilibrium construction and my main results in Section 3 and Section 4. In Section 5, I will discuss how the common knowledge of free-rider problem may fail to be established in cyclic networks. The conclusion is made in Section 6. All the missing proofs are left in the Appendix.

1.1 Related Literature

This paper bears on questions in several fields. My paper is related to studies in political economy on how collective actions serve as an information aggregation device. The closest studies might be [Lohmann, 1994] and [Shadmehr and Bernhardt, 2019].[5] [Lohmann, 1994] solves a free-rider problem in a two-stage collective action problem by identifying pivotal players, where participating in the first stage has an explicit cost but is publicly

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[5]Pioneered works on information aggregation induced by collective actions can be also seen in [Austen-Smith and Banks, 1996], [Olson, 1965], or [Feddersen and Pesendorfer, 1997]. Two-stage collective actions are also seen in [Lohmann, 1993], [Shadmehr, 2015], or [Battaglini, 2016]; sequential collective actions is also seen in [Dekel and Piccione, 2000] and [Piketty, 2000]. Nevertheless, none of these studies puts private monitoring into consideration.

observable. [Shadmehr and Bernhardt, 2019] studies another two-stage collective action problem involving only two players, in which the first mover's participation has an implicit cost but is also public-observable. The rationale behind my equilibrium construction and their two-stage approach is similar. My reporting phase, like their first stage, is meant to aggregate the pay-off relevant information. On the other hand, my coordination phase can be viewed as a decision-making stage for the public, like their second stage. My model, however, emphasizes that players' actions are privately observable under the social network constraint. What follows is that players need to coordinate *when* to move together (to achieve the ex-post efficient outcome) by some continuation strategies in the coordination phase. The remedy for my free-rider problem, as aforementioned, is not only to identify pivotal players but to avoid them from exploiting those continuation strategies.

In the economics of networks, my paper considers games played on networks. To the best of my knowledge, my paper is the first article on repeated games with incomplete information in this field. Having players with types, I borrow the k-threshold game from [Chwe, 2000]. [Chwe, 2000] identifies the minimum sufficient network over which players communicate and in which ex-post efficiency can be sustained in a one-shot k-threshold game. I explore what kinds of networks induce ex-post efficiency in repeated k-threshold games through communication by actions.

Among the studies on communication over networks, [Calvó-Armengol et al., 2015] provides a network-centrality result in characterizing individual's efforts on a pre-play costly strategic communication that facilitates coordination.[6] The substantial difference is that I view communication as a learning-in-equilibrium property instead of being an additional information channel.

Recent papers do consider repeated games played on networks but with complete information. Without additional communication channel, [Xue, 2004] and [Nava and Piccione, 2014] construct sequential equilibria that sustain full cooperation in prisoners' dilemmas between neighbors.[7] [Xue, 2004] studies the line-shaped network, and, there, the punishment

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[6]Strategic communication over networks is also seen in [Hagenbach and Koessler, 2010] and [Galeotti et al., 2010]. They adopt cheap talks as the communication device.

[7]In random matching with anonymous players, [Deb and González-Díaz, 2019] sustain cooperation be-

scheme basically contains several patterns that could either activate further punishment or lead players back to full cooperation. By assuming unknown network structure, the punishment scheme in [Nava and Piccione, 2014] is rationalized by a off-path belief weighing on the event that possible deviations are acyclic and only happen locally. This sort of belief-based rationalization reduces the complexity of tracking history of actions. This paper considers weak sequential equilibrium and adopt a belief-based punishment scheme, which considerably simplifies the equilibrium construction off the path.[8] The difference is that my off-path belief is over the pay-off relevant information instead of unobservable histories of actions.

This paper is broadly related to the literature of repeated games with incomplete information. The learning-in-equilibrium phenomenon is well-noted. Along with this line, this article is amongst studies on imperfect monitoring.[9] Much more general games are considered in this literature than those in mine. Notable results in this literature include the provision of folks theorem and the proof of learning-in-equilibrium, by characterizing sufficient signal structures generated by actions and states. Without discounting, [Renault and Tomala, 2004] characterizes a network monitoring under which players can learn the true state in equilibrium. My result is not covered by them since players in my model concern short-term incentive, due to discounting. With discounting, the learning-in-equilibrium is provided in [Fudenberg and Yamamoto, 2011], [Yamamoto, 2014], or [Sugaya and Yamamoto] by employing ex-post equilibria ([Hörner and Lovo, 2009]; [Hörner et al., 2011]).[10]

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yond prisoner's dilemma without additional communication channel. There are papers employing additional communication technology to sustain their desirable outcomes in different settings. For instance, [Kandori, 1992] employs labelling, and [Ellison, 1994] uses public randomization device to restore a [Kandori, 1992]'s kind of robustness in random matching. Also in random matching, [Takahashi, 2010] assumes a retrievable collection of opponent's past history of actions, and [Deb] allows cheap talks. The usage of cheap talks is also seen in [Ben-Porath and Kahneman, 1996], [Tomala, 2011], [Lippert and Spagnolo, 2011], [Ali and Miller, 2016], [Laclau, 2014] and [Laclau, 2012]. [Wolitzky, 2014] considers the usage of tokens.

[8]My off-path belief will be stated in Section 4.3.

[9]On perfect monitoring, readers may refer to [Forges, 1992], [Hart, 1985], [Aumann and Maschler, 1995], [Jordan, 1995], [Cripps and Thomas, 2003], [Gossner and Vieille, 2003], [Hörner and Lovo, 2009], [Hörner et al., 2011], or [Wiseman, 2012].

[10][Fudenberg and Yamamoto, 2011] provides the folk theorem under imperfect public monitoring. [Yamamoto, 2014] shows the individual learning under private monitoring. [Sugaya and Yamamoto] attains

A result in my paper, Lemma 2.1, shows that players will learn the relevant information—whether the number of Rebels exceeds the threshold—in every APEX equilibrium. My main results, Theorem 1 and 2, then show this aspect of learning, and this aspect of learning generally cannot be obtained by employing ex-post equilibria. In other words, the APEX equilibrium constructed in this study is generally not an ex-post equilibrium, and there must be a pair of a threshold k and a network under which no ex-post equilibrium can induce players to learn the relevant information.

2   Model

There is a set of players $N = \{1, ..., n\}$. They constitute a network $G = (V, E)$ so that the vertices are players ($V = N$) and an edge is a pair of them (E is a subset of the set containing all two-element subsets of N). Throughout this paper, G is assumed to be finite, commonly known, fixed, undirected, and connected.[11]

Time is discrete and denoted by $S = \{0, 1, ...\}$ with index s. Each player could be either type R or type I assigned by the nature at $s = 0$ according to a common prior π; R or I represents a Rebel or an Inert respectively. Call $\theta \in \Theta := \{R, I\}^n$ a state of nature. At each $s \geq 1$, players play a normal form game, the *k-threshold game*, infinitely repeated played with common discounted factor $\delta \in (0, 1)$. In the k-threshold game, $A_R = \{\textbf{revolt}, \textbf{stay}\}$ is the set of actions for R and $A_I = \{\textbf{stay}\}$ is that for I. Denote by $\#X$ the cardinality of an set X. A Rebel i's stage-game payoff function is defined as below, while an Inert's stage-game payoff is equal to 1 no matter how other players play.

$$u_R(a_i, a_{-i}) = 1 \quad \text{if } a_i = \textbf{revolt} \text{ and } \#\{j : a_j = \textbf{revolt}\} \geq k$$
$$u_R(a_i, a_{-i}) = -1 \quad \text{if } a_i = \textbf{revolt} \text{ and } \#\{j : a_j = \textbf{revolt}\} < k \ .$$
$$u_R(a_i, a_{-i}) = 0 \quad \text{if } a_i = \textbf{stay}$$

Let $[R](\theta)$ be the set of Rebels given θ and the notion *relevant information* indicate

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[11]the common learning ([Cripps et al., 2008]) as well as the folk theorem under private signalling.

[11]A path in G from i to j is a finite sequence $(l_1, l_2, ..., l_L)$ without repetition so that $l_1 = i$, $l_L = j$, and $\{l_q, l_{q+1}\} \in E$ for all $1 \leq q < L$. G is fixed if G is not random, and G is undirected if there is no order relation over each edge. G is connected if, for all $i, j \in N$, $i \neq j$, there is a path from i to j.

whether or not $\#[R](\theta) \geq k$. Note that the ex-post efficient outcome in the stage game is that every Rebel plays **revolt** whenever $\#[R](\theta) \geq k$, and plays **stay** otherwise.[12]

During the game, every player can observe his and his neighbors' types and his and their histories of actions, but no more. A history of actions played by i from period one to period $s \geq 1$ is denoted by $h_i^s \in H_i^s := \bigtimes_{\varsigma=1}^s A_{\theta_i}$. Let $G_i := \{j : \{i, j\} \in E\}$ be i's *neighbors*. Denote $\theta_{G_i} \in \Theta_{G_i} := \{R, I\}^{G_i}$ as the type profile of i's neighbors. Let $h_i^0 = \emptyset$, and denote $h_{G_i}^s \in H_{G_i}^s := \bigtimes_{j \in G_i} \bigtimes_{\varsigma=1}^s H_j^\varsigma$ as a history of actions played by i's neighbors from period one to period $s \geq 1$. The information set of i about θ at every period is the cylinder $p(\theta) = \{\theta_{G_i}\} \times \{R, I\}^{N \setminus G_i}$, and the information set about histories of action from period one to period $s \geq 1$ is $\{h_{G_i}^s\} \times H_{N \setminus G_i}^s$. A player i's pure behavior strategy τ_i is a measurable function with respect to i's information partition if τ_i maps $\{\theta_{G_i}\} \times \{R, I\}^{N \setminus G_i} \times \{h_{G_i}^s\} \times H_{N \setminus G_i}^s$ to a single action in his action set for every $s \in \{1, 2, ...\}$ and every $\theta \in \Theta$. I assume that payoffs are hidden to emphasize that observing neighbors' actions are the only channel to infer other players' types and actions.[13]

Likewise, define $H^s := \bigtimes_{j \in N} H_j^s$ as the set of histories of actions from period one to period $s \geq 1$ and $H := \bigcup_{\varsigma=0}^\infty H^\varsigma$ as the collection of histories of actions. By abusing the notation a bit, let $h(\tau, \theta) \in H$ denote the realized history of actions generated by strategy profile $\tau = (\tau_1, \tau_2, ..., \tau_n)$ given θ. Designate $\alpha_{G_i}^{\pi,\tau}(\theta, h^s | \theta_{G_i}, h_{G_i}^s)$ as the conditional distribution over $\Theta \times H^s$ induced by π and τ conditional on i's information up to period $s \geq 1$. The belief of i over Θ induced by π and τ up to period $s \geq 1$ is defined by

$$\beta_{G_i}^{\pi,\tau}(\theta | h_{G_i}^s) := \sum_{h^s \in H^s} \alpha_{G_i}^{\pi,\tau}(\theta, h^s | \theta_{G_i}, h_{G_i}^s).$$

The equilibrium concept is the weak sequential equilibrium.[14] My objective is looking for the existence of *approaching ex-post efficient equilibrium* or *APEX equilibrium*, which is defined below.

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[12]Moreover, at every θ and every k, the ex-post efficient outcome is unique and gives the maximum as well as the same payoff to every Rebel.

[13]Such restriction will be relaxed in the Section 5.

[14]A weak sequential equilibrium is an assessment $\{\tau^*, \alpha^*\}$, where α^* is a collection of distributions over players' information sets with the property that, for all $i \in N$ and for all $s = 1, 2, ..., \alpha_{G_i}^*(\theta, h^s | \theta_{G_i}, h_{G_i}^s) = \alpha_{G_i}^{\pi,\tau^*}(\theta, h^s | \theta_{G_i}, h_{G_i}^s)$ whenever the information set is reached with positive probability given τ^*. Moreover,

Definition 2.1 (APEX strategy). *A behavior strategy τ is APEX if for all θ, there is a terminal period $T^\theta < \infty$ such that the actions in h_θ^τ after T^θ repeats the static ex-post Pareto efficient outcome.*

Definition 2.2 (APEX equilibrium). *An equilibrium (τ^*, α^*) is APEX if τ^* is APEX.*

In an APEX strategy, all Rebels will play **revolt** forever after some period only if $\#[R](\theta) \geq k$; otherwise, Rebels will play **stay** forever after some period. It is as if the Rebels will learn the relevant information in the equilibrium because they will play the ex-post efficient outcome after a certain point of time and keep on doing so. Notice that, in an APEX equilibrium, it is not only as if the Rebels will learn the relevant information: they must learn that. Lemma 2.1 articulates this fact.

Lemma 2.1 (Learning in the APEX equilibrium). *If the assessment (τ^*, μ^*) is an APEX equilibrium, then for all $\theta \in \Theta$, there is a finite time T_i^θ for every Rebel i so that*

$$\sum_{\theta \in \{\theta : [R](\theta) \geq k\}} \beta_{G_i}^{\pi, \tau^*}(\theta | h_{G_i}^s) = \text{ either } 1 \text{ or } 0$$

whenever $s \geq T_i^\theta$.

Definition 2.3 (Learning the relevant information). *A Rebel i learns the relevant information at period ς according to strategy τ if $\sum_{\theta \in \{\theta : [R](\theta) \geq k\}} \beta_{G_i}^{\pi, \tau}(\theta | h_{G_i}^s) = $ either 1 or 0 whenever $s \geq \varsigma$.*

It is clear that an APEX equilibrium exists when $k = 1$. As for other cases, let us start with the case of $k = n$ and then continue on to the case of $1 < k < n$. The proof is by construction. In the case of $k = n$, the constructed APEX equilibrium is intuitive and is a sequential equilibrium. My main result tackles the case of $1 < k < n$. In such case, my constructed APEX equilibrium is not trivial and can only works for acyclic networks. Section 5.2 discusses why my constructed equilibrium is intractable in cyclic networks.

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for all $i \in N$ and for all $s = 1, 2, ..., \tau_i^*$ maximizes i's continuation expected payoff of

$$E_G^\delta(u_{\theta_i}(\tau_i, \tau_{-i}^*) | \alpha_{G_i}^{\pi, \tau_i, \tau_{-i}^*}(\theta, h^s | \theta_{G_i}, h_{G_i}^s))$$

conditional on θ_{G_i} and $h_{G_i}^s$ for all $h_{G_i}^s \in H_{G_i}^s$.

3   Equilibrium: APEX for $k = n$

In this section, my objective is to show the existence of APEX equilibrium for the case of $k = n$. In this case, notice that a Rebel can get a better payoff from playing **revolt** than from **stay** *only if* all players are Rebels. Two consequences follow. Firstly, if a Rebel has an Inert neighbor, this Rebel will always play **revolt** in the equilibrium. Secondly, at any period $s \geq 1$, it is credible for every Rebel to punish a deviation by playing **stay** forever *if* there is another one who also plays **stay** forever, independently from the belief held by the punisher. These two features constitute an APEX equilibrium and further transform itself to a sequential equilibrium.

Theorem 1 (APEX equilibrium for the case of $k = n$). *For any n-person repeated k-Threshold game with parameter $k = n$ played in a network, there is a $\delta^* \in (0, 1)$ such that a sequential APEX equilibrium exists whenever $\delta > \delta^*$.*

Image that there are an Inert somewhere as well a Rebel i somewhere. Since the network is connected, there is a path connecting these two players. Along with this path, consider the "closest" Inert from Rebel i; this is an Inert who can be reached by the least number of consecutive edges from i. Note that this Inert's Rebel neighbors will play **stay** forever since $k = n$. Consider a strategy for Rebels on this path: a Rebel will play **stay** only after observing his neighbor plays **stay**. On this path and according to this strategy, Rebel i will know the existence of such Inert eventually since the network is finite. This contagion argument suggests the following APEX strategy. Every Rebel plays **revolt** initially except for he has an Inert neighbor. Each of them will continuously play **revolt** but switch to **stay** instantly if he observes any of his neighbor plays **stay**. Upon observing a n consecutive **revolt**, a Rebels knows that no Inert exists; otherwise, he knows some Inert exists. The above strategy is an APEX strategy if all Rebels play ex-post efficient outcome after peiord n. To extend it to be an APEX equilibrium, let the deviant play **stay** forever and the punisher who detects it also play **stay** forever. This off-path strategy is credible for both the deviant and the punisher, independent from the belief held by the punisher, and hence
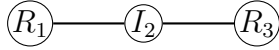
Figure 1: The state and the network in which the APEX equilibrium does not exist when $k = 2$.

it is also sequential rational.[15]

4   Equilibrium: APEX for $1 < k < n$

I show the existence of APEX equilibrium for the case of $1 < k < n$ in this section. In contrast to the case of $k = n$, a Rebel still has the incentive to play **revolt** even if he has an Inert neighbor. This opens a possibility for the non-existence of APEX equilibrium. Example 1 below demonstrates it.

