Coordination in Social Networks: Communication by Actions

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Abstract

1 Introduction

2 Model

There is a set of players $N = \{1, ..., n\}$. They constitute a network $G = (V, E)$ so that the vertices are players ($V = N$) and an edge is a pair of them (E is a subset of the set containing all two-element subsets of N). Throughout this paper, G is assumed to be finite, commonly known, fixed, undirected, and connected.[1]

Time is discrete with index $s \in \{0, 1, ...\}$. Each player could be either type R or type I assigned by the nature at $s = 0$ according to a common prior π; R or I represents a Rebel or an Inert respectively. Call $\theta \in \Theta \equiv \{R, I\}^n$ a state of nature. At each $s \geq 1$, players play a normal form game, the *k-threshold game*, infinitely repeated played with common discounted factor $\delta \in (0, 1)$. In the k-threshold game, $A_R = \{\mathbf{revolt}, \mathbf{stay}\}$ is the set of

[1] A path in G from i to j is a finite sequence $(l_1, l_2, ..., l_L)$ without repetition so that $l_1 = i$, $l_L = j$, and $\{l_q, l_{q+1}\} \in E$ for all $1 \leq q < L$. G is fixed if G is not random, and G is undirected if there is no order relation over each edge. G is connected if, for all $i, j \in N$, $i \neq j$, there is a path from i to j.

actions for R and $A_I = \{\textbf{stay}\}$ is that for I. Denote by $\#X$ the cardinality of an set X. A Rebel i's stage-game payoff function is defined as below, while an Inert's stage-game payoff is equal to 1 no matter how other players play.

$$\begin{array}{ll} u_R(a_i, a_{-i}) = 1 & \text{if } a_i = \textbf{revolt} \text{ and } \#\{j : a_j = \textbf{revolt}\} \geq k \\ u_R(a_i, a_{-i}) = -1 & \text{if } a_i = \textbf{revolt} \text{ and } \#\{j : a_j = \textbf{revolt}\} < k \\ u_R(a_i, a_{-i}) = 0 & \text{if } a_i = \textbf{stay} \end{array} .$$

Let $[R](\theta)$ be the set of Rebels given θ and the notion *relevant information* indicate whether or not $\#[R](\theta) \geq k$. Note that the ex-post efficient outcome in the stage game is that every Rebel plays **revolt** whenever $\#[R](\theta) \geq k$, and plays **stay** otherwise.[2]

During the game, every player can observe his and his neighbors' types and his and their histories of actions, but no more. A history of actions played by i from period one to period $s \geq 1$ is denoted by $h_i^s \in H_i^s \equiv \times_{\varsigma=1}^{s} A_{\theta_i}$. Let $G_i \equiv \{j : \{i, j\} \in E\}$ be i's neighbors. Denote $\theta_{G_i} \in \Theta_{G_i} \equiv \{R, I\}^{G_i}$ as the type profile of i's neighbors. Let $h_i^0 = \emptyset$, and denote $h_{G_i}^s \in H_{G_i}^s \equiv \times_{j \in G_i} \times_{\varsigma=1}^{s} H_j^{\varsigma}$ as a history of actions played by i's neighbors from period one to period $s \geq 1$. The information set of i about θ at every period is defined by the cylinder $\{\theta_{G_i}\} \times \{R, I\}^{N \setminus G_i}$, and the information set about histories of action from period one to period $s \geq 1$ is $\{h_{G_i}^s\} \times H_{N \setminus G_i}^s$. A player i's pure behavior strategy τ_i is a measurable function with respect to i's information partition if τ_i maps $\{\theta_{G_i}\} \times \{R, I\}^{N \setminus G_i} \times \{h_{G_i}\} \times H_{N \setminus G_i}^s$ to a single action in his action set for every $s \in \{1, 2, ...\}$ and every $\theta \in \Theta$. I assume that payoffs are hidden to emphasize that observing neighbors' actions are the only channel to infer other players' types and actions.[3]

Likewise, define $H^s \equiv \times_{j \in N} H_j^s$ as the set of histories of actions from period one to period $s \geq 1$ and $H \equiv \bigcup_{\varsigma=0}^{\infty} H^{\varsigma}$ as the collection of histories of actions. By abusing the notation a bit, let $h(\tau, \theta) \in H$ denote the realized history of actions generated by strategy profile $\tau = (\tau_1, \tau_2, ..., \tau_n)$ given θ. Designate $\alpha_{G_i}^{\pi,\tau}(\theta, h^s | \theta_{G_i}, h_{G_i}^s)$ as the conditional distribution over $\Theta \times H^s$ induced by π and τ conditional on i's information up to period $s \geq 1$. The belief

[2] Moreover, at every θ and every k, the ex-post efficient outcome is unique and gives the maximum as well as the same payoff to every Rebel.

[3] Such restriction will be relaxed in the Section 5.

of i over Θ induced by π and τ up to period $s \geq 1$ is defined by

$$\beta_{G_i}^{\pi,\tau}(\theta|h_{G_i}^s) \equiv \sum_{h^s \in H^s} \alpha_{G_i}^{\pi,\tau}(\theta, h^s|\theta_{G_i}, h_{G_i}^s).$$

The equilibrium concept is the weak sequential equilibrium.[4] My objective is looking for the existence of *approaching ex-post efficient equilibrium* or *APEX equilibrium*, which is defined below.

Definition 2.1 (APEX strategy). *A behavior strategy τ is APEX if for all θ, there is a terminal period $T^\theta < \infty$ such that the actions in h_θ^τ after T^θ repeats the static ex-post Pareto efficient outcome.*

Definition 2.2 (APEX equilibrium). *An equilibrium (τ^*, α^*) is APEX if τ^* is APEX.*

In an APEX strategy, all Rebels will play **revolt** forever after some period only if $\#[R](\theta) \geq k$; otherwise, Rebels will play **stay** forever after some period. It is as if the Rebels will learn the relevant information in the equilibrium because they will play the ex-post efficient outcome after a certain point of time and keep on doing so. Notice that, in an APEX equilibrium, it is not only as if the Rebels will learn the relevant information: they must learn that. Lemma 2.1 articulates this fact.

Lemma 2.1 (Learning in the APEX equilibrium). *If the assessment (τ^*, μ^*) is an APEX equilibrium, then for all $\theta \in \Theta$, there is a finite time T_i^θ for every Rebel i so that*

$$\sum_{\theta \in \{\theta:[R](\theta)\geq k\}} \beta_{G_i}^{\pi,\tau^*}(\theta|h_{G_i}^s) = \text{ either } 1 \text{ or } 0$$

whenever $s \geq T_i^\theta$.

[4] A weak sequential equilibrium is an assessment $\{\tau^*, \alpha^*\}$, where α^* is a collection of distributions over players' information sets with the property that, for all $i \in N$ and for all $s = 1, 2, ...$, $\alpha_{G_i}^*(\theta, h^s|\theta_{G_i}, h_{G_i}^s) = \alpha_{G_i}^{\pi,\tau^*}(\theta, h^s|\theta_{G_i}, h_{G_i}^s)$ whenever the information set is reached with positive probability given τ^*. Moreover, for all $i \in N$ and for all $s = 1, 2, ...$, τ_i^* maximizes i's continuation expected payoff of

$$E_G^\delta(u_{\theta_i}(\tau_i, \tau_{-i}^*)|\alpha_{G_i}^{\pi,\tau_i,\tau_{-i}^*}(\theta, h^s|\theta_{G_i}, h_{G_i}^s))$$

conditional on θ_{G_i} and $h_{G_i}^s$ for all $h_{G_i}^s \in H_{G_i}^s$.

Definition 2.3 (Learning the relevant information)**.** *A Rebel i learns the relevant information at period ς if $\sum_{\theta \in \{\theta : [R](\theta) \geq k\}} \beta_{G_i}^{\pi,\tau^*}(\theta | h_{G_i}^s) =$ either 1 or 0 whenever $s \geq \varsigma$.*

It is clear that an APEX equilibrium exists when $k = 1$. As for other cases, let us start with the case of $k = n$ and then continue on to the case of $1 < k < n$. The proof is by construction. In the case of $k = n$, the constructed APEX equilibrium is intuitive and satisfies a stronger equilibrium concept. My main result tackles the case of $1 < k < n$. In such case, my constructed APEX equilibrium is not trivial and can only works for acyclic networks. Section 5.1 discusses why my constructed equilibrium is intractable in cyclic networks.

3 Equilibrium: APEX for $k = n$

In this section, my objective is to show the existence of APEX equilibrium for the case of $k = n$. In this case, notice that a Rebel can get a better payoff from playing **revolt** than from **stay** *only if* all players are Rebels. Two consequences follow. Firstly, if a Rebel has an Inert neighbor, this Rebel will always play **revolt** in the equilibrium. Secondly, at any period $s \geq 1$, it is credible for every Rebel to punish a deviation by playing **stay** forever *if* there is another one who also plays **stay** forever, independently from the belief held by the punisher. These two features constitute an APEX equilibrium and further transform itself to a sequential equilibrium.

Theorem 1 (APEX equilibrium for the case of $k = n$)**.** *For any n-person repeated k-Threshold game with parameter $k = n$ played in a network, there is a δ^* such that a sequential APEX equilibrium exists whenever $\delta > \delta^*$.*

Image that there are an Inert somewhere as well a Rebel i somewhere. Since the network is connected, there is a path connecting these two players. Along with this path, consider the "closest" Inert from Rebel i; this is an Inert who can be reached by the least number of consecutive edges from i. Note that this Inert's Rebel neighbors will play **stay** forever since $k = n$. Consider a strategy for Rebels on this path: a Rebel will play **stay** only after

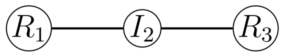

Figure 1: The state and the network in which the APEX equilibrium does not exist when $k = 2$.

observing his neighbor plays **stay**. On this path and according to this strategy, Rebel i will know the existence of such Inert eventually since the network is finite. This contagion argument suggests the following APEX strategy. Every Rebel plays **revolt** initially except for he has an Inert neighbor. Each of them will continuously play **revolt** but switch to **stay** instantly if he observes any of his neighbor plays **stay**. Upon observing a n consecutive **revolt**, a Rebels knows that no Inert exists; otherwise, he knows some Inert exists. The above strategy is an APEX strategy if all Rebels play ex-post efficient outcome after peiord n. To extend it to be an APEX equilibrium, let the deviant play **stay** forever and the punisher who detects it also play **stay** forever. This out-of-path strategy is credible for both the deviant and the punisher, independent from the belief held by the punisher, and hence it is also sequential rational.[5]

4 Equilibrium: $1 < k < n$

In this section, my objective is to show the existence of APEX equilibrium for the case of $1 < k < n$. In contrast to the case of $k = n$, a Rebel still has the incentive to play **revolt** even if he has an Inert neighbor. This opens a possibility for the non-existence of APEX equilibrium. Example 1 below demonstrates it.

Example 1. Suppose that $k = 2$ and $\theta = (R, I, R)$. The state and the network is represented in Figure 1. Rebel 1 never learns the type of player 3 since Inert 2 cannot reveal it. Therefore no APEX equilibrium exists in this scenario.

The following assumption on the prior—*full support on strong connectedness*—excludes the possibility of nonexistence of APEX equilibrium. To this end, I begin with the definition

[5]This sequential rationality is in the sense of [Kreps and Wilson, 1982].

of *strong connectedness.*

Definition 4.1 (Strong connectedness). *Given G, a state θ has strong connectedness if, for every two Rebels, there is a path consisting of Rebels to connect them.*

In the language of graph theory, this definition is equivalent to: given G, θ has strong connectedness if the induced graph by $[R](\theta)$ is connected.

Definition 4.2 (Full support on strong connectedness). *Given G, π has full support on strong connectedness if*

$$\pi(\theta) > 0 \Leftrightarrow \ \theta \text{ has strong connectedness}$$

As a remark, the definition of the full support on strong connectedness is stronger than common knowledge about that every state has strong connectedness. This marginal requirement is subtle and is more convenient in constructing equilibrium.[6]

I am ready to state the main characterization of this paper:

Theorem 2 (APEX equilibrium for the case of $1 < k < n$). *For any n-person repeated k-Threshold game with parameter $1 < k < n$ played in networks, if networks are acyclic and if π has full support on strong connectedness, then there is a δ^* such that an APEX equilibrium exists whenever $\delta > \delta^*$.*[7]

Constructing an APEX equilibrium in this case is convoluted. I illustrate the proof idea throughout this paper while leaving the formal proof in Appendix. Moreover, since the case of $k = 2$ is trivial given that θ has strong connectedness, I focus on $2 < k < n$ cases.[8]

[6]The main result only requires a weaker version: $\pi(\theta) > 0 \Rightarrow \ \theta$ has strong connectedness . However, working on this weaker version is at the expense of much tedious proof. Throughout this paper, I will stick to the original definition.

