Coordination in Social Networks: Communication by Actions

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draft: v4.0

Abstract

1 Introduction

This paper studies repeated collective actions within social networks. Players hold local information but can perfectly observe their neighbors' actions. I explore how players share information through their past play in order to achieve the ex-post efficient outcome in the future. Though the motive of this study is to understand the dynamic of social movements, a general interest centers on the collective action behaviors within social structures.

Consider pro-democracy movements. Strong discontents overthrowing a regime may exist, but it is difficult to organize around these discontents because information about the existence of such discontents is not always transparent. For instance, in East Germany, the government had control over the electoral system and the mass media, and the eavesdropping by secret agents had impeded people from showing their discontents. As [Karl-Dieter and Christiane, 1993] or [Chwe, 2000] have suggested, such discontents may be revealed only to someone whom you trust or have an intimate relationship with, but are hardly revealed publicly. This lack of common knowledge about the existence of strong discontent impedes people from conducting a one-shot uprising due to the fear of possible failure. However, an event may trigger a later event. When rebels are aware of the scale to share relevant information about the level of collective discontent through their actions, they might be willing to take actions although it is at risk of facing failure. I view the dynamic of their actions as an equilibrium so that the entire movement is as of a learning process.

I models such dynamic collective actions in the following way. Players infinitely play repeated k-*threshold games* within their social network ([Chwe, 2000]), where k is a commonly known parameter, with a common discount factor. There are two types of players, one we called them *Rebel* and one we called them *Inert*. The network constrains players' information structure in that players' types as well as their past play can be observed only by their neighbors. A Rebel has two actions, which are **revolt** or **stay**, while an Inert has only one action, which is **stay**. If a Rebel chooses **revolt**, he will get the payoff of 1 if at least k players also choose **revolt**; he will get -1 otherwise. If a Rebel chooses **stay**, he will get the payoff of 0 despite how many players choose **revolt**. Rebels have a common prior over type profiles but do not necessarily know how many Rebels exist in this society. Therefore, the parameter k represents the threshold for overthrowing a regime, and the payoff structure mimics that overthrowing a regime is risky. Cheap talk is not allowed, payoff is hidden, no outside mechanism serves as an communication device.

Rebels communicate by playing actions, while Inerts block the communication between Rebels. For different k and different network structures, I am looking for a weak sequential equilibrium that has the property of *approaching ex-post efficient* or *APEX*.[1] An equilibrium is APEX if the tails of actions in the equilibrium path repeats the static ex-post efficient outcome after some finite period T. This notion serves to check whether players learn the

[1]The APEX equilibrium I constructed would not satisfy full consistency ([Kreps and Wilson, 1982]) for some k, but does satisfy updating consistency ([Perea, 2002]). Updating consistency requires that, for every player, for every player's strategies, for every information sets s^1, s^2, where s^2 follows s^1, if s^2 happens with positive probability given s^1 and given players' strategies contingent on s^1, then the belief over s^2 should satisfy Bayesian updating conditional on the belief over s^1 and players' strategies contingent on s^1. In other words, the updating consistency requires that players hold belief in every information set and hold updated belief that follows the previous belief. This requirement imposes restrictions on off-path beliefs that induce sequential rationality, although it is weaker than full consistency in that full consistency implies updating consistency.

relevant information on the equilibrium path in finite time: if there are at least k Rebels in this society, then all Rebels should **revolt** after T as if they are aware that at least k Rebels exist; otherwise, all Rebels should **stay** after T.

I focus my exploration on the networks that are finite, commonly known, fixed, undirected, and connected, or *networks* henceforth. On all *acyclic networks*, this paper's main result Theorem 2 affirms that Rebels can coordinate themselves to achieve the ex-post efficient outcome irrespective of the overthrowing threshold, given that they are sufficiently patient and under a *strong connectedness condition* on type profiles. This strong connectedness condition is meant to assure that no Inert can block the communication between Rebels. To be precise, the strong connectedness condition says that a type profile occurs if and only if it induces a path consisting of Rebels to connect every two Rebels. The "only-if" part in this condition is in essence necessary in that if it fails, there must be some overthrowing threshold for which no APEX equilibrium exists.[2] This condition and the acyclic networks restriction can be dispensed together when the overthrowing threshold is the number of players. My Theorem 1 shows that, if the overthrowing threshold is the number of players, there exists an APEX equilibrium in any network if Rebels are sufficiently patient. In the aspect of characterizing the network structure that sustains the ex-post efficient outcome, Theorem 1 and Theorem 2 are in sharp contrast to the sufficient network result of [Chwe, 2000]: there always exists a network and a prior so that the ex-post efficient outcome cannot be an equilibrium in the one-shot k-threshold game when $k \geq 3$, but that can be achieved in the long term in an equilibrium if this game is played repeatedly and infinitely.

The existence of APEX equilibrium is proved by construction. I let two phases, the *reporting phase* and the *coordination phase*, consecutively alternate on the equilibrium path, each of which endures a specified length of periods. Ignoring incentive compatibility, this

[2] The "only-if" part is equivalent to say the network induced by Rebels is not connected. It implies some Rebels never learn each other's type. When the threshold is high enough, it then violates Lemma 2.1, which states that in every APEX equilibrium under some threshold, every Rebel must learn whether the number of Rebels exceeds this threshold. The "if" part is a technical assumption for the proof easiness but can be dispensed without changing my main result.

construction idea is simple.[3] Rebels truthfully inform their neighbors *which* Rebels they have been aware of by varying their actions in reporting phases. They can do so because the network is commonly known: players are not anonymous; their locations are equivalent to their indexes. Rebels moreover can observe their neighbors' past play. What follows is that Rebels are aware of each other iteratively as reporting phases go by. In the coordination phase, several sequences of actions are specified to represent three kinds of knowledge: (1) "I am certain that the number of Rebels exceeds k." (2) "I am certain that the number of Rebels cannot exceed k." (3) "I am still uncertain." Rebels then play these specified sequence to disseminate their knowledge until a consensus is reached. Whenever a consensus is reached, they either play the ex-post efficient outcome or together enter the next reporting phase.

When the incentive compatibility is taken into consideration, this construction, however, constitutes a free-rider problem. This problem stems from the fact that a Rebel concerns short-term incentive and always seeks the opportunity to manipulate his sequence of actions. Consider a scenario where two nearby Rebels vary actions to share information in a reporting phase.[4] Suppose that these two Rebels can learn the true state from each other's truthful information sharing. Further suppose that each of them has a continuation strategy on the equilibrium path to initiate the ex-post efficient outcome. Then sharing truthful information is not the best response since either Rebel can wait for the other's information sharing. This is to say the coordination of ex-post outcome is as a public good. This public good can only be made by information sharing, which incurs costs in that playing **revolt** is risky. To solve the free-rider problem implied by this simple construction idea, the assumption of acyclic network becomes crucial. If the networks is acyclic, I can show that, there are at most two Rebels subject to the free rider problem. Moreover, these two Rebels are neighbors and commonly know that they are subject to the free rider problem. The equilibrium then prescribes that one of them is the free rider, who chooses his most profitable sequence of actions in the reporting phase, and the other one shares truthful information. However, this

[3]Indeed, if cheap talk is allowed, Theorem 2 immediately holds since Rebels have no incentive to lie by that the k-threshold game is a coordination game without conflict of interest.

[4]See Example 5.

common knowledge property does not hold in cyclic networks. I leave a concrete example and a discussion for this issue in Section 5.2.

The paper is organized as follows. I introduce the model in Section 2. I illustrate my equilibrium construction and my main results in Section 3 and Section 4. In Section 5, I discuss how the common knowledge of free-rider problem may fail to be established in cyclic networks. The conclusion is made in Section 6. All the missing proofs are left in the Appendix.

1.1 Related Literature

This paper bears on questions in several fields. Firstly, my paper is related to studies in political economy on how collective actions serves in aggregating information.[5] Departing from the conventional setup in which the dynamic of collective actions is exogenous, such as [Lohmann, 1994] or [Shadmehr and Bernhardt, 2019], I emphasizes that this dynamic emerges endogenously as an equilibrium from a dynamic problem. to that these acts might not be simultaneously observed due to the social network constraint and, therefore, how to aggregate information is a dynamic problem per se. The most closest study might be [Lohmann, 1994] and [Shadmehr and Bernhardt, 2019]. The departure from these two studies is that the timing of actions in not exogenous and the actions in previous stage is not public known. While the timing is exogenous, Lohmann partially solve the free-rider problem and the key idea there is the action is public known. In this paper, the free rider problem lies in players need to coordinate their information and therefore players might wait.

[Lohmann, 1994], [Dekel and Piccione, 2000], [Shadmehr, 2015], [Shadmehr and Bernhardt, 2019], or [Battaglini, 2016], which also consider sequential acts, I emphasizes that these acts might not be simultaneously observed due to the social network constraint and, therefore, how to aggregate information is a dynamic problem per se. [6]

[5]See also [Austen-Smith and Banks, 1996], [OLSON, 1965], [Feddersen and Pesendorfer, 1997]

[6][Lohmann, 1994] solves the free-rider problem resulted from costly informative acting. [Shadmehr, 2015] and [Shadmehr and Bernhardt, 2019] characterizes the likelihood of regime change in a two-period leader-follower coordination game under two-side uncertainty and show interpretation for using moderate

Second, in the economics of networks, my paper is in a strain that considering games played on networks. When players have types, I borrow the game from [Chwe, 2000] as aforementioned, but I am not aware of pioneer studies in repeated games.

- Cheap talk or costly talk: [Barberà Sàndez and Jackson, 2019], [Shadmehr and Bernhardt, 2019], [Calvó-Armengol and Beltran, 2009], [Calvó-Armengol et al., 2015]

In complete information, a growing body of literature studies the repeated games.

- random matching: [Ellison, 1994], [Kandori, 1992], [Ali and Miller, 2016], [Takaki, 1984], Debs.

- network monitoring: [Laclau, 2012], [Wolitzky, 2013] [Wolitzky, 2014], [Ambrus et al., 2014], [Kandori, 1992], [Xue, 2004]

My paper is also related to the literature of discounted repeated games with incomplete information. In this literature, they consider more general games than the games adopted here.

- [Fudenberg and Yamamoto, 2010] [Fudenberg and Yamamoto, 2011] [Wiseman, 2012], [Yamamoto, 2014], [Hrner et al., 2011]

For some k and some network, any belief free equilibrium is not an APEX equilibrium.

2 Model

There is a set of players $N = \{1, ..., n\}$. They constitute a network $G = (V, E)$ so that the vertices are players ($V = N$) and an edge is a pair of them (E is a subset of the set

punishment and for allowing the existence of radical leader might be beneficial for the current regime. [Battaglini, 2016] shows that informative acting is impossible to be an equilibrium if policy maker's interest is not aligned with the protesters. All of the above model The departure is that the protesters do not observe the same information aggregation stage at the same time, and there is a delayed advantage. The preference is the same but the sequence of movers is endogenous.

containing all two-element subsets of N). Throughout this paper, G is assumed to be finite, commonly known, fixed, undirected, and connected.[7]

Time is discrete and denoted by $S = \{0, 1, ...\}$ with index s. Each player could be either type R or type I assigned by the nature at $s = 0$ according to a common prior π; R or I represents a Rebel or an Inert respectively. Call $\theta \in \Theta := \{R, I\}^n$ a state of nature. At each $s \geq 1$, players play a normal form game, the *k-threshold game*, infinitely repeated played with common discounted factor $\delta \in (0, 1)$. In the k-threshold game, $A_R = \{\textbf{revolt}, \textbf{stay}\}$ is the set of actions for R and $A_I = \{\textbf{stay}\}$ is that for I. Denote by $\#X$ the cardinality of an set X. A Rebel i's stage-game payoff function is defined as below, while an Inert's stage-game payoff is equal to 1 no matter how other players play.

$$\begin{aligned} u_R(a_i, a_{-i}) &= 1 && \text{if } a_i = \textbf{revolt} \text{ and } \#\{j : a_j = \textbf{revolt}\} \geq k \\ u_R(a_i, a_{-i}) &= -1 && \text{if } a_i = \textbf{revolt} \text{ and } \#\{j : a_j = \textbf{revolt}\} < k \\ u_R(a_i, a_{-i}) &= 0 && \text{if } a_i = \textbf{stay} \end{aligned} .$$

Let $[R](\theta)$ be the set of Rebels given θ and the notion *relevant information* indicate whether or not $\#[R](\theta) \geq k$. Note that the ex-post efficient outcome in the stage game is that every Rebel plays **revolt** whenever $\#[R](\theta) \geq k$, and plays **stay** otherwise.[8]

During the game, every player can observe his and his neighbors' types and his and their histories of actions, but no more. A history of actions played by i from period one to period $s \geq 1$ is denoted by $h_i^s \in H_i^s := \times_{\varsigma=1}^{s} A_{\theta_i}$. Let $G_i := \{j : \{i, j\} \in E\}$ be i's *neighbors*. Denote $\theta_{G_i} \in \Theta_{G_i} := \{R, I\}^{G_i}$ as the type profile of i's neighbors. Let $h_i^0 = \emptyset$, and denote $h_{G_i}^s \in H_{G_i}^s := \times_{j \in G_i} \times_{\varsigma=1}^{s} H_j^{\varsigma}$ as a history of actions played by i's neighbors from period one to period $s \geq 1$. The information set of i about θ at every period is the cylinder $p(\theta) = \{\theta_{G_i}\} \times \{R, I\}^{N \setminus G_i}$, and the information set about histories of action from period one to period $s \geq 1$ is $\{h_{G_i}^s\} \times H_{N \setminus G_i}^s$. A player i's pure behavior strategy τ_i is a measurable function with respect to i's information partition if τ_i maps

[7]A path in G from i to j is a finite sequence $(l_1, l_2, ..., l_L)$ without repetition so that $l_1 = i$, $l_L = j$, and $\{l_q, l_{q+1}\} \in E$ for all $1 \leq q < L$. G is fixed if G is not random, and G is undirected if there is no order relation over each edge. G is connected if, for all $i, j \in N$, $i \neq j$, there is a path from i to j.

[8]Moreover, at every θ and every k, the ex-post efficient outcome is unique and gives the maximum as well as the same payoff to every Rebel.

