Coordination in Social Networks

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MOTIVATION

- Collective action may fail in the presence of incomplete information.
  - Example of collective action
    - Pro-democracy revolution
    - Raising fund for start-ups

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- How to make collective action successful if people can act repeatedly?

Background

East Germany 1989-1990.

- **Collective action is not static**
  - Protest leads revolution.
- **Public Information is noisy**
  - Mass media is controlled by government.
- **Information is transmitted within social networks**:
  - Church networks

What this paper does?

- **Dynamics of collective action on networks.**

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- **Dynamics of collective action on networks.**
- **How people obtain sufficient information over time to coordinate their actions.**

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- Players linked in a fixed and exogenous network.
- Players of two types (**Rebel**,**Inert**). They can **observe own/neighbor's type**.
- Type-contingent action.
- Pay-off contingent on global type distribution.
- Players choose simultaneously and repeatedly. They can **observe own/neighbor's actions**.

Look for

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Result

- Such equilibrium can be constructed under some assumptions.

Related Literature

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  - **This paper adds network-monitoring**
- Repeated game in networks.
  - **This paper consider incomplete information and imperfect monitoring**

Model

Network

- Let $N = \{1, ..., n\}$ be the set of players.
- G_i is i's neighborhood; G_i is a subset of N such that $i \in G_i$.
- $G = \{G_i\}_i$ is the network.

Assumption

G is fixed (not random), finite, connected, commonly known, and undirected.

Static k-threshold game [Chwe 2000]

- $1 \leq k \leq n$
- $\theta_i \in \Theta_i = \{Rebel, Inert\}$: i's type
- $\theta \in \Theta = \times_{i \in N} \Theta_i$: type profile
- $\pi \in \Delta\Theta$: the prior

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- $\pi \in \Delta\Theta$: the prior
- $A_{Rebel} = \{\textbf{revolt}, \textbf{stay}\}$; $A_{Inert} = \{\textbf{stay}\}$

Static k-threshold game [Chwe 2000], **In this presentation,** Or

- Static game payoff for Rebel i: $u_{Rebel_i}(a_{Rebel_i}, a_{-\theta_i})$

$$\begin{aligned} u_{Rebel_i}(a_{Rebel_i}, a_{-\theta_i}) &= 1 \quad \text{if } a_{Rebel_i} = \textbf{revolt} \text{ and } \#\{j : a_{\theta_j} = \textbf{revolt}\} \geq k \\ u_{Rebel_i}(a_{Rebel_i}, a_{-\theta_i}) &= \text{-1} \quad \text{if } a_{Rebel_i} = \textbf{revolt} \text{ and } \#\{j : a_{\theta_j} = \textbf{revolt}\} < k \end{aligned}$$

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- **stay** is a safe arm; **revolt** is a risky arm.
- Ex-post (Pareto) efficient outcome:
  - If there are at least k Rebels, all Rebels play **revolt**.
  - Otherwise, all Rebels play **stay**.

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- **stay** is a safe arm; **revolt** is a risky arm.
- Ex-post (Pareto) efficient outcome:
  - If there are at least k Rebels, all Rebels play **revolt**.
  - Otherwise, all Rebels play **stay**.
- Relevant information: Whether or not at least k Rebels exist.

Time line (Time is infinite, discrete)

- Nature choose θ initially according to π.
- Players play the static k-threshold game infinitely repeatedly.

ASSUMPTION

- *Players know their neighbors' types.*
- *Players perfectly observe their neighbors' actions.*
- π *has full support*
- *Common* δ*.*
- *Pay-off is hidden (in this presentation)*

- Pay-off could also be noisy or perfectly observable.

Notations:

- $[\textit{Rebels}](\theta) = \{j : \theta_j = \textit{Rebel}\}$ for all $\theta \in \Theta$.
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- θ_{G_i}: i's private information about the state. ($\theta_{G_i} \in \Theta_{G_i} = \prod_{j \in G_i} \Theta_j$)
- $h^m_{G_i}$: the history observed by i up to period m. ($h^m_{G_i} \in H^m_{G_i} = \prod_{s=1}^{m} \prod_{j \in G_i} A_{\theta_j}$)
- h: an infinite sequence of players' actions. ($h \in H = \prod_{s=1}^{\infty} \prod_{j \in N} A_{\theta_j}$)

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- $\tau = (\tau_1, ..., \tau_i, ..., \tau_n)$: a strategy profile.

