Coordination in Social Networks: Communication by Actions

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draft: v4.0

Abstract

1 Introduction

2 Model

There is a set of players $N = \{1, ..., n\}$. They constitute a network $G = (V, E)$ so that the vertices are players ($V = N$) and an edge is a pair of them (E is a subset of the set containing all two-element subsets of N). Throughout this paper, G is assumed to be finite, commonly known, fixed, undirected, and connected.[1]

Time is discrete and denoted by $S = \{0, 1, ...\}$ with index s. Each player could be either type R or type I assigned by the nature at $s = 0$ according to a common prior π; R or I represents a Rebel or an Inert respectively. Call $\theta \in \Theta \equiv \{R, I\}^n$ a state of nature. At each $s \geq 1$, players play a normal form game, the *k-threshold game*, infinitely repeated played with common discounted factor $\delta \in (0, 1)$. In the k-threshold game, $A_R = \{\textbf{revolt}, \textbf{stay}\}$

[1]A path in G from i to j is a finite sequence $(l_1, l_2, ..., l_L)$ without repetition so that $l_1 = i$, $l_L = j$, and $\{l_q, l_{q+1}\} \in E$ for all $1 \leq q < L$. G is fixed if G is not random, and G is undirected if there is no order relation over each edge. G is connected if, for all $i, j \in N$, $i \neq j$, there is a path from i to j.

is the set of actions for R and $A_I = \{\textbf{stay}\}$ is that for I. Denote by $\#X$ the cardinality of an set X. A Rebel i's stage-game payoff function is defined as below, while an Inert's stage-game payoff is equal to 1 no matter how other players play.

$$\begin{array}{ll} u_R(a_i, a_{-i}) = 1 & \text{if } a_i = \textbf{revolt} \text{ and } \#\{j : a_j = \textbf{revolt}\} \geq k \\ u_R(a_i, a_{-i}) = -1 & \text{if } a_i = \textbf{revolt} \text{ and } \#\{j : a_j = \textbf{revolt}\} < k \\ u_R(a_i, a_{-i}) = 0 & \text{if } a_i = \textbf{stay} \end{array} .$$

Let $[R](\theta)$ be the set of Rebels given θ and the notion *relevant information* indicate whether or not $\#[R](\theta) \geq k$. Note that the ex-post efficient outcome in the stage game is that every Rebel plays **revolt** whenever $\#[R](\theta) \geq k$, and plays **stay** otherwise.[2]

During the game, every player can observe his and his neighbors' types and his and their histories of actions, but no more. A history of actions played by i from period one to period $s \geq 1$ is denoted by $h_i^s \in H_i^s \equiv \times_{\varsigma=1}^{s} A_{\theta_i}$. Let $G_i := \{j : \{i,j\} \in E\}$ be i's *neighbors*. Denote $\theta_{G_i} \in \Theta_{G_i} \equiv \{R, I\}^{G_i}$ as the type profile of i's neighbors. Let $h_i^0 = \emptyset$, and denote $h_{G_i}^s \in H_{G_i}^s \equiv \times_{j \in G_i} \times_{\varsigma=1}^{s} H_j^{\varsigma}$ as a history of actions played by i's neighbors from period one to period $s \geq 1$. The information set of i about θ at every period is the cylinder $p(\theta) = \{\theta_{G_i}\} \times \{R, I\}^{N \setminus G_i}$, and the information set about histories of action from period one to period $s \geq 1$ is $\{h_{G_i}^s\} \times H_{N \setminus G_i}^s$. A player i's pure behavior strategy τ_i is a measurable function with respect to i's information partition if τ_i maps $\{\theta_{G_i}\} \times \{R, I\}^{N \setminus G_i} \times \{h_{G_i}^s\} \times H_{N \setminus G_i}^s$ to a single action in his action set for every $s \in \{1, 2, ...\}$ and every $\theta \in \Theta$. I assume that payoffs are hidden to emphasize that observing neighbors' actions are the only channel to infer other players' types and actions.[3]

Likewise, define $H^s := \times_{j \in N} H_j^s$ as the set of histories of actions from period one to period $s \geq 1$ and $H := \bigcup_{\varsigma=0}^{\infty} H^{\varsigma}$ as the collection of histories of actions. By abusing the notation a bit, let $h(\tau, \theta) \in H$ denote the realized history of actions generated by strategy profile $\tau = (\tau_1, \tau_2, ..., \tau_n)$ given θ. Designate $\alpha_{G_i}^{\pi,\tau}(\theta, h^s | \theta_{G_i}, h_{G_i}^s)$ as the conditional distribution over $\Theta \times H^s$ induced by π and τ conditional on i's information up to period $s \geq 1$. The belief

[2]Moreover, at every θ and every k, the ex-post efficient outcome is unique and gives the maximum as well as the same payoff to every Rebel.

[3]Such restriction will be relaxed in the Section 5.

of i over Θ induced by π and τ up to period $s \geq 1$ is defined by

$$\beta^{\pi,\tau}_{G_i}(\theta | h^s_{G_i}) := \sum_{h^s \in H^s} \alpha^{\pi,\tau}_{G_i}(\theta, h^s | \theta_{G_i}, h^s_{G_i}).$$

The equilibrium concept is the weak sequential equilibrium.[4] My objective is looking for the existence of *approaching ex-post efficient equilibrium* or *APEX equilibrium*, which is defined below.

Definition 2.1 (APEX strategy)**.** *A behavior strategy τ is APEX if for all θ, there is a terminal period $T^\theta < \infty$ such that the actions in h^τ_θ after T^θ repeats the static ex-post Pareto efficient outcome.*

Definition 2.2 (APEX equilibrium)**.** *An equilibrium (τ^*, α^*) is APEX if τ^* is APEX.*

In an APEX strategy, all Rebels will play **revolt** forever after some period only if $\#[R](\theta) \geq k$; otherwise, Rebels will play **stay** forever after some period. It is as if the Rebels will learn the relevant information in the equilibrium because they will play the ex-post efficient outcome after a certain point of time and keep on doing so. Notice that, in an APEX equilibrium, it is not only as if the Rebels will learn the relevant information: they must learn that. Lemma 2.1 articulates this fact.

Lemma 2.1 (Learning in the APEX equilibrium)**.** *If the assessment (τ^*, μ^*) is an APEX equilibrium, then for all $\theta \in \Theta$, there is a finite time T^θ_i for every Rebel i so that*

$$\sum_{\theta \in \{\theta : [R](\theta) \geq k\}} \beta^{\pi,\tau^*}_{G_i}(\theta | h^s_{G_i}) = \textit{ either } 1 \textit{ or } 0$$

whenever $s \geq T^\theta_i$.

[4] A weak sequential equilibrium is an assessment $\{\tau^*, \alpha^*\}$, where α^* is a collection of distributions over players' information sets with the property that, for all $i \in N$ and for all $s = 1, 2, ...$, $\alpha^*_{G_i}(\theta, h^s | \theta_{G_i}, h^s_{G_i}) = \alpha^{\pi,\tau^*}_{G_i}(\theta, h^s | \theta_{G_i}, h^s_{G_i})$ whenever the information set is reached with positive probability given τ^*. Moreover, for all $i \in N$ and for all $s = 1, 2, ...$, τ^*_i maximizes i's continuation expected payoff of

$$E^\delta_G(u_{\theta_i}(\tau_i, \tau^*_{-i}) | \alpha^{\pi,\tau_i,\tau^*_{-i}}_{G_i}(\theta, h^s | \theta_{G_i}, h^s_{G_i}))$$

conditional on θ_{G_i} and $h^s_{G_i}$ for all $h^s_{G_i} \in H^s_{G_i}$.

Definition 2.3 (Learning the relevant information). *A Rebel i learns the relevant information at period ς according to strategy τ if $\sum_{\theta \in \{\theta:[R](\theta) \geq k\}} \beta^{\pi,\tau}_{G_i}(\theta | h^s_{G_i}) =$ either 1 or 0 whenever $s \geq \varsigma$.*

It is clear that an APEX equilibrium exists when $k = 1$. As for other cases, let us start with the case of $k = n$ and then continue on to the case of $1 < k < n$. The proof is by construction. In the case of $k = n$, the constructed APEX equilibrium is intuitive and satisfies a stronger equilibrium concept. My main result tackles the case of $1 < k < n$. In such case, my constructed APEX equilibrium is not trivial and can only works for acyclic networks. Section 5.2 discusses why my constructed equilibrium is intractable in cyclic networks.

3 Equilibrium: APEX for $k = n$

In this section, my objective is to show the existence of APEX equilibrium for the case of $k = n$. In this case, notice that a Rebel can get a better payoff from playing **revolt** than from **stay** *only if* all players are Rebels. Two consequences follow. Firstly, if a Rebel has an Inert neighbor, this Rebel will always play **revolt** in the equilibrium. Secondly, at any period $s \geq 1$, it is credible for every Rebel to punish a deviation by playing **stay** forever *if* there is another one who also plays **stay** forever, independently from the belief held by the punisher. These two features constitute an APEX equilibrium and further transform itself to a sequential equilibrium.

Theorem 1 (APEX equilibrium for the case of $k = n$). *For any n-person repeated k-Threshold game with parameter $k = n$ played in a network, there is a $\delta^* \in (0, 1)$ such that a sequential APEX equilibrium exists whenever $\delta > \delta^*$.*

Image that there are an Inert somewhere as well a Rebel i somewhere. Since the network is connected, there is a path connecting these two players. Along with this path, consider the "closest" Inert from Rebel i; this is an Inert who can be reached by the least number of consecutive edges from i. Note that this Inert's Rebel neighbors will play **stay** forever

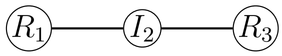

Figure 1: The state and the network in which the APEX equilibrium does not exist when $k = 2$.

since $k = n$. Consider a strategy for Rebels on this path: a Rebel will play **stay** only after observing his neighbor plays **stay**. On this path and according to this strategy, Rebel i will know the existence of such Inert eventually since the network is finite. This contagion argument suggests the following APEX strategy. Every Rebel plays **revolt** initially except for he has an Inert neighbor. Each of them will continuously play **revolt** but switch to **stay** instantly if he observes any of his neighbor plays **stay**. Upon observing a n consecutive **revolt**, a Rebels knows that no Inert exists; otherwise, he knows some Inert exists. The above strategy is an APEX strategy if all Rebels play ex-post efficient outcome after peiord n. To extend it to be an APEX equilibrium, let the deviant play **stay** forever and the punisher who detects it also play **stay** forever. This out-of-path strategy is credible for both the deviant and the punisher, independent from the belief held by the punisher, and hence it is also sequential rational.[5]

4 Equilibrium: APEX for $1 < k < n$

In this section, my objective is to show the existence of APEX equilibrium for the case of $1 < k < n$. In contrast to the case of $k = n$, a Rebel still has the incentive to play **revolt** even if he has an Inert neighbor. This opens a possibility for the non-existence of APEX equilibrium. Example 1 below demonstrates it.

Example 1. Suppose that $k = 2$ and $\theta = (R, I, R)$. The state and the network is represented in Figure 1. Rebel 1 never learns the type of player 3 since Inert 2 cannot reveal it. Therefore no APEX equilibrium exists in this scenario.

The following assumption on the prior—*full support on strong connectedness*—excludes

[5] This sequential rationality is in the sense of [Kreps and Wilson, 1982].

the possibility of nonexistence of APEX equilibrium. To this end, I begin with the definition of *strong connectedness*.

Definition 4.1 (Strong connectedness)**.** *Given G, a state θ has strong connectedness if, for every two Rebels, there is a path consisting of Rebels to connect them.*

In the language of graph theory, the following definition is equivalent: given G, θ has strong connectedness if the induced graph by $[R](\theta)$ is connected.

Definition 4.2 (Full support on strong connectedness)**.** [6] *Given G, π has full support on strong connectedness if*

$$\pi(\theta) > 0 \Leftrightarrow \ \theta \textit{ has strong connectedness.}$$

I state the main characterization of this paper:

Theorem 2 (APEX equilibrium for the case of $1 < k < n$)**.** *For any n-person repeated k-Threshold game with parameter $1 < k < n$ played in networks, if networks are acyclic and if π has full support on strong connectedness, then there is a $\delta^* \in (0, 1)$ such that an APEX equilibrium exists whenever $\delta > \delta^*$.*[7]

Constructing an APEX equilibrium in this case is convoluted. I illustrate the proof idea throughout this paper while leaving the formal proof in Appendix. Moreover, since the case of $k = 2$ is trivial given that θ has strong connectedness, I focus on $2 < k < n$ cases.[8]

[6]The full support on strong connectedness is stronger than common knowledge about that every state has strong connectedness. This marginal requirement is subtle and is more convenient in constructing equilibrium. The main result only requires a weaker version: $\pi(\theta) > 0 \Rightarrow \ \theta$ has strong connectedness . However, working on this weaker version is at the expense of much tedious proof. Throughout this paper, I will stick to the original definition.

[7]A network is acyclic if the path from i to j for all $i \neq j$ is unique.