$\{\theta_{G_i}\} \times \{R, I\}^{N\setminus G_i} \times \{h^s_{G_i}\} \times H^s_{N\setminus G_i}$ to a single action in his action set for every $s \in \{1, 2, ...\}$ and every $\theta \in \Theta$. I assume that payoffs are hidden to emphasize that observing neighbors' actions are the only channel to infer other players' types and actions.[9]

Likewise, define $H^s := \times_{j\in N} H^s_j$ as the set of histories of actions from period one to period $s \geq 1$ and $H := \bigcup_{\varsigma=0}^{\infty} H^{\varsigma}$ as the collection of histories of actions. By abusing the notation a bit, let $h(\tau, \theta) \in H$ denote the realized history of actions generated by strategy profile $\tau = (\tau_1, \tau_2, ..., \tau_n)$ given θ. Designate $\alpha^{\pi,\tau}_{G_i}(\theta, h^s|\theta_{G_i}, h^s_{G_i})$ as the conditional distribution over $\Theta \times H^s$ induced by π and τ conditional on i's information up to period $s \geq 1$. The belief of i over Θ induced by π and τ up to period $s \geq 1$ is defined by

$$\beta^{\pi,\tau}_{G_i}(\theta|h^s_{G_i}) := \sum_{h^s\in H^s} \alpha^{\pi,\tau}_{G_i}(\theta, h^s|\theta_{G_i}, h^s_{G_i}).$$

The equilibrium concept is the weak sequential equilibrium.[10] My objective is looking for the existence of *approaching ex-post efficient equilibrium* or *APEX equilibrium*, which is defined below.

Definition 2.1 (APEX strategy)**.** *A behavior strategy τ is APEX if for all θ, there is a terminal period $T^\theta < \infty$ such that the actions in h^τ_θ after T^θ repeats the static ex-post Pareto efficient outcome.*

Definition 2.2 (APEX equilibrium)**.** *An equilibrium (τ^*, α^*) is APEX if τ^* is APEX.*

In an APEX strategy, all Rebels will play **revolt** forever after some period only if $\#[R](\theta) \geq k$; otherwise, Rebels will play **stay** forever after some period. It is as if the Rebels will learn the relevant information in the equilibrium because they will play the

[9]Such restriction will be relaxed in the Section 5.

[10]A weak sequential equilibrium is an assessment $\{\tau^*, \alpha^*\}$, where α^* is a collection of distributions over players' information sets with the property that, for all $i \in N$ and for all $s = 1, 2, ...$, $\alpha^*_{G_i}(\theta, h^s|\theta_{G_i}, h^s_{G_i}) = \alpha^{\pi,\tau^*}_{G_i}(\theta, h^s|\theta_{G_i}, h^s_{G_i})$ whenever the information set is reached with positive probability given τ^*. Moreover, for all $i \in N$ and for all $s = 1, 2, ...$, τ^*_i maximizes i's continuation expected payoff of

$$E^\delta_G(u_{\theta_i}(\tau_i, \tau^*_{-i})|\alpha^{\pi,\tau_i,\tau^*_{-i}}_{G_i}(\theta, h^s|\theta_{G_i}, h^s_{G_i}))$$

conditional on θ_{G_i} and $h^s_{G_i}$ for all $h^s_{G_i} \in H^s_{G_i}$.

ex-post efficient outcome after a certain point of time and keep on doing so. Notice that, in an APEX equilibrium, it is not only as if the Rebels will learn the relevant information: they must learn that. Lemma 2.1 articulates this fact.

Lemma 2.1 (Learning in the APEX equilibrium)**.** *If the assessment (τ^*, μ^*) is an APEX equilibrium, then for all $\theta \in \Theta$, there is a finite time T_i^θ for every Rebel i so that*

$$\sum_{\theta \in \{\theta : [R](\theta) \geq k\}} \beta_{G_i}^{\pi, \tau^*}(\theta | h_{G_i}^s) = \text{ either } 1 \text{ or } 0$$

whenever $s \geq T_i^\theta$.

Definition 2.3 (Learning the relevant information)**.** *A Rebel i learns the relevant information at period ς according to strategy τ if $\sum_{\theta \in \{\theta : [R](\theta) \geq k\}} \beta_{G_i}^{\pi, \tau}(\theta | h_{G_i}^s) = \text{either } 1 \text{ or } 0$ whenever $s \geq \varsigma$.*

It is clear that an APEX equilibrium exists when $k = 1$. As for other cases, let us start with the case of $k = n$ and then continue on to the case of $1 < k < n$. The proof is by construction. In the case of $k = n$, the constructed APEX equilibrium is intuitive and satisfies a stronger equilibrium concept. My main result tackles the case of $1 < k < n$. In such case, my constructed APEX equilibrium is not trivial and can only works for acyclic networks. Section 5.2 discusses why my constructed equilibrium is intractable in cyclic networks.

3 Equilibrium: APEX for $k = n$

In this section, my objective is to show the existence of APEX equilibrium for the case of $k = n$. In this case, notice that a Rebel can get a better payoff from playing **revolt** than from **stay** *only if* all players are Rebels. Two consequences follow. Firstly, if a Rebel has an Inert neighbor, this Rebel will always play **revolt** in the equilibrium. Secondly, at any period $s \geq 1$, it is credible for every Rebel to punish a deviation by playing **stay** forever *if* there is another one who also plays **stay** forever, independently from the belief held by the punisher. These two features constitute an APEX equilibrium and further transform itself to a sequential equilibrium.

Theorem 1 (APEX equilibrium for the case of $k = n$)**.** *For any n-person repeated k-Threshold game with parameter $k = n$ played in a network, there is a $\delta^* \in (0, 1)$ such that a sequential APEX equilibrium exists whenever $\delta > \delta^*$.*

Image that there are an Inert somewhere as well a Rebel i somewhere. Since the network is connected, there is a path connecting these two players. Along with this path, consider the "closest" Inert from Rebel i; this is an Inert who can be reached by the least number of consecutive edges from i. Note that this Inert's Rebel neighbors will play **stay** forever since $k = n$. Consider a strategy for Rebels on this path: a Rebel will play **stay** only after observing his neighbor plays **stay**. On this path and according to this strategy, Rebel i will know the existence of such Inert eventually since the network is finite. This contagion argument suggests the following APEX strategy. Every Rebel plays **revolt** initially except for he has an Inert neighbor. Each of them will continuously play **revolt** but switch to **stay** instantly if he observes any of his neighbor plays **stay**. Upon observing a n consecutive **revolt**, a Rebels knows that no Inert exists; otherwise, he knows some Inert exists. The above strategy is an APEX strategy if all Rebels play ex-post efficient outcome after peiord n. To extend it to be an APEX equilibrium, let the deviant play **stay** forever and the punisher who detects it also play **stay** forever. This off-path strategy is credible for both the deviant and the punisher, independent from the belief held by the punisher, and hence it is also sequential rational.[11]

4 Equilibrium: APEX for $1 < k < n$

I show the existence of APEX equilibrium for the case of $1 < k < n$ in this section. In contrast to the case of $k = n$, a Rebel still has the incentive to play **revolt** even if he has an Inert neighbor. This opens a possibility for the non-existence of APEX equilibrium. Example 1 below demonstrates it.

Example 1. Suppose $k = 2$ and $\theta = (R, I, R)$. The state and the network is represented in Figure 1. Rebel 1 never learns the type of player 3 since Inert 2 cannot reveal it, and

[11] This sequential rationality is in the sense of [Kreps and Wilson, 1982].

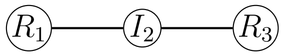

Figure 1: The state and the network in which the APEX equilibrium does not exist when $k = 2$.

hence no APEX equilibrium exists in this scenario.

The following assumption on the prior—*full support on strong connectedness*—excludes the possibility of nonexistence of APEX equilibrium. To this end, I begin with the definition of *strong connectedness.*

Definition 4.1 (Strong connectedness)**.** *Given G, a state θ has strong connectedness if, for every two Rebels, there is a path consisting of Rebels to connect them.*

In the language of graph theory, the following definition is equivalent: given G, θ has strong connectedness if the induced graph by $[R](\theta)$ is connected.

Definition 4.2 (Full support on strong connectedness)**.** [12] *Given G, π has full support on strong connectedness if*

$$\pi(\theta) > 0 \Leftrightarrow \ \theta \textit{ has strong connectedness.}$$

I state the main characterization of this paper:

Theorem 2 (APEX equilibrium for the case of $1 < k < n$)**.** *For any n-person repeated k-Threshold game with parameter $1 < k < n$ played in networks, if networks are acyclic and if π has full support on strong connectedness, then there is a $\delta^* \in (0, 1)$ such that an APEX equilibrium exists whenever $\delta > \delta^*$.*[13]

[12] The prior has full support on strong connectedness is stronger than every state has strong connectedness is common knowledge. This marginal requirement is convenient in constructing equilibrium. The main result only requires a weaker version—$\pi(\theta) > 0 \Rightarrow \ \theta$ has strong connectedness . Working on this weaker version is at the expense of much tedious proof, however.

[13] A network is acyclic if the path from i to j for all $i \neq j$ is unique.

Constructing an APEX equilibrium in this case is convoluted. I illustrate the proof idea throughout this paper while leaving the formal proof in Appendix. Moreover, since the case of $k = 2$ is trivial given that θ has strong connectedness, I focus on $2 < k < n$ cases.[14]

I start with considering a specific APEX strategy as the framework in constructing an APEX equilibrium in Section 4.1; incentive compatibility is not incorporated at this moment. Then I introduce an auxiliary scenario to illuminate the discussion of incentive compatibility in Section 4.2. I then go back to present essential details of the constructed APEX equilibrium in Section 4.3.

4.1 APEX strategy

To begin, the following example is examined.

Example 2. Let $k = 5$ and consider the configurations in Figure 2 and Figure 3. Note that there are 6 players in both of these configurations; 5 Rebels are in Figure 2 and 4 Rebels are in Figure 3. The below is an APEX strategy.

Strategy for Example 2:

1. At period j, $1 \leq j \leq 6$, each Rebel plays **revolt** if himself is j or any of his Rebel neighbors is j; otherwise, he plays **stay**. For instance, from period one to period 6, what Rebel 3 plays is (**revolt**, **stay**, **revolt**, **revolt**, **stay**, **revolt**) in the configuration in Figure 2. Consequently, implied by Rebel 3's actions, in the configuration in Figure 2, right after period 1, 2, 3, or 4, Rebel 4 is still aware the existence of Rebel 1, 2, 3, and 4; right after period 5, Rebel 4 is aware that player 5 is an Inert; right after period 6, Rebel 4 is aware that player 6 is a Rebel. Similar behavior applies to other players.

[14] Suppose $[R](\theta) \geq k = 2$, by the full support on strong connectedness, each Rebel have a Rebel neighbor. The following strategy is an APEX strategy. A Rebel plays **revolt** from period one afterwards if he has a Rebel neighbor; otherwise, he plays **stay** forever from period one. It extends to an APEX equilibrium by letting the off-path belief be assigning probability one to the event that all non-neighbors are Inerts.

2. Right after period 6, at period 7, by observing his neighbors' actions, a Rebel who has learnt the number of Rebels exceeding or equal to k plays $(\textbf{revolt}, \textbf{revolt})$ consecutively. Then he plays **revolt** from period 19 afterwards. Likewise, a Rebel who has observed the number of Rebels, which is implied by his neighbors' actions, is less than k, he plays $(\textbf{stay}, \textbf{stay})$ consecutively. Then he plays **revolt** from period 19 afterwards. A Rebel whose observation does not belong to the above two categories plays the sequence $(\textbf{revolt}, \textbf{stay})$ consecutively and plays **stay** forever after period 19.

3. From period 7 to period 19, a Rebel plays $(\textbf{revolt}, \textbf{revolt})$ consecutively right after observing the same sequence played by any of his neighbor; he then plays **revolt** from period 19 afterwards. Correspondingly, a Rebel plays $(\textbf{stay}, \textbf{stay})$ consecutively right after observing the same sequence played by any of his neighbor and then plays **stay** from period 19 afterwards.

4. The off-path strategy is simple: ignoring the deviation and following the on-path strategy in which more Rebels can be implied.

By following the on-path strategy of the above, both Rebel 1 and 4 learn the relevant information right after period 7, and all Rebels play the ex-post outcome from period 19 afterwards. This strategy is thus an APEX strategy with a straightforward idea: from period one to period 6, "a reporting phase", Rebels reports who their Rebel neighbors are by sequences of actions, and, from period 7 to period 19, "a coordination phase", they coordinate themselves to disseminate the knowledge whether the relevant information is learnt.

The above strategy, nevertheless, calls for revising to seemly integrate incentive compatibility. Notice that Rebel 6's actions are redundant: the ex-post efficient outcome will be achieved anyway despite how Rebel 6 plays. This is because Rebel 3's neighbors include Rebel 6's and how Rebel 6 plays cannot be observed by other Rebels outside Rebel 3's neighborhood. The notion "redundant" will be replaced by and formally defined as *inactive* later to emphasize that Rebel 6 can actually report nothing. What is more, Rebel 4 has no

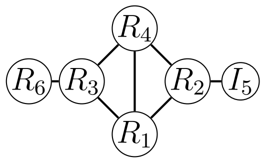

Figure 2: A configuration of the state and the network in which players 1, 2, 3, 4, 5 are Rebels and players 6 is an Inerts.

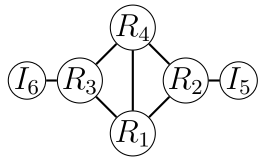

Figure 3: A configuration of the state and the network in which players 1, 2, 3, 4 are Rebels and player 5, 6 are Inerts.

incentive to report. This is because Rebel 1 reports the identical information to the same Rebels. In other words, the ex-post efficient outcome will be played anyway despite how Rebel 4 plays. Rebel 1's position is exactly the same as Rebel 4's, and thus both of them have no incentive to report. A free-rider problem exists.[15]

As follows, a revision of Strategy for Example 2 is designed and meant to be the foundation framework in constructing the APEX equilibrium for Theorem 2. Let $\mathbf{I}$ be all possible sets of Rebels amongst states in the network.[16] For convenience, construct a set W that consists of sequences of actions, in which all sequences have equal length, so that there is a one-to-one mapping between $\mathbf{I}$ and W. This W exists. For an example, which is used in the constructed APEX equilibrium for Theorem 2, let the length of each sequence in W be the multiplication of a series of prime numbers. In this series, each prime number is distinct and assigned to a distinct player. Denote x_i as the prime number assigned to i. The length

[15]It is worth noting that whether a free-rider problem could happen is not solely determined by whether a network is cyclic. It depends on how an APEX strategy is constructed and exists in the constructed APEX equilibrium for Theorem 2, in which only acyclic networks are of concern. The formal definition of free-rider problem is in Section 4.3, and a deeper issue will be discussed in Section 5.2.

