0.1 Proof for equilibrium

Claim 0.1.0.1. *For $\#Ex_{I_i^{m,t}} \cup I_i^{m,t}| \geq k$, where m is a period in reporting period. If i report $\langle 1 \rangle$, then i has known $\#[Rebels](\theta) \geq s$ or $\#[Rebels](\theta) < s$ after reporting period, and thus the coordination can be either initiated in t-block or be never initiated.*

Proof. By directly checking the equilibrium path, we have

1. if $\#I_i^{|RP^t|,t} \geq s$, then the coordination can be initiated by such i.

2. if $\#I_i^{|RP^t|,t} = k-1$, and if there is one more node who reported $\langle 1 \rangle$, then the coordination can be initiated by i.

3. if $\#I_i^{|RP^t|,t} = k-1$, and if there are no nodes who reported in current period, then $\#I_i^{|RP^t|,t} = \#I_i^t = k-1$. We now check the conditions guiding i to **POST-CHECK**.

   - If i is coming from the conditions in **MAIN**, it means that there is no further H-node outside I_i^{t-1}, and thus outside $\bigcup_{k \in I_i^{t-1}} G_k$.
   - If i is coming from the conditions in **CHECK.0**, it means that there is no further H-node outside $\bigcup_{k \in I_i^{t-1}} G_k \cap R^0$, and thus outside $\bigcup_{k \in I_i^{t-1}} G_k$.
   - If i is coming from the conditions in **CHECK.m**, it means that there is no further H-node outside $\bigcup_{k \in I_i^{t-1}} G_k \cap R^0$, and thus outside $\bigcup_{k \in I_i^{t-1}} G_k$.

   Then $\#I_i^t < k$, but $I_i^t = \bigcup_{k \in I_i^{t-1}} N_k \cap R^0$, and hence $\#R^0 < k$, and thus the coordination can never happen.

□

Lemma 0.1.1. *If the state has strong connectivity, then for all n-person repeated k-Threshold game with parameter $1 \leq k \leq n$ played in any finite connected undirected network without circle, the equilibrium path is approaching efficient.*

Proof. We want to show that when θ satisfying $\#[Rebels](\theta) \geq k$, all the Rebels play **revolt** eventually; when θ satisfying $\#[Rebels](\theta) < k$, all the Rebels play **stay** eventually.

1. If all the Rebels only play $\langle I^{t-1} \rangle$ or $\langle$**stay**$\rangle$ in reporting period for all $t \geq 1$ block, then by the equilibrium path, those nodes played $\langle I^{t-1} \rangle$ are R^t-node, and those nodes played $\langle$**stay**$\rangle$ are not-R^t nodes.

   If there are some Rebels play $\langle$**stay**$\rangle$ in the first division in t-block, then all the Rebels play **stay** eventually; If R^t Rebels play $\langle$**stay**$\rangle$ in the first sub-block in second division in t-block, then all the Rebels will play **stay** after third division in this block. Otherwise, all the Rebels go to the next reporting period.

   By Theorem **??**, there is a t^* such that there is a R^{t^*} node knows θ, and therefore he knows if θ satisfying $\#[Rebels](\theta) \geq k$ or $\#[Rebels](\theta) < k$. In equilibrium path, such node play $\langle$**stay**$\rangle$ either in the first sub-block in first division or in the first sub-block in second division in coordination period.Thus the equilibrium path is approaching efficient.

2. If there are some Rebels play $\langle 1 \rangle$ in reporting period for a $t \geq 1$ block, then by Claim 0.1.0.1, such nodes will knows if θ satisfying $\#[Rebels](\theta) \geq k$ or $\#[Rebels](\theta) < k$ after reporting period in this t-block. $\langle$**stay**$\rangle$ either in the first sub-block in first division or in the first sub-block in second division in coordination period. Thus the equilibrium path is approaching efficient.

$\square$

0.1.1 Main claims in reporting period

We show the main claims here. The details of the other claims in equilibrium path will be in appendix.

