Coordination in Social Networks: Communication by Actions

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Abstract

1 Introduction

2 Model

Given a finite set X, denote $\#X$ as its cardinality .

There is a set of players $N = \{1, ..., n\}$. They constitute a network $G = (V, E)$ so that the vertices are players ($V = N$) and an edge is a pair of them (E is a subset of the set containing all two-element subsets of N). Throughout this paper, G is assumed to be finite, commonly known, fixed, undirected, and connected.[1]

Time is discrete with index $s \in \{0, 1, ...\}$. At $s = 0$, the nature chooses a state $\theta \in \Theta = \{R, I\}^n$ once and for all according to a common prior π. R and I represent as Rebel and Inert respectively. After the nature moves, players play a normal form game, the *k-threshold game*, infinitely repeated played with common discounted factor δ. In the k-threshold game,

[1]A path in G from i to j is a finite sequence $(l_1, l_2, ..., l_L)$ without repetition so that $l_1 = i$, $l_L = j$, and $\{l_q, l_{q+1}\} \in E$ for all $1 \leq q < L$. G is fixed if G is not random, and G is undirected if there is no order relation over each edge. G is connected if, for all $i, j \in N$, $i \neq j$, there is a path from i to j.

$A_R = \{\textbf{revolt}, \textbf{stay}\}$ is the set of actions for R while $A_I = \{\textbf{stay}\}$ is that for I. A Rebel's static payoff function is defined as follows.

- $u_R(a_R, a_{-i}) = 1$ if $a_R = \textbf{revolt}$ and $\#\{j : a_j = \textbf{revolt}\} \geq k$
- $u_R(a_R, a_{-i}) = -1$ if $a_R = \textbf{revolt}$ and $\#\{j : a_j = \textbf{revolt}\} < k$
- $u_R(a_R, a_{-i}) = 0$ if $a_R = \textbf{stay}$

. An Inert's static payoff is equal to 1 no matter how other players play.

For convenience, let $[R](\theta)$ be the set of Rebels given θ and the notion *relevant information* indicate to the information about whether or not $\#[R](\theta) \geq k$. Note that in the k-threshold game, the ex-post efficient outcome is that every Rebel plays **revolt** when $\#[R](\theta) \geq k$, and plays **stay** otherwise.[2]

During the game, any player, say i, can observe information only from himself and from his direct neighbors $G_i = \{j : \{i, j\} \in E\}$. These pieces of information include his and his neighbors' types ($\theta_{G_i} \in \Theta_{G_i} = \{R, I\}^{G_i}$) and his and their histories of actions up to period s ($h^s_{G_i} \in H^s_{G_i} \equiv \times_{t=1}^{s}(\times_{j \in G_i} H^t_j)$). I assume that payoffs are hidden to emphasize that observing neighbors' actions are the only channel to infer other players' types and actions.[3] To be precise, when θ is realized at $s = 0$, i's information set about θ is $P_i(\theta) \equiv \{\theta_{G_i}\} \times \{R, I\}^{N \setminus G_i}$. For the information sets about players' actions, the sets of histories of actions are given to be empty at $s = 0$. At $s > 0$, a history of actions played by i is $h^s_i \in H^s_i \equiv \emptyset \times A^s_i$ while a history of actions played by all players is $h^s \in H^s \equiv \times_{t=1}^{s}(\times_{j \in N} H^t_j)$. i's information set about other players' histories of actions up to $s > 0$ is $\{h^s_{G_i}\} \times H^s_{N \setminus G_i}$. A player i's pure behavior strategy τ_i is a measurable function with respect to his information partition if it maps $P_i(\theta) \times \{h_{G_i}\} \times H^s_{N \setminus G_i}$ to a single action in his action set for every s and for every θ.

By abusing the notation a bit, let h^τ_θ denote the realized sequence of actions generated by $\tau = (\tau_1, \tau_2, ..., \tau_n)$ given θ. Define $\mu^{\pi,\tau}_{G_i}(\theta, h^s | \theta_{G_i}, h^s_{G_i})$ as the conditional distribution over

[2]Moreover, at every θ and every k, the ex-post efficient outcome is unique and gives the maximum as well as the same payoff to every Rebel.

[3]Such restriction will be relaxed in the Section 5.

