Coordination in Social Networks: Communication by Actions

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Abstract

1 Introduction

2 Model

Given a finite set X, denote $\#X$ as its cardinality .

There is a set of players $N = \{1, 2, ..., n\}$. They constitute a network $G = (V, E)$ so that the vertices are players ($V = N$) and an edge is a pair of them (E is a subset of the set containing all two-element subsets of N). Throughout this paper, G is assumed to be finite, commonly known, fixed, undirected, and connected.[1]

Time is discrete with index $s \in \{0, 1, ...\}$. At $s = 0$, the nature chooses a state $\theta \in \Theta = \{R, I\}^n$ once and for all according to a common prior π. R and I represent as Rebel and Inert respectively. After the nature moves, players play a normal form game, the *k-threshold game*, infinitely repeated played with common discounted factor δ. In the k-threshold game,

[1]A path in G from i to j is a finite sequence $(l_1, l_2, ..., l_L)$ without repetition so that $l_1 = i$, $l_L = j$, and $\{l_q, l_{q+1}\} \in E$ for all $1 \leq q < L$. G is fixed if G is not random, and G is undirected if there is no order relation over each edge. G is connected if, for all $i, j \in N$, $i \neq j$, there is a path from i to j.

$A_R = \{\textbf{revolt}, \textbf{stay}\}$ is the set of actions for R while $A_I = \{\textbf{stay}\}$ is that for I. A Rebel's static payoff function is defined as follows.

- $u_R(a_R, a_{-i}) = 1$ if $a_R = \textbf{revolt}$ and $\#\{j : a_j = \textbf{revolt}\} \geq k$
- $u_R(a_R, a_{-i}) = -1$ if $a_R = \textbf{revolt}$ and $\#\{j : a_j = \textbf{revolt}\} < k$
- $u_R(a_R, a_{-i}) = 0$ if $a_R = \textbf{stay}$

. An Inert's static payoff is equal to 1 no matter how other players play.

For the sake of convenience, let $[R](\theta)$ be the set of Rebels given θ and the notion *relevant information* indicate to the information about whether or not $\#[R](\theta) \geq k$.

In the stage game, the ex-post efficient outcome is that every Rebel plays **revolt** if and only if $\#[R](\theta) \geq k$; otherwise, the one is that every Rebel plays **stay**. Moreover, at every θ and every k, the ex-post efficient outcome is unique and gives the maximum as well as the same payoff to every Rebel .

During the game, any player, say i, can observe information only from himself and from his direct neighbors $G_i = \{j|\{i,j\} \in E\}$. These pieces of information include his and his neighbors' types ($\theta_{G_i} \in \Theta_{G_i} = \{R, I\}^{G_i}$) and his and their histories of actions up to period s ($h^s_{G_i} \in H^s_{G_i} \equiv \times^s_{t=1}(\times_{j \in G_i} H^t_j)$). I assume that payoffs are hidden to emphasize that observing neighbors' actions are the only channel to infer other players' types and actions.[2] To be precise, when θ is realized at $s = 0$, i's information set about θ is $P_i(\theta) \equiv \{\theta_{G_i}\} \times \{R, I\}^{N \setminus G_i}$. For the information sets about players' actions, the sets of histories of actions are given to be empty at $s = 0$. At $s > 0$, a history of actions played by i is $h^s_i \in H^s_i \equiv \emptyset \times A^s_i$ while a history of actions played by all players is $h^s \in H^s \equiv \times^s_{t=1}(\times_{j \in N} H^t_j)$. i's information set about other players' histories of actions up to $s > 0$ is $\{h_{G_i}\} \times H^s_{N \setminus G_i}$. A player i's pure behavior strategy τ_i is a measurable function with respect to his information partition if it maps $P_i(\theta) \times \{h_{G_i}\} \times H^s_{N \setminus G_i}$ to a single action in his action set for every s and for every θ.

By abusing the notation a bit, let h^τ_θ denote the realized sequence of actions generated by $\tau = (\tau_1, \tau_2, ..., \tau_n)$ given θ. Define $\mu^{\pi;\tau}_{G_i}(\theta, h^s | \theta_{G_i}, h^s_{G_i})$ as the conditional distribution over

[2] Such restriction will be relaxed in the Section 5.

$\Theta \times H^s$ conditional on i's information up to s, which is induced by π and τ. The belief of i over θ up to s is then $\beta_{G_i}^{\pi,\tau}(\theta|\theta_{G_i}, h_{G_i}^s) \equiv \sum_{h^s \in H^s} \alpha_{G_i}^{\pi,\tau}(\theta, h^s|\theta_{G_i}, h_{G_i}^s)$.

The equilibrium concept is the weak sequential equilibrium.[3] I am looking for the existence of approaching ex-post efficient equilibrium (*APEX equilibrium henceforth*), which is formally defined below.

Definition 2.1 (APEX strategy). *A behavior strategy τ is APEX if, for all θ, there is a terminal period $T^\theta < \infty$ such that the actions in h_θ^τ after T^θ repeats the static ex-post Pareto efficient outcome.*

Definition 2.2 (APEX equilibrium). *An equilibrium (τ^*, α^*) is APEX if τ^* is APEX.*

In an APEX strategy, all Rebels will play **revolt** forever after some period only if there are more than k Rebels; otherwise, Rebels will play **stay** forever after some period. It is as if the Rebels will learn the relevant information in the equilibrium. This is because, they will play the ex-post efficient outcome after a certain point of time and keep on doing so. Note that there are some implications based on the definition. Firstly, it is an equilibrium if every player plays **stay** forever. Secondly, in an APEX equilibrium, it is not only as if the Rebels will learn the relevant information: they must learn that by the following lemma.

Lemma 2.1 (Learning in the APEX equilibrium). *If the assessment (τ^*, μ^*) is an APEX equilibrium, then for all $\theta \in \Theta$, there is a finite time T_i^θ for every Rebel i so that*

$$\sum_{\theta \in \{\theta : [R](\theta) \geq k\}} \beta_{G_i}^{\pi,\tau^*}(\theta|h_{G_i}^s) = \text{ either } 1 \text{ or } 0$$

whenever $s \geq T_i^\theta$.

[3]A weak sequential equilibrium is an assessment $\{\tau^*, \mu^*\}$, where μ^* is a collection of distributions over players' information sets with the property that, for all i and for all s, $\mu_{G_i}^*(\theta, h^s|\theta_{G_i}, h_{G_i}^s) = \mu_{G_i}^{\pi,\tau^*}(\theta, h^s|\theta_{G_i}, h_{G_i}^s)$ whenever the information set is reached with positive probability given τ^*. Moreover, for all i and for all s, τ_i^* maximizes i's continuation expected payoff conditional on θ_{G_i} and $h_{G_i}^s$ of

$$E_G^\delta(u_{\theta_i}(\tau_i, \tau_{-i}^*)|\alpha_{G_i}^{\pi,\tau_i,\tau_{-i}^*}(\theta, h^s|\theta_{G_i}, h_{G_i}^s))$$

for all $h_{G_i}^s$.

Proof. In Appendix. □

Definition 2.3 (Learning the relevant information)**.** *A Rebel i learns the relevant information at period $\dot{s}$ if $\sum_{\theta\in\{\theta:[R](\theta)\geq k\}}\beta^{\pi,\tau^*}_{G_i}(\theta|h^s_{G_i}) =$ either 1 or 0 whenever $s\geq\dot{s}$.*

It is clear that an APEX equilibrium exists when $k=1$. As for other cases, let us start with the case of $k=n$ and then continue on to the case of $1<k<n$.

3 APEX equilibrium for $k=n$

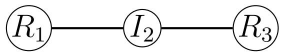
In the case of $k=n$, a Rebel can get a better payoff from playing **revolt** than that from **stay** *only if* all players are Rebels. Two consequences follow. Firstly, if a Rebel has an Inert neighbor, this Rebel will always play **revolt** in the equilibrium. Secondly, at any period of time, it is credible for every Rebel to use playing stay forever afterwards as a punishment to a deviation if there is another player who also plays **stay** forever afterwards, independently from the belief held by the punisher. These two features constitute an APEX equilibrium and further transform itself to a sequential equilibrium.