Claim 0.1.1.1. *For $|Ex_{I_i^{m,t}} \cup I_i^{m,t}| \geq s$. Denote D be the set of H-neighbours who detect i's deviation. If $|I_i^{m,t}| < s$, and if $D \neq \emptyset$, then there is a $M < \infty$ and an event E such that i's expected continuation pay-off is less than that in equilibrium path by at least*

$$\delta^M \frac{\beta_i(E|h_{N_i}^m)}{1-\delta}$$

, where $\beta_i(E|h_{N_i}^m) > 0$

Proof. Denote D be the set of neighbours who detect i's deviation. Let the events be

$$\begin{aligned} E_1 &= \{\theta : \#[Rebels](\theta) < k\} \\ E_2 &= \{\theta : k \leq \#[Rebels](\theta) < k + \#D\} \\ E_3 &= \{\theta : \#[Rebels](\theta) \geq k + \#D\} \end{aligned}$$

In equilibrium path, there are periods t^s (t^f) such that if θ satisfying $\#[\text{Rebels}](\theta) \geq k$ ($\#[\text{Rebels}](\theta) < k$) then Rebels play **revolt** (**stay**) forever. If i follows the equilibrium path, the expected static pay-off after $\max\{t^s, t^f\}$[1] is

$$\beta_i(E_2|h_{N_i}^m) + \beta_i(E_3|h_{N_i}^m)$$

If i deviate, the expected static pay-off after $\max\{t^s, t^f\}$ is

$$\beta_i(E_3|h_{N_i}^m)$$

Therefore there is a loss in expected static pay-off of

$$\beta_i(E_2|h_{N_i}^m)$$

Thus, there is a loss in expected continuation pay-off contingent on E by

$$\delta^{\max\{t^s, t^f\}} \frac{\beta_i(E_2|h_{N_i}^m)}{1-\delta}$$

$\square$

[1]There is t^s or t^f for each θ. The maximum is among those possible θ.

Claim 0.1.1.2. *For $|Ex_{I_i^{m,t}} \cup I_i^{m,t}| \geq s$. If $|I_i^{m,t}| < s$, there is a δ such that i will not deviate by reporting $\bar{I}_i^{t-1} \neq I_i^{t-1}$ if such deviation is not detected by i's neighbour.*

Proof. Assume $\bar{I}_i^{t-1} \neq I_i^{t-1}$. Since a detection of deviation has not occur, it must be the case that there is a non-empty set $F = \{j \in \bar{I}_i^{t-1} : \theta_j = Inerts\}$[2].

Let the set

$$E_1 = \{\bar{\theta} : \bar{\theta}_j = Rebel \text{ if } j \in F \text{ and } \bar{\theta}_j = \theta_j \text{ if } j \notin F\}$$

be the set of pseudo events by changing θ_j where $j \in F$. And let

$$E_2 = \{\theta : \theta_j = Inert \text{ if } j \in F \text{ and } \bar{\theta}_j = \theta_j \text{ if } j \notin F\}$$

be the set of true event.

Then consider the event

$$\begin{aligned} E &= \{\bar{\theta} \in E_1 : \#[Rebels](\bar{\theta}) \geq k\} \\ &= \{\theta \in E_2 : \#[Rebels](\theta) \geq k - \#F\} \end{aligned}$$

Partition E as sub events

$$\begin{aligned} E_3 &= \{\theta \in E_2 : \#[Rebels](\theta) \geq k\} \\ E_4 &= \{\theta \in E_2 : k > \#[Rebels](\theta) \geq k - \#F\} \end{aligned}$$

By Lemma and following the strategies in equilibrium path (since i have not been detected), there is a block $\bar{t}^s$ with respect to $\bar{\theta}$ such that if $\bar{\theta} \in E$ then there some $R^{\bar{t}^s}$ Rebel js, says J, will initiate the coordination, and then Rebels play **revolt** forever after $\bar{t}^s$-block. Note that such j is with $\#I_i^{\bar{t}^s} \geq k$ by Claim.

We have several cases:

1. Case 1: If $i \in J$, his own initiation will only depends on $\#I_i^{\bar{t}^s}$ by Claim, which is the same as he has reported $\langle I_i^{t-1} \rangle$. It is strictly better by not deviating since playing $\langle \bar{I}_i^{t-1} \rangle$ is more costly than $\langle \bar{I}_i^{t-1} \rangle$ ($X_{\bar{I}_i^{t-1}} > X_{I_i^{t-1}}$).

2. Case 2: If there is another j who $\bar{I}_i^{t-1} \not\subset I_j^{\bar{t}^s}$[3], then such j's initiation of coordination dependent of his own information about θ, $\subset I_j^{\bar{t}^s}$, by Claim and i's deviation did not change j's information. It is strictly better by not deviating since playing $\langle \bar{I}_i^{t-1} \rangle$ is more costly than $\langle \bar{I}_i^{t-1} \rangle$.