[16]To be precise, $\mathbf{I} = \{X \subseteq N : \theta_j = R \Leftrightarrow j \in X \text{ for some } \theta \in \Theta\}$

$$0 < \underbrace{(\text{coordination phase}) < (\text{reporting phase})}_{\text{1-block}} < \underbrace{(\text{coordination phase}) < (\text{reporting phase})}_{\text{2-block}} < ...$$

Figure 4: The partition of the time in the repeated k-threshold game. $<$ is the linear order relation over the time.

of a sequence in W is therefore $\bigotimes_{j\in N} x_j = x_1 \otimes ... \otimes x_n$, where $\otimes$ is the usual multiplication operator. If $I \in \mathbf{I}$, the corresponding $\langle I \rangle \in W$ is crafted to be:

$$(\overbrace{\textbf{stay}, ..., \textbf{stay}, \underbrace{\textbf{revolt}, \textbf{stay}, ..., \textbf{stay}}_{\bigotimes_{j\in I} x_j}}^{\bigotimes_{j\in N} x_j}).$$

A one-to-one mapping between $\mathbf{I}$ and W exists since a multiplication of prime numbers can be uniquely factorized.

Fix W and partition the time into two consecutively alternating phases: *coordination phase* and *reporting phase*, while the time is starting from the coordination phase. The t-th completion of two consecutive phases is called the t-block. The length of each reporting phase is equal to the length of $w \in W$, and the length of each coordination phase is twice of the total number of players. Figure 4 depicts this partition.[17]

Given θ, denote the set I_i as i's Rebel neighbors. If i is a Rebel, let $I_i^1 = I_i$, and let $I_i^t = \bigcup_{j\in G_i} I_j^{t-1}$ for $t \geq 2$. If j is an Inert, let $I_j^t = \emptyset$ for $t \geq 1$. Put it differently, I_i^t is the set of Rebels who can be reached from Rebel i by a path consisting of Rebels and of which the length is at most t. Let us then identify a set of Rebels, *active Rebels*, who are crucial in the information sharing process. For this purpose, first define G_i^t for each t: if i is a Rebel, let $G_i^1 = G_i$, and let $G_i^t = \bigcup_{j\in G_i} G_j^{t-1}$ for $t \geq 2$; if j is an Inert, let $G_j^t = \emptyset$ for $t \geq 1$. That is, G_i^t is the set of players who can be reached from Rebel i by a path consisting of Rebels and of which the length is at most t. Finally, define the active Rebels at t-block as follows.

[17] More precisely, partition the time by $\{\{0\}, \{1, ..., s_{l_1}, s_{l_1}+1, ...s_1\}, \{s_1+1, ..., s_{l_2}, s_{l_2}+1, ..., s_2\}, ..., \{s_{t-1}+1, ..., s_{l_t}, s_{l_t}+1, ..., s_t\}, ...\}$, where $t = 1, 2, ...$ and $s_0 = 0$, so that the length of $\{s_{l_t}+1, ..., s_t\}$ is equal to the length of $w \in W$ for each t, while the length of $\{s_{t-1}+1, ..., s_{l_t}\}$ is $2n$. Call $\{s_{t-1}+1, ..., s_t\}$ the t-block. call $\{s_{l_t}+1, ..., s_t\}$ the *reporting phase* at t-block and call $\{s_{t-1}+1, ..., s_{l_t}\}$ the *coordination phase* at t-block.

Definition 4.3 (Active Rebel at t-block)**.** *Set $R^0 = [R](\theta)$. The set of active Rebels in the t-block is*

$$R^t := \{i \in R^{t-1} : \nexists j \in G_i \text{ such that } I_i^t \subseteq G_j^t\}.$$

Otherwise speaking, an active Rebel in the t-block is a Rebel whose information about θ, I_i^t, is not a subset of any other Rebel's same information. In the configuration in Figure 2 for Example 2, the set of Rebels is $\{1, 2, 3, 4, 6\}$, the active Rebels in the 1-block are Rebel 2 and 3, and no Rebel is active in the t-block when $t \geq 2$. Notice that the active Rebels have to be also the active ones in the previous block; they are fewer and fewer as t goes by. Let us call a Rebel *inactive Rebel* if he is not an active Rebel.

It is sufficient to reveal the relevant information by letting only active Rebels share information about θ, given that the network is acyclic and given that θ has strong connectedness. Theorem 3 articulates this.

Theorem 3. *If the network is acyclic and if the θ has strong connectedness, then there is a strategy so that there exists an active Rebel in the t-block who can learn the relevant information at* $t + 1$*-block.*

Note that $I_i^t \in \mathbf{I}$. The following strategy is for Theorem 3.[18]

Strategy for Theorem 3:

1. In each reporting phase in the t-block, each active Rebel i at t-block plays $\langle I_i^t \rangle \in W$.
2. At each period in each coordination phase in the t-block, all Rebels play **stay**.

Proof. Here, I show the proof that there is t and there exists Rebel i who can learn the relevant information in the $t + 1$-block. The proof that i has to be active in the t-block is in Appendix. Following the above strategy, right after t-block, Rebel i assigns probability one to the event

$$\{\theta \in \Theta : \theta_j = R \Leftrightarrow j \in I_i^{t+1}\}$$

[18]Theorem 3 is equivalent to the following statement: if the network is acyclic and if the θ has strong connectedness, then there exists $t \geq 0$ and $i \in R^t$ so that $I_i^{t+1} = [R](\theta)$.

by Bayesian rule. To conclude the proof, it is sufficient is to show there exists a t so that $I_i^{t+1} = [R](\theta)$. By definition, I_i^t is the set of Rebels who can be reached by at most t consecutive edges from Rebel i, in each of which the endpoints are Rebels. Since θ has strong connectedness, there exists a t_i so that $I_i^{t_i} = [R](\theta)$. What follows is i learns θ at t_i. □

Remark. Theorem 3 is not true if the network is cyclic. Take the configuration in Figure 2 for Example 2 as an example. By following Strategy for Theorem 3, in the 2-block, there is no Rebel who learns the relevant information. Then, in the 3-block, either Rebel 1 or 4 learns the relevant information, but both Rebel 1 and 4 are inactive at 2-block. Theorem 3 thus captures a free-rider problem in cyclic networks as the above mentioned: Rebel 1 and Rebel 4 are redundant mutually.

Modifying Strategy for Theorem 3, Strategy for Proposition 4.1 below is an APEX strategy.

Proposition 4.1. *If the network is acyclic and if the θ has strong connectedness, then there is an APEX strategy.*

Consider the following strategy.

Strategy for Proposition 4.1:

1. Suppose T^θ has arrived, Rebels play the ex-post efficient outcome.

2. Suppose T^θ has not yet arrived. In the reporting phase, Rebels follow Strategy for Theorem 3. In the coordination phase, as below, the strategy is a contagion process.

   (a) If a Rebel has been certain that there are k or more Rebels, he plays sequence of actions $(\mathbf{stay}, \mathbf{stay})$ continuously starting right after he was certain that. He plays **stay** forever after this phase; T^θ arrives right after this phase.

   (b) If a Rebel has learnt that there are less Rebels than k Rebels, he plays sequence of actions $(\mathbf{revolt}, \mathbf{revolt})$ continuously starting right after he learnt that. He plays **revolt** forever after this phase; T^θ arrives right after this phase.

(c) If a Rebel has observed the sequence of actions $(\mathbf{stay}, \mathbf{stay})$, he plays $(\mathbf{stay}, \mathbf{stay})$ continuously starting right after he observed that and plays **stay** forever after this phase; T^θ arrives right after this phase.

(d) If a Rebel has observed the sequence of actions $(\mathbf{revolt}, \mathbf{revolt})$, he plays $(\mathbf{revolt}, \mathbf{revolt})$ continuously starting right after he observed that and plays **revolt** forever after this phase; T^θ arrives right after this phase.

(e) If a Rebel has not learnt the relevant information, he plays sequence of actions $(\mathbf{revolt}, \mathbf{stay})$ continuously.

This is a simple extension of Strategy for Example 2. Rebels share information in reporting phase. If a Rebel has learnt the relevant information, he disseminates it to all Rebels contagiously in coordination phase; otherwise, he continues to the next phase—a reporting phase.

If incentive compatibility is under consideration, this logic, however, brings another free-rider problem. Suppose that there are two Rebels who share information to each other in a reporting phase, and each of them is certain that he will learn the relevant information if the other one shares *truthful* information to him. Because sharing information is done by alternating actions, which incurs expected payoff, they will not truthfully share their information. This is because each of them will choose his most profitable way of sharing information without impeding learning the relevant information provided that the other one share the truthful information. The free-rider problem turns out to be the main challenge in the construction of an APEX equilibrium. The proof solves it by arguing that if the network is acyclic, the free-rider problem only occurs between two Rebel neigbors who *commonly know it*, while this argument does not hold for cyclic network.[19] With the help from this argument, the constructed equilibrium solves the free-rider problem by arbitrarily assigning one of them to be the free rider, who can choose his most profitable way in sharing information, while letting the other one share truthful information.

To make the discussion of incentive compatibility more transparent, I introduce T*-round*

[19]Section 5.2 provides an example that the free-rider problem is not commonly known between the Rebels who involve.

writing game as an auxiliary scenario. In the T-round writing game, T is fixed, players are endowed a writing technology so that they can write to share information about θ for T rounds. They then play a one-shot k-threshold game at round $T+1$. This game is a reduced form of the original game by fixing T, so that I can pay attention only to the incentive compatibility in information sharing process and ignore the incentive compatibility in the coordination phase.[20] This writing technology represent a way of communication by *writing sentences that are composed by letters according to a fixed grammar.* Though this auxiliary game will be soon introduced in the next section, I draw Table 1 to shed light on the parallel between it and the original game.

Table 1: The analogue between T-round writing game and the repeated k-threshold game

T-round writing game	Repeated k-threshold game
A round	A range of periods
A sentence	A sequence of actions
The length of a sentence in a round	The length of a range of periods
A chosen letter in a sentence	A chosen action
The cost of writing a sentence	The expected payoff occurring in a sequence of actions
The fixed grammar	The equilibrium path

4.2 T-round writing game

The network, the set of states, and the set of players follow exactly the same definitions defined in Section 2. In the T-round writing game, each player endows a *writing technology.* A writing technology for player i is a pair of (W, M_i), in which $W = \{\mathbf{r}, \mathbf{s}\}^L$, $L \in \mathbb{N}$, and

[20]There is another free-rider problem in the coordination phase in Strategy for Theorem 3: a Rebel who has learnt the relevant information might deviate to play $(\mathbf{revolt}, \mathbf{stay})$ from playing $(\mathbf{revolt}, \mathbf{revolt})$. This strategy will be further modified to incorporate the incentive compatibility in Section 4.3.2.

$M := \bigtimes_{i\in N} \bigtimes_{t=1}^{T} M_i^t$ recursively defined by

$$M_i^1 = \{f | f : \Theta_{G_i} \to W\} \cup \{\emptyset\}$$

$$\text{for } 2 \le t \le T \text{ , } M_i^t = \{f | f : \bigtimes_{j\in G_i} M_j^{t-1} \to W\} \cup \{\emptyset\}.$$

W is interpreted as the set of sentence composed by letters **r** or **s** with length L, while M_i is understood as i's grammar. $\emptyset$ represents remaining silent. The phrase of "i writes a sentence to all his neighbors at round t" is equivalent to "i selects an $f \in M_i^t$ to get an element $w \in W$ according to f, which can be observed by all i's neighbors".

The time line for the deterministic T-round writing game is as follows.

1. Nature chooses θ according to the prior π.

2. θ is then fixed throughout rounds.

3. At $t = 1, ..., T$ round, players write to their neighbors.

4. At $T + 1$ round, players play a one-shot k-Threshold game.

5. The game ends.

A Rebel's payoff is the summation of his stage payoff across stages, while an Inert's payoff is set to be 1. The equilibrium concept is weak sequential equilibrium. An APEX strategy is a strategy that induces the ex-post outcome in the one-shot k-threshold game at $T + 1$ round. The definition of APEX equilibrium is adapted accordingly. In the examples below, let us focus on the configuration represented in Figure 5 and Figure 6 with $n = L = 8$. I.e. there are 8 players and the length of a sentence is also 8. Note that the differences between configurations in Figure 5 and Figure 6 are: (1) there are 6 Rebels in Figure 5 but are 5 Rebels in Figure 6; (2) player 8 is a Rebel in Figure 5 but he is an Inert in Figure 5.

Example 3 (T-round writing and the free-rider problem)**.** Let $k = 6$ and $T = 2$. Suppose that remaining silent incurs no cost, but writing incurs an extremely small cost $\epsilon > 0$ so

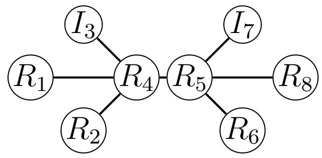

Figure 5: A configuration of the state and the network in which players 1, 2, 4, 5, 6, 8 are Rebels while players 3, 7 are Inerts.

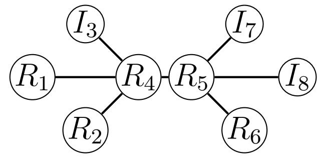

Figure 6: A configuration of the state and the network in which players 1, 2, 4, 5, 6 are Rebels while players 3, 7, 8 are Inerts.

that ϵ is strictly decreasing with the number of $\mathbf{r}$ in a sentence. This is to say writing $(\mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r})$ incurs the least cost while writing $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{s})$ incurs the largest.[21]

Let us consider the following strategy ϕ.

At $t = 1$, the peripheral Rebels remain silent. Rebel 4 (or 5)'s grammar for writing a sentence is that if player i is a Rebel and known to him, he writes $\mathbf{r}$ in the i-th component in the sentence; otherwise, he writes $\mathbf{s}$ in that component. According to this grammar, the central player Rebel 4 writes the sentence $(\mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s}, \mathbf{s})$ on both configurations in Figure 5 and in Figure 6. The central player Rebel 5 writes $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r})$ in the configuration in Figure 5 and writes $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s})$ in the configuration in Figure 6. Rebels 4 and 5's sentences thus reveals who are Rebels and who are not. Notice that the common knowledge of the network contributes to the ability of revealing players' types.