Theorem 1 (APEX equilibrium for the case of $k=n$)**.** *For any n-person repeated k-Threshold game with parameter $k=n$ played in a network, there is a δ^* such that a sequential APEX equilibrium exists whenever $\delta>\delta^*$.*

Proof. In Appendix. □

The proof is a contagion argument. Suppose a Rebel plays **revolt** at any period except: (1) he has an Inert neighbor, or (2) he has observed his Rebel neighbor played **stay** once. Since the network is finite and connected, a Rebel is certain that there is an Inert somewhere if he has seen his neighbor played **stay**; otherwise, he continues to believe that all players are Rebels. Upon observing a n consecutive **revolt**, it implies that no Inert exists; if not, it implies some Inert exist. The above strategy is an APEX strategy and therefore ready for the equilibrium path for an APEX equilibrium. For any deviation from the above strategy, let the out-of-path strategy be playing **stay** forever for both of the deviant and the Rebel

Figure 1: The state and the network in which the APEX equilibrium does not exist when $k = 2$.



(the punisher) who detects that. This out-of-path strategy is credible for both the deviant and the punisher and is independent from the belief held by the punisher. Hence, it is also sequential rational.

4 APEX equilibrium for $1 < k < n$

In contrast to the case of $k = n$, a Rebel still has the incentive to play **revolt** even if he has an Inert neighbor. This opens a possibility of non-existence of APEX equilibrium. Let us consider Example 1 below.

Example 1. Suppose that $k = 2$ and $\theta = (R, I, R)$. The state and the network is represented in Figure 1. Rebel 1 never learns θ_3 since Inert 2 cannot reveal information about θ_3. The APEX equilibrium does not exist in this scenario.

The following condition that works on the prior *full support on strong connectedness* excludes the possibility of non-existence of APEX equilibrium. To this end, I begin with the definition of *strong connectedness*.

Definition 4.1 (Strong connectedness)**.** *Given G, a state θ has strong connectedness if, for every two Rebels, there is a path consisting of Rebels to connect them.*

In the language of graph theory, this definition is equivalent to: given G, θ has strong connectedness if the induced graph by $[R](\theta)$ is connected.

Definition 4.2 (Full support on strong connectedness)**.** *Given G, π has full support on strong connectedness if*

$$\pi(\theta) > 0 \Leftrightarrow \ \theta \textit{ has strong connectedness}$$

As a remark, the definition of the full support on strong connectedness is stronger than common knowledge about that every state has strong connectedness. This marginal requirement is subtle and is more convenient in constructing equilibrium.[4]

I am ready to state the main characterization of this paper:

Theorem 2 (APEX equilibrium for the case of $1 < k < n$)**.** *For any n-person repeated k-Threshold game with parameter $1 < k < n$ played in networks, if networks are acyclic and if π has full support on strong connectedness, then there is a δ^* such that an APEX equilibrium exists whenever $\delta > \delta^*$.*[5]

Constructing an APEX equilibrium, in this case, is more convoluted than that in the case of $k = n$. I illustrate the proof idea throughout this paper till Section, while leaving the formal proof in Appendix. Moreover, since the case of $k = 2$ is trivial, the depictions throughout this paper below are all for $2 < k < n$ cases.

In the case of $k = n$, T^θ can be determined independently from θ by setting $T^\theta = n$, but it is not obvious how to obtain T^θ before an equilibrium has been constructed.[6] Moreover, the free-rider problem might exist in the current case (as demonstrated in Introduction), but this problem never occurs in the proposed APEX equilibrium for Theorem 1. As for the punishment scheme, playing **stay** forever is not effective anymore since a deviation might only be seen by parts of players (network monitoring), and thus group punishment is hard to be executed.

Before we delve further into the logic of the proof of Theorem 2, I would like to introduce a game in Section 4.1, *T-round writing game*, to be an auxiliary scenario that is simpler but mimics relevant features in the original game to shed light on the equilibrium construction. In this simpler version, I allow players to write to each other. To be more precise, they will be endowed a writing technology so that they can write without cost (cheap talk), with a fixed cost, or with a cost function before they play actions.

[4]The main result only requires a weaker version: $\pi(\theta) > 0 \Rightarrow$ θ has strong connectedness . However, working on this weaker version is at the expense of much tedious proof. Throughout this paper, I will stick to the original definition.

[5]A network is acyclic if the path from i to j for all $i \neq j$ is unique.

[6]Readers might refer to the proof of Theorem 1.

The equilibrium construction of T-round writing game will be studied by means of examples and intentionally serve to demonstrate how to solve the free-rider problem. I then argue the equilibrium construction in the original game to be an analogue to the one in the T-round writing game.

Roughly speaking, in the T-round writing game, players can write to share information about θ for T-round. They then play a one-shot k-threshold game at round $T+1$. Note that in an APEX equilibrium path in the original game, players would stop update their belief after some finite time and keep playing the same action in the k-threshold game. The game form of the T-round writing game mimics the structure of the APEX equilibrium path in the original game game. I consider in order the case of writing without cost, writing with a fixed cost, and then writing with cost function. I then modify the T-round writing game to allow that T can be endogenously determined in the equilibrium, which is further analogous to the original game.

4.1 Deterministic T-round writing game

The network, the set of states, and the set of players follow exactly the same definitions defined in Section 2. In the deterministic T-round writing game, each player endows a *writing technology.* A writing technology for player i is a pair of (W, M_i), in which $W = \{\mathbf{r}, \mathbf{s}\}^L$, $L \in \mathbb{N}$, and $M \equiv \times_{t=1}^{T} M_i^t$ recursively defined by

$$M_i^1 = \{f | f : \Theta_{G_i} \to W\} \cup \{\emptyset\}$$

$$\text{for } 2 \leq t \leq T \text{ , } M_i^t = \{f | f : \times_{j \in G_i} M_j^{t-1} \to W\} \cup \{\emptyset\}.$$

W is interpreted as the set of sentence composed by letters $\mathbf{r}$ or $\mathbf{s}$ with length L, while M_i can be understood as i's grammar. The $\emptyset$ is interpreted as keeping silent. The meaning of "i writes to his neighbors at round t" is equivalent to "i selects an $f \in M_i^t$ to get an element $w \in W$ according to f. Moreover, m can be observed by all i's neighbors". A sentence combined by $w, w' \in W$ is denoted as $w \oplus w'$ with the property that $(w \oplus w')_l = \mathbf{r}$ if and only if $w_l = \mathbf{r}$ or $w'_l = \mathbf{r}$ for all $l \in \{1, 2, ..., L\}$.

The time line for the deterministic T-round writing game is as follows.

Figure 2: A configuration of the state and the network in which player 1,2,4,5,6,8 are a Rebel while player 3,7 are Inerts.

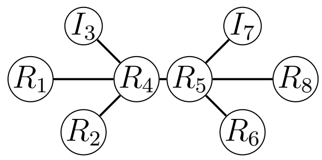
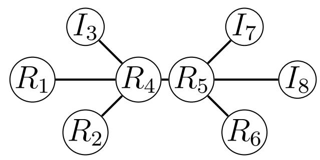


Figure 3: A configuration of the state and the network in which player 1,2,4,5,6 are a Rebel while player 3,7,8 are Inerts.



1. Nature choose θ according to π.

2. Types are then fixed over time.

3. At $t = 1, ..., T$ round, players write to their neighbors.

4. At $T + 1$ round, players play a one-shot k-Threshold game.

5. Game ends.

There is no discounting. A Rebel's payoff is the summation of his stage payoff across stages, while an Inert's payoff is set to be 1. The equilibrium concept is weak sequential equilibrium. The definition of APEX strategy is adapted as the strategy that induces ex-post outcome in the k-threshold game at $T + 1$ round, and the definition of APEX equilibrium is adapted accordingly. In examples below, let us focus on the configuration represented in Figure 2 and Figure 3 with $n = 8$ and $L = 8$. Note that the difference in player 8's type is the only difference between these two configurations. There are specific k and T in each example, and I characterize an APEX equilibrium there.