[7]A network is acyclic if the path from i to j for all $i \neq j$ is unique.

[8]When $k = 2$, if $[R](\theta) \geq k = 2$, each Rebel must have a Rebel neighbor by the full support on strong connectedness. They play **revolt** forever from period one. Otherwise, only a single Rebel exists, and he plays **stay** forever. The above strategy is an APEX strategy. It can be extended to an APEX equilibrium by setting the out-of-path belief as: each Rebel is certain that all players outside his neighborhood are Inerts.

Image that players play an APEX equilibrium. Starting from the terminal period T^θ, by the definition of APEX equilibrium, players in fact play a one-shot k-threshold game with its payoff has been discounted to its present value; a Rebel gets the payoff of $1/(1-\delta)$ or $-1/(1-\delta)$ by playing **revolt** or **stay** forever respectively. Since they keep playing the same action, they cannot update their belief about θ after T^θ. In addition, every Rebel has to know the relevant information after T^θ by Lemma 2.1. This implies that any information sharing process leads to revealing the relevant information must be completed before T^θ. Players have no communication technologies, say cheap talk, to share information, and therefore this information sharing process can be only through playing series of actions that incurs expected payoff. Constructing an APEX equilibrium amounts to developing a framework for players in sharing information before T^θ and determining the terminal period T^θ by themselves. Substantially, the following scenario describes the essence of an APEX equilibrium: "Share information accompanying positive or negative payoff for T^θ periods to reveal the relevant information. Then play a one-shot discounted k-threshold game at period $T^\theta + 1$."

There remains a challenge: how to manage players' higher-order belief over others' actions in information sharing, which in turns induce players' belief over states, due to imperfect monitoring? The proof solves it by exploiting a crucial but implicit assumption underlying the infinite repeated game: players move synchronously. At every period, players commonly know whether or not they are in that period. What follows is that the time horizontal line can be partitioned, such partition is commonly known to players, and players' behavior in different parts in the partition can induce different belief to players. Players can then share information by playing actions part by part along with such partition. Furthermore, if there are parts for players to disseminate what the relevant information is and whether they have learnt it, then they have the chance to reach a consensus about when to terminate their belief updating. The logic of the proof for Theorem 2 is further summarized: "Players share information according to the partition of time before T^θ, and then play a one-shot discounted k-threshold game at period $T^\theta + 1$."

This logic, however, brings another challenge. Suppose that there are two players who share information to each other in a part in the partition, and each of them is certain that he will learn the relevant information if the other one shares information to him. They are also certain that the relevant information will be spread if it is learned in the next part and the ex-post outcome will be played after then. Due to sharing information incurs positive or negative payoff, will they *truthfully* share their information? This is a free-rider problem. Each of them will choose his most profitable way of sharing information without impeding learning the relevant information provided that the other one share the truthful information. Therefore, none of them will share truthful information. The free-rider problem turns out to be the main challenge in the construction of an APEX equilibrium. The proof solves it by first arguing that if the network is acyclic, the free-rider problem only occurs between two Rebel neigbors who *commonly know it*. With the help from this argument, the constructed equilibrium solves the free-rider problem by arbitrarily assigning one of them to be the free rider, who can choose his most profitable way in sharing information, while letting the other one share truthful information.

I introduce *deterministic T-round writing game* and *indeterministic T-round writing game* as simpler auxiliary scenarios to represent the main structure of the constructed equilibrium as well as to display the free rider problem. In these games, players' behavior virtually follows the above described logic of the constructed APEX equilibrium. In the deterministic T-round writing game, T is fixed, players are endowed a writing technology so that they can write to share information about θ for T rounds. They then play a one-shot k-threshold game at round $T+1$. This game is a reduced form of the original game by fixing T, in which I can pay attention only to how players share information and ignore how players determine the terminal period. In the indeterministic T-round writing game, the setting is the same as deterministic T-round writing game, except for that players have to jointly decide what T will be. This game is thus a simpler version of the original game, in which players have an additional communication technology.

4.1 Deterministic T-round writing game

The network, the set of states, and the set of players follow exactly the same definitions defined in Section 2. In the deterministic T-round writing game, each player endows a *writing technology.* A writing technology for player i is a pair of (W, M_i), in which $W = \{\mathbf{r}, \mathbf{s}\}^L$, $L \in \mathbb{N}$, and $M \equiv \times_{i\in N} \times_{t=1}^{T} M_i^t$ recursively defined by

$$M_i^1 = \{f | f : \Theta_{G_i} \to W\} \cup \{\emptyset\}$$

$$\text{for } 2 \leq t \leq T\ ,\ M_i^t = \{f | f : \times_{j\in G_i} M_j^{t-1} \to W\} \cup \{\emptyset\}.$$

W is interpreted as the set of sentence composed by letters $\mathbf{r}$ or $\mathbf{s}$ with length L, while M_i is understood as i's grammar. $\emptyset$ represents remaining silent. The phrase of "i writes a sentence to all his neighbors at round t" is equivalent to "i selects an $f \in M_i^t$ to get an element $w \in W$ according to f, which can be observed by all i's neighbors".

The time line for the deterministic T-round writing game is as follows.

1. Nature chooses θ according to the prior π.

2. θ is then fixed throughout rounds.

3. At $t = 1, ..., T$ round, players write to their neighbors.

4. At $T + 1$ round, players play a one-shot k-Threshold game.

5. The game ends.

A Rebel's payoff is the summation of his stage payoff across stages, while an Inert's payoff is set to be 1. The equilibrium concept is weak sequential equilibrium. An APEX strategy is a strategy that induces the ex-post outcome in the one-shot k-threshold game at $T + 1$ round. The definition of APEX equilibrium is adapted accordingly. In the examples below, let us focus on the configuration represented in Figure 2 and Figure 3 with $n = L = 8$. I.e. there are 8 players and the length of a sentence is also 8. Note that the differences between configurations in Figure 2 and Figure 3 are: (1) $\#[R](\theta) = 6$ in Figure 2 but $\#[R](\theta) = 5$ in Figure 3; (2) player 8 is a Rebel in Figure 2 but he is an Inert in Figure 2.

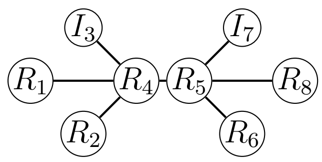

Figure 2: A configuration of the state and the network in which players 1, 2, 4, 5, 6, 8 are Rebels while players 3, 7 are Inerts.

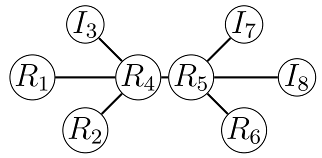

Figure 3: A configuration of the state and the network in which players 1, 2, 4, 5, 6 are Rebels while players 3, 7, 8 are Inerts.

Example 2 (Deterministic T-round writing without cost—cheap talk)**.** Let $k = 6$, $T = 2$, and suppose writing is costless. This is, there are two rounds for writing, while writing is the same as cheap talk.

Consider the following strategy ϕ.

At $t = 1$, the peripheral Rebels remain silent. Rebel 4 (or 5)'s grammar for writing a sentence is: if player i is a Rebel and known to him, he writes $\mathbf{r}$ in the i-th component in the sentence; otherwise, he writes $\mathbf{s}$ in that component. According to this grammar, the central player Rebel 4 writes the sentence $(\mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s}, \mathbf{s})$ on both configurations in Figure 2 and in Figure 3. The central player Rebel 5 writes $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r})$ in the configuration in Figure 2 and writes $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s})$ in the configuration in Figure 3.[9] Rebels 4 and 5's sentences thus reveals who are Rebels and who are not. Notice that the common knowledge of the network contributes to the ability of revealing players' types.

At $t = 2$, the peripheral Rebels still remain silent. Rebel 4 writes $(\mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r})$ in the configuration in Figure 2 and writes $(\mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s})$ in the configuration in Figure 3. Rebel 5 writes exactly the same sentence as Rebel 4. This is to say Rebel 4 and 5 share

[9]The notion of "peripheral" and "central" will be formalized in Section 4.3.1

information at $t = 1$ and then coordinate to announce a mixture sentence at $t = T$.

At $T+1$, by counting $\mathbf{r}$ in Rebel 4 or 5's mixture sentence, all Rebels know whether the number of Rebels outnumbers k. This leads all Rebels to play the ex-post efficient outcome in the one-shot k-threshold game.

Remark. An assessment (ϕ, o), where o is a belief system, is made to be an APEX equilibrium for Example 2 above as follows. The out-of-path strategy of ϕ specifies that if there is a detectable deviation, then the Rebels who detect this deviation will remain silent until $t = T$ and then play **stay** at $T + 1$.[10] The out-of-path belief of o is to believe that all players who are not neighbors are Inerts. The in-path belief of o is the belief induced by ϕ. Since any deviation by Rebel 4 or 5 would strictly decrease the possibility of achieving an ex-post efficient outcome, and writing is costless, the assessment (ϕ, o) constitutes an APEX equilibrium.

If writing is costly instead, the next example shows that (ϕ, o) is not an APEX equilibrium. This is due to the free-rider problem.

Example 3 (Deterministic T-round writing with cost function and the free-rider problem)**.** Let $k = 6$ and $T = 2$. Suppose that remaining silent incurs no cost, but writing incurs an extremely small cost $\epsilon > 0$ that is strictly decreasing with the number of $\mathbf{r}$ in a sentence. This is writing $(\mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r})$ incurs the least cost while writing $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{s})$ incurs the largest.

If so, that assessment (ϕ, o) in the previous example will no longer be an APEX equilibrium. To show this, first note that Rebels 4 and 5 truthfully reveals their information at $t = 1$. According to the above-mentioned, Rebel 4 will know the relevant information at $t = 2$ even if he deviates to writing that all his neighbors are Rebels, which incurs less cost than his truthful writing.[11][12] Rebel 5 is in the same situation as Rebel 4 and therefore

[10]A deviation could be, for instance, a wrong sentence that is not grammatical, is deviating from the in-path ϕ,..., etc

[11]This sentence is $(\mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s}, \mathbf{s})$, which incurs less cost than the truthfully reporting sentence $(\mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s}, \mathbf{s})$.

[12]If he remains silent, then this behavior will be considered as a deviation, and therefore he will never get the maximum payoff of 1. Hence, he will avoid doing so.

also writes the sentence that indicates that all his neighbors are Rebels. However, these sentences are uninformative. It turns out that both of them will deviate, and neither of them can know the relevant information at $t = 2$.

Fortunately, the following example shows that the free-rider problem can be solved.

Example 4 (Deterministic T-round writing with cost function and solving the free-rider problem)**.** The solution to solve the free-rider problem in the previous example is to extend T. It would open the possibility of the existence of a free rider at some round, while letting this free rider reveals relevant information at the next round. To this end, let $k = 6$ and $T = 3$. Consider the following strategy ρ and focus on the interaction between Rebels 4 and 5.

At $t = 1$, the lowest-index Rebel between Rebels 4 and 5 is the free rider, while the other one truthfully writes down his information. This is to say Rebel 4 will be the free rider and he writes the least-cost sentence. Rebel 5 writes $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r})$ in the configuration of Figure 2and writes $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s})$ in the configuration of Figure 3.

At $t = 2$, Rebel 4 has known the relevant information. Rebel 4 writes the least-cost sentence if $\#[R](\theta) \geq k$ but remains silent otherwise. The consequence is Rebel 4's behavior reveals the relevant information to his neighbors at this round. Rebel 5 remains silent instead.

At $t = T$, Rebel 5 has known the relevant information since he is Rebele 4's neighbor. He writes the least-cost sentence if $\#[R](\theta) \geq k$ but remains silent otherwise. Therefore Rebel 5's behavior at this round reveals the relevant information to his neighbors. Rebel 4 remains silent instead.

At $T+1$, all Rebels know the relevant information by observing Rebels 4 and 5's behavior. They play the ex-post efficient outcome accordingly.