Example 1. Suppose $k = 2$ and $\theta = (R, I, R)$. The state and the network is represented in Figure 1. Rebel 1 never learns the type of player 3 since Inert 2 cannot reveal it, and hence no APEX equilibrium exists in this scenario.

The following assumption on the prior—*full support on strong connectedness*—excludes the possibility of nonexistence of APEX equilibrium. To this end, I begin with the definition of *strong connectedness.*

Definition 4.1 (Strong connectedness)**.** *Given G, a state θ has strong connectedness if, for every two Rebels, there is a path consisting of Rebels to connect them.*

In the language of graph theory, the following definition is equivalent: given G, θ has strong connectedness if the induced graph by $[R](\theta)$ is connected.

Definition 4.2 (Full support on strong connectedness)**.** [16] *Given G, π has full support on*

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[15]This sequential rationality is in the sense of [Kreps and Wilson, 1982].

[16]The prior has full support on strong connectedness is stronger than every state has strong connectedness is common knowledge. This marginal requirement is convenient in constructing equilibrium. The main result only requires a weaker version—$\pi(\theta) > 0 \Rightarrow \theta$ has strong connectedness . Working on this weaker version is at the expense of much tedious proof, however.

strong connectedness if

$$\pi(\theta) > 0 \Leftrightarrow \ \theta \ has \ strong \ connectedness.$$

I state the main characterization of this paper:

Theorem 2 (APEX equilibrium for the case of $1 < k < n$). *For any n-person repeated k-Threshold game with parameter $1 < k < n$ played in networks, if networks are acyclic and if π has full support on strong connectedness, then there is a $\delta^* \in (0, 1)$ such that an APEX equilibrium exists whenever $\delta > \delta^*$.*[17]

Constructing an APEX equilibrium in this case is convoluted. I illustrate the proof idea throughout this paper while leaving the formal proof in Appendix. Moreover, since the case of $k = 2$ is trivial given that θ has strong connectedness, I focus on $2 < k < n$ cases.[18]

I start with considering a specific APEX strategy as the framework in constructing an APEX equilibrium in Section 4.1; incentive compatibility is not incorporated at this moment. Then I introduce an auxiliary scenario to illuminate the discussion of incentive compatibility in Section 4.2. I then go back to present essential details of the constructed APEX equilibrium in Section 4.3.

4.1 APEX strategy

To begin, the following example is examined.

Example 2. Let $k = 5$ and consider the configurations in Figure 2 and Figure 3. Note that there are 6 players in both of these configurations; 5 Rebels are in Figure 2 and 4 Rebels are in Figure 3. The below is an APEX strategy.

Strategy for Example 2:

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[17]A network is acyclic if the path from i to j for all $i \neq j$ is unique.

[18]Suppose $[R](\theta) \geq k = 2$, by the full support on strong connectedness, each Rebel have a Rebel neighbor. The following strategy is an APEX strategy. A Rebel plays **revolt** from period one afterwards if he has a Rebel neighbor; otherwise, he plays **stay** forever from period one. It extends to an APEX equilibrium by letting the off-path belief be assigning probability one to the event that all non-neighbors are Inerts.

1. At period j, $1 \leq j \leq 6$, each Rebel plays **revolt** if himself is j or any of his Rebel neighbors is j; otherwise, he plays **stay**. For instance, from period one to period 6, what Rebel 3 plays is (**revolt**, **stay**, **revolt**, **revolt**, **stay**, **revolt**) in the configuration in Figure 2. Consequently, implied by Rebel 3's actions, in the configuration in Figure 2, right after period 1, 2, 3, or 4, Rebel 4 is still aware the existence of Rebel 1, 2, 3, and 4; right after period 5, Rebel 4 is aware that player 5 is an Inert; right after period 6, Rebel 4 is aware that player 6 is a Rebel. Similar behavior applies to other players.

2. Right after period 6, at period 7, by observing his neighbors' actions, a Rebel who has learnt the number of Rebels exceeding or equal to k plays (**revolt**, **revolt**) consecutively. Then he plays **revolt** from period 19 afterwards. Likewise, a Rebel who has observed the number of Rebels, which is implied by his neighbors' actions, is less than k, he plays (**stay**, **stay**) consecutively. Then he plays **revolt** from period 19 afterwards. A Rebel whose observation does not belong to the above two categories plays the sequence (**revolt**, **stay**) consecutively and plays **stay** forever after period 19.

3. From period 7 to period 19, a Rebel plays (**revolt**, **revolt**) consecutively right after observing the same sequence played by any of his neighbor; he then plays **revolt** from period 19 afterwards. Correspondingly, a Rebel plays (**stay**, **stay**) consecutively right after observing the same sequence played by any of his neighbor and then plays **stay** from period 19 afterwards.

4. The off-path strategy is simple: ignoring the deviation and following the on-path strategy in which more Rebels can be implied.

By following the on-path strategy of the above, both Rebel 1 and 4 learn the relevant information right after period 7, and all Rebels play the ex-post outcome from period 19 afterwards. This strategy is thus an APEX strategy with a straightforward idea: from period one to period 6, "a reporting phase", Rebels reports who their Rebel neighbors are by sequences of actions, and, from period 7 to period 19, "a coordination phase", they
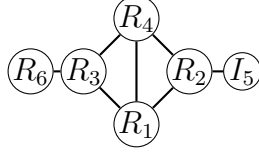
Figure 2: A configuration of the state and the network in which players 1, 2, 3, 4, 5 are Rebels and players 6 is an Inerts.
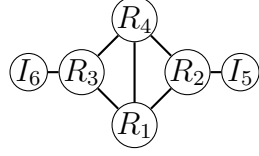

Figure 3: A configuration of the state and the network in which players 1, 2, 3, 4 are Rebels and player 5, 6 are Inerts.

coordinate themselves to disseminate the knowledge whether the relevant information is learnt.

The above strategy, nevertheless, calls for revising to seemly integrate incentive compatibility. Notice that Rebel 6's actions are redundant: the ex-post efficient outcome will be achieved anyway despite how Rebel 6 plays. This is because Rebel 3's neighbors include Rebel 6's and how Rebel 6 plays cannot be observed by other Rebels outside Rebel 3's neighborhood. The notion "redundant" will be replaced by and formally defined as *inactive* later to emphasize that Rebel 6 can actually report nothing. What is more, Rebel 4 has no incentive to report. This is because Rebel 1 reports the identical information to the same Rebels. In other words, the ex-post efficient outcome will be played anyway despite how Rebel 4 plays. Rebel 1's position is exactly the same as Rebel 4's, and thus both of them have no incentive to report. A free-rider problem exists.[19]

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[19]It is worth noting that whether a free-rider problem could happen is not solely determined by whether a network is cyclic. It depends on how an APEX strategy is constructed and exists in the constructed APEX equilibrium for Theorem 2, in which only acyclic networks are of concern. The formal definition of free-rider problem is in Section 4.3, and a deeper issue will be discussed in Section 5.2.

As follows, a revision of Strategy for Example 2 is designed and meant to be the foundation framework in constructing the APEX equilibrium for Theorem 2. Let $\mathbf{I}$ be all possible sets of Rebels amongst states in the network.[20] For convenience, construct a set W that consists of sequences of actions, in which all sequences have equal length, so that there is a one-to-one mapping between $\mathbf{I}$ and W. This W exists. For an example, which is used in the constructed APEX equilibrium for Theorem 2, let the length of each sequence in W be the multiplication of a series of prime numbers. In this series, each prime number is distinct and assigned to a distinct player. Denote x_i as the prime number assigned to i. The length of a sequence in W is therefore $\bigotimes_{j \in N} x_j = x_1 \otimes ... \otimes x_n$, where $\otimes$ is the usual multiplication operator. If $I \in \mathbf{I}$, the corresponding $\langle I \rangle \in W$ is crafted to be:

$$\overbrace{(\mathbf{stay}, ..., \mathbf{stay}, \underbrace{\mathbf{revolt}, \mathbf{stay}, ..., \mathbf{stay}}_{\bigotimes_{j \in I} x_j})}^{\bigotimes_{j \in N} x_j}.$$

A one-to-one mapping between $\mathbf{I}$ and W exists since a multiplication of prime numbers can be uniquely factorized.

Fix W and partition the time into two consecutively alternating phases: *coordination phase* and *reporting phase*, while the time is starting from the coordination phase. The t-th completion of two consecutive phases is called the t-block. The length of each reporting phase is equal to the length of $w \in W$, and the length of each coordination phase is twice of the total number of players. Figure 4 depicts this partition.[21]

Given θ, denote the set I_i as i's Rebel neighbors. If i is a Rebel, let $I_i^1 = I_i$, and let $I_i^t = \bigcup_{j \in G_i} I_j^{t-1}$ for $t \geq 2$. If j is an Inert, let $I_j^t = \emptyset$ for $t \geq 1$. Put it differently, I_i^t is the set of Rebels who can be reached from Rebel i by a path consisting of Rebels and of which the length is at most t. Let us then identify a set of Rebels, *active Rebels*, who are crucial in the information sharing process. For this purpose, first define G_i^t for each t: if i is a Rebel,

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[20]To be precise, $\mathbf{I} = \{X \subseteq N : \theta_j = R \Leftrightarrow j \in X \text{ for some } \theta \in \Theta\}$

[21]More precisely, partition the time by $\{\{0\}, \{1, ..., s_{l_1}, s_{l_1}+1, ...s_1\}, \{s_1+1, ..., s_{l_2}, s_{l_2}+1, ..., s_2\}, ..., \{s_{t-1}+1, ..., s_{l_t}, s_{l_t}+1, ..., s_t\}, ...\}$, where $t = 1, 2, ...$ and $s_0 = 0$, so that the length of $\{s_{l_t}+1, ..., s_t\}$ is equal to the length of $w \in W$ for each t, while the length of $\{s_{t-1}+1, ..., s_{l_t}\}$ is $2n$. Call $\{s_{t-1}+1, ..., s_t\}$ the t-block. call $\{s_{l_t}+1, ..., s_t\}$ the *reporting phase* at t-block and call $\{s_{t-1}+1, ..., s_{l_t}\}$ the *coordination phase* at t-block.

$$0 < \underbrace{(\text{coordination phase}) < (\text{reporting phase})}_{1\text{-block}} < \underbrace{(\text{coordination phase}) < (\text{reporting phase})}_{2\text{-block}} < \ldots$$

Figure 4: The partition of the time in the repeated k-threshold game. $<$ is the linear order relation over the time.

let $G_i^1 = G_i$, and let $G_i^t = \bigcup_{j \in G_i} G_j^{t-1}$ for $t \geq 2$; if j is an Inert, let $G_j^t = \emptyset$ for $t \geq 1$. That is, G_i^t is the set of players who can be reached from Rebel i by a path consisting of Rebels and of which the length is at most t. Finally, define the active Rebels at t-block as follows.

Definition 4.3 (Active Rebel at t-block). *Set $R^0 = [R](\theta)$. The set of active Rebels in the t-block is*

$$R^t := \{i \in R^{t-1} : \nexists j \in G_i \text{ such that } I_i^t \subseteq G_j^t\}.$$

Otherwise speaking, an active Rebel in the t-block is a Rebel whose information about θ, I_i^t, is not a subset of any other Rebel's same information. In the configuration in Figure 2 for Example 2, the set of Rebels is $\{1, 2, 3, 4, 6\}$, the active Rebels in the 1-block are Rebel 2 and 3, and no Rebel is active in the t-block when $t \geq 2$. Notice that the active Rebels have to be also the active ones in the previous block; they are fewer and fewer as t goes by. Let us call a Rebel *inactive Rebel* if he is not an active Rebel.

It is sufficient to reveal the relevant information by letting only active Rebels share information about θ, given that the network is acyclic and given that θ has strong connectedness. Theorem 3 articulates this.

Theorem 3. *If the network is acyclic and if the θ has strong connectedness, then there is a strategy so that there exists an active Rebel in the t-block who can learn the relevant information at $t + 1$-block.*

Note that $I_i^t \in \mathbf{I}$. The following strategy is for Theorem 3.[22]

<div align="center">

Strategy for Theorem 3:

</div>

---

[22]Theorem 3 is equivalent to the following statement: if the network is acyclic and if the θ has strong connectedness, then there exists $t \geq 0$ and $i \in R^t$ so that $I_i^{t+1} = [R](\theta)$.

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1. In each reporting phase in the t-block, each active Rebel i at t-block plays $\langle I_i^t \rangle \in W$.

2. At each period in each coordination phase in the t-block, all Rebels play **stay**.

Proof. Here, I show the proof that there is t and there exists Rebel i who can learn the relevant information in the $t+1$-block. The proof that i has to be active in the t-block is in Appendix. Following the above strategy, right after t-block, Rebel i assigns probability one to the event

$$\{\theta \in \Theta : \theta_j = R \Leftrightarrow j \in I_i^{t+1}\}$$

by Bayesian rule. To conclude the proof, it is sufficient is to show there exists a t so that $I_i^{t+1} = [R](\theta)$. By definition, I_i^t is the set of Rebels who can be reached by at most t consecutive edges from Rebel i, in each of which the endpoints are Rebels. Since θ has strong connectedness, there exists a t_i so that $I_i^{t_i} = [R](\theta)$. What follows is i learns θ at t_i. $\qquad\square$

Remark. Theorem 3 is not true if the network is cyclic. Take the configuration in Figure 2 for Example 2 as an example. By following Strategy for Theorem 3, in the 2-block, there is no Rebel who learns the relevant information. Then, in the 3-block, either Rebel 1 or 4 learns the relevant information, but both Rebel 1 and 4 are inactive at 2-block. Theorem 3 thus captures a free-rider problem in cyclic networks as the above mentioned: Rebel 1 and Rebel 4 are redundant mutually.

Modifying Strategy for Theorem 3, Strategy for Proposition 4.1 below is an APEX strategy.

Proposition 4.1. *If the network is acyclic and if the θ has strong connectedness, then there is an APEX strategy.*

Consider the following strategy.

<div align="center">

Strategy for Proposition 4.1:

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1. Suppose T^θ has arrived, Rebels play the ex-post efficient outcome.

2. Suppose T^θ has not yet arrived. In the reporting phase, Rebels follow Strategy for Theorem 3. In the coordination phase, as below, the strategy is a contagion process.

   (a) If a Rebel has been certain that there are k or more Rebels, he plays sequence of actions (**stay**, **stay**) continuously starting right after he was certain that. He plays **stay** forever after this phase; T^θ arrives right after this phase.

   (b) If a Rebel has learnt that there are less Rebels than k Rebels, he plays sequence of actions (**revolt**, **revolt**) continuously starting right after he learnt that. He plays **revolt** forever after this phase; T^θ arrives right after this phase.

   (c) If a Rebel has observed the sequence of actions (**stay**, **stay**), he plays (**stay**, **stay**) continuously starting right after he observed that and plays **stay** forever after this phase; T^θ arrives right after this phase.

   (d) If a Rebel has observed the sequence of actions (**revolt**, **revolt**), he plays (**revolt**, **revolt**) continuously starting right after he observed that and plays **revolt** forever after this phase; T^θ arrives right after this phase.

