Coordination in Social Networks

Chun-ting Chen

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Motivation

- A collective action may fails due to incomplete information about participants' inclination.
  - A revolution needs rebels' joint contribution.
  - People's "types", aggressiveness or inertness are not commonly known.
- Moreover, transmitting relevant information is difficult.
  1. No cheap talk.
  2. No (uncensored) discussion forum, etc.
- A collective action: a joint investment, a project, etc
- Participant: investors, co-workers, etc,

Motivation

However, history tell us:

- **A collective action is not static**: An event may trigger later events.
  - Consecutive uprisings in East Germany 1989-1990.
- **Information is transmitted within social networks**:
  - Ex., Gangster networks (1911 Revolution); Church networks (1989 Berlin Uprising), etc.

Motivation

Question

- *If rational rebels know that a "tiny" event can trigger later events, how do they conduct a decisive collective action within their social network?*

Model

1. No cheap talk. Communication is taking actions.
2. Communication is facing expected cost.
3. Players communicate repeatedly in a network.

Motivation

Looking for

- An equilibrium, where the ex-post efficient outcome played repeatedly after a finite time T in the equilibrium path when δ is high enough.

Related Literature

- Public good provision.
  - One strand: [Chwe 2000], [Lohmann, 1993,1994], [Bolton and Harris, 1999], [Bramoullé and Kranton, 2007]
  - **This paper adds network-monitoring**
- Social learning.
  - One strand: [Goyal, 2012], [Acemoglu et al., 2011], [Chatterjee and Dutta, 2011].
  - **This paper considers farsighted-learning in the game**
- Repeated game.
  - One strand: [Laclau, 2012], [Wolitzky, 2013], [Wolitzky, 2014]
  - **This paper consider incomplete information and imperfect monitoring**
  - One strand: [Fudenberg and Yamamoto, 2010] [Fudenberg and Yamamoto, 2011] [Wiseman, 2012] [Yamamoto 2014]
  - **This paper consider n-person game without full-rank conditions on public or private signals generated by single-period actions.**

Model

Network

- Let $N = \{1, \ldots, n\}$ be the set of players.
- G_i is i's neighborhood, G_i is a subset of N and $i \in G_i$.
- $G = \{G_i\}_i$ is the network.

Assumption

G is fixed (not random), finite, connected, commonly known, and undirected.

Model

Static k-threshold game [Chwe 2000]

- θ_i: i's type
- $\theta_i \in \Theta_i = \{Rebel, Inert\}$
- $\Theta = \times_{i \in N} \Theta_i$; $\theta \in \Theta$
- $A_{Rebel_i} = \{\textbf{revolt}, \textbf{stay}\}$; $A_{Inert_i} = \{\textbf{stay}\}$
- $1 \leq k \leq n$

Model

Here, Static k-threshold game [Chwe 2000]

- Static game payoff for player i: $u_{\theta_i}(a_{\theta_i}, a_{-\theta_i})$

$$
\begin{array}{rcll}
u_{Inert_i}(a_{Inert_i}, a_{-\theta_i}) & = & 1 & \text{if } a_{Inert_i} = \textbf{stay} \\
\\
u_{Rebel_i}(a_{Rebel_i}, a_{-\theta_i}) & = & 1 & \text{if } a_{Rebel_i} = \textbf{revolt} \text{ and } \#\{j : a_{\theta_j} = \textbf{revolt}\} \geq k \\
u_{Rebel_i}(a_{Rebel_i}, a_{-\theta_i}) & = & -1 & \text{if } a_{Rebel_i} = \textbf{revolt} \text{ and } \#\{j : a_{\theta_j} = \textbf{revolt}\} < k \\
\\
u_{Rebel_i}(a_{Rebel_i}, a_{-\theta_i}) & = & 0 & \text{if } a_{Rebel_i} = \textbf{stay}
\end{array}
$$

- **stay** is a safe arm; **revolt** is a risky arm.
- Ex-post (Pareto) efficient outcome:
  - Inerts play **stay**.
  - If there are more than k Rebels, all Rebels play **revolt**.
  - Otherwise, all Rebels play **stay**.