Remark. Why does Rebel 5 *know* that he is not a free rider and therefore behaves not like a free rider? The following is the reason. He *knows* that, by common knowledge of the network, he and Rebel 4 are in a free-rider problem. Moreover, by common knowledge of the network, he knows that Rebel 4 knows that he and Rebel 4 are in a free-rider problem, he knows that Rebels 4 knows that he knows that,...,and so forth. Consequently, Rebel 5 and 4 commonly know that they are engaged in a free-rider problem. Section 4.3.2 articulates this fact: the common knowledge about engaging in a free-rider problem exists for any acyclic network in the constructed APEX equilibrium due to the common knowledge of the network.

Next, I allow T is not fixed; players have to determine what T is by themselves. I organize the discussion in the next section.

4.2 Indeterministic T-round writing game

The setting here is exactly the same as in the deterministic T-round writing game, except for that there are infinite rounds and players have to jointly decide when they will play the one-shot k-threshold game. They have to reach an agreement—the common knowledge of which round the terminal round T is.

The set of rounds is countably infinite with generic element t and linearly ordered. The writing technology is the same as that in deterministic T-round writing game, except for there are sentences different in their length: $W = \{\mathbf{r}, \mathbf{s}\}^L \cup \{\mathbf{r}, \mathbf{s}\}^{L'}$.

Conceptually, there could be two kinds of rounds. In the first kind, players write to share information about θ. In the second kind, players write to form the common knowledge about T. Let us partition the set of rounds into two parts, Γ and Γ', which represent the first kind and the second kind respectively. The round in Γ is labelled $\gamma \in \{1, 2, ...\}$. The round in Γ' is labelled γ'_{l_γ}, where $\gamma' \in \{0', 1', ...\}$ and $l_\gamma \in \{1, 2, ...\}$ for $\gamma = 0, 1, ...$ The rounds are linearly ordered by $<$. Specifically, the rounds and their labelling are ordered as shown in Figure 4.

Example 5 (Indeterministic T-round writing with cost function)**.** Let $k = 6$ and $l_\gamma = 2$ for $\gamma = 0, 1, ...$ Let us consider the following strategy ψ. At a round in Γ', ψ specifies that,

- if a Rebel thinks "He is certain that $\#[R](\theta) \geq k$ and the nearest forthcoming round in Γ is the terminal round," he will write $(\mathbf{r})$;
- if a Rebel thinks "He is uncertain about $\#[R](\theta) \geq k$," he will write $(\mathbf{s})$;

$$0_1^{'} < 0_2^{'} < ... < 0_{l_0}^{'} < 1 < 1_1^{'} < 1_2^{'} < ... < 1_{l_1}^{'} < 2 < ...,$$

where $l_0, l_1, ...$ are all finite numbers.

Figure 4: The linearly ordered rounds in the indeterministic T-round writing game.

- otherwise, if a Rebel think "$\#[R](\theta) \geq k$ is impossible and the nearest forthcoming round in Γ is the terminal round," he will write $\emptyset$.

According to this strategy, $t = 1$ is not terminal in the configuration in Figure 2 or Figure 3 since all Rebels writes (**s**) at $t = 0_1^{'}, 0_2^{'}$.

At $t = 1$, Rebels 4 and 5 are in a free-rider problem at $t = 1$. ψ solves it by letting Rebel 4 be the free rider and Rebel 5 write his information truthfully.

At $t = 1_1^{'}$, Rebel 4 knows $\#[R](\theta) \geq k$ in the configuration in Figure 2 and knows $\#[R](\theta) < k$ in the configuration in Figure 3. Therefore, he writes (**r**) in the configuration in Figure 2 and $\emptyset$ in the configuration in Figure 3. As for other Rebels, they write (**s**).

At $t = 1_2^{'}$, all Rebels will learn the relevant information by observing what Rebel 4 writes at $t = 1_1^{'}$. They terminate their writing at $t = 2$ since they have learnt the relevant information. Therefore, $t = 2$ is the terminal round, and the Rebels play a one-shot k-threshold game at $t = 3$.

The strategy ψ can be made to be an APEX equilibrium strategy in the same way as Example 2.

4.3 Dispensability of writing technology

In fact, writing technology is dispensable, and repeated actions are sufficient to serve as a communication protocol to achieve an ex-post outcome in an equilibrium. To fix the idea, I draw analogue between the writing game and the original game in Table 1.

In the equilibrium construction in the original game, let us partition the periods so that it is analogous to partitioning the rounds. The analogue of Γ is the set of *periods for*

$$\underbrace{(\text{periods for coordination})}_{\text{0-block}} < \underbrace{(\text{periods for reporting}) < (\text{periods for coordination})}_{\text{1-block}} < ...$$

Figure 5: The linearly ordered period-sections in the repeated k-threshold game.

reporting in the original game to emphasize that these periods are for reporting information about θ. The analogue of $\Gamma^{'}$ is the set of *periods for coordination* in the original game to emphasize that these periods are for coordinating to play the ex-post efficient outcome. The partition of periods is linearly ordered by $<$, and let us define a coarser partition as *t-blocks* indexed by $t \in \{0, 1, ...\}$ along with the order of partition of periods as shown in Figure 5.

Table 1: The analogue between the indeterministic T-round writing game and the repeated k-threshold game

Indeterministic T-round writing game	Repeated k-threshold game
A round	A range of periods
A sentence	A sequence of actions
The length of a sentence in a round	The length of a range of periods
A chosen letter in a sentence	A chosen action
The cost of writing a sentence	The expected payoff occurring in a sequence of actions
The fixed grammar	The equilibrium path

I am ready to delve into the details of the equilibrium construction for the original game. I begin with defining *information hierarchy* among players for each t-block. Information hierarchy identifies a smaller set of Rebels by whom the information sharing process leading to reveal the relevant information is played. I can pay attention to this smaller set of Rebels instead of the whole set of Rebels to reduce complexity in equilibrium construction.

4.3.1 Information hierarchy

The information hierarchy across Rebels presents Rebels' information *before* entering the periods for reporting at t-block. That is a tuple

$$(\{G_i^t\}_{i\in N}, \{I_i^t\}_{i\in N}, R^t, \theta).$$

G_i^t represents *the extended neighbors*: $j \in G_i^t$ if j can be reached by at most t consecutive edges from a Rebel i, in which the endpoints of $t-1$ edges are both Rebels. I_i^t represents *the extended Rebel neighbors*: the set of Rebels in G_i^t. R^t represents *the active Rebels*: those Rebels who are *active* in the sense that their information about Rebels does not contained the information of any of their neighbor. Those objects are defined by:

At 0-block,

$$\text{if } \theta_i = I,\ G_i^0 \equiv \emptyset,\ I_i^0 \equiv \emptyset.$$

$$\text{if } \theta_i = R,\ G_i^0 \equiv \{i\},\ I_i^0 \equiv \{i\}.$$

$$R^0 \equiv [R](\theta).$$

At 1-block,

$$\text{if } \theta_i = I,\ G_i^1 \equiv \emptyset,\ I_i^1 \equiv \emptyset.$$

$$\text{if } \theta_i = R,\ G_i^1 \equiv G_i,\ I_i^1 \equiv G_i \cap R^0.$$

$$R^1 \equiv \{i \in R^0 : \nexists j \in G_i \text{ such that } I_i^1 \subseteq G_j^1\}.$$

At t-block for $t > 1$,

$$\text{if } \theta_i = I,\ G_i^t \equiv \emptyset,\ I_i^t \equiv \emptyset.$$

$$\text{if } \theta_i = R,\ G_i^t \equiv \bigcup_{j\in G_i} G_j^{t-1},\ I_i^t \equiv \bigcup_{j\in G_i} I_j^{t-1},$$

$$I_{ij}^t \equiv I_i^{t-1} \cap I_j^{t-1} \text{ if } j \in G_i.$$

$$R^t \equiv \{i \in R^{t-1} : \nexists j \in G_i \text{ such that } I_i^t \subseteq G_j^t\}.$$

According to the definition above, the peripheral Rebels in the configuration of Figure 2 are active in 0-block (in R^0) but not active in 1-block (not in R^1), while the central players are active in both 0-block and 1-block. It can be shown that $R^t \subseteq R^{t-1}$ as follows.

Lemma 4.1. *If the θ has strong connectedness, then*

$$R^t \subseteq R^{t-1}$$

for all $t \geq 1$.

Is it sufficient to let only active Rebels share information about θ while θ can be revealed eventually? The answer is affirmative proven by Theorem 3 below, if the network is acyclic and if the θ has strong connectedness.

Theorem 3. *If the network is acyclic and if the θ has strong connectedness, then*

$$[R](\theta) \neq \emptyset \Rightarrow \text{ there exists } t \geq 0 \text{ and } i \in R^t \text{ such that } I_i^{t+1} = [R](\theta).$$

The strategy on the equilibrium path will specify how Rebels act according to the information hierarchy. R^t will be those Rebels who are active in terms of sharing information in the periods for reporting in the t-block.

4.3.2 The equilibrium path in the periods for reporting

If there is no further mention, all the description in this section is for the APEX equilibrium path *before* the terminal period T^θ. Let us shorten "periods for reporting in t-block" by O^t, denote $|\mathrm{O}^t|$ as the length of O^t, and shorten **revolt** and **stay** to **r** and **s** receptively henceforth.

$|\mathrm{O}^t|$ is independent from t and determines only the set of players. Firstly, assign each player i a distinguishing prime number x_i starting from 3 (by exploiting the common knowledge of the network). Then let $|\mathrm{O}^t| = x_1 \otimes x_2 \otimes ... \otimes x_n$, where $\otimes$ is the usual multiplication operator. The sequence of actions in O^t is with length $|\mathrm{O}^t|$ and would take one of the forms specified in the right column in Table 2. There, if $I \subseteq N$, then $x_I \equiv \otimes_{i \in I} x_i$. The abbreviations of these sequences are listed in the left column. Since these sequences in the periods for reporting are meant to share information about θ, the terms "playing the sequence" and "reporting the information" are interchangeable and will be alternatively used.

It is worth noting that this sequence constructed by prime numbers brings two benefits. Firstly, since the multiplication of distinguishing prime numbers can be uniquely factorized,

Table 2: The notations for the sequences of actions in O^t on the path

Notations		The sequences of actions
$\langle I \rangle$	$\equiv$	$\langle \mathbf{s}, ..., \mathbf{s}, \underbrace{\mathbf{r}, \mathbf{s}, ..., \mathbf{s}}_{x_I} \rangle$
$\langle 1 \rangle$	$\equiv$	$\langle \mathbf{s}, ..., \mathbf{s}, \mathbf{r} \rangle$
$\langle$**all stay**$\rangle$	$\equiv$	$\langle \mathbf{s}, ..., \mathbf{s}, \mathbf{s} \rangle$

the Rebels can utilize such sequence to precisely report players' identities. Secondly, the un-discounted expected payoff of playing $\langle I \rangle$ for some $I \subseteq N$ is always equal to -1, and therefore it is relatively easy to calculate. This is because, at any period in O^t, at most one player would play $\mathbf{r}$ by the property of prime number multiplication.[13]

The sequences $\langle I \rangle$ or $\langle 1 \rangle$ are meant to distinguished themselves from $\langle$**all stay**$\rangle$. The sequence $\langle$**all stay**$\rangle$ is for the inactive Rebels to report nothing. The sequence $\langle I \rangle$ is used for reporting $I \neq \emptyset$ when I is a set of Rebels. The sequence $\langle 1 \rangle$ is intentionally crafted to tackle the free-rider problem. To see how $\langle 1 \rangle$ works, let us formally define the *pivotal Rebel* and the *free-rider problem.*

Definition 4.3 (Pivotal Rebels in O^t)**.** *A Rebel p is pivotal in O^t if $p \in R^t$, $I_p^t \subset [R](\theta)$, $\#I_p^t < k$, and p is certain that he will learn the relevant information in the end of O^t, given that each $i \in R^t$ reports $\langle I_i^t \rangle$.*

From the definition, a pivotal Rebel in O^t is one who can learn the relevant information if all of his active Rebel neighbors truthfully report their information about θ to him. The pivotal Rebels can be further classified into two kinds: ones who can learn the true state, and ones who learn only the relevant information. When $k = 6$, in the configuration in Figure 2, only Rebels 4 and 5 are pivotal, and they are of the first kind; in the configuration in Figure 6, only Rebel 5 is pivotal, and he is of the first kind; in the configuration in Figure 7, only Rebel 4 pivotal, and he is of the second kind.