3. Case 3: If there is another j who $\bar{I}_i^{t-1} \subset I_j^{\bar{t}^s}$ such that $\#I_i^{\bar{t}^s} \geq k$. If i did not follow j's initiation of coordination, then there is a detection of deviation by checking the equilibrium path. Such detection will let i's continuation expected pay-off down to zero, and therefore i

[2]Otherwise, there is a detection of deviation. Recall the definition in information hierarchy: $I_i^{-1} \subset I_i^0 \subset ... \subset I_i^{t-1}$ for all $i \in R^0$

[3]Recall that

should follow this initiation as Claim shows. If i follows, and $\#I_i^{\bar{t}^s} \geq s$, we are in the Case 1. If i follows, but $\#I_i^{\bar{t}^s} < s$, then i's expected static pay-off after $\bar{t}^s$ is at most

$$\max\{\beta_i(E_3|h^m_{N_i}) \times 1 + \beta_i(E_4|h^m_{N_i}) \times (-1), 0\}$$

However, if i follow the equilibrium path, there is are t^s, t^f such that the expected static pay-off after $\max\{t^s, t^f\}$ is

$$\max\{\beta_i(E_3|h^{m'}_{N_i}), 0\}$$

Thus, there is a loss in expected continuation pay-off contingent on E by

$$\delta^{\max\{t^s,t^f\}} \frac{\min\{\beta_i(E_3|h^m_{G_i}), \beta_i(E_4|h^m_{G_i})\}}{1-\delta}$$

$\square$

Claim 0.1.1.3. *For $\#Ex_{I_i^{m,t}} \cup I_i^{m,t} \geq s$. If $\#I_i^{m,t} \leq s-1$, and if $i \notin C$ or i did not satisfy the condition to play $\langle 1 \rangle$, i will not play $\langle 1 \rangle$.*

Proof. Let

$$E' = \{\theta : \#I_i^{RP^t,t} \leq k-1\}$$

The event is not empty by checking the timing where i deviated. We have two case:

1. If i has a neighbour $j \in C$, then $j \notin O_i^{RP^t,t}$, and then suppose all other neighbour are not in R^t.
2. If
$$\exists j \in R^{t-1} \cap \bar{G}_i \text{ such that } \exists k_1, k_2 \in Tr_{ij}[[k_1 \in N_j^{t-1} \backslash I_i^{t-1}] \wedge [k_2 \in \bar{G}_{k_2}]]$$
, then just let $E = \{\theta : N_i^t \cap R^0 \leq k-1\} = \{\theta : I_i^t \leq k-1\} = E'$[4].

Next, let

$$\begin{aligned} E_1 &= \{\theta : \#[Reble](\theta) < k\} \cap E' \\ E_2 &= \{\theta : \#[Reble](\theta) \geq k\} \cap E' \end{aligned}$$

be the event contingent on i's information $I_i^{RP^t,t}$. Since i deviate to play $\langle 1 \rangle$ and note that this deviation can not be detected, his behaviour, $\langle \textbf{stay} \rangle$ and $\langle \mathbf{1}_i \rangle$, in the first sub-block at first division in coordination period will decide his neighbours' belief as if his neighbours think he is still on the path. In that sub-block, we have two case:

1. If i play $\langle \textbf{stay} \rangle$, then the coordination to **stay** starts.

[4] note that $I_i^t = I_i^{RP^t,t}$

2. If i play $\langle \mathbf{1}_i \rangle$, then the coordination to **revolt** starts.

But due to E_1 and E_2 still have positive probability (due to his own prior and others' strategies), i's expected static pay-off after the coordination period in this t-block is at most

$$\max\{\beta_i(E_2|h^m_{N_i}) \times 1 + \beta_i(E_1|h^m_{N_i}) \times (-1), 0\}$$

However, if he stay in the equilibrium, there is a t^s (t^f) such that Rebels play **revolt** (**stay**) contingent on E_2 (E_1), and thus after $t^* = \max\{t^s, t^f\}$ he get the expected pay-off as

$$\max\{\beta_i(E_2|h^m_{N_i}) \times 1, 0\}$$

After some calculation, after t^*, there is a loss of

$$\delta^{t^*} \frac{\min\{\beta_i(E_2|h^m_{G_i}), \beta_i(E_1|h^m_{G_i})\}}{1-\delta}$$

□

Claim 0.1.1.4. *For $|Ex_{I_i^{m,t}} \cup I_i^{m,t}| \geq s$. If $\beta_i(|[H]| \geq s|h_{N_i}^{||RP^t-|1|+1|}) > 0$, then if i can report $\langle 1 \rangle$, then i will not report $\langle l \rangle$ when δ is high enough.*

Proof. There are two cases when i play $\langle 1 \rangle$.