Model

OR, Static k-threshold game [Chwe 2000]

- Static game payoff for player i: $u_{\theta_i}(a_{\theta_i}, a_{-\theta_i})$

$$
\begin{aligned}
u_{Inert_i}(a_{Inert_i}, a_{-\theta_i}) &= 1 && \text{if } a_{Inert_i} = \textbf{stay} \\[1em]
u_{Rebel_i}(a_{Rebel_i}, a_{-\theta_i}) &= 1 && \text{if } a_{Rebel_i} = \textbf{revolt} \text{ and } \#\{j : a_{\theta_j} = \textbf{revolt}\} \geq k \\
u_{Rebel_i}(a_{Rebel_i}, a_{-\theta_i}) &= -1 && \text{if } a_{Rebel_i} = \textbf{revolt} \text{ and } \#\{j : a_{\theta_j} = \textbf{revolt}\} < k \\[1em]
u_{Rebel_i}(a_{Rebel_i}, a_{-\theta_i}) &= 1 && \text{if } a_{Rebel_i} = \textbf{stay} \text{ and } \#\{j : a_{\theta_j} = \textbf{revolt}\} \geq k \\
u_{Rebel_i}(a_{Rebel_i}, a_{-\theta_i}) &= 0 && \text{if } a_{Rebel_i} = \textbf{stay} \text{ and } \#\{j : a_{\theta_j} = \textbf{revolt}\} < k
\end{aligned}
$$

- **stay** is a safe arm; **revolt** is a risky arm;
- Ex-post (Pareto) efficient outcome:
  - Inerts play **stay**.
  - If there are more than k Rebels, at least some k Rebels play **revolt**.
  - Otherwise, all Rebels play **stay**.

Model

Repeated k-threshold game

- Time is infinite, discrete.
- Nature choose θ at 0 period according to π.
- Players play the static k-threshold game infinitely repeatedly.

Assumption

- *Players know their neighbors' types.*
- *Players perfectly observe their neighbors' actions.*
- *π has full support*
- *Common δ.*

Model

Notations:

- $[Rebels](\theta) = \{j : \theta_j = Rebel\}$ for all $\theta \in \Theta$.
- θ_{G_i}: i's private information about the state. $(\theta_{G_i} \in \Theta_{G_i} = \prod_{j \in G_i} \Theta_j)$
- $h^m_{G_i}$: the history observed by i up to period m. $(h^m_{G_i} \in H^m_{G_i} = \prod_{s=1}^{m} \prod_{j \in G_i} A_{\theta_j})$
- $h \in H = \prod_{s=1}^{\infty} \prod_{j \in N} A_{\theta_j}$: a infinite sequence of players' actions
- $\tau_i : \Theta_{G_i} \times \bigcup_1^\infty H^m_{G_i} \to A_{\theta_i}$, i's strategy.
- $\tau = (\tau_1, ..., \tau_i, ..., \tau_n)$: a strategy profile
- $\beta_i^{\pi,\tau}(\theta | h^m_{G_i})$: i's belief for a θ at period m given τ.

APEX

Notations:

- h_θ^τ : a realized h generated by τ given θ.
- Call h_θ^τ a τ_θ-path.
- Call $\{\tau_\theta\}_{\theta\in\Theta}$ the τ-path

Definition

The τ-path is **approaching ex-post efficient** ($APEX$) $\Leftrightarrow$

$$\forall\,\theta, \text{ there is a finite time } T^\theta$$

such that the actions after T^θ in τ_θ repeats the static ex-post efficient outcome.

APEX

Definition

$h^m_{G_i}$ is **reached by τ-path**

$$\Leftrightarrow$$

$\exists\theta$ such that $h^m_{G_i}$ is in τ_θ-path.

Lemma

If the τ-path is APEX $\Rightarrow$

$$\forall\theta\ \forall i, \text{ there is a finite time } T^\theta_i$$

such that $\sum_{\theta:\#[Rebels](\theta)\geq k} \beta^{\pi,\tau}_{G_i}(\theta|h^s_{G_i}) = 1$ *or* $= 0$ *if* $s \geq T^\theta_i$ *and if* $h^s_{G_i}$ *reached by τ-path.*

APEX

Definition

A sequential equilibrium (τ^*, β^*) is APEX $\Leftrightarrow$ τ^*-path is APEX, and β^* is the belief system consistent with τ^*.

Leading Example

If pay-off is observable, an Apex Equilibrium for $k = n = 3$ in

- At 1st period
  - All Rebels choose **revolt**.
- After 1st period
  - If the pay-off is observed as 1, choose **revolt** afterwards.
  - Otherwise, choose **stay** afterwards.
- Any deviation $\Rightarrow$
  - Choosing **stay** forever.