[13] This statement holds given that there is no player who plays $\langle 1 \rangle$.

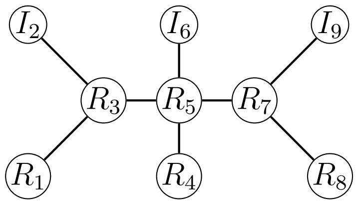

Figure 6: A configuration of the state and the network in which player 1, 3, 4, 5, 7, 8 are Rebels while players 2, 4, 9 are Inerts.

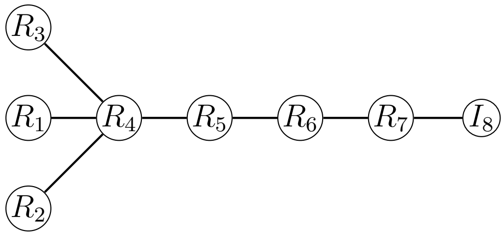

Figure 7: A configuration of the state and the network in which player 1, 2, 3, 4, 5, 6, 7 are Rebels while player 8 is an Inert.

Call p of the first kind by *θ-pivotal*. For the second kind, if the network is acyclic and if the prior has full support on strong connectedness, p is the second kind in O^t only if $I_p^t = k - 1$. Call the one with $I_p^t = k - 1$ by $k-1$-*pivotal* Rebel.[14]

Below is the defined free-rider problem in O^t .

Definition 4.4. *A free-rider problem exists in* O^t *if there are multiple θ-pivotal Rebels in* O^t.

The following lemma is crucial.

Lemma 4.2. *If the network is acyclic and if π has full support on strong connectedness, there are at most two θ-pivotal Rebels in the t-block. Moreover, they are neighbors.*[15]

Notably,

[14]To show that a pivotal Rebel is the second kind in O^t only if $I_p^t = k - 1$, one can follow the same argument in Lemma 4.1 and Theorem 3.

[15]As a remark, Lemma 4.2 is not true when the network is cyclic. To see this, consider a 4-player circle when $\theta = (R, R, R, R)$.

Lemma 4.3. *If the network is acyclic and if π has full support on strong connectedness, when there are two θ-pivotal Rebels $p, p^{'}$ in the t-block, then they commonly know that they are θ-pivotal Rebels at the beginning of t-block.*

By Lemma 4.3, θ-pivotal Rebels in O^t can identify themselves at the beginning of O^t. This importance cannot be further emphasized. If the free-rider problem occurs in O^t, the strategy can specify that the lowest indexed θ-pivotal Rebel p in the free-rider problem will play $\langle 1 \rangle$, while the other one $p^{'}$ will play $\langle I^t_{p^{'}} \rangle$ *beforehand.* In short, this knowledge is encoded in the belief system of an APEX equilibrium.

The assumption of acyclic network in Lemma 4.3 is indispensable. In Section 5.1, there is a configuration in a cycle where there is no common knowledge of the free-rider problem if players still follow the equilibrium path working for Theorem 2.

Overall, the sequences played in O^t on the path are listed in Table 3.

Table 3: The sequences of actions played in O^t on the path

Rebel i	i plays
$i \notin R^t$	$\langle$**all stay**$\rangle$
$i \in R^t$ but i is not pivotal	$\langle I^t_i \rangle$
i is $k-1$-pivotal	$\langle 1 \rangle$
i is θ-pivotal but not in the free-rider problem	$\langle 1 \rangle$
i is in the free-rider problem with the lowest index	$\langle 1 \rangle$
i is in the free-rider problem without the lowest index	$\langle I^t_i \rangle$

4.3.3 The equilibrium path in the periods for coordination

In this section, I discuss the sequences of actions in the periods for coordination on the path. The term "periods for coordination in t-block" is shorten by K^t. If there is no further mention, all the description in this section is for the APEX equilibrium path *before* the terminal period T^{θ}.

The main feature in the periods of coordination is that, whenever a Rebel i is considered inactive starting at some t-block (i.e. $i \notin R^t$ for some $t \in \mathbb{N}$), there is no strategy for i to convince all Rebels that $\#[R](\theta) \geq k$ even though i knows it and wants to propagandize it. The structure in the periods of coordination is much intrigued, and the periods are further partitioned by *divisions* and *sub-blocks*. I depict that below, where (K) represents a certain range of periods for coordination.

In K^0,

$$\overbrace{\underbrace{(\mathrm{K})}_{\text{one sub-block}}}^{\text{1-division}}\ \overbrace{\underbrace{(\mathrm{K})}_{\text{one sub-block}}}^{\text{2-division}}\ \overbrace{\underbrace{(\mathrm{K})\cdots(\mathrm{K})}_{n\text{ sub-blocks}}}^{\text{3-division}}$$

In K^t, $t > 0$,

$$\overbrace{\underbrace{(\mathrm{K})\cdots(\mathrm{K})}_{n\text{ sub-blocks}}}^{\text{1-division}}\ \overbrace{\underbrace{(\mathrm{K})\cdots(\mathrm{K})}_{t+1\text{ sub-blocks}}}^{\text{2-division}}\ \overbrace{\underbrace{(\mathrm{K})\cdots(\mathrm{K})}_{n\text{ sub-blocks}}}^{\text{3-division}}$$

In the t-block, denote $\mathrm{K}^t_{u,v}$ as the v-th sub-block in u-division and denote $|\mathrm{K}^t_{u,v}|$ as the length of $\mathrm{K}^t_{u,v}$. Similarly, denote K^t_u as the u-division and denote $|\mathrm{K}^t_u|$ as the length of K^t_u. Let us shorten **revolt** and **stay** to **r** and **s** receptively henceforth. On the path, the length of $|\mathrm{K}^t_{u,v}|$ is determined. For all $v \in \{1, ..., n\}$ let $|\mathrm{K}^t_{u,v}| = n$ for $u = 1, 2$ and $|\mathrm{K}^t_{u,v}| = 1$ for $u = 3$. The notations for the sequences of actions on the path are shown in Table 4 shows.[16] For a Rebel i on the equilibrium path, either $\langle i \rangle$ or $\langle \textbf{all stay} \rangle$ is seen. Note that a Rebel who is certain $\#[R](\theta) < k$ will always play $\langle \textbf{all stay} \rangle$ in every K^t_u for all $t \geq 0, u = 1, 2$. $\langle i \rangle$ is contrasted with it in the representation of uncertainty about $\#[R](\theta)$. $\langle i \rangle$ is also crafted to simplify expected payoff calculation: there is at most one **r** played in K^t_u for $t \geq 0$ and $u = 1, 2$.

The equilibrium behavior on the path in K^0 Since the 0-block has a simpler structure, I begin with depicting the equilibrium path in K^0, which is shown in Table 5. I describe Rebel i's behavior by contingent on whether i has learnt the relevant information. It is not standard but more intuitive, while detailing it in Appendix. Notice that if Rebel i is certain

[16] Because, in the 3-division, the length of the sequence of actions is 1, i.e. playing an action, I dispense notations in the 3-division for conciseness.

Table 4: The notations for the sequences of actions in K_u^t for $u = 1, 2$, on the path

Notations		The sequences of actions
$\langle i \rangle$	$\equiv$	$\langle \mathbf{s}, ..., \mathbf{s}, \underbrace{\mathbf{r}, \mathbf{s}, ..., \mathbf{s}}_{i} \rangle$
$\langle$**all stay**$\rangle$	$\equiv$	$\langle \mathbf{s}, ..., \mathbf{s}, \mathbf{s} \rangle$

that $\#[R](\theta) < k$, it must be the case that all Rebels are i's neighbors by the full support on strong connectedness. All Rebels will also be certain about that immediately after K_1^0.

The intriguing part in Table 5 might be "How Rebel i initiates the common knowledge about $\#[R](\theta) \geq k$." i does so by *first play* $\langle i \rangle$ *in* K_1^0 *and then play* $\langle$**all stay**$\rangle$ *in* K_2^0. His behavior is then distinguishable from other kinds of Rebels. His neighbors will know $\#[R](\theta) \geq k$ immediately after K_2^0, and then all Rebels will know that by playing **r** contagiously in K_3^0.

i will not deviate to play $\langle$**all stay**$\rangle$ even though it might be undetectable. This is by the assumption of acyclic network. If i does so, i will be considered as an inactive Rebel afterwards by all of his neighbors. From this point on, he cannot update the belief held by his neighbors whose information hierarchy rank is no less than the one he pretends to be. More precisely, if he is a R^τ Rebel, but he pretends not to be one, he cannot influence the belief updating of the Rebels who are in R^{t-1}, $t \geq \tau$.[17] He then faces a positive probability that not enough Rebels can be informed of $\#[R](\theta) \geq k$. If this event happens, he will only get zero payoff. However, he can surely get the maximum payoff of 1 afterwards and forever after K_2^0. Sufficiently high discount factor will deter this deviation. In essence, one major part of proof for Theorem 2 follows the same argument.

[17]This argument is due to Lemma 4.1 and the belief updating on the path described in Table 6, Table 7, Table 8, Table 9, and Table 10.

Table 5: The sequences of actions played in K^0 on the path

The sequences of actions played in K^0_1 on the path

Rebel i	i plays
i is certain $\#[R](\theta) < k$	$\langle$**all stay**$\rangle$
$i \notin R^1$ and is uncertain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$
$i \in R^1$ and is uncertain $\#[R](\theta) \geq k$	$\langle i \rangle$
i is certain $\#[R](\theta) \geq k$	$\langle i \rangle$

The sequences of actions played in K^0_2 on the path

Rebel i	i plays
i is certain $\#[R](\theta) < k$	$\langle$**all stay**$\rangle$
$i \notin R^1$ and is uncertain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$
$i \in R^1$ and is uncertain $\#[R](\theta) \geq k$	$\langle i \rangle$
i is certain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$

The sequences of actions played in K^0_3 on the path

Rebel i	i plays
i is certain $\#[R](\theta) < k$	$\langle \mathbf{s} \rangle$
$i \notin R^1$ and is uncertain $\#[R](\theta) \geq k$	$\langle \mathbf{s} \rangle$
$i \in R^1$ and is uncertain $\#[R](\theta) \geq k$	$\langle \mathbf{s} \rangle$
i is certain $\#[R](\theta) \geq k$	$\langle \mathbf{r} \rangle$

How Rebels update their belief that is being consistent with the equilibrium path after K_1^0 and K_2^0 is listed in Table 6 and Table 7. One can track the evolution of information filtrations through Table 6 to Table 7.

Table 6: The belief of $j \in G_i$ after observing i's previous actions immediately after K_1^0

i plays	The event $j \in G_i$ assigns with probability one
In K_1^0	
$\langle$**all stay**$\rangle$	$i \notin R^1$ if $j \in R^1$
$\langle$**all stay**$\rangle$	$\#[Rebels](\theta) < k$ if $j \notin R^1$
$\langle i \rangle$	$i \in R^1$ or $\#[Rebels](\theta) \geq k$

Table 7: The belief of $j \in G_i$ after observing i's previous actions immediately after K_2^0

i plays		The event $j \in G_i$ assigns with probability one
In K_1^0	In K_2^0	
$\langle$**all stay**$\rangle$	$\langle$**all stay**$\rangle$	$i \notin R^1$ if $j \in R^1$
$\langle$**all stay**$\rangle$	$\langle$**all stay**$\rangle$	$\#[Rebels](\theta) < k$ if $j \notin R^1$
$\langle i \rangle$	$\langle$**all stay**$\rangle$	$\#[Rebels](\theta) \geq k$
$\langle i \rangle$	$\langle i \rangle$	$i \in R^1$

The equilibrium behavior on the path in K^t for $t \geq 1$ Next, I describe the equilibrium behavior on the path in K^t whenever $t \geq 1$. Players' belief over states will now be contingent on the behavior in O^t since Rebels share information in O^t. As an analogue to the grammar interpretation in the deterministic (or indeterministic) T-round writing game, I first illustrate how players update their belief after observing the equilibrium strategy on the path. After that, the in-path strategy contingent on players' belief is introduced. Players' information filtrations evolve from Table 8 through Table 9 to Table 10.