$\Theta \times H^s$ conditional on i's information up to s, which is induced by π and τ. The belief of i over θ up to s is then $\beta^{\pi,\tau}_{G_i}(\theta|\theta_{G_i}, h^s_{G_i}) \equiv \sum_{h^s \in H^s} \alpha^{\pi,\tau}_{G_i}(\theta, h^s|\theta_{G_i}, h^s_{G_i})$.

The equilibrium concept is the weak sequential equilibrium.[4] I am looking for the existence of approaching ex-post efficient equilibrium (*APEX equilibrium henceforth*), which is formally defined below.

Definition 2.1 (APEX strategy). *A behavior strategy τ is APEX if, for all θ, there is a terminal period $T^\theta < \infty$ such that the actions in h^τ_θ after T^θ repeats the static ex-post Pareto efficient outcome.*

Definition 2.2 (APEX equilibrium). *An equilibrium (τ^*, α^*) is APEX if τ^* is APEX.*

In an APEX strategy, all Rebels will play **revolt** forever after some period only if there are more than k Rebels; otherwise, Rebels will play **stay** forever after some period. It is as if the Rebels will learn the relevant information in the equilibrium. This is because, they will play the ex-post efficient outcome after a certain point of time and keep on doing so. Note that there are some implications based on the definition. Firstly, it is an equilibrium if every player plays **stay** forever. Secondly, in an APEX equilibrium, it is not only as if the Rebels will learn the relevant information: they must learn that by the following lemma.

Lemma 2.1 (Learning in the APEX equilibrium). *If the assessment (τ^*, μ^*) is an APEX equilibrium, then for all $\theta \in \Theta$, there is a finite time T^θ_i for every Rebel i so that*

$$\sum_{\theta \in \{\theta : [R](\theta) \geq k\}} \beta^{\pi,\tau^*}_{G_i}(\theta|h^s_{G_i}) = \text{ either } 1 \text{ or } 0$$

whenever $s \geq T^\theta_i$.

[4] A weak sequential equilibrium is an assessment $\{\tau^*, \mu^*\}$, where μ^* is a collection of distributions over players' information sets with the property that, for all i and for all s, $\mu^*_{G_i}(\theta, h^s|\theta_{G_i}, h^s_{G_i}) = \mu^{\pi,\tau^*}_{G_i}(\theta, h^s|\theta_{G_i}, h^s_{G_i})$ whenever the information set is reached with positive probability given τ^*. Moreover, for all i and for all s, τ^*_i maximizes i's continuation expected payoff conditional on θ_{G_i} and $h^s_{G_i}$ of

$$E^\delta_G(u_{\theta_i}(\tau_i, \tau^*_{-i})|\alpha^{\pi,\tau_i,\tau^*_{-i}}_{G_i}(\theta, h^s|\theta_{G_i}, h^s_{G_i}))$$

for all $h^s_{G_i}$.

Definition 2.3 (Learning the relevant information)**.** *A Rebel i learns the relevant information at period $\dot{s}$ if $\sum_{\theta \in \{\theta : [R](\theta) \geq k\}} \beta_{G_i}^{\pi, \tau^*}(\theta | h_{G_i}^s) =$ either 1 or 0 whenever $s \geq \dot{s}$.*

It is clear that an APEX equilibrium exists when $k = 1$. As for other cases, let us start with the case of $k = n$ and then continue on to the case of $1 < k < n$.

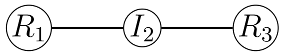
3 APEX equilibrium for $k = n$

In the case of $k = n$, a Rebel can get a better payoff from playing **revolt** than that from **stay** *only if* all players are Rebels. Two consequences follow. Firstly, if a Rebel has an Inert neighbor, this Rebel will always play **revolt** in the equilibrium. Secondly, at any period of time, it is credible for every Rebel to use playing stay forever afterwards as a punishment to a deviation if there is another player who also plays **stay** forever afterwards, independently from the belief held by the punisher. These two features constitute an APEX equilibrium and further transform itself to a sequential equilibrium.