   (e) If a Rebel has not learnt the relevant information, he plays sequence of actions (**revolt**, **stay**) continuously.

This is a simple extension of Strategy for Example 2. Rebels share information in reporting phase. If a Rebel has learnt the relevant information, he disseminates it to all Rebels contagiously in coordination phase; otherwise, he continues to the next phase—a reporting phase.

If incentive compatibility is under consideration, this logic, however, brings another free-rider problem. Suppose that there are two Rebels who share information to each other in a reporting phase, and each of them is certain that he will learn the relevant information if the other one shares *truthful* information to him. Because sharing information is done by alternating actions, which incurs expected payoff, they will not truthfully share their information. This is because each of them will choose his most profitable way of sharing information without impeding learning the relevant information provided that the other one share the truthful information. The free-rider problem turns out to be the main challenge

in the construction of an APEX equilibrium. The proof solves it by arguing that if the network is acyclic, the free-rider problem only occurs between two Rebel neigbors who *commonly know it*, while this argument does not hold for cyclic network.[23] With the help from this argument, the constructed equilibrium solves the free-rider problem by arbitrarily assigning one of them to be the free rider, who can choose his most profitable way in sharing information, while letting the other one share truthful information.

To make the discussion of incentive compatibility more transparent, I introduce *T-round writing game* as an auxiliary scenario. In the T-round writing game, T is fixed, players are endowed a writing technology so that they can write to share information about θ for T rounds. They then play a one-shot k-threshold game at round $T + 1$. This game is a reduced form of the original game by fixing T, so that I can pay attention only to the incentive compatibility in information sharing process and ignore the incentive compatibility in the coordination phase.[24] This writing technology represent a way of communication by *writing sentences that are composed by letters according to a fixed grammar*. Though this auxiliary game will be soon introduced in the next section, I draw Table 1 to shed light on the parallel between it and the original game.

4.2 T-round writing game

The network, the set of states, and the set of players follow exactly the same definitions defined in Section 2. In the T-round writing game, each player endows a *writing technology*. A writing technology for player i is a pair of (W, M_i), in which $W = \{\mathbf{r}, \mathbf{s}\}^L$, $L \in \mathbb{N}$, and $M := \bigtimes_{i \in N} \bigtimes_{t=1}^{T} M_i^t$ recursively defined by

$$M_i^1 = \{f | f : \Theta_{G_i} \to W\} \cup \{\emptyset\}$$

$$\text{for } 2 \leq t \leq T \ , \ M_i^t = \{f | f : \bigtimes_{j \in G_i} M_j^{t-1} \to W\} \cup \{\emptyset\}.$$

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[23]Section 5.2 provides an example that the free-rider problem is not commonly known between the Rebels who involve.

[24]There is another free-rider problem in the coordination phase in Strategy for Theorem 3: a Rebel who has learnt the relevant information might deviate to play (**revolt**, **stay**) from playing (**revolt**, **revolt**). This strategy will be further modified to incorporate the incentive compatibility in Section 4.3.2.

Table 1: The analogue between T-round writing game and the repeated k-threshold game

T-round writing game	Repeated k-threshold game
A round	A range of periods
A sentence	A sequence of actions
The length of a sentence in a round	The length of a range of periods
A chosen letter in a sentence	A chosen action
The cost of writing a sentence	The expected payoff occurring in a sequence of actions
The fixed grammar	The equilibrium path

W is interpreted as the set of sentence composed by letters $\mathbf{r}$ or $\mathbf{s}$ with length L, while M_i is understood as i's grammar. $\emptyset$ represents remaining silent. The phrase of "i writes a sentence to all his neighbors at round t" is equivalent to "i selects an $f \in M_i^t$ to get an element $w \in W$ according to f, which can be observed by all i's neighbors".

The time line for the deterministic T-round writing game is as follows.

1. Nature chooses θ according to the prior π.

2. θ is then fixed throughout rounds.

3. At $t = 1, ..., T$ round, players write to their neighbors.

4. At $T + 1$ round, players play a one-shot k-Threshold game.

5. The game ends.

A Rebel's payoff is the summation of his stage payoff across stages, while an Inert's payoff is set to be 1. The equilibrium concept is weak sequential equilibrium. An APEX strategy is a strategy that induces the ex-post outcome in the one-shot k-threshold game at $T + 1$ round. The definition of APEX equilibrium is adapted accordingly. In the examples below, let us focus on the configuration represented in Figure 5 and Figure 6 with $n = L = 8$. I.e. there are 8 players and the length of a sentence is also 8. Note that the differences
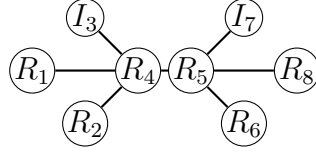
Figure 5: A configuration of the state and the network in which players 1, 2, 4, 5, 6, 8 are Rebels while players 3, 7 are Inerts.
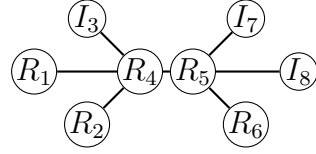

Figure 6: A configuration of the state and the network in which players 1, 2, 4, 5, 6 are Rebels while players 3, 7, 8 are Inerts.

between configurations in Figure 5 and Figure 6 are: (1) there are 6 Rebels in Figure 5 but are 5 Rebels in Figure 6; (2) player 8 is a Rebel in Figure 5 but he is an Inert in Figure 5.

Example 3 (T-round writing and the free-rider problem)**.** Let $k = 6$ and $T = 2$. Suppose that remaining silent incurs no cost, but writing incurs an extremely small cost $\epsilon > 0$ so that ϵ is strictly decreasing with the number of $\mathbf{r}$ in a sentence. This is to say writing $(\mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r})$ incurs the least cost while writing $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{s})$ incurs the largest.[25]

Let us consider the following strategy ϕ.

At $t = 1$, the peripheral Rebels remain silent. Rebel 4 (or 5)'s grammar for writing a sentence is that if player i is a Rebel and known to him, he writes $\mathbf{r}$ in the i-th component in the sentence; otherwise, he writes $\mathbf{s}$ in that component. According to this grammar, the central player Rebel 4 writes the sentence $(\mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s}, \mathbf{s})$ on both configurations in Figure 5 and in Figure 6. The central player Rebel 5 writes $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r})$ in the configuration in Figure 5 and writes $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s})$ in the configuration in Figure 6.

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[25]This cost is not directly linking to the expected payoff in the original game, but is meant to be a metaphor to emphasize that alternating actions might incurs negative expected payoff in the original game and to illustrate the potential free-rider problem.

Rebels 4 and 5's sentences thus reveals who are Rebels and who are not. Notice that the common knowledge of the network contributes to the ability of revealing players' types.

At $t = 2$, the peripheral Rebels still remain silent. Rebel 4 writes $(\mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r})$ in the configuration in Figure 5 and writes $(\mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s})$ in the configuration in Figure 6. Rebel 5 writes exactly the same sentence as Rebel 4. This is to say Rebel 4 and 5 share information at $t = 1$ and then coordinate to announce a mixture sentence at $t = T$.

At $T + 1$, by counting $\mathbf{r}$ in Rebel 4 or 5's mixture sentence, all Rebels know whether the number of Rebels outnumbers k. This leads all Rebels to play the ex-post efficient outcome in the one-shot k-threshold game.

The above ϕ is not an APEX equilibrium. According to the above-mentioned, Rebel 4 will know the relevant information at $t = 2$ even if he deviates to writing that all his neighbors are Rebels, which incurs less cost than his truthful writing.[26][27] Rebel 5 is in the same situation as Rebel 4 and therefore also writes the sentence that indicates that all his neighbors are Rebels. However, these sentences are uninformative. It turns out that both of them will deviate, and neither of them can know the relevant information at $t = 2$.

Fortunately, the following example shows that the free-rider problem can be solved.

Example 4 (T-round writing and solving the free-rider problem)**.** The solution to solve the free-rider problem in the previous example is to extend T. It would open the possibility of the existence of a free rider at some round, while letting this free rider reveals relevant information at the next round. To this end, let $k = 6$ and $T = 3$. Consider the following strategy ρ and focus on the interaction between Rebels 4 and 5.

At $t = 1$, the lowest-index Rebel between Rebels 4 and 5 is the free rider, while the other one truthfully writes down his information. This is to say Rebel 4 will be the free rider and he writes the least-cost sentence. Rebel 5 writes $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r})$ in the configuration of Figure 5and writes $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s})$ in the configuration of Figure 6.

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[26]This sentence is $(\mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s}, \mathbf{s})$, which incurs less cost than the truthfully reporting sentence $(\mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s}, \mathbf{s})$.

[27]If he remains silent, then this behavior will be considered as a deviation, and therefore he will never get the maximum payoff of 1. Hence, he will avoid doing so.

At $t = 2$, Rebel 4 has known the relevant information. Rebel 4 writes the least-cost sentence if there are k or more Rebels but remains silent otherwise. The consequence is Rebel 4's behavior reveals the relevant information to his neighbors at this round. Rebel 5 remains silent instead.

At $t = T$, Rebel 5 has known the relevant information since he is Rebele 4's neighbor. He writes the least-cost sentence if there are k or more Rebels but remains silent otherwise. Therefore Rebel 5's behavior at this round reveals the relevant information to his neighbors. Rebel 4 remains silent instead.

At $T+1$, all Rebels know the relevant information by observing Rebels 4 and 5's behavior. They play the ex-post efficient outcome accordingly.

Remark. Why does Rebel 5 *know* that he is not a free rider and therefore behaves not like a free rider? The following is the reason. He *knows* that, by common knowledge of the network, he and Rebel 4 are in a free-rider problem. Moreover, by common knowledge of the network, he knows that Rebel 4 knows that he and Rebel 4 are in a free-rider problem, he knows that Rebels 4 knows that he knows that,...,and so forth. Consequently, Rebel 5 and 4 commonly know that they are engaged in a free-rider problem.

In the next section, Section 4.3, I delver into substantial ideas in constructing an APEX equilibrium and leave actual details in Appendix. Separately, I depict the construction in the reporting phase and in the coordination phase. The depiction might seem repetitive, but deliberately tailored. Especially, so far, the attention is devoutly paid to integrating incentive compatibility in the reporting phase, but in the lack of the coordination phase. Section 4.3.2 manages the incentive compatibility in the coordination phase.

4.3 Equilibrium construction

The framework is adapted from Proposition 4.1, and the time is partitioned by two consecutively alternating phases as shown in Figure 4. Recall that the time is partitioned into

$$0 < \underbrace{(\text{coordination phase}) < (\text{reporting phase})}_{\text{1-block}} < \underbrace{(\text{coordination phase}) < (\text{reporting phase})}_{\text{2-block}} < ...$$

The off-path belief is simple and serves as a grim trigger. Whenever Rebel i detects a deviation at period ς, he forms the following belief:

$$\sum_{\theta \in \{\theta : \theta_j = I, j \notin G_i\}} \beta_{G_i}^{\pi,\tau}(\theta | h_{G_i}^s) = 1, \text{ for all } s \geq \varsigma. \tag{1}$$

This is to say i believe all players outside his neighborhood are Inerts. Thus, if $\#I_i^\varsigma < k$, he will play **stay** forever after he detects a deviation.

4.3.1 The equilibrium path in the reporting phase

The description in this section is for the APEX equilibrium path before T^θ. Let us shorten "reporting phase in t-block" by $\mathbf{O}^t$, denote $|\mathbf{O}^t|$ as the length of $\mathbf{O}^t$, and shorten **revolt** and **stay** to $\mathbf{r}$ and $\mathbf{s}$ receptively. For simplicity, the term "reporting phase" refers to "reporting phase in t-block" in this section.

$|\mathbf{O}^t|$ is independent from t and determined only by the set of players. Firstly, assign each player i a distinct prime number x_i starting from 3. Then let $|\mathbf{O}^t| = \bigotimes_{i \in N} x_i = x_1 \otimes x_2 \otimes ... \otimes x_n$, where $\otimes$ is the usual multiplication operator. The sequence of actions in $\mathbf{O}^t$ is with length $|\mathbf{O}^t|$ and would take one of the forms specified in the right column in Table 2. The abbreviations of these sequences are listed in the left column. Since these sequences in the reporting phase are meant to share information about θ, the terms "playing the sequence" and "reporting the information" are interchangeable and will be alternatively used.

Table 2: The notations for the sequences of actions in $\mathbf{O}^t$ on the path

Notations		The sequences of actions
$\langle I \rangle$	:=	$(\overbrace{\mathbf{s}, ..., \mathbf{s}, \underbrace{\mathbf{r}, \mathbf{s}, ..., \mathbf{s}}_{\bigotimes_{j \in I} x_j}}^{\bigotimes_{i \in N} x_i})$
$\langle 1 \rangle$	:=	$(\overbrace{\mathbf{s}, ..., \mathbf{s}, \mathbf{r}}^{\bigotimes_{i \in N} x_i})$
$\langle \text{all stay} \rangle$	:=	$(\overbrace{\mathbf{s}, ..., \mathbf{s}, \mathbf{s}}^{\bigotimes_{i \in N} x_i})$

It is worth noting that the sequence constructed by prime numbers brings two benefits. Firstly, since the multiplication of distinguishing prime numbers can be uniquely factorized, the Rebels can use such sequence to precisely report players' identities. Secondly, the undiscounted expected payoff of playing $\langle I_i^t \rangle$ for some I_i^t for an active Rebel i is always equal to -1, and therefore it is relatively easy to calculate. This is because only active Rebels will report $\langle I \rangle$ for some I. Since this I is not reported by any other Rebel, at most one Rebel would play **r** at any period in the reporting phase by the property of prime number multiplication.[28]

I list the sequences played in the reporting phase on the path in Table 3. The terms θ-pivotal, $k - 1$-pivotal, and free-rider problem will be defined one-paragraph later.

Table 3: The sequences of actions played in $\mathbf{O}^t$ on the path

Rebel i	i plays
i is inactive	$\langle \textbf{all stay} \rangle$
i is active but i is not pivotal	$\langle I_i^t \rangle$
i is $k - 1$-pivotal	$\langle 1 \rangle$
i is θ-pivotal but not in the free-rider problem	$\langle 1 \rangle$
i is in the free-rider problem with the lowest index	$\langle 1 \rangle$
i is in the free-rider problem without the lowest index	$\langle I_i^t \rangle$

On the path, the sequences $\langle I \rangle$ or $\langle 1 \rangle$ are meant to differentiate themselves from $\langle \textbf{all stay} \rangle$. The sequence $\langle \textbf{all stay} \rangle$ is for the inactive Rebels at t-block to report nothing. The sequence $\langle I \rangle$ is for active Rebels at t-block to report I if I is a set of Rebels. Although the definitions of *pivotal Rebel* and *free-rider problem* has not yet formally defined at this present, the sequence $\langle 1 \rangle$ is intentionally crafted to tackle the free-rider problem. To see how $\langle 1 \rangle$ works, I turn to formally defining the pivotal Rebel and the free-rider problem.

Definition 4.4 (Pivotal Rebels in $\mathbf{O}^t$). *A Rebel p is pivotal in $\mathbf{O}^t$ if p is uncertain the*

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[28]This statement holds if there is no Rebel who plays $\langle 1 \rangle$.
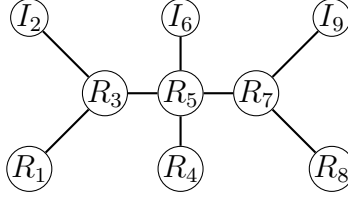
Figure 7: A configuration of the state and the network in which player 1, 3, 4, 5, 7, 8 are Rebels while players 2, 4, 9 are Inerts.
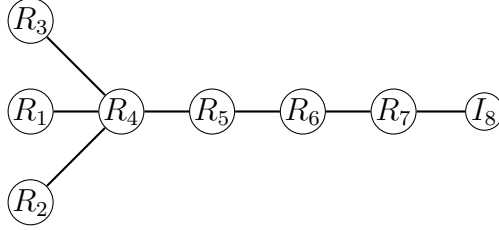

Figure 8: A configuration of the state and the network in which player 1, 2, 3, 4, 5, 6, 7 are Rebels while player 8 is an Inert.

relevant information at t-block, and p is certain that he will learn the relevant information right after $\mathbf{O}^t$, *given that each* $i \in R^t$ *reports* $\langle I_i^t \rangle$.