Leading Example

If pay-off is hidden, an Apex Equilibrium for $k = n = 3$ in

- At 1st period
  - Rebel 2 chooses **revolt** if he observes $\theta = (Rebel, Rebel, Rebel)$; Otherwise, chooses **stay** forever.
  - Rebel 1 (or Rebel 3) choose **stay**.
- After 1st period
  - If Rebel 2 chooses **revolt** in the last period, then Rebel 1 (or Rebel 3) chooses **revolt** forever;
  - If Rebel 2 chooses **stay** in the last period, then Rebel 1 (or Rebel 3) chooses **stay** forever.
- Any deviation $\Rightarrow$
  - Choosing **stay** forever.

Goal

Goal
Can we generalize the above result?

Assumption

Payoff is hidden (or noisy).

Results

- $k = n$: we can.
- $k < n$: with additional assumptions,
    - acyclic networks: we can .
    - all networks: open question.

$k = n$: Result

Theorem

In *any network,* ***if*** *the prior has full support,* ***then*** *for repeated $k = n$ Threshold game,* ***there is*** *a δ such that a sequential equilibrium which is APEX* ***exists****.*

Proof:

1. Some Inerts neighbors $\Rightarrow$ play **stay** forever.
2. No Inert neighbor $\Rightarrow$ play **revolt** until **stay** is observed, and then play **stay** forever.
3. Any deviation $\Rightarrow$ play **stay** forever.
4. There is a finite time T^{θ} such that ex-post efficient outcome repeats afterwards.

$k = n$: Result

Comments for $k = n$:

1. **stay** means "some Inerts are out there."
2. **revolt** means "some Inerts may not be there."
3. Any deviation $\Rightarrow$ punished by shifting to **stay** forever by single player
   - Group punishment is not necessary.

$k < n$: Result and Conjecture

Since a Inert always play **stay**, define

Definition

Strong connectedness$\Leftrightarrow$ for every pair of Rebels, there is a path consisting of Rebels to connect them.

Definition

Full support on strong connectedness$\Leftrightarrow$

$$\pi(\theta) > 0 \text{ if and only if } \theta \text{ has strong connectedness.}$$

to not reduce the game to incomp. info. game without communication.

$k < n$: Result and Conjecture

Theorem

***In** any acyclic network, **if** π has full support on strong connectedness, **then** for repeated $1 \leq k \leq n$ Threshold game, **there is** a δ such that a weak sequential equilibrium which is APEX **exists**.*

Conjecture

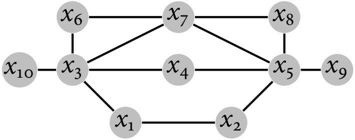
***In** any cyclic network, ...[same as above]...*

Outline

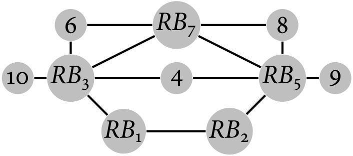
1. Communication by actions
2. Communication in the equilibrium
    1. Communication protocol
    2. Reporting and coordination messages in the protocol
    3. Information hierarchy in communication
    4. In-the-path belief updating
    5. Off-path belief
    6. Sketch of proof

Communication by binary actions

1. Indexing each node i as a distinct prime number x_i. For instance,



2. Then, If



Rebel 3 report $x_1 \times x_7 \times x_3$ to Rebel 1 by sending a finite sequence

$$\textbf{stay}, \ldots, \textbf{stay}, \underbrace{\textbf{revolt}, \textbf{stay}, \ldots, \textbf{stay}}_{x_1 \times x_7 \times x_3}$$

Communication protocol

Communication phases
Two phases, RP and CD, alternate in time horizontal line

$$\underbrace{\langle\text{coordination period}\rangle}_{0-block}\ \underbrace{\langle\text{reporting period}\rangle\langle\text{coordination period}\rangle}_{1-block}\ \ldots$$

- Reporting period (RP): talking about θ
  - Cheap talking: θ will be revealed.
- Why do I need coordination period (CD) ?

Communication protocol

Coordination period

Why do I need coordination period ?

- Ans: Since higher-order belief is hard to track.
  - APEX: to find T^θ for all θ.
  - When is T^θ?.
- Sol: Let CD be long enough

$$\overbrace{\langle \ldots \rangle}^{\text{RP}} \overbrace{\langle \langle \cdot \rangle \langle \cdot \rangle \ldots \langle \cdot \rangle \rangle}^{\text{CD}}$$

Communication protocol

Coordination period

Why do I need coordination period ?

- Ans: Since higher-order belief is hard to track.
  - APEX: to find T^θ for all θ.
  - When is T^θ?.
- Sol: Let CD be long enough

$$\underbrace{\color{red}{\langle \ldots \rangle}}^{RP}\ \overbrace{\langle \langle \cdot \rangle \langle \cdot \rangle \ldots \langle \cdot \rangle \rangle}^{CD}$$

  - If a Rebel knows the relevant info. after RP,

Communication protocol

Coordination period

Why do I need coordination period ?