Theorem 1 (APEX equilibrium for the case of $k = n$)**.** *For any n-person repeated k-Threshold game with parameter $k = n$ played in a network, there is a δ^* such that a sequential APEX equilibrium exists whenever $\delta > \delta^*$.*

The proof is a contagion argument. Suppose a Rebel plays **revolt** at any period except: (1) he has an Inert neighbor, or (2) he has observed his Rebel neighbor played **stay** once. Since the network is finite and connected, a Rebel is certain that there is an Inert somewhere if he has seen his neighbor played **stay**; otherwise, he continues to believe that all players are Rebels. Upon observing a n consecutive **revolt**, it implies that no Inert exists; if not, it implies some Inert exist. The above strategy is an APEX strategy and therefore ready for the equilibrium path for an APEX equilibrium. For any deviation from the above strategy, let the out-of-path strategy be playing **stay** forever for both of the deviant and the Rebel (the punisher) who detects that. This out-of-path strategy is credible for both the deviant and the punisher and is independent from the belief held by the punisher. Hence, it is also sequential rational.

Figure 1: The state and the network in which the APEX equilibrium does not exist when $k = 2$.



4 APEX equilibrium for $1 < k < n$

In contrast to the case of $k = n$, a Rebel still has the incentive to play **revolt** even if he has an Inert neighbor. This opens a possibility of non-existence of APEX equilibrium. Let us consider Example 1 below.

Example 1. Suppose that $k = 2$ and $\theta = (R, I, R)$. The state and the network is represented in Figure 1. Rebel 1 never learns θ_3 since Inert 2 cannot reveal information about θ_3. The APEX equilibrium does not exist in this scenario.

The following condition that works on the prior *full support on strong connectedness* excludes the possibility of non-existence of APEX equilibrium. To this end, I begin with the definition of *strong connectedness.*

Definition 4.1 (Strong connectedness)**.** *Given G, a state θ has strong connectedness if, for every two Rebels, there is a path consisting of Rebels to connect them.*

In the language of graph theory, this definition is equivalent to: given G, θ has strong connectedness if the induced graph by $[R](\theta)$ is connected.

Definition 4.2 (Full support on strong connectedness)**.** *Given G, π has full support on strong connectedness if*

$$\pi(\theta) > 0 \Leftrightarrow \ \theta \text{ has strong connectedness}$$

As a remark, the definition of the full support on strong connectedness is stronger than common knowledge about that every state has strong connectedness. This marginal requirement is subtle and is more convenient in constructing equilibrium.[5]

I am ready to state the main characterization of this paper:

[5]The main result only requires a weaker version: $\pi(\theta) > 0 \Rightarrow \ \theta$ has strong connectedness . However,

Theorem 2 (APEX equilibrium for the case of $1 < k < n$)**.** *For any n-person repeated k-Threshold game with parameter $1 < k < n$ played in networks, if networks are acyclic and if π has full support on strong connectedness, then there is a δ^* such that an APEX equilibrium exists whenever $\delta > \delta^*$.*[6]

Constructing an APEX equilibrium, in this case, is more convoluted than that in the case of $k = n$. I illustrate the proof idea throughout this paper till Section, while leaving the formal proof in Appendix. Moreover, since the case of $k = 2$ is trivial, the depictions throughout this paper below are all for $2 < k < n$ cases.

In the case of $k = n$, T^θ can be determined independently from θ by setting $T^\theta = n$, but it is not obvious how to obtain T^θ before an equilibrium has been constructed.[7] Moreover, the free-rider problem might exist in the current case (as demonstrated in Introduction), but this problem never occurs in the proposed APEX equilibrium for Theorem 1. As for the punishment scheme, playing **stay** forever is not effective anymore since a deviation might only be seen by parts of players (network monitoring), and thus group punishment is hard to be executed.

Before we delve further into the logic of the proof of Theorem 2, I would like to introduce a game in Section 4.1: *T-round writing game*. This is a simpler auxiliary scenario that mimics relevant features in the original game. In the T-round writing game, I allow players to write to each other. They will be endowed a writing technology so that they can write without cost (cheap talk) or with a cost function before they take actions.

Roughly speaking, in the T-round writing game, players can write to share information about θ for T rounds. They then play a one-shot k-threshold game at round $T + 1$. Note that in an APEX equilibrium path in the original game, players would stop updating their belief after some finite time and keep playing the same action in the k-threshold game. The game form of the T-round writing game mimics the structure of the APEX equilibrium path in the original game. In Section 4.2, I then further modify the T-round writing game

working on this weaker version is at the expense of much tedious proof. Throughout this paper, I will stick to the original definition.