By the definition, a pivotal Rebel in the reporting phase is one who can learn the relevant information if all of his active Rebel neighbors truthfully report their information about θ to him. The pivotal Rebels can be further classified into two kinds: ones who can learn θ, and ones who learn only the relevant information. When $k = 6$, in the configuration in Figure 5, only Rebels 4 and 5 are pivotal; they are of the first kind. In the configuration in Figure 7, only Rebel 5 is pivotal; he is of the first kind. In the configuration in Figure 8, only Rebel 4 pivotal; he is of the second kind.

Call p of the first kind the θ-*pivotal* Rebel. For the second kind, if the network is acyclic and if the prior has full support on strong connectedness, p is the second kind only if $I_p^t = k - 1$. Call this second kind $k - 1$-*pivotal* Rebel.[29] Below is the defined free-rider

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[29]To show that a pivotal Rebel is the second kind in $\mathbf{O}^t$ only if $I_p^t = k - 1$, one can follow the same argument in Theorem 3.

problem in the reporting phase.

Definition 4.5. *A free-rider problem exists in $\mathbf{O}^t$ if there are multiple θ-pivotal Rebels in $\mathbf{O}^t$.*

The following lemma is crucial.

Lemma 4.1. *If the network is acyclic and if π has full support on strong connectedness, there are at most two active θ-pivotal Rebels in the t-block. Moreover, they are neighbors.*[30]

And notably,

Lemma 4.2. *If the network is acyclic and if π has full support on strong connectedness, when there are two active θ-pivotal Rebels $p, p^{'}$ in the t-block, then they commonly know that they are θ-pivotal Rebels at the beginning of t-block.*

By Lemma 4.2, θ-pivotal Rebels in the reporting phase can identify themselves at the beginning of this phase. This importance cannot be further emphasized. If the free-rider problem will occurs in the reporting phase, the strategy can, before the reporting phase, prescribes that the lowest indexed θ-pivotal Rebel p involving in the free-rider problem will play $\langle 1 \rangle$, while the other one will play $\langle I_{p'}^t \rangle$. Otherwise speaking, this knowledge is encoded in the belief system of an APEX equilibrium.

Remark. As the above mentioned, the assumption of acyclic network in Lemma 4.2 is indispensable. Lemma 4.2 does not hold if the network is cyclic. Section 5.2 demonstrates it.

4.3.2 The equilibrium path in the coordination phase

The descriptions in this section is for the APEX equilibrium path before T^θ. The term "coordination phase in t-block" is shorten by $\mathbf{C}^t$.

It is elaborate to spell out the coordination phase structure, but this phase is actually a simple contagion scenario: Rebels jointly decide to terminate or continue their information

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[30]As a remark, Lemma 4.1 is not true when the network is cyclic. To see this, consider a 4-player circle given that $\theta = (R, R, R, R)$.

sharing during this phase. For that, a coordination phase is partitioned into three *divisions*. In the first division, if there is a Rebel has learnt that $\#[R](\theta) < k$, all Rebels will play **stay** forever right after this division, and T^θ arrives; otherwise, they continue to the next one. In the second division, if there is a Rebel has learnt that $\#[R](\theta) \geq k$, a portion of Rebels, at least k Rebels, will play **revolt** forever right after this division; otherwise, they continue to the next one. In the third division, if there is a Rebel has learnt that $\#[R](\theta) \geq k$ in the previous divisions, all Rebel will play **revolt** forever right after this division, and T^θ arrives; otherwise, they continue to the next phase—a reporting phase.

To fulfil the above contagion argument, a set of sequences of actions played on the path is prescribed so that Rebels update their belief according to it. For this task, further partition a division into *sub-blocks*. I depict the whole partition in the coordination phase below, where □ represents a sub-block in a coordination phase.

In $\mathbf{C}^1$,
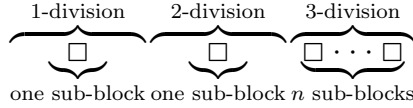
In $\mathbf{C}^t$, $t \geq 2$,
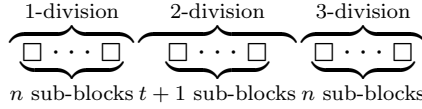
For each t, denote $\mathbf{C}^t(u, \cdot)$ as the u-division and $|\mathbf{C}^t(u, \cdot)|$ as the length of $\mathbf{C}^t(u, \cdot)$. Likewise, denote $\mathbf{C}^t(u, v)$ as the v-th sub-block in u-division and $|\mathbf{C}^t(u, v)|$ as the length of $\mathbf{C}^t(u, v)$. Let us shorten **revolt** and **stay** to **r** and **s** receptively.

Let

$$|\mathbf{C}^t(u, v)| = \begin{cases} n & \text{if} \quad u = 1, 2, v = 1, ..., n. \\ 1 & \text{if} \quad u = 3, v = 1, ..., n. \end{cases}$$

Let $|\mathbf{C}^t(u, v)| = n$ if $u = 1, 2$, $v = 1, ..., n$. Let $|\mathbf{C}^t(u, v)| = 1$ if $u = 3$, $v = 1, ..., n$. I list the the sequences of actions on the path and their notations in Table 4, where i is the original index of player i.

Table 4: The notations for the sequences of actions in $\mathbf{C}^t(u,v)$ for $u = 1,2$, $v = 1,...,n$, on the path

Notations		The sequences of actions
$\mathbf{s}$	:=	$(\mathbf{s})$
$\mathbf{r}$	:=	$(\mathbf{r})$
$\langle i \rangle$	:=	$(\overbrace{\mathbf{s},...,\mathbf{s},\underbrace{\mathbf{r},\mathbf{s},...,\mathbf{s}}_{i}}^{n})$
$\langle \mathbf{all\ stay} \rangle$	:=	$(\overbrace{\mathbf{s},...,\mathbf{s},\mathbf{s}}^{n})$

4.3.3 The equilibrium behavior on the path in $\mathbf{C}^1$

I begin with depicting the equilibrium path in $\mathbf{C}^1$, which is shown in Table 5. The belief updating after $\mathbf{C}^1(1,\cdot)$ and $\mathbf{C}^1(2,\cdot)$ on the path is listed in Table 6. The evolution of players' information filtrations can be tracked throughout in this table. If there is no confusing, the term "coordination phase" refers to "coordination phase in the 1-block" in this section.

In the first division in the coordination phase, if Rebel i is certain $\#[R](\theta) < k$, i play $\langle \mathbf{all\ stay} \rangle$. It implies that i is certain that there is no Rebels outside G_i and therefore learns θ by strong connectedness. This is to say all Rebels are i's neighbors and thus all i's Rebel neighbors are inactive. Since $\langle \mathbf{all\ stay} \rangle$ is played only by an inactive Rebel or a Rebel who is certain $\#[R](\theta) < k$, all i's Rebel neighbors learn $\#[R](\theta) < k$ right after the first division in the coordination phase. Since all Rebels are i's neighbors, all Rebels learn $\#[R](\theta) < k$ right after the first division in the coordination phase. T^θ arrives then. Likewise, if Rebel i is inactive and all i's neighbors play $\langle \mathbf{all\ stay} \rangle$, all Rebels learn $\#[R](\theta) < k$ right after the first division in the coordination phase, and T^θ arrives.

In the second division in the coordination phase, there is a non-trivial construction to let Rebel i disseminates the knowledge about $\#[R](\theta) \geq k$ if i has learnt that. Rebel i does so by playing $\langle i \rangle$ in the first division in the coordination phase and then play $\langle \mathbf{all\ stay} \rangle$ in the second division in the coordination phase. His behavior is thus different from other

Table 5: The sequences of actions played in $\mathbf{C}^1$ on the path

The sequences of actions played in $\mathbf{C}^1(1,1)$ on the path

Rebel i	i plays
i is certain $\#[R](\theta) < k$	$\langle \mathbf{all\ stay} \rangle$
i is inactive and is uncertain $\#[R](\theta) \geq k$	$\langle \mathbf{all\ stay} \rangle$
i is active and is uncertain $\#[R](\theta) \geq k$	$\langle i \rangle$
i is certain $\#[R](\theta) \geq k$	$\langle i \rangle$

The sequences of actions played in $\mathbf{C}^1(2,1)$ on the path

Rebel i	i plays
i is certain $\#[R](\theta) < k$	$\langle \mathbf{all\ stay} \rangle$
i is inactive and is uncertain $\#[R](\theta) \geq k$	$\langle \mathbf{all\ stay} \rangle$
i is active and is uncertain $\#[R](\theta) \geq k$	$\langle i \rangle$
i is certain $\#[R](\theta) \geq k$	$\langle \mathbf{all\ stay} \rangle$

The sequences of actions played in $\mathbf{C}^1(3,v)$, where $v = 1, ..., n$, on the path

Rebel i	i plays
i is certain $\#[R](\theta) < k$	$\mathbf{s}$
i is inactive and is uncertain $\#[R](\theta) \geq k$	$\mathbf{s}$
i is active and is uncertain $\#[R](\theta) \geq k$	$\mathbf{s}$
i is certain $\#[R](\theta) \geq k$	$\mathbf{r}$

kinds of Rebels. His neighbors will know $\#[R](\theta) \geq k$ right after the second division in the coordination phase and play **r** forever afterwards. Other Rebels will observe **r** being played in the third division in the coordination phase and thus know $\#[R](\theta) \geq k$ as well. All Rebels learn $\#[R](\theta) \geq k$ right after the third division in the coordination phase, and T^{θ} arrives.

Note that Rebel i who has learnt $\#[R](\theta) \geq k$ will not deviate to play ⟨**all stay**⟩ in the first division in the coordination phase even though it might be undetectable. This is because the network is acyclic. If i does so, i will be considered as an inactive Rebel by all his neighbors from the point onwards right after the first division in the coordination phase. Two consequences follow. The first is that each of i's neighbor is certain there is no more Rebel "behind" i.[31] The second is that i keeps reporting nothing in the forthcoming reporting phases. Thus, all i's neighbors report strictly less Rebels than they are supposed to do if i follows the equilibrium path. i then faces the possibility that no Rebel can know $\#[R](\theta) \geq k$ even if the total number of Rebels indeed exceeds k. If this event happens, i will only get zero payoff. However, i can surely get stage-game payoff as 1 afterwards right after the second division in the coordination phase. Sufficiently high $\delta \in (0, 1)$ will deter this deviation.

4.3.4 The equilibrium behavior on the path in $\mathbf{C}^t$ for $t \geq 2$

The on-path strategy contingent on players' belief is introduced in Table 7. The evolution of information filtrations can be tracked throughout in Table 8.

The intrigued part in $\mathbf{C}^t$ is how a pivotal Rebel p in $\mathbf{O}^{t-1}$ disseminates the relevant information. For convenience, in this section, let $I_{ij}^{t+1} = I_i^t \cup I_j^t$, and the term "coordination phase" refers to "coordination phase in the t-block." I begin with the case when p is certain $\#[R](\theta) < k$.

If p is certain $\#[R](\theta) < k$, p plays ⟨**all stay**⟩ in the first sub-block of the first division in the coordination phase. Consequently, all p's neighbors know $\#[R](\theta) < k$ right after that since p has played ⟨1⟩ in the just finished reporting phase to announce he is pivotal.

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[31]To be more precise, there is no more Rebel in a sub-tree that excludes j and roots at i.

Table 6: In $\mathbf{C}^1$, on the path, the belief of i's neighbor j after observing i's previous actions.

i plays	The event to which j assigns probability one right after $\mathbf{C}^1(1,1)$
In $\mathbf{C}^1(1,1)$	
$\langle$**all stay**$\rangle$	i is inactive if j is inactive
$\langle$**all stay**$\rangle$	$\#[R](\theta) < k$ if j is inactive
$\langle i \rangle$	i is active or $\#[R](\theta) \geq k$

i plays		The event to which j assigns probability one right after $\mathbf{C}^1(2,1)$
In $\mathbf{C}^1(1,1)$	In $\mathbf{C}^1(2,1)$	
$\langle$**all stay**$\rangle$	$\langle$**all stay**$\rangle$	i is inactive if j is inactive
$\langle$**all stay**$\rangle$	$\langle$**all stay**$\rangle$	$\#[R](\theta) < k$ if j is inactive
$\langle i \rangle$	$\langle$**all stay**$\rangle$	$\#[R](\theta) \geq k$
$\langle i \rangle$	$\langle i \rangle$	i is active

Table 7: The sequences of actions played in $\mathbf{C}^t$, $t \geq 2$ on the path

The sequences of actions played in $\mathbf{C}^t(1, v)$, where $v = 1, ..., n$, for $t \geq 2$ on the path

Rebel i	i plays
i is certain $\#[R](\theta) < k$	$\langle$**all stay**$\rangle$
i is inactive and is uncertain $\#[R](\theta) \geq k$	$\langle i \rangle$
i is active and is uncertain $\#[R](\theta) \geq k$	$\langle i \rangle$
i is certain $\#[R](\theta) \geq k$	$\langle i \rangle$

The sequences of actions played in $\mathbf{C}^t(2, v)$, where $v = 1$, for $t \geq 2$ on the path

Rebel i	i plays
i is certain $\#[R](\theta) < k$	$\langle$**all stay**$\rangle$
i is inactive and is uncertain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$
i is active and is uncertain $\#[R](\theta) \geq k$	$\langle i \rangle$
i is certain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$

The sequences of actions played in $\mathbf{C}^t(2, v)$, where $v = 2, ..., t+1$, for $t \geq 2$ on the path

Rebel i	i plays
i is certain $\#[R](\theta) < k$	$\langle$**all stay**$\rangle$
i is inactive and is uncertain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$
i is active and is uncertain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$
i is certain $\#[R](\theta) \geq k$	$\langle i \rangle$

The sequences of actions played in $\mathbf{C}^t(3, v)$, where $v = 1, ..., n$, for $t \geq 2$ on the path

Rebel i	i plays
i is certain $\#[R](\theta) < k$	**s**
i is inactive and is uncertain $\#[R](\theta) \geq k$	**s**
i is active and is uncertain $\#[R](\theta) \geq k$	**s**
i is certain $\#[R](\theta) \geq k$	**r**

Table 8: In $\mathbf{C}^t$, $t \geq 2$, on the path, the belief of i's neighbor j after observing i's previous actions.

i plays	The event to which j assigns probability one right after $\mathbf{O}^t$
In $\mathbf{O}^t$	
$\langle\mathbf{all\ stay}\rangle$	i is inactive and $I_{ji}^{t+1} = I_j^t$
$\langle I_i^t\rangle$	i is active and $I_{ji}^{t+1} = I_j^t \cap I_i^t$
$\langle 1\rangle$	i is pivotal

i plays		The event to which j assigns probability one right after $\mathbf{C}^t(1,1)$
In $\mathbf{O}^t$	In $\mathbf{C}^t(1,1)$	
$\langle\mathbf{all\ stay}\rangle$	$\langle i\rangle$	i is inactive and $I_{ji}^{t+1} = I_j^t$
$\langle I_i^t\rangle$	$\langle\mathbf{all\ stay}\rangle$	$\#[R](\theta) < k$
$\langle I_i^t\rangle$	$\langle i\rangle$	$(\#[R](\theta) \geq k)$ or (i is active and $I_{ji}^{t+1} = I_j^t \cap I_i^t$)
$\langle 1\rangle$	$\langle\mathbf{all\ stay}\rangle$	$\#[R](\theta) < k$
$\langle 1\rangle$	$\langle i\rangle$	$\#[R](\theta) \geq k$

i plays			The event to which j assigns probability one right after $\mathbf{C}^t(2,1)$
In $\mathbf{O}^t$	In $\mathbf{C}^t(1,1)$	In $\mathbf{C}^t(2,1)$	
$\langle\mathbf{all\ stay}\rangle$	$\langle i\rangle$	$\langle\mathbf{all\ stay}\rangle$	i is inactive and $I_{ji}^{t+1} = I_j^t$
$\langle I_i^t\rangle$	$\langle\mathbf{all\ stay}\rangle$	$\langle\mathbf{all\ stay}\rangle$	$\#[R](\theta) < k$
$\langle I_i^t\rangle$	$\langle i\rangle$	$\langle\mathbf{all\ stay}\rangle$	$\#[R](\theta) \geq k$
$\langle I_i^t\rangle$	$\langle i\rangle$	$\langle i\rangle$	$i \in R^t$ and $I_{ji}^{t+1} = I_j^t \cap I_i^t$
$\langle 1\rangle$	$\langle\mathbf{stay}\rangle$	$\langle\mathbf{all\ stay}\rangle$	$\#[R](\theta) < k$
$\langle 1\rangle$	$\langle i\rangle$	$\langle\mathbf{all\ stay}\rangle$	$\#[R](\theta) \geq k$

p's neighbors then play $\langle$**all stay**$\rangle$ continuously in each sub-block in the first division in the coordination phase, and therefore all Rebels know $\#[R](\theta) < k$ contagiously by observing $\langle$**all stay**$\rangle$ being played. All Rebels play **s** forever after the first division in the coordination phase, and T^θ arrives.