At $t = 2$, the peripheral Rebels still remain silent. Rebel 4 writes $(\mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r})$ in the configuration in Figure 5 and writes $(\mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s})$ in the configuration in Figure 6. Rebel 5 writes exactly the same sentence as Rebel 4. This is to say Rebel 4 and 5 share

[21] This cost is not directly linking to the expected payoff in the original game, but is meant to be a metaphor to emphasize that alternating actions might incurs negative expected payoff in the original game and to illustrate the potential free-rider problem.

information at $t = 1$ and then coordinate to announce a mixture sentence at $t = T$.

At $T+1$, by counting $\mathbf{r}$ in Rebel 4 or 5's mixture sentence, all Rebels know whether the number of Rebels outnumbers k. This leads all Rebels to play the ex-post efficient outcome in the one-shot k-threshold game.

The above ϕ is not an APEX equilibrium. According to the above-mentioned, Rebel 4 will know the relevant information at $t = 2$ even if he deviates to writing that all his neighbors are Rebels, which incurs less cost than his truthful writing.[22][23] Rebel 5 is in the same situation as Rebel 4 and therefore also writes the sentence that indicates that all his neighbors are Rebels. However, these sentences are uninformative. It turns out that both of them will deviate, and neither of them can know the relevant information at $t = 2$.

Fortunately, the following example shows that the free-rider problem can be solved.

Example 4 (T-round writing and solving the free-rider problem)**.** The solution to solve the free-rider problem in the previous example is to extend T. It would open the possibility of the existence of a free rider at some round, while letting this free rider reveals relevant information at the next round. To this end, let $k = 6$ and $T = 3$. Consider the following strategy ρ and focus on the interaction between Rebels 4 and 5.

At $t = 1$, the lowest-index Rebel between Rebels 4 and 5 is the free rider, while the other one truthfully writes down his information. This is to say Rebel 4 will be the free rider and he writes the least-cost sentence. Rebel 5 writes $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r})$ in the configuration of Figure 5and writes $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s})$ in the configuration of Figure 6.

At $t = 2$, Rebel 4 has known the relevant information. Rebel 4 writes the least-cost sentence if there are k or more Rebels but remains silent otherwise. The consequence is Rebel 4's behavior reveals the relevant information to his neighbors at this round. Rebel 5 remains silent instead.

At $t = T$, Rebel 5 has known the relevant information since he is Rebele 4's neighbor.

[22]This sentence is $(\mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s}, \mathbf{s})$, which incurs less cost than the truthfully reporting sentence $(\mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s}, \mathbf{s})$.

[23]If he remains silent, then this behavior will be considered as a deviation, and therefore he will never get the maximum payoff of 1. Hence, he will avoid doing so.

He writes the least-cost sentence if there are k or more Rebels but remains silent otherwise. Therefore Rebel 5's behavior at this round reveals the relevant information to his neighbors. Rebel 4 remains silent instead.

At $T+1$, all Rebels know the relevant information by observing Rebels 4 and 5's behavior. They play the ex-post efficient outcome accordingly.

Remark. Why does Rebel 5 *know* that he is not a free rider and therefore behaves not like a free rider? The following is the reason. He *knows* that, by common knowledge of the network, he and Rebel 4 are in a free-rider problem. Moreover, by common knowledge of the network, he knows that Rebel 4 knows that he and Rebel 4 are in a free-rider problem, he knows that Rebels 4 knows that he knows that,...,and so forth. Consequently, Rebel 5 and 4 commonly know that they are engaged in a free-rider problem.

In the next section, Section 4.3, I delver into substantial ideas in constructing an APEX equilibrium and leave actual details in Appendix. Separately, I depict the construction in the reporting phase and in the coordination phase. The depiction might seem repetitive, but deliberately tailored. Especially, so far, the attention is devoutly paid to integrating incentive compatibility in the reporting phase, but in the lack of the coordination phase. Section 4.3.2 manages the incentive compatibility in the coordination phase.

4.3 Equilibrium construction

The framework is adapted from Proposition 4.1, and the time is partitioned by two consecutively alternating phases as shown in Figure 4. Recall that the time is partitioned into

$$0 < \underbrace{\text{(coordination phase)} < \text{(reporting phase)}}_{\text{1-block}} < \underbrace{\text{(coordination phase)} < \text{(reporting phase)}}_{\text{2-block}} < \ldots$$

The off-path belief is simple and serves as a grim trigger. Whenever Rebel i detects a deviation at period ς, he forms the following belief:

$$\sum_{\theta \in \{\theta : \theta_j = I, j \notin G_i\}} \beta_{G_i}^{\pi,\tau}(\theta | h_{G_i}^s) = 1, \text{ for all } s \geq \varsigma. \tag{1}$$

This is to say i believe all players outside his neighborhood are Inerts. Thus, if $\#I_i^{\varsigma} < k$, he will play **stay** forever after he detects a deviation.

4.3.1 The equilibrium path in the reporting phase

The description in this section is for the APEX equilibrium path before T^{θ}. Let us shorten "reporting phase in t-block" by $\mathbf{O}^t$, denote $|\mathbf{O}^t|$ as the length of $\mathbf{O}^t$, and shorten **revolt** and **stay** to $\mathbf{r}$ and $\mathbf{s}$ receptively. For simplicity, the term "reporting phase" refers to "reporting phase in t-block" in this section.

$|\mathbf{O}^t|$ is independent from t and determined only by the set of players. Firstly, assign each player i a distinct prime number x_i starting from 3. Then let $|\mathbf{O}^t| = \bigotimes_{i\in N} x_i = x_1 \otimes x_2 \otimes ... \otimes x_n$, where $\otimes$ is the usual multiplication operator. The sequence of actions in $\mathbf{O}^t$ is with length $|\mathbf{O}^t|$ and would take one of the forms specified in the right column in Table 2. The abbreviations of these sequences are listed in the left column. Since these sequences in the reporting phase are meant to share information about θ, the terms "playing the sequence" and "reporting the information" are interchangeable and will be alternatively used.

Table 2: The notations for the sequences of actions in $\mathbf{O}^t$ on the path

Notations		The sequences of actions
$\langle I \rangle$	:=	$(\overbrace{\mathbf{s}, ..., \mathbf{s}, \underbrace{\mathbf{r}, \mathbf{s}, ..., \mathbf{s}}_{\bigotimes_{j\in I} x_j}}^{\bigotimes_{i\in N} x_i})$
$\langle 1 \rangle$	:=	$(\overbrace{\mathbf{s}, ..., \mathbf{s}, \mathbf{r}}^{\bigotimes_{i\in N} x_i})$
$\langle$**all stay**$\rangle$	:=	$(\overbrace{\mathbf{s}, ..., \mathbf{s}, \mathbf{s}}^{\bigotimes_{i\in N} x_i})$

It is worth noting that the sequence constructed by prime numbers brings two benefits. Firstly, since the multiplication of distinguishing prime numbers can be uniquely factorized, the Rebels can use such sequence to precisely report players' identities. Secondly, the undiscounted expected payoff of playing $\langle I_i^t \rangle$ for some I_i^t for an active Rebel i is always equal to -1, and therefore it is relatively easy to calculate. This is because only active Rebels will report $\langle I \rangle$ for some I. Since this I is not reported by any other Rebel, at most one Rebel would play $\mathbf{r}$ at any period in the reporting phase by the property of prime number multiplication.[24]

I list the sequences played in the reporting phase on the path in Table 3. The terms θ-pivotal, $k-1$-pivotal, and free-rider problem will be defined one-paragraph later.

Table 3: The sequences of actions played in $\mathbf{O}^t$ on the path

Rebel i	i plays
i is inactive	$\langle$**all stay**$\rangle$
i is active but i is not pivotal	$\langle I_i^t \rangle$
i is $k-1$-pivotal	$\langle 1 \rangle$
i is θ-pivotal but not in the free-rider problem	$\langle 1 \rangle$
i is in the free-rider problem with the lowest index	$\langle 1 \rangle$
i is in the free-rider problem without the lowest index	$\langle I_i^t \rangle$

On the path, the sequences $\langle I \rangle$ or $\langle 1 \rangle$ are meant to differentiate themselves from $\langle$**all stay**$\rangle$. The sequence $\langle$**all stay**$\rangle$ is for the inactive Rebels at t-block to report nothing. The sequence $\langle I \rangle$ is for active Rebels at t-block to report I if I is a set of Rebels. Although the definitions of *pivotal Rebel* and *free-rider problem* has not yet formally defined at this present, the sequence $\langle 1 \rangle$ is intentionally crafted to tackle the free-rider problem. To see how $\langle 1 \rangle$ works, I turn to formally defining the pivotal Rebel and the free-rider problem.

Definition 4.4 (Pivotal Rebels in $\mathbf{O}^t$). *A Rebel p is pivotal in $\mathbf{O}^t$ if p is active at t-block, p is uncertain the relevant information, and p is certain that he will learn the relevant information right after $\mathbf{O}^t$, given that each $i \in R^t$ reports $\langle I_i^t \rangle$.*

By the definition, a pivotal Rebel in the reporting phase is one who can learn the relevant information if all of his active Rebel neighbors truthfully report their information about θ

[24]This statement holds if there is no Rebel who plays $\langle 1 \rangle$.

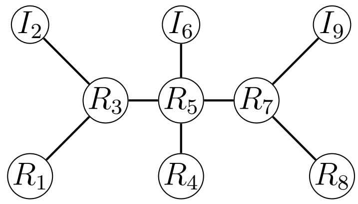

Figure 7: A configuration of the state and the network in which player 1, 3, 4, 5, 7, 8 are Rebels while players 2, 4, 9 are Inerts.

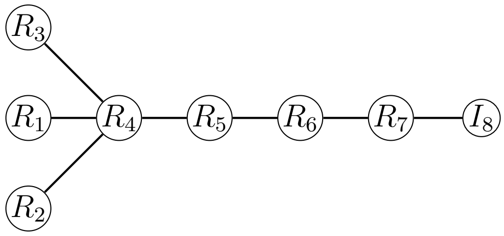

Figure 8: A configuration of the state and the network in which player 1, 2, 3, 4, 5, 6, 7 are Rebels while player 8 is an Inert.

to him. The pivotal Rebels can be further classified into two kinds: ones who can learn θ, and ones who learn only the relevant information. When $k = 6$, in the configuration in Figure 5, only Rebels 4 and 5 are pivotal; they are of the first kind. In the configuration in Figure 7, only Rebel 5 is pivotal; he is of the first kind. In the configuration in Figure 8, only Rebel 4 pivotal; he is of the second kind.

Call p of the first kind the *θ-pivotal* Rebel. For the second kind, if the network is acyclic and if the prior has full support on strong connectedness, p is the second kind only if $I_p^t = k - 1$. Call this second kind *$k - 1$-pivotal* Rebel.[25] Below is the defined free-rider problem in the reporting phase.

Definition 4.5. *A free-rider problem exists in $\mathbf{O}^t$ if there are multiple θ-pivotal Rebels in $\mathbf{O}^t$.*

The following lemma is crucial.

[25] To show that a pivotal Rebel is the second kind in $\mathbf{O}^t$ only if $I_p^t = k - 1$, one can follow the same argument in Theorem 3.

Lemma 4.1. *If the network is acyclic and if π has full support on strong connectedness, there are at most two θ-pivotal Rebels in the t-block. Moreover, they are neighbors.*[26]

And notably,

Lemma 4.2. *If the network is acyclic and if π has full support on strong connectedness, when there are two θ-pivotal Rebels $p, p^{'}$ in the t-block, then they commonly know that they are θ-pivotal Rebels at the beginning of t-block.*

By Lemma 4.2, θ-pivotal Rebels in the reporting phase can identify themselves at the beginning of this phase. This importance cannot be further emphasized. If the free-rider problem will occurs in the reporting phase, the strategy can, before the reporting phase, prescribes that the lowest indexed θ-pivotal Rebel p involving in the free-rider problem will play $\langle 1 \rangle$, while the other one will play $\langle I^t_{p'} \rangle$. Otherwise speaking, this knowledge is encoded in the belief system of an APEX equilibrium.

Remark. As the above mentioned, the assumption of acyclic network in Lemma 4.2 is indispensable. Lemma 4.2 does not hold if the network is cyclic. Section 5.2 demonstrates it.

4.3.2 The equilibrium path in the coordination phase

The descriptions in this section is for the APEX equilibrium path before T^θ. The term "coordination phase in t-block" is shorten by $\mathbf{C}^t$.

It is elaborate to spell out the coordination phase structure, but this phase is actually a simple contagion scenario: Rebels jointly decide to terminate or continue their information sharing during this phase. For that, a coordination phase is partitioned into three *divisions*. In the first division, if there is a Rebel has learnt that $\#[R](\theta) < k$, all Rebels will play **stay** forever right after this division, and T^θ arrives; otherwise, they continue to the next one. In the second division, if there is a Rebel has learnt that $\#[R](\theta) \geq k$, a portion of Rebels, at least k Rebels, will play **revolt** forever right after this division; otherwise, they continue

[26] As a remark, Lemma 4.1 is not true when the network is cyclic. To see this, consider a 4-player circle given that $\theta = (R, R, R, R)$.

to the next one. In the third division, if there is a Rebel has learnt that $\#[R](\theta) \geq k$ in the previous divisions, all Rebel will play **revolt** forever right after this division, and T^θ arrives; otherwise, they continue to the next phase—a reporting phase.

To fulfil the above contagion argument, a set of sequences of actions played on the path is prescribed so that Rebels update their belief according to it. For this task, further partition a division into *sub-blocks*. I depict the whole partition in the coordination phase below, where $\square$ represents a sub-block in a coordination phase.