[6] A network is acyclic if the path from i to j for all $i \neq j$ is unique.

[7] Readers might refer to the proof of Theorem 1.

to allow T to be endogenously determined in the equilibrium, which is seamlessly analogous to the original game.[8]

4.1 Deterministic T-round writing game

The network, the set of states, and the set of players follow exactly the same definitions defined in Section 2. In the deterministic T-round writing game, each player endows a *writing technology.* A writing technology for player i is a pair of (W, M_i), in which $W = \{\mathbf{r}, \mathbf{s}\}^L$, $L \in \mathbb{N}$, and $M \equiv \times_{i \in N} \times_{t=1}^{T} M_i^t$ recursively defined by

$$M_i^1 = \{f | f : \Theta_{G_i} \to W\} \cup \{\emptyset\}$$

$$\text{for } 2 \leq t \leq T \text{ , } M_i^t = \{f | f : \times_{j \in G_i} M_j^{t-1} \to W\} \cup \{\emptyset\}.$$

W is interpreted as the set of sentence composed by letters $\mathbf{r}$ or $\mathbf{s}$ with length L, while M_i can be understood as i's grammar. The $\emptyset$ is interpreted as keeping silent. The meaning of "i writes to his neighbors at round t" is equivalent to "i selects an $f \in M_i^t$ to get an element $w \in W$ according to f. Moreover, m can be observed by all i's neighbors". A sentence being a mixture of two sentences $w, w' \in W$ is denoted as $w \oplus w'$ with the property that $(w \oplus w')_l = \mathbf{r}$ if and only if $w_l = \mathbf{r}$ or $w_l' = \mathbf{r}$ for all $l \in \{1, ..., L\}$.

The time line for the deterministic T-round writing game is as follows.

1. Nature chooses θ according to π.

2. θ is then fixed throughout rounds.

3. At $t = 1, ..., T$ round, players write to their neighbors.

4. At $T + 1$ round, players play a one-shot k-Threshold game.

5. The game ends.

There is no discounting. A Rebel's payoff is the summation of his stage payoff across stages, while an Inert's payoff is set to be 1. The equilibrium concept is a weak sequential

[8] TBA.

Figure 2: A configuration of the state and the network in which players 1,2,4,5,6,8 are Rebels while players 3,7 are Inerts.

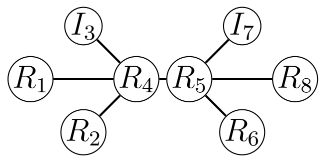
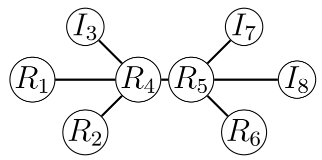


Figure 3: A configuration of the state and the network in which players 1,2,4,5,6 are Rebels while players 3,7,8 are Inerts.



equilibrium. The definition of APEX strategy is adapted as the strategy that induces ex-post outcome in the k-threshold game at $T+1$ round, and the definition of APEX equilibrium is adapted accordingly. In the examples below, let us focus on the configuration represented in Figure 2 and Figure 3 with $n = 8$ and $L = 8$.

Example 2 (Deterministic T-round writing without cost—cheap talk)**.** Let $k = 6$,$T = 2$, and writing be costless. Consider the following strategy ϕ.

At $t = 1$, ϕ specifies that the peripheral Rebels 1,2,6,8 keep silent. The central player Rebel 4 writes $(\mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s}, \mathbf{s})$. The central player Rebel 5 writes $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r})$ in the configuration in Figure 2 and writes $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s})$ in the configuration in Figure 3.[9] On the path of ϕ, Rebel 4's sentence thus reveals that players 1,2,4,5 are Rebels while player 3 is an Inert. It reveals so by the grammar: $\mathbf{r}$ is written in the i-th component if player i is a Rebel while $\mathbf{s}$ is written in the j-th component if player j is Inert or unknown to Rebel 4. Rebel 5's sentence is written according to the same grammar. Note that the common knowledge of the network contributes to the ability in revealing the players' types.