- Ans: Since higher-order belief is hard to track.
  - APEX: to find T^{θ} for all θ.
  - When is T^{θ}?.
- Sol: Let CD be long enough

$$\underbrace{\langle \dots \rangle}_{RP}\ \underbrace{\langle \langle \cdot \rangle \langle \cdot \rangle \dots \langle \cdot \rangle \rangle}_{CD}$$

  - If a Rebel knows the relevant info. after RP,$\Rightarrow$ sending messages to let his neighbors know that

Communication protocol

Coordination period

Why do I need coordination period ?

- Ans: Since higher-order belief is hard to track.
  - APEX: to find T^{θ} for all θ.
  - When is T^{θ}?.
- Sol: Let CD be long enough

$$\overbrace{\langle \ldots \rangle}^{RP}\ \overbrace{\langle \langle \cdot \rangle \langle \cdot \rangle \ldots \langle \cdot \rangle \rangle}^{CD}$$

  - If a Rebel knows the relevant info. after $RP \Rightarrow$ sending messages to let neighbors know that $\Rightarrow$ neighbors send msg. to let their neighbors know $\Rightarrow$....$\Rightarrow$ all the Rebels commonly know relevant information.

Equilibrium path

Coordination period and messages

Idea

- At least “three” messages to coordinate Rebels
  1. to **revolt**
  2. to **stay**
  3. to continue to next block
- Create these distinguishable messages by binary actions

Equilibrium path

Coordination period and messages

- CD^t: the CD in t-block

$$\overbrace{\langle \underbrace{\langle\cdot\rangle \cdots \langle\cdot\rangle}_{\text{1st division}} \rangle \langle \underbrace{\langle\cdot\rangle \cdots \langle\cdot\rangle}_{\text{2nd division}} \rangle}^{CD^t}$$

- $CD^t_{p,q}$: the p sub-block in q division.
- $\langle CD^t_{p,q} \rangle$: the messages in $CD^t_{p,q}$ are distinguishable

$$\begin{array}{ll} \langle \textbf{stay} \rangle & \mathbf{s}, \ldots, \mathbf{s}, \mathbf{s}, \mathbf{s}, \ldots, \mathbf{s} \\ \langle x_i \rangle & \mathbf{s}, \ldots, \mathbf{s}, \underbrace{\mathbf{r}, \mathbf{s}, \ldots, \mathbf{s}}_{x_i} \end{array}$$

- 1st division: sending **message to stay**; otherwise **continue**
- 2nd division: sending **message to revolt**; otherwise **continue**

Equilibrium path

1st division in CD

- Message to **stay**: Whenever a Rebel i knows $\#[Rebels](\theta) < k$, he plays **stay** afterward.

$$\overbrace{\langle \underbrace{\langle \textbf{stay} \rangle \ldots}_{\text{1st division}} \rangle \langle \quad \underbrace{\langle \ldots \rangle}_{\text{2nd division}} \quad \rangle}^{CD^t}$$

Equilibrium path

1st division in CD

- Message to **stay**: ... then nearby Rebel j plays **stay** afterward

$$\overbrace{\langle \underbrace{\langle\cdot\rangle\langle\textbf{stay}\rangle\ldots}_{\text{1st division}} \rangle\langle \quad \underbrace{\langle\ldots\rangle}_{\text{2nd division}} \quad \rangle}^{CD^t}$$

Equilibrium path

1st division in CD

- Otherwise,

$$\overbrace{\langle \underbrace{\langle x_i\rangle\langle x_i\rangle\ldots\langle\cdot\rangle}_{\text{1st division}}\rangle\langle\underbrace{\langle\cdot\rangle\cdots\langle\cdot\rangle}_{\text{2nd division}}\rangle}^{CD^t}$$

Equilibrium path

2nd division

- Message to **revolt**: Whenever a Rebel i know $\#[Rebels](\theta) \geq k$, he play

$$\overbrace{\underbrace{\langle\langle\cdot\rangle\cdots\langle\cdot\rangle\rangle}_{\text{1st division}}\underbrace{\langle\langle\textbf{stay}\rangle\langle\cdot\rangle\cdot\cdot\langle\cdot\rangle\rangle}_{\text{2nd division}}}^{CD^t}$$

  in the first sub-block.