[8]Suppose $[R](\theta) \geq k = 2$, by the full support on strong connectedness, each Rebel have a Rebel neighbor. The following strategy is an APEX strategy. A Rebel plays **revolt** forever from period one if he has a Rebel neighbor; otherwise, he plays **stay** forever from period one. It can be extended to an APEX equilibrium by letting the out-of-path belief be assigning probability one on the event that all non-neighbor players are Inerts.

I organize the proof idea in the following manner. To begin, I consider a specific APEX strategy to be the framework in constructing the APEX equilibirum for Theorem 2; incentive compatibility is not incorporated at this moment. Next, I introduce an auxiliary framework to illuminate the incentive compatibility issue in Section 4.1. I then go back to present essential details of the constructed APEX equilibrium in Section 4.2. To this end, let us first exam an example below.

Example 2. Let $k = 5$, and consider the configurations in Figure 2 and Figure 3. Note that there are 6 players in both of these configurations, 5 Rebels are in Figure 2 and 4 Rebels are in Figure 3. An APEX strategy can be constructed below.

At period j, $1 \leq j \leq 6$, each Rebel plays **revolt** if his own or any of his Rebel neighbor's index is j. Otherwise, he plays **stay**. As a consequence, from period one to period 6, what Rebel 3 plays is (**revolt**, **stay**, **revolt**, **revolt**, **stay**, **revolt**) in the configuration in Figure 2 and is (**revolt**, **stay**, **revolt**, **revolt**, **stay**, **stay**) in the configuration in Figure 3. Take another example, what Rebel 6 plays is (**stay**, **stay**, **revolt**, **stay**, **stay**, **revolt**) in the configuration in Figure 2.

Right after period 6, at period 7, a Rebel who has learnt the number of Rebels exceeding or equal to k, implied by observing his neighbors' actions, plays (**revolt**, **revolt**) consecutively. And then he plays **revolt** from period 19 afterwards. Likewise, a Rebel who has observed the number of Rebels implied by his neighbors' actions is less than k, he plays (**stay**, **stay**) consecutively. And then he plays **revolt** from period 19 afterwards. If a Rebel's observation does not belong to the above two categories, he plays the sequence (**revolt**, **stay**) consecutively and plays **stay** forever after period 19.

From period 7 to period 19, a Rebel plays (**revolt**, **revolt**) consecutively right after observing the same sequence played by any of his neighbor and then plays **revolt** from period 19 afterwards. Similarly, a Rebel plays (**stay**, **stay**) consecutively right after observing the same sequence played by any of his neighbor and then plays (**stay** from period 19 afterwards.

The out-of-path strategy is simple: following the strategy by which more Rebels would be implied. Along with the in-path strategy of the above strategy, both Rebel 1 and 4 learn the relevant information right after period 7; all Rebels play the ex-post outcome from period 19 afterwards. Thus this strategy is an APEX strategy.

The above in-path APEX strategy is straightforward: in a *reporting phase*, during period one to period 6, Rebels "truthfully reports" their Rebel neighbors by alternating their actions, and, in a *coordination phase*, during period 7 to period 19, they "coordinate" the knowledge whether the relevant information is learnt. This strategy might, nevertheless, be too primitive to be integrate incentive compatibility. For instance, Rebel 6's actions are "redundant" comparing to Reble 3's: the ex-post efficient outcome will be achieved anyway despite how Rebel 6 plays, for that the Rebel neighbors of Rebel 6 are also Rebel3's, and, morover, how Rebel 6 plays cannot be observed by other Rebels other than Rebel 3. This notion redundant will be replaced and formally defined by *inactive* later soon to distinguish it from the notion of *active*. On the other hand, this strategy is "too short," for the specification of action variation only up to period 17. In larger networks with larger k, say a chain with 200 players and $k = 100$, a strategy with a specification of longer action variation will be needed. Lastly but not inclusively, Rebel 4 has incentive to deviate from truthful reporting since his Rebel neighbors are the same as Rebel 1's, and, moreover, his actions can be observed by the same set of players as Rebel 1's. Consequently, the ex-post efficient outcome will be played no matter how Rebel 6 plays. Note that Rebel 1's position is identical to Rebel 4's, and thus both of them have no incentive to reporting truthfully—a "free-rider problem." Note that, whether a free-rider problem could happen is not solely determined by whether a network is cyclic. It depends on how an APEX strategy is constructed and still exists in the constructed APEX equilibrium for Theorem 2 in which only acyclic networks are of concern. The formal definition of free-rider problem is in Section 4.2, and the in-depth issue about it will be in Section 5.2.

An APEX equilibrium construction thus calls for revising the above strategy. For this purpose, I begin with setting up a framework in constructing APEX strategy in which active and inactive Rebels are defined. Let us construct a set W that consists of sequences of actions, in which all sequences have equal length, so that there is a one-to-one mapping between Θ and W. This W exists because the network and the states are finite. For instance,

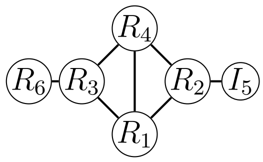

Figure 2: A configuration of the state and the network in which players 1, 2, 3, 4, 5 are Rebels and players 6 is an Inerts.

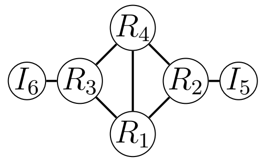

Figure 3: A configuration of the state and the network in which players 1, 2, 3, 4 are Rebels and player 5, 6 are Inerts.

the length of each sequence in W is n. Given θ, the i-th component in the corresponding w_θ is **revolt** if i is a Rebel; otherwise, it is **stay**. Take another example, which is used in the constructed APEX equilibrium for Theorem 2, the length of each sequence in W is the multiplication of a series of prime numbers. In this series, each prime number is distinct and assigned to distinct player. Denote x_i as the prime number assigned to i. The length of a sequence is therefore $\bigotimes_{i\in N} x_i = x_1 \otimes ... \otimes x_n$, where $\otimes$ is the usual multiplication operator. A θ has $[R](\theta)$ Rebels, and the corresponding w_θ is crafted to be:

$$(\overbrace{\textbf{stay}, ..., \textbf{stay}, \underbrace{\textbf{revolt}, \textbf{stay}, ..., \textbf{stay}}_{\bigotimes_{i\in[R](\theta)} x_i}}^{\bigotimes_{i\in N} x_i}).$$

There is a one-to-one mapping between Θ and W since a multiplication of prime numbers can be uniquely factorized. The above observation is organized as follows.

Proposition 4.1. *There is a set $W \subseteq \{\textbf{revolt}, \textbf{stay}\}^L$, where $L \in \mathbb{N}$, so that there is a bijective mapping $f : \Theta \to W$.*

Fix W and then partition the time by $\{\{0\}, \{1, ..., s_1\}, \{s_1+1, ..., s_2\}, ..., \{s_{t-1}+1, ..., s_t\}, ...\}$, where $t = 1, 2, ...$ and $s_0 = 0$, so that the length of $\{s_{t-1} + 1, ..., s_t\}$ is equal to the length of $w \in W$ for each t. Call $\{s_{t-1}+1, ..., s_t\}$ the t-block. Let $\langle I \rangle \in W$ be the sequence in W that represents the state in which I is the set of Rebels. Note that this $\langle I \rangle$ is unique given I.

Given θ, denote the set I_i as i's Rebel neighbors. If i is a Rebel, let $I_i^1 = I_i$, and let $I_i^t = \bigcup_{j \in G_i} I_j^{t-1}$ for $t \geq 2$. If j is an Inert, let $I_j^t = \emptyset$ for $t \geq 1$. In short, I_i^t is the set of Rebels who can be reached from Rebel i by a path consisting of Rebels and of which the length is at most t.

The phrase i *learns* θ indicates there is a period $s \geq 1$ so that i assigns probability one to the event $\{\theta\}$ by observing histories of actions. The following proposition is immediately obtained.

Proposition 4.2. *If θ has strong connectedness, then there is a strategy so that there exists a Rebel who can learn θ.*

Proof. The strategy is as follows.

Strategy 4.2: At each t-block, each Rebel i plays $\langle I_i^t \rangle$

Following the above strategy, right after t-block, Rebel i assigns probability one to the event

$$\{\theta : \theta_j = R \text{ and } j \in I_i^{t+1}\}$$

by Bayesian rule. To conclude the proof, the remaining is to show there exists a t so that $I_i^{t+1} = [R](\theta)$. By definition, I_i^t is the set of Rebels who can be reached by at most t consecutive edges from Rebel i, in each of which the endpoints are Rebels. Since θ has strong connectedness, there exists a t_i so that $I_i^{t_i} = [R](\theta)$. What follows is i learns θ at t_i. $\square$

Remark. The strong connectedness assumption in Proposition 4.2 is indispensable as Example 1 demonstrates. Proposition 4.2 is essentially an if-and-only-if result.

The above Strategy 4.2 would be troublesome if incentive compatibility is incorporated. This is because tracing the expected payoff of every player in the network is a laboring task. To reduce the complexity, I identify a smaller set of Rebels, *active Rebels*, who are crucial

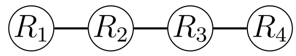

Figure 4: A configuration of the state and the network in which players 1, 2, 3, 4 are Rebels.

in the information sharing process and thus needed to be traced. First define G_i^t for each t: if i is a Rebel, let $G_i^1 = G_i$, and let $G_i^t = \bigcup_{j \in G_i} G_j^{t-1}$ for $t \geq 2$; if j is an Inert, let $G_j^t = \emptyset$ for $t \geq 1$. Simply put, G_i^t is the set of players who can be reached from Rebel i by a path consisting of Rebels and of which the length is at most t. Then define the active Rebels at t-block as follows.

Definition 4.3 (Active Rebel at t-block)**.** *Set $R^0 = [R](\theta)$. The set of active Rebels at t-block is*

$$R^t := \{i \in R^{t-1} : \nexists j \in G_i \text{ such that } I_i^t \subseteq G_j^t\}.$$

Otherwise speaking, an active Rebel in the t-block is a Rebel whose information about θ, I_i^t, is not a subset of any other Rebel's same information. For instance, in the configuration in Figure 4, $R^0 = \{1, 2, 3, 4\}$, $R^1 = \{2, 3\}$, and $R^2 = R^3 = ... = \emptyset$. Furthermore, the active Rebels have to be also the active ones in the previous block; they are fewer and fewer as t goes by.

It is indeed sufficient to reveal the relevant information by letting only active Rebels share information about θ, given that the network is acyclic and given that θ has strong connectedness. Theorem 3 articulates this.

Theorem 3. *If the network is acyclic and if the θ has strong connectedness, then there is a strategy so that there exists a R^t Rebel who can learn θ at $t+1$-block.*

The following strategy is for Theorem 3.[9]

Strategy 3: At each t-block, each active Rebel i at t-block plays $\langle I_i^t \rangle$.

[9] Comparing to Strategy 4.2, only fewer Rebels play actions to share information. Theorem 3 is equivalent to the following statement: if the network is acyclic and if the θ has strong connectedness, then there exists $t \geq 0$ and $i \in R^t$ so that $I_i^{t+1} = [R](\theta)$.

$$0 < \underbrace{(\text{coordination phase}) < (\text{reporting phase})}_{\text{1-block}} < \underbrace{(\text{coordination phase}) < (\text{reporting phase})}_{\text{2-block}} < ...$$

Figure 5: The partition of the time in the repeated k-threshold game. $<$ is the linear order relation over the time.

Remark. Theorem 3 is not true if the network is cyclic. Take the configuration in Figure**??** as an example. There, $R^0 = \{1, 2, 3, 4, 5, 6\}$, $R^1 = \{2, 3\}$, and $R^2 = R^3 = ... = \emptyset$. In the 2-block, there is no Rebel who learns θ. In the 3-block, both Rebel 3 and 4 learn θ, but neither Rebel 4 or 5 is active at 2-block.

Next, I construct an APEX strategy based on a modification of Strategy 3. Fix W again, but then partition the time slightly differently from the above mentioned. Partition the time into two consecutively alternating phases: *coordination phase* and *reporting phase*, while the time is starting from the coordination phase. The t-th completion of two consecutive phases is called the t-block. Figure 5 depicts this partition. The length of each reporting phase is equal to the length of $w \in W$, and the length of each coordination phase is $2n$. The usage of reporting phase is information sharing, and the usage of coordination phase is to coordinate when the ex-post efficient outcome will be played.[10]

Proposition 4.3. *If the network is acyclic and if the θ has strong connectedness, then there is an APEX strategy.*

The following is an APEX strategy for this proposition. Suppose T^θ has arrived, Rebels play the ex-post efficient outcome. Suppose T^θ has not yet arrived. In the reporting phase, Rebels follow Strategy 3. In the coordination phase, as below, the strategy is a contagion process.