- Case 1: If $\#I_i^{||RP^t|,t} \geq k$, let the event E be

$$E = \{\theta : \#[Rebel](\theta) = \#I_i^{||RP^t-|2|+1|,t}\}$$

  That is, the event that no more Rebels outside i's information about Rebels. Contingent on E, there is no more Rebel can initiate the coordination. This is because for all $j \in O_i^{|RP^t|,t}$, j is with $|I_j^{t-1}| < k-1$, and for all $j \in \bar{G}_i$ who have not yet reported, $j \notin R^t$ since all the Rebels are in $|I_i^{||RP^t|,t}|$. Since only i can initiate the coordination, if i deviated, compared to equilibrium, there is a loss in expected continuation pay-off as

$$\delta^t \frac{\beta_i(E|h^m_{N_i})}{1-\delta}$$

- Case 2: If $\#I_i^{||RP^t|,t} = k-1$, since $\beta_i(\#[Rebels](\theta) \geq s|h_{G_i}^{|RP^t|}) > 0$, the following event E_1 must have positive probability; otherwise, since no neighbours can report after current period, and thus $\beta_i(\#[Rebels](\theta) \geq s|h_{G_i}^{|RP^t|}) = 0$.

  Let

$$E_1 = \{\theta : \exists j \in \bar{G}_i, j \notin O_i^{|RP^t|,t}[\#I_j^{|RP^t|,t} \geq s-1]\}$$

  Let sub-events $E_1' \subset E_1$ as

$$E_1^{'} = \{\theta: \text{ exist a unique} j \in \bar{G}_i, j \notin O_i^{|RP^t|,t}[\#I_j^{|RP^t|,t} \geq s-1]\}$$

Note that this $E_1^{'}$ can be constructed since the network is tree. If there is θ admits 2 or more js in the definition E_1, these js must be not each others' neighbour. Suppose there are two js, says j, $j^{'}$, there must be at least one node in $I_j^{|RP^t|,t}$ but outside of $I_{j^{'}}^{|RP^t|,t}$. We then pick a j, and suppose those nodes outside of $I_j^{|RP^t|,t}$ are Inert.

Now, dependent on such j, let

$$E = \{\theta: \#[Rebel](\theta) = \#I_j^{|RP^t|,t} \cup I_i^{|RP^t|,t}\}$$

If i report $\langle l \rangle$, there are following consequences.

- i will be consider as $\notin R^t$ by j, and thus i can not initiate the coordination.
- Such j will have $\#I_j^{|RP^t|} = \#I_j^t < s$. Since there is no more H-nodes outside $I_j^{||RP^t-|2|+1|,t} \cup I_i^{||RP^t-|2|+1|,t}$, contingent on E, such j will then play stay forever after coordination period in t-block.
- Without the extra Rebels in $I_j^{|RP^t|}$, only $\#I_i^{||RP^t|,t} = k-1$ may play **revolt**, and therefore there is no coordination to success.

However, if i play $\langle 1 \rangle$, coordination can be initiated by himself in the following coordination period. Thus, there is a loss in expected continuation pay-off by

$$\delta^{|t|} \frac{\beta_i(E|h_{N_i}^m)}{1-\delta}$$

$\square$

0.1.2 Main claims in coordination period

We show the main claims here. The details of the other claims in equilibrium path will be in appendix.