Alternatively, if p is certain $\#[R](\theta) \geq k$, p plays $\langle p \rangle$ in each sub-block in the first division in the coordination phase. To reveal $\#[R](\theta) \geq k$, p plays $\langle$**all stay**$\rangle$ in the first sub-block of the second division in the coordination phase. Notice that $\langle$**all stay**$\rangle$ is a costless sequence of actions. It might not seem intuitive at first sight, but playing $\langle$**all stay**$\rangle$ effectively prevents another free-rider problem. Suppose there are two pivotal Rebels, say p and p', who have already known $\#[R](\theta) \geq k$ right after the previous reporting phase. If initiation to disseminate knowledge about $\#[R](\theta) \geq k$ incurs negative payoff, p or p' will have the incentive, again, to wait for the other one initiates it. Playing $\langle$**all stay**$\rangle$ in the first sub-block of the second division in the coordination phase proudly becomes the initiation sequence by its cheapness. By the same contagion argument as the above mentioned, all Rebels play **r** after the third division in the coordination phase, and T^θ arrives.

The remaining question is why a non-pivotal Rebel, say i, does not mimic a pivotal Rebel's behavior by playing $\langle 1 \rangle$ in the just finished reporting phase even though it might be undetectable. The reason is the following. If i plays $\langle 1 \rangle$, i's neighbor will think i is pivotal. According to the equilibrium path, it implies that all players play either **r** or **s** forever after the third division in the coordination phase; therefore, the belief updating is also terminated. What follows is i cannot learn the relevant information after the third division in the coordination phase. If i does not deviate, i will learn the relevant information eventually and choose the best response to get the maximum payoff at every θ. He prefers not to deviate if $\delta \in (0,1)$ is high enough.[32]

4.4   Sketch of the proof of Theorem 2

In the above I have listed Rebels' behavior in Table 3, Table 5, Table 8, and their belief updating in Table 6, Table 8, as the blueprint for the constructed equilibrium path. I sketch

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[32]By Lemma A.1 in Appendix, the relevant information is learnt by every Rebel on the path eventually.

the proof of Theorem 2 as follows.

I use off-path belief to prevent players from making detectable deviations, such as deviating from playing the specified forms of sequences that are listed in Table 2 or Table 4. This off-path belief serves as a grim trigger: the punisher will play **stay** forever if he has not yet learnt the relevant information. Due to this, if a deviant who has not yet learnt the relevant information, from his perspective, making detectable deviation will strictly reduce the possibility of achieving the ex-post efficient outcome. Remind that the ex-post efficient outcome gives him the highest ex-post stage payoff in the k-threshold game. As for making an undetectable deviation, though grim trigger is not effective, an undetectable deviation will too strictly reduce the deviant's expected continuation payoff. This is caused by two reasons. The first is that, in the reporting phase, an undetectable deviation that untruthfully reports less Rebels diminishes the possibility of achieving the ex-post outcome. The second is that an undetectable deviation that adds "noises" into information sharing process strictly reduces the deviant's expected continuation payoff as argued in Section 4.3.4. Since the stage-game payoff after T^θ is maximum ex-post for every θ by following the constructed equilibrium, a sufficiently high $\delta \in (0, 1)$ will deter Rebels from deviating. I then conclude the proof.

5 Discussion

In the above APEX equilibrium construction, players act as if acting a sequence. Nevertheless, the actual description of an APEX equilibrium should prescribe how they act period by period and how they update belief at every period. This description is tedious; it is left in the Appendix.

Instead of providing further details in equilibrium construction, I discuss the scenario when pay-off is a signal and why my constructed APEX equilibrium may fail in cyclic networks.

5.1 Payoff as signals

The hidden payoff assumption can be relaxed without changing the main result. One may consider a situation in which the stage payoff depends not only on players' joint efforts but also on a random shock, say the weather. To fix the idea, there is a public signal $y \in \{r, s\}$ generated according to the action profile. Let a Rebel's payoff function be $u_R(a_R, y)$ such that $u_R(\textbf{stay}, r) = u_R(\textbf{stay}, s) = u_0$. y is drawn from the distribution of

$$p_{rr} = \Pr(y = r | \#\{j : a_j \textbf{revolt}\} \geq k)$$

$$p_{sr} = 1 - p_{rr} = \Pr(y = s | \#\{j : a_j \textbf{revolt}\} \geq k)$$

$$p_{ss} = \Pr(y = s | \#\{j : a_j \textbf{revolt}\} < k)$$

$$p_{rs} = 1 - p_{ss} = \Pr(y = r | \#\{j : a_j \textbf{revolt}\} < k)$$

such that

$$p_{rr} u_R(\textbf{revolt}, r) + p_{sr} u_R(\textbf{revolt}, s) > u_0 > p_{rr} u_R(\textbf{revolt}, r) + p_{ss} u_R(\textbf{revolt}, s),$$

and

$$0 \leq p_{rs} \leq 1, 0 \leq p_{ss} \leq 1.$$

The APEX equilibrium constructed for Theorem 2 is still a one in this scenario. Note that in that APEX equilibrium path, at most one **revolt** can occur at every period before some Rebel plays $\langle 1 \rangle$. This implies that the signal y is completely uninformative before some Rebel plays $\langle 1 \rangle$. If a Rebel i deviates to play $\langle 1 \rangle$ in $\textbf{O}^t$ at some t in the hope gathering information from y, he will not learn the relevant information after $\textbf{O}^t$ since the terminal period will come right after t-block. He will, however, learn the relevant information and play the ex-post efficient outcome if he is on the path, and hence he will not deviate.

5.2 Cyclic networks

Scenarios in cyclic networks substantially differ from the acyclic counterpart. The free-rider problem could become intractable in cyclic networks. Let us consider the configuration in Figure 9, and suppose $k = 4$.
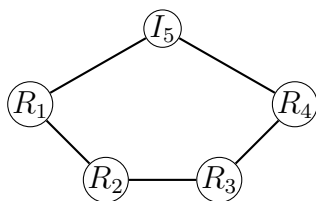
Figure 9: A configuration of the state and the network in which player 1, 2, 3, 4 are Rebels while player 5 is an Inert.

In Figure 9, Rebels 2 and 3 are θ-pivotal by definition. From the perspective of Rebel 2, the type of player 5 could be Inert. Therefore, Rebel 2 does not know that Rebel 1 is pivotal. Similarly, Rebel 2 does not know that Rebel 3 is pivotal, *even though* player 3 is indeed θ-pivotal. Therefore there is no common knowledge of the free-rider problem at the beginning of 1-block.

However, the common knowledge of engaging in a free-rider problem is restored when we cut the edge between player 4 and 5; Rebel 2 knows that he is the only θ-pivotal Rebel.

I leave a conjecture in this paper and end this section.

Conjecture 5.1. *For any n-person repeated k-Threshold game with parameter $k < n$ played in any network, if π has full support on strong connectedness, then there exists a $\delta^* \in (0, 1)$ such that an APEX equilibrium exists whenever $\delta > \delta^*$.*

6 Conclusion

I model a coordination game and illustrate the learning processes generated by strategies in a sequential equilibrium to answer the question proposed in the beginning: what kind of networks can conduct coordination in a collective action with information barrier. In the equilibrium, players transmit the relevant information by encoding such information by their actions as time goes by. Since there might be an negative expected payoff in coding information, the potential free-rider problems might occur to impede the learning process. My result show that if the network is acyclic, players can always learn the underlying relevant information and conduct the coordination only by actions. In cyclic networks,

however, what kinds of equilibrium strategies can lead to learning the relevant information still remains to be answered.

The construction of the communication protocol by actions exploits the assumption of the common knowledge of the network and the finite type space. Since the relevant information has been parametrized as a threshold in the stage game, players can acquire this information by jointly incrementally reporting their own private information period by period. The major punishment to deter deviation is then the joint shifting to play that same action as the stopping to update information. The threshold game thus seems a potential model in proofing that a communication protocol by actions not only leads a learning process but also constitutes an equilibrium to reveal the relevant information in finite time.

Existing literatures in political science and sociology have recognized the importance of social network in influencing individual's behavior in participating social movements ([Passy, 2003][McAdam, 2003][Siegel, 2009]). This paper views networks as routes for communication in which rational individuals initially have local information but they can influence nearby individuals by taking actions. Such influence may take long time to travel across individuals and the whole process incurs inefficient outcomes in many periods. A characterization in the speed of information transmission across a network is not answered here, although it is an important topic in investigating the most efficient way to let the information be spread. This question would remain for the future research.

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A Appendix

A.1 The APEX equilibrium for Theorem 2

A.1.1 Equilibrium path

By definition of information hierarchy,

$$
\begin{aligned}
I_i^t &= \bigcup_{k_1 \in G_i} \bigcup_{k_2 \in G_{k_0}} ... \bigcup_{k_t \in G_{k_{t-1}}} I_{k_t}^1 \\
&= \{ j \in [R](\theta) : \exists \text{ a path } (i, k_1...k_l, j) \text{ s.t. } 0 \le l \le t-1 \text{ and } \theta_i = \theta_{k_1} = ... = \theta_{k_l} = R \}
\end{aligned}
$$

Let us define several notions.

Notation A.1 (Extended vertices by I_i^t).

$$
\begin{aligned}
X_i^t := \{ j \in N : \\
\exists \text{ a path } (i, k_1...k_l, k_{l+1}) \text{ s.t. } k_{l+1} = j, \, l \ge t-1, \, \{i, k_1, ..., k_t\} \subset I_i^t \} \cup I_i^t
\end{aligned}
$$

Namely, X_i^t is the set of all possible Rebels in G given information I_i^t.

Notation A.2 (The vertices induced by a sub-tree rooting in j but excluding i).

$$
TR_{ij} := \{ v \in N : \text{there is a path from } i \text{ to } v \text{ through } j, \, j \in G_i \} \cup \{ j \}
$$

Notation A.3 (The vertices that are in X_i^t and TR_{ij}, but outside of I_i^t).

$$
Y_{ij}^t := TR_{ij} \cap (X_i^t \setminus I_i^t)
$$

Notation A.4.

$$
D_i^t := \{ j \in G_i : Y_{ij}^t \neq \emptyset \}
$$

Definition A.1 (Finite register machine)**.** *A finite register machine for i consists finite registers Σ. A register $\sigma \in Sigma$ is a tuple*

$$
(\Omega, \times_{G_i} A_R, f, \lambda),
$$

in which

1. Ω is the set of all information sets over Θ induced by $\times_{s=0}^{\infty} H_i^s$.

2. $\times_{G_i} A_R$ is the set of inputs, which is equal to the set of action profiles of i's neighbors.

3. $f : \Omega \to A_R$ is the outcome function, which maps each information set to an action.

4. $\lambda : \Omega \times \times_{G_i} A_R \to \Sigma$ is the transition function, which prescribes the transition between registers.

There are initial registers.

Note that this definition of finite register machine is non-standard in the following sense. An i's register prescribes i's action according to his information at a certain period but does not characterize i's information transition. The information of i, up to period s, is specified by $P_i(\theta) \times \{h_{G_i}^s\} \times H_{N \setminus G_i}^s$, which is characterized in Section 2.

Definition A.2 (m-register in t-block)**.** *A m-register in a (sub)block or a division is the register in the m-th period in that (sub)block or division.*

Notation A.5 ($m \dashv \Gamma$)**.**

$$m \dashv \Gamma \text{ is the m-register in the (sub)block or division } \Gamma.$$

Definition A.3 (Terminal **r**)**.** *The terminal **r** is a register such that the image of f is* $\{\boldsymbol{revolt}\}$ *and the image of transition λ is a singleton containing the register machine itself.*

Definition A.4 (Terminal **s**)**.** *The terminal **s** is a register such that the image of f is* $\{\boldsymbol{stay}\}$ *and the image of transition λ is a singleton containing the register machine itself.*

The equilibrium will be represented as a finite register machine. Note that, though players act as if they are acting a whole sequence, they in fact act period by period. For convenience, for any finite sequence of action $\langle \rangle$, denote $\langle \rangle_m$ as the m-th (counting from the beginning) component in $\langle \rangle$, and denote $\langle \rangle(m)$ as the prefix of $\langle \rangle$ with length m. Let us also shorten action **revolt** to be **r** and **stay** to be **s**.

Initial registers   The initial register for each Rebel is $1 \dashv \mathbf{C}^1(1,1)$, which is defined in the next section.