- Otherwise ,

$$\overbrace{\underbrace{\langle\langle\cdot\rangle\cdots\langle\cdot\rangle\rangle}_{\text{1st division}}\underbrace{\langle\langle x_i\rangle\langle\cdot\rangle\cdot\cdot\langle\cdot\rangle\rangle}_{\text{2nd division}}}^{CD^t}$$

Equilibrium path

2nd division

- Message to **revolt**:... then nearby Rebel j play $\langle x_j \rangle$ to inform nearby Rebels, etc

$$\overbrace{\langle \underbrace{\langle\cdot\rangle \cdots \langle\cdot\rangle}_{\text{1st division}} \rangle \langle \underbrace{\langle\cdot\rangle \langle x_j \rangle \cdots \langle\cdot\rangle}_{\text{2nd division}} \rangle}^{CD^t}$$

- Otherwise ,

$$\overbrace{\langle \underbrace{\langle\cdot\rangle \cdots \langle\cdot\rangle}_{\text{1st division}} \rangle \langle \underbrace{\langle\cdot\rangle \langle \textbf{stay} \rangle \cdots \langle\cdot\rangle}_{\text{2nd division}} \rangle}^{CD^t}$$

Equilibrium path

After coordination period

- Either stopping or continuing communication
  1. Stopping: if relevant info. is revealed $\Rightarrow$ messages will be sent $\Rightarrow$ all Rebels play the ex-post eff. outcome afterward.
  2. Continuing: otherwise, go to the next block.

Observation

Either stopping or continuing belief updating.

- "a grim-trigger"

Equilibrium path

Lemma

Before a Rebel knows $\#[Rebels](\theta) < k$ or $\#[Rebels](\theta) \geq k$, he will not send ***Message to stay*** *or* ***Message to revolt*** *if δ is high enough.*

1. If he send, then information updating stops (grim-trigger).
2. If he does not send, he can learn the relevant information.

Equilibrium path

Coordination messages

- No expected cost to send **Message to stay** or **Message to revolt**
- The player who knows the relevant info. is willing to send messages.

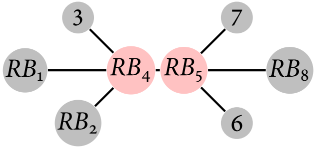
Equilibrium path

- However, sending message in RP is costly.
- A free rider problem in PR may occur.

Equilibrium path

Free rider problem

1. $k = 5$
2. Only one block (RP and then CD).
3. Free riders:



Why?

1. No expected cost to send **Message to stay** or **Message to revolt**,
2. Rebel 4 will not report truthfully given that Rebel 5 report truthfully.
3. Rebel 4 will not report truthfully given that Rebel 5 report truthfully.

Equilibrium path

Reporting period and messages

Idea

- "Burning moneys" before sending **message to revolt**.
  1. Gives incentives to report θ.
  2. Prevent potential free rider problems.
- Characterizing "how much money a Rebel should burn"
  - Building Information Hierarchy

Equilibrium path

Reporting period and messages

- RP^t: the reporting period at t block

$$\overbrace{\langle\langle\cdot\rangle\rangle}^{RP^t}$$

- $\langle RP^t\rangle$: the reporting message

Burning moneys	$\neg\langle \textbf{stay}\rangle$	**s, ..., s, r, s, ..., s**
Not burning money	$\langle \textbf{stay}\rangle$	**s, ..., s, s, s, ..., s**

- Burning moneys+**message to revolt**:
  - Rebels believe that $\#[Rebels](\theta) \geq k$
- Not burning moneys+**message to revolt**:
  - Rebels don't believe that $\#[Rebels](\theta) \geq k$

Equilibrium path

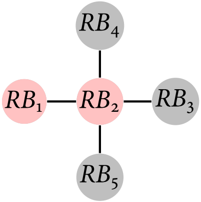
How much money should a Rebel burns?

- Burning money is to convince Rebels to coordination to revolt.
- Information Hierarchy: how much money should be burned?.

Information Hierarchy

Main goal of **Information Hierarchy**

- Characterizing Rebels' incentives in money burning.
- Ex: $k = 4$ and

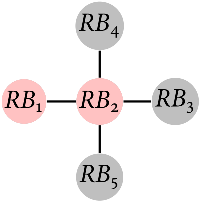


1. Rebel 1's information can be reported by Rebel 2.

Information Hierarchy

Main goal of **Information Hierarchy**

- Easing the punishment scheme when monitoring is imperfect.
- Note that $k < n$, punishment by single player is not enough.
- Ex: $k = 4$ and



1. Rebel 1 can only be monitored by Rebel 2.
2. Suppose Rebel 2,3,4,5 can coordinate at period T and play **revolt** forever.
3. If Rebel 1 did not burn money at period $T - 1$, Rebel 2 has no incentive to punish him.