Claim 0.1.2.1. *In **COORDINATION**. Suppose there is no $j \in G_i$ has played $\langle 1 \rangle$ in reporting period, Suppose $|I_i^t| < s$, Suppose $\beta_i(\#[Rebel](\theta)|h_{G_i}^m) > 0$. Then*

- *if i has not observed $\langle$**stay**$\rangle$ played by $j \in G_i$ in the first sub-block at second division, or*
- *if i has not observed $\langle \mathbf{1}_j \rangle$ played by $j \in G_i$ after first sub-block at second division*

, then i will not play

- *$\langle$**stay**$\rangle$ in the first sub-block at second division and*

- $\langle \mathbf{1}_j \rangle$ *after first sub-block at second division*

Proof. Since $|I_i^t| < s$ and due to the equilibrium strategies played by i's neighbours, we have

$$0 < \beta_i(\#[Rebel](\theta)| \geq s|h_{G_i}^m) < 1$$

If i deviate, all i's neighbour who did not detect the deviation will play **revolt** after coordination period in this block; if i's deviation is detected by some neighbours, we are in the case of Claim and so that i will not deviate. We then check if i deviate but no neighbour detect it. Let

$$E^{'} = \{\theta : \#I_i^t \leq k-1\}$$

and let

$$\begin{aligned} E_1 &= \{\theta : \#[Reble](\theta) < k\} \cap E^{'} \\ E_2 &= \{\theta : \#[Reble](\theta) \geq k\} \cap E^{'} \end{aligned}$$

E_1 and E_2 have positive probability (due to his own prior and others' strategies). Since after i deviated, all the Rebels will play **revolt** after this block, i's expected static pay-off after the coordination period in this t-block is at most

$$\max\{\beta_i(E_2|h_{N_i}^m) \times 1 + \beta_i(E_1|h_{N_i}^m) \times (-1), 0\}$$

However, if he stay in the equilibrium, there is a t^s (t^f) such that Rebels play **revolt** (**stay**) contingent on E_2 (E_1), and thus after $t^* = \max\{t^s, t^f\}$ he get the expected pay-off as

$$\max\{\beta_i(E_2|h_{N_i}^m) \times 1, 0\}$$

After some calculation, after t^*, there is a loss of

$$\delta^{t^*} \frac{\min\{\beta_i(E_2|h_{G_i}^m), \beta_i(E_1|h_{G_i}^m)\}}{1-\delta}$$

$\square$

A Proof

A.1 Proof for Lemma ??

Proof. The proof is by induction. We first show that the statement is true for $t = 1$.

Claim A.1.0.2. ***Base:*** $i \in R^1 \Leftrightarrow [i \in R^0] \wedge [\exists k_1, k_2 \in (R^0 \cap N_i \setminus i)]$.

Proof. $\Rightarrow$: Since $i \in R^1$, then $i \in R^0$ and $\forall j \in N_i \setminus i[I_i^0 \not\subseteq N_j^0]$ by definition. Since $I_i^0 = N_i \cap R^0$ and $i \in N_j^0$, then $\forall j \in N_i \setminus i[\exists k \in (R^0 \cap N_i \setminus i)[k \notin N_j^0]]$. Since the $j \in N_i \setminus i$ is arbitrary, we then have a pair of $k_1, k_2 \in (R^0 \cap N_i \setminus i)$ such that both $k_1 \notin N_{k_2}^0$ and $k_2 \notin N_{k_1}^0$.

$\Leftarrow$: Pick $k \in \{k_1, k_2\} \subseteq N_i \cap R^0$, and pick an arbitrary $j \in (N_i \setminus \{i,k\})$. Note that $k \notin D_j^0$, otherwise there is a circle from i to i since $i \in N_j^0 \subseteq D_j^0$. Hence $[k \in N_i \cap R^0] \wedge [k \notin D_j^0]$, and therefore $[k \in I_i^0] \wedge [k \notin N_j^0]$. Then we have $I_i^0 \not\subseteq N_j^0$ for arbitrary $j \in N_i \setminus i$, and thus $i \in R^1$. $\square$

Induction hypothesis: the statement is true up to t and $t \geq 1$.

Claim A.1.0.3. *If the hypothesis is true, then*

$$i \in R^{t+1} \Leftrightarrow [i \in R^t] \wedge [\exists k_1, k_2 \in R^t \cap N_i \setminus i]$$

Proof. $\Rightarrow$: since $i \in R^{t+1}$, then $i \in R^t$ and $\forall j \in N_i \setminus i[I_i^t \not\subseteq N_j^t]$ by definition. Recall Equation (**??**) and Equation (**??**), then for every $m \in I_i^{t-1}$, we can find a path connecting i to m (the existence of such path is by the induction hypothesis). If $j \in N_i \setminus i$, then we can find a path connecting j to m by connecting j to i, and then connecting i to m. Thus, if $m \in I_i^{t-1}$ then $m \in N_J^t$, and hence $I_i^{t-1} \subseteq N_j^t$ if $j \in N_i \setminus i$.