1. If a Rebel has been certain $\#[R](\theta) < k$, he plays sequence of actions $(\textbf{stay}, \textbf{stay})$

[10] More precisely, partition the time by $\{\{0\}, \{1, ..., s_{l_1}, s_{l_1}+1, ...s_1\}, \{s_1+1, ..., s_{l_2}, s_{l_2}+1, ..., s_2\}, ..., \{s_{t-1}+1, ..., s_{l_t}, s_{l_t}+1, ..., s_t\}, ...\}$, where $t = 1, 2, ...$ and $s_0 = 0$, so that the length of $\{s_{l_t}+1, ..., s_t\}$ is equal to the length of $w \in W$ for each t, while the length of $\{s_{t-1}+1, ..., s_{l_t}\}$ is $2n$. Call $\{s_{t-1}+1, ..., s_t\}$ the t-block. call $\{s_{l_t}+1, ..., s_t\}$ the *reporting phase* at t-block and call $\{s_{t-1}+1, ..., s_{l_t}\}$ the *coordination phase* at t-block.

continuously starting right after he was certain that. He plays **stay** forever after this phase; T^θ arrives right after this phase.

2. If a Rebel has learnt $\#[R](\theta) \geq k$, he plays sequence of actions $(\mathbf{revolt}, \mathbf{revolt})$ continuously starting right after he learnt that. He plays **revolt** forever after this phase; T^θ arrives right after this phase.

3. If a Rebel has observed the sequence of actions $(\mathbf{stay}, \mathbf{stay})$, he plays $(\mathbf{stay}, \mathbf{stay})$ continuously starting right after he observed that and plays **stay** forever after this phase; T^θ arrives right after this phase.

4. If a Rebel has observed the sequence of actions $(\mathbf{revolt}, \mathbf{revolt})$, he plays $(\mathbf{revolt}, \mathbf{revolt})$ continuously starting right after he observed that and plays **revolt** forever after this phase; T^θ arrives right after this phase.

5. If a Rebel is uncertain $\#[R](\theta) \geq k$, he plays sequence of actions $(\mathbf{revolt}, \mathbf{stay})$ continuously.

The idea is simple. Rebels share information in reporting phase. If a Rebel has learnt the relevant information, he disseminates it to all Rebels contagiously in coordination phase; otherwise, he continues to the next phase—a reporting phase.

If incentive compatibility is a concern, this logic, however, brings a free-rider problem. Suppose that there are two Rebels who share information to each other in a reporting phase, and each of them is certain that he will learn the relevant information if the other one shares *truthful* information to him. Due to sharing information incurs positive or negative payoff, they will not truthfully share their information. This is because each of them will choose his most profitable way of sharing information without impeding learning the relevant information provided that the other one share the truthful information. The free-rider problem turns out to be the main challenge in the construction of an APEX equilibrium. The proof solves it by arguing that if the network is acyclic, the free-rider problem only occurs between two Rebel neigbors who *commonly know it*, while this argument does not hold for cyclic network.[11] With the help from this argument, the constructed equilibrium solves the free-rider problem by arbitrarily assigning one of them to be the free rider, who can choose his most profitable way in sharing information, while letting the other one share truthful information.

To make the discussion of incentive compatibility more transparent, I introduce *T-round writing game* as an auxiliary scenario. In the T-round writing game, T is fixed, players are endowed a writing technology so that they can write to share information about θ for T rounds. They then play a one-shot k-threshold game at round $T+1$. This game is a reduced form of the original game by fixing T, so that I can pay attention only to the incentive compatibility in information sharing process and ignore how players determine the terminal period. This writing technology represent a way of communication by *writing sentences that are composed by letters according to a fixed grammar*. Though this auxiliary game will be soon introduced in the next section, I draw Table 1 to shed light on the parallel between it and the original game.

Table 1: The analogue between T-round writing game and the repeated k-threshold game

T-round writing game	Repeated k-threshold game
A round	A range of periods
A sentence	A sequence of actions
The length of a sentence in a round	The length of a range of periods
A chosen letter in a sentence	A chosen action
The cost of writing a sentence	The expected payoff occurring in a sequence of actions
The fixed grammar	The equilibrium path

[11] Section 5.2 provides an example that the free-rider problem is not commonly known between the Rebels who involve.

4.1 T-round writing game

The network, the set of states, and the set of players follow exactly the same definitions defined in Section 2. In the T-round writing game, each player endows a *writing technology.* A writing technology for player i is a pair of (W, M_i), in which $W = \{\mathbf{r}, \mathbf{s}\}^L$, $L \in \mathbb{N}$, and $M := \times_{i \in N} \times_{t=1}^{T} M_i^t$ recursively defined by

$$M_i^1 = \{f | f : \Theta_{G_i} \to W\} \cup \{\emptyset\}$$

$$\text{for } 2 \leq t \leq T \text{ , } M_i^t = \{f | f : \times_{j \in G_i} M_j^{t-1} \to W\} \cup \{\emptyset\}.$$

W is interpreted as the set of sentence composed by letters $\mathbf{r}$ or $\mathbf{s}$ with length L, while M_i is understood as i's grammar. $\emptyset$ represents remaining silent. The phrase of "i writes a sentence to all his neighbors at round t" is equivalent to "i selects an $f \in M_i^t$ to get an element $w \in W$ according to f, which can be observed by all i's neighbors".

The time line for the deterministic T-round writing game is as follows.

1. Nature chooses θ according to the prior π.
2. θ is then fixed throughout rounds.
3. At $t = 1, ..., T$ round, players write to their neighbors.
4. At $T + 1$ round, players play a one-shot k-Threshold game.
5. The game ends.

A Rebel's payoff is the summation of his stage payoff across stages, while an Inert's payoff is set to be 1. The equilibrium concept is weak sequential equilibrium. An APEX strategy is a strategy that induces the ex-post outcome in the one-shot k-threshold game at $T + 1$ round. The definition of APEX equilibrium is adapted accordingly. In the examples below, let us focus on the configuration represented in Figure 6 and Figure 7 with $n = L = 8$. I.e. there are 8 players and the length of a sentence is also 8. Note that the differences between configurations in Figure 6 and Figure 7 are: (1) $\#[R](\theta) = 6$ in Figure 6 but $\#[R](\theta) = 5$ in Figure 7; (2) player 8 is a Rebel in Figure 6 but he is an Inert in Figure 6.

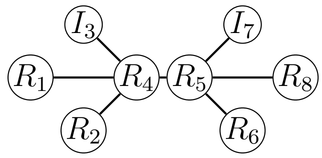

Figure 6: A configuration of the state and the network in which players 1, 2, 4, 5, 6, 8 are Rebels while players 3, 7 are Inerts.

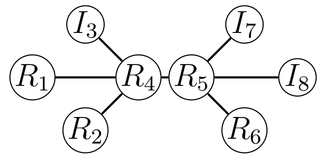

Figure 7: A configuration of the state and the network in which players 1, 2, 4, 5, 6 are Rebels while players 3, 7, 8 are Inerts.

Example 3 (T-round writing and the free-rider problem)**.** Let $k = 6$ and $T = 2$. Suppose that remaining silent incurs no cost, but writing incurs an extremely small cost $\epsilon > 0$ so that ϵ is strictly decreasing with the number of $\mathbf{r}$ in a sentence. This is to say writing $(\mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r})$ incurs the least cost while writing $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{s})$ incurs the largest. Consider the following strategy ϕ.

At $t = 1$, the peripheral Rebels remain silent. Rebel 4 (or 5)'s grammar for writing a sentence is that if player i is a Rebel and known to him, he writes $\mathbf{r}$ in the i-th component in the sentence; otherwise, he writes $\mathbf{s}$ in that component. According to this grammar, the central player Rebel 4 writes the sentence $(\mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s}, \mathbf{s})$ on both configurations in Figure 6 and in Figure 7. The central player Rebel 5 writes $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r})$ in the configuration in Figure 6 and writes $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s})$ in the configuration in Figure 7. Rebels 4 and 5's sentences thus reveals who are Rebels and who are not. Notice that the common knowledge of the network contributes to the ability of revealing players' types.

At $t = 2$, the peripheral Rebels still remain silent. Rebel 4 writes $(\mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r})$ in the configuration in Figure 6 and writes $(\mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s})$ in the configuration in Figure 7. Rebel 5 writes exactly the same sentence as Rebel 4. This is to say Rebel 4 and 5 share information at $t = 1$ and then coordinate to announce a mixture sentence at $t = T$.

At $T+1$, by counting $\mathbf{r}$ in Rebel 4 or 5's mixture sentence, all Rebels know whether the number of Rebels outnumbers k. This leads all Rebels to play the ex-post efficient outcome in the one-shot k-threshold game.

The above ϕ is not an APEX equilibrium. According to the above-mentioned, Rebel 4 will know the relevant information at $t = 2$ even if he deviates to writing that all his neighbors are Rebels, which incurs less cost than his truthful writing.[12][13] Rebel 5 is in the same situation as Rebel 4 and therefore also writes the sentence that indicates that all his neighbors are Rebels. However, these sentences are uninformative. It turns out that both of them will deviate, and neither of them can know the relevant information at $t = 2$.

Fortunately, the following example shows that the free-rider problem can be solved.

Example 4 (T-round writing and solving the free-rider problem)**.** The solution to solve the free-rider problem in the previous example is to extend T. It would open the possibility of the existence of a free rider at some round, while letting this free rider reveals relevant information at the next round. To this end, let $k = 6$ and $T = 3$. Consider the following strategy ρ and focus on the interaction between Rebels 4 and 5.

At $t = 1$, the lowest-index Rebel between Rebels 4 and 5 is the free rider, while the other one truthfully writes down his information. This is to say Rebel 4 will be the free rider and he writes the least-cost sentence. Rebel 5 writes $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r})$ in the configuration of Figure 6and writes $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s})$ in the configuration of Figure 7.

At $t = 2$, Rebel 4 has known the relevant information. Rebel 4 writes the least-cost sentence if $\#[R](\theta) \geq k$ but remains silent otherwise. The consequence is Rebel 4's behavior reveals the relevant information to his neighbors at this round. Rebel 5 remains silent instead.

At $t = T$, Rebel 5 has known the relevant information since he is Rebele 4's neighbor.

[12] This sentence is $(\mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s}, \mathbf{s})$, which incurs less cost than the truthfully reporting sentence $(\mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s}, \mathbf{s})$.

[13] If he remains silent, then this behavior will be considered as a deviation, and therefore he will never get the maximum payoff of 1. Hence, he will avoid doing so.

He writes the least-cost sentence if $\#[R](\theta) \geq k$ but remains silent otherwise. Therefore Rebel 5's behavior at this round reveals the relevant information to his neighbors. Rebel 4 remains silent instead.

At $T+1$, all Rebels know the relevant information by observing Rebels 4 and 5's behavior. They play the ex-post efficient outcome accordingly.

Remark. Why does Rebel 5 *know* that he is not a free rider and therefore behaves not like a free rider? The following is the reason. He *knows* that, by common knowledge of the network, he and Rebel 4 are in a free-rider problem. Moreover, by common knowledge of the network, he knows that Rebel 4 knows that he and Rebel 4 are in a free-rider problem, he knows that Rebels 4 knows that he knows that,...,and so forth. Consequently, Rebel 5 and 4 commonly know that they are engaged in a free-rider problem. Lemma 4.2 articulate a general result: engaging in a free-rider problem is a common knowledge between Rebels who involve for any acyclic network in the constructed APEX equilibrium. Lemma 4.2 is, however, false in cyclic networks. Section 5.2 discusses this issue.

4.2 Equilibrium construction

I then delver into substantial ideas in constructing an APEX equilibrium in the following sections, while leaving actual details in Appendix. The framework is adapted from Proposition 4.3, and the time is partitioned by two consecutively alternating phases as shown in Figure 5. Recall that the time is partitioned into

$$0 < \underbrace{(\text{coordination phase}) < (\text{reporting phase})}_{\text{1-block}} < \underbrace{(\text{coordination phase}) < (\text{reporting phase})}_{\text{2-block}} < \ldots$$

The out-of-path belief is simple and serves as a grim trigger. Whenever Rebel i detects a deviation at period ς, he forms the following belief:

$$\sum_{\theta \in \{\theta : \theta_j = I, j \notin G_i\}} \beta^{\pi,\tau}_{G_i}(\theta | h^s_{G_i}) = 1, \text{ for all } s \geq \varsigma. \tag{1}$$

This is to say i believe all players outside his neighborhood are Inerts. Thus, if $\#I^{\varsigma}_i < k$, he will play **stay** forever after he detects a deviation. This out-of-path belief thus serves as a grim trigger.

4.2.1 The equilibrium path in the reporting phase

The description in this section is for the APEX equilibrium path before T^θ. Let us shorten "reporting phase in t-block" by $\mathcal{O}^t$, denote $|\mathcal{O}^t|$ as the length of $\mathcal{O}^t$, and shorten **revolt** and **stay** to **r** and **s** receptively. For simplicity, the term "reporting phase" refers to "reporting phase in t-block" in this section.

$|\mathcal{O}^t|$ is independent from t and determined only by the set of players. Firstly, assign each player i a distinct prime number x_i starting from 3. Then let $|\mathcal{O}^t| = \bigotimes_{i\in N} x_i = x_1 \otimes x_2 \otimes ... \otimes x_n$, where $\otimes$ is the usual multiplication operator. The sequence of actions in $\mathcal{O}^t$ is with length $|\mathcal{O}^t|$ and would take one of the forms specified in the right column in Table 2. The abbreviations of these sequences are listed in the left column. Since these sequences in the reporting phase are meant to share information about θ, the terms "playing the sequence" and "reporting the information" are interchangeable and will be alternatively used.