Example 2 (Deterministic T-round writing without cost—cheap talk)**.** Let $k = 6$ and $T = 2$. Assume that writing is costless. Let us consider the following strategy ϕ on its path. Suppose the state and the network are represented in Figure 2.

At $t = 1$, ϕ specifies that the peripheral Rebels 1,2,6,8 keep silent; the central player Rebel 4 writes $(\mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s}, \mathbf{s})$; the central player Rebel 5 writes $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r})$.[7] On the path of ϕ, Rebel 4's sentence thus reveals that players 1,2,4,5 are Rebels while players 3 is an Inert. It reveals so by the grammar that $\mathbf{r}$ is written in the i-th component if player i is a Rebel while $\mathbf{s}$ is written in the j-th component if player j's type is Inert or unknown to Rebel 4. Rebel 5's sentence is written according to the same grammar. Note that the common knowledge of the network structure contributes to the ability in revealing players' types.

At $t = 2$, ϕ specifies that the peripheral Rebels 1,2,6,8 still keep silent; Rebel 4 writes $(\mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r})$; Rebel 5 writes $(\mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r})$. This is to say Rebel 4 and 5 share information at $t = 1$ and then coordinate to announce a combined sentence at $t = 2 = T$.

At $t = 3 = T + 1$, all Rebels knows that the number of Rebels (by counting $\mathbf{r}$ in Rebel 4 or 5's combined sentence) is greater than or equal to $k = 6$. This leads all Rebels to play the ex-post efficient outcome **revolt** in the k-threshold game.

Suppose the configuration is that in Figure 3. At $t = 1$, similarly, ϕ specifies that Rebels 1,2,6,8 keep silent; Rebel 4 writes $(\mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s}, \mathbf{s})$; Rebel 5 writes $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s})$. At $t = 2$, however, ϕ specifies that the peripheral Rebels 1,2,6,8 keep silent; Rebel 4 keeps silent; Rebel 5 keeps silent as well. On the path, keeping silent by Rebel 4 (or 5) reflects that Rebel 4 (or 5) knows that the total number of Rebels is less than $k = 6$. At $t = 3$, all Rebels know this relevant information and play the ex-post efficient outcome **stay**.

Hence ϕ is a candidate for an APEX equilibrium path. Let ν be the belief system and the in-path belief of ν is induced by ϕ. For the assessment off the path, the out-of-path strategy of ϕ could be made as follows. If a Rebel make a detectable deviation detected by some others, then the Rebels who detect that deviation keeps silent until $t = T$ and then

[7] The notion of "peripheral" and "center" will be formalized in Section 4.3.1

play **stay** at $t = T + 1$.[8] The out-of-path belief of ν is to believe that all players who are not neighbors are all Inerts.

Since writing is costless, and any deviation by Rebel 4 or 5 would strictly decrease the possibility to achieve ex-post efficient outcome, the assessment (ϕ, ν) constitutes an APEX equilibrium.

Example 3 (Deterministic T-round writing with a fixed cost)**.** Let $k = 6$ and $T = 2$. Suppose that writing incurs a fixed cost $\epsilon > 0$ while keeping silent does not. Let us consider the assessment (ϕ, ν) in the above example. Since any deviation by Rebel 4 or 5 would strictly decrease the possibility to achieve the ex-post efficient outcome while the ex-post efficient outcome will give the maximum expected payoff for every Rebel at $t = 3 = T+1$ if the relevant information can be revealed then, if ϵ is sufficiently small, (ϕ, ν) also constitutes an APEX equilibrium.

Example 4 (Deterministic T-round writing with cost function—free-rider problem)**.** Let $k = 6$ and $T = 2$. Suppose that keeping silent incurs no cost, but writing incurs a cost $\epsilon > 0$ that is strictly decreasing with the number of $\mathbf{r}$ in a sentence. This is to say writing $(\mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r})$ incurs the least cost while writing $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{s})$ incurs the largest.

If so, that assessment (ϕ, ν) in the previous two examples will no longer be an APEX equilibrium. To see this, first note that the sentence of either Rebel 4 or 5's truthfully reveals their information at $t = 1$ on the path of ϕ. Since that, Rebel 4 will know the relevant information after $t = 1$ (by common knowledge of the network structure) even if he deviates to writing the sentence that indicates that all his neighbors are Rebels.[9][10] Rebel 5 is in the same situation as Rebel 4 and therefore also write the sentence that indicates that all his neighbors are Rebels. However, these sentences are uninformative. It turns out that both of them will deviate, and neither of them can know relevant information after $t = 1$.

However, free-rider problem will be solved by the following example.

[8] For instance, a wrong sentence that is not according to any grammar, deviating from the in-path ϕ, etc

[9] This sentence is $(\mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s}, \mathbf{s})$, which incurs less cost than the truthfully reporting sentence $(\mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s}, \mathbf{s})$.

[10] If he keeps silent, then this benavior wil be considered as a deviation, and therefore he will never get the maximum payoff of 1. Hence, he will avoid doing so.

Example 5 (Deterministic T-round writing with cost function—solving free-rider problem)**.** The free-rider problem occurs in the previous example can be solved. The solution is to add more rounds to the game and exploit the assumption of common knowledge about the network. More precisely, let $k = 6$ and $T = 3$.

Consider a strategy $\phi^{'}$ and focus on the interaction between Rebel 4 and 5. On the path of $\phi^{'}$, at $t = 1$, $\phi^{'}$ specifies that the Rebel with lowest index between Rebel 4 and 5 is the "free rider", while the other Rebel write his information truthfully.

This is to say, at $t = 2$, Rebel 4 will be the free rider—who writes the least-cost sentence; Rebel 5 writes $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r})$ in the configuration of Figure 2 and writes $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s})$ in the configuration of Figure 3.

At $t = 2$, Rebel 5 keeps silent; Rebel 4 writes the least-cost sentence if Rebel 5 writes $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r})$ at $t = 1$ but keeps silent if Rebel 5 writes $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s})$ then. Thus, Rebel 4' behavior at $t = 2$ reveals the relevant information.

At $t = 3 = T$, Rebel 4 keeps silent; Rebel 5 writes the least-cost sentence if Rebel 4 writes the least-cost sentence at $t = 2$ and keeps silent if Rebel 4 keeps silent then.

It is straightforward to check that at $t = 4 = T + 1$, all Rebels know the relevant information and play the ex-post efficient outcome accordingly. To construct an APEX equilibrium from $\phi^{'}$, recall (ϕ, ν) and let the in-path belief of $\nu^{'}$ be induced by $\phi^{'}$. The out-of-path strategy follow that in ϕ, and the out-of-path belief follow that of ν.

An observation is worthy to be noted: why is Rebel 5 willing to concede that Rebel 4 is chosen to be the free rider at $t = 1$? The reason is as follows. He *knows* that, by common knowledge about the network, he and Rebel 4 are in a free-rider problem. Moreover, again by common knowledge about the network, he knows that Rebel 4 knows this, Rebels 4 knows he knows that Rebel 4 knows this,..., and so forth to infinite order of belief hierarchy. This is to say, at least in this case, Rebel 4 and 5 commonly known that they are engaged in a free-rider problem due to the common knowledge assumption on network. In Section 4.3.2, this property of common knowledge about engaging in a free-rider problem will be formally characterized. This property is not a merely special case. It holds for any acyclic network in the constructed APEX equilibrium in the original game.

Figure 4: The linearly ordered labelled rounds in the indeterministic T-round writing game after θ is realized.

$$0_1^{'} < 0_2^{'} < ... < 0_{l_0}^{'} < 1 < 1_1^{'} < 1_2^{'} < ... < 1_{l_1}^{'} < 2 < ...,$$

where $l_0, l_1, ...$ are all finite numbers.