Further note that $I_i^t = \bigcup_{k \in N_i \cap R^t} I_k^{t-1}$, then we must have $\forall j \in N_i \setminus i[\exists k \in (R^t \cap N_i \setminus i)[I_k^{t-1} \not\subseteq N_j^t]]$, since $I_i^{t-1} \subseteq N_j^t$. Since the $j \in N_i \setminus i$ is arbitrary, we then have a pair of $k_1, k_2 \in (R^t \cap N_i \setminus i)$ such that both $k_1 \notin N_{k_2}^t$ and $k_2 \notin N_{k_1}^t$.

$\Leftarrow$: By the induction hypothesis, we have a chain $k_{1_0}, ..., k_1, i, k_2, ..., k_{2_0}$ with $k_{1_0} \in R^0$,..., $k_1 \in R^t$, $i \in R^t$, $k_2 \in R^t$,...,and $k_{1_0} \in R^0$. Note that $k_{1_0} \notin D_j^t$ whenever $j \in N_i \setminus i$, otherwise there is a circle from i to i since $\{i, k_2, ..., k_{2_0}\} \in N_j^t \subseteq D_j^t$. Hence $[k_{1_0} \in I_{k_1}^{t-1}] \wedge [k_{1_0} \notin D_j^t]$, and therefore $[I_{k_1}^{t-1} \in I_i^t] \wedge [I_{k_1}^{t-1} \notin N_j^t]$. Then we have $I_i^t = \bigcup_{k \in N_i \cap R^t} I_k^{t-1} \not\subseteq N_j^t$ for arbitrary $j \in N_i \setminus i$, and thus $i \in R^1$.

$\square$

We can then conclude that the statement is true by induction.

$\square$

A.2 Proof for Lemma ??

Proof.
1. The proof is by induction, and by Lemma **??**. Since the state has strong connectivity, all the R^0 nodes are connected, and thus we have a R^0-path connecting each pair of R^0 nodes. Since all pairs of R^0 nodes are connected by a R^0-path, then for all pairs of R^1 nodes must be in some of such paths by Lemma **??**, and then connected by a R^0-path. But then all the R^0-nodes in such path are all R^1 nodes by Lemma **??** again and by $R^t \subseteq R^{t-1}$. Thus, for all pairs of R^1 nodes has a R^1-path connecting them. The similar argument holds for $t > 1$, we then get the result.

2. The uniqueness is by the fact that the network is a tree, and therefore the path connecting all distinguish nodes is unique.

$\square$

A.3 Proof for Lemma ??

Proof. We have to show that $R^{t-1} \supseteq \bigcup_{i\in R^t} N_i \cap [H]$ and $R^{t-1} \subseteq \bigcup_{i\in R^t} N_i \cap [H]$.

- $\supseteq$: Since R^t is not empty, we can pick a node $m \in \bigcup_{i\in R^t} N_i \cap [H]$. By Lemma **??**, $m \in R^t \cup R^{t-1} = R^{t-1}$, and therefore $m \in R^{t-1}$.
- $\subseteq$: Since both R^{t-1} and R^t are not empty, we can pick nodes $m_1 \in R^{t-1}$ and $m_2 \in R^t$. Since the state has strong connectivity, there is a R^{t-1} path connecting them by Lemma **??**. But then the nodes (expect for m_1, m_2) in this path are all R^t nodes by Lemma **??**, and then there is $m_1^{'} \in N_{m_1} \cap R^t$. Since the $m_1 \in R^{t-1}$ we picked is arbitrary, therefore it means for all $m \in R^{t-1}$ there is a $m^{'} \in N_m \cap R^t$, and hence $m \in N_{m^{'}} \cap [H]$ while $m^{'} \in R^t$. We then get the result.

□

A.4 Proof for Lemma ??