Information Hierarchy

Information Hierarchy

$$0 - 1 - RB_2 - RB_3 - RB_4 - RB_5 - RB_6 - 7$$

At 0-block, let

$$R^0 = [Rebels](\theta)$$

Information Hierarchy

$$0 - 1 - RB_2 - RB_3 - RB_4 - RB_5 - RB_6 - 7$$

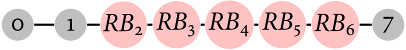
At 1-block, first let

$$
\begin{aligned}
G_i^0 &\equiv G_i \\
I_i^0 &\equiv G_i \cap R^0
\end{aligned}
$$

For instance,

$$
\begin{aligned}
I_2^0 &= \{2, 3\} & G_2^0 &= \{1, 2, 3\} \\
I_3^0 &= \{2, 3, 4\} & G_3^0 &= \{2, 3, 4\}
\end{aligned}
$$

Information Hierarchy



Then define

$$\leq^0$$

by

$$i \in \leq^0 \Leftrightarrow \exists j \in \bar{G}_i (I_i^0 \subseteq G_j^0 \cap R^0)$$

- For instance,

$$2 \in \leq^0, 3 \notin \leq^0$$

- Since

$$\begin{array}{ll} I_2^0 = \{2, 3\} & G_2^0 \cap R^0 = \{2, 3\} \\ I_3^0 = \{2, 3, 4\} & G_3^0 \cap R^0 = \{2, 3, 4\} \end{array}$$

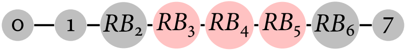
Information Hierarchy

$$0 - 1 - RB_2 - RB_3 - RB_4 - RB_5 - RB_6 - 7$$

At 1-block, let

$$R^1 \equiv \{ i \in R^0 | i \notin \leq^0 \} = \{ \quad 3, 4, 5 \quad \}$$

Information Hierarchy



At 2-block, let

$$
\begin{aligned}
G_i^1 &\equiv \bigcup_{k \in I_i^0} G_k \\
I_i^1 &\equiv \bigcup_{k \in G_i \cap R^1} I_k^0
\end{aligned}
$$

For instance,

$$
\begin{aligned}
I_3^1 &= \{2, 3, 4, 5\} \qquad & G_3^1 &= \{1, 2, 3, 4, 5\} \\
I_4^1 &= \{2, 3, 4, 5, 6\} \qquad & G_4^1 &= \{2, 3, 4, 5, 6\}
\end{aligned}
$$

Information Hierarchy

$0 - 1 - RB_2 - RB_3 - RB_4 - RB_5 - RB_6 - 7$

Then define

$$\leq^1$$

by

$$i \in \leq^1 \Leftrightarrow \exists j \in \bar{G}_i \left(I_i^1 \subseteq G_j^1 \cap R^0 \right)$$

- For instance,

$$3 \in \leq^1, 4 \notin \leq^0$$

- Since

$$\begin{array}{ll} I_3^1 = \{2,3,4,5\} & G_3^1 \cap R^0 = \{2,3,4,5\} \\ I_4^1 = \{2,3,4,5,6\} & G_4^1 \cap R^0 = \{2,3,4,5,6\} \end{array}$$

Information Hierarchy

$$0 - 1 - RB_2 - RB_3 - RB_4 - RB_5 - RB_6 - 7$$

At 2-block, let

$$R^2 \equiv \{ i \in R^1 | i \notin \leq^1 \} = \{ \quad 4 \quad \}$$

Information Hierarchy

Theorem

Given θ, if

1. *the network is FFCCU and acyclic*
2. *the state has strong connectedness*

$\Rightarrow \exists t^{\theta}$ *and* $\exists i \in R^{t^{\theta}}$ *such that* $I_i^{t^{\theta}} \supset [Rebels](\theta)$.

So, APEX can be attained by

R^t Rebels	play	$\langle I_i^{t-1} \rangle$	$\mathbf{s}, \ldots, \mathbf{s}, \overbrace{\mathbf{r}, \mathbf{s}, \ldots, \mathbf{s}}^{\prod_{j \in I_i^{t-1}} x_j}$
non-R^t Rebels	play	$\langle \mathbf{stay} \rangle$	$\mathbf{s}, \ldots, \mathbf{s}, \mathbf{s}, \mathbf{s}, \ldots, \mathbf{s}$

However, "Pivotal Rebels" will deviate.