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- $\beta_i^{\pi,\tau}(\theta | h^m_{G_i})$: i's belief for a θ at period m given τ.

Notations:

- h_θ^τ : a history generated by τ given θ.
- Call h_θ^τ a τ_θ-path.
- Call $\{h_\theta^\tau\}_{\theta\in\Theta}$ the τ-path

Definition

The τ-path is **approaching ex-post efficient** (*APEX*) $\Leftrightarrow$

$$\forall\theta\text{, there is a finite time } T^\theta$$

such that the actions after T^θ in τ_θ repeats the static ex-post efficient outcome.

Definition

$h^m_{G_i}$ is **reached by τ-path**

$$\Leftrightarrow$$

$\exists\theta$ such that $h^m_{G_i}$ is in τ_θ-path.

Lemma

If the τ-path is APEX $\Rightarrow \forall\theta\ \forall i$, there is a finite time T_i^θ such that

$$\sum_{\theta:\#[Rebels](\theta)\geq k} \beta_i^{\pi,\tau}(\theta | h^s_{G_i}) = 1 \ or \ = 0, \ if\ s \geq T_i^\theta$$

whenever $h^s_{G_i}$ is reached by τ-path.

Definition (weak APEX equilibrium)

A weak sequential equilibrium (τ^*, β^*) is APEX $\Leftrightarrow$ τ^*-path is APEX, and β^* is the belief system consistent with τ^*.

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Definition (APEX equilibrium)

A sequential equilibrium (τ^*, β^*) is APEX $\Leftrightarrow$ (τ^*, β^*) is a weak APEX equilibrium and β^* is fully consistent with τ^*[Krep and Wilson 1982].

APEX

- $k = n$: For all networks, an APEX equilibrium can be found.

CASE OF $k = n$

THEOREM ($k = n$)

In any network, if the prior has full support, then for repeated $k = n$ Threshold game, an APEX equilibrium exists whenever δ is sufficiently high.

Sketch of proof:

1. Some Inerts neighbors $\Rightarrow$ play **stay** forever.
2. No Inert neighbor $\Rightarrow$ play **revolt** until **stay** is observed, and then play **stay** forever.
3. There is a finite time T^{θ} such that ex-post efficient outcome repeats afterwards.
4. Any deviation $\Rightarrow$ play **stay** forever.

APEX

- $k < n$: with additional assumptions,
    - acyclic networks (tree networks): a weak APEX equilibrium can be found.
    - cyclic networks: open question.

CASE OF $k < n$

DEFINITION (PATH IN A NETWORK)

A **path** from node i to node j is a sequence of nodes

$$\{i, m_1, m_2, ..., m_n, j\} \text{ without repetition}$$

such that $i \in G_{m_1}, m_1 \in G_{m_2}, ..., m_n \in G_j$.

DEFINITION (ACYCLIC NETWORK (TREE))

A network is **acyclic** $\Leftrightarrow$ the path from node i to node j is unique for all nodes i, j.

CASE OF $k < n$

DEFINITION

θ has **Strong connectedness**$\Leftrightarrow$ for every pair of Rebels, there is a path consisting of Rebels to connect them.

DEFINITION

π has **full support on strong connectedness**$\Leftrightarrow$

$$\pi(\theta) > 0 \text{ if and only if } \theta \text{ has strong connectedness.}$$

- I.e. Commonly certainty of strong connectedness.

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ASSUMPTION

*π has **full support on strong connectedness**.*

- Without this assumption, the game is reduced to incomplete information game without communication.

Case of $k < n$

Theorem ($k \leq n$)

In any acyclic network, if π has full support on strong connectedness, then for repeated $1 \leq k \leq n$ Threshold game, a weak APEX equilibrium exists whenever δ is sufficiently high.