Table 2: The notations for the sequences of actions in $\mathcal{O}^t$ on the path

Notations		The sequences of actions
$\langle I\rangle$	:=	$(\overbrace{\mathbf{s}, ..., \mathbf{s}, \underbrace{\mathbf{r}, \mathbf{s}, ..., \mathbf{s}}_{\bigotimes_{j\in I} x_j}}^{\bigotimes_{i\in N} x_i})$
$\langle 1\rangle$	:=	$(\overbrace{\mathbf{s}, ..., \mathbf{s}, \mathbf{r}}^{\bigotimes_{i\in N} x_i})$
$\langle$**all stay**$\rangle$	:=	$(\overbrace{\mathbf{s}, ..., \mathbf{s}, \mathbf{s}}^{\bigotimes_{i\in N} x_i})$

It is worth noting that the sequence constructed by prime numbers brings two benefits. Firstly, since the multiplication of distinguishing prime numbers can be uniquely factorized, the Rebels can use such sequence to precisely report players' identities. Secondly, the undiscounted expected payoff of playing $\langle I_i^t\rangle$ for some I_i^t for an active Rebel i is always equal to -1, and therefore it is relatively easy to calculate. This is because only active Rebels will report $\langle I\rangle$ for some I. Since this I is not reported by any other Rebel, at most one

Rebel would play **r** at any period in the reporting phase by the property of prime number multiplication.[14]

I list the sequences played in the reporting phase on the path in Table 3.

Table 3: The sequences of actions played in $\mathcal{O}^t$ on the path

Rebel i	i plays
$i \notin R^t$	$\langle$**all stay**$\rangle$
$i \in R^t$ but i is not pivotal	$\langle I_i^t \rangle$
i is $k-1$-pivotal	$\langle 1 \rangle$
i is θ-pivotal but not in the free-rider problem	$\langle 1 \rangle$
i is in the free-rider problem with the lowest index	$\langle 1 \rangle$
i is in the free-rider problem without the lowest index	$\langle I_i^t \rangle$

On the path, the sequences $\langle I \rangle$ or $\langle 1 \rangle$ are meant to differentiate themselves from $\langle$**all stay**$\rangle$. The sequence $\langle$**all stay**$\rangle$ is for the inactive Rebels at t-block to report nothing. The sequence $\langle I \rangle$ is for active Rebels at t-block to report I if I is a set of Rebels. Although the definitions of *pivotal Rebel* and *free-rider problem* has not yet formally defined at this present, the sequence $\langle 1 \rangle$ is intentionally crafted to tackle the free-rider problem. To see how $\langle 1 \rangle$ works, I turn to formally defining the pivotal Rebel and the free-rider problem.

Definition 4.4 (Pivotal Rebels in $\mathcal{O}^t$)**.** *A Rebel p is pivotal in $\mathcal{O}^t$ if p is active at t-block, p is uncertain the relevant information, and p is certain that he will learn the relevant information right after $\mathcal{O}^t$, given that each $i \in R^t$ reports $\langle I_i^t \rangle$.*

By the definition, a pivotal Rebel in the reporting phase is one who can learn the relevant information if all of his active Rebel neighbors truthfully report their information about θ to him. The pivotal Rebels can be further classified into two kinds: ones who can learn θ, and ones who learn only the relevant information. When $k = 6$, in the configuration in Figure 6, only Rebels 4 and 5 are pivotal; they are of the first kind. In the configuration in

[14]This statement holds if there is no Rebel who plays $\langle 1 \rangle$.

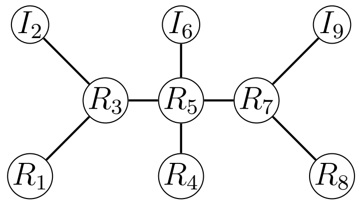

Figure 8: A configuration of the state and the network in which player 1, 3, 4, 5, 7, 8 are Rebels while players 2, 4, 9 are Inerts.

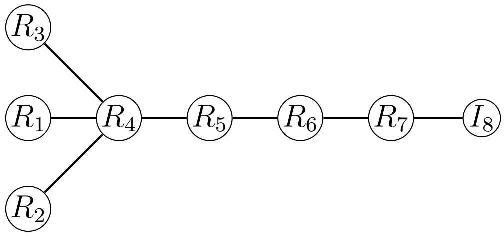

Figure 9: A configuration of the state and the network in which player 1, 2, 3, 4, 5, 6, 7 are Rebels while player 8 is an Inert.

Figure 8, only Rebel 5 is pivotal; he is of the first kind. In the configuration in Figure 9, only Rebel 4 pivotal; he is of the second kind.

Call p of the first kind the *θ-pivotal* Rebel. For the second kind, if the network is acyclic and if the prior has full support on strong connectedness, p is the second kind in the reporting phase only if $I_p^t = k-1$. Call the one with $I_p^t = k-1$ by *$k-1$-pivotal* Rebel.[15] Below is the defined free-rider problem in the reporting phase.

Definition 4.5. *A free-rider problem exists in $\mathcal{O}^t$ if there are multiple θ-pivotal Rebels in $\mathcal{O}^t$.*

The following lemma is crucial.

Lemma 4.1. *If the network is acyclic and if π has full support on strong connectedness, there are at most two θ-pivotal Rebels in the t-block. Moreover, they are neighbors.*[16]

[15]To show that a pivotal Rebel is the second kind in $\mathcal{O}^t$ only if $I_p^t = k-1$, one can follow the same argument in Theorem 3.

[16]As a remark, Lemma 4.1 is not true when the network is cyclic. To see this, consider a 4-player circle

And notably,

Lemma 4.2. *If the network is acyclic and if π has full support on strong connectedness, when there are two θ-pivotal Rebels p, p' in the t-block, then they commonly know that they are θ-pivotal Rebels at the beginning of t-block.*

By Lemma 4.2, θ-pivotal Rebels in the reporting phase can identify themselves at the beginning of this phase. This importance cannot be further emphasized. If the free-rider problem will occurs in the reporting phase, the strategy can specify that the lowest indexed θ-pivotal Rebel p involving in the free-rider problem will play $\langle 1 \rangle$, while the other one will play $\langle I^t_{p'} \rangle$ before the reporting phase. In summary, this knowledge is encoded in the belief system of an APEX equilibrium.

Remark. It is worth noting that the assumption of acyclic network in Lemma 4.2 is indispensable. Lemma 4.2 does not hold if the network is cyclic. Section 5.2 demonstrates it.

4.2.2 The equilibrium path in the coordination phase

The descriptions in this section is for the APEX equilibrium path before T^θ. The term "coordination phase in t-block" is shorten by $\mathcal{C}^t$.

It is elaborate to spell out the coordination phase structure, but this phase is actually a simple contagion scenario: Rebels jointly decide to terminate or continue their information sharing during this phase. For that, a coordination phase is partitioned into three *divisions*. In the first division, if there is a Rebel has learnt that $\#[R](\theta) < k$, all Rebels will play **stay** forever right after this division, and T^θ arrives; otherwise, they continue to the next one. In the second division, if there is a Rebel has learnt that $\#[R](\theta) \geq k$, a portion of Rebels, at least k Rebels, will play **revolt** forever right after this division; otherwise, they continue to the next one. In the third division, if there is a Rebel has learnt that $\#[R](\theta) \geq k$ in the previous divisions, all Rebel will play **revolt** forever right after this division, and T^θ arrives; otherwise, they continue to the next phase—a reporting phase.

when $\theta = (R, R, R, R)$.

To fulfil the above contagion argument, a set of sequences of actions played on the path is specified so that Rebels update their belief according to it. For this task, further partition a division into *sub-blocks*. I depict the whole partition in the coordination phase below, where $\square$ represents a sub-block in a coordination phase.

In $\mathcal{C}^1$,

$$\overbrace{\underbrace{\square}_{\text{one sub-block}}}^{\text{1-division}}\ \overbrace{\underbrace{\square}_{\text{one sub-block}}}^{\text{2-division}}\ \overbrace{\underbrace{\square\cdots\square}_{n\text{ sub-blocks}}}^{\text{3-division}}$$

In $\mathcal{C}^t$, $t \geq 2$,

$$\overbrace{\underbrace{\square\cdots\square}_{n\text{ sub-blocks}}}^{\text{1-division}}\ \overbrace{\underbrace{\square\cdots\square}_{t+1\text{ sub-blocks}}}^{\text{2-division}}\ \overbrace{\underbrace{\square\cdots\square}_{n\text{ sub-blocks}}}^{\text{3-division}}$$

For each t, denote $\mathcal{C}^t(u,\cdot)$ as the u-division and $|\mathcal{C}^t(u,\cdot)|$ as the length of $\mathcal{C}^t(u,\cdot)$. Likewise, denote $\mathcal{C}^t(u,v)$ as the v-th sub-block in u-division and $|\mathcal{C}^t(u,v)|$ as the length of $\mathcal{C}^t(u,v)$. Let us shorten **revolt** and **stay** to **r** and **s** receptively. Let $|\mathcal{C}^t(u,v)| = n$ if $u = 1,2$, $v = 1,...,n$. Let $|\mathcal{C}^t(u,v)| = 1$ if $u = 3$, $v = 1,...,n$. I list the the sequences of actions on the path and their notations in Table 4, where i is the original index of player i.[17]

Table 4: The notations for the sequences of actions in $\mathcal{C}^t(u,v)$ for $u = 1,2$, $v = 1,...,n$, on the path

Notations		The sequences of actions
$\langle i \rangle$	$\equiv$	$(\overbrace{\mathbf{s},...,\mathbf{s},\underbrace{\mathbf{r},\mathbf{s},...,\mathbf{s}}_{i}}^{n})$
$\langle$**all stay**$\rangle$	$\equiv$	$(\overbrace{\mathbf{s},...,\mathbf{s},\mathbf{s}}^{n})$

4.2.3 The equilibrium behavior on the path in $\mathcal{C}^1$

I begin with depicting the equilibrium path in $\mathcal{C}^1$, which is shown in Table 5. The belief updating after $\mathcal{C}^1(1,\cdot)$ and $\mathcal{C}^1(2,\cdot)$ on the path is listed in Table 6. The evolution of players'

[17] In the 3-division, since the sequence of actions which length is 1 is equivalent to playing a single action, I do not provide additional notations for conciseness.

information filtrations can be tracked throughout in this table. Since there is only one sub-block in $\mathcal{C}^1(1,\cdot)$ or $\mathcal{C}^1(2,\cdot)$, $\mathcal{C}^1(1,\cdot)$ or $\mathcal{C}^1(2,\cdot)$ is interchangeable with $\mathcal{C}^1(1,1)$ or $\mathcal{C}^1(2,1)$ respectively. If there is no confusing, the term "coordination phase" refers to "coordination phase in the 1-block" in this section.

Table 5: The sequences of actions played in $\mathcal{C}^1$ on the path

The sequences of actions played in $\mathcal{C}^1(1,\cdot)$ on the path

Rebel i	i plays
i is certain $\#[R](\theta) < k$	$\langle$**all stay**$\rangle$
$i \notin R^1$ and is uncertain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$
$i \in R^1$ and is uncertain $\#[R](\theta) \geq k$	$\langle i \rangle$
i is certain $\#[R](\theta) \geq k$	$\langle i \rangle$

The sequences of actions played in $\mathcal{C}^1(2,\cdot)$ on the path

Rebel i	i plays
i is certain $\#[R](\theta) < k$	$\langle$**all stay**$\rangle$
$i \notin R^1$ and is uncertain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$
$i \in R^1$ and is uncertain $\#[R](\theta) \geq k$	$\langle i \rangle$
i is certain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$

The sequences of actions played in $\mathcal{C}^1(3_v)$, where $v = 1, ..., n$, on the path

Rebel i	i plays
i is certain $\#[R](\theta) < k$	**s**
$i \notin R^1$ and is uncertain $\#[R](\theta) \geq k$	**s**
$i \in R^1$ and is uncertain $\#[R](\theta) \geq k$	**s**
i is certain $\#[R](\theta) \geq k$	**r**

In the first division in the coordination phase, if Rebel i is certain $\#[R](\theta) < k$, i play $\langle$**all stay**$\rangle$. It implies that i is certain that there is no Rebels outside G_i and therefore learns

θ by strong connectedness. This is to say all Rebels are i's neighbors and thus all i's Rebel neighbors are inactive. Since $\langle$**all stay**$\rangle$ is played only by an inactive Rebel or a Rebel who is certain $\#[R](\theta) < k$, all i's Rebel neighbors learn $\#[R](\theta) < k$ right after the first division in the coordination phase. Since all Rebels are i's neighbors, all Rebels learn $\#[R](\theta) < k$ right after the first division in the coordination phase. T^θ arrives then. Likewise, if Rebel i is inactive and all i's neighbors play $\langle$**all stay**$\rangle$, all Rebels learn $\#[R](\theta) < k$ right after the first division in the coordination phase, and T^θ arrives.