In $\mathbf{C}^1$,

$$\overbrace{\underbrace{\square}_{\text{one sub-block}}}^{\text{1-division}}\ \overbrace{\underbrace{\square}_{\text{one sub-block}}}^{\text{2-division}}\ \overbrace{\underbrace{\square\cdots\square}_{n\text{ sub-blocks}}}^{\text{3-division}}$$

In $\mathbf{C}^t$, $t \geq 2$,

$$\overbrace{\underbrace{\square\cdots\square}_{n\text{ sub-blocks}}}^{\text{1-division}}\ \overbrace{\underbrace{\square\cdots\square}_{t+1\text{ sub-blocks}}}^{\text{2-division}}\ \overbrace{\underbrace{\square\cdots\square}_{n\text{ sub-blocks}}}^{\text{3-division}}$$

For each t, denote $\mathbf{C}^t(u,\cdot)$ as the u-division and $|\mathbf{C}^t(u,\cdot)|$ as the length of $\mathbf{C}^t(u,\cdot)$. Likewise, denote $\mathbf{C}^t(u,v)$ as the v-th sub-block in u-division and $|\mathbf{C}^t(u,v)|$ as the length of $\mathbf{C}^t(u,v)$. Let us shorten **revolt** and **stay** to **r** and **s** receptively. Let $|\mathbf{C}^t(u,v)| = n$ if $u = 1, 2$, $v = 1, ..., n$. Let $|\mathbf{C}^t(u,v)| = 1$ if $u = 3$, $v = 1, ..., n$. I list the the sequences of actions on the path and their notations in Table 4, where i is the original index of player i.[27]

Table 4: The notations for the sequences of actions in $\mathbf{C}^t(u,v)$ for $u = 1, 2$, $v = 1, ..., n$, on the path

Notations		The sequences of actions
$\langle i \rangle$	:=	$(\overbrace{\mathbf{s}, ..., \mathbf{s}, \underbrace{\mathbf{r}, \mathbf{s}, ..., \mathbf{s}}_{i}}^{n})$
$\langle$**all stay**$\rangle$	:=	$(\overbrace{\mathbf{s}, ..., \mathbf{s}, \mathbf{s}}^{n})$

[27]In the 3-division, since the sequence of actions which length is 1 is equivalent to playing a single action, I do not provide additional notations for conciseness.

4.3.3 The equilibrium behavior on the path in $\mathbf{C}^1$

I begin with depicting the equilibrium path in $\mathbf{C}^1$, which is shown in Table 5. The belief updating after $\mathbf{C}^1(1,\cdot)$ and $\mathbf{C}^1(2,\cdot)$ on the path is listed in Table 6. The evolution of players' information filtrations can be tracked throughout in this table. Since there is only one sub-block in $\mathbf{C}^1(1,\cdot)$ or $\mathbf{C}^1(2,\cdot)$, $\mathbf{C}^1(1,\cdot)$ or $\mathbf{C}^1(2,\cdot)$ is interchangeable with $\mathbf{C}^1(1,1)$ or $\mathbf{C}^1(2,1)$ respectively. If there is no confusing, the term "coordination phase" refers to "coordination phase in the 1-block" in this section.

In the first division in the coordination phase, if Rebel i is certain $\#[R](\theta) < k$, i play $\langle$**all stay**$\rangle$. It implies that i is certain that there is no Rebels outside G_i and therefore learns θ by strong connectedness. This is to say all Rebels are i's neighbors and thus all i's Rebel neighbors are inactive. Since $\langle$**all stay**$\rangle$ is played only by an inactive Rebel or a Rebel who is certain $\#[R](\theta) < k$, all i's Rebel neighbors learn $\#[R](\theta) < k$ right after the first division in the coordination phase. Since all Rebels are i's neighbors, all Rebels learn $\#[R](\theta) < k$ right after the first division in the coordination phase. T^θ arrives then. Likewise, if Rebel i is inactive and all i's neighbors play $\langle$**all stay**$\rangle$, all Rebels learn $\#[R](\theta) < k$ right after the first division in the coordination phase, and T^θ arrives.

In the second division in the coordination phase, there is a non-trivial construction to let Rebel i disseminates the knowledge about $\#[R](\theta) \geq k$ if i has learnt that. Rebel i does so by playing $\langle i \rangle$ in the first division in the coordination phase and then play $\langle$**all stay**$\rangle$ in the second division in the coordination phase. His behavior is thus different from other kinds of Rebels. His neighbors will know $\#[R](\theta) \geq k$ right after the second division in the coordination phase and play **r** forever afterwards. Other Rebels will observe **r** being played in the third division in the coordination phase and thus know $\#[R](\theta) \geq k$ as well. All Rebels learn $\#[R](\theta) \geq k$ right after the third division in the coordination phase, and T^θ arrives.

Note that Rebel i who has learnt $\#[R](\theta) \geq k$ will not deviate to play $\langle$**all stay**$\rangle$ in the first division in the coordination phase even though it might be undetectable. This is because the network is acyclic. If i does so, i will be considered as an inactive Rebel by all his neighbors from the point onwards right after the first division in the coordination

Table 5: The sequences of actions played in $\mathbf{C}^1$ on the path

The sequences of actions played in $\mathbf{C}^1(1, \cdot)$ on the path

Rebel i	i plays
i is certain $\#[R](\theta) < k$	$\langle$**all stay**$\rangle$
i is inactive and is uncertain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$
i is active and is uncertain $\#[R](\theta) \geq k$	$\langle i \rangle$
i is certain $\#[R](\theta) \geq k$	$\langle i \rangle$

The sequences of actions played in $\mathbf{C}^1(2, \cdot)$ on the path

Rebel i	i plays
i is certain $\#[R](\theta) < k$	$\langle$**all stay**$\rangle$
i is inactive and is uncertain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$
i is active and is uncertain $\#[R](\theta) \geq k$	$\langle i \rangle$
i is certain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$

The sequences of actions played in $\mathbf{C}^1(3_v)$, where $v = 1, ..., n$, on the path

Rebel i	i plays
i is certain $\#[R](\theta) < k$	**s**
i is inactive and is uncertain $\#[R](\theta) \geq k$	**s**
i is active and is uncertain $\#[R](\theta) \geq k$	**s**
i is certain $\#[R](\theta) \geq k$	**r**

phase. Two consequences follow. The first is that each of i's neighbor is certain there is no more Rebel "behind" i.[28] The second is that i keeps reporting nothing in the forthcoming reporting phases. Thus, all i's neighbors report strictly less Rebels than they are supposed to do if i follows the equilibrium path. i then faces the possibility that no Rebel can know $\#[R](\theta) \geq k$ even if the total number of Rebels indeed exceeds k. If this event happens, i will only get zero payoff. However, i can surely get stage-game payoff as 1 afterwards right after the second division in the coordination phase. Sufficiently high $\delta \in (0,1)$ will deter this deviation.

Table 6: In $\mathbf{C}^1$, on the path, the belief of i's neighbor j after observing i's previous actions.

i plays	The event to which j assigns probability one right after $\mathbf{C}^1(1,\cdot)$
In $\mathbf{C}^1(1,\cdot)$	
$\langle$**all stay**$\rangle$	i is inactive if j is inactive
$\langle$**all stay**$\rangle$	$\#[R](\theta) < k$ if j is inactive
$\langle i \rangle$	i is active or $\#[R](\theta) \geq k$

i plays		The event to which j assigns probability one right after $\mathbf{C}^1(2,\cdot)$
In $\mathbf{C}^1(1,\cdot)$	In $\mathbf{C}^1(2,\cdot)$	
$\langle$**all stay**$\rangle$	$\langle$**all stay**$\rangle$	i is inactive if j is inactive
$\langle$**all stay**$\rangle$	$\langle$**all stay**$\rangle$	$\#[R](\theta) < k$ if j is inactive
$\langle i \rangle$	$\langle$**all stay**$\rangle$	$\#[R](\theta) \geq k$
$\langle i \rangle$	$\langle i \rangle$	i is active

4.3.4 The equilibrium behavior on the path in $\mathbf{C}^t$ for $t \geq 2$

The on-path strategy contingent on players' belief is introduced in Table 7. The evolution of information filtrations can be tracked throughout in Table 8.

[28] To be more precise, there is no more Rebel in a sub-tree that excludes j and roots at i.

The intrigued part in $\mathbf{C}^t$ is how a pivotal Rebel p in $\mathbf{O}^{t-1}$ disseminates the relevant information. For convenience, in this section, let $I_{ij}^{t+1} = I_i^t \cup I_j^t$, and the term "coordination phase" refers to "coordination phase in the t-block." I begin with the case when p is certain $\#[R](\theta) < k$.

If p is certain $\#[R](\theta) < k$, p plays $\langle \textbf{all stay} \rangle$ in the first sub-block of the first division in the coordination phase. Consequently, all p's neighbors know $\#[R](\theta) < k$ right after that since p has played $\langle 1 \rangle$ in the just finished reporting phase to announce he is pivotal. p's neighbors then play $\langle \textbf{all stay} \rangle$ continuously in each sub-block in the first division in the coordination phase, and therefore all Rebels know $\#[R](\theta) < k$ contagiously by observing $\langle \textbf{all stay} \rangle$ being played. All Rebels play $\mathbf{s}$ forever after the first division in the coordination phase, and T^θ arrives.

Alternatively, if p is certain $\#[R](\theta) \geq k$, p plays $\langle p \rangle$ in each sub-block in the first division in the coordination phase. To reveal $\#[R](\theta) \geq k$, p plays $\langle \textbf{all stay} \rangle$ in the first sub-block of the second division in the coordination phase. Notice that $\langle \textbf{all stay} \rangle$ is a costless sequence of actions. It might not seem intuitive at first sight, but playing $\langle \textbf{all stay} \rangle$ effectively prevents another free-rider problem. Suppose there are two pivotal Rebels, say p and p', who have already known $\#[R](\theta) \geq k$ right after the previous reporting phase. If initiation to disseminate knowledge about $\#[R](\theta) \geq k$ incurs negative payoff, p or p' will have the incentive, again, to wait for the other one initiates it. Playing $\langle \textbf{all stay} \rangle$ in the first sub-block of the second division in the coordination phase proudly becomes the initiation sequence by its cheapness. By the same contagion argument as the above mentioned, all Rebels play $\mathbf{r}$ after the third division in the coordination phase, and T^θ arrives.

The remaining question is why a non-pivotal Rebel, say i, does not mimic a pivotal Rebel's behavior by playing $\langle 1 \rangle$ in the just finished reporting phase even though it might be undetectable. The reason is the following. If i plays $\langle 1 \rangle$, i's neighbor will think i is pivotal. According to the equilibrium path, it implies that all players play either $\mathbf{r}$ or $\mathbf{s}$ forever after the third division in the coordination phase; therefore, the belief updating is also terminated. What follows is i cannot learn the relevant information after the third division in the coordination phase. If i does not deviate, i will learn the relevant information

Table 7: The sequences of actions played in $\mathbf{C}^t$, $t \geq 2$ on the path

The sequences of actions played in $\mathbf{C}^t(1, v)$ for $t \geq 2$ and for $v = 1, 2, ..., n$ on the path

Rebel i	i plays
i is certain $\#[R](\theta) < k$	$\langle$**all stay**$\rangle$
i is inactive and is uncertain $\#[R](\theta) \geq k$	$\langle i \rangle$
i is active and is uncertain $\#[R](\theta) \geq k$	$\langle i \rangle$
i is certain $\#[R](\theta) \geq k$	$\langle i \rangle$

The sequences of actions played in $\mathbf{C}^t(2, v)$ for $t \geq 2$ for $v = 1$ on the path

Rebel i	i plays
i is certain $\#[R](\theta) < k$	$\langle$**all stay**$\rangle$
i is inactive and is uncertain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$
i is active and is uncertain $\#[R](\theta) \geq k$	$\langle i \rangle$
i is certain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$

The sequences of actions played in $\mathbf{C}^t(2, v)$ for $t \geq 2$ for $v = 2, ..., t+1$ on the path

Rebel i	i plays
i is certain $\#[R](\theta) < k$	$\langle$**all stay**$\rangle$
i is inactive and is uncertain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$
i is active and is uncertain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$
i is certain $\#[R](\theta) \geq k$	$\langle i \rangle$

The sequences of actions played in $\mathbf{C}^t(3, \cdot)$ for $t \geq 2$ on the path

Rebel i	i plays
i is certain $\#[R](\theta) < k$	**s**
i is inactive and is uncertain $\#[R](\theta) \geq k$	**s**
i is active and is uncertain $\#[R](\theta) \geq k$	**s**
i is certain $\#[R](\theta) \geq k$	**r**

Table 8: In $\mathbf{C}^t$, on the path, the belief of i's neighbor j after observing i's previous actions.

i plays	The event to which j assigns probability one right after $\mathbf{O}^t$
In $\mathbf{O}^t$	
$\langle$**all stay**$\rangle$	i is inactive and $I_{ji}^{t+1} = I_j^t$
$\langle I_i^t \rangle$	i is active and $I_{ji}^{t+1} = I_j^t \cap I_i^t$
$\langle 1 \rangle$	i is pivotal

i plays		The event to which j assigns probability one right after $\mathbf{C}^t(1,1)$
In $\mathbf{O}^t$	In $\mathbf{C}^t(1,1)$	
$\langle$**all stay**$\rangle$	$\langle i \rangle$	i is inactive and $I_{ji}^{t+1} = I_j^t$
$\langle I_i^t \rangle$	$\langle$**all stay**$\rangle$	$\#[R](\theta) < k$
$\langle I_i^t \rangle$	$\langle i \rangle$	$(\#[R](\theta) \geq k)$ or (i is active and $I_{ji}^{t+1} = I_j^t \cap I_i^t$)
$\langle 1 \rangle$	$\langle$**all stay**$\rangle$	$\#[R](\theta) < k$
$\langle 1 \rangle$	$\langle i \rangle$	$\#[R](\theta) \geq k$

i plays			The event to which j assigns probability one right after $\mathbf{C}^t(2,1)$
In $\mathbf{O}^t$	In $\mathbf{C}^t(1,1)$	In $\mathbf{C}^t(2,1)$	
$\langle$**all stay**$\rangle$	$\langle i \rangle$	$\langle$**all stay**$\rangle$	i is inactive and $I_{ji}^{t+1} = I_j^t$
$\langle I_i^t \rangle$	$\langle$**all stay**$\rangle$	$\langle$**all stay**$\rangle$	$\#[R](\theta) < k$
$\langle I_i^t \rangle$	$\langle i \rangle$	$\langle$**all stay**$\rangle$	$\#[R](\theta) \geq k$
$\langle I_i^t \rangle$	$\langle i \rangle$	$\langle i \rangle$	$i \in R^t$ and $I_{ji}^{t+1} = I_j^t \cap I_i^t$
$\langle 1 \rangle$	$\langle$**stay**$\rangle$	$\langle$**all stay**$\rangle$	$\#[R](\theta) < k$
$\langle 1 \rangle$	$\langle i \rangle$	$\langle$**all stay**$\rangle$	$\#[R](\theta) \geq k$

eventually and choose the best response to get the maximum payoff at every θ. He prefers not to deviate if $\delta \in (0,1)$ is high enough.[29]

4.4 Sketch of the proof of Theorem 2

In the above I have listed Rebels' behavior in Table 3, Table 5, Table 8, and their belief updating in Table 6, Table 8, as the blueprint for the constructed equilibrium path. I sketch the proof of Theorem 2 as follows.