4.2 Indeterministic T-round writing game

In this section, the setting is exactly the same as that in the deterministic T-round writing game, except for that players can now jointly decide the round in which they will play the one-shot k-threshold game and then end the game. In other words, before they play the one-shot k-threshold game, they have to *reach an agreement*—the common knowledge about which round is the terminal round T.

The set of rounds is countably infinite and linearly ordered with generic element t. The writing technology is the same as that in deterministic T-round writing game, except for letting $W = \{\mathbf{r}, \mathbf{s}\}^L \cup \{\mathbf{r}, \mathbf{s}\}^{L^{'}}$ now. In the example below, let $L = 8$ and $L^{'} = 1$.

Conceptually, there could be two kinds of rounds. In the first kind, players write to share information about θ (as they do in the deterministic T-round writing game). In the second kind, players write to form the common knowledge of T. Let us partition the set of rounds into two parts, Γ and $\Gamma^{'}$, which represent the first kind and the second kind respectively. The round in Γ is labelled by γ while the round in $\Gamma^{'}$ is labelled by $\gamma_l^{'}$. The rounds is linearly ordered by $<$ (after θ is realized). To be more precise, the rounds is ordered as shown in Figure 4. As an indeterministic T-round writing game is illustrated below, an APEX equilibrium is constructed .

Example 6 (Indeterministic T-round writing with cost function)**.** Let $l_j = 2$ for $j = 0, 1, ...$ Suppose that the setting is exactly the same as that in Example 5, except for that T is not deterministic. Let us consider the path of a strategy ψ. At a round in $\Gamma^{'}$, ψ specifies that, if a Rebel thinks "it is certain that the total number of Rebels outnumbers k and the nearest forthcoming round in Γ is the terminal round," he write ($\mathbf{r}$); if a Rebel thinks “it is possible but not certain that $\#[R](\theta) \geq k$,” he writes $(\mathbf{s})$; otherwise, he writes $\emptyset$ to show that “it is impossible that $\#[R](\theta) \geq k$ and the nearest forthcoming round in Γ is the terminal round.”

According to this strategy, $t = 1$ is not terminal if no Rebel has written $(\mathbf{r})$ or $\emptyset$ before that. For instance, $t = 1$ is not terminal in the configuration in Figure 2 and Figure 3.

If $t = 1$ is not terminal, at $t = 1$, Rebel 4 and 5 are in a free-rider problem as Example 4 shows. ψ solves it by specifying Rebel 4 is the free rider and Rebel 5 writes his information about θ truthfully (as what ϕ' does in Example 5).

At $t = 1'_1$, Rebel 4 knows $\#[R](\theta) \geq k$ in the configuration in Figure 2 and knows $\#[R](\theta) < k$ in the configuration in Figure 3. Therefore he writes $(\mathbf{r})$ and $\emptyset$ respectively for these two configuration; as for other Rebels, they writes $(\mathbf{s})$.

After $t = 1'_1$, it is straightforward to check that all the Rebels will learn the relevant information (by seeing the writing of Rebel 4 at $t = 1'_1$) and will terminate their writing at $t = 2$. Therefore, $t = 1$ is the terminal round, and Rebels play a one-shot k-threshold game at $t = 2$.

Denote the belief system as υ. The strategy ψ can be made to be an APEX equilibrium strategy in a usual way by setting that the in-path belief is induced by ψ and adopting the out-of-path assessment of (ϕ, ν) from the previous example.

4.3 Dispensability of writing technology

In essence, writing technology is dispensable, and repeated actions are sufficient to serve as a communication protocol to achieve ex-post outcome in an equilibrium. In this section, I draw the analogue between the writing game and the original game in Table 1.

More precisely, in the equilibrium construction in the original game, let us partition the periods, and each part in the partition is analogous to a round in the writing game. The length of periods in a part is analogous to the length of sentence. Since actions played in a certain part of periods will incur an expected payoff, it is an analogue that writing is costly at a certain round in the writing game. The disjoint unions of parts of periods also constitute a coarser partition of periods, which is analogous to partitioning the rounds. As

Figure 5: The linearly ordered partitions of periods in the repeated k-threshold game after θ is realized.

$$\underbrace{\text{(periods for coordination)}}_{\text{0-block}} < \underbrace{\text{(periods for reporting)} < \text{(periods for coordination)}}_{\text{1-block}} < \ldots$$

an analogue of the partition of $\Gamma \cup \Gamma'$ in the indeterministic T-round writing game, the analogue of Γ is the set of *periods for reporting* in the original game to emphasize that these periods are for reporting information about θ; Γ' is the set of *periods for coordination* in the original game to emphasize that these periods are for coordinating to play the ex-post efficient outcome. The partition of periods is linearly ordered by $<$ (after θ is realized), and let us define a coarser partition with parts *t-blocks* indexed by $t \in \{0, 1, ...\}$ along with the order of partition of periods as shown in Figure 5.

One could see that Figure 4 and Figure 5 are seamlessly analogous to each other.

Table 1: The analogue between indeterministic T-round writing game and repeated k-threshold game

Indeterministic T-round writing game	Repeated k-threshold game
A round	A range of periods
A sentence	A sequence of actions
The length of a sentence in a round	The length of a part of periods
A chosen digit in a sentence	A chosen action
The cost of writing a sentence	The expected payoff in a part of periods
The fixed grammar	The equilibrium path

Note that the notions of *peripheral* and *central* in Example 2 is not yet formalized as well as analogizing to the original game. I generalize these notions in the original game by defining *information hierarchy* among players for each t-block below.

4.3.1 Information hierarchy

The information hierarchy across Rebels in G presents Rebels' information *before entering the periods for reporting at t-block.* That is designated by a tuple

$$(\{G_i^t\}_{i\in N}, \{I_i^t\}_{i\in N}, R^t, \theta).$$

G_i^t represents *the extended neighbors*: $j \in G_i^t$ if j can be reached by t consecutive edges from i, in which the endpoints of $t-1$ edges are both Rebels, while the remaining one consists of a Rebel and i himself; I_i^t represents as *the extended Rebel neighbors*—the set of Rebels in G_i^t; R^t represents as *the active Rebels*—those Rebels who are *active* in the sense that their extended Rebel neighbors are not a subset their direct neighbors' extended Rebel neighbors. Those objects are defined by:

At $t = 0$,

$$\text{if } \theta_i = I,\ G_i^0 \equiv \emptyset,\ I_i^0 \equiv \emptyset.$$

$$\text{if } \theta_i = R,\ G_i^0 \equiv \{i\},\ I_i^0 \equiv \{i\}.$$

$$R^0 \equiv [R](\theta).$$

At $t = 1$,

$$\text{if } \theta_i = I,\ G_i^1 \equiv \emptyset,\ I_i^1 \equiv \emptyset.$$

$$\text{if } \theta_i = R,\ G_i^1 \equiv G_i,\ I_i^0 \equiv G_i \cap R^0.$$

$$R^1 \equiv \{i \in R^0 : \nexists j \in G_i \text{ such that } I_i^1 \subseteq G_j^1\}.$$

At $t > 1$,

$$\text{if } \theta_i = I,\ G_i^t \equiv \emptyset,\ I_i^t \equiv \emptyset.$$

$$\text{if } \theta_i = R,\ G_i^t \equiv \bigcup_{j\in G_i} G_j^{t-1},\ I_i^t \equiv \bigcup_{j\in G_i} I_j^{t-1},$$

$$I_{ij}^t \equiv I_i^{t-1} \cap I_j^{t-1} \text{ if } j \in G_i.$$

$$R^t \equiv \{i \in R^{t-1} : \nexists j \in G_i \text{ such that } I_i^t \subseteq G_j^t\}.$$

According to the above definition, the peripheral Rebels in the configuration in Figure 2 are active in 0-block (in R^0) but not active in 1-block (not in R^1), while the central players are active in both 0-block and 1-block. It can be shown that $R^t \subseteq R^{t-1}$ by the following lemma.