In the second division in the coordination phase, there is a non-trivial construction: how Rebel i disseminates the knowledge about $\#[R](\theta) \geq k$ if i has learnt that. Rebel i does so by playing $\langle i \rangle$ in the first division in the coordination phase and then play $\langle$**all stay**$\rangle$ in the second division in the coordination phase. His behavior is thus different from other kinds of Rebels. His neighbors will know $\#[R](\theta) \geq k$ right after the second division in the coordination phase and play $\mathbf{r}$ forever afterwards. Other Rebels will observe $\mathbf{r}$ being played in the third division in the coordination phase and thus know $\#[R](\theta) \geq k$ as well. All Rebels learn $\#[R](\theta) \geq k$ right after the third division in the coordination phase, and T^θ arrives.

Note that Rebel i who has learnt $\#[R](\theta) \geq k$ will not deviate to play $\langle$**all stay**$\rangle$ in the first division in the coordination phase even though it might be undetectable. This is because the network is acyclic. If i does so, i will be considered as an inactive Rebel by all his neighbors from the point onwards right after the first division in the coordination phase. Two consequences follow. The first is that each of i's neighbor is certain there is no more Rebel "behind" i.[18] The second is that i keeps reporting nothing in the forthcoming reporting phases. Thus, all i's neighbors report strictly less Rebels than they are supposed to do if i follows the equilibrium path. i then faces the possibility that no Rebel can know $\#[R](\theta) \geq k$ even if the total number of Rebels indeed exceeds k. If this event happens, i will only get zero payoff. However, i can surely get stage-game payoff as 1 afterwards right after the second division in the coordination phase. Sufficiently high $\delta \in (0, 1)$ will deter this deviation.

[18] To be more precise, this is to say there is no more Rebel in a sub-tree which excludes j and roots at i.

Table 6: In $\mathcal{C}^1$, on the path, the belief of i's neighbor j after observing i's previous actions.

i plays	The event to which j assigns probability one right after $\mathcal{C}^1(1,\cdot)$
In $\mathcal{C}^1(1,\cdot)$	
$\langle$**all stay**$\rangle$	$i \notin R^1$ if $j \in R^1$
$\langle$**all stay**$\rangle$	$\#[R](\theta) < k$ if $j \notin R^1$
$\langle i \rangle$	$i \in R^1$ or $\#[R](\theta) \geq k$

i plays		The event to which j assigns probability one right after $\mathcal{C}^1(2,\cdot)$
In $\mathcal{C}^1(1,\cdot)$	In $\mathcal{C}^1(2,\cdot)$	
$\langle$**all stay**$\rangle$	$\langle$**all stay**$\rangle$	$i \notin R^1$ if $j \in R^1$
$\langle$**all stay**$\rangle$	$\langle$**all stay**$\rangle$	$\#[R](\theta) < k$ if $j \notin R^1$
$\langle i \rangle$	$\langle$**all stay**$\rangle$	$\#[R](\theta) \geq k$
$\langle i \rangle$	$\langle i \rangle$	$i \in R^1$

4.2.4 The equilibrium behavior on the path in $\mathcal{C}^t$ for $t \geq 2$

The in-path strategy contingent on players' belief is introduced in Table 7. The evolution of information filtrations can be tracked throughout in Table 8. The delicate part in $\mathcal{C}^t$ is how a pivotal Rebel p in $\mathcal{O}^{t-1}$ disseminates the relevant information. For convenience, in this section, let $I_{ij}^{t+1} = I_i^t \cap I_j^t$ and the term "coordination phase" refer to "coordination phase in the t-block". I begin with the case when p is certain $\#[R](\theta) < k$.

If p is certain $\#[R](\theta) < k$, p plays $\langle$**all stay**$\rangle$ in the first sub-block of the first division in the coordination phase. Consequently, all p's neighbors know $\#[R](\theta) < k$ right after that since p has played $\langle 1 \rangle$ in the just finished reporting phase to announce he is pivotal. p's neighbors then play $\langle$**all stay**$\rangle$ continuously in each sub-block in the first division in the coordination phase, and therefore all Rebels know $\#[R](\theta) < k$ contagiously by observing $\langle$**all stay**$\rangle$ being played. All Rebels play **s** forever after the first division in the coordination phase, and T^θ arrives.

On the other hand, suppose p is certain $\#[R](\theta) \geq k$, p plays $\langle p \rangle$ in each sub-block in the first division in the coordination phase. To reveal $\#[R](\theta) \geq k$, p plays $\langle$**all stay**$\rangle$ in the first sub-block of the second division in the coordination phase. Notice that $\langle$**all stay**$\rangle$ is a costless sequence of actions. It might not seem intuitive at first sight, but playing $\langle$**all stay**$\rangle$ effectively prevents another free-rider problem. Suppose there are two pivotal Rebels, say p and p', who have already known $\#[R](\theta) \geq k$ right after the just finished reporting phase. If initiation to disseminate knowledge about $\#[R](\theta) \geq k$ incurs negative payoff, p or p' will have the incentive, again, to wait for the other one initiates it. Playing $\langle$**all stay**$\rangle$ in the first sub-block of the second division in the coordination phase proudly becomes the initiation sequence by its cheapness. By the same contagion argument as the above mentioned, all Rebels play **r** after the third division in the coordination phase, and T^θ arrives.

The remaining question is why a non-pivotal Rebel, say i, does not mimic a pivotal Rebel's behavior by playing $\langle 1 \rangle$ in the just finished reporting phase even though it might be undetectable. The reason is the following. If i plays $\langle 1 \rangle$, i's neighbor will think i is pivotal. According to the equilibrium path, it implies that all players play either **r** or **s**

Table 7: The sequences of actions played in $\mathcal{C}^t$, $t \geq 2$ on the path

The sequences of actions played in $\mathcal{C}^t(1, v)$ for $t \geq 2$ and for $v = 1, 2, ..., n$ on the path

Rebel i	i plays
i is certain $\#[R](\theta) < k$	$\langle$**all stay**$\rangle$
$i \notin R^t$ and is uncertain $\#[R](\theta) \geq k$	$\langle i \rangle$
$i \in R^t$ and is uncertain $\#[R](\theta) \geq k$	$\langle i \rangle$
i is certain $\#[R](\theta) \geq k$	$\langle i \rangle$

The sequences of actions played in $\mathcal{C}^t(2, v)$ for $t \geq 2$ for $v = 1$ on the path

Rebel i	i plays
i is certain that $\#[R](\theta) < k$	$\langle$**all stay**$\rangle$
$i \notin R^t$ and is uncertain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$
$i \in R^t$ and is uncertain $\#[R](\theta) \geq k$	$\langle i \rangle$
i is certain that $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$

The sequences of actions played in $\mathcal{C}^t(2, v)$ for $t \geq 2$ for $v = 2, ..., t + 1$ on the path

Rebel i	i plays
i is certain that $\#[R](\theta) < k$	$\langle$**all stay**$\rangle$
$i \notin R^t$ and is uncertain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$
$i \in R^t$ and is uncertain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$
i is certain that $\#[R](\theta) \geq k$	$\langle i \rangle$

The sequences of actions played in $\mathcal{C}^t(3, \cdot)$ for $t \geq 2$ on the path

Rebel i	i plays
i is certain that $\#[R](\theta) < k$	**s**
$i \notin R^1$ and is uncertain $\#[R](\theta) \geq k$	**s**
$i \in R^1$ and is uncertain $\#[R](\theta) \geq k$	**s**
i is certain that $\#[R](\theta) \geq k$	**r**

Table 8: In $\mathcal{C}^t$, on the path, the belief of i's neighbor j after observing i's previous actions.

i plays	The event to which j assigns probability one right after $\mathcal{O}^t$
In $\mathcal{O}^t$	
$\langle$**all stay**$\rangle$	$i \notin R^t$ and $I_{ji}^{t+1} = I_j^t$
$\langle I_i^t \rangle$	$i \in R^t$ and $I_{ji}^{t+1} = I_j^t \cap I_i^t$
$\langle 1 \rangle$	i is pivotal

i plays		The event to which j assigns probability one right after $\mathcal{C}^t(1,1)$
In $\mathcal{O}^t$	In $\mathcal{C}^t(1,1)$	
$\langle$**all stay**$\rangle$	$\langle i \rangle$	$i \notin R^t$ and $I_{ji}^{t+1} = I_j^t$
$\langle I_i^t \rangle$	$\langle$**all stay**$\rangle$	$\#[R](\theta) < k$
$\langle I_i^t \rangle$	$\langle i \rangle$	$(\#[R](\theta) \geq k)$ or ($i \in R^t$ and $I_{ji}^{t+1} = I_j^t \cap I_i^t$)
$\langle 1 \rangle$	$\langle$**all stay**$\rangle$	$\#[R](\theta) < k$
$\langle 1 \rangle$	$\langle i \rangle$	$\#[R](\theta) \geq k$

i plays			The event to which j assigns probability one right after $\mathcal{C}^t(2,1)$
In $\mathcal{O}^t$	In $\mathcal{C}^t(1,1)$	In $\mathcal{C}^t(2,1)$	
$\langle$**all stay**$\rangle$	$\langle i \rangle$	$\langle$**all stay**$\rangle$	$i \notin R^t$ and $I_{ji}^{t+1} = I_j^t$
$\langle I_i^t \rangle$	$\langle$**all stay**$\rangle$	$\langle$**all stay**$\rangle$	$\#[R](\theta) < k$
$\langle I_i^t \rangle$	$\langle i \rangle$	$\langle$**all stay**$\rangle$	$\#[R](\theta) \geq k$
$\langle I_i^t \rangle$	$\langle i \rangle$	$\langle i \rangle$	$i \in R^t$ and $I_{ji}^{t+1} = I_j^t \cap I_i^t$
$\langle 1 \rangle$	$\langle$**stay**$\rangle$	$\langle$**all stay**$\rangle$	$\#[R](\theta) < k$
$\langle 1 \rangle$	$\langle i \rangle$	$\langle$**all stay**$\rangle$	$\#[R](\theta) \geq k$

forever after the third division in the coordination phase; therefore, the belief updating is also terminated. What follows is i cannot learn the relevant information after the third division in the coordination phase. If i does not deviate, i will learn the relevant information eventually and choose the best response to get the maximum payoff at every θ. He prefers not to deviate if $\delta \in (0, 1)$ is high enough.[19]

4.3 Sketch of the proof of Theorem 2

In the above I have listed Rebels' behavior in Table 3, Table 5, Table 8, and their belief updating in Table 6, Table 8, as the blueprint for the constructed equilibrium path. I sketch the proof of Theorem 2 as follows.

First, I use out-of-path belief to prevent players from making detectable deviations, such as deviating from playing the specified forms of sequences that are listed in Table 2 or Table 4. This out-of-path belief serves as a grim trigger: the punisher will play **stay** forever if he has not yet learnt the relevant information. Despite a detectable deviation might be detected by only a few Rebels owing to the network monitoring, I argue that any detectable deviation made by a deviant before he learns the relevant information will strictly reduce the possibility of achieving the ex-post efficient outcome. For any undetectable deviation made by a deviant before he learns the relevant information, I prove that it will also strictly reduce the deviant's expected continuation payoff. Suppose this undetectable deviation is meant to report less Rebels in a reporting phase, it will diminish the possibility of achieving the ex-post outcome. Suppose this one introduces "noises" in information sharing process so that the deviant will never learn the relevant information, his expected continuation payoff will be strictly reduced, as argued in the last paragraph in Section 4.2.4. Since the stage-game payoff after T^{θ} is maximum for every θ by following the equilibrium path, a high enough $\delta \in (0, 1)$ will deter Rebels from deviating. I then conclude the proof.

[19]By Lemma A.1 in Appendix, the relevant information is learnt by every Rebel on the path eventually.

5 Discussion

In the above APEX equilibrium construction, players act as if acting a sequence. Nevertheless, the actual description of an APEX equilibrium should specify how they act period-by-period and how they update belief at every period. This description is awfully tedious; it is left in the Appendix.

Instead of providing further details in equilibrium construction, I discuss the scenario when pay-off is a signal and why my constructed APEX equilibrium may fail in cyclic networks.

5.1 Payoff as signals

The hidden payoff assumption can be relaxed without changing the main result. One may consider a situation in which the stage payoff depends not only on players' joint efforts but also on a random shock, say the weather. To fix the idea, there is a public signal $y \in \{r, s\}$ generated according to the action profile. Let a Rebel's payoff function be $u_R(a_R, y)$ such that $u_R(\mathbf{stay}, r) = u_R(\mathbf{stay}, s) = u_0$. y is drawn from the distribution of

$$\begin{aligned}
p_{rr} &= \Pr(y = r | \#\{j : a_j \mathbf{revolt}\} \geq k) \\
p_{sr} = 1 - p_{rr} &= \Pr(y = s | \#\{j : a_j \mathbf{revolt}\} \geq k) \\
p_{ss} &= \Pr(y = s | \#\{j : a_j \mathbf{revolt}\} < k) \\
p_{rs} = 1 - p_{ss} &= \Pr(y = r | \#\{j : a_j \mathbf{revolt}\} < k)
\end{aligned}$$

such that

$$p_{rr} u_R(\mathbf{revolt}, r) + p_{sr} u_R(\mathbf{revolt}, s) > u_0 > p_{rr} u_R(\mathbf{revolt}, r) + p_{ss} u_R(\mathbf{revolt}, s),$$

and

$$0 \leq p_{rs} \leq 1, 0 \leq p_{ss} \leq 1.$$

The APEX equilibrium constructed for Theorem 2 is still a one in this scenario. Note that in that APEX equilibrium path, at most one **revolt** can occur at every period before some Rebel plays $\langle 1 \rangle$. This implies that the signal y is completely uninformative before some Rebel plays $\langle 1 \rangle$. If a Rebel i deviates to play $\langle 1 \rangle$ in $\mathcal{O}^t$ at some t in the hope gathering information from y, he will not learn the relevant information after $\mathcal{O}^t$ since the terminal period will come right after t-block. He will, however, learn the relevant information and play the ex-post efficient outcome if he is on the path, and hence he will not deviate.