Table 8: The belief of $j \in G_i$ after observing i's previous actions immediately after O^t

i plays	The event $j \in G_i$ assigns with probability one
In O^t	
$\langle$**all stay**$\rangle$	$i \notin R^t$ and $I_{ji}^{t+1} = I_j^t$
$\langle I_i^t \rangle$	$i \in R^t$ and $I_{ji}^{t+1} = I_j^t \cap I_i^t$
$\langle 1 \rangle$	i is pivotal

Table 9: The belief of $j \in G_i$ after observing i's previous actions immediately after K_1^t contingent on O^t

i plays		The event $j \in G_i$ assigns with probability one
In O^t	In $\mathrm{K}_{1,1}^t$	
$\langle$**all stay**$\rangle$	$\langle i \rangle$	$i \notin R^t$ and $I_{ji}^{t+1} = I_j^t$
$\langle I_i^t \rangle$	$\langle$**all stay**$\rangle$	$\#[Rebels](\theta) < k$
$\langle I_i^t \rangle$	$\langle i \rangle$	$i \in R^t$ and $I_{ji}^{t+1} = I_j^t \cap I_i^t$, or $\#[Rebels](\theta) \geq k$
$\langle 1 \rangle$	$\langle$**all stay**$\rangle$	$\#[Rebels](\theta) < k$
$\langle 1 \rangle$	$\langle i \rangle$	$\#[Rebels](\theta) \geq k$

Table 10: The belief of $j \in G_i$ after observing i's previous actions immediately after K_2^t contingent on O^t, K_1^t, and K_2^t

i plays			The event $j \in G_i$ assigns with probability one
In O^t	In $\mathrm{K}_{1,1}^t$	In $\mathrm{K}_{2,1}^t$	
$\langle$**all stay**$\rangle$	$\langle i \rangle$	$\langle$**all stay**$\rangle$	$i \notin R^t$ and $I_{ji}^{t+1} = I_j^t$
$\langle I_i^t \rangle$	$\langle$**all stay**$\rangle$	$\langle$**all stay**$\rangle$	$\#[Rebels](\theta) < k$
$\langle I_i^t \rangle$	$\langle i \rangle$	$\langle$**all stay**$\rangle$	$\#[Rebels](\theta) \geq k$
$\langle I_i^t \rangle$	$\langle i \rangle$	$\langle i \rangle$	$i \in R^t$ and $I_{ji}^{t+1} = I_j^t \cap I_i^t$
$\langle 1 \rangle$	$\langle$**stay**$\rangle$	$\langle$**all stay**$\rangle$	$\#[Rebels](\theta) < k$
$\langle 1 \rangle$	$\langle i \rangle$	$\langle$**all stay**$\rangle$	$\#[Rebels](\theta) \geq k$

The delicate part in K^t is how a pivotal Rebel p propagandizes the relevant information. p does so by playing $\langle \textbf{all stay} \rangle$ in $\mathrm{K}^t_{1,1}$ in the case of $\#[R](\theta) < k$, while playing $\langle p \rangle$ in the case of $\#[R](\theta) \geq k$. Notice that playing $\langle \textbf{all stay} \rangle$ in $\mathrm{K}^t_{1,1}$ by p will immediately inform p's neighbors that $\#[R](\theta) < k$. On the contrary, playing sequence other than $\langle \textbf{all stay} \rangle$ has not yet revealed $\#[R](\theta) \geq k$ since that might be played by other non-pivotal Rebels.

In $\mathrm{K}^t_{2,1}$, however, p reveals $\#[R](\theta) \geq k$ by playing $\langle \textbf{all stay} \rangle$, which is a costless sequence of actions. It might not seem intuitive at first sight, but it effectively prevents a potential free-rider problem: there are two pivotal Rebels, say p and $p^{'}$, who have already known $\#[R](\theta) \geq k$ in K^t. If initiating the common knowledge about $\#[R](\theta) \geq k$ incurs negative payoff, p or $p^{'}$ will have the incentive (again!) to let the other one initiate it. Playing $\langle \textbf{all stay} \rangle$ in $\mathrm{K}^t_{2,1}$ proudly becomes the initiation sequence by its cheapness.

The remaining argument is why other non-pivotal Rebels, say i, do not mimic the pivotal Rebels' behavior to play $\langle 1 \rangle$ in O^t. If i plays $\langle 1 \rangle$, all Rebels will learn the relevant information immediately after K^t_2 based on the belief updating on the path. It implies that the beginning of $t+1$-block is the terminal period. He will not learn the relevant information after that because the belief updating will be also terminated. However, he is still uncertain whether he can learn the relevant information in O^t since he is not pivotal. Since that the ex-post efficient outcome gives him the maximum payoff at every θ, and that he will learn the relevant information eventually on the equilibrium path, he prefers not to deviate given that the discount factor is high enough.[18] The proof of Theorem 2 heavily depends on arguments of this kind.

As a complement, I depict equilibrium strategy contingent on players' belief in Table 11. The idea in Table 11 is as follows. K^t_1 is the periods for a Rebel to advocate the knowledge of $\#[R](\theta) < k$. A Rebel i will initiate coordination to $\mathbf{s}$ by playing $\langle \textbf{all stay} \rangle$ in $\mathrm{K}^t_{1,1}$ if he knows $\#[R](\theta) < k$; otherwise, he will play $\langle i \rangle$. K^t_2 is the periods for a Rebel to advocate the knowledge of $\#[R](\theta) \geq k$. A Rebel i will initiate coordination to $\mathbf{r}$ by playing $\langle \textbf{all stay} \rangle$

[18]Since R^t Rebels share information on the equilibrium path, by Theorem 3, the belief updating in Table 8, Table 9, and Table 10, and the in-path behavior in Table 11, the relevant information is learnt by every Rebel eventually on the path.

in $\mathrm{K}_{2,1}^t$ if he knows $\#[R](\theta) \geq k$ and has played $\langle 1 \rangle$ in the previous periods O^t; otherwise, he will play $\langle i \rangle$. K_3^t is used for coordinating to play $\mathbf{r}$ or $\mathbf{s}$ contagiously.

Table 11: The sequences of actions played in K^t, $t \geq 1$ on the path

The sequences of actions played in $\mathrm{K}^t_{1,v}$ for $t \geq 1$ and for $v = 1, 2, ..., n$ on the path

Rebel i	i plays
i is certain $\#[R](\theta) < k$	$\langle$**all stay**$\rangle$
$i \notin R^t$ and is uncertain $\#[R](\theta) \geq k$	$\langle i \rangle$
$i \in R^t$ and is uncertain $\#[R](\theta) \geq k$	$\langle i \rangle$
i is certain $\#[R](\theta) \geq k$	$\langle i \rangle$

The sequences of actions played in $\mathrm{K}^t_{2,v}$ for $t \geq 1$ for $v = 1$ on the path

Rebel i	i plays
i is certain that $\#[R](\theta) < k$	$\langle$**all stay**$\rangle$
$i \notin R^t$ and is uncertain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$
$i \in R^t$ and is uncertain $\#[R](\theta) \geq k$	$\langle i \rangle$
i is certain that $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$

The sequences of actions played in $\mathrm{K}^t_{2,v}$ for $t \geq 1$ for $v = 2, ..., t+1$ on the path

Rebel i	i plays
i is certain that $\#[R](\theta) < k$	$\langle$**all stay**$\rangle$
$i \notin R^t$ and is uncertain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$
$i \in R^t$ and is uncertain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$
i is certain that $\#[R](\theta) \geq k$	$\langle i \rangle$

The sequences of actions played in K^t_3 for $t \geq 1$ on the path

Rebel i	i plays
i is certain that $\#[R](\theta) < k$	**s**
$i \notin R^1$ and is uncertain $\#[R](\theta) \geq k$	**s**
$i \in R^1$ and is uncertain $\#[R](\theta) \geq k$	**s**
i is certain that $\#[R](\theta) \geq k$	**r**

4.3.4 Out-of-path belief and learning on the path

In this section, I demonstrate examples to show the learning process on the equilibrium path and illustrate the out-of-path belief. Whenever Rebel i detects a deviation at period s, he forms the following belief:

$$\sum_{\theta\in\{\theta:\theta_j=I,j\notin G_i\}} \beta^{\pi,\tau}_{G_i}(\theta|h^{\dot{s}}_{G_i}) = 1, \text{ for all } \dot{s}\geq s. \tag{1}$$

Thus, if $\#I^0_i < k$, he will play **stay** forever after he detects a deviation. This out-of-path belief serves as a grim trigger.

The example below demonstrates how players learn on the equilibrium path. Let us take the configuration in Figure 2.[19]

Example 6. Let $k = 6$. Players have their prime numbers as $(x_1, x_2, ..., x_8) = (3, 5, 7, 11, 13, 17, 19, 23)$.

At K^0_1, Rebels 1, 2, 6, and 8 play $\langle$**all stay**$\rangle$; Rebels 4 and 5 play $\langle 4\rangle$ and $\langle 5\rangle$ respectively.

At K^0_2, Rebels 1, 2, 6, and 8 play $\langle$**all stay**$\rangle$; Rebels 4 and 5 play $\langle 4\rangle$ and $\langle 5\rangle$ respectively.

At $\mathrm{K}^0_{3,v}$, for $v = 1, .., n$, all Rebels play **s**.

Immediately after K^0_3, all Rebels are uncertain about $\#[R](\theta) \geq k$.

At O^1, $|\mathrm{O}^1| = 111546435$. Rebels 1, 2, 6, and 8 play

$$\langle \textbf{all stay}\rangle = \langle\, \overbrace{\mathbf{s}, ..., \mathbf{s}}^{111546435}\,\rangle.$$

Rebel 4 plays

$$\langle 1\rangle = \langle \overbrace{\mathbf{s}, ..., \mathbf{s}, \mathbf{r}}^{111546435}\rangle.$$

Rebel 5 plays

$$\langle \{4, 5, 6, 8\}\rangle = \langle \overbrace{\mathbf{s}, ..., \mathbf{s}, \underbrace{\mathbf{r}, \mathbf{s}, ..., \mathbf{s}}_{55913}}^{111546435}\rangle.$$

Immediately after O^1, Rebel 4 is certain $\#[R](\theta) \geq k$ while Rebels 1,2,5,6,8 are not.

At $\mathrm{K}^1_{1,v}$ for $v = 1, ..., n$, Rebel i plays $\langle i\rangle$. Rebels 1,2,4,5 are certain $\#[R](\theta) \geq k$, while the others are not.

[19] As Example 5 does.

At $\mathrm{K}^1_{2,1}$, Rebels 6 and 8 play $\langle 6 \rangle$ and $\langle 8 \rangle$ respectively. Rebels 1,2,4,5 play $\langle$**all stay**$\rangle$. Immediately after $\mathrm{K}^1_{2,1}$, Rebels 6 and 8 also know $\#[R](\theta) \geq k$.

At $\mathrm{K}^1_{2,2}$, all Rebels play $\langle$**all stay**$\rangle$.

At $\mathrm{K}^1_{2,v}$ for $v = 1, ..., n$, all Rebels play $\mathbf{r}$.

After $\mathrm{K}^1_{2,n}$, all Rebels play $\mathbf{r}$ forever.

5 Discussion

5.1 Cyclic networks

Scenarios in cyclic networks substantially differ from the acyclic counterpart. The free-rider problem could become intractable in cyclic networks. Let us consider the configuration in Figure 5.1.

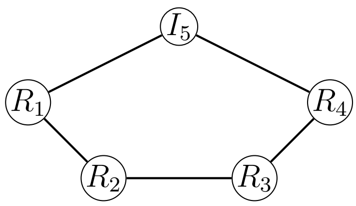

Figure 8: A configuration of the state and the network in which player 1,2,3,4 are Rebels while player 5 is an Inert.

Suppose $k = 4$ and the period at the beginning of 1-block is not terminal. Rebels 2 and 3 are θ-pivotal. From the perspective of Rebel 2's view, the type of player 5 could be Inert. Therefore, Rebel 2 does not know whether or not Rebel 1 is pivotal. Similarly, Rebel 2 does not know whether or not Rebel 3 is pivotal, *even though* player 3 is indeed θ-pivotal. Therefore there is no common knowledge of the free-rider problem at at the beginning of 1-block.

However, the common knowledge of the free-rider problem is restored when we cut the edge between player 4 and 5: Rebel 2 knows that he is the only θ-pivotal Rebel.