Lemma 4.1. *If the θ has strong connectedness, then*

$$R^t \subseteq R^{t-1}$$

for all $t \geq 1$

Proof. I show that if $i \notin R^{t-1}$ then $i \notin R^t$ for all t. Given $t = \dot{t}$, by definition, if $i \notin R^{\dot{t}}$, then there is a j such that Rebels, who can be reached by $\dot{t}$ consecutive edges from i, can be also reached by $\dot{t}$ consecutive edges from j. Then if i can reach new Rebels at any $\ddot{t} > \dot{t}$ by $\ddot{t}$ consecutive edges, those new Rebels can be also be reached by $\ddot{t}$ consecutive edges by j, and hence $i \notin R^{\ddot{t}}$. □

Is it enough to let only active Rebels share information about θ while θ can be revealed eventually? The answer is affirmative by Theorem A.2 below, if the network is acyclic and if the θ has strong connectedness.

Theorem 3. *If the network is acyclic and if the θ has strong connectedness, then*

$$[R](\theta) \neq \emptyset \Rightarrow \text{ there exists } t \geq 0 \text{ and } i \in R^t \text{ such that } I_i^{t+1} = [R](\theta).$$

Proof. In Appendix. □

The strategy on the equilibrium path will specify how Rebels act according to the $(\{G_i^t\}_{i \in N}, \{I_i^t\}_{i \in N}, R^t, \theta)$; R^t will be those Rebels who are central in terms of sharing information in periods for reporting in the t-block.

4.3.2 The equilibrium path in periods for reporting

If there is no further mentioned, all the description in this section is for the APEX equilibrium path *before* the terminal period T^θ is reached. For conciseness, let us shorten "periods for reporting in t-block" by RP^t, denote $|RP^t|$ as the length of RP^t, and shorten **revolt** and **stay** to **r** and **s** receptively henceforth.

$|RP^t|$ is independent from t and determined only the set of players by the following procedure. Firstly, assign each player i a distinguished prime number x_i starting from 3 (by exploiting the common knowledge about network structure). Then let $|RP^t| = x_1 \otimes x_2 \otimes ... \otimes x_n$, where $\otimes$ is the usual multiplication operator. The sequence of actions in RP^t is with length $|RP^t|$ and would take one of the forms specifies in the right column in Table 2. There, if $I \subseteq N$, then $x_I \equiv \otimes_{i \in I} x_i$. The abbreviations for these sequences are listed in the left column. Since these sequences in the periods for reporting are meant to share information about θ, I would alternate using "playing the sequence" with "reporting the information" to mean the same behavior in the periods for reporting.

Table 2: The notations for the sequences of actions in RP^t on the path

Notations		The sequence of actions
$\langle I \rangle$	$\equiv$	$\langle \mathbf{s}, ..., \mathbf{s}, \underbrace{\mathbf{r}, \mathbf{s}, ..., \mathbf{s}}_{X_I} \rangle$
$\langle 1 \rangle$	$\equiv$	$\langle \mathbf{s}, ..., \mathbf{s}, \mathbf{r} \rangle$
$\langle$**all stay**$\rangle$	$\equiv$	$\langle \mathbf{s}, ..., \mathbf{s}, \mathbf{s} \rangle$

It is worth noting that this sequence constructed by prime numbers brings two benefits. Firstly, since the multiplication of distinguish prime numbers can be uniquely factorized, the Rebels can utilize such sequence to precisely report players' indexes. Secondly and ultimately conveniently, the un-discounted expected payoff of playing $\langle I \rangle$ for some $I \subseteq N$ is always equal to -1. This is because, at any period in RP^t, if there is no player playing $\langle 1 \rangle$, there is at most one player would play $\mathbf{r}$ by the property of prime number multiplication.

The sequence $\langle 1 \rangle$ seems odd, but It is intentionally created to deal with the free-rider problem. To see how it functions, let us formally define the free-rider problem by first defining the *pivotal Rebel* below.

Definition 4.3 (Pivotal Rebels in RP^t)**.** *A Rebel is pivotal in RP^t if he is in R^t and certain that he will learn the relevant information in the end of RP^t, given that each Rebel in R^t, say i, reports $\langle I_i^t \rangle$.*

Figure 6: A configuration of the state and the network in which player 1,3,5,6,7,8 are Rebels while players 2,4,9 are Inerts.

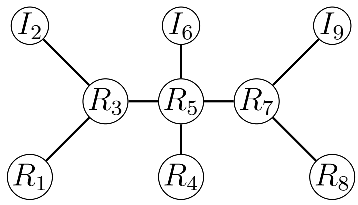
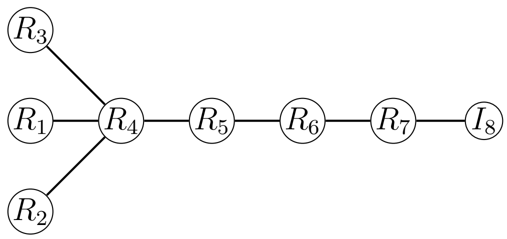


Figure 7: A configuration of the state and the network in which player 1,2,3,4,5,6,7 are Rebels while players 8 is an Inert.



From the definition, a pivotal Rebel p in RP^t is the one who can learn the relevant information if all of his active Rebel neighbors truthfully report their information about θ to him. He can be further classified into tow kinds. The first kind is the one who can learn the true state, while the second kind is the one who can learn the relevant information only.

In fact, if the network is acyclic and the prior has full support on strong connectedness, it can be shown that p in RP^t is the second kind only if $I_p^t = k - 1$. In other words, p is either with $I_p^t = k-1$ or the one who can learn the true state, or both. For conciseness, call p of the first kind by *θ-pivotal*; call the one with $I_p^t = k - 1$ by *$k - 1$-pivotal.* As instances, when $k = 6$ and in RP^1, in the configuration in Figure 2, only Rebels 4 and 5 are pivotal (θ-pivotal); in the configuration in Figure 6, only Rebels 5 is pivotal (θ-pivotal); in the configuration in Figure 7, only Rebels 4 pivotal ($k - 1$-pivotal).

Below is the free-rider problem in RP^t defined.

Definition 4.4. *There is a free-rider problem in RP^t if there are multiple θ-pivotal Rebels in RP^t.*

The following crucial lemma says that there are at most two θ-pivotal Rebels in a R^t for all t. Therefore, if there is a free-rider problem, it occurs between two θ-pivotal Rebels. Moreover, if there are two of them, they are neighbors.

Lemma 4.2. *If the network is acyclic and if π has full support on strong connectedness, then for each t-block, there are at most two θ-pivotal Rebels. Moreover, if there are two of them, they are neighbors.*[11]

Proof. In Appendix. □

Notably,

Lemma 4.3. *If the network is acyclic and if π has full support on strong connectedness, then for each t-block, if there are two θ-pivotal Rebels p and p', then they commonly know that they are θ-pivotal Rebels at the beginning of t-block.*

Proof. In Appendix. □

By Lemma 4.3, θ-pivotal Rebels in RP^t can identify themselves at the beginning of RP^t. It is crucial because, on the APEX equilibrium path, if the free-rider problem will occur RP^t, the strategy will be specified beforehand. It will be specified that the lowest indexed θ-pivotal Rebel p in the free-rider problem plays $\langle 1 \rangle$, while the other one p' plays $\langle I^t_{p'} \rangle$.

The assumption of acyclic network in Lemma 4.3 is indispensable. In Section 5.1, there is a configuration in a cycle such that there is no common knowledge of free-rider problem by my equilibrium construction.

Overall, the sequences played in RP^t on the path are listed in Table 3.