5.2 Cyclic networks

Scenarios in cyclic networks substantially differ from the acyclic counterpart. The free-rider problem could become intractable in cyclic networks. Let us consider the configuration in Figure 10, and suppose $k = 4$.

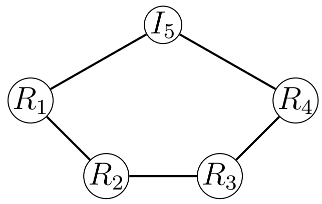

Figure 10: A configuration of the state and the network in which player 1, 2, 3, 4 are Rebels while player 5 is an Inert.

In Figure 10, Rebels 2 and 3 are θ-pivotal by definition. From the perspective of Rebel 2, the type of player 5 could be Inert. Therefore, Rebel 2 does not know that Rebel 1 is pivotal. Similarly, Rebel 2 does not know that Rebel 3 is pivotal, *even though* player 3 is indeed θ-pivotal. Therefore there is no common knowledge of the free-rider problem at the beginning of 1-block.

However, the common knowledge of engaging in a free-rider problem is restored when we cut the edge between player 4 and 5; Rebel 2 knows that he is the only θ-pivotal Rebel.

I leave a conjecture in this paper and end this section.

Conjecture 5.1. *For any n-person repeated k-Threshold game with parameter $k < n$ played in any network, if π has full support on strong connectedness, then there exists a $\delta^* \in (0, 1)$ such that an APEX equilibrium exists whenever $\delta > \delta^*$.*

6 Conclusion

I model a coordination game and illustrate the learning processes generated by strategies in a sequential equilibrium to answer the question proposed in the beginning: what kind of networks can conduct coordination in a collective action with information barrier. In the equilibrium, players transmit the relevant information by encoding such information by their actions as time goes by. Since there might be an negative expected payoff in coding information, the potential free-rider problems might occur to impede the learning process. My result show that if the network is acyclic, players can always learn the underlying relevant information and conduct the coordination only by actions. In cyclic networks, however, what kinds of equilibrium strategies can lead to learning the relevant information still remains to be answered.

The construction of the communication protocol by actions exploits the assumption of the common knowledge of the network and the finite type space. Since the relevant information has been parametrized as a threshold in the stage game, players can acquire this information by jointly incrementally reporting their own private information period by period. The major punishment to deter deviation is then the joint shifting to play that same action as the stopping to update information. The threshold game thus seems a potential model in proofing that a communication protocol by actions not only leads a learning process but also constitutes an equilibrium to reveal the relevant information in finite time.

Existing literatures in political science and sociology have recognized the importance of social network in influencing individual's behavior in participating social movements ([Passy, 2003][McAdam, 2003][Siegel, 2009]). This paper views networks as routes for communication in which rational individuals initially have local information but they can influence nearby individuals by taking actions. Such influence may take long time to travel across individuals and the whole process incurs inefficient outcomes in many periods. A characterization in the speed of information transmission across a network is not answered here, although it is an important topic in investigating the most efficient way to let the information be spread. This question would remain for the future research.

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A Appendix

A.1 The APEX equilibrium for Theorem 2

A.1.1 Equilibrium path

By definition of information hierarchy,

$$\begin{aligned} I_i^t &= \bigcup_{k_1 \in G_i} \bigcup_{k_2 \in G_{k_0}} \cdots \bigcup_{k_t \in G_{k_{t-1}}} I_{k_t}^1 \\ &= \{j \in [R](\theta) : \exists \text{ a path } (i, k_1 ... k_l, j) \text{ s.t. } 0 \leq l \leq t-1 \text{ and } \theta_i = \theta_{k_1} = ... = \theta_{k_l} = R\} \end{aligned}$$

Let us define several notions.

Definition A.1 (Extended tree by I_i^t)**.**

$$\begin{aligned} X_i^t \;\equiv\; & \{j \in N : \\ & \exists \textit{ a path } (i, k_1 ... k_l, k_{l+1}) \textit{ s.t. } k_{l+1} = j,\ l \geq t-1,\ \{i, k_1, ..., k_t\} \subset I_i^t\} \cup I_i^t \end{aligned}$$

Namely, X_i^t is the set of all possible Rebels in G given information I_i^t.

Definition A.2 (The sub-tree rooting in j but excluding i)**.**

$$TR_{ij} \equiv \{v \in N : \textit{there is a path from } i \textit{ to } v \textit{ through } j,\ j \in G_i\} \cup \{i, j\}$$

Definition A.3 (Extended vertices outside I_i^t in TR_{ij})**.**

$$Y_{ij}^t \equiv TR_{ij} \cap (X_i^t \setminus I_i^t)$$

Definition A.4.

$$D_i^t \equiv \{j \in G_i : Y_{ij}^t \neq \emptyset\}$$

Definition A.5 (Finite register machine)**.** *A finite register machine for i consists of finite registers Σ. A register is a tuple*

$$(\Omega, \times_{G_i} A_R, f, \lambda),$$

in which Ω are sets of events induced by H_i. $\times_{G_i} A_R$ is the sets of input. $f : \Omega \to A_R$ assigns an action to each event. $\lambda : \Omega \times \times_{G_i} A_R \to \Sigma$ is the transition function. There is a set of initial registers.

i's register specifies i's action according his information at a certain period but does not characterize i's information transition. The register machine here is more like the *switch function* instead of the finite automata. The information of i up to period s is $P_i(\theta) \times \{h_{G_i}^s\} \times H_{N \setminus G_i}^s$ characterized in Section 2.

Definition A.6 (m-register in t-block)**.** *A m-register in a (sub)block or a division is the register in the m-th period in that (sub)block or division.*

To shorten the notation, denote $m \dashv \Gamma$ as the m-register in the (sub)block or division Γ.

Definition A.7 (Terminal **r**)**.** *The terminal **r** is a register such that the image of f is $\{$**revolt**$\}$ and the image of λ is a singleton containing itself.*

Definition A.8 (Terminal **s**)**.** *The terminal **s** is a register such that the image of f is $\{$**stay**$\}$ and the image of λ is a singleton containing itself.*

The equilibrium will be represented as a finite register machine. Moreover, though players act as if acting a whole sequence, they in fact act period by period. For convenience, for any finite sequence of action $\langle\rangle$, denote $\langle\rangle_m$ as the m-th (counting from the beginning) component in $\langle\rangle$, and denote $\langle\rangle(m)$ as the prefix of $\langle\rangle$ with length m. Let us also shorten action **revolt** to be **r** and **stay** to be **s**.

Initial registers The initial register for each Rebel is $1 \dashv \mathcal{C}^1(1, \cdot)$, which is defined in the next section.

Registers in $\mathcal{C}^1$

Table 9: The $m \dashv \mathcal{C}^1_1$ on the path

$1 \leq m \leq |\mathcal{C}^1(1,\cdot)| - 1$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$\#X^1_i < k$	$\langle$**all stay**$\rangle_m$		terminal **s**
$i \notin R^1$, $\#X^1_i \geq k$, $I^1_i < k$	$\langle$**all stay**$\rangle_m$		$m+1 \dashv \mathcal{C}^1_1$
$i \in R^1$, $\#X^1_i \geq k$, $I^1_i < k$	$\langle i \rangle_m$		$m+1 \dashv \mathcal{C}^1_1$
$i \in R^1$, $\#X^1_i \geq k$, $I^1_i \geq k$	$\langle i \rangle_m$		$m+1 \dashv \mathcal{C}^1_1$

$m = |\mathcal{C}^1(1,\cdot)|$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$\#X^1_i < k$		$\langle$**all stay**$\rangle_m$		terminal **s**
$i \notin R^1$, $\#X^1_i \geq k$, $I^1_i < k$	all j play $\langle$**all stay**$\rangle(m-1)$	$\langle$**all stay**$\rangle_m$	all j play **s**	terminal **s**
$i \notin R^1$, $\#X^1_i \geq k$, $I^1_i < k$	$\exists j$ plays $\langle j \rangle(m-1)$	$\langle$**all stay**$\rangle_m$	such j plays $\langle j \rangle_m$	$1 \dashv \mathcal{C}^1(2,\cdot)$
$i \in R^1$, $\#X^1_i \geq k$, $I^1_i < k$		$\langle i \rangle_m$		$1 \dashv \mathcal{C}^1(2,\cdot)$
$i \in R^1$, $\#X^1_i \geq k$, $I^1_i \geq k$		$\langle i \rangle_m$		$1 \dashv \mathcal{C}^1(2,\cdot)$

Table 10: The $m \dashv \mathcal{C}_2^1$ on the path

$1 \leq m < |\mathcal{C}^1(2, \cdot)|$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$i \notin R^1$		$\langle \textbf{all stay} \rangle_m$		$m+1 \dashv \mathcal{C}_2^1$
$i \in R^1$, $I_i^1 < k$	$\exists j \in G_i$, j plays $< j >_j = s$	$\langle \textbf{all stay} \rangle_m$		$m+1 \dashv \mathcal{C}_2^1$
$i \in R^1$, $I_i^1 < k$	$\forall j \in G_i$, j plays $< j >_j = r$	$\langle i \rangle_m$		$m+1 \dashv \mathcal{C}_2^1$
$i \in R^1$, $I_i^1 \geq k$		$\langle \textbf{all stay} \rangle_m$		$m+1 \dashv \mathcal{C}_2^1$

$m = |\mathcal{C}^1(2, \cdot)|$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$i \notin R^1$	$\forall j \in G_i$, j plays $\langle j \rangle (m-1)$	$\langle \textbf{all stay} \rangle_m$	$\forall j \in G_i$, j plays $\langle j \rangle_m$	$1 \dashv \mathcal{C}_3^1$
$i \notin R^1$	$\exists j \in G_i$, j plays $\langle \textbf{all stay} \rangle (m-1)$	$\langle \textbf{all stay} \rangle_m$	such j plays $\langle \textbf{all stay} \rangle_m$	terminal $\mathbf{r}$
$i \in R^1$, $I_i^1 < k$	$\forall j \in G_i$, j play $\langle j \rangle (m-1)$	$\langle i \rangle_m$	$\forall j \in G_i$, j plays $\langle j \rangle_m$	$1 \dashv \mathcal{C}_3^1$
$i \in R^1$, $I_i^1 < k$	$\exists j \in G_i$, j plays $\langle \textbf{all stay} \rangle (m-1)$	$\langle i \rangle_m$	such j plays $\langle \textbf{all stay} \rangle_m$	terminal $\mathbf{r}$
$i \in R^t$, $I_i^1 \geq k$		$\langle \textbf{all stay} \rangle_m$		terminal $\mathbf{r}$

Table 11: The $m \dashv \mathcal{C}_3^1$ on the path

$1 \leq m < |\mathcal{C}^1(3,\cdot)|$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
	s	$\forall j$ play **s**	$m+1 \dashv \mathcal{C}_3^1$
	s	$\exists j$ play **r**	terminal **r**

$1 \leq m = |\mathcal{C}^1(3,\cdot)|$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
	s	$\forall j$ play **s**	$1 \dashv \mathcal{O}^1$
	s	$\exists j$ play **r**	terminal **r**

Registers in $\mathcal{O}^t$ Let $m_i = |\mathcal{O}^t| - x_{I_i^t}$ be the period in which i report I_i^t. I.e. m_i is the period where $\mathbf{r}$ occurs in $\langle I_i^t \rangle$. Denote $G_i(m) = \{j \in G_i : m_j < m\}$. Define $I_i^{t+1}(m) \equiv I_i^t \cup \bigcup_{j \in G_i(m)} I_j^t$ to be the information of i up to the m-th period in $\mathcal{O}^t$. Define $X_i^{t+1}(m)$ to be the extended tree from $I_i^t(m)$ in the same way as that in Definition A.1, and define $Y_{ij}^t(m)$ and $D_i^t(m)$ accordingly.