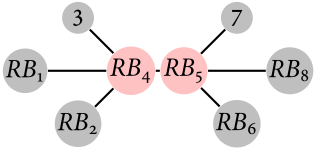
Information Hierarchy

Pivotal players

Definition

i is pivotal in $RP^t \Leftrightarrow i \in R^t$ and i **will** know $\#[Rebels](\theta) \geq k$ or $\#[Rebels](\theta) < k$ after RP^t **before** I_i^{t-1} is reported.

1. Ex. $k = 5$
2. Rebel 4 and Rebel 5 are pivotal (**Free Rider problem**)



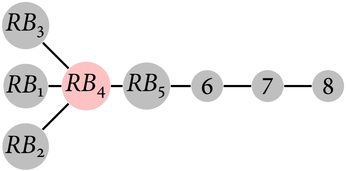
3. They will manipulate their reporting to save costs.
   - By reporting some other number.

Pivotal players

Definition

i is pivotal in $RP^t \Leftrightarrow i \in R^t$ and i **will** know that $\#[Rebels](\theta) \geq k$ or $\#[Rebels](\theta) < k$ after RP^t **before** I_i^{t-1} is reported.

1. Ex. $k = 6$
2. Rebel 4 is pivotal



3. He will manipulate his reporting to save costs.
   - By reporting some other number.

Information Hierarchy

Solving Pivotal-player problem. Step 1.

Definition

Free Rider Problem A FRP in a t-block is that $\exists i, j \in R^t$, $i \neq j$ such that

1. i, j is pivotal in RP^t
2. i, j **will** know the $\#[Rebels](\theta)$ after RP^t **before** I_i^{t-1} is reported.

Solving Pivotal-player problem. Step 1.

Lemma

If networks are acyclic, then

- *there is a unique block B^t where FRP may occur.*
- *there are only two $i, j \in R^t$ are involved, and $i \in G_j$.*
- *Moreover, both of i, j know that they will be involved before B^t and after B^{t-1}.*

Thus, before B^t and after B^{t-1}, pick one of them be pivotal player.

- By their prim number.

Information Hierarchy

Solving Pivotal-player problem. Step 2.

Non-pivotal R^t Rebels	play	$\langle I_i^{t-1} \rangle$	$\mathbf{s}, \ldots, \mathbf{s}, \overbrace{\mathbf{r}, \mathbf{s}, \ldots, \mathbf{s}}^{\prod_{j \in I_i^{t-1}} x_j}$
Pivotal R^t Rebels	may play	$\langle 1 \rangle$	$\mathbf{s}, \ldots, \mathbf{s}, \mathbf{s}, \mathbf{s}, \ldots, \mathbf{r}$
non-R^t Rebels	play	$\langle \textbf{stay} \rangle$	$\mathbf{s}, \ldots, \mathbf{s}, \mathbf{s}, \mathbf{s}, \ldots, \mathbf{s}$

I.e. Add $\langle 1 \rangle$ into the equilibrium path.

Information Hierarchy

Solving Pivotal-player problem. Step 3.
In the equilibrium path,

Lemma

If networks are acyclic, in RP^t, before i plays I_i^{t-1}

$$i \text{ knows that } \#[Rebels](\theta) \geq k - 1$$

$\Leftrightarrow$

$$i \text{ is pivotal but } i \text{ may not know } \#[Rebels](\theta) \text{ after } RP^t$$

Lemma

If networks are acyclic,

$$i \text{ knows that } \#[Rebels](\theta) \geq k - 1$$

$\Leftrightarrow$

$$i \text{ play } \langle 1 \rangle$$

Information Hierarchy

Solving Pivotal-player problem. Step 3.
Consequently, in the path,

i has played	i in FRP	$j \in G_i$ play	i knows
$\langle 1 \rangle$	yes	$\langle \cdot \rangle$	$\#[Rebels](\theta) \geq k$
$\langle 1 \rangle$	no	$\langle 1 \rangle$	$\#[Rebels](\theta) \geq k$
$\langle 1 \rangle$	no	$\langle \textbf{stay} \rangle$	$\#[Rebels](\theta) < k$

i has to play **message to revolt** or **message to revolt** if he played $\langle 1 \rangle$

Table : Equilibrium path if i played $\langle 1 \rangle$

In RP^t	In $CD^t_{1,1}$	In $CD^t_{1,2}$	After
i plays	i plays	i plays	
$\langle 1 \rangle$	$\langle \textbf{stay} \rangle$	$\langle \textbf{stay} \rangle$	**coordination to stay**
$\langle 1 \rangle$	$\langle \mathbf{x}_i \rangle$	$\langle \textbf{stay} \rangle$	**coordination to revolt**