Proof. 1. If $1 \leq |R^t| \leq 2$, then by Lemma **??** and by Lemma **??**, we have a spanning tree consisting the nodes in R^{t-1},...,R^0. Since the state has strong connectivity, all the H-nodes are in this tree. By Lemma **??**, we have

$$R^0 = \bigcup_{k_1\in R^1} N_{k_1} \cap [H] = \bigcup_{k_1\in N_{k_2}\cap R^1} \bigcup_{k_2\in N_{k_3}\cap R^2} \cdots \bigcup_{k_{t-1}\in N_{k_t}\cap R^t} N_{k_t} \cap [H]$$

Then by Equation (14), if $i \in R^t$ we have

$$I_i^t = \bigcup_{k_0\in N_i\cap R^t} \bigcup_{k_1\in N_{k_0}\cap R^{t-1}} \cdots \bigcup_{k_{t-1}\in N_{k_{t-2}}\cap R^1} N_{k_{t-1}} \cap R^0$$

Now note that $R^0 = [H]$, and compare the above two equations, we got $[H] = I_i^t$ if $i \in R^t$.

2. For a given t-block, in the case when $R^t \neq \emptyset$ and $R^{t+1} \neq \emptyset$, the proof is a direct application of Lemma **??**, and we continue taking the union of nodes' information set. Since the network is finite, the $[H]$ will be a subset of some nodes' information set eventually.

   We then only consider the case when $R^t \neq \emptyset$ and $R^{t+1} = \emptyset$. But in such case, it means that there is no R^t node which has more than two distinguish R^t neighbours by Lemma **??**, and then $1 \leq |R^t| \leq 2$ since all pairs of R^t nodes are connected by R^t-path by Lemma **??**. The first part of this Lemma **??** then shows the result.

□

A.5 Proof for Lemma ??

Proof. Suppose there are three or more R^t-nodes in C, then pick any three nodes of them, and denote them as i_1, i_2, i_3. Let's say i_2 is in the $(i_1 i_3)$-path, and therefore $i_2 \in Tr_{i_1 i_2}$ and $i_3 \in Tr_{i_2 i_3}$. First we show that $i_1 \in N_{i_2}$ (or $i_3 \in N_{i_2}$). Suppose $i_1 \notin N_{i_2}$, since $i_1, i_2 \in R^t$, then the $(i_1 i_2)$-path is a R^t-path by Lemma ??. Let this $(i_1 i_2)$-path be $(i_1, j_1, ..., j_n, i_2)$. Since $i_1, j_1, ..., j_n, i_2 \in R^t$, we then have $I_{i_1}^{t-1} \nsubseteq N_{j_1}^{t-1}, ..., I_{j_n}^{t-1} \nsubseteq N_{i_2}^{t-1}$ and $I_{j_1}^{t-1} \nsubseteq N_{i_1}^{t-1}, ..., I_{i_2}^{t-1} \nsubseteq N_{j_n}^{t-1}$. Since $I_{i_1}^{t-1} \subseteq N_{i_1}^{t-1}, ..., I_{i_2}^{t-1} \subseteq N_{i_2}^{t-1}$ by Lemma ??, we further have $\exists k_1 \in [H][k_1 \in N_{j_1}^{t-1} \setminus I_{i_1}^{t-1}]$,...,$\exists k_n \in [H][k_n \in N_{j_n}^{t-1} \setminus I_{i_2}^{t-1}]$. Since the state has Strong Connectivity, such $k_1, ..., k_n$ are connected. But then we have already found k_1, k_2 such that $k_1 \in N_{j_1}^{t-1} \setminus I_{i_1}^{t-1}$ and $k_2 \in N_{k_1} \setminus k_1$. It is a contradiction that $i_1 \in C$.

Now, i_1, i_2, i_3 will form a R^t-path as (i_1, i_2, i_3). With the same argument as the above, we then have $\exists k_1 \in [H][k_1 \in N_{i_2}^{t-1} \setminus I_{i_1}^{t-1}]$ and $\exists k_2 \in [H][k_2 \in N_{i_3}^{t-1} \setminus I_{i_2}^{t-1}]$, and thus i_1 is not in C. □

A.6 Proof for Lemma ??

Proof. Since $i \in R^t$, there is a $j \in R^{t-1}$ and $j \in N_i \setminus i$ by Lemma ??. Given any $j \in R^{t-1} \cap (N_i \setminus i)$, first note that $N_j^{t-1} \subseteq \bigcup_{k \in N_i^{t-1}} N_k$ by $N_j^{t-1} \equiv \bigcup_{k \in I_j^{t-2}} N_k$, and $I_j^{t-2} \subseteq I_i^{t-1} \subseteq N_i^{t-1}$. If there is another node outside $\bigcup_{k \in N_i^{t-1}} N_k$ in Tr_{ij}, then there must be another node connected to N_j^{t-1} since the network is connected. It is a contradiction that $i \in C$.

□