I use off-path belief to prevent players from making detectable deviations, such as deviating from playing the specified forms of sequences that are listed in Table 2 or Table 4. This off-path belief serves as a grim trigger: the punisher will play **stay** forever if he has not yet learnt the relevant information. Due to this, if a deviant who has not yet learnt the relevant information, from his perspective, making detectable deviation will strictly reduce the possibility of achieving the ex-post efficient outcome. As for making an undetectable deviation, though grim trigger is not effective, an undetectable deviation will too strictly reduce the deviant's expected continuation payoff. This is caused by two reasons. The first is that, in the reporting phase, an undetectable deviation that untruthfully reports less Rebels diminishes the possibility of achieving the ex-post outcome. The second is that an undetectable deviation that adds "noises" into information sharing process strictly reduces the deviant's expected continuation payoff as argued in Section 4.3.4. Since the stage-game payoff after T^{θ} is maximum ex-post for every θ by following the constructed equilibrium, a sufficiently high $\delta \in (0,1)$ will deter Rebels from deviating. I then conclude the proof.

5 Discussion

In the above APEX equilibrium construction, players act as if acting a sequence. Nevertheless, the actual description of an APEX equilibrium should prescribe how they act period by period and how they update belief at every period. This description is tedious; it is left in the Appendix.

[29]By Lemma A.1 in Appendix, the relevant information is learnt by every Rebel on the path eventually.

Instead of providing further details in equilibrium construction, I discuss the scenario when pay-off is a signal and why my constructed APEX equilibrium may fail in cyclic networks.

5.1 Payoff as signals

The hidden payoff assumption can be relaxed without changing the main result. One may consider a situation in which the stage payoff depends not only on players' joint efforts but also on a random shock, say the weather. To fix the idea, there is a public signal $y \in \{r, s\}$ generated according to the action profile. Let a Rebel's payoff function be $u_R(a_R, y)$ such that $u_R(\mathbf{stay}, r) = u_R(\mathbf{stay}, s) = u_0$. y is drawn from the distribution of

$$\begin{aligned} p_{rr} &= \Pr(y = r | \#\{j : a_j \mathbf{revolt}\} \geq k) \\ p_{sr} = 1 - p_{rr} &= \Pr(y = s | \#\{j : a_j \mathbf{revolt}\} \geq k) \\ p_{ss} &= \Pr(y = s | \#\{j : a_j \mathbf{revolt}\} < k) \\ p_{rs} = 1 - p_{ss} &= \Pr(y = r | \#\{j : a_j \mathbf{revolt}\} < k) \end{aligned}$$

such that

$$p_{rr} u_R(\mathbf{revolt}, r) + p_{sr} u_R(\mathbf{revolt}, s) > u_0 > p_{rr} u_R(\mathbf{revolt}, r) + p_{ss} u_R(\mathbf{revolt}, s),$$

and

$$0 \leq p_{rs} \leq 1, 0 \leq p_{ss} \leq 1.$$

The APEX equilibrium constructed for Theorem 2 is still a one in this scenario. Note that in that APEX equilibrium path, at most one **revolt** can occur at every period before some Rebel plays $\langle 1 \rangle$. This implies that the signal y is completely uninformative before some Rebel plays $\langle 1 \rangle$. If a Rebel i deviates to play $\langle 1 \rangle$ in $\mathbf{O}^t$ at some t in the hope gathering information from y, he will not learn the relevant information after $\mathbf{O}^t$ since the terminal period will come right after t-block. He will, however, learn the relevant information and play the ex-post efficient outcome if he is on the path, and hence he will not deviate.

5.2 Cyclic networks

Scenarios in cyclic networks substantially differ from the acyclic counterpart. The free-rider problem could become intractable in cyclic networks. Let us consider the configuration in Figure 9, and suppose $k = 4$.

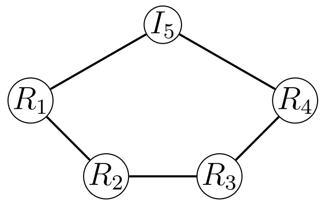

Figure 9: A configuration of the state and the network in which player 1, 2, 3, 4 are Rebels while player 5 is an Inert.

In Figure 9, Rebels 2 and 3 are θ-pivotal by definition. From the perspective of Rebel 2, the type of player 5 could be Inert. Therefore, Rebel 2 does not know that Rebel 1 is pivotal. Similarly, Rebel 2 does not know that Rebel 3 is pivotal, *even though* player 3 is indeed θ-pivotal. Therefore there is no common knowledge of the free-rider problem at the beginning of 1-block.

However, the common knowledge of engaging in a free-rider problem is restored when we cut the edge between player 4 and 5; Rebel 2 knows that he is the only θ-pivotal Rebel.

I leave a conjecture in this paper and end this section.

Conjecture 5.1. *For any n-person repeated k-Threshold game with parameter $k < n$ played in any network, if π has full support on strong connectedness, then there exists a $\delta^* \in (0, 1)$ such that an APEX equilibrium exists whenever $\delta > \delta^*$.*

6 Conclusion

I model a coordination game and illustrate the learning processes generated by strategies in a sequential equilibrium to answer the question proposed in the beginning: what kind of networks can conduct coordination in a collective action with information barrier. In the equilibrium, players transmit the relevant information by encoding such information by their actions as time goes by. Since there might be an negative expected payoff in coding information, the potential free-rider problems might occur to impede the learning process. My result show that if the network is acyclic, players can always learn the underlying relevant information and conduct the coordination only by actions. In cyclic networks, however, what kinds of equilibrium strategies can lead to learning the relevant information still remains to be answered.

The construction of the communication protocol by actions exploits the assumption of the common knowledge of the network and the finite type space. Since the relevant information has been parametrized as a threshold in the stage game, players can acquire this information by jointly incrementally reporting their own private information period by period. The major punishment to deter deviation is then the joint shifting to play that same action as the stopping to update information. The threshold game thus seems a potential model in proofing that a communication protocol by actions not only leads a learning process but also constitutes an equilibrium to reveal the relevant information in finite time.

Existing literatures in political science and sociology have recognized the importance of social network in influencing individual's behavior in participating social movements ([Passy, 2003][McAdam, 2003][Siegel, 2009]). This paper views networks as routes for communication in which rational individuals initially have local information but they can influence nearby individuals by taking actions. Such influence may take long time to travel across individuals and the whole process incurs inefficient outcomes in many periods. A characterization in the speed of information transmission across a network is not answered here, although it is an important topic in investigating the most efficient way to let the information be spread. This question would remain for the future research.

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A Appendix

A.1 The APEX equilibrium for Theorem 2

A.1.1 Equilibrium path

By definition of information hierarchy,

$$\begin{aligned} I_i^t &= \bigcup_{k_1 \in G_i} \bigcup_{k_2 \in G_{k_0}} \cdots \bigcup_{k_t \in G_{k_{t-1}}} I_{k_t}^1 \\ &= \{j \in [R](\theta) : \exists \text{ a path } (i, k_1 ... k_l, j) \text{ s.t. } 0 \le l \le t-1 \text{ and } \theta_i = \theta_{k_1} = ... = \theta_{k_l} = R\} \end{aligned}$$

Let us define several notions.

Definition A.1 (Extended tree by I_i^t)**.**

$$\begin{aligned} X_i^t &:= \{j \in N : \\ &\quad \exists \textit{ a path } (i, k_1 ... k_l, k_{l+1}) \textit{ s.t. } k_{l+1} = j,\ l \ge t-1,\ \{i, k_1, ..., k_t\} \subset I_i^t\} \cup I_i^t \end{aligned}$$

Namely, X_i^t is the set of all possible Rebels in G given information I_i^t.

Definition A.2 (The sub-tree rooting in j but excluding i)**.**

$$TR_{ij} := \{v \in N : \textit{there is a path from } i \textit{ to } v \textit{ through } j,\ j \in G_i\} \cup \{i, j\}$$

Definition A.3 (Extended vertices outside I_i^t in TR_{ij})**.**

$$Y_{ij}^t := TR_{ij} \cap (X_i^t \setminus I_i^t)$$

Definition A.4.

$$D_i^t := \{j \in G_i : Y_{ij}^t \neq \emptyset\}$$

Definition A.5 (Finite register machine)**.** *A finite register machine for i consists of finite registers Σ. A register is a tuple*

$$(\Omega, \times_{G_i} A_R, f, \lambda),$$

in which Ω are sets of events induced by H_i. $\times_{G_i} A_R$ is the sets of input. $f : \Omega \to A_R$ assigns an action to each event. $\lambda : \Omega \times \times_{G_i} A_R \to \Sigma$ is the transition function. There is a set of initial registers.

i's register prescribes i's action according his information at a certain period but does not characterize i's information transition. The register machine here is more like the *switch function* instead of the finite automata. The information of i up to period s is $P_i(\theta) \times \{h^s_{G_i}\} \times H^s_{N \setminus G_i}$ characterized in Section 2.

Definition A.6 (m-register in t-block)**.** *A m-register in a (sub)block or a division is the register in the m-th period in that (sub)block or division.*

To shorten the notation, denote $m \dashv \Gamma$ as the m-register in the (sub)block or division Γ.

Definition A.7 (Terminal $\mathbf{r}$)**.** *The terminal $\boldsymbol{r}$ is a register such that the image of f is $\{$**revolt**$\}$ and the image of λ is a singleton containing itself.*

Definition A.8 (Terminal $\mathbf{s}$)**.** *The terminal $\boldsymbol{s}$ is a register such that the image of f is $\{$**stay**$\}$ and the image of λ is a singleton containing itself.*

The equilibrium will be represented as a finite register machine. Moreover, though players act as if acting a whole sequence, they in fact act period by period. For convenience, for any finite sequence of action $\langle\rangle$, denote $\langle\rangle_m$ as the m-th (counting from the beginning) component in $\langle\rangle$, and denote $\langle\rangle(m)$ as the prefix of $\langle\rangle$ with length m. Let us also shorten action **revolt** to be $\mathbf{r}$ and **stay** to be $\mathbf{s}$.

Initial registers The initial register for each Rebel is $1 \dashv \mathbf{C}^1(1, \cdot)$, which is defined in the next section.

Registers in $\mathbf{C}^1$

Table 9: The $m \dashv \mathbf{C}^1_1$ on the path

$1 \le m \le |\mathbf{C}^1(1,\cdot)| - 1$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$\#X^1_i < k$	$\langle\textbf{all stay}\rangle_m$		terminal $\mathbf{s}$
$i \notin R^1$, $\#X^1_i \ge k$, $I^1_i < k$	$\langle\textbf{all stay}\rangle_m$		$m+1 \dashv \mathbf{C}^1_1$
$i \in R^1$, $\#X^1_i \ge k$, $I^1_i < k$	$\langle i\rangle_m$		$m+1 \dashv \mathbf{C}^1_1$
$i \in R^1$, $\#X^1_i \ge k$, $I^1_i \ge k$	$\langle i\rangle_m$		$m+1 \dashv \mathbf{C}^1_1$

$m = |\mathbf{C}^1(1,\cdot)|$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$\#X^1_i < k$		$\langle\textbf{all stay}\rangle_m$		terminal $\mathbf{s}$
$i \notin R^1$, $\#X^1_i \ge k$, $I^1_i < k$	all j play $\langle\textbf{all stay}\rangle(m-1)$	$\langle\textbf{all stay}\rangle_m$	all j play $\mathbf{s}$	terminal $\mathbf{s}$
$i \notin R^1$, $\#X^1_i \ge k$, $I^1_i < k$	$\exists j$ plays $\langle j\rangle(m-1)$	$\langle\textbf{all stay}\rangle_m$	such j plays $\langle j\rangle_m$	$1 \dashv \mathbf{C}^1(2,\cdot)$
$i \in R^1$, $\#X^1_i \ge k$, $I^1_i < k$		$\langle i\rangle_m$		$1 \dashv \mathbf{C}^1(2,\cdot)$
$i \in R^1$, $\#X^1_i \ge k$, $I^1_i \ge k$		$\langle i\rangle_m$		$1 \dashv \mathbf{C}^1(2,\cdot)$

Table 10: The $m \dashv \mathbf{C}_2^1$ on the path

$1 \leq m < |\mathbf{C}^1(2,\cdot)|$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$i \notin R^1$		$\langle \textbf{all stay} \rangle_m$		$m+1 \dashv \mathbf{C}_2^1$
$i \in R^1$, $I_i^1 < k$	$\exists j \in G_i$, j plays $< j >_j = s$	$\langle \textbf{all stay} \rangle_m$		$m+1 \dashv \mathbf{C}_2^1$
$i \in R^1$, $I_i^1 < k$	$\forall j \in G_i$, j plays $< j >_j = r$	$\langle i \rangle_m$		$m+1 \dashv \mathbf{C}_2^1$
$i \in R^1$, $I_i^1 \geq k$		$\langle \textbf{all stay} \rangle_m$		$m+1 \dashv \mathbf{C}_2^1$

$m = |\mathbf{C}^1(2,\cdot)|$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$i \notin R^1$	$\forall j \in G_i$, j plays $\langle j \rangle(m-1)$	$\langle \textbf{all stay} \rangle_m$	$\forall j \in G_i$, j plays $\langle j \rangle_m$	$1 \dashv \mathbf{C}_3^1$
$i \notin R^1$	$\exists j \in G_i$, j plays $\langle \textbf{all stay} \rangle(m-1)$	$\langle \textbf{all stay} \rangle_m$	such j plays $\langle \textbf{all stay} \rangle_m$	terminal $\mathbf{r}$
$i \in R^1$, $I_i^1 < k$	$\forall j \in G_i$, j play $\langle j \rangle(m-1)$	$\langle i \rangle_m$	$\forall j \in G_i$, j plays $\langle j \rangle_m$	$1 \dashv \mathbf{C}_3^1$
$i \in R^1$, $I_i^1 < k$	$\exists j \in G_i$, j plays $\langle \textbf{all stay} \rangle(m-1)$	$\langle i \rangle_m$	such j plays $\langle \textbf{all stay} \rangle_m$	terminal $\mathbf{r}$
$i \in R^t$, $I_i^1 \geq k$		$\langle \textbf{all stay} \rangle_m$		terminal $\mathbf{r}$

Table 11: The $m \dashv \mathbf{C}_3^1$ on the path

$1 \leq m < |\mathbf{C}^1(3, \cdot)|$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
	$\mathbf{s}$	$\forall j$ play $\mathbf{s}$	$m+1 \dashv \mathbf{C}_3^1$
	$\mathbf{s}$	$\exists j$ play $\mathbf{r}$	terminal $\mathbf{r}$

$1 \leq m = |\mathbf{C}^1(3, \cdot)|$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
	$\mathbf{s}$	$\forall j$ play $\mathbf{s}$	$1 \dashv \mathbf{O}^1$
	$\mathbf{s}$	$\exists j$ play $\mathbf{r}$	terminal $\mathbf{r}$

Registers in $\mathbf{O}^t$ Let $m_i = |\mathbf{O}^t| - x_{I_i^t}$ be the period in which i report I_i^t. I.e. m_i is the period where $\mathbf{r}$ occurs in $\langle I_i^t \rangle$. Denote $G_i(m) = \{j \in G_i : m_j < m\}$. Define $I_i^{t+1}(m) := I_i^t \cup \bigcup_{j \in G_i(m)} I_j^t$ to be the information of i up to the m-th period in $\mathbf{O}^t$. Define $X_i^{t+1}(m)$ to be the extended tree from $I_i^t(m)$ in the same way as that in Definition A.1, and define $Y_{ij}^t(m)$ and $D_i^t(m)$ accordingly.