Outline:

1. Communication by actions

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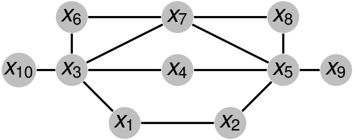
1. Communication by actions
2. Communication in the equilibrium
    1. Communication protocol
    2. In-the-path belief
    3. Off-path belief
    4. Sketch of proof

1. Indexing each node i as a distinct prime number x_i. For instance,



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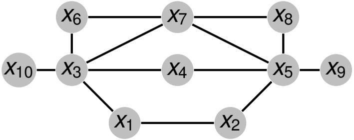
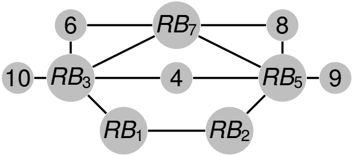


2. Then, in the case of



Rebel 3 report $x_1 \times x_7 \times x_3$ to Rebel 1 by sending a finite sequence

$$\textbf{stay}, \ldots, \textbf{stay}, \underbrace{\textbf{revolt}, \textbf{stay}, \ldots, \textbf{stay}}_{x_1 \times x_7 \times x_3}$$

Communication Phases

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Phases

1. **RP** (Reporting period): revealing the information about θ.
2. **CD** (Coordination period): coordinating the future actions.
3. RP and CD alternate finitely.

$$\underbrace{\langle RP \rangle \langle CD \rangle}\ldots$$

Phases

1. **RP** (Reporting period): revealing the information about θ.
2. **CD** (Coordination period): coordinating the future actions.
3. RP and CD alternate finitely.

$$\underbrace{\langle RP\rangle\langle CD\rangle}_{\textbf{block}}\ldots$$

4. Call a complete two phases, $\langle RP\rangle\langle CD\rangle$, a **block**.

COORDINATION PERIOD AND MESSAGES

In coordination period,

- “three” messages coordinate actions
    1. to **revolt**
    2. to **stay**
    3. to continue to next block

- CD^t: the CD in t-block

$$\overbrace{\underbrace{\langle\langle\cdot\rangle\cdots\langle\cdot\rangle\rangle}_{\text{1st division}}\underbrace{\langle\langle\cdot\rangle\cdots\langle\cdot\rangle\rangle}_{\text{2nd division}}}^{CD^t}$$

- 1st division: sending **message to stay**; otherwise **continue**
- 2nd division: sending **message to revolt**; otherwise **continue**

- CD^t: the CD in t-block

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- 1st division: sending **message to stay**; otherwise **continue**
- 2nd division: sending **message to revolt**; otherwise **continue**
- $CD^t_{p,q}$: the p sub-block in q division.
- $\langle CD^t_{p,q}\rangle$: the messages in $CD^t_{p,q}$ are

$$\begin{array}{ll} \langle \textbf{stay} \rangle & \mathbf{s}, \ldots, \mathbf{s}, \mathbf{s}, \mathbf{s}, \ldots, \mathbf{s} \\ \langle x_i \rangle & \mathbf{s}, \ldots, \mathbf{s}, \underbrace{\mathbf{r}, \mathbf{s}, \ldots, \mathbf{s}}_{x_i} \end{array}$$

- Whenever a Rebel i knows $\#[Rebels](\theta) < k$, he plays **stay** afterward.

$$\overbrace{\langle \underbrace{\langle \textbf{stay} \rangle \ldots}_{\text{1st division}} \rangle \langle \underbrace{\langle \ldots \rangle}_{\text{2nd division}} \rangle}^{CD^t}$$

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$$\overbrace{\langle \underbrace{\langle x_i \rangle \langle x_i \rangle \ldots \langle \cdot \rangle}_{\text{1st division}} \rangle \langle \underbrace{\langle \cdot \rangle \cdots \langle \cdot \rangle}_{\text{2nd division}} \rangle}^{CD^t}$$

Coordination period and messages

2nd division in CD

- Whenever a Rebel i know $\#[Rebels](\theta) \geq k$, he plays

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- ... then nearby Rebel j play $\langle x_j\rangle$ to inform nearby Rebels, and so on.

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- Communication either stops or continues after a CD.
  1. Stopping: If some Rebels learn the relevant information $\Rightarrow$ all Rebels coordinate to play same actions.
  2. Continuing: Otherwise, go to the next block.

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Lemma

*Before a Rebel knows $\#[Rebels](\theta) < k$ or $\#[Rebels](\theta) \geq k$, he will not send **Message to stay** or **Message to revolt** if δ is high enough.*

- a “grim trigger”.

Comment

Reporting period and messages

- RP^t: the reporting period at t block
- $\langle RP^t \rangle$: the reporting message

Burning money	$\neg\langle \textbf{stay} \rangle$	**s**, ..., **s**, **r**, **s**, ..., **s**
Not burning money	$\langle \textbf{stay} \rangle$	**s**, ..., **s**, **s**, **s**, ..., **s**

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- Gives incentive to burn money between.
  1. Burning moneys+**message to revolt**: coordination to **revolt**
  2. Otherwise, no coordination to **revolt**
- How much money should a Rebel burn? — Characterization in the next slides.