At $t = 2$, ϕ specifies that the peripheral Rebels 1,2,6,8 still remain silent. Rebel 4

[9]The notion of "peripheral" and "center" will be formalized in Section 4.3.1

writes $(\mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r})$ in the configuration in Figure 2 and writes $(\mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s})$ in the configuration in Figure 3. Rebel 5 writes exactly the same way as Rebel 4. This is to say Rebel 4 and 5 share information at $t = 1$ and then coordinate to announce a mixture sentence at $t = 2 = T$.

At $t = 3 = T + 1$, all Rebels know whether or not the number of Rebels (by counting $\mathbf{r}$ in Rebel 4 or 5's combined sentence) is greater than or equal to $k = 6$. This leads all Rebels to play the ex-post efficient outcome in the one-shot k-threshold game.

Let o be the belief system and the in-path belief be induced by ϕ. The out-of-path strategy of ϕ specifies that, if a detectable deviation is made, then the Rebels who detect that deviation will keep silent until $t = T$ and then play **stay** at $t = T + 1$.[10] The out-of-path belief of o is to believe that all players who are not neighbors are all Inerts, which rationalizes the out-of-path strategy. Since writing is costless, and any deviation by Rebel 4 or 5 would strictly decrease the possibility of achieving an ex-post efficient outcome, the assessment (ϕ, o) constitutes an APEX equilibrium.

Example 3 (Deterministic T-round writing with cost function and the free-rider problem)**.** Let $k = 6$ and $T = 2$. Suppose that keeping silent incurs no cost, but writing incurs a cost of $\epsilon > 0$ that is strictly decreasing with the number of $\mathbf{r}$ in a sentence. This is to say writing $(\mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r})$ incurs the least cost while writing $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{s})$ incurs the largest.

If so, that assessment (ϕ, o) in the previous example will no longer be an APEX equilibrium. To prove this, first note that the sentence of either Rebel 4 or 5's truthfully reveals their information at $t = 1$ on the path of ϕ. According to the above-mentioned, Rebel 4 will know the relevant information after $t = 1$ (by common knowledge of the network) even if he deviates to writing that all his neighbors are Rebels.[11][12] Rebel 5 is in the same situation as Rebel 4 and therefore will also write the sentence that indicates that all his neighbors

[10] A deviation could be, for instance, a wrong sentence that is not grammatical, is deviating from the in-path ϕ, ..., etc

[11] This sentence is $(\mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s}, \mathbf{s})$, which incurs less cost than the truthfully reporting sentence $(\mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s}, \mathbf{s})$.

[12] If he keeps silent, then this behavior will be considered as a deviation, and therefore he will never get the maximum payoff of 1. Hence, he will avoid doing so.

are Rebels. However, these sentences are uninformative. It turns out that both of them will deviate, and neither of them can know the relevant information after $t = 1$.

Fortunately, the following example shows the free-rider problem can be solved.

Example 4 (Deterministic T-round writing with cost function and solving the free-rider problem)**.** The free-rider problem that occurs in the previous example can be solved. The solution is to add more rounds to the game and exploit the assumption of common knowledge of the network. More precisely, let $k = 6$ and $T = 3$.

Consider a strategy $\dot{\phi}$ and focus on the interaction between Rebels 4 and 5. On the path of $\dot{\phi}$, at $t = 1$, $\dot{\phi}$ specifies that the lowest-index Rebel between Rebels 4 and 5 is chosen to be a free rider, while the other one truthfully writes down his information. To be precise, at $t = 2$, Rebel 4 will be the free rider—the person who writes the least-cost sentence; Rebel 5 writes $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r})$ in the configuration of Figure 2 while writing $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s})$ in the configuration of Figure 3.

At $t = 2$, Rebel 5 keeps silent. Rebel 4 writes the least-cost sentence if Rebel 5 writes $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r})$ at $t = 1$ but keeps silent if Rebel 5 writes $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s})$. Thus, the behavior of Rebel 4 at $t = 2$ reveals the relevant information.

At $t = 3 = T$, Rebel 4 keeps silent. Rebel 5 writes the least-cost sentence if Rebel 4 writes the least-cost sentence at $t = 2$ and keeps silent if Rebel 4 keeps silent then.