5.2 Payoff as signals

The hidden payoff assumption can be relaxed without changing the main result. One may consider a situation in which the stage payoff depends not only on players' joint efforts but also on a random shock, say the weather. To fix the idea, there is a public signal $y \in \{r, s\}$ generated according to the action profile. Let a Rebel's payoff function be $u_R(a_R, y)$ such that $u_R(\mathbf{stay}, r) = u_R(\mathbf{stay}, s) = u_0$. y is drawn from the distribution of

$$\begin{aligned}
p_{rr} &= \Pr(y = r | \#\{j : a_j \mathbf{revolt}\} \geq k) \\
p_{sr} = 1 - p_{rr} &= \Pr(y = s | \#\{j : a_j \mathbf{revolt}\} \geq k) \\
p_{ss} &= \Pr(y = s | \#\{j : a_j \mathbf{revolt}\} < k) \\
p_{rs} = 1 - p_{ss} &= \Pr(y = r | \#\{j : a_j \mathbf{revolt}\} < k)
\end{aligned}$$

such that

$$p_{rr} u_R(\mathbf{revolt}, r) + p_{sr} u_R(\mathbf{revolt}, s) > u_0 > p_{rr} u_R(\mathbf{revolt}, r) + p_{ss} u_R(\mathbf{revolt}, s),$$

and

$$0 \leq p_{rs} \leq 1, 0 \leq p_{ss} \leq 1.$$

The APEX equilibrium constructed for Theorem 2 is still a one in this scenario. Note that in that APEX equilibrium path, at most one **revolt** can occur at every period before some Rebel plays $\langle 1 \rangle$. This implies that the signal y is completely uninformative before some Rebel plays $\langle 1 \rangle$. If a Rebel i deviates to play $\langle 1 \rangle$ in O^t at some t in the hope gathering information from y, he will not learn the relevant information after O^t since the terminal period will come immediately after t-block. He will, however, learn the relevant information and play the ex-post efficient outcome if he is on the path, and hence he will not deviate.

6 Conclusion

I model a coordination game and illustrate the learning processes generated by strategies in a (weak) sequential equilibrium to answer the question proposed in the beginning: what kind of networks can conduct coordination in a collective action with information barrier. In the equilibrium, players transmit the relevant information by encoding such information by their actions as time goes by. Since there might be an negative expected payoff in coding information, the potential free-rider problems might occur to impede the learning process. My result show that if the network is acyclic, players can always learn the underlying relevant information and conduct the coordination only by actions. In cyclic networks, however, what kinds of equilibrium strategies can lead to learning the relevant information still remains to be answered.

The construction of the communication protocol by actions exploits the assumption of the common knowledge of the network and the finite type space. Since the relevant information has been parametrized as a threshold in the stage game, players can acquire this information by jointly incrementally reporting their own private information period by period. The major punishment to deter deviation is then the joint shifting to play that same action as the stopping to update information. The threshold game thus seems a potential model in proofing that a communication protocol by actions not only leads a learning process but also constitutes an equilibrium to reveal the relevant information in finite time.

Existing literatures in political science and sociology have recognized the importance of social network in influencing individual's behavior in participating social movements ([Passy, 2003][McAdam, 2003][Siegel, 2009]). This paper views networks as routes for communication in which rational individuals initially have local information but they can influence nearby individuals by taking actions. Such influence may take long time to travel across individuals and the whole process incurs inefficient outcomes in many periods. A characterization in the speed of information transmission across a network is not answered here, although it is an important topic in investigating the most efficient way to let the information be spread. This question would remain for the future research.

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A Appendix

A.1 The APEX equilibrium for Theorem 2

A.1.1 Equilibrium path

By definition of information hierarchy,

$$\begin{aligned} I_i^t &= \bigcup_{k_1 \in G_i} \bigcup_{k_2 \in G_{k_0}} \cdots \bigcup_{k_t \in G_{k_{t-1}}} I_{k_t}^1 \\ &= \{j \in [R](\theta) : \exists \text{ a path } (i, k_1 ... k_l, j) \text{ s.t. } 0 \leq l \leq t-1 \text{ and } \theta_i = \theta_{k_1} = ... = \theta_{k_l} = R\} \end{aligned}$$

Let us define several notions.

Definition A.1 (Extended tree by I_i^t)**.**

$$\begin{aligned} X_i^t \equiv \{ & j \in N : \\ & \exists \textit{ a path } (i, k_1 ... k_l, k_{l+1}) \textit{ s.t. } k_{l+1} = j,\ l \geq t-1,\ \{i, k_1, ..., k_t\} \subset I_i^t\} \cup I_i^t \end{aligned}$$

X_i^t is the set of all possible Rebels in G given information I_i^t.

Definition A.2 (The tree rooted in i and spanning in the direction toward j)**.**

$$TR_{ij} \equiv \{v \in N : \textit{there is a path from } i \textit{ to } v \textit{ through } j,\ j \in G_i\} \cup \{i, j\}$$

Definition A.3 (Extended vertices outside I_i^t in TR_{ij})**.**

$$Y_{ij}^t \equiv TR_{ij} \cap (X_i^t \setminus I_i^t)$$

Definition A.4 (i's capable neighbors by I_i^t)**.**

$$D_i^t \equiv \{j \in G_i : Y_{ij}^t \neq \emptyset\}$$

Definition A.5 (Finite register machine)**.** *A finite register machine for i consists of finite registers Σ. A register is a tuple*

$$(\Omega, \times_{G_i} A_R, f, \lambda),$$

in which Ω are sets of events induced by H_i. $\times_{G_i} A_R$ is the sets of input. $f : \Omega \to A_R$ assigns an action to each event. $\lambda : \Omega \times \times_{G_i} A_R \to \Sigma$ is the transition function. There is a set of initial registers.

i's register specifies i's action according his information at a certain period but does not characterize i's information transition. The register machine here is more like the *switch function* instead of the finite automata. The information of i up to period s is $P_i(\theta) \times \{h_{G_i}^s\} \times H_{N \setminus G_i}^s$ characterized in Section 2.

Definition A.6 (m-register in t-block)**.** *A m-register in a (sub)block or a division is the register in the m-th period in that (sub)block or division.*

To shorten the notation, denote $m \dashv \Gamma$ as the m-register in the (sub)block or division Γ.

Definition A.7 (Terminal **r**)**.** *The terminal **r** is a register such that the image of f is $\{$**revolt**$\}$ and the image of λ is a singleton containing itself.*

Definition A.8 (Terminal **s**)**.** *The terminal **s** is a register such that the image of f is $\{$**stay**$\}$ and the image of λ is a singleton containing itself.*

The equilibrium will be represented as a finite register machine. Moreover, though players act as if acting a whole sequence, they in fact act period by period. For convenience, for any finite sequence of action $\langle\rangle$, denote $\langle\rangle_m$ as the m-th (counting from the beginning) component in $\langle\rangle$, and denote $\langle\rangle(m)$ as the prefix of $\langle\rangle$ with length m. Let us also shorten action **revolt** to be **r** and **stay** to be **s**.

Initial registers The initial register for each Rebel is $1 \dashv \mathrm{K}^0_1$, which is defined in the next section.

Registers in K^0

Table 12: The $m \dashv \mathrm{K}_1^0$ on the path

$1 \leq m \leq |\mathrm{K}_1^0| - 1$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$\#X_i^1 < k$	$\langle$**all stay**$\rangle_m$		terminal **s**
$i \notin R^1$, $\#X_i^1 \geq k$, $I_i^1 < k$	$\langle$**all stay**$\rangle_m$		$m+1 \dashv \mathrm{K}_1^0$
$i \in R^1$, $\#X_i^1 \geq k$, $I_i^1 < k$	$\langle i \rangle_m$		$m+1 \dashv \mathrm{K}_1^0$
$i \in R^1$, $\#X_i^1 \geq k$, $I_i^1 \geq k$	$\langle i \rangle_m$		$m+1 \dashv \mathrm{K}_1^0$

$m = |\mathrm{K}_1^0|$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$\#X_i^1 < k$		$\langle$**all stay**$\rangle_m$		terminal **s**
$i \notin R^1$, $\#X_i^1 \geq k$, $I_i^1 < k$	all j play $\langle$**all stay**$\rangle(m-1)$	$\langle$**all stay**$\rangle_m$	all j play **s**	terminal **s**
$i \notin R^1$, $\#X_i^1 \geq k$, $I_i^1 < k$	$\exists j$ plays $\langle j \rangle(m-1)$	$\langle$**all stay**$\rangle_m$	such j plays $\langle j \rangle_m$	$1 \dashv \mathrm{K}_2^0$
$i \in R^1$, $\#X_i^1 \geq k$, $I_i^1 < k$		$\langle i \rangle_m$		$1 \dashv \mathrm{K}_2^0$
$i \in R^1$, $\#X_i^1 \geq k$, $I_i^1 \geq k$		$\langle i \rangle_m$		$1 \dashv \mathrm{K}_2^0$

Table 13: The $m \dashv \mathrm{K}_2^0$ on the path

$1 \leq m < |\mathrm{K}_2^0|$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$i \notin R^1$		$\langle\textbf{all stay}\rangle_m$		$m+1 \dashv \mathrm{K}_2^0$
$i \in R^1$, $I_i^1 < k$	$\exists j \in G_i$, j plays $< j >_j = s$	$\langle\textbf{all stay}\rangle_m$		$m+1 \dashv \mathrm{K}_2^0$
$i \in R^1$, $I_i^1 < k$	$\forall j \in G_i$, j plays $< j >_j = r$	$\langle i\rangle_m$		$m+1 \dashv \mathrm{K}_2^0$
$i \in R^1$, $I_i^1 \geq k$		$\langle\textbf{all stay}\rangle_m$		$m+1 \dashv \mathrm{K}_2^0$

$m = |\mathrm{K}_2^0|$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$i \notin R^1$	$\forall j \in G_i$, j plays $\langle j\rangle(m-1)$	$\langle\textbf{all stay}\rangle_m$	$\forall j \in G_i$, j plays $\langle j\rangle_m$	$1 \dashv \mathrm{K}_3^0$
$i \notin R^1$	$\exists j \in G_i$, j plays $\langle\textbf{all stay}\rangle(m-1)$	$\langle\textbf{all stay}\rangle_m$	such j plays $\langle\textbf{all stay}\rangle_m$	terminal $\mathbf{r}$
$i \in R^1$, $I_i^1 < k$	$\forall j \in G_i$, j play $\langle j\rangle(m-1)$	$\langle i\rangle_m$	$\forall j \in G_i$, j plays $\langle j\rangle_m$	$1 \dashv \mathrm{K}_3^0$
$i \in R^1$, $I_i^1 < k$	$\exists j \in G_i$, j plays $\langle\textbf{all stay}\rangle(m-1)$	$\langle i\rangle_m$	such j plays $\langle\textbf{all stay}\rangle_m$	terminal $\mathbf{r}$
$i \in R^t$, $I_i^1 \geq k$		$\langle\textbf{all stay}\rangle_m$		terminal $\mathbf{r}$

Table 14: The $m \dashv \mathrm{K}_3^0$ on the path

$1 \leq m < |\mathrm{K}_3^0|$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
	$\mathbf{s}$	$\forall j$ play $\mathbf{s}$	$m+1 \dashv \mathrm{K}_3^0$
	$\mathbf{s}$	$\exists j$ play $\mathbf{r}$	terminal $\mathbf{r}$

$1 \leq m = |\mathrm{K}_3^0|$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
	$\mathbf{s}$	$\forall j$ play $\mathbf{s}$	$1 \dashv \mathrm{O}^1$
	$\mathbf{s}$	$\exists j$ play $\mathbf{r}$	terminal $\mathbf{r}$

Registers in O^t Let $m_i = |\mathrm{O}^t| - x_{I_i^t}$ be the period in which i report I_i^t. I.e. m_i is the period where $\mathbf{r}$ occurs in $\langle I_i^t \rangle$. Denote $G_i(m) = \{j \in G_i : m_j < m\}$. Define $I_i^{t+1}(m) \equiv I_i^t \cup \bigcup_{j \in G_i(m)} I_j^t$ to be the information of i up to the m-th period in O^t. Define $X_i^{t+1}(m)$ to be the extended tree from $I_i^t(m)$ in the same way as that in Definition A.1, and define $Y_{ij}^t(m)$ and $D_i^t(m)$ accordingly.