4.3.3 The equilibrium path in periods for coordination

In this section, I discuss the sequences of action in periods of coordination on the path. Let us shorten "periods for coordination in t-block" by CD^t. If there is no further mentioned,

[11] As a remark, the above lemma is not true when the network is cyclic. To see this, consider a 4-player circle when $\theta = (R, R, R, R)$.

Table 3: The sequences of actions played in RP^t on the path

Rebel i	i plays
$i \notin R^t$	$\langle$**all stay**$\rangle$
$i \in R^t$ but i is not pivotal	$\langle I_i^t \rangle$
i is $k-1$-pivotal	$\langle 1 \rangle$
i is θ-pivotal but not in the free-rider problem	$\langle 1 \rangle$
i is in the free-rider problem with the lowest index	$\langle 1 \rangle$
i is in the free-rider problem without the lowest index	$\langle I_i^t \rangle$

all the description in this section is for the APEX equilibrium path *before* the terminal period T^θ is reached.

The crucial feature in periods of coordination is that, in short, whenever a Rebel i has been thought to be not active starting at some t-block (i.e. $i \notin R^t$ for some $t \in \mathbb{N}$), there is no strategy for i to convince all the Rebels that $\#[R](\theta) \geq k$, even though i might want to propagandize it.

The structure in the periods for coordination is more intrigued than that in periods for reporting, as the counterpart of Γ' in indeterministic T-round writing game. In periods of coordination, these periods are further partitioned by *divisions* and *sub-blocks*. I depict that in details below, where (CD) represents a certain range of periods for coordination.

In CD^0,

$$\overbrace{\underbrace{(CD)}_{\text{one sub-block}}}^{\text{1-division}}\overbrace{\underbrace{(CD)}_{\text{one sub-block}}}^{\text{2-division}}\overbrace{\underbrace{(CD)\cdots(CD)}_{n\text{ sub-blocks}}}^{\text{3-division}}$$

In CD^t, $t > 0$,

$$\overbrace{\underbrace{(CD)\cdots(CD)}_{n\text{ sub-blocks}}}^{\text{1-division}}\overbrace{\underbrace{(CD)\cdots(CD)}_{t+1\text{ sub-blocks}}}^{\text{2-division}}\overbrace{\underbrace{(CD)\cdots(CD)}_{n\text{ sub-blocks}}}^{\text{3-division}}$$

For convenience, in the t-block, denote $CD_{u,v}^t$ as the v-th sub-block in u-division; denote $|CD_{u,v}^t|$ as the length of $CD_{u,v}^t$. Similarly, denote CD_u^t as the u-division; denote $|CD_u^t|$ as the length of CD_u^t. Let us shorten **revolt** and **stay** to **r** and **s** receptively henceforth. On the path, for all $v \in \mathbb{N}$, $|CD^t_{u,v}| = n$ for $u = 1, 2$ and $|CD^t_{u,v}| = 1$ for $u = 3$. The notations for the sequences of actions on the path are shown in Table 4 shows.[12]

Table 4: The notations for the sequences of actions in CD^t_u for $u = 1, 2$, on the path

Notations		The sequence of actions
$\langle i \rangle$	$\equiv$	$\langle \mathbf{s}, ..., \mathbf{s}, \underbrace{\mathbf{r}, \mathbf{s}, ..., \mathbf{s}}_{i} \rangle$
$\langle$**all stay**$\rangle$	$\equiv$	$\langle \mathbf{s}, ..., \mathbf{s}, \mathbf{s} \rangle$

The equilibrium behavior on the path in CD^0 Since the 0-block has a simpler structure, I begin with depicting the equilibrium path in CD^0 as shown in Table 5, Table 6, and Table 7. The description for a Rebel i there is whether or not i has learnt the relevant information. Notice that the Rebel i might learn the relevant information by observing his neighbors' behavior. If a Rebel i is certain that $\#[R](\theta) < k$, by the full support on strong connectedness, it must be the case that all Rebels are i's neighbors. All Rebels will be also certian about that immediately after $CD^0_{1,1}$.

The intriguing part might be "how a Rebel i initiates the common knowledge about $\#[R](\theta) \geq k$." i does so by first play $\langle i \rangle$ in $CD^0_{1,1}$ and then play $\langle$**all stay**$\rangle$ in $CD^0_{2,1}$. His behavior is then distinguishable from Rebels of other kinds. His neighbors will know $\#[R](\theta) \geq k$ immediately after $CD^0_{2,1}$, and then all the Rebels will know that by playing **r** contagiously in CD^0_3.

Notice that, by the assumption of acyclic network, i will not deviate to play $\langle$**all stay**$\rangle$ even though it might be undetectable. This is because, if so, i will be considered as an inactive Rebels, as a "dead end", afterwards by *all* of his neighbors. He will no longer to be able to influence his neighbors' belief. He then faces a positive probability that not enough Rebels can be informed of $\#[R](\theta) \geq k$. in that scenario, he will only get zero payoff.

[12]Since, in 3-division, the length of the sequence of actions is one, i.e. playing an action, I dispense notations there for conciseness.

Taking sufficiently high discount factor impedes his deviation. In essence, all the proofs for the equilibrium behavior on the path follow this argument in Appendix.

Table 5: The sequences of actions played in $CD^0_{1,1}$ on the path

Rebel i	i plays
i is certain that $\#[R](\theta) < k$	$\langle$**all stay**$\rangle$
$i \notin R^1$ and is uncertain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$
$i \in R^1$ and is uncertain $\#[R](\theta) \geq k$	$\langle i \rangle$
i is certain that $\#[R](\theta) \geq k$	$\langle i \rangle$

Table 6: The sequences of actions played in $CD^0_{2,1}$ on the path

Rebel i	i plays
i is certain that $\#[R](\theta) < k$	$\langle$**all stay**$\rangle$
$i \notin R^1$ and is uncertain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$
$i \in R^1$ and is uncertain $\#[R](\theta) \geq k$	$\langle i \rangle$
i is certain that $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$

Table 7: The sequences of actions played in CD^0_3 on the path

Rebel i	i plays
i is certain that $\#[R](\theta) < k$	$\langle \mathbf{s} \rangle$
$i \notin R^1$ and is uncertain $\#[R](\theta) \geq k$	$\langle \mathbf{s} \rangle$
$i \in R^1$ and is uncertain $\#[R](\theta) \geq k$	$\langle \mathbf{s} \rangle$
i is certain that $\#[R](\theta) \geq k$	$\langle \mathbf{r} \rangle$

It is useful to list Rebels' updated beliefs consistent with the equilibrium path after CD_1^0 and CD_2^0, as Table 8 and Table 9 shows.

Table 8: The belief of $j \in G_i$ after observing i's previous actions immediately after CD_1^0

i plays	The event $j \in G_i$ assigns with probability one
In $CD_{1,1}^0$	
$\langle$**all stay**$\rangle$	$i \notin R^1$ if $j \in R^1$
$\langle$**all stay**$\rangle$	$\#[Rebels](\theta) < k$ if $j \notin R^1$
$\langle i \rangle$	$i \in R^1$ or $\#[Rebels](\theta) \geq k$

Table 9: The belief of $j \in G_i$ after observing i's previous actions immediately after CD_2^0

i plays		The event $j \in G_i$ assigns with probability one
In $CD_{1,1}^0$	In $CD_{2,1}^0$	
$\langle$**all stay**$\rangle$	$\langle$**all stay**$\rangle$	$i \notin R^1$ if $j \in R^1$
$\langle$**all stay**$\rangle$	$\langle$**all stay**$\rangle$	$\#[Rebels](\theta) < k$ if $j \notin R^1$
$\langle i \rangle$	$\langle$**all stay**$\rangle$	$\#[Rebels](\theta) \geq k$
$\langle i \rangle$	$\langle i \rangle$	$i \in R^1$

The equilibrium behavior on the path in CD^t for $t \geq 1$ Next, I describe the equilibrium behavior on the path in CD^t when $t \geq 1$. As in Example 6, players' belief over states will now contingent on others' bahavior in RP^t. After all, RP^t is meant to exchanging information about states. Contrary to introducing equilibrium strategy consistent with players' belief, I first illustrate how players update their belief after observing the equilibrium strategy. The belief updating in CR^t is shown in Table 10, Table 11, and Table 12.