It is straightforward to check that at $t = 4 = T + 1$, all Rebels know the relevant information and play the ex-post efficient outcome accordingly. To construct an APEX equilibrium from $\dot{\phi}$, recall (ϕ, o) and let the in-path belief of $\dot{o}$ be induced by $\dot{\phi}$. The out-of-path strategy follows that in ϕ, and the out-of-path belief follows that of o.

An observation is worthy to be noted: why is Rebel 5 willing to concede that Rebel 4 is chosen to be the free rider at $t = 1$? The reason is as follows. He *knows* that, by common knowledge of the network, he and Rebel 4 are in a free-rider problem. Moreover, again by common knowledge of the network, he knows that Rebel 4 knows this, Rebels 4 knows he knows that Rebel 4 knows this, ..., and so forth to infinite order of belief hierarchy. This is to say, at least in this case, Rebel 4 and 5 commonly know that they are engaged in a free-rider problem due to the common knowledge of the network. In Section 4.3.2, this

Figure 4: The linearly ordered labelled rounds in the indeterministic T-round writing game.

$$0_1' < 0_2' < ... < 0_{l_0}' < 1 < 1_1' < 1_2' < ... < 1_{l_1}' < 2 < ...,$$

where $l_0, l_1, ...$ are all finite numbers.

property of common knowledge about engaging in a free-rider problem will be formally characterized. This property is not merely a special case. It holds for any acyclic network in the constructed APEX equilibrium in the original game.

4.2 Indeterministic T-round writing game

In this section, the setting is exactly the same as that in the deterministic T-round writing game, except for that players will now jointly decide the round in which they will play the one-shot k-threshold game and then end the game. In other words, before they play the one-shot k-threshold game, they have to *reach an agreement*—the common knowledge about which round is the terminal round T.

The set of rounds is countably infinite and linearly ordered with generic element t. The writing technology is the same as that in deterministic T-round writing game, except for letting $W = \{\mathbf{r}, \mathbf{s}\}^L \cup \{\mathbf{r}, \mathbf{s}\}^{L'}$ now. In the example below, let $L = 8$ and $L' = 1$.

Conceptually, there could be two kinds of rounds. In the first kind, players write to share information about θ (as they do in the deterministic T-round writing game). In the second kind, players write to form the common knowledge about T. Let us partition the set of rounds into two parts, Γ and Γ', which represent the first kind and the second kind respectively. The round in Γ is labelled γ while the round in Γ' is labelled γ_l'. The rounds are linearly ordered by $<$. To be more precise, the rounds are ordered as shown in Figure 4. An indeterministic T-round writing game is illustrated below. Along with it, an APEX equilibrium will be constructed.

Example 5 (Indeterministic T-round writing with cost function)**.** Let $l_j = 2$ for $j = 0, 1, ...$ Suppose that the setting is exactly the same as that in Example 4, except for that T is not deterministic. Let us consider the path of a strategy ψ. At a round in Γ', ψ specifies that, if a Rebel thinks "it is certain that the total number of Rebels outnumbers k and the nearest forthcoming round in Γ is the terminal round," he will write ($\mathbf{r}$). If a Rebel thinks "it is possible but not certain that $\#[R](\theta) \geq k$," he will write ($\mathbf{s}$). Otherwise, he will write $\emptyset$ to show that "it is impossible that $\#[R](\theta) \geq k$ and the nearest forthcoming round in Γ is the terminal round."

According to this strategy, $t = 1$ is not terminal if no Rebel has written ($\mathbf{r}$) or $\emptyset$ before that. For instance, $t = 1$ is not terminal in the configuration in Figure 2 and Figure 3.

If $t = 1$ is not terminal, at $t = 1$, Rebels 4 and 5 are in a free-rider problem as Example 3 shows. ψ solves it by specifying that Rebel 4 is the free rider and that Rebel 5 writes his information truthfully (as what $\dot{\phi}$ does in Example 4).

At $t = 1_1'$, Rebel 4 knows $\#[R](\theta) \geq k$ in the configuration in Figure 2 and knows $\#[R](\theta) < k$ in the configuration in Figure 3. Therefore, he writes ($\mathbf{r}$) and $\emptyset$ respectively for these two configurations. As for other Rebels, they write ($\mathbf{s}$).