Table 15: The $m \dashv \mathrm{O}^t$ on the path, where $1 \leq m < |\mathrm{O}^t|$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$i \notin R^t$		$\langle \textbf{all stay} \rangle_m$		$m+1 \dashv \mathrm{O}^t$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I_i^{t+1}(m) < k$, $X_i^{t+1}(m) < k$	$\langle \textbf{all stay} \rangle_m$		terminal $\mathbf{s}$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I_i^{t+1}(m) < k$, $X_i^{t+1}(m) \geq k$	$\langle I_i^t \rangle_m$		$m+1 \dashv \mathrm{O}^t$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I_i^{t+1}(m) \geq k$, $X_i^{t+1}(m) \geq k$	$\langle 1 \rangle_m$		$m+1 \dashv \mathrm{O}^t$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I_i^{t+1}(m) \geq k-1$, $X_i^{t+1}(m) \geq k$, $D_i^t = 1$	$\langle 1 \rangle_m$		$m+1 \dashv \mathrm{O}^t$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I_i^{t+1}(m) \geq k-1$, $X_i^{t+1}(m) \geq k$, $D_i^t > 1$	$\langle I_i^t \rangle_m$		$m+1 \dashv \mathrm{O}^t$
$i \in R^t$, the free rider	$X_i^{t+1}(m) \geq k$	$\langle 1 \rangle_m$		$m+1 \dashv \mathrm{O}^t$
$i \in R^t$, the free rider	$X_i^{t+1}(m) < k$	$\langle \textbf{all stay} \rangle_m$		terminal $\mathbf{s}$
$i \in R^t$, i is $k-1$-pivotal	$X_i^{t+1}(m) \geq k$	$\langle 1 \rangle_m$		$m+1 \dashv \mathrm{O}^t$
$i \in R^t$, i is $k-1$-pivotal	$X_i^{t+1}(m) < k$	$\langle \textbf{all stay} \rangle_m$		terminal $\mathbf{s}$

Table 16: The $m \dashv \mathrm{O}^t$ on the path, where $m = |\mathrm{O}^t|$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$i \notin R^t$		$\langle \textbf{all stay} \rangle_m$		$1 \dashv \mathrm{K}^t_{1,1}$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I_i^{t+1}(m) < k$, $X_i^{t+1}(m) < k$	$\langle \textbf{all stay} \rangle_m$		terminal $\mathbf{s}$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I_i^{t+1}(m) < k-1$, $X_i^{t+1}(m) \geq k$	$\langle I_i^t \rangle_m$		$1 \dashv \mathrm{K}^t_{1,1}$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I_i^{t+1}(m) \geq k$, $X_i^{t+1}(m) \geq k$	$\langle 1 \rangle_m$		$1 \dashv \mathrm{K}^t_{1,1}$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I_i^{t+1}(m) \geq k-1$, $X_i^{t+1}(m) \geq k$, $D_i^t = 1$	$\langle 1 \rangle_m$		$1 \dashv \mathrm{K}^t_{1,1}$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I_i^{t+1}(m) \geq k-1$, $X_i^{t+1}(m) \geq k$, $D_i^t > 1$	$\langle I_i^t \rangle_m$		$1 \dashv \mathrm{K}^t_{1,1}$
$i \in R^t$, the free rider	$X_i^{t+1}(m) \geq k$	$\langle 1 \rangle_m$		$1 \dashv \mathrm{K}^t_{1,1}$
$i \in R^t$, the free rider	$X_i^{t+1}(m) < k$	$\langle \textbf{all stay} \rangle_m$		terminal $\mathbf{s}$
$i \in R^t$, i is $k-1$-pivotal	$X_i^{t+1}(m) \geq k$	$\langle 1 \rangle_m$		$1 \dashv \mathrm{K}^t_{1,1}$
$i \in R^t$, i is $k-1$-pivotal	$X_i^{t+1}(m) < k$	$\langle \textbf{all stay} \rangle_m$		terminal $\mathbf{s}$

Registers in K^t for $t \geq 1$

Table 17: The $m \dashv \mathrm{K}^t_{1,v}$ for $v = 1, ..., n$ on the path

$1 \leq m < |\mathrm{K}^t_{1,v}|$, where $v = 1, ..., n$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$X_i^{t+1} < k$	$\langle$**all stay**$\rangle_m$		terminal $\mathbf{s}$
$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\exists j \in G_i, j = m$ such that $a_j = \mathbf{s}$	terminal $\mathbf{s}$
$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\forall j \in G_i$ such that $a_j = \langle j \rangle_m$	$m + 1 \dashv \mathrm{K}^t_{1,v}$

$m = |\mathrm{K}^t_{1,v}|$, where $v = 1, ..., n - 1$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$X_i^{t+1} < k$	$\langle$**all stay**$\rangle_m$		terminal $\mathbf{s}$
$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\exists j \in G_i, j = m$ such that $a_j = \mathbf{s}$	terminal $\mathbf{s}$
$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\forall j \in G_i$ such that $a_j = \langle j \rangle_m$	$1 \dashv \mathrm{K}^t_{1,v+1}$

$1 \leq m < |\mathrm{K}^t_{1,n}|$

$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\exists j \in G_i, j = m$ such that $a_j = \mathbf{s}$	terminal $\mathbf{s}$
$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\forall j \in G_i$ such that $a_j = \langle j \rangle_m$	$m + 1 \dashv \mathrm{K}^t_{1,n}$

$m = |\mathrm{K}^t_{1,n}|$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\exists j \in G_i, j = m$ such that $a_j = \mathbf{s}$	terminal $\mathbf{s}$
$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\forall j \in G_i$ such that $a_j = \langle j \rangle_m$	$1 \dashv \mathrm{K}^t_{2,1}$

Table 18: The $m \dashv \mathrm{K}^t_{2,v}$ for $v = 1, ..., t+1$ on the path

$1 \leq m < |\mathrm{K}^t_{2,v}|$, where $v = 1, ..., t+1$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$I_i^{t+1} < k$	$\exists j,\ \langle j \rangle_j = \mathbf{s}$	$\langle \textbf{all stay} \rangle_m$		$m+1 \dashv \mathrm{K}^t_{2,v}$
$I_i^{t+1} < k$	$\forall j,\ \langle j \rangle_j = \mathbf{r}$	$\langle i \rangle_m$		$m+1 \dashv \mathrm{K}^t_{2,v}$
$I_i^{t+1} \geq k$		$\langle \textbf{all stay} \rangle_m$		$m+1 \dashv \mathrm{K}^t_{2,v}$

$m = |\mathrm{K}^t_{2,v}|$, where $v = 1, ..., t$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$I_i^{t+1} < k$	$\exists j \in G_i,\ \langle j \rangle_j = \mathbf{s}$	$\langle \textbf{all stay} \rangle_m$		$1 \dashv \mathrm{K}^t_{2,v+1}$
$I_i^{t+1} < k$	$\forall j \in G_i,\ \langle j \rangle_j = \mathbf{r}$	$\langle i \rangle_m$		$1 \dashv \mathrm{K}^t_{2,v+1}$
$I_i^{t+1} \geq k$		$\langle \textbf{all stay} \rangle_m$		$1 \dashv \mathrm{K}^t_{2,v+1}$

$1 \leq m < |\mathrm{K}^t_{2,t+1}|$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$I_i^{t+1} < k$	$\exists j \in G_i,\ \langle j \rangle_j = \mathbf{s}$	$\langle \textbf{all stay} \rangle_m$		$m+1 \dashv \mathrm{K}^t_{2,t+1}$
$I_i^{t+1} < k$	$\forall j \in G_i,\ \langle j \rangle_j = \mathbf{r}$	$\langle i \rangle_m$		$m+1 \dashv \mathrm{K}^t_{2,t+1}$
$I_i^{t+1} \geq k$		$\langle \textbf{all stay} \rangle_m$		$m+1 \dashv \mathrm{K}^t_{2,t+1}$

$m = |\mathrm{K}^t_{2,t+1}|$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$I_i^{t+1} < k$	$\exists j \in G_i,\ \langle j \rangle_j = \mathbf{s}$	$\langle \textbf{all stay} \rangle_m$		terminal **r**
$I_i^{t+1} < k$	$\forall j \in G_i,\ \langle j \rangle_j = \mathbf{r}$	$\langle i \rangle_m$		$1 \dashv \mathrm{K}^t_{3,1}$
$I_i^{t+1} \geq k$		$\langle \textbf{all stay} \rangle_m$		terminal **r**

Table 19: The $m \dashv \mathrm{K}_3^t$ on the path

$1 \leq m < |\mathrm{K}_3^t|$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
	$\mathbf{s}$	$\forall j \in G_i$, j plays $\mathbf{s}$	$m + 1 \dashv \mathrm{K}_3^t$
	$\mathbf{s}$	$\exists j \in G_i$, j plays $\mathbf{r}$	terminal $\mathbf{r}$

$m = |\mathrm{K}_3^t|$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
	$\mathbf{s}$	$\forall j \in G_i$, j plays $\mathbf{s}$	$1 \dashv \mathrm{O}^{t+1}$
	$\mathbf{s}$	$\exists j \in G_i$, j play $\mathbf{r}$	terminal $\mathbf{r}$

A.2 Missing proofs

Proof of Lemma 2.1

Proof. The proof is done by contraposition. Suppose Rebels' strategies constitute an APEX equilibrium. By definition of the APEX equilibrium, at every θ, there is a period T^θ when all Rebels' actions start to repeat themselves. Let $T = \max_{\theta\in\Theta} T^\theta$. For Rebel i, let $T_i = T + 1$, and suppose $0 < \sum_{\theta:\#[R](\theta)\geq k} \beta^{\pi,\tau^*}_{G_i}(\theta|h^s_{G_i}) < 1$ for some $s \geq T_i$. Then this Rebel assigns positive weight at some $\dot{\theta} \in \{\theta : \#[R](\theta) < k\}$ and some positive weight at some $\ddot{\theta} \in \{\theta : \#[R](\theta) \geq k\}$ at period s. Note that i has already known θ_j if $j \in G_i$, and therefore i assigns positive weight at some $\theta' \in \{\theta : \#[R](\theta) < k, \theta_l = R, l \notin G_i\}$ and positive weight at some $\theta'' \in \{\theta : \#[R](\theta) < k, \theta_l = I, l \notin G_i\}$. Since all Rebels' actions start to repeat themselves at period T, i cannot update information afterwards. Suppose i's continuation strategy is to continuously play **revolt**, then this is not ex-post efficient when $\#[R](\theta) < k$; suppose i's continuation strategy is to continuously play **stay**, then this is not ex-post efficient when $\#[R](\theta) \geq k$. □

Proof of Theorem 1

Proof. Let τ^* be the following strategy. After the nature moves, a Rebel i plays **revolt** if he has no Inert neighbor; i plays **stay** forever if he has an Inert neighbor. After the first period, if i has not detected a deviation and observes that all his Rebel neighbors play **revolt** continuously previously, he plays **revolt** in the current period; otherwise, he plays **stay** afterwards and forever. If a Rebel j deviates, then j plays **stay** afterwards and forever.

At period s, according to τ^*, if i has not detected a deviation, but he observe his Rebel neighbors plays **stay** in the current period, he forms the belief of

$$\sum_{\theta:\#[R](\theta)\geq k} \beta^{\pi,\tau^*}_{G_i}(\theta|h^s_{G_i}) = 0$$

afterwards and forever. Therefore, he plays **stay** afterwards and forever as his best response.

At period s, if a Rebel detects a deviation, or he has deviated, to play **stay** afterwards and forever is his best response since at least one player will play **stay** afterwards and forever.

Since the network is finite with n vertices, if all players do not deviate, after period n, each Rebel plays **revolt** and gets payoff 1 forever if $\theta \in \{\theta : \#[R](\theta) \geq k\}$; each Rebels plays **stay** and gets payoff 0 forever if $\theta \in \{\theta : \#[R](\theta) < k\}$. However, a Rebel who has deviated surely gets payoff 0 forever after period n. Therefore, there is a $0 < \delta < 1$ large enough to impede Rebels to deviate.

To check if τ^* and $\{\beta_{G_i}^{\pi,\tau^*}(\theta|h_{G_i}^s)\}_{i\in N}$ satisfy full consistency[20], take any $0 < x < 1$ such that Rebels play τ^* with probability $1-x$ and play other behavior strategies with probability x. Clearly, when $x \to 0$, the belief converges to $\{\beta_{G_i}^{\pi,\tau^*}(\theta|h_{G_i}^s)\}_{i\in N}$. Since the out-of-path strategy is the best response for both of the Rebel who detects deviation and the Rebel who makes deviation, for arbitrary beliefs they hold, τ^* is a sequential equilibrium. $\square$

Proof of Lemma 4.1

Proof. I show that if $i \notin R^{t-1}$ then $i \notin R^t$ for all t.