Beliefs in equilibrium path

In the equilibrium path

Table : In RP^t

R^t	either play	$\langle I_i^{t-1}\rangle$	$\mathbf{s}, \ldots, \mathbf{s}, \overbrace{\mathbf{r}, \mathbf{s}, \ldots, \mathbf{s}}^{\prod_{j\in I_i^{t-1}} x_j}$
R^t	or play	$\langle 1\rangle$	$\mathbf{s}, \ldots, \mathbf{s}, \mathbf{s}, \mathbf{s}, \ldots, \mathbf{r}$
R^t	play	$\langle \mathbf{stay}\rangle$	$\mathbf{s}, \ldots, \mathbf{s}, \mathbf{s}, \mathbf{s}, \ldots, \mathbf{s}$

Table : Belief updating after CD^t, $t > 0$

In RP^t	In $CD^t_{1,1}$	In $CD^t_{1,2}$	
i plays	i plays	i plays	The events j believe with probability one
$\langle \mathbf{stay}\rangle$	$\langle \mathbf{x}_i\rangle$	$\langle \mathbf{stay}\rangle$	$i \notin R^t$
$\langle I_i^{t-1}\rangle$	$\langle \mathbf{stay}\rangle$	$\langle \mathbf{stay}\rangle$	$\#[Rebels](\theta) < k$
$\langle I_i^{t-1}\rangle$	$\langle \mathbf{x}_i\rangle$	$\langle \mathbf{stay}\rangle$	$\#[Rebels](\theta) \geq k$
$\langle I_i^{t-1}\rangle$	$\langle \mathbf{x}_i\rangle$	$\langle \mathbf{x}_i\rangle$	$i \in R^t$
$\langle 1\rangle$	$\langle \mathbf{stay}\rangle$	$\langle \mathbf{stay}\rangle$	$\#[Rebels](\theta) < k$
$\langle 1\rangle$	$\langle \mathbf{x}_i\rangle$	$\langle \mathbf{stay}\rangle$	$\#[Rebels](\theta) \geq k$

Off-path Belief

Whenever i detects a deviation, he believes that

$$\text{for all } j \notin G_i,\ \theta_j \neq \text{Rebel}$$

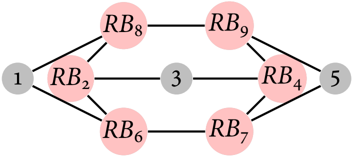
1. If $\#(G_i \cap [Rebels](\theta)) < k$, he will play **stay** forever.
2. This off-path belief then also serve as a grim trigger - belief-grim-trigger.

Sketch of proof

1. The equilibrium path is APEX.
2. If game enters B^t, all Rebels have not know relevant info. before B^t.
3. Detectable deviation $\Rightarrow$ APEX **may** fail by belief-grim-trigger.
4. Undetectable deviation $\Rightarrow$ APEX **may** fail by protocol-grim-trigger
   - pivotal R^t, non-pivotal R^t, non-R^t, will not mimic each other.
5. Ex-post outcome gives maximum ex-post static pay-off.
6. Sufficiently high δ will impede deviation.

Discussion

1. From the above steps, an APEX equilibrium for acyclic networks is constructed.
2. Solving Pivotal-player problem for cyclic networks need more elaboration



Discussion

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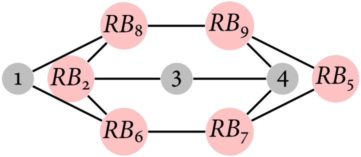

Discussion

1. payoff is perfectly observed
   - Play **revolt** in the first period, then the relevant information revealed.
2. payoff is noisy
   - With full support assumption, the existing equilibrium is APEX.
   - Ex.

$$
\begin{aligned}
p_{1s} &= \Pr(y = y_1 | \#\textbf{revolt} \geq k) \\
p_{1f} &= \Pr(y = y_1 | \#\textbf{revolt} < k) \\
p_{2s} &= \Pr(y = y_2 | \#\textbf{revolt} \geq k) \\
p_{2f} &= \Pr(y = y_2 | \#\textbf{revolt} < k)
\end{aligned}
$$

$$
1 > p_{1s} > 0, 1 > p_{2s} > 0, p_{1f} = 1 - p_{1s}, p_{2f} = 1 - p_{2s} \tag{1}
$$

Further works

1. For the networks with circles, the proof for an APEX equilibrium is still open.
2. There should be a general model in which players can communicate only by their actions to learn the relevant information in finite time when $\delta < 1$, while the communication protocol itself is an equilibrium.
3. Communication in network could serve as a criteria in equilibrium selection.