INFORMATION HIERARCHY

Information Hierarchy

- Characterizing Rebels' incentives in money burning. ▸ other reason

Ex:

$$0 - 1 - RB_2 - RB_3 - RB_4 - RB_5 - RB_6 - 7$$

Information Hierarchy

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$$0 - 1 - RB_2 - RB_3 - RB_4 - RB_5 - RB_6 - 7$$

- **Rebel 2 has less incentive**: Rebel 2's information can be reported by Rebel 3 to Rebel 4.

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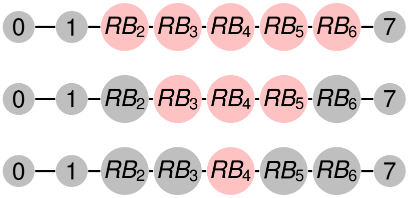

1. At 0-block, let $R^0 = \{2, 3, 4, 5, 6\}$
2. At 1-block, let $R^1 = \{\quad 3, 4, 5 \quad\}$
3. At 2-block, let $R^2 = \{\qquad 4 \qquad\}$

details

The Rebels known by i after t-block: I_i^t.

Theorem

Given θ, if

1. *the network is acyclic*
2. *the state has strong connectedness*

$\Rightarrow \exists t^\theta$ *and* $\exists i \in R^{t^\theta}$ *such that* $I_i^{t^\theta} \supset [Rebels](\theta)$.

Thus, ideally, APEX can be attained by

- At t block

R^t Rebels	play	$\langle I_i^{t-1} \rangle$	$\mathbf{s}, ..., \mathbf{s}, \overbrace{\mathbf{r}, \mathbf{s}, ..., \mathbf{s}}^{\prod_{j \in I_i^{t-1}} x_j}$
non-R^t Rebels	play	$\langle$**stay**$\rangle$	$\mathbf{s}, ..., \mathbf{s}, \mathbf{s}, \mathbf{s}, ..., \mathbf{s}$

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Thus, ideally, APEX can be attained by

- At t block

R^t Rebels	play	$\langle I_i^{t-1} \rangle$	$\mathbf{s}, \ldots, \mathbf{s}, \overbrace{\mathbf{r}, \mathbf{s}, \ldots, \mathbf{s}}^{\prod_{j \in I_i^{t-1}} x_j}$
non-R^t Rebels	play	$\langle \textbf{stay} \rangle$	$\mathbf{s}, \ldots, \mathbf{s}, \mathbf{s}, \mathbf{s}, \ldots, \mathbf{s}$

- However, "Pivotal Rebels" will deviate.

Relevant information: $\#[Rebels](\theta) \geq k$ or $\#[Rebels](\theta) < k$.

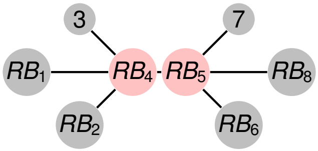
DEFINITION (PIVOTAL PLAYER IN RP^t)

i is **pivotal** in RP^t

$$\Leftrightarrow$$

$i \in R^t$ and i will learn the relevant info before I_i^{t-1} is reported given others' truthful reporting.

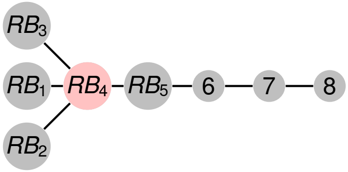
Ex. $k = 5$,



1. Rebel 4 and Rebel 5 are pivotal (**Free Rider problem**)
2. They can manipulate their reporting to save costs.

Go to discussion

Ex. $k = 6$,



1. Rebel 4 is pivotal (given Rebel 5's reporting)
2. He can manipulate his reporting to save costs.

Definition (Free rider in RP^t)

i is a **free rider** in $RP^t \Leftrightarrow$

1. i is pivotal in RP^t
2. i will learn $\#[Rebels](\theta)$ before I_i^{t-1} is reported.

Definition (Free Rider Problem in RP^t)

A **free rider problem** occurs in $RP^t \Leftrightarrow$ There are more than 2 free riders in RP^t.

LEMMA

If networks are acyclic, then

- *there is a unique PR^t where Free Rider Problem may occur.*
- *there are only two free riders i, j are involved. Moreover $i \in G_j$.*
- *Moreover, before PR^t and after CD^{t-1}, i, j both certain that they will be involved in free rider problem.*

Thus, before RP^t and after CD^{t-1}, pick one of them as a free rider.