Registers in C[1]

Table 9: The $m \dashv \mathbf{C}^1(1,1)$ on the path

$$1 \le m \le |\mathbf{C}^1(1,1)| - 1$$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$\#X_i^1 < k$	$\langle\text{all stay}\rangle_m$		terminal $\mathbf{s}$
$i \notin R^1,\ \#X_i^1 \ge k,\ I_i^1 < k$	$\langle\text{all stay}\rangle_m$		$m+1 \dashv \mathbf{C}^1(1,1)$
$i \in R^1,\ \#X_i^1 \ge k,\ I_i^1 < k$	$\langle i\rangle_m$		$m+1 \dashv \mathbf{C}^1(1,1)$
$i \in R^1,\ \#X_i^1 \ge k,\ I_i^1 \ge k$	$\langle i\rangle_m$		$m+1 \dashv \mathbf{C}^1(1,1)$

$$m = |\mathbf{C}^1(1,1)|$$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$\#X_i^1 < k$	$\langle\text{all stay}\rangle_m$		terminal $\mathbf{s}$
$i \notin R^1,\ \#X_i^1 \ge k,\ I_i^1 < k$ all j play $\langle\text{all stay}\rangle(m-1)$	$\langle\text{all stay}\rangle_m$	all j play $\mathbf{s}$	terminal $\mathbf{s}$
$i \notin R^1,\ \#X_i^1 \ge k,\ I_i^1 < k$ $\exists j$ plays $\langle j\rangle(m-1)$	$\langle\text{all stay}\rangle_m$	such j plays $\langle j\rangle_m$	$1 \dashv \mathbf{C}^1(2,1)$
$i \in R^1,\ \#X_i^1 \ge k,\ I_i^1 < k$	$\langle i\rangle_m$		$1 \dashv \mathbf{C}^1(2,1)$
$i \in R^1,\ \#X_i^1 \ge k,\ I_i^1 \ge k$	$\langle i\rangle_m$		$1 \dashv \mathbf{C}^1(2,1)$

Table 10: The $m \dashv \mathbf{C}^1(2,1)$ on the path

$$1 \le m < |\mathbf{C}^1(2,1)|$$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$i \notin R^1$		$\langle$**all stay**$\rangle_m$		$m+1 \dashv \mathbf{C}^1(2,1)$
$i \in R^1,\ I_i^1 < k$	$\exists j \in G_i,\ j$ plays $< j >_j = s$	$\langle$**all stay**$\rangle_m$		$m+1 \dashv \mathbf{C}^1(2,1)$
$i \in R^1,\ I_i^1 < k$	$\forall j \in G_i,\ j$ plays $< j >_j = r$	$\langle i \rangle_m$		$m+1 \dashv \mathbf{C}^1(2,1)$
$i \in R^1,\ I_i^1 \ge k$		$\langle$**all stay**$\rangle_m$		$m+1 \dashv \mathbf{C}^1(2,1)$

$$m = |\mathbf{C}^1(2,1)|$$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$i \notin R^1$	$\forall j \in G_i,\ j$ plays $\langle j \rangle (m-1)$	$\langle$**all stay**$\rangle_m$	$\forall j \in G_i,\ j$ plays $\langle j \rangle_m$	$1 \dashv \mathbf{C}^1(3, \cdot)$
$i \notin R^1$	$\exists j \in G_i,\ j$ plays $\langle$**all stay**$\rangle (m-1)$	$\langle$**all stay**$\rangle_m$	such j plays $\langle$**all stay**$\rangle_m$	terminal $\mathbf{r}$
$i \in R^1,\ I_i^1 < k$	$\forall j \in G_i,\ j$ play $\langle j \rangle (m-1)$	$\langle i \rangle_m$	$\forall j \in G_i,\ j$ plays $\langle j \rangle_m$	$1 \dashv \mathbf{C}^1(3, \cdot)$
$i \in R^1,\ I_i^1 < k$	$\exists j \in G_i,\ j$ plays $\langle$**all stay**$\rangle (m-1)$	$\langle i \rangle_m$	such j plays $\langle$**all stay**$\rangle_m$	terminal $\mathbf{r}$
$i \in R^t,\ I_i^1 \ge k$		$\langle$**all stay**$\rangle_m$		terminal $\mathbf{r}$

Table 11: The $m \dashv \mathbf{C}^1(3, \cdot)$ on the path

$$1 \le m < |\mathbf{C}^1(3, \cdot)|$$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
	s	$\forall j$ play **s**	$m + 1 \dashv \mathbf{C}^1(3, \cdot)$
	s	$\exists j$ play **r**	terminal **r**

$$1 \le m = |\mathbf{C}^1(3, \cdot)|$$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
	s	$\forall j$ play **s**	$1 \dashv \mathbf{O}^1$
	s	$\exists j$ play **r**	terminal **r**

Registers in $\mathbf{O}^t$ First denote $m_i = |\mathbf{O}^t| - x_{I_i^t}$ as the period in which i reports that I_i^t. I.e. m_i is the period where $\mathbf{r}$ occurs in $\langle I_i^t \rangle$. Then define $G_i(m) := \{j \in G_i : m_j < m\}$. Define $I_i^{t+1}(m) := I_i^t \cup \bigcup_{j \in G_i(m)} I_j^t$ as the information of i up to the m-th period in $\mathbf{O}^t$. Define $X_i^{t+1}(m)$ to be the vertices induced by the sub-tree by $I_i^t(m)$ in the same way as that in Definition A.1, and define $Y_{ij}^t(m)$ and $D_i^t(m)$ accordingly.

Table 12: The $m \dashv \mathbf{O}^t$ on the path, where $1 \leq m < |\mathbf{O}^t|$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$i \notin R^t$		$\langle \mathbf{all\ stay} \rangle_m$		$m + 1 \dashv \mathbf{O}^t$
$i \in R^t$, not a free rider, not $k-1$-pivotal	$\#I_i^{t+1}(m) < k$, $\#X_i^{t+1}(m) < k$	$\langle \mathbf{all\ stay} \rangle_m$		terminal $\mathbf{s}$
$i \in R^t$, not a free rider, not $k-1$-pivotal	$\#I_i^{t+1}(m) < k$, $\#X_i^{t+1}(m) \geq k$	$\langle I_i^t \rangle_m$		$m + 1 \dashv \mathbf{O}^t$
$i \in R^t$, not a free rider, not $k-1$-pivotal	$\#I_i^{t+1}(m) \geq k$, $\#X_i^{t+1}(m) \geq k$	$\langle 1 \rangle_m$		$m + 1 \dashv \mathbf{O}^t$
$i \in R^t$, not a free rider, not $k-1$-pivotal	$\#I_i^{t+1}(m) \geq k-1$, $\#X_i^{t+1}(m) \geq k$, $\#D_i^t = 1$	$\langle 1 \rangle_m$		$m + 1 \dashv \mathbf{O}^t$
$i \in R^t$, not a free rider, not $k-1$-pivotal	$\#I_i^{t+1}(m) \geq k-1$, $\#X_i^{t+1}(m) \geq k$, $\#D_i^t > 1$	$\langle I_i^t \rangle_m$		$m + 1 \dashv \mathbf{O}^t$
$i \in R^t$, the free rider	$\#X_i^{t+1}(m) \geq k$	$\langle 1 \rangle_m$		$m + 1 \dashv \mathbf{O}^t$
$i \in R^t$, the free rider	$\#X_i^{t+1}(m) < k$	$\langle \mathbf{all\ stay} \rangle_m$		terminal $\mathbf{s}$
$i \in R^t$, i is $k-1$-pivotal	$\#X_i^{t+1}(m) \geq k$	$\langle 1 \rangle_m$		$m + 1 \dashv \mathbf{O}^t$
$i \in R^t$, i is $k-1$-pivotal	$\#X_i^{t+1}(m) < k$	$\langle \mathbf{all\ stay} \rangle_m$		terminal $\mathbf{s}$

Table 13: The $m \dashv \mathbf{O}^t$ on the path, where $m = |\mathbf{O}^t|$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$i \notin R^t$		$\langle \mathbf{all\ stay} \rangle_m$		$1 \dashv \mathbf{C}^t(1,1)$
$i \in R^t$, not a free rider, not $k-1$-pivotal	$\#I_i^{t+1}(m) < k$, $\#X_i^{t+1}(m) < k$	$\langle \mathbf{all\ stay} \rangle_m$		terminal $\mathbf{s}$
$i \in R^t$, not a free rider, not $k-1$-pivotal	$\#I_i^{t+1}(m) < k-1$, $\#X_i^{t+1}(m) \geq k$	$\langle I_i^t \rangle_m$		$1 \dashv \mathbf{C}^t(1,1)$
$i \in R^t$, not a free rider, not $k-1$-pivotal	$\#I_i^{t+1}(m) \geq k$, $\#X_i^{t+1}(m) \geq k$	$\langle 1 \rangle_m$		$1 \dashv \mathbf{C}^t(1,1)$
$i \in R^t$, not a free rider, not $k-1$-pivotal	$\#I_i^{t+1}(m) \geq k-1$, $\#X_i^{t+1}(m) \geq k$, $\#D_i^t = 1$	$\langle 1 \rangle_m$		$1 \dashv \mathbf{C}^t(1,1)$
$i \in R^t$, not a free rider, not $k-1$-pivotal	$\#I_i^{t+1}(m) \geq k-1$, $\#X_i^{t+1}(m) \geq k$, $\#D_i^t > 1$	$\langle I_i^t \rangle_m$		$1 \dashv \mathbf{C}^t(1,1)$
$i \in R^t$, the free rider	$\#X_i^{t+1}(m) \geq k$	$\langle 1 \rangle_m$		$1 \dashv \mathbf{C}^t(1,1)$
$i \in R^t$, the free rider	$\#X_i^{t+1}(m) < k$	$\langle \mathbf{all\ stay} \rangle_m$		terminal $\mathbf{s}$
$i \in R^t$, i is $k-1$-pivotal	$\#X_i^{t+1}(m) \geq k$	$\langle 1 \rangle_m$		$1 \dashv \mathbf{C}^t(1,1)$
$i \in R^t$, i is $k-1$-pivotal	$\#X_i^{t+1}(m) < k$	$\langle \mathbf{all\ stay} \rangle_m$		terminal $\mathbf{s}$

Registers in $\mathbf{C}^t$ for $t \geq 2$

Table 14: The $m \dashv \mathbf{C}^t(1, v)$ for $v = 1, ..., n$ on the path

$$1 \leq m < |\mathbf{C}^t(1, v)|, \text{ where } v = 1, ..., n$$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$X_i^{t+1} < k$	$\langle \textbf{all stay} \rangle_m$		terminal $\mathbf{s}$
$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\exists j \in G_i, j = m$ such that $a_j = \mathbf{s}$	terminal $\mathbf{s}$
$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\forall j \in G_i$ such that $a_j = \langle j \rangle_m$	$m + 1 \dashv \mathbf{C}^t(1, v)$

$$m = |\mathbf{C}^t(1, v)|, \text{ where } v = 1, ..., n - 1$$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$X_i^{t+1} < k$	$\langle \textbf{all stay} \rangle_m$		terminal $\mathbf{s}$
$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\exists j \in G_i, j = m$ such that $a_j = \mathbf{s}$	terminal $\mathbf{s}$
$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\forall j \in G_i$ such that $a_j = \langle j \rangle_m$	$1 \dashv \mathbf{C}^t(1, v + 1)$

$$1 \leq m < |\mathbf{C}^t(1, n)|$$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\exists j \in G_i, j = m$ such that $a_j = \mathbf{s}$	terminal $\mathbf{s}$
$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\forall j \in G_i$ such that $a_j = \langle j \rangle_m$	$m + 1 \dashv \mathbf{C}^t(1, n)$

$$m = |\mathbf{C}^t(1, n)|$$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\exists j \in G_i, j = m$ such that $a_j = \mathbf{s}$	terminal $\mathbf{s}$
$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\forall j \in G_i$ such that $a_j = \langle j \rangle_m$	$1 \dashv \mathbf{C}^t(2, 1)$

Table 15: The $m \dashv \mathbf{C}^t(2, v)$ for $v = 1, ..., t+1$ on the path

$1 \le m < |\mathbf{C}^t(2, v)|$, where $v = 1, ..., t+1$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$I_i^{t+1} < k$	$\exists j,\ \langle j \rangle_j = \mathbf{s}$	$\langle \text{all stay} \rangle_m$		$m + 1 \dashv \mathbf{C}^t(2, v)$
$I_i^{t+1} < k$	$\forall j,\ \langle j \rangle_j = \mathbf{r}$	$\langle i \rangle_m$		$m + 1 \dashv \mathbf{C}^t(2, v)$
$I_i^{t+1} \ge k$		$\langle \text{all stay} \rangle_m$		$m + 1 \dashv \mathbf{C}^t(2, v)$

$m = |\mathbf{C}^t(2, v)|$, where $v = 1, ..., t$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$I_i^{t+1} < k$	$\exists j \in G_i,\ \langle j \rangle_j = \mathbf{s}$	$\langle \text{all stay} \rangle_m$		$1 \dashv \mathbf{C}^t(2, v+1)$
$I_i^{t+1} < k$	$\forall j \in G_i,\ \langle j \rangle_j = \mathbf{r}$	$\langle i \rangle_m$		$1 \dashv \mathbf{C}^t(2, v+1)$
$I_i^{t+1} \ge k$		$\langle \text{all stay} \rangle_m$		$1 \dashv \mathbf{C}^t(2, v+1)$

$1 \le m < |\mathbf{C}^t(2, t+1)|$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$I_i^{t+1} < k$	$\exists j \in G_i,\ \langle j \rangle_j = \mathbf{s}$	$\langle \text{all stay} \rangle_m$		$m + 1 \dashv \mathbf{C}^t(2, t+1)$
$I_i^{t+1} < k$	$\forall j \in G_i,\ \langle j \rangle_j = \mathbf{r}$	$\langle i \rangle_m$		$m + 1 \dashv \mathbf{C}^t(2, t+1)$
$I_i^{t+1} \ge k$		$\langle \text{all stay} \rangle_m$		$m + 1 \dashv \mathbf{C}^t(2, t+1)$

$m = |\mathbf{C}^t(2, t+1)|$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$I_i^{t+1} < k$	$\exists j \in G_i,\ \langle j \rangle_j = \mathbf{s}$	$\langle \text{all stay} \rangle_m$		terminal $\mathbf{r}$
$I_i^{t+1} < k$	$\forall j \in G_i,\ \langle j \rangle_j = \mathbf{r}$	$\langle i \rangle_m$		$1 \dashv \mathbf{C}^t(3, 1)$
$I_i^{t+1} \ge k$		$\langle \text{all stay} \rangle_m$		terminal $\mathbf{r}$

Table 16: The $m \dashv \mathbf{C}^t(3, 1)$ on the path

$$1 \leq m < |\mathbf{C}^t(3, 1)|$$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
	$\mathbf{s}$	$\forall j \in G_i,\ j$ plays $\mathbf{s}$	$m + 1 \dashv \mathbf{C}^t(3, 1)$
	$\mathbf{s}$	$\exists j \in G_i,\ j$ plays $\mathbf{r}$	terminal $\mathbf{r}$

$$m = |\mathbf{C}^t(3, 1)|$$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
	$\mathbf{s}$	$\forall j \in G_i,\ j$ plays $\mathbf{s}$	$1 \dashv \mathbf{O}^{t+1}$
	$\mathbf{s}$	$\exists j \in G_i,\ j$ play $\mathbf{r}$	terminal $\mathbf{r}$

A.2 Missing proofs

Proof of Lemma 2.1

Proof. The proof is done by contraposition. Suppose Rebels' strategies constitute an APEX equilibrium. By definition of the APEX equilibrium, at every θ, there is a period T^θ when all Rebels' actions start to repeat themselves. Let $T = \max_{\theta \in \Theta} T^\theta$. For Rebel i, let $T_i = T + 1$, and suppose $0 < \sum_{\theta : \#[R](\theta) \geq k} \beta_{G_i}^{\pi, \tau^*}(\theta | h_{G_i}^s) < 1$ for some $s \geq T_i$. Then this Rebel assigns positive weight at some $\theta' \in \{\theta : \#[R](\theta) < k\}$ and some positive weight at some $\theta'' \in \{\theta : \#[R](\theta) \geq k\}$ at period s. Note that i has already known θ_j if $j \in G_i$, and therefore i assigns positive weight at some $\theta' \in \{\theta : \#[R](\theta) < k, \theta_l = R, l \notin G_i\}$ and positive weight at some $\theta'' \in \{\theta : \#[R](\theta) < k, \theta_l = I, l \notin G_i\}$. Since all Rebels' actions start to repeat themselves at period T, i cannot update information afterwards. Suppose i's continuation strategy is to continuously play **revolt**, then this is not ex-post efficient when $\#[R](\theta) < k$. Suppose i's continuation strategy is to continuously play **stay**, then this is not ex-post efficient when $\#[R](\theta) \geq k$. $\qquad\square$

Proof of Theorem 1

Proof. Let τ^* be the following strategy. After the nature moves, a Rebel i plays **revolt** if he has no Inert neighbor, and plays **stay** forever if he has an Inert neighbor. After the first period, if i has not detect a deviation and observes that all his Rebel neighbors play **revolt** continuously previously, he will play **revolt** in the current period; otherwise, he will play **stay** afterwards and forever. If a Rebel j deviates, then j plays **stay** afterwards and forever.

At period s, according to τ^*, if i has not detect a deviation, but observes that his Rebel neighbors play **stay** in the current period, he will form the belief of

$$\sum_{\theta : \#[R](\theta) \geq k} \beta_{G_i}^{\pi, \tau^*}(\theta | h_{G_i}^s) = 0$$

afterwards and forever. Therefore, he plays **stay** afterwards and forever as his best response.

At period s, if a Rebel detects a deviation or has deviated, then to play **stay** afterwards and forever is his best response since at least one player will play **stay** afterwards and forever.

Since the network is finite with n vertices, if all players do not deviate, after period n, each Rebel plays **revolt** and get payoff 1 forever if $\theta \in \{\theta : \#[R](\theta) \geq k\}$ or plays **stay** and gets payoff 0 forever if $\theta \in \{\theta : \#[R](\theta) < k\}$. However, a Rebel who has deviated surely gets payoff 0 forever after period n. Therefore, there is a $0 < \delta < 1$ large enough to impede Rebels to deviate.