By definition,

$$\begin{aligned} G_i^t &= \bigcup_{k_1\in G_i} \bigcup_{k_2\in G_{k_0}} \cdots \bigcup_{k_t\in G_{k_{t-1}}} G_{k_t}^1 \\ &= \{j \in N : \exists \text{ a path } (i, k_1 \ldots k_l, j) \text{ such that } l \leq t-1 \text{ and } \theta_i = \theta_{k_1} = \ldots = \theta_{k_l} = R\}, \end{aligned}$$

while

$$\begin{aligned} I_i^t &= \bigcup_{k_1\in G_i} \bigcup_{k_2\in G_{k_0}} \cdots \bigcup_{k_t\in G_{k_{t-1}}} I_{k_t}^1 \\ &= \{j \in [R](\theta) : \exists \text{ a path } (i, k_1 \ldots k_l, j) \text{ such that } l \leq t-1 \text{ and } \theta_i = \theta_{k_1} = \ldots = \theta_{k_l} = R\}. \end{aligned}$$

The above equality says that, at $t = \dot{t}$, if $i \notin R^{\dot{t}}$, then there is a j such that the Rebels, who can be reached by $\dot{t}$ consecutive edges from i, can be also reached by $\dot{t}$ consecutive edges from j. Therefore, if there are new Rebels who can be reached from i at any $\ddot{t} > \dot{t}$

[20]Krep and Wilson (1982)

by $\ddot{t}$ consecutive edges, those new ones can be also be reached by $\ddot{t}$ consecutive edges by j. Hence, $i \notin R^{\ddot{t}}$. □

Proof of Theorem 3

Proof. Since the network is finite, θ has strong connectedness, and $[R](\theta) \neq \emptyset$, there is a minimum t_i such that $I_i^{t_i} = [R](\theta)$ for each i by the definition of I_i^t. Let $P = \arg\min_{i \in N}\{t_1, ..., t_n\}$ with generic element p. Therefore $I_p^{t_p} = [R](\theta)$. I show that $p \in R^{t_p-1}$ to complete the proof. I prove it by contradiction. If $p \notin R^{t_p-1}$, then $I_p^{t_p-1} \subseteq G_j^{t_p-1}$ for some $j \in G_p$. Then, all the Rebels in TR_{jp} are in $G_j^{t_p-1}$, but there exist Rebels in TR_{pj} who are in $G_j^{t_p-1}$ but not in $I_p^{t_p-1}$. This is because the network is acyclic and $I_p^{t_p-1} \subset [R](\theta)$. But then $p \notin P$ since $I_j^{t_j-1} = [R](\theta)$ already. I then conclude that $p \in R^{t_p-1}$.

□

Proof of Lemma 4.2

Proof. The proof is by contradiction. Suppose that, at t-block and before T^θ, there are three or more θ-pivotal Rebels. Since θ has strong connectedness, there are three of them, p_1, p_2, p_3, with the property $p_1 \in G_{p_2}$ and $p_2 \in G_{p_3}$.

Since the network is acyclic, $p_1 \notin G_{p_3}$ and $p_3 \notin G_{p_1}$. Since p_1 is θ-pivotal, $I^t \subset [R](\theta)$ and $I_p^{t+1} = [R](\theta)$. It implies that, in $TR_{p_1p_2}$, p_1 can reach all Rebels by $t+1$ edges, but cannot reach all of them by t edges. The same situation applies to p_3. However, it means that p_2 can reach all Rebels in $TR_{p_1p_1}$ by t edges and reach all Rebels in $TR_{p_1p_1}$ by t edges, and hence $I_{p_2}^t = [R](\theta)$. It contradict to the definition of θ-pivotal Rebel.

□

Proof of Lemma 4.3

Proof. A θ-pivotal p knows that $p^{'} \in G_i$ if $p^{'}$ is another one. p picks a neighbor p' and checks whether or not $[R](\theta) \subseteq I_p^t \cup I_{p'}^t$ for all possible $I_{p'}^t$. By common knowledge of the network, p knows $G_{p'}^t$. Since p is θ-pivotal, he is certain that all the Rebel in the direction from p toward $p^{'}$ is in $G_{p^{'}}^t$ and hence in $I_{p^{'}}^t$. Then p can check whether or not $[R](\theta) \subseteq I_p^t \cup I_{p^{'}}^t$ for all possible $I^t_{p'}$. If so, then p knows p' is also θ-pivotal by the definition of θ-pivotal. Similarly, a θ-pivotal p' can do the same procedure. Therefore, if there are two θ-pivotal p and p', they commonly know that they are θ-pivotal. They commonly know this at the beginning of t-block since they know I^t_p and $I^t_{p'}$ by the construction of information hierarchy. $\square$

Proof of Theorem 2 I begin with the following lemmas stating that all Rebels eventually learn the relevant information on the path.

Lemma A.1. *If the network is acyclic and if the θ has strong connectedness, then the equilibrium path specified in Section A.1.1 is an APEX strategy.*

Proof. Firstly, suppose θ satisfies $\#[R](\theta) < k$. I show that, all Rebels will enter terminal **s** eventually without entering terminal **r**. Let p be the Rebel defined in the proof of Theorem 3 so that $I^{t_p}_p = [R](\theta)$, where p is one of the earliest Rebels who knows $\#[R](\theta) < k$. I claim that $\#X^{t_p}_p < k$ if and only if $I^{t_p}_p = [R](\theta)$. For the only if part, the proof is by way of contradiction. If not, by the full support on strong connectedness, there is a possible Rebel outside $I^{t_p}_p$, and therefore p is uncertain $\#[R](\theta) < k$. For the if part, note that $I^{t_p}_p \subset X^{t_p}_p$ and therefore $\#I^{t_p}_p < \#X^{t_p}_p < k$. I also claim that $\#X^{t_p}_p(m) < k$ if and only $I^{t_p}_p(m) = [R](\theta)$. The proof is exactly the same as the noted above by replacing $I^{t_p}_p$ to $I^{t_p}_p$ and $I^{t_p}_p(m)$ to $I^{t_p}_P(m)$

Referring to Table 12 to Table 16, whenever there is a p so that $\#X^{t_p}_p < k$, p plays **stay** forever. It implies that all Rebels enter terminal **s** immediately after $\mathrm{K}^t_1, t \geq 0$. Notice that Rebels entering to terminal **r** only after some period after K^t_1 and therefore all Rebels will enter terminal **s** before terminal **r**.

Secondly, suppose θ satisfies $\#[R](\theta) \geq k$. I show that all Rebels will enter terminal **r** eventually. Note first that if there is a Rebel p so that $\#I^1_p \geq k$, all Rebels enter terminal **r** after K^0_3 by referring to Table 12, Table 13, and Table 14. At $t > 0$, if there is a Rebel p who has play $\langle 1 \rangle$ at O^t, by the postulate of $\#[R](\theta) \geq k$, after K^t_3, all Rebels enter terminal **r** according to the equilibrium path specified in Table 15, Table 16, Table 17, Table 18, and Table 19. There must be some Rebel $p \in R^t$ who plays $\langle 1 \rangle$ at O^t for some t by the same argument in the proof of Theorem 3. $\square$

Due to Lemma A.1, define $T^{\theta}_{\tau^*}$ as the earliest period at which all Rebels play ex-post efficient outcome afterwards according to an APEX equilibrium τ^*. For simplicity, I suppress the notation $\beta^{\pi,\tau}_{G_i}(\theta|h^s_{G_i})$ to $\beta^{\tau}_{G_i}(\theta|h^s_{G_i})$ and the notation $\alpha^{\pi,\tau}_{G_i}(\theta, h^s|\theta_{G_i}, h^s_{G_i})$ to $\alpha^{\tau}_{G_i}(\theta, h^s|h^s_{G_i})$. If $P(\theta)$ is a property of θ, define

$$\beta^{\tau}_{G_i}(P(\theta)|h^s_{G_i}) \equiv \sum_{\theta\in\{\theta:P(\theta)\}} \beta^{\tau}_{G_i}(\theta|h^s_{G_i}).$$

Furthermore, if m, s are periods and $m > s$, denote $h^{m|s}$ as a history in H^m so that its sequence til period s is h^s. Denote $\tau'|^s_{\tau}$ is a strategy following τ til period s.

Claim 1. *Suppose Rebel i follows an APEX equilibrium τ^* til period s. If there is a strategy $(\tau_i, \tau_{-i})|^s_{\tau^*}$ generating a history $h^{m|s}, \infty > m > s$ so that i will be uncertain about the relevant information and stop belief updating after m, then Rebel i will not deviate to $(\tau|^s_{\tau^*}$ if $\delta \in (0,1)$ is sufficiently high.*

Proof. Denote $\beta^{\tau|^s_{\tau^*}}_{G_i}(\theta|h^{m|s}, h^s_{G_i})$ as i's belief about θ at m following $h^{m|s}$ induced by $\tau|^s_{\tau^*}$. By the postulate, $0 < \beta^{\tau|^s_{\tau^*}}_{G_i}(\#[R](\theta)|h^{m|s}, h^s_{G_i}) < 1$. From perspective that i holds a belief of $\beta^{\tau^*}_{G_i}(\#[R](\theta) \geq k|h^s_{G_i})$ at period s, $h^{m|s}$ can be thought of an imperfect signal at period m to infer whether or not $\#[R](\theta) \geq k$: if $\#[R](\theta) \geq k$, $h^{m|s}$ occurs with probability η and does not with probability $1-\eta$, where $0 < \eta < 1$; if $\#[R](\theta) < k$, $h^{m|s}$ occurs with probability μ and does not with probability $1-\mu$, where $0 < \mu < 1$. Denote $M = \max\{m, T^{\theta}_{\tau^*}\}$. Rebel i's maximum expected stage-game payoff starting from M by following $h^{m|s}$ calculated at period s is

$$V = \max\{\eta\beta^{\tau^*}_{G_i}(\#[R](\theta) \geq k|h^s_{G_i}) - \mu\beta^{\tau^*}_{G_i}(\#[R](\theta) < k|h^s_{G_i}), 0\}.$$

The first term $\eta\beta^{\tau^*}_{G_i}(\#[R](\theta) \geq k|h^s_{G_i}) - \mu\beta^{\tau^*}_{G_i}(\#[R](\theta) < k|h^s_{G_i})$ is i's expected stage-game payoff if all Rebels play **revolt** afterwards starting from M. The second term 0 is the one by playing **stay** afterwards. Rebel i's expected stage-game payoff starting from M by following τ^* calculated at period s is

$$\beta^{\tau^*}_{G_i}(\#[R](\theta) \geq k|h^s_{G_i}) > V.$$

The inequality above is due to $0 < \eta < 1, 0 < \mu < 1$. There is a difference in present value of

$$W(\delta) = \frac{\delta^{M-s}(\beta_{G_i}^{\tau^*}(\#[R](\theta) \geq k | h_{G_i}^s) - V)}{1-\delta}.$$

Denote L as the summation of all gains from deviation calculated from period s to period M. L is finite since the stage-game payoff is finite and $M - s$ is finite. Taking sufficiently high $\delta \in (0,1)$ so that $W(\delta) > L$ will deter this deviation. □

Claim 2.

Proof. □

If Rebel i makes detectable deviation. $I_i^t \geq k$. In O^t, playing $\langle I \rangle$ for some $I \subset N$ incurs more negative expected payoff than does $\langle 1 \rangle$. In K_1^t after O^t, detectable deviation is the sequence other than $\langle i \rangle$ or $\langle$**all stay**$\rangle$. deviation will be detected by all neighbors, all neighbors play **s** contagiously, the expected continuation payoff after some period S is zero. On the path, the one is $1/(1-\delta)$. In K_2^t, detectable deviation is the sequence other than $\langle i \rangle$ or $\langle$**all stay**$\rangle$. deviation will be detected by all neighbors, all neighbors play **s** contagiously, the expected continuation payoff after some period S is zero. In K_1^0, detectable deviation is the sequence other than $\langle i \rangle$ or $\langle$**all stay**$\rangle$. deviation will be detected by all neighbors, all neighbors play **s** contagiously, the expected continuation payoff after some period S is zero.

If $I_i^t < k$. suppose there are $m > 0$ Rebel neighbors detects this deviation. The expected continuation payoff is at most $\beta_{G_i}(\#[R](\theta) \geq k + m) | h_{G_i}^s)$