Table 10: The belief of $j \in G_i$ after observing i's previous actions immediately after RP^t

i plays	The event $j \in G_i$ assigns with probability one
In RP^t	
$\langle$**all stay**$\rangle$	$i \notin R^t$ and $I_{ji}^{t+1} = I_j^t$
$\langle I_i^t \rangle$	$i \in R^t$ and $I_{ji}^{t+1} = I_j^t \cap I_i^t$
$\langle 1 \rangle$	i is pivotal

Table 11: The belief of $j \in G_i$ after observing i's previous actions immediately after CD_1^t contingent on RP^t

i plays		The event $j \in G_i$ assigns with probability one
In RP^t	In $CD_{1,1}^t$	
$\langle$**all stay**$\rangle$	$\langle i \rangle$	$i \notin R^t$ and $I_{ji}^{t+1} = I_j^t$
$\langle I_i^t \rangle$	$\langle$**all stay**$\rangle$	$\#[Rebels](\theta) < k$
$\langle I_i^t \rangle$	$\langle i \rangle$	$i \in R^t$ and $I_{ji}^{t+1} = I_j^t \cap I_i^t$, or $\#[Rebels](\theta) \geq k$
$\langle 1 \rangle$	$\langle$**all stay**$\rangle$	$\#[Rebels](\theta) < k$
$\langle 1 \rangle$	$\langle i \rangle$	$\#[Rebels](\theta) \geq k$

Table 12: The belief of $j \in G_i$ after observing i's previous actions immediately after CD_2^0 contingent on RP^t and CD_2^0

i plays			The event $j \in G_i$ assigns with probability one
In RP^t	In $CD_{1,1}^t$	In $CD_{2,1}^t$	
$\langle$**all stay**$\rangle$	$\langle i \rangle$	$\langle$**all stay**$\rangle$	$i \notin R^t$ and $I_{ji}^{t+1} = I_j^t$
$\langle I_i^t \rangle$	$\langle$**all stay**$\rangle$	$\langle$**all stay**$\rangle$	$\#[Rebels](\theta) < k$
$\langle I_i^t \rangle$	$\langle i \rangle$	$\langle$**all stay**$\rangle$	$\#[Rebels](\theta) \geq k$
$\langle I_i^t \rangle$	$\langle i \rangle$	$\langle i \rangle$	$i \in R^t$ and $I_{ji}^{t+1} = I_j^t \cap I_i^t$
$\langle 1 \rangle$	$\langle$**stay**$\rangle$	$\langle$**all stay**$\rangle$	$\#[Rebels](\theta) < k$
$\langle 1 \rangle$	$\langle i \rangle$	$\langle$**all stay**$\rangle$	$\#[Rebels](\theta) \geq k$

The delicate part in CD^t is how a pivotal Rebel p propagandizes the relevant information. p does so by playing $\langle$**all stay**$\rangle$ in $CR^t_{1,1}$ in the case of $\#[R](\theta) < k$, while playing $\langle 1 \rangle$ in the case of $\#[R](\theta) \geq k$. Notice that playing $\langle$**all stay**$\rangle$ in $CR^t_{1,1}$ by p will immediately inform p's neighbors that $\#[R](\theta) < k$. On the contrary, playing $\langle 1 \rangle$ in $CR^t_{1,1}$ by p has not yet revealed $\#[R](\theta) \geq k$ since $\langle 1 \rangle$ can be also played by non-pivotal Rebels.

In $CR^t_{2,1}$, however, he reveals $\#[R](\theta) \geq k$ by playing $\langle$**all stay**$\rangle$, a no-cost action. It might not seem intuitive at its first glance, but it effectively prevents a potential free-rider problem: there are two pivotal Rebels, say p and $p^{'}$, who have already known $\#[R](\theta) \geq k$ in CR^t. If initiating the common knowledge about $\#[R](\theta) \geq k$ incurs negative payoff, p or $p^{'}$ has incentive to let the other initiate it. Playing $\langle$**all stay**$\rangle$ in $CR^t_{2,1}$ proudly becomes the initiation sequence since it incurs *no* cost.

The remaining argument is why other non-pivotal Rebels, say i, do not mimic pivotal Rebels' behavior to play $\langle 1 \rangle$ in RP^t. He has no incentive to do so because, based on the belief updating on the path, if they play $\langle 1 \rangle$, all the Rebels will learn the relevant information after CD^t_2. It implies that the period at the beginning of $t+1$-block is a terminal period. However, he is uncertain whether or not he could learn the relevant information in RP^t since he is not pivotal. He will not learn the relevant information since the belief updating is also terminated after t-block. Since the ex-post efficient outcome gives him the maximum payoff at every θ, and he will learn the relevant information eventually on the equilibrium path (this fact will be shown in Appendix), he will prefer not to deviate given that the discount factor is high enough. This argument is a major one in the proof of Theorem 2.

As a complementary, I depict equilibrium strategy consistent with players' belief in Table 13, Table 14, Table 15, and Table 16.

Table 13: The sequences of actions played in $CD^t_{1,v}$ for $t \geq 1$ and for $v = 1, 2, ..., n$ on the path

Rebel i	i plays
i is certain that $\#[R](\theta) < k$	$\langle$**all stay**$\rangle$
$i \notin R^t$ and is uncertain $\#[R](\theta) \geq k$	$\langle i \rangle$
$i \in R^t$ and is uncertain $\#[R](\theta) \geq k$	$\langle i \rangle$
i is certain that $\#[R](\theta) \geq k$	$\langle i \rangle$

Table 14: The sequences of actions played in $CD^t_{2,v}$ for $t \geq 1$ for $v = 1$ on the path

Rebel i	i plays
i is certain that $\#[R](\theta) < k$	$\langle$**all stay**$\rangle$
$i \notin R^t$ and is uncertain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$
$i \in R^t$ and is uncertain $\#[R](\theta) \geq k$	$\langle i \rangle$
i is certain that $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$

Table 15: The sequences of actions played in $CD^t_{2,v}$ for $t \geq 1$ for $v = 2, ..., t+1$ on the path

Rebel i	i plays
i is certain that $\#[R](\theta) < k$	$\langle$**all stay**$\rangle$
$i \notin R^t$ and is uncertain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$
$i \in R^t$ and is uncertain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$
i is certain that $\#[R](\theta) \geq k$	$\langle i \rangle$

Table 16: The sequences of actions played in CD_3^t for $t \geq 1$ on the path

Rebel i	i plays
i is certain that $\#[R](\theta) < k$	$\langle \mathbf{s} \rangle$
$i \notin R^1$ and is uncertain $\#[R](\theta) \geq k$	$\langle \mathbf{s} \rangle$
$i \in R^1$ and is uncertain $\#[R](\theta) \geq k$	$\langle \mathbf{s} \rangle$
i is certain that $\#[R](\theta) \geq k$	$\langle \mathbf{r} \rangle$

4.3.4 Learning on the path and the out-o-path Belief

Whenever Rebel i detects a deviation at period s, he forms the following belief:

$$\sum_{\theta \in \{\theta : \theta_j = I, j \notin G_i\}} \beta_{G_i}^{\pi,\tau}(\theta | h_{G_i}^{\dot{s}}) = 1, \text{ for all } \dot{s} \geq s \tag{1}$$

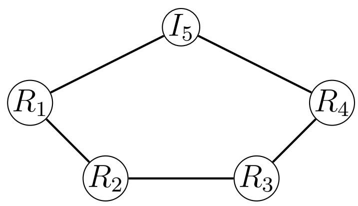
. Thus, if $\#I_i^0 < k$, he will play **stay** forever after he detects a deviation. This out-of-path belief serves as a grim trigger.