To check if τ^* and $\{\beta_{G_i}^{\pi,\tau^*}(\theta|h_{G_i}^s)\}_{i \in N}$ satisfy full consistency[33], take any $0 < x < 1$ such that Rebels play τ^* with probability $1 - x$ and play other behavior strategies with probability x. Clearly, when $x \to 0$, the belief converges to $\{\beta_{G_i}^{\pi,\tau^*}(\theta|h_{G_i}^s)\}_{i \in N}$. Since the off-path strategy is the best response for both the Rebel who detects deviation and the Rebel who deviates, for the arbitrary beliefs they hold, τ^* is a sequential equilibrium. $\qquad \square$

Proof of Theorem 3

Proof. Since the network is finite, θ has strong connectedness, and $[R](\theta) \neq \emptyset$, there is a minimum t_i such that $I_i^{t_i} = [R](\theta)$ for each i by the definition of I_i^t. Let $P = \underset{i \in N}{\arg\min}\{t_i\}$ with generic element p. Therefore $I_p^{t_p} = [R](\theta)$. I show that $p \in R^{t_p-1}$ to complete the proof. I prove it by contradiction. If $p \notin R^{t_p-1}$, then $I_p^{t_p-1} \subseteq G_j^{t_p-1}$ for some $j \in G_p$. Then, all the Rebels in TR_{jp} are in $G_j^{t_p-1}$, but there exist Rebels in TR_{pj} who are in $G_j^{t_p-1}$ but not in $I_p^{t_p-1}$. This is because the network is acyclic and $I_p^{t_p-1} \subset [R](\theta)$. But then $p \notin P$ since $I_j^{t_j-1} = [R](\theta)$ already. I then conclude that $p \in R^{t_p-1}$.

$\qquad \square$

Proof of Lemma 4.1

Proof. The proof is by contradiction. Suppose that, at t-block and before T^θ, there are three or more active θ-pivotal Rebels. Since θ has strong connectedness, there are three of them, p_1, p_2, p_3, with the property $p_1 \in G_{p_2}$ and $p_2 \in G_{p_3}$.

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[33]Krep and Wilson (1982)

Since the network is acyclic, $p_1 \notin G_{p_3}$ and $p_3 \notin G_{p_1}$. Since p_1 is an active θ-pivotal Rebel, p_1 is uncertain $I_{p_1}^t = [R](\theta)$ and certain $I_{p_1}^{t+1} = [R](\theta)$. This implies that, in $TR_{p_1 p_2}$, p_1 can reach all Rebels by $t + 1$ edges, but might not reach all of them by t edges. The same situation applies to p_3. However, this means that p_2 can reach all Rebels in $TR_{p_1 p_1}$ by t edges and reach all Rebels in $TR_{p_1 p_1}$ by t edges, and hence p_2 is certain $I_{p_2}^t = [R](\theta)$ at t-block. It contradicts to the definition of the θ-pivotal Rebel.

$\square$

Proof of Lemma 4.2

Proof. Since there are two active θ-pivotal Rebels and since the network is acyclic, all Rebels in $TR_{p'p}$ is in I_p^t, and all Rebels in $TR_{pp'}$ is in $I_{p'}^t$. Moreover, all vertices in $Y_{p'p}^t$ can be reached by $t + 1$ edges from p', and all vertices in $Y_{pp'}^t$ can be reached by $t + 1$ edges from p. Furthermore, some vertices in $Y_{p'p}^t$ must only be reached by $t + 1$ edges from p', and some vertices in $Y_{pp'}^t$ must only be reached by $t + 1$ edges from p.

By the network being commonly known, p is therefore certain that he himself is θ-pivotal by concluding it might be $I_p^t \subset [R](\theta)$ and $I_p^{t+1} = [R](\theta)$. By the network being commonly known and p being active, p is certain that p' is θ-pivotal by concluding $I_{p'}^t \subset [R](\theta)$ and $I_{p'}^{t+1} = [R](\theta)$. The above arguments hold true for p' by replacing p by p' and p' by p. Then p and p' commonly know that they are θ-pivotal by applying the common knowledge of network. Note that, though, they are uncertain whether they are both active. $\square$

Proof of Theorem 2 I begin with the following lemmas stating that all Rebels eventually learn the relevant information on the path.

Lemma A.1. *If the network is acyclic and if the θ has strong connectedness, then the equilibrium path prescribed in Section A.1.1 is an APEX strategy.*

Proof. Firstly, suppose θ satisfies $\#[R](\theta) < k$. I show that, all Rebels will enter terminal **s** eventually without entering terminal **r**. Let p be the Rebel defined in the proof of Theorem 3 so that $I_p^{t_p} = [R](\theta)$, where p is one of the earliest Rebels who knows $\#[R](\theta) < k$. I claim

that $\#X_p^{t_p} < k$ if and only if $I_p^{t_p} = [R](\theta)$. For the "only if" part, the proof is by way of contradiction. If not, by the full support of strong connectedness, there is a possible Rebel outside $I_p^{t_p}$, and therefore p is uncertain that $\#[R](\theta) < k$. For the "if" part, note that $I_p^{t_p} \subset X_p^{t_p}$ and therefore $\#I_p^{t_p} < \#X_p^{t_p} < k$. I also claim that $\#X_p^{t_p}(m) < k$ if and only if $I_p^{t_p}(m) = [R](\theta)$. The proof is exactly the same as the noted above by replacing $I_p^{t_p}$ to $I_p^{t_p}$ and $I_p^{t_p}(m)$ to $I_P^{t_p}(m)$.

Referring to Table 9 to Table 13, whenever there is a p so that $\#X_p^{t_p} < k$, p plays **stay** forever. It implies that all Rebels enter terminal **s** right after $\mathbf{C}_1^t, t \geq 0$. Notice that Rebels enter terminal **r** only after some period after $\mathbf{C}_1^t$ and therefore all Rebels will enter terminal **s** before terminal **r**.

Secondly, suppose θ satisfies $\#[R](\theta) \geq k$. I show that all Rebels will enter terminal **r** eventually. Note first that if there is a Rebel p that $\#I_p^1 \geq k$, all Rebels enter terminal **r** after $\mathbf{C}^1(3, \cdot)$ referring to Table 9, Table 10, and Table 11. At $t > 0$, if there is a Rebel p who has played $\langle 1 \rangle$ at $\mathbf{O}^t$, by the postulate of $\#[R](\theta) \geq k$, after $\mathbf{C}^t(3, \cdot)$, all Rebels enter terminal **r** according to the equilibrium path prescribed in Table 12, Table 13, Table 14, Table 15, and Table 16. There must be some Rebel $p \in R^t$ who plays $\langle 1 \rangle$ at $\mathbf{O}^t$ for some t by the same argument in the proof of Theorem 3. $\qquad\square$

Due to Lemma A.1, define $T_{\tau*}^\theta$ as the earliest period at which all Rebels play ex-post efficient outcome afterwards according to an APEX equilibrium τ^*. I prepare the following claims to proceed the proof. These claims will hold for π having full support on strong connectedness. For conciseness, in the following claims, I only consider non-trivial cases. I only consider $2 < k < n$ as aforementioned and the period s when $s < T_{\tau*}^\theta$. For the case of $s > T_{\tau*}^\theta$, it should be clear that Rebels will not deviate from the equilibrium path.[34] Furthermore, I only consider the case that i still believes $\#[R](\theta) \geq k$ with positive probability. Otherwise i will play **stay** forever on the equilibrium path.

Some notations are introduced. I suppress the notation $\beta_{G_i}^{\pi,\tau}(\theta | h_{G_i}^s)$ to $\beta_{G_i}^{\tau}(\theta | h_{G_i}^s)$ and

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[34]This is because the ex-post efficient outcome in the k-threshold game is being played if $s > T_{\tau*}^\theta$.

the notation $\alpha_{G_i}^{\pi,\tau}(\theta, h^s|\theta_{G_i}, h_{G_i}^s)$ to $\alpha_{G_i}^{\tau}(\theta, h^s|h_{G_i}^s)$. If $\mathbf{Prop}(\theta)$ is a property of θ, define

$$\beta_{G_i}^{\tau}(\mathbf{Prop}(\theta)|h_{G_i}^s) := \sum_{\theta \in \{\theta : \mathbf{Prop}(\theta)\}} \beta_{G_i}^{\tau}(\theta|h_{G_i}^s).$$

Furthermore, if s' and s are periods and $s' > s$, denote $h^{s'|s}$ as a history in $H^{s'}$ so that $(h^s, h^{s'|s}) \in H^{s'}$. Denote $\tau'|_{\tau}^s$ as a strategy following τ till period s. Finally, define

$$I_i^s := \begin{cases} I_i^{t+1}(m) & \text{if } s \in \mathbf{O}^t, s = m + \sum_{\gamma=1}^{t}(|\mathbf{C}^{\gamma}| + |\mathbf{O}^{\gamma-1}|). \\ I_i^t & \text{if } s \in \mathbf{C}^t. \end{cases}$$

Lemma A.2. *Suppose Rebel i follows an APEX equilibrium τ^* till period s when $0 < \beta_{G_i}^{\tau^*}(\#[R](\theta) \geq k|h_{G_i}^s) < 1$. Let $\tau|_{\tau^*}^s = (\tau_i, \tau_{-i})|_{\tau^*}^s$ be a continuation strategy.*

(1) *If $\tau|_{\tau^*}^s$ generates a history $h^{s'|s}$ in which i will be uncertain about the relevant information and stop belief-updating after some period s', then i will not deviate to $\tau|_{\tau^*}^s$ given sufficient high δ.*

(2) *If, conditional on $E_1 \subseteq \text{supp}(\beta_i)$, $\tau|_{\tau^*}^s$ generates the stage payoff of 0 after $\infty > s' > s$, then Rebel i will not deviate to $\tau|_{\tau^*}^s$ if δ is sufficiently high.*

(3) *If $s \notin \mathbf{C}^1$, then any Rebel has less than k Rebel neighbors.*

(4) *If $s \notin \mathbf{C}^1$ and $\tau|_{\tau^*}^s$ generates detectable deviations, then Rebel i will not deviate to $\tau|_{\tau^*}^s$ if δ is sufficiently high.*

Lemma A.3. *Suppose Rebel i follows an APEX equilibrium τ^* till period s when $0 < \beta_{G_i}^{\tau^*}(\#[R](\theta) \geq k|h_{G_i}^s) < 1$ and $s < T_{\tau^*}^{\theta}$. If $\tau|_{\tau^*}^s = (\tau_i, \tau_{-i})|_{\tau^*}^s$ is a strategy that generates a history $h^{s'|s}$ with positive probability in which i will be uncertain about the relevant information and stop belief-updating after $\infty > s' > s$, then Rebel i will not deviate to $\tau|_{\tau^*}^s$ if δ is sufficiently high.*

Proof. Denote $\beta_{G_i}^{\tau|_{\tau^*}^s}(\theta|h^{s'|s}, h_{G_i}^s)$ as i's belief about θ at s' following $h^{s'|s}$ induced by $\tau|_{\tau^*}^s$. By the postulate, $0 < \beta_{G_i}^{\tau|_{\tau^*}^s}(\#[R](\theta)|h^{s'|s}, h_{G_i}^s) < 1$. From the perspective that i holds a belief of $0 < \beta_{G_i}^{\tau^*}(\#[R](\theta) \geq k|h_{G_i}^s) < 1$ at period s, $h^{s'|s}$ can be thought of a signal at period s' to infer whether or not $\#[R](\theta) \geq k$: if $\#[R](\theta) \geq k$, $h^{s'|s}$ occurs with probability η and does

not occur with probability $1 - \eta$, or if $\#[R](\theta) < k$, $h^{s'|s}$ occurs with probability μ and does not occur with probability $1 - \mu$, subject to $0 \leq \eta \leq 1$, $0 \leq \mu \leq 1$ and $0 < \eta/\mu < \infty$. The last inequality follows the postulate that $h^{s'|s}$ is generated with positive probability by $\tau|^s_{\tau^*}$. Denote $S = \max\{s', T^\theta_{\tau^*}\}$. Following $h^{s'|s}$, Rebel i's maximum expected stage-game payoff starting from S and calculated at period s is

$$w = \max\{\eta\beta^{\tau^*}_{G_i}(\#[R](\theta) \geq k|h^s_{G_i}) - \mu\beta^{\tau^*}_{G_i}(\#[R](\theta) < k|h^s_{G_i}), 0\}.$$

The first term of the above bracket, $\eta\beta^{\tau^*}_{G_i}(\#[R](\theta) \geq k|h^s_{G_i}) - \mu\beta^{\tau^*}_{G_i}(\#[R](\theta) < k|h^s_{G_i})$, is i's expected stage-game payoff if all Rebels play **revolt** afterwards starting from S. The second term 0 is that by playing **stay** afterwards. Rebel i's expected stage-game payoff starting from S by following τ^*, in present value, is

$$\beta^{\tau^*}_{G_i}(\#[R](\theta) \geq k|h^s_{G_i}) > w.$$

The inequality above is due to $0 < \eta < 1, 0 < \mu < 1$ by postulate. There is a difference in present value of

$$v(\delta) = \frac{\delta^{S-s}(\beta^{\tau^*}_{G_i}(\#[R](\theta) \geq k|h^s_{G_i}) - w)}{1 - \delta}.$$

Denote z as the summation of all gains from deviation calculated from period s to period S. z is finite since the stage-game payoff is finite and $S - s$ is finite. Taking sufficiently high $\delta \in (0, 1)$ so that $v(\delta) > z$ will deter this deviation. $\qquad\square$

Lemma A.4. *Suppose Rebel i follows an APEX equilibrium τ^* till period s, in which $0 < \beta^{\tau^*}_{G_i}(\#[R](\theta) \geq k|h^s_{G_i}) \leq 1$ and $s < T^\theta_{\tau^*}$. Let* **supp**(β_i) *be the support of $\beta^{\tau^*}_{G_i}(\#[R](\theta) \geq k|h^s_{G_i})$ and $\tau|^s_{\tau^*} = (\tau_i, \tau_{-i})|^s_{\tau^*}$ be a continuation strategy. If, conditional on $E_1 \subseteq$ supp(β_i), $\tau|^s_{\tau^*}$ generates the stage payoff of 0 after $\infty > s' > s$, then Rebel i will not deviate to $\tau|^s_{\tau^*}$ if δ is sufficiently high.*

Proof. Note that the equilibrium path bring i the highest ex-post payoff for every $\theta' \in$ **supp**(β_i) after $T^{\theta'}_{\tau^*}$. Comparing to the equilibrium path, There is a difference in expected payoff in present value as $-\delta^{S-s}\epsilon/(1 - \delta)$ conditional on E. Conditional on **supp**$(\beta_i) \setminus E$, comparing to the equilibrium path, i's expected payoff in present value is at most $\infty > z > 0$. Taking sufficiently high δ will deter i's deviation. $\qquad\square$

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Lemma A.5. *Suppose Rebel i follows the constructed APEX equilibrium τ^* till period s and $s < T^\theta_{\tau^*}$. If $s \notin \mathbf{C}^1$, then any Rebel has less than k Rebel neighbors.*

Proof. According to the equilibrium path, if there is a Rebel who has k or more Rebel neighbors, then $T^\theta_{\tau^*} \in \mathbf{C}^1$ and $s \in \mathbf{C}^1$ since $s < T^\theta_{\tau^*}$. $\qquad\square$

Lemma A.6. *Suppose Rebel i follows an APEX equilibrium τ^* till period s, in which $0 < \beta^{\tau^*}_{G_i}(\#[R](\theta) \geq k | h^s_{G_i}) < 1$ and $s \notin \mathbf{C}^1$. Let $\mathbf{supp}(\beta_i)$ be the support of $\beta^{\tau^*}_{G_i}(\#[R](\theta) \geq k | h^s_{G_i})$ and $\tau|^s_{\tau^*} = (\tau_i, \tau_{-i})|^s_{\tau^*}$ be a continuation strategy. If $\tau|^s_{\tau^*}$ generates detectable deviations, then Rebel i will not deviate to $\tau|^s_{\tau^*}$ if δ is sufficiently high.*