Table 12: The $m \dashv \mathbf{O}^t$ on the path, where $1 \leq m < |\mathbf{O}^t|$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$i \notin R^t$		$\langle \textbf{all stay} \rangle_m$		$m+1 \dashv \mathbf{O}^t$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I_i^{t+1}(m) < k$, $X_i^{t+1}(m) < k$	$\langle \textbf{all stay} \rangle_m$		terminal $\mathbf{s}$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I_i^{t+1}(m) < k$, $X_i^{t+1}(m) \geq k$	$\langle I_i^t \rangle_m$		$m+1 \dashv \mathbf{O}^t$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I_i^{t+1}(m) \geq k$, $X_i^{t+1}(m) \geq k$	$\langle 1 \rangle_m$		$m+1 \dashv \mathbf{O}^t$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I_i^{t+1}(m) \geq k-1$, $X_i^{t+1}(m) \geq k$, $D_i^t = 1$	$\langle 1 \rangle_m$		$m+1 \dashv \mathbf{O}^t$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I_i^{t+1}(m) \geq k-1$, $X_i^{t+1}(m) \geq k$, $D_i^t > 1$	$\langle I_i^t \rangle_m$		$m+1 \dashv \mathbf{O}^t$
$i \in R^t$, the free rider	$X_i^{t+1}(m) \geq k$	$\langle 1 \rangle_m$		$m+1 \dashv \mathbf{O}^t$
$i \in R^t$, the free rider	$X_i^{t+1}(m) < k$	$\langle \textbf{all stay} \rangle_m$		terminal $\mathbf{s}$
$i \in R^t$, i is $k-1$-pivotal	$X_i^{t+1}(m) \geq k$	$\langle 1 \rangle_m$		$m+1 \dashv \mathbf{O}^t$
$i \in R^t$, i is $k-1$-pivotal	$X_i^{t+1}(m) < k$	$\langle \textbf{all stay} \rangle_m$		terminal $\mathbf{s}$

Table 13: The $m \dashv \mathbf{O}^t$ on the path, where $m = |\mathbf{O}^t|$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$i \notin R^t$		$\langle \textbf{all stay} \rangle_m$		$1 \dashv \mathbf{C}^t_{1,1}$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I_i^{t+1}(m) < k$, $X_i^{t+1}(m) < k$	$\langle \textbf{all stay} \rangle_m$		terminal $\mathbf{s}$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I_i^{t+1}(m) < k-1$, $X_i^{t+1}(m) \geq k$	$\langle I_i^t \rangle_m$		$1 \dashv \mathbf{C}^t_{1,1}$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I_i^{t+1}(m) \geq k$, $X_i^{t+1}(m) \geq k$	$\langle 1 \rangle_m$		$1 \dashv \mathbf{C}^t_{1,1}$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I_i^{t+1}(m) \geq k-1$, $X_i^{t+1}(m) \geq k$, $D_i^t = 1$	$\langle 1 \rangle_m$		$1 \dashv \mathbf{C}^t_{1,1}$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I_i^{t+1}(m) \geq k-1$, $X_i^{t+1}(m) \geq k$, $D_i^t > 1$	$\langle I_i^t \rangle_m$		$1 \dashv \mathbf{C}^t_{1,1}$
$i \in R^t$, the free rider	$X_i^{t+1}(m) \geq k$	$\langle 1 \rangle_m$		$1 \dashv \mathbf{C}^t_{1,1}$
$i \in R^t$, the free rider	$X_i^{t+1}(m) < k$	$\langle \textbf{all stay} \rangle_m$		terminal $\mathbf{s}$
$i \in R^t$, i is $k-1$-pivotal	$X_i^{t+1}(m) \geq k$	$\langle 1 \rangle_m$		$1 \dashv \mathbf{C}^t_{1,1}$
$i \in R^t$, i is $k-1$-pivotal	$X_i^{t+1}(m) < k$	$\langle \textbf{all stay} \rangle_m$		terminal $\mathbf{s}$

Registers in $\mathbf{C}^t$ for $t \geq 2$

Table 14: The $m \dashv \mathbf{C}^t(1, v)$ for $v = 1, ..., n$ on the path

$1 \leq m < |\mathbf{C}^t(1, v)|$, where $v = 1, ..., n$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$X_i^{t+1} < k$	$\langle \textbf{all stay} \rangle_m$		terminal $\mathbf{s}$
$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\exists j \in G_i, j = m$ such that $a_j = \mathbf{s}$	terminal $\mathbf{s}$
$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\forall j \in G_i$ such that $a_j = \langle j \rangle_m$	$m + 1 \dashv \mathbf{C}^t(1, v)$

$m = |\mathbf{C}^t(1, v)|$, where $v = 1, ..., n - 1$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$X_i^{t+1} < k$	$\langle \textbf{all stay} \rangle_m$		terminal $\mathbf{s}$
$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\exists j \in G_i, j = m$ such that $a_j = \mathbf{s}$	terminal $\mathbf{s}$
$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\forall j \in G_i$ such that $a_j = \langle j \rangle_m$	$1 \dashv \mathbf{C}^t_{1,v+1}$

$1 \leq m < |\mathbf{C}^t(1, n)|$

$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\exists j \in G_i, j = m$ such that $a_j = \mathbf{s}$	terminal $\mathbf{s}$
$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\forall j \in G_i$ such that $a_j = \langle j \rangle_m$	$m + 1 \dashv \mathbf{C}^t(1, n)$

$m = |\mathbf{C}^t(1, n)|$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\exists j \in G_i, j = m$ such that $a_j = \mathbf{s}$	terminal $\mathbf{s}$
$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\forall j \in G_i$ such that $a_j = \langle j \rangle_m$	$1 \dashv \mathbf{C}^t_{2,1}$

Table 15: The $m \dashv \mathbf{C}^t(2,v)$ for $v = 1,...,t+1$ on the path

$1 \leq m < |\mathbf{C}^t(2,v)|$, where $v = 1,...,t+1$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$I_i^{t+1} < k$	$\exists j,\ \langle j \rangle_j = \mathbf{s}$	$\langle$**all stay**$\rangle_m$		$m+1 \dashv \mathbf{C}^t(2,v)$
$I_i^{t+1} < k$	$\forall j,\ \langle j \rangle_j = \mathbf{r}$	$\langle i \rangle_m$		$m+1 \dashv \mathbf{C}^t(2,v)$
$I_i^{t+1} \geq k$		$\langle$**all stay**$\rangle_m$		$m+1 \dashv \mathbf{C}^t(2,v)$

$m = |\mathbf{C}^t(2,v)|$, where $v = 1,....,t$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$I_i^{t+1} < k$	$\exists j \in G_i,\ \langle j \rangle_j = \mathbf{s}$	$\langle$**all stay**$\rangle_m$		$1 \dashv \mathbf{C}^t_{2,v+1}$
$I_i^{t+1} < k$	$\forall j \in G_i,\ \langle j \rangle_j = \mathbf{r}$	$\langle i \rangle_m$		$1 \dashv \mathbf{C}^t_{2,v+1}$
$I_i^{t+1} \geq k$		$\langle$**all stay**$\rangle_m$		$1 \dashv \mathbf{C}^t_{2,v+1}$

$1 \leq m < |\mathbf{C}^t(2,t+1)|$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$I_i^{t+1} < k$	$\exists j \in G_i,\ \langle j \rangle_j = \mathbf{s}$	$\langle$**all stay**$\rangle_m$		$m+1 \dashv \mathbf{C}^t(2,t+1)$
$I_i^{t+1} < k$	$\forall j \in G_i,\ \langle j \rangle_j = \mathbf{r}$	$\langle i \rangle_m$		$m+1 \dashv \mathbf{C}^t(2,t+1)$
$I_i^{t+1} \geq k$		$\langle$**all stay**$\rangle_m$		$m+1 \dashv \mathbf{C}^t(2,t+1)$

$m = |\mathbf{C}^t(2,t+1)|$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$I_i^{t+1} < k$	$\exists j \in G_i,\ \langle j \rangle_j = \mathbf{s}$	$\langle$**all stay**$\rangle_m$		terminal **r**
$I_i^{t+1} < k$	$\forall j \in G_i,\ \langle j \rangle_j = \mathbf{r}$	$\langle i \rangle_m$		$1 \dashv \mathbf{C}^t_{3,1}$
$I_i^{t+1} \geq k$		$\langle$**all stay**$\rangle_m$		terminal **r**

Table 16: The $m \dashv \mathbf{C}^t(3,\cdot)$ on the path

$1 \leq m < |\mathbf{C}^t(3,\cdot)|$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
	$\mathbf{s}$	$\forall j \in G_i$, j plays $\mathbf{s}$	$m+1 \dashv \mathbf{C}^t(3,\cdot)$
	$\mathbf{s}$	$\exists j \in G_i$, j plays $\mathbf{r}$	terminal $\mathbf{r}$

$m = |\mathbf{C}^t(3,\cdot)|$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
	$\mathbf{s}$	$\forall j \in G_i$, j plays $\mathbf{s}$	$1 \dashv \mathbf{O}^{t+1}$
	$\mathbf{s}$	$\exists j \in G_i$, j play $\mathbf{r}$	terminal $\mathbf{r}$

A.2 Missing proofs

Proof of Lemma 2.1

Proof. The proof is done by contraposition. Suppose Rebels' strategies constitute an APEX equilibrium. By definition of the APEX equilibrium, at every θ, there is a period T^θ when all Rebels' actions start to repeat themselves. Let $T = \max_{\theta\in\Theta} T^\theta$. For Rebel i, let $T_i = T + 1$, and suppose $0 < \sum_{\theta:\#[R](\theta)\geq k} \beta^{\pi,\tau^*}_{G_i}(\theta|h^s_{G_i}) < 1$ for some $s \geq T_i$. Then this Rebel assigns positive weight at some $\theta' \in \{\theta : \#[R](\theta) < k\}$ and some positive weight at some $\theta'' \in \{\theta : \#[R](\theta) \geq k\}$ at period s. Note that i has already known θ_j if $j \in G_i$, and therefore i assigns positive weight at some $\theta' \in \{\theta : \#[R](\theta) < k, \theta_l = R, l \notin G_i\}$ and positive weight at some $\theta'' \in \{\theta : \#[R](\theta) < k, \theta_l = I, l \notin G_i\}$. Since all Rebels' actions start to repeat themselves at period T, i cannot update information afterwards. Suppose i's continuation strategy is to continuously play **revolt**, then this is not ex-post efficient when $\#[R](\theta) < k$; suppose i's continuation strategy is to continuously play **stay**, then this is not ex-post efficient when $\#[R](\theta) \geq k$. □

Proof of Theorem 1

Proof. Let τ^* be the following strategy. After the nature moves, a Rebel i plays **revolt** if he has no Inert neighbor; i plays **stay** forever if he has an Inert neighbor. After the first period, if i has not detected a deviation and observes that all his Rebel neighbors play **revolt** continuously previously, he plays **revolt** in the current period; otherwise, he plays **stay** afterwards and forever. If a Rebel j deviates, then j plays **stay** afterwards and forever.

At period s, according to τ^*, if i has not detected a deviation, but he observe his Rebel neighbors plays **stay** in the current period, he forms the belief of

$$\sum_{\theta:\#[R](\theta)\geq k} \beta^{\pi,\tau^*}_{G_i}(\theta|h^s_{G_i}) = 0$$

afterwards and forever. Therefore, he plays **stay** afterwards and forever as his best response.

At period s, if a Rebel detects a deviation, or he has deviated, to play **stay** afterwards and forever is his best response since at least one player will play **stay** afterwards and forever.

Since the network is finite with n vertices, if all players do not deviate, after period n, each Rebel plays **revolt** and gets payoff 1 forever if $\theta \in \{\theta : \#[R](\theta) \geq k\}$; each Rebels plays **stay** and gets payoff 0 forever if $\theta \in \{\theta : \#[R](\theta) < k\}$. However, a Rebel who has deviated surely gets payoff 0 forever after period n. Therefore, there is a $0 < \delta < 1$ large enough to impede Rebels to deviate.