Non-pivotal R^t Rebels	play	$\langle I_i^{t-1} \rangle$	$\mathbf{s}, ..., \mathbf{s}, \overbrace{\mathbf{r}, \mathbf{s}, ..., \mathbf{s}}^{\prod_{j \in I_i^{t-1}} x_j}$
Pivotal R^t Rebels	may play	$\langle 1 \rangle$	$\mathbf{s}, ..., \mathbf{s}, \mathbf{s}, \mathbf{s}, ..., \mathbf{r}$
non-R^t Rebels	play	$\langle \textbf{stay} \rangle$	$\mathbf{s}, ..., \mathbf{s}, \mathbf{s}, \mathbf{s}, ..., \mathbf{s}$

I.e. Add $\langle 1 \rangle$ into the equilibrium path.

In the equilibrium path,

Lemma

If networks are acyclic,

$$i \text{ is pivotal but } i \text{ is not free rider in } RP^t$$

$\Rightarrow$

$$i \text{ has learned that } \#[Rebels](\theta) \geq k-1 \text{ in } RP^t$$

Lemma

If networks are acyclic,

$$i \text{ play } \langle 1 \rangle \text{ in } RP^t$$

$\Leftrightarrow$

$$i \text{ has learned that } \#[Rebels](\theta) \geq k-1 \text{ in } RP^t$$

Consequently, if i play $\langle 1 \rangle$ in the path

In RP^t, i plays	is i a free rider?	In RP^t, $j \in G_i$ plays	After RP^t, i knows
$\langle 1 \rangle$	yes	$\langle \cdot \rangle$	$\#[Rebels](\theta) \geq k$

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$\langle 1 \rangle$	no	$\langle 1 \rangle$	$\#[Rebels](\theta) \geq k$
$\langle 1 \rangle$	no	$\langle \textbf{stay} \rangle$	$\#[Rebels](\theta) < k$

$\Rightarrow$ i can tell the relevant info. after RP^t.

Consequently, pivotal i has to play **message to stay** or **message to revolt**

Table : Equilibrium path if i played $\langle 1 \rangle$

In RP^t	In $CD^t_{1,1}$	In $CD^t_{1,2}$	After CD^t
i plays	i plays	i plays	
$\langle 1 \rangle$	$\langle$**stay**$\rangle$	$\langle$**stay**$\rangle$	**stay**
$\langle 1 \rangle$	$\langle \mathbf{x}_i \rangle$	$\langle$**stay**$\rangle$	**revolt**

Table : Belief updating after CD^t, $t > 0$

In RP^t	In $CD^t_{1,1}$	In $CD^t_{1,2}$	
i plays	i plays	i plays	The events $j \in G_i$ believes with probability one
$\langle I_i^{t-1} \rangle$	$\langle \mathbf{stay} \rangle$	$\langle \mathbf{stay} \rangle$	$\#[Rebels](\theta) < k$
$\langle I_i^{t-1} \rangle$	$\langle \mathbf{x}_i \rangle$	$\langle \mathbf{stay} \rangle$	$\#[Rebels](\theta) \geq k$
$\langle 1 \rangle$	$\langle \mathbf{stay} \rangle$	$\langle \mathbf{stay} \rangle$	$\#[Rebels](\theta) < k$
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$\langle \textbf{stay} \rangle$	$\langle \mathbf{x}_i \rangle$	$\langle \textbf{stay} \rangle$	$i \notin R^t$
$\langle I_i^{t-1} \rangle$	$\langle \mathbf{x}_i \rangle$	$\langle \mathbf{x}_i \rangle$	$i \in R^t$

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$\langle 1 \rangle$	$\langle \textbf{stay} \rangle$	$\langle \textbf{stay} \rangle$	$\#[Rebels](\theta) < k$
$\langle 1 \rangle$	$\langle \mathbf{x}_i \rangle$	$\langle \textbf{stay} \rangle$	$\#[Rebels](\theta) \geq k$

OFF-PATH BELIEF

Whenever i detects a deviation, he believes that

$$\text{for all } j \notin G_i,\ \theta_j \neq \mathit{Rebel}$$

1. If he has less than k Rebel-neighbors, he will play **stay** forever.

Off-path Belief

Whenever i detects a deviation, he believes that

$$\text{for all } j \notin G_i,\ \theta_j \neq \mathit{Rebel}$$

1. If he has less than k Rebel-neighbors, he will play **stay** forever.
2. This off-path belief then also serve as another “grim trigger” (belief-grim-trigger).

