Coordination in Social Networks

Chun-Ting Chen

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Motivation

- Repeated game is a standard model to understand strategic learning.
  - Farsighted-learning, Reputation, etc.
- An exogenous network (social network) models information structure.
  - [Renault an Tomala 1998], [Chwe 2000], [Wolitzky 2012, 2014], [Laclau 2012, 2014], etc.
- Will people solve the uncertainty and act collectively in networks eventually?

What this paper does?

- I model a repeated game with incomplete information and network-monitoring with discount factor.
  - Players can only observe own/neighbors' types and own/neighbors' actions.
- Look for an equilibrium in which the pay-off relevant information become commonly known in finite time.
  - A strong requirement.
- Such equilibrium can be constructed under some assumptions.

Time line

1. Players are allocated in a fixed and finite network.
2. Nature choose players' types according to a probability distribution.
3. Types are then fixed over time.
4. Players play a stage game infinitely repeatedly with common discount factor.

- A fixed and finite network
  - n players; $N = \{1, \ldots, n\}$ is the set of players.
  - G_i is i's neighborhood; G_i is a subset of N such that $i \in G_i$.
  - $G = \{G_i\}_i$ is the network.
- Players of two types
  - i's type: $\theta_i \in \Theta_i = \{S, B\}$
  - Type profile: $\theta \in \Theta = \times_{i \in N} \Theta_i$

- Stage game—K-threshold game: a protest ([Chwe 2000])
  - S-type's action set$= \{\mathbf{p}, \mathbf{n}\}$
  - B-type's action set$= \{\mathbf{n}\}$
  - Pay-off function:

$$
\begin{aligned}
u_{S_i}(a_{S_i}, a_{-\theta_i}) &= 1 \quad \text{if } a_{S_i} = \mathbf{p} \text{ and } \#\{j : a_{\theta_j} = \mathbf{p}\} \geq k \\
u_{S_i}(a_{S_i}, a_{-\theta_i}) &= \text{-1} \quad \text{if } a_{S_i} = \mathbf{p} \text{ and } \#\{j : a_{\theta_j} = \mathbf{p}\} < k \\
\\
u_{S_i}(a_{S_i}, a_{-\theta_i}) &= 0 \quad \text{if } a_{S_i} = \mathbf{n}
\end{aligned}
$$

Model

- Player i's strategy: $\tau_i : \Theta_{G_i} \times \bigcup_{m=0}^{\infty} H_{G_i}^m \to A_{\theta_i}$, where
  - $\Theta_{G_i} = \prod_{j \in G_i} \Theta_j$
  - $H_{G_i}^m = \{\emptyset\} \times \prod_1^m \prod_{j \in G_i} A_{\theta_j}$

MODEL

- Assumptions:
  - Pay-off is hidden.
    - Viewing the pay-off as an expected pay-off: [Aumann and Maschiler 1990], [Miyahara and Sekiguchi 2013], [Wolitzky 2013], etc.
  - Network G is commonly known, connected, and undirected.

Equilibrium Concept

- WPBE (weak perfect Bayesian equilibrium)
- Sequential equilibrium
- APEX Equilibrium: a refinement of the above concepts.

EQUILIBRIUM CONCEPT

APEX (*approaching ex-post efficient*) equilibrium

1. It is a refinement of equilibrium, by first defining:

DEFINITION (APEX STRATEGY)

A strategy, τ, is APEX $\Leftrightarrow$

$$\forall\theta, \text{ there is a finite time } T^{\theta}$$

such that the actions after T^{θ} in the path generated by τ repeats the static ex-post efficient outcome.

2. , then defining:

DEFINITION (APEX EQUILIBRIUM)

An equilibrium is APEX $\Leftrightarrow$ the equilibrium strategy is APEX.

EQUILIBRIUM CONCEPT

- Under some assumption, the result shows that an APEX strategy exists such that
  1. the "relevant information" to attain ex-post efficient outcome is commonly known after a finite T, and
  2. an APEX equilibrium can be constructed from this APEX strategy.
- "relevant information": *whether or not at least k S-types exist*.