5 Discussion

5.1 Cyclic networks

Such scenario substantially differs from the cyclic counterpart. The free-rider problem becomes intractable in cyclic networks, at least in the way of my equilibrium construction. Let us consider the configuration in Figure 8.

Figure 8: A configuration of the state and the network in which player 1,2,3,4 are Rebels while player 5 is an Inert.



Suppose $k = 4$ and the period s at the beginning of 1-block is not terminal. By the definition of pivotal Rebel in Section 4.3.2, Rebel 2 and 3 are θ-pivotal. From the perspective of Rebel 2's view, the type of player 5 could be Inert. Therefore, Rebel 2 does not know whether or not Rebel 1 is pivotal. Similarly, Rebel 2 does not know whether or not Rebel 3 is pivotal, *even though* player 3 is indeed θ-pivotal. Therefore there is no common knowledge of free-rider problem at period s.

However, if let us cut the edge between player 4 and 5, Rebel 2 knows that he is the only θ-pivotal Rebel.

6 Conclusion

A Appendix

A.1 Notations

A.2 Missing proofs

proof for Lemma 2.1

Lemma (Learning in the APEX equilibrum). *If the assessment* (τ^*, μ^*) *is an APEX equilibrium, then for all* $\theta \in \Theta$*, there is a finite time* T_i^θ *for every Rebel* i *such that* $\sum_{\theta \in \{\theta : [R](\theta) \geq k\}} \beta_{G_i}^{\pi,\tau^*}(\theta | h_{G_i}^s) =$ *either* 1 *or* 0 *whenever* $s \geq T_i^\theta$*.*

Proof. The proof is done by contraposition. Suppose Rebels' strategies constitute an APEX equilibrium. By definition of APEX equilibrium, at every θ, there is a period T^θ when all Rebels' actions start to repeat themselves. Let $T = \max_{\theta \in \Theta} T^\theta$. For Rebel i, let $T_i = T + 1$, and suppose $0 < \sum_{\theta : \#[R](\theta) \geq k} \beta_{G_i}^{\pi,\tau^*}(\theta | h_{G_i}^s) < 1$ for some $s \geq T_i$. Then this Rebel assigns positive weight at some $\dot{\theta} \in \{\theta : \#[R](\theta) < k\}$ and some positive weight at some $\ddot{\theta} \in \{\theta : \#[R](\theta) \geq k\}$ at period s. Note that i has already known θ_j if $j \in G_i$, and therefore i assigns positive weight at some $\theta' \in \{\theta : \#[R](\theta) < k, \theta_l = R, l \notin G_i\}$ and positive weight at some $\theta'' \in \{\theta : \#[R](\theta) < k, \theta_l = I, l \notin G_i\}$. Since all Rebels' actions start to repeat themselves at period T, i cannot update information afterwards. Suppose i's continuation strategy is to continuously play **revolt**, then this is not ex-post efficient when $\#[Rebels](\theta) < k$; suppose i's continuation strategy is to continuously play **stay**, then this is not ex-post efficient when $\#[Rebels](\theta) \geq k$. $\square$

proof for Theorem 1

Theorem (APEX equilibrium for the case of $k = n$). *For any* n*-person repeated* k*-Threshold game with parameter* $k = n$ *played in a network, there is a* δ^* *such that a sequential APEX equilibrium exists whenever* $\delta > \delta^*$*.*

Proof. Let τ^* be the following strategy. After the nature moves, a Rebel plays **revolt** if he has no Inert neighbor; a Rebel plays **stay** forever if he has an Inert neighbor. After the first period, if a Rebel has not detected a deviation and observes that his Rebel neighbors play **revolt** continuously in the previous periods, he plays **revolt** in the current period; otherwise, he plays **stay** forever. If a Rebel deviates, then he will plays **stay** forever after the period he deviates.

At period s, according to τ^*, if a Rebel has not detected a deviation but observe his Rebel neighbors plays **stay** in the current period, he forms the belief of

$$\sum_{\theta:\#[Rebels](\theta)\geq k} \beta_{G_i}^{\pi,\tau^*}(\theta|h_{G_i}^s) = 0$$

and will not change this belief afterwards. Therefore, he plays **stay** afterwards as his best response.

At period s, if a Rebel detects a deviation or he has deviated, to play **stay** is the best response since at least one player will play **stay**.

Since the network is finite with n vertices, if all players do not deviate, after period n, all Rebels play **revolt** forever if $\theta \in \{\theta : \#[Rebels](\theta) \geq k\}$, and all Rebels play **stay** forever if $\theta \in \{\theta : \#[Rebels](\theta) < k\}$. Thus, in the equilibrium path, a Rebel gets $\max\{1, 0\}$ after period n. However, a Rebel who has deviated at most get 0 after period n. Therefore, there is a $0 < \delta < 1$ large enough to impede Rebels to deviate.

To check if τ^* and $\{\beta_{G_i}^{\pi,\tau^*}(\theta|h_{G_i}^s)\}_{i\in N}$ satisfy full consistency[13], take any $0 < \eta < 1$ such that Rebels play τ^* with probability $1-\eta$ and play other behavior strategies with probability η. Clearly, when $\eta \to 0$, the belief converges to $\{\beta_{G_i}^{\pi,\tau^*}(\theta|h_{G_i}^s)\}_{i\in N}$. Since the out-of-path strategy is optimal for the Rebel who detects deviation and the Rebel who makes deviation, for arbitrary beliefs they hold, τ^* is a sequential equilibrium. □

proof for Theorem A.2

Theorem. *If the network is acyclic and if the θ has strong connectedness, then*

$$[R](\theta) \neq \emptyset \Rightarrow \text{ there exists } t \geq 0 \text{ and } i \in R^t \text{ such that } I_i^{t+1} = [R](\theta).$$

[13] Krep and Wilson (1982)

Proof. First define

$$TR_{ij} \equiv \{v \in N : \text{there is a path from } i \text{ to } v \text{ through } j\}.$$

By definition,

$$\begin{aligned} G_i^t &= \bigcup_{k_1 \in G_i} \bigcup_{k_2 \in G_{k_0}} \cdots \bigcup_{k_t \in G_{k_{t-1}}} G_{k_t} \\ &= \{j \in N : \text{ there is a path } (i, k_1 ... k_l, j) \text{ such that } l \leq t-1 \text{ and } \theta_i = \theta_{k_1} = ... = \theta_{k_l} = R\}, \end{aligned}$$

while

$$\begin{aligned} I_i^t &= \bigcup_{k_1 \in G_i} \bigcup_{k_2 \in G_{k_0}} \cdots \bigcup_{k_t \in G_{k_{t-1}}} G_{k_t} \cap [R](\theta) \\ &= \{j \in [R](\theta) : \text{ there is a path } (i, k_1 ... k_l, j) \text{ such that } l \leq t-1 \text{ and } \theta_i = \theta_{k_1} = ... = \theta_{k_l} = R\}. \end{aligned}$$

Since the θ has strong connectedness and since $[R](\theta) \neq \emptyset$, for each i, there is a minimum t_i such that $I_i^{t_i} = [R](\theta)$. Let $P = \arg\min_{i \in N}\{t_1, ..., t_n\}$ with generic element p. Therefore $I_p^{t_p} = [R](\theta)$. I show that $p \in R^{t_p - 1}$ to complete the proof. I prove it by contradiction. If $p \notin R^{t_p-1}$, then $I_p^{t_p-1} \subseteq G_j^{t_p-1}$ for some $j \in G_p$. Then, all the Rebels in TR_{jp} are in $G_j^{t_p-1}$, but there exist Rebels in TR_{pj} who are in $G_j^{t_p-1}$ but not in $I_p^{t_p-1}$. This is because the network is acyclic and $I_p^{t_p-1} \subset [R](\theta)$. However, $p \notin P$, since $I_j^{t_j-1} = [R](\theta)$ already. I then conclude that $p \in R^{t_p-1}$.

□