After $t = 1_1'$, eventually, all Rebels will learn the relevant information (by seeing the writing of Rebel 4 at $t = 1_1'$) and terminate their writing at $t = 2$. Therefore, $t = 1$ is the terminal round, and the Rebels play a one-shot k-threshold game at $t = 2$.

The strategy ψ can be made to be an APEX equilibrium strategy in the same way as the previous examples.

4.3 Dispensability of writing technology

In essence, writing technology is dispensable, and repeated actions are sufficient to serve as a communication protocol to achieve an ex-post outcome in an equilibrium. In this section, I draw analogue between the writing game and the original game in Table 1.

More precisely, in the equilibrium construction in the original game, let us partition the periods so that each part in the partition is analogous to a round in the writing game. The length of periods in a part is analogous to the length of a sentence. Since the actions played in a certain part of periods will incur an expected payoff, it is an analogue that writing is costly at a certain round in the writing game. The disjoint unions of period-sections also

Figure 5: The linearly ordered period-sections in the repeated k-threshold game.

$$\underbrace{\text{(periods for coordination)}}_{\text{0-block}} < \underbrace{\text{(periods for reporting)} < \text{(periods for coordination)}}_{\text{1-block}} < ...$$

constitute a coarser partition of periods, which is analogous to partitioning the rounds. The analogue of Γ is the set of *periods for reporting* in the original game to emphasize that these periods are for reporting information about θ. The analogue of Γ' is the set of *periods for coordination* in the original game to emphasize that these periods are for coordinating to play the ex-post efficient outcome. The partition of periods is linearly ordered by $<$, and let us define a coarser partition with parts *t-blocks* indexed by $t \in \{0, 1, ...\}$ along with the order of partition of periods as shown in Figure 5.

One could see that Figure 4 and Figure 5 are seamlessly analogous to each other as Table 1 shows.

Table 1: The analogue between indeterministic T-round writing game and repeated k-threshold game

Indeterministic T-round writing game	Repeated k-threshold game
A round	A range of periods
A sentence	A sequence of actions
The length of a sentence in a round	The length of a part of periods
A chosen digit in a sentence	A chosen action
The cost of writing a sentence	The expected payoff in a part of periods
The fixed grammar	The equilibrium path

Note that the notions of *peripheral* and *central* in Example 2 is not yet formalized as well as analogized to the original game. I generalize these notions in the original game by defining *information hierarchy* among players for each t-block below.

4.3.1 Information hierarchy

The information hierarchy across Rebels in G presents Rebels' information *before entering the periods for reporting at t-block*. That is designated by a tuple

$$(\{G_i^t\}_{i\in N}, \{I_i^t\}_{i\in N}, R^t, \theta).$$

G_i^t represents *the extended neighbors*: $j \in G_i^t$ if j can be reached by t consecutive edges from i, in which the endpoints of $t-1$ edges are both Rebels, while the remaining one consists of a Rebel and i himself. I_i^t represents as *the extended Rebel neighbors*: the set of Rebels in G_i^t. R^t represents as *the active Rebels*: those Rebels who are *active* in the sense that their extended Rebel neighbors are not a subset of their direct neighbors' extended Rebel neighbors. Those objects are defined by:

At $t = 0$,

$$\text{if } \theta_i = I,\ G_i^0 \equiv \emptyset,\ I_i^0 \equiv \emptyset.$$

$$\text{if } \theta_i = R,\ G_i^0 \equiv \{i\},\ I_i^0 \equiv \{i\}.$$

$$R^0 \equiv [R](\theta).$$

At $t = 1$,

$$\text{if } \theta_i = I,\ G_i^1 \equiv \emptyset,\ I_i^1 \equiv \emptyset.$$

$$\text{if } \theta_i = R,\ G_i^1 \equiv G_i,\ I_i^0 \equiv G_i \cap R^0.$$

$$R^1 \equiv \{i \in R^0 : \nexists j \in G_i \text{ such that } I_i^1 \subseteq G_j^1\}.$$

At $t > 1$,

$$\text{if } \theta_i = I,\ G_i^t \equiv \emptyset,\ I_i^t \equiv \emptyset.$$

$$\text{if } \theta_i = R,\ G_i^t \equiv \bigcup_{j\in G_