APEX equilibrium: Outline

1. An example for APEX sequential equilibrium
2. Result 1: APEX sequential equilibrium for $k = n$.
3. Result 2: APEX WPBE for $k < n$.
    1. Consider cheap talk.
    2. Consider "costly" talk.
    3. Sketch of proof.
4. Further works

EXAMPLE

Let $k = n = 3$, when discount factor is high enough, an APEX sequential equilibrium can be constructed by

- Period 1
    - S-type 2: choose **p** if $\theta = (S, S, S)$;
    - S-type 2: choose **n** if $\theta \neq (S, S, S)$, and then choose **n** forever
    - S-type 1 (or S-type 3): **p**.
- After period 1
    - If S-type 2 chooses **p** in the last period $\Rightarrow$ S-type 1 (or S-type 3) chooses **p** forever;
    - If S-type 2 chooses **n** in the last period $\Rightarrow$ S-type 1 (or S-type 3) chooses **n** forever

      .
- Any deviation $\Rightarrow$ Choosing **n** forever .

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- Period 1
  - S-type 2: choose **p** if $\theta = (S, S, S)$;
  - S-type 2: choose **n** if $\theta \neq (S, S, S)$, and then choose **n** forever (the state is revealed)
  - S-type 1 (or S-type 3): **p**.
- After period 1
  - If S-type 2 chooses **p** in the last period $\Rightarrow$ S-type 1 (or S-type 3) chooses **p** forever;
  - If S-type 2 chooses **n** in the last period $\Rightarrow$ S-type 1 (or S-type 3) chooses **n** forever (undetectable deviation).
- Any deviation $\Rightarrow$ Choosing **n** forever (detectable deviation).

Main features in equilibrium construction

- The 1st-period actions serve as "messages" to reveal the relevant information.
- The "timing", 2nd-period ,to coordinate is part of equilibrium strategy (commonly known).
- Playing **n** forever serves as a "grim trigger".

Theorem ($k = n$)

In any network, for repeated k $=$ n Threshold game, an APEX sequential equilibrium exists whenever discount factor is sufficiently high.

Sketch of proof:

1. "messages" to reveal the relevant information.
   - Some B-types neighbors $\Rightarrow$ play **n** forever.
   - No B-type neighbor $\Rightarrow$ play **p** unless **n** is observed, and then play **n** forever.

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2. "Timing" to coordinate.
   - Finite network $\Rightarrow$ there is a finite time T such that players coordinate to the ex-post efficient outcome.

Result 1: APEX for $k = n$

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2. "Timing" to coordinate.
   - Finite network $\Rightarrow$ there is a finite time T such that players coordinate to the ex-post efficient outcome.
3. Any deviation $\Rightarrow$ play **n** forever.
4. A fully consistent belief system can be chosen.

Result 2: APEX for $k < n$

Theorem ($k < n$)

In any acyclic network, if prior π has full support on strong connectedness, then for repeated $k < n$ Threshold game, an APEX WPBE exists whenever discount factor is sufficiently high.

- acyclic network $\Leftrightarrow$ Tree network $\Leftrightarrow$ the path from any two nodes is unique.
- full support on strong connectedness: next slide.

Strong connectedness

Definition

θ has **strong connectedness**$\Leftrightarrow$ for every pair of S-types, there is a path consisting of S-types to connect them.

Definition

π has **full support on strong connectedness**$\Leftrightarrow$

$\pi(\theta) > 0$ **if and only if** θ has strong connectedness.

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π has **full support on strong connectedness** $\Leftrightarrow$

$$\pi(\theta) > 0 \textbf{ if and only if } \theta \text{ has strong connectedness.}$$

- Why do I need this assumption?

STRONG CONNECTEDNESS

- An B-type will not reveal information.
- **Without** full support on strong connectedness, in general, an Apex equilibrium does not exist when pay-off (as a signal) is hidden or noisy.

Equilibrium construction for $k < n$

- Difficulties:
  - Only two actions.
  - A discount factor.
  - Network-monitoring.
- Idea
  1. Consider an augmented T-period cheap talk phase.
  2. Consider an augmented T-period "costly" talk phase.

Time line

- Nature choose θ according to π.
- Types are then fixed over time.
- At the first T periods, players play T-period cheap talk.
- At $T+1$ period, players play a one-shot k-Threshold game.
- Game ends.

k-THRESHOLD GAME AUGMENTED BY T-PERIOD CHEAP TALK

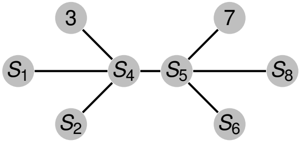
- T is a big number.
- A "letter-writing technology" for player i:
  - A set of sentences: $W = \{n, p\}^L$, where L is a big number.
  - A fixed grammar:

$$M_i^1 = \{f | f : \Theta_{G_i} \to W\} \cup \{\} \text{ ; } M_i^{t+1} = \{f | f \text{ is a selection from } \prod_{j \in G_i} M_j^t\} \text{ for } T \geq t \geq 1$$

k-THRESHOLD GAME AUGMENTED BY T-PERIOD CHEAP TALK

Example of a WPBE construction:

- $k = 5$, $T = 2$.
- G and θ=



- Equilibrium path
  - At $t = 1$,

S-type 4	(**send**	$(\overbrace{n, n, n, p, p, n, p, p}^{8})$)
S-type 5	(**send**	$(\overbrace{p, n, p, p, p, n, n, n}^{8})$)
S-type 1,2,6,8	(**send**	$(\overbrace{n, n, n, n, n, n, n, n}^{8})$)

k-Threshold game augmented by T-period cheap talk

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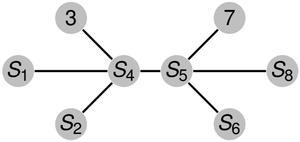


- Equilibrium path
  - At $t = 2$,

$$\text{S-type 4} \quad (\textbf{send} \quad (\overbrace{n, n, n, p, p, n, p, p}^{8}), (\overbrace{p, n, p, p, p, n, n, n}^{8}))$$

$$\text{S-type 5} \quad (\textbf{send} \quad (\overbrace{p, n, p, p, p, n, n, n}^{8}), (\overbrace{n, n, n, p, p, n, p, p}^{8}))$$

$$\text{S-type 1,2,6,8} \quad (\textbf{send} \quad (\overbrace{n, n, n, n, n, n, n, n}^{8}), (\overbrace{n, n, n, n, n, n, n, n}^{8}))$$

  - At $t = 3$, all S-types play **p**, then game ends.

- Off-path strategy
  - If S-type 4 (or 5) make detectable deviation (e.g. wrong sentence)
    $\Rightarrow$ others play **n** and then **n**.
  - If S-type 4 (or 5) make undetectable deviation $\Rightarrow$ he is facing a possibility of failure to coordinate.
- Off-path belief
  - Detectable deviation $\Rightarrow$ believing that all players outside neighborhood are B-types.

If there is a fixed cost ϵ to send the letter...

- Off-path strategy
  - If S-type 4 (or 5) make detectable deviation (e.g. wrong sentence or **not send**) $\Rightarrow$ others play **not send** and then **n**.
  - If S-type 4 (or 5) make undetectable deviation $\Rightarrow$ he is facing a possibility of failure to coordinate.
- Off-path belief
  - Detectable deviation $\Rightarrow$ believing that all players outside neighborhood are B-types.

If there is a fixed cost ϵ to send the letter...

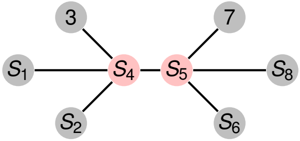
- Off-path strategy
  - If S-type 4 (or 5) make detectable deviation (e.g. wrong sentence or **not send**)

    $\Rightarrow$ others play **not send** and then **n**.
  - If S-type 4 (or 5) make undetectable deviation $\Rightarrow$ he is facing a possibility of failure to coordinate.
- Off-path belief
  - Detectable deviation $\Rightarrow$ believing that all players outside neighborhood are B-types.

So, when ϵ is small enough and T is large enough, an weak equilibrium can be constructed when ϵ is independent from messages.

However, if ϵ is not independent from messages, then a Free Rider Problem may occur.

- Suppose $\epsilon \downarrow$ when announce more S-types in the 1^{st} period.
- $k = 5$, $T = 2$.
- G and θ=



1. S-type 4 and S-type 5 will deviate from truthfully announce (Free Rider Problem).
2. Why? They will report more S-types to save costs.

THEOREM ($k < n$)

In any acyclic network, if prior π has full support on strong connectedness, then for repeated $k < n$ Threshold game, a weak APEX equilibrium exists whenever discount factor is sufficiently high.

Sketch of proof:

1. The Free Rider Problem can be solved in acyclic networks.
2. An Apex equilibrium path can be constructed.
3. APEX outcome gives maximum ex-post continuation pay-off after T.
4. Detectable deviation $\Rightarrow$ playing **n** forever (by off-path belief).
5. Undetectable deviation $\Rightarrow$ facing a possibility of coordination failure.
6. Any deviation will let APEX fail with positive probability.
7. Sufficiently high discount factor will impede deviation.

Further works

1. Cyclic networks.
2. Look for a general model in which finite-time communication protocol such that itself can be extended to an equilibrium.
3. Equilibrium selection.