Sketch of proof

1. The equilibrium path is APEX.
2. APEX outcome gives maximum ex-post continuation pay-off after some T.
3. Detectable deviation $\Rightarrow$ belief-grim-trigger. ▸ belief-grim-trigger
4. Undetectable deviation $\Rightarrow$ protocol-grim-trigger. ▸ protocol-grim-trigger
5. Any deviation will let APEX fail in a positive probability.
6. Sufficiently high δ will impede deviation.

1. From the above steps, an APEX equilibrium for **acyclic** networks is constructed.
    - At most **2** free riders will occur. ▸ example
2. Solving Pivotal-player problem for **cyclic** networks need more elaboration.
    - More than **3** free riders will occur. ▸ example

1. payoff is perfectly observed
    - Play **revolt** in the first period, then the relevant information revealed.
2. payoff is noisy
    - With full support assumption, the existing equilibrium is APEX.
    - Ex.

$$
\begin{aligned}
p_{1s} &= \Pr(y = y_1 | \#\textbf{revolt} \geq k) \\
p_{1f} &= \Pr(y = y_1 | \#\textbf{revolt} < k) \\
p_{2s} &= \Pr(y = y_2 | \#\textbf{revolt} \geq k) \\
p_{2f} &= \Pr(y = y_2 | \#\textbf{revolt} < k)
\end{aligned}
$$

$$
1 > p_{1s} > 0, 1 > p_{2s} > 0, p_{1f} = 1 - p_{1s}, p_{2f} = 1 - p_{2s} \tag{1}
$$

Further works

1. Cyclic networks.
2. A general model in which players can communicate only by their actions to learn the relevant information in finite time when $\delta < 1$, while the communication protocol itself is an equilibrium.
3. Equilibrium selection.

OR, Static k-threshold game [Chwe 2000]

- Static game payoff for Rebel i: $u_{Rebel_i}(a_{Rebel_i}, a_{-\theta_i})$

$$
\begin{aligned}
u_{Rebel_i}(a_{Rebel_i}, a_{-\theta_i}) &= 1 && \text{if } a_{Rebel_i} = \textbf{revolt} \text{ and } \#\{j : a_{\theta_j} = \textbf{revolt}\} \geq k \\
u_{Rebel_i}(a_{Rebel_i}, a_{-\theta_i}) &= \text{-1} && \text{if } a_{Rebel_i} = \textbf{revolt} \text{ and } \#\{j : a_{\theta_j} = \textbf{revolt}\} < k
\end{aligned}
$$

$$
\begin{aligned}
u_{Rebel_i}(a_{Rebel_i}, a_{-\theta_i}) &= 1 && \text{if } a_{Rebel_i} = \textbf{stay} \text{ and } \#\{j : a_{\theta_j} = \textbf{revolt}\} \geq k \\
u_{Rebel_i}(a_{Rebel_i}, a_{-\theta_i}) &= 0 && \text{if } a_{Rebel_i} = \textbf{stay} \text{ and } \#\{j : a_{\theta_j} = \textbf{revolt}\} < k
\end{aligned}
$$

EXAMPLE: PAY-OFF IS OBSERVABLE

If pay-off is observable, an Apex Equilibrium for $k = n = 3$ in

- At 1st period

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- After 1st period
  - If the pay-off is observed as 1, choose **revolt** afterwards.
  - Otherwise, choose **stay** afterwards.

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  - Rebel 2 chooses **revolt** if he observes $\theta = (\textit{Rebel}, \textit{Rebel}, \textit{Rebel})$; Otherwise, chooses **stay** forever.
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  - If Rebel 2 chooses **revolt** in the last period, then Rebel 1 (or Rebel 3) chooses **revolt** forever;

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- The player who knows the relevant info. is willing to send messages.

- However, sending message to reveal information in RP is costly.
- A free rider problem in PR may occur.

Appendix: From CD to PR

1. $k = 5$
2. Only one block (RP and then CD).
3. No expected cost in CD.