Table 12: The $m \dashv \mathcal{O}^t$ on the path, where $1 \leq m < |\mathcal{O}^t|$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$i \notin R^t$		$\langle \textbf{all stay} \rangle_m$		$m+1 \dashv \mathcal{O}^t$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I_i^{t+1}(m) < k$, $X_i^{t+1}(m) < k$	$\langle \textbf{all stay} \rangle_m$		terminal $\mathbf{s}$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I_i^{t+1}(m) < k$, $X_i^{t+1}(m) \geq k$	$\langle I_i^t \rangle_m$		$m+1 \dashv \mathcal{O}^t$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I_i^{t+1}(m) \geq k$, $X_i^{t+1}(m) \geq k$	$\langle 1 \rangle_m$		$m+1 \dashv \mathcal{O}^t$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I_i^{t+1}(m) \geq k-1$, $X_i^{t+1}(m) \geq k$, $D_i^t = 1$	$\langle 1 \rangle_m$		$m+1 \dashv \mathcal{O}^t$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I_i^{t+1}(m) \geq k-1$, $X_i^{t+1}(m) \geq k$, $D_i^t > 1$	$\langle I_i^t \rangle_m$		$m+1 \dashv \mathcal{O}^t$
$i \in R^t$, the free rider	$X_i^{t+1}(m) \geq k$	$\langle 1 \rangle_m$		$m+1 \dashv \mathcal{O}^t$
$i \in R^t$, the free rider	$X_i^{t+1}(m) < k$	$\langle \textbf{all stay} \rangle_m$		terminal $\mathbf{s}$
$i \in R^t$, i is $k-1$-pivotal	$X_i^{t+1}(m) \geq k$	$\langle 1 \rangle_m$		$m+1 \dashv \mathcal{O}^t$
$i \in R^t$, i is $k-1$-pivotal	$X_i^{t+1}(m) < k$	$\langle \textbf{all stay} \rangle_m$		terminal $\mathbf{s}$

Table 13: The $m \dashv \mathcal{O}^t$ on the path, where $m = |\mathcal{O}^t|$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$i \notin R^t$		$\langle \textbf{all stay} \rangle_m$		$1 \dashv \mathcal{C}^t_{1,1}$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I^{t+1}_i(m) < k$, $X^{t+1}_i(m) < k$	$\langle \textbf{all stay} \rangle_m$		terminal $\mathbf{s}$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I^{t+1}_i(m) < k-1$, $X^{t+1}_i(m) \geq k$	$\langle I^t_i \rangle_m$		$1 \dashv \mathcal{C}^t_{1,1}$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I^{t+1}_i(m) \geq k$, $X^{t+1}_i(m) \geq k$	$\langle 1 \rangle_m$		$1 \dashv \mathcal{C}^t_{1,1}$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I^{t+1}_i(m) \geq k-1$, $X^{t+1}_i(m) \geq k$, $D^t_i = 1$	$\langle 1 \rangle_m$		$1 \dashv \mathcal{C}^t_{1,1}$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I^{t+1}_i(m) \geq k-1$, $X^{t+1}_i(m) \geq k$, $D^t_i > 1$	$\langle I^t_i \rangle_m$		$1 \dashv \mathcal{C}^t_{1,1}$
$i \in R^t$, the free rider	$X^{t+1}_i(m) \geq k$	$\langle 1 \rangle_m$		$1 \dashv \mathcal{C}^t_{1,1}$
$i \in R^t$, the free rider	$X^{t+1}_i(m) < k$	$\langle \textbf{all stay} \rangle_m$		terminal $\mathbf{s}$
$i \in R^t$, i is $k-1$-pivotal	$X^{t+1}_i(m) \geq k$	$\langle 1 \rangle_m$		$1 \dashv \mathcal{C}^t_{1,1}$
$i \in R^t$, i is $k-1$-pivotal	$X^{t+1}_i(m) < k$	$\langle \textbf{all stay} \rangle_m$		terminal $\mathbf{s}$

Registers in $\mathcal{C}^t$ for $t \geq 2$

Table 14: The $m \dashv \mathcal{C}^t(1, v)$ for $v = 1, ..., n$ on the path

$1 \leq m < |\mathcal{C}^t(1, v)|$, where $v = 1, ..., n$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$X_i^{t+1} < k$	$\langle$**all stay**$\rangle_m$		terminal $\mathbf{s}$
$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\exists j \in G_i, j = m$ such that $a_j = \mathbf{s}$	terminal $\mathbf{s}$
$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\forall j \in G_i$ such that $a_j = \langle j \rangle_m$	$m + 1 \dashv \mathcal{C}^t(1, v)$

$m = |\mathcal{C}^t(1, v)|$, where $v = 1, ..., n - 1$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$X_i^{t+1} < k$	$\langle$**all stay**$\rangle_m$		terminal $\mathbf{s}$
$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\exists j \in G_i, j = m$ such that $a_j = \mathbf{s}$	terminal $\mathbf{s}$
$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\forall j \in G_i$ such that $a_j = \langle j \rangle_m$	$1 \dashv \mathcal{C}^t_{1,v+1}$

$1 \leq m < |\mathcal{C}^t(1, n)|$

$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\exists j \in G_i, j = m$ such that $a_j = \mathbf{s}$	terminal $\mathbf{s}$
$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\forall j \in G_i$ such that $a_j = \langle j \rangle_m$	$m + 1 \dashv \mathcal{C}^t(1, n)$

$m = |\mathcal{C}^t(1, n)|$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\exists j \in G_i, j = m$ such that $a_j = \mathbf{s}$	terminal $\mathbf{s}$
$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\forall j \in G_i$ such that $a_j = \langle j \rangle_m$	$1 \dashv \mathcal{C}^t_{2,1}$

Table 15: The $m \dashv \mathcal{C}^t(2,v)$ for $v = 1, ..., t+1$ on the path

$1 \leq m < |\mathcal{C}^t(2,v)|$, where $v = 1, ..., t+1$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$I_i^{t+1} < k$	$\exists j,\ \langle j \rangle_j = \mathbf{s}$	$\langle$**all stay**$\rangle_m$		$m+1 \dashv \mathcal{C}^t(2,v)$
$I_i^{t+1} < k$	$\forall j,\ \langle j \rangle_j = \mathbf{r}$	$\langle i \rangle_m$		$m+1 \dashv \mathcal{C}^t(2,v)$
$I_i^{t+1} \geq k$		$\langle$**all stay**$\rangle_m$		$m+1 \dashv \mathcal{C}^t(2,v)$

$m = |\mathcal{C}^t(2,v)|$, where $v = 1, ..., t$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$I_i^{t+1} < k$	$\exists j \in G_i,\ \langle j \rangle_j = \mathbf{s}$	$\langle$**all stay**$\rangle_m$		$1 \dashv \mathcal{C}^t_{2,v+1}$
$I_i^{t+1} < k$	$\forall j \in G_i,\ \langle j \rangle_j = \mathbf{r}$	$\langle i \rangle_m$		$1 \dashv \mathcal{C}^t_{2,v+1}$
$I_i^{t+1} \geq k$		$\langle$**all stay**$\rangle_m$		$1 \dashv \mathcal{C}^t_{2,v+1}$

$1 \leq m < |\mathcal{C}^t(2,t+1)|$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$I_i^{t+1} < k$	$\exists j \in G_i,\ \langle j \rangle_j = \mathbf{s}$	$\langle$**all stay**$\rangle_m$		$m+1 \dashv \mathcal{C}^t(2,t+1)$
$I_i^{t+1} < k$	$\forall j \in G_i,\ \langle j \rangle_j = \mathbf{r}$	$\langle i \rangle_m$		$m+1 \dashv \mathcal{C}^t(2,t+1)$
$I_i^{t+1} \geq k$		$\langle$**all stay**$\rangle_m$		$m+1 \dashv \mathcal{C}^t(2,t+1)$

$m = |\mathcal{C}^t(2,t+1)|$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$I_i^{t+1} < k$	$\exists j \in G_i,\ \langle j \rangle_j = \mathbf{s}$	$\langle$**all stay**$\rangle_m$		terminal **r**
$I_i^{t+1} < k$	$\forall j \in G_i,\ \langle j \rangle_j = \mathbf{r}$	$\langle i \rangle_m$		$1 \dashv \mathcal{C}^t_{3,1}$
$I_i^{t+1} \geq k$		$\langle$**all stay**$\rangle_m$		terminal **r**

Table 16: The $m \dashv \mathcal{C}^t(3,\cdot)$ on the path

$1 \leq m < |\mathcal{C}^t(3,\cdot)|$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
	$\mathbf{s}$	$\forall j \in G_i$, j plays $\mathbf{s}$	$m+1 \dashv \mathcal{C}^t(3,\cdot)$
	$\mathbf{s}$	$\exists j \in G_i$, j plays $\mathbf{r}$	terminal $\mathbf{r}$

$m = |\mathcal{C}^t(3,\cdot)|$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
	$\mathbf{s}$	$\forall j \in G_i$, j plays $\mathbf{s}$	$1 \dashv \mathcal{O}^{t+1}$
	$\mathbf{s}$	$\exists j \in G_i$, j play $\mathbf{r}$	terminal $\mathbf{r}$

A.2 Missing proofs

Proof of Lemma 2.1

Proof. The proof is done by contraposition. Suppose Rebels' strategies constitute an APEX equilibrium. By definition of the APEX equilibrium, at every θ, there is a period T^θ when all Rebels' actions start to repeat themselves. Let $T = \max_{\theta\in\Theta} T^\theta$. For Rebel i, let $T_i = T + 1$, and suppose $0 < \sum_{\theta:\#[R](\theta)\geq k} \beta^{\pi,\tau^*}_{G_i}(\theta|h^s_{G_i}) < 1$ for some $s \geq T_i$. Then this Rebel assigns positive weight at some $\theta' \in \{\theta : \#[R](\theta) < k\}$ and some positive weight at some $\theta'' \in \{\theta : \#[R](\theta) \geq k\}$ at period s. Note that i has already known θ_j if $j \in G_i$, and therefore i assigns positive weight at some $\theta' \in \{\theta : \#[R](\theta) < k, \theta_l = R, l \notin G_i\}$ and positive weight at some $\theta'' \in \{\theta : \#[R](\theta) < k, \theta_l = I, l \notin G_i\}$. Since all Rebels' actions start to repeat themselves at period T, i cannot update information afterwards. Suppose i's continuation strategy is to continuously play **revolt**, then this is not ex-post efficient when $\#[R](\theta) < k$; suppose i's continuation strategy is to continuously play **stay**, then this is not ex-post efficient when $\#[R](\theta) \geq k$. □

Proof of Theorem 1

Proof. Let τ^* be the following strategy. After the nature moves, a Rebel i plays **revolt** if he has no Inert neighbor; i plays **stay** forever if he has an Inert neighbor. After the first period, if i has not detected a deviation and observes that all his Rebel neighbors play **revolt** continuously previously, he plays **revolt** in the current period; otherwise, he plays **stay** afterwards and forever. If a Rebel j deviates, then j plays **stay** afterwards and forever.

At period s, according to τ^*, if i has not detected a deviation, but he observe his Rebel neighbors plays **stay** in the current period, he forms the belief of

$$\sum_{\theta:\#[R](\theta)\geq k} \beta^{\pi,\tau^*}_{G_i}(\theta|h^s_{G_i}) = 0$$

afterwards and forever. Therefore, he plays **stay** afterwards and forever as his best response.

At period s, if a Rebel detects a deviation, or he has deviated, to play **stay** afterwards and forever is his best response since at least one player will play **stay** afterwards and forever.

Since the network is finite with n vertices, if all players do not deviate, after period n, each Rebel plays **revolt** and gets payoff 1 forever if $\theta \in \{\theta : \#[R](\theta) \geq k\}$; each Rebels plays **stay** and gets payoff 0 forever if $\theta \in \{\theta : \#[R](\theta) < k\}$. However, a Rebel who has deviated surely gets payoff 0 forever after period n. Therefore, there is a $0 < \delta < 1$ large enough to impede Rebels to deviate.

To check if τ^* and $\{\beta^{\pi,\tau^*}_{G_i}(\theta|h^s_{G_i})\}_{i\in N}$ satisfy full consistency[20], take any $0 < x < 1$ such that Rebels play τ^* with probability $1-x$ and play other behavior strategies with probability x. Clearly, when $x \to 0$, the belief converges to $\{\beta^{\pi,\tau^*}_{G_i}(\theta|h^s_{G_i})\}_{i\in N}$. Since the out-of-path strategy is the best response for both of the Rebel who detects deviation and the Rebel who makes deviation, for arbitrary beliefs they hold, τ^* is a sequential equilibrium. □

Proof of Theorem 3

Proof. Since the network is finite, θ has strong connectedness, and $[R](\theta) \neq \emptyset$, there is a minimum t_i such that $I^{t_i}_i = [R](\theta)$ for each i by the definition of I^t_i. Let $P = \arg\min_{i\in N}\{t_1, ..., t_n\}$ with generic element p. Therefore $I^{t_p}_p = [R](\theta)$. I show that $p \in R^{t_p-1}$ to complete the proof. I prove it by contradiction. If $p \notin R^{t_p-1}$, then $I^{t_p-1}_p \subseteq G^{t_p-1}_j$ for some $j \in G_p$. Then, all the Rebels in TR_{jp} are in $G^{t_p-1}_j$, but there exist Rebels in TR_{pj} who are in $G^{t_p-1}_j$ but not in $I^{t_p-1}_p$. This is because the network is acyclic and $I^{t_p-1}_p \subset [R](\theta)$. But then $p \notin P$ since $I^{t_j-1}_j = [R](\theta)$ already. I then conclude that $p \in R^{t_p-1}$.