Proof. Since $s \notin \mathbf{C}^1$, any Rebel has less than k Rebel neighbors by Lemma?. According to the off-path belief, i's detectable deviations will let $n > d > 0$ detectors play **stay** forever in every $\theta' \in \mathbf{supp}(\beta_i)$. Since $0 < \beta_i < 1$, there is an event E consistent with $k + d > \#[R](\theta) \geq k$ and assigned by β_i with positive probability. Conditional on E_3, all continuation histories will bring the stage payoff at most 0 to i after some period s', since no more than $\#[R](\theta) - d$ Rebels will choose **revolt** after s'. If he has followed τ^*, i's stage-game payoff is 1 after T^θ. Therefore, after $S = \max\{T^\theta, s'\}$, there is a difference as least 1 on stage payoff. Lemma? applies to this case, and therefore i will not deviate provided that δ is sufficiently high. $\qquad\square$

Lemma A.7. *Suppose Rebel i plays $\langle$**all stay**$\rangle$ in $\mathbf{O}^t$ for some $t \geq 1$ has not been detected. If there is a difference between i's continuation play and that of a on-path t-block inactive Rebel, then at least one of i's neighbors will detect this difference.*

Proof. If i's playing $\langle$**all stay**$\rangle$ has been detected. If i has not been detected by j. Let j be an active neignobor in the t'-block. Since the network is acyclic, i will not choose *textbfr* before j according to the equilibrium path. If all of j is inactive in the t'-block, since the network is acyclic, j is certain that $\#[R](\theta) < k$, and therefore i should play **stay** forever. $\qquad\square$

Lemma A.8. *Suppose Rebel i plays $\langle 1 \rangle$ in $\mathbf{O}^t$ for some $t \geq 1$. If i does not mimic the equilibrium strategy of a t-block pivotal Rebel afterward, then i's deviation will be detected.*

Proof. If i has not been detected by j. i's on-path continuation play is independent from all of his neighbor. Thus the difference is detected by all of i's neighbors. □

Lemma A.9. *If p is θ-pivotal at t-block, then all t-block active Rebels are in p's neighborhood.*

Proof. The acyclic network assumption is relevant here. The proof is done by contradiction. Assume there is an active Rebel j outside p's neighborhood. Let $j^{'} \in G_j$ on the path from p to j. Since j is active, there is a Rebel $i \in I_j^t$ who cannot be reached from $j^{'}$ by t edges. Since the network is acyclic, the path from p to i through $j^{'}$ and j is unique, and therefore it consists of at least $t+2$ edges. But then p is uncertain $I_p^{t+1} = [R](\theta)$. It contradicts that p is θ-pivotal. □

Lemma A.10. *Suppose Rebel i follows the constructed APEX equilibrium τ^* till period s and $s < T_{\tau^*}^\theta$. Further suppose $s \in \mathbf{O}^t$ and i is pivotal. Rebel i will not deviate to a strategy $\tau|_{\tau^*}^s = (\tau_i, \tau_{-i})|_{\tau^*}^s$ in which i plays $\langle$**all stay**$\rangle$ during $\mathbf{O}^t$, provided that δ is sufficiently high.*

Proof. Since $s \notin \mathbf{C}^1$, any of i's neighbors must have less than k Rebel neighbors by Lemma A.5. Let E_1 be the event consistent with I_i^s and $\#[R](\theta) \geq k$, and let E_2 be the event that all Rebels can be reached by $t+1$ edges from i. Since the prior has full support on strong connectedness, i assigns positive probability on $E_1 \cap E_2$. Since the network is acyclic, conditional on $E_1 \cap E_2$, for every Rebel $j \neq i$, all Rebels in TR_{ij} are in I_i^s while there is a Rebel in TR_{ji} who is not in I_j^s. If i plays $\langle$**all stay**$\rangle$ in $\mathbf{O}^t$, then every Rebel $j \neq i$ will believe with probability 1 that $\#[R](\theta) < k$ after some period $s^{'}$, no matter whether i makes further deviation after $\mathbf{O}^t$. If i does not make detectable deviation after $\mathbf{O}^t$, then i will behave as an inactive Rebel in every $t^{'}$-block where $t^{'} \geq t$. But then j will believe that $\#I_j^{t^{'}+1} < k$ for all $t^{'} \geq t$, and believe with probability 1 that $\#[R](\theta) < k$ after some period, following the equilibrium path and by Lemma A.1. On the other hand, if i makes detectable deviation after $\mathbf{O}^t$, *all* i's neighbors will detect that. They believe with probability 1 that $\#[R](\theta) < k$ according to the off-path belief (Equation 1), and then all other Rebels (other than i) will contagiously believe with probability 1 that $\#[R](\theta) < k$.

Conditional on $E_1 \cap E_2$, all other Rebels will play **stay** after some period s', and thus i's stage payoff is 0 after s'. If i has followed the equilibrium path, i's stage payoff is 1 after $T_{\tau*}^\theta$. Take $S = \max\{T_{\tau*}^\theta, s'\}$. There is a payoff difference in present value as $\delta^{S-s}/(1-\delta)$. Denote z as the summation of all gains from deviation calculated from period s to period S. z is finite since the stage-game payoff is finite and $S - s$ is finite. Conditional on other events, i's stage payoff after some periods will be at most the same as that on the path if i makes no detectable deviation. The payoff will be 0 if i makes detectable deviation by that all i's deviation is detected by all i's neighbors. Since i assigns positive probability on $E_1 \cap E_2$, taking sufficiently high δ will deter i's deviation.

$\square$

Lemma A.11. *Suppose Rebel i has not detected others' deviations according to the constructed APEX equilibrium τ^* till period s. Rebel i will not deviate to a strategy in which $d > 0$ Rebels can detect his deviation, if δ is sufficiently high.*

Proof. Suppose the diameter (the longest length among all shortest paths between vertices) of the network equals to 1. This is an $n = 2$ case, and therefore I omit it.

Suppose the diameter of the network equals to 2. The network is star-like, and therefore $T_{\tau*}^\theta$ must be reached in $\mathbf{C}^1$ on the equilibrium path. Peripheral Rebels will not deviate from the equilibrium path, since, there, they play **stay** until all Rebels play **revolt** when $\#[R](\theta) \geq k$. The central Rebel will not deviate either. This is because, when $\#[R](\theta) \geq k$, any deviation will let all peripheral Rebels play **stay** forever, and when $\#[R](\theta) < k$, any deviation will cost the central Rebel a series of negative stage payoffs.

The following argument is then for the network which the diameter is larger than 2. Remind that a Rebel who detects deviations will choose **stay** forever according to the off-path belief (Equation 1) if this Rebel has less than k Rebel neighbors. Since that, two cases will be discussed: $\#I_i^s < k$ or $\#I_i^s \geq k$. Also, recall that if $s \in \mathbf{C}^1$, it might be the case that any Rebel who detects i's deviation will not choose **stay** forever since such detector has k or more Rebel neighbors. However, if $s \notin \mathbf{C}^1$, then this detector must have less than k Rebel neighbors by Lemma A.5. I first discuss the $s \in \mathbf{C}^1$ case and then proceed to others.

Suppose $I_i^s < k$. Let us first consider $s \in \mathbf{C}^1$ while i is inactive in the 1-block. Inactive

i's continuation strategy $\tau|_{\tau^*}^s$ will not deviate from the equilibrium path during $\mathbf{C}^1$ by the following reason. The state space can be partitioned to two events. In the event that any detector has at least k Rebel neighbors, i can guarantee himself the highest stage payoff of 1 forever immediately after $\mathbf{C}^1(3,1)$ on the path. It is clear that i will not deviate after $\mathbf{C}^1(3,1)$. Before $\mathbf{C}^1(3,1)$, any deviation of i brings negative stage payoff to i, while i can guarantee himself a stage payoff of 0 on the path. Therefore, i will not deviate conditional on this event. In the other event that some detector has less than k Rebel neighbors, i will not deviate by the argument given below, which is for active Rebels in the 1-block. Since i will not be deterred in either of the above two event, i will be on the path during $\mathbf{C}^1$.

Next consider $s \in \mathbf{C}^1$ while i is active in the 1-block. Let E_1 be the event consistent with I_i^s and $\#[R](\theta) \geq k$. Further let E_2 be the event that all detectors each of whom has less than k Rebel neighbors. Note that i assigns positive probability to $E_1 \cap E_2$. This comes from the assumptions that (1) the prior has full support on strong connectedness, (2) the diameter of the network is larger than 3 and (3) every neighbor of i is not fully aware of Rebel neighbors of i by activeness of i. Assumptions (1), (2) and (3) together imply that some Rebels might be outside i's neighborhood while the total of Rebels exceeds k. Denote $E_3 \subseteq E_1 \cap E_2$ as the sub-event that is consistent with $k + d > \#[R](\theta) \geq k$. Assumptions (1), (2) and (3) together also imply that i assigns positive probability on E_3. Conditional on E_3, all continuation histories will bring the stage payoff at most 0 to i after some period s', since no more than $\#[R](\theta) - d$ Rebels will choose **revolt** after s'. If he has followed τ^*, i's stage-game payoff is 1 after T^θ. Therefore, after $S = \max\{T^\theta, s'\}$, there is a difference as least 1 on stage payoff. As for other sub-events in $E_1 \cap E_2$ so that $\#[R](\theta) \geq k + d$ or $\#[R](\theta) < k$, after some periods, i's stage-game payoff is at most the same as that by following τ^*. Denote z as the summation of all gains from deviation calculated from period s to period S. z is finite since the stage-game payoff is finite and $S - s$ is finite. Taking sufficiently high discount factor δ will then deter i's deviation during $\mathbf{C}^1$.

i will not deviate when $s \in \mathbf{C}^1$ as shown above. I proceed to consider $s \notin \mathbf{C}^1$. If $s \notin \mathbf{C}^1$, then the detector must have less than k Rebel neighbors as aforementioned. Applying the above no-deviation argument for the 1-block active Rebels, sufficiently high δ deters i's

deviation at $s \notin \mathbf{C}^1$.

Suppose $I_i^s \geq k$. If $s \in \mathbf{O}^t$, then i is pivotal, and therefore i is on the way of playing $\langle 1 \rangle$. Moreover, on the path, i anticipates $T_{\tau*}^\theta \in \mathbf{C}^{t+1}$ and that the highest stage payoff will be achieved after $\mathbf{C}^{t+1}$. i will not deviate to play $\langle \mathbf{all\ stay} \rangle$ as proven in Lemma A.10. Playing other sequences than $\langle 1 \rangle$ and $\langle \mathbf{all\ stay} \rangle$ will incur less payoff during $\mathbf{O}^t$ in present value *and* will weakly diminish the probability to achieve stage payoff of 1 after $\mathbf{C}^{t+1}$.[35] Hence, i will not deviate when $s \in \mathbf{O}^t$. If $s \in \mathbf{C}^t$, i's detectable deviation is to play a sequence other than what Table 4 prescribes. This implies that this deviation is detected by *all* i's neighbors. Since i must be active, by assumptions (1), (2) and (3) and following the same argument for 1-block active Rebels, i assigns positive probability to the event that all i's neighbors will play **stay** forever. All Rebels will play **stay** contagiously, and thus i's stage payoff is 0 after some s'. If i has followed the equilibrium path, i's stage payoff is 1 after $T_{\tau*}^\theta$. Take $S = \max\{T_{\tau*}^\theta, s'\}$. There is a payoff difference in present value as $\delta^{S-s}/(1-\delta)$. Denote z as the summation of all gains from deviation calculated from period s to period S. z is finite since the stage-game payoff is finite and $S - s$ is finite. Taking sufficiently high δ will deter i's deviation.

$\square$

Lemma A.12. *Suppose Rebel i follows the constructed APEX equilibrium τ^* til period s and $s < T_{\tau*}^\theta$.*

If $s \in \mathbf{O}^t$, then i will not deviates from playing

(o_1) $\langle I_i^t \rangle$ *to* $\langle 1 \rangle$,

(o_2) $\langle I_i^t \rangle$ *to* $\langle \mathbf{all\ stay} \rangle$,

(o_3) $\langle 1 \rangle$ *to* $\langle \mathbf{all\ stay} \rangle$,

(o_4) $\langle 1 \rangle$ *to* $\langle I_i^t \rangle$,

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[35]Incurring less payoff in present value is due to discounting and the prescription that at most one Rebel plays **revolt** in a given period in $\mathbf{O}^t$ on the equilibrium path. The probability to achieve the highest stage payoff is weakly diminished, because there might be some detectors who start to play **stay** forever, by applying the same argument for the 1-block active Rebels.

(o_5) $\langle$ **all stay** $\rangle$ to $\langle I_i^t \rangle$, or

(o_6) $\langle$ **all stay** $\rangle$ to $\langle 1 \rangle$,

during $\mathbf{O}^t$ given that δ is sufficiently high.

If $s \in \mathbf{C}^t$ for $t \geq 1$, then i will not deviates from playing

(c_1) $\langle i \rangle$ to $\langle$ **all stay** $\rangle$,

(c_2) $\langle$ **all stay** $\rangle$ to $\langle i \rangle$,

(c_3) $\mathbf{s}$ to $\mathbf{r}$, or

(c_4) $\mathbf{r}$ to $\mathbf{s}$,

during $\mathbf{C}^t$ given that δ is sufficiently high.

Proof. If these deviation are detectable, they will not made by i by Lemma?. Since that, the pre-assumption for these deviations is that they are undetectable.

I first suppose $s \in \mathbf{O}^t$. Let (o_1) be the case. According to the equilibrium path, i is active and not a free rider in this case. Let E_1 be the event that no Rebel is pivotal and $T_{\tau^*}^\theta$ will not arrive in $\mathbf{C}^{t+1}$. Let E_2 be the event that there is one of i's neighbors, p, who is θ-pivotal, a free rider but not $k-1$-pivotal. Note that if there is a θ-pivotal Rebel, this Rebel should be in i's neighborhood and i's other neighbors are all inactive. This is because the network is acyclic and i is active. Note also that i assigns positive probabilities on either E_1 or E_2, or both. This is because, otherwise, i is certain that there is a $k-1$-pivotal Rebel who is not θ-pivotal. But then i is certain that $\#[R](\theta) \geq k$ since i is active, and hence i's prescribed sequence of actions on the path will not be $\langle I_i^t \rangle$. In both E_1 and E_2, i will be considered to be $k-1$-pivotal if i mimics a $k-1$-pivotal Rebel's behavior after $\mathbf{O}^t$. It will lead i to a history that i will be uncertain about the relevant information forever. Conditional on E_1, in $\mathbf{C}^{t+1}$, i will not be uncertain whether $\#[R](\theta)$ since $T_{\tau^*}^\theta$ does not arrive then. Conditional on E_2, in $\mathbf{C}^{t+1}$, i will be uncertain about p's I_p^t and therefore about whether $\#[R](\theta)$. Since all Rebels will play the same action after $\mathbf{C}^{t+1}$, i will not learn the relevant information in this continuation history. Note that other continuation

histories will make detectable deviations in $\mathbf{C}^t$ or afterword are detectable by Lemma? and cause more Rebels to play **stay** forever. i will not deviate to these continuation histories either.

will not be $k-1$-pivotal. This deviation induces a possible continuation history $h^{s'|s}$ in which i will be uncertain the relevant information after some periods. Lemma A.3 then applies.[36] $\qquad\square$

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[36]A careful examination is as follows. Let E be the event that there are some Rebels who are outside player is $k-1$-pivotal in the t-block. i assigns positive probability on E since If i makes no detectable deviations after $\mathbf{O}^t$, $h^{s'|s}$ happens with positive probability since all i's neighbors play the same action after $\mathbf{C}^{t+1}$. If i makes detectable deviation after $\mathbf{O}^t$