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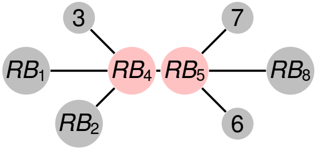
3. No expected cost in CD.
4. Free riders:



1. $k = 5$
2. Only one block (RP and then CD).
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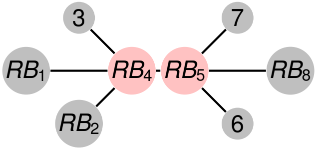


Why? By backward induction,

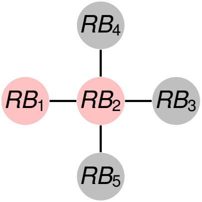
1. No expected cost to send **Message to stay** or **Message to revolt** in CD.
2. If RB_5 report truthfully, RB_4 can wait for that.
3. If RB_4 report truthfully, RB_5 can wait for that.

Go back to CD

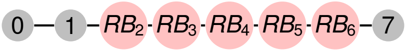
Main goal of **Information Hierarchy**

- Easing the punishment scheme when monitoring is imperfect.

Ex: $k = 4$,



1. **Rebel 1 can only be monitored by Rebel 2.**
2. Suppose Rebel 2,3,4,5 can coordinate at period T and play **revolt** forever.
3. If Rebel 1 did not burn money at period $T-1$, Rebel 2 has no incentive to punish him.



At 1-block, first let

$$
\begin{aligned}
G_i^0 &\equiv G_i \\
I_i^0 &\equiv G_i \cap R^0
\end{aligned}
$$

For instance,

$$
\begin{aligned}
I_2^0 &= \{2,3\} \qquad & G_2^0 &= \{1,2,3\} \\
I_3^0 &= \{2,3,4\} \qquad & G_3^0 &= \{2,3,4\}
\end{aligned}
$$

$$0 - 1 - RB_2 - RB_3 - RB_4 - RB_5 - RB_6 - 7$$

Then define

$$\leq^0$$

by

$$i \in \leq^0 \Leftrightarrow \exists j \in \bar{G}_i (I_i^0 \subseteq G_j^0 \cap R^0)$$

- For instance,

$$2 \in \leq^0, 3 \notin \leq^0$$

- Since

$$I_2^0 = \{2,3\} \qquad G_2^0 \cap R^0 = \{2,3\}$$
$$I_3^0 = \{2,3,4\} \qquad G_3^0 \cap R^0 = \{2,3,4\}$$

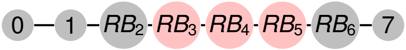
Appendix-Information Hierarchy

$$0 - 1 - RB_2 - RB_3 - RB_4 - RB_5 - RB_6 - 7$$

At 1-block, let

$$R^1 \equiv \{i \in R^0 | i \notin \leq^0\} = \{ \quad 3, 4, 5 \quad \}$$

APPENDIX-INFORMATION HIERARCHY



At 2-block, let

$$
\begin{aligned}
G_i^1 &\equiv \bigcup_{k \in I_i^0} G_k \\
I_i^1 &\equiv \bigcup_{k \in G_i \cap R^1} I_k^0
\end{aligned}
$$

For instance,

$$
\begin{aligned}
I_3^1 &= \{2,3,4,5\} \qquad & G_3^1 &= \{1,2,3,4,5\} \\
I_4^1 &= \{2,3,4,5,6\} \qquad & G_4^1 &= \{2,3,4,5,6\}
\end{aligned}
$$

$$0 - 1 - RB_2 - RB_3 - RB_4 - RB_5 - RB_6 - 7$$

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- Since

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$$I_4^1 = \{2,3,4,5,6\} \qquad G_4^1 \cap R^0 = \{2,3,4,5,6\}$$

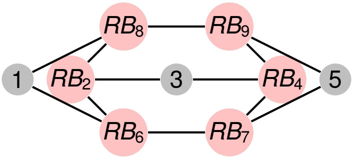
$$0 - 1 - RB_2 - RB_3 - RB_4 - RB_5 - RB_6 - 7$$

At 2-block, let

$$R^2 \equiv \{i \in R^1 | i \notin \leq^1\} = \{ \quad 4 \quad \}$$

Go back to IH

More than 3 free riders will occur **at** a block in cyclic network.



We may pick one of free riders.

Go to discussion

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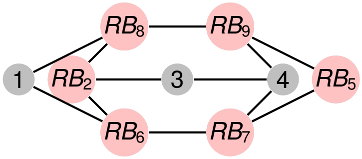

We may pick one of free riders. How to pick?

Go to discussion