To check if τ^* and $\{\beta^{\pi,\tau^*}_{G_i}(\theta|h^s_{G_i})\}_{i\in N}$ satisfy full consistency[30], take any $0 < x < 1$ such that Rebels play τ^* with probability $1 - x$ and play other behavior strategies with probability x. Clearly, when $x \to 0$, the belief converges to $\{\beta^{\pi,\tau^*}_{G_i}(\theta|h^s_{G_i})\}_{i\in N}$. Since the off-path strategy is the best response for both of the Rebel who detects deviation and the Rebel who makes deviation, for arbitrary beliefs they hold, τ^* is a sequential equilibrium. $\square$

Proof of Theorem 3

Proof. Since the network is finite, θ has strong connectedness, and $[R](\theta) \neq \emptyset$, there is a minimum t_i such that $I^{t_i}_i = [R](\theta)$ for each i by the definition of I^t_i. Let $P = \arg\min_{i\in N}\{t_1, ..., t_n\}$ with generic element p. Therefore $I^{t_p}_p = [R](\theta)$. I show that $p \in R^{t_p-1}$ to complete the proof. I prove it by contradiction. If $p \notin R^{t_p-1}$, then $I^{t_p-1}_p \subseteq G^{t_p-1}_j$ for some $j \in G_p$. Then, all the Rebels in TR_{jp} are in $G^{t_p-1}_j$, but there exist Rebels in TR_{pj} who are in $G^{t_p-1}_j$ but not in $I^{t_p-1}_p$. This is because the network is acyclic and $I^{t_p-1}_p \subset [R](\theta)$. But then $p \notin P$ since $I^{t_j-1}_j = [R](\theta)$ already. I then conclude that $p \in R^{t_p-1}$.

$\square$

Proof of Lemma 4.1

Proof. The proof is by contradiction. Suppose that, at t-block and before T^θ, there are three or more θ-pivotal Rebels. Since θ has strong connectedness, there are three of them, p_1, p_2, p_3, with the property $p_1 \in G_{p_2}$ and $p_2 \in G_{p_3}$.

[30] Krep and Wilson (1982)

Since the network is acyclic, $p_1 \notin G_{p_3}$ and $p_3 \notin G_{p_1}$. Since p_1 is θ-pivotal, $I^t \subset [R](\theta)$ and $I_p^{t+1} = [R](\theta)$. It implies that, in $TR_{p_1p_2}$, p_1 can reach all Rebels by $t+1$ edges, but cannot reach all of them by t edges. The same situation applies to p_3. However, it means that p_2 can reach all Rebels in $TR_{p_1p_1}$ by t edges and reach all Rebels in $TR_{p_1p_1}$ by t edges, and hence $I_{p_2}^t = [R](\theta)$. It contradict to the definition of θ-pivotal Rebel.

□

Proof of Lemma 4.2

Proof. A θ-pivotal p knows that $p^{'} \in G_i$ if $p^{'}$ is another one. p picks a neighbor p' and checks whether or not $[R](\theta) \subseteq I_p^t \cup I_{p^{'}}^t$ for all possible $I_{p^{'}}^t$. By common knowledge of the network, p knows $G_{p'}^t$. Since p is θ-pivotal, he is certain that all the Rebel in the direction from p toward $p^{'}$ is in $G_{p^{'}}^t$ and hence in $I_{p^{'}}^t$. Then p can check whether or not $[R](\theta) \subseteq I_p^t \cup I_{p^{'}}^t$ for all possible $I_{p^{'}}^t$. If so, then p knows $p^{'}$ is also θ-pivotal by the definition of θ-pivotal. Similarly, a θ-pivotal $p^{'}$ can do the same procedure. Therefore, if there are two θ-pivotal p and $p^{'}$, they commonly know that they are θ-pivotal. They commonly know this at the beginning of t-block since they know I_p^t and $I_{p^{'}}^t$ by the construction of information hierarchy. □

Proof of Theorem 2 I begin with the following lemmas stating that all Rebels eventually learn the relevant information on the path.

Lemma A.1. *If the network is acyclic and if the θ has strong connectedness, then the equilibrium path prescribed in Section A.1.1 is an APEX strategy.*

Proof. Firstly, suppose θ satisfies $\#[R](\theta) < k$. I show that, all Rebels will enter terminal **s** eventually without entering terminal **r**. Let p be the Rebel defined in the proof of Theorem 3 so that $I_p^{t_p} = [R](\theta)$, where p is one of the earliest Rebels who knows $\#[R](\theta) < k$. I claim that $\#X_p^{t_p} < k$ if and only if $I_p^{t_p} = [R](\theta)$. For the only if part, the proof is by way of contradiction. If not, by the full support on strong connectedness, there is a possible Rebel outside $I_p^{t_p}$, and therefore p is uncertain $\#[R](\theta) < k$. For the if part, note that $I_p^{t_p} \subset X_p^{t_p}$ and therefore $\#I_p^{t_p} < \#X_p^{t_p} < k$. I also claim that $\#X_p^{t_p}(m) < k$ if and only

$I_p^{t_p}(m) = [R](\theta)$. The proof is exactly the same as the noted above by replacing $I_p^{t_p}$ to $I_P^{t_p}$ and $I_p^{t_p}(m)$ to $I_P^{t_p}(m)$

Referring to Table 9 to Table 13, whenever there is a p so that $\#X_p^{t_p} < k$, p plays **stay** forever. It implies that all Rebels enter terminal **s** right after $\mathbf{C}_1^t, t \geq 0$. Notice that Rebels entering to terminal **r** only after some period after $\mathbf{C}_1^t$ and therefore all Rebels will enter terminal **s** before terminal **r**.

Secondly, suppose θ satisfies $\#[R](\theta) \geq k$. I show that all Rebels will enter terminal **r** eventually. Note first that if there is a Rebel p so that $\#I_p^1 \geq k$, all Rebels enter terminal **r** after $\mathbf{C}^1(3, \cdot)$ by referring to Table 9, Table 10, and Table 11. At $t > 0$, if there is a Rebel p who has play $\langle 1 \rangle$ at $\mathbf{O}^t$, by the postulate of $\#[R](\theta) \geq k$, after $\mathbf{C}^t(3, \cdot)$, all Rebels enter terminal **r** according to the equilibrium path prescribed in Table 12, Table 13, Table 14, Table 15, and Table 16. There must be some Rebel $p \in R^t$ who plays $\langle 1 \rangle$ at $\mathbf{O}^t$ for some t by the same argument in the proof of Theorem 3. □

Due to Lemma A.1, define $T_{\tau^*}^{\theta}$ as the earliest period at which all Rebels play ex-post efficient outcome afterwards according to an APEX equilibrium τ^*. For simplicity, I suppress the notation $\beta_{G_i}^{\pi,\tau}(\theta|h_{G_i}^s)$ to $\beta_{G_i}^{\tau}(\theta|h_{G_i}^s)$ and the notation $\alpha_{G_i}^{\pi,\tau}(\theta, h^s|\theta_{G_i}, h_{G_i}^s)$ to $\alpha_{G_i}^{\tau}(\theta, h^s|h_{G_i}^s)$. If $P(\theta)$ is a property of θ, define

$$\beta_{G_i}^{\tau}(P(\theta)|h_{G_i}^s) := \sum_{\theta \in \{\theta : P(\theta)\}} \beta_{G_i}^{\tau}(\theta|h_{G_i}^s).$$

Furthermore, if m, s are periods and $m > s$, denote $h^{m|s}$ as a history in H^m so that $(h^s, h^{m|s}) \in H^m$. Denote $\tau'|_{\tau}^s$ as a strategy following τ til period s.

Claim 1. *Suppose Rebel i follows an APEX equilibrium τ^* til period s. If there is a strategy $\tau|_{\tau^*}^s = (\tau_i, \tau_{-i})|_{\tau^*}^s$ generating a history $h^{m|s}, \infty > m > s$ so that i will be uncertain about the relevant information and stop belief updating after m, then Rebel i will not deviate to $\tau|_{\tau^*}^s$ if $\delta \in (0, 1)$ is sufficiently high.*

Proof. Denote $\beta_{G_i}^{\tau|_{\tau^*}^s}(\theta|h^{m|s}, h_{G_i}^s)$ as i's belief about θ at m following $h^{m|s}$ induced by $\tau|_{\tau^*}^s$. By the postulate, $0 < \beta_{G_i}^{\tau|_{\tau^*}^s}(\#[R](\theta)|h^{m|s}, h_{G_i}^s) < 1$. From the perspective that i holds a belief of $\beta_{G_i}^{\tau^*}(\#[R](\theta) \geq k|h_{G_i}^s)$ at period s, $h^{m|s}$ can be thought of an imperfect signal at period m to infer whether or not $\#[R](\theta) \geq k$: if $\#[R](\theta) \geq k$, $h^{m|s}$ occurs with probability η and does not occur with probability $1-\eta$; if $\#[R](\theta) < k$, $h^{m|s}$ occurs with probability μ and does not occur with probability $1-\mu$ so that $0 \leq \eta \leq 1$, $0 \leq \mu \leq 1$, and $0 < \eta/\mu < \infty$. Denote $M = \max\{m, T^{\theta}_{\tau^*}\}$. Rebel i's maximum expected stage-game payoff starting from M by following $h^{m|s}$ calculated at period s is

$$V = \max\{\eta\beta^{\tau^*}_{G_i}(\#[R](\theta) \geq k|h^s_{G_i}) - \mu\beta^{\tau^*}_{G_i}(\#[R](\theta) < k|h^s_{G_i}), 0\}.$$

The first term $\eta\beta^{\tau^*}_{G_i}(\#[R](\theta) \geq k|h^s_{G_i}) - \mu\beta^{\tau^*}_{G_i}(\#[R](\theta) < k|h^s_{G_i})$ is i's expected stage-game payoff if all Rebels play **revolt** afterwards starting from M. The second term 0 is the one by playing **stay** afterwards. Rebel i's expected stage-game payoff starting from M by following τ^* calculated at period s is

$$\beta^{\tau^*}_{G_i}(\#[R](\theta) \geq k|h^s_{G_i}) > V.$$

The inequality above is due to $0 < \eta < 1, 0 < \mu < 1$. There is a difference in present value of

$$W(\delta) = \frac{\delta^{M-s}(\beta^{\tau^*}_{G_i}(\#[R](\theta) \geq k|h^s_{G_i}) - V)}{1-\delta}.$$

Denote L as the summation of all gains from deviation calculated from period s to period M. L is finite since the stage-game payoff is finite and $M - s$ is finite. Taking sufficiently high $\delta \in (0, 1)$ so that $W(\delta) > L$ will deter this deviation. □

Claim 2. *Suppose Rebel i follows an APEX equilibrium τ^* til period s. If i deviates to a strategy $\tau|^s_{\tau^*} = (\tau_i, \tau_{-i})|^s_{\tau^*}$ so that there are d Rebels, $d > 0$, detects this deviation, then Rebel i will not deviate to $\tau|^s_{\tau^*}$ if $\delta \in (0, 1)$ is sufficiently high.*

Proof. There are two cases: $I^s_i < k$ or $I^s_i \geq k$. If $s \in \mathbf{O}^t$ for some $t \geq 1$, then I^s_i refers to $I^s_i = I^{t+1}_i(m)$ such that m is the m-th period in $\mathbf{O}^t$ and $s = m + \sum_{\gamma=1}^{t}(|\mathbf{C}^\gamma| + |\mathbf{O}^{\gamma-1}|)$, where $|\mathbf{O}^0| = 0$.

Suppose $I^s_i < k$. If $\tau|^s_{\tau^*}$ leads to a history so that i will never learn the relevant information, i will not deviate, by Claim 1. If $\tau|^s_{\tau^*}$ leads to a history so that i is certain that there are k' Rebels, $k \leq k' < k + d$, play some action forever after some period m_d,

i's stage-game payoff is at most 0 after m_d. If Rebel i follows τ^*, i's stage-game payoff is 1 after T^θ. Therefore after $M = \max\{T^\theta, m_d\}$, there is a difference as least $1/(1-\delta)$ as the present value at M. Since $I_i^s < k$, $0 < \beta_{G_i}^{\tau^*}(\#[R](\theta) \geq k | h_{G_i}^s) < 1$, and therefore $\eta = \beta_{G_i}^{\tau^*}(k \leq \#[R](\theta) < k + d | h_{G_i}^s)$ is positive. It implies there is a difference at least $\eta\delta^{M-s}/(1-\delta)$ as the present value at s. As for the case $\tau|_{\tau^*}^s$ leads to a history so that i is certain that there are k' Rebels, $k' \geq k + d$ or $k' < k$, i's stage-game payoff after $M = \max\{T^\theta, m_d\}$ is as same as that if i follows τ^*. Denote L as the summation of all gains from deviation calculated from period s to period M. L is finite since both stage-game payoff and $M - s$ are finite. Taking sufficiently high $\delta \in (0, 1)$ so that $\eta\delta^{M-s}/(1-\delta) > L$ will deter this deviation.

Suppose $I_i^s \geq k$. Since i follows τ^* til s, there are two cases: $s \in \mathbf{O}^t$ for some $t \geq 1$ or $s \in \mathbf{C}^t$ for some $t \geq 1$. Suppose $s \in \mathbf{O}^t$, i will be ready to play $\langle 1 \rangle$ in $\mathbf{O}^t$. Because playing $\langle I \rangle$, where I is a subset of N, incurs more negative payoff than $\langle 1 \rangle$ since there is discounting and there is at most one another player might play $\langle I \rangle$, i will not deviate in this case. Suppose $s \in \mathbf{C}^t$. If $s \in \mathbf{C}^t(1, v)$, where $v = 1, ..., n$, i's detectable deviation is to play a sequence other than what Table 4 specifies, it will be detected by all i's neighbors. There is a positive probability that all i's neighbors will play **stay** forever according to the off-path belief and then all Rebels will contagiously play **stay** forever. i's expected continuation payoff is 0 from some timing after s. The same situation happens if $s \in \mathbf{C}^t(2, v)$, where $v = 1, ..., t + 1$, or $s \in \mathbf{C}^t(3, v)$, where $v = 1, ..., n$. Take m_s as the period when all Rebels play **stay** forever. m_s is finite since the network is finite. Then take $M = \max\{T^\theta, m_s\}$. There is a difference at least $\eta\delta^{M-s}/(1-\delta)$ as the present value at s. Denote L as the summation of all gains from deviation calculated from period s to period M. L is finite since both stage-game payoff and $M - s$ are finite. Taking sufficiently high $\delta \in (0, 1)$ so that $\eta\delta^{M-s}/(1-\delta) > L$ will deter this deviation. $\square$