□

Proof of Lemma 4.1

Proof. The proof is by contradiction. Suppose that, at t-block and before T^θ, there are three or more θ-pivotal Rebels. Since θ has strong connectedness, there are three of them, p_1, p_2, p_3, with the property $p_1 \in G_{p_2}$ and $p_2 \in G_{p_3}$.

[20] Krep and Wilson (1982)

Since the network is acyclic, $p_1 \notin G_{p_3}$ and $p_3 \notin G_{p_1}$. Since p_1 is θ-pivotal, $I^t \subset [R](\theta)$ and $I_p^{t+1} = [R](\theta)$. It implies that, in $TR_{p_1 p_2}$, p_1 can reach all Rebels by $t+1$ edges, but cannot reach all of them by t edges. The same situation applies to p_3. However, it means that p_2 can reach all Rebels in $TR_{p_1 p_1}$ by t edges and reach all Rebels in $TR_{p_1 p_1}$ by t edges, and hence $I_{p_2}^t = [R](\theta)$. It contradict to the definition of θ-pivotal Rebel.

□

Proof of Lemma 4.2

Proof. A θ-pivotal p knows that $p^{'} \in G_i$ if $p^{'}$ is another one. p picks a neighbor p' and checks whether or not $[R](\theta) \subseteq I_p^t \cup I_{p^{'}}^t$ for all possible $I_{p^{'}}^t$. By common knowledge of the network, p knows $G_{p'}^t$. Since p is θ-pivotal, he is certain that all the Rebel in the direction from p toward $p^{'}$ is in $G_{p^{'}}^t$ and hence in $I_{p^{'}}^t$. Then p can check whether or not $[R](\theta) \subseteq I_p^t \cup I_{p^{'}}^t$ for all possible $I_{p^{'}}^t$. If so, then p knows $p^{'}$ is also θ-pivotal by the definition of θ-pivotal. Similarly, a θ-pivotal $p^{'}$ can do the same procedure. Therefore, if there are two θ-pivotal p and $p^{'}$, they commonly know that they are θ-pivotal. They commonly know this at the beginning of t-block since they know I_p^t and $I_{p^{'}}^t$ by the construction of information hierarchy. □

Proof of Theorem 2 I begin with the following lemmas stating that all Rebels eventually learn the relevant information on the path.

Lemma A.1. *If the network is acyclic and if the θ has strong connectedness, then the equilibrium path specified in Section A.1.1 is an APEX strategy.*

Proof. Firstly, suppose θ satisfies $\#[R](\theta) < k$. I show that, all Rebels will enter terminal **s** eventually without entering terminal **r**. Let p be the Rebel defined in the proof of Theorem 3 so that $I_p^{t_p} = [R](\theta)$, where p is one of the earliest Rebels who knows $\#[R](\theta) < k$. I claim that $\#X_p^{t_p} < k$ if and only if $I_p^{t_p} = [R](\theta)$. For the only if part, the proof is by way of contradiction. If not, by the full support on strong connectedness, there is a possible Rebel outside $I_p^{t_p}$, and therefore p is uncertain $\#[R](\theta) < k$. For the if part, note that $I_p^{t_p} \subset X_p^{t_p}$ and therefore $\#I_p^{t_p} < \#X_p^{t_p} < k$. I also claim that $\#X_p^{t_p}(m) < k$ if and only

$I_p^{t_p}(m) = [R](\theta)$. The proof is exactly the same as the noted above by replacing $I_p^{t_p}$ to $I_P^{t_p}$ and $I_p^{t_p}(m)$ to $I_P^{t_p}(m)$

Referring to Table 9 to Table 13, whenever there is a p so that $\#X_p^{t_p} < k$, p plays **stay** forever. It implies that all Rebels enter terminal **s** right after $\mathcal{C}_1^t, t \geq 0$. Notice that Rebels entering to terminal **r** only after some period after $\mathcal{C}_1^t$ and therefore all Rebels will enter terminal **s** before terminal **r**.

Secondly, suppose θ satisfies $\#[R](\theta) \geq k$. I show that all Rebels will enter terminal **r** eventually. Note first that if there is a Rebel p so that $\#I_p^1 \geq k$, all Rebels enter terminal **r** after $\mathcal{C}^1(3, \cdot)$ by referring to Table 9, Table 10, and Table 11. At $t > 0$, if there is a Rebel p who has play $\langle 1 \rangle$ at $\mathcal{O}^t$, by the postulate of $\#[R](\theta) \geq k$, after $\mathcal{C}^t(3, \cdot)$, all Rebels enter terminal **r** according to the equilibrium path specified in Table 12, Table 13, Table 14, Table 15, and Table 16. There must be some Rebel $p \in R^t$ who plays $\langle 1 \rangle$ at $\mathcal{O}^t$ for some t by the same argument in the proof of Theorem 3. $\square$

Due to Lemma A.1, define $T_{\tau^*}^{\theta}$ as the earliest period at which all Rebels play ex-post efficient outcome afterwards according to an APEX equilibrium τ^*. For simplicity, I suppress the notation $\beta_{G_i}^{\pi,\tau}(\theta|h_{G_i}^s)$ to $\beta_{G_i}^{\tau}(\theta|h_{G_i}^s)$ and the notation $\alpha_{G_i}^{\pi,\tau}(\theta, h^s|\theta_{G_i}, h_{G_i}^s)$ to $\alpha_{G_i}^{\tau}(\theta, h^s|h_{G_i}^s)$. If $P(\theta)$ is a property of θ, define

$$\beta_{G_i}^{\tau}(P(\theta)|h_{G_i}^s) \equiv \sum_{\theta \in \{\theta : P(\theta)\}} \beta_{G_i}^{\tau}(\theta|h_{G_i}^s).$$

Furthermore, if m, s are periods and $m > s$, denote $h^{m|s}$ as a history in H^m so that $(h^s, h^{m|s}) \in H^m$. Denote $\tau'|_{\tau}^{s}$ as a strategy following τ til period s.

Claim 1. *Suppose Rebel i follows an APEX equilibrium τ^* til period s. If there is a strategy $\tau|_{\tau^*}^{s} = (\tau_i, \tau_{-i})|_{\tau^*}^{s}$ generating a history $h^{m|s}, \infty > m > s$ so that i will be uncertain about the relevant information and stop belief updating after m, then Rebel i will not deviate to $\tau|_{\tau^*}^{s}$ if $\delta \in (0, 1)$ is sufficiently high.*

Proof. Denote $\beta_{G_i}^{\tau|_{\tau^*}^{s}}(\theta|h^{m|s}, h_{G_i}^s)$ as i's belief about θ at m following $h^{m|s}$ induced by $\tau|_{\tau^*}^{s}$. By the postulate, $0 < \beta_{G_i}^{\tau|_{\tau^*}^{s}}(\#[R](\theta)|h^{m|s}, h_{G_i}^s) < 1$. From the perspective that i holds a belief of $\beta_{G_i}^{\tau^*}(\#[R](\theta) \geq k|h_{G_i}^s)$ at period s, $h^{m|s}$ can be thought of an imperfect signal at period m to infer whether or not $\#[R](\theta) \geq k$: if $\#[R](\theta) \geq k$, $h^{m|s}$ occurs with probability η and does not occur with probability $1-\eta$; if $\#[R](\theta) < k$, $h^{m|s}$ occurs with probability μ and does not occur with probability $1-\mu$ so that $0 \leq \eta \leq 1$, $0 \leq \mu \leq 1$, and $0 < \eta/\mu < \infty$. Denote $M = \max\{m, T^{\theta}_{\tau^*}\}$. Rebel i's maximum expected stage-game payoff starting from M by following $h^{m|s}$ calculated at period s is

$$V = \max\{\eta\beta^{\tau^*}_{G_i}(\#[R](\theta) \geq k|h^s_{G_i}) - \mu\beta^{\tau^*}_{G_i}(\#[R](\theta) < k|h^s_{G_i}), 0\}.$$

The first term $\eta\beta^{\tau^*}_{G_i}(\#[R](\theta) \geq k|h^s_{G_i}) - \mu\beta^{\tau^*}_{G_i}(\#[R](\theta) < k|h^s_{G_i})$ is i's expected stage-game payoff if all Rebels play **revolt** afterwards starting from M. The second term 0 is the one by playing **stay** afterwards. Rebel i's expected stage-game payoff starting from M by following τ^* calculated at period s is

$$\beta^{\tau^*}_{G_i}(\#[R](\theta) \geq k|h^s_{G_i}) > V.$$

The inequality above is due to $0 < \eta < 1, 0 < \mu < 1$. There is a difference in present value of

$$W(\delta) = \frac{\delta^{M-s}(\beta^{\tau^*}_{G_i}(\#[R](\theta) \geq k|h^s_{G_i}) - V)}{1-\delta}.$$

Denote L as the summation of all gains from deviation calculated from period s to period M. L is finite since the stage-game payoff is finite and $M - s$ is finite. Taking sufficiently high $\delta \in (0, 1)$ so that $W(\delta) > L$ will deter this deviation. $\square$

Claim 2. *Suppose Rebel i follows an APEX equilibrium τ^* til period s. If i deviates to a strategy $\tau|^s_{\tau^*} = (\tau_i, \tau_{-i})|^s_{\tau^*}$ so that there are d Rebels, $d > 0$, detects this deviation, then Rebel i will not deviate to $\tau|^s_{\tau^*}$ if $\delta \in (0, 1)$ is sufficiently high.*

Proof. There are two cases: $I^s_i < k$ or $I^s_i \geq k$. If $s \in \mathcal{O}^t$ for some $t \geq 1$, then I^s_i refers to $I^s_i = I^{t+1}_i(m)$ such that m is the m-th period in $\mathcal{O}^t$ and $s = m + \sum_{\gamma=1}^{t}(|\mathcal{C}^{\gamma}| + |\mathcal{O}^{\gamma-1}|)$, where $|\mathcal{O}^0| = 0$.

Suppose $I^s_i < k$. If $\tau|^s_{\tau^*}$ leads to a history so that i will never learn the relevant information, i will not deviate, by Claim 1. If $\tau|^s_{\tau^*}$ leads to a history so that i is certain that there are k' Rebels, $k \leq k' < k + d$, play some action forever after some period m_d,

i's stage-game payoff is at most 0 after m_d. If Rebel i follows τ^*, i's stage-game payoff is 1 after T^θ. Therefore after $M = \max\{T^\theta, m_d\}$, there is a difference as least $1/(1-\delta)$ as the present value at M. Since $I_i^s < k$, $0 < \beta_{G_i}^{\tau^*}(\#[R](\theta) \geq k|h_{G_i}^s) < 1$, and therefore $\eta = \beta_{G_i}^{\tau^*}(k \leq \#[R](\theta) < k + d|h_{G_i}^s)$ is positive. It implies there is a difference at least $\eta\delta^{M-s}/(1-\delta)$ as the present value at s. As for the case $\tau|_{\tau^*}^s$ leads to a history so that i is certain that there are k' Rebels, $k' \geq k + d$ or $k' < k$, i's stage-game payoff after $M = \max\{T^\theta, m_d\}$ is as same as that if i follows τ^*. Denote L as the summation of all gains from deviation calculated from period s to period M. L is finite since both stage-game payoff and $M - s$ are finite. Taking sufficiently high $\delta \in (0, 1)$ so that $\eta\delta^{M-s}/(1-\delta) > L$ will deter this deviation.

Suppose $I_i^s \geq k$. Since i follows τ^* til s, there are two cases: $s \in \mathcal{O}^t$ for some $t \geq 1$ or $s \in \mathcal{C}^t$ for some $t \geq 1$. Suppose $s \in \mathcal{O}^t$, i will be ready to play $\langle 1 \rangle$ in $\mathcal{O}^t$. Because playing $\langle I \rangle$, where I is a subset of N, incurs more negative payoff than $\langle 1 \rangle$ since there is discounting and there is at most one another player might play $\langle I \rangle$, i will not deviate in this case. Suppose $s \in \mathcal{C}^t$. If $s \in \mathcal{C}^t(1, v)$, where $v = 1, ..., n$, i's detectable deviation is to play a sequence other than what Table 4 specifies, it will be detected by all i's neighbors. There is a positive probability that all i's neighbors will play **stay** forever according to the out-of-path belief and then all Rebels will contagiously play **stay** forever. i's expected continuation payoff is 0 from some timing after s. The same situation happens if $s \in \mathcal{C}^t(2, v)$, where $v = 1, ..., t + 1$, or $s \in \mathcal{C}^t(3, v)$, where $v = 1, ..., n$. Take m_s as the period when all Rebels play **stay** forever. m_s is finite since the network is finite. Then take $M = \max\{T^\theta, m_s\}$. There is a difference at least $\eta\delta^{M-s}/(1-\delta)$ as the present value at s. Denote L as the summation of all gains from deviation calculated from period s to period M. L is finite since both stage-game payoff and $M - s$ are finite. Taking sufficiently high $\delta \in (0, 1)$ so that $\eta\delta^{M-s}/(1-\delta) > L$ will deter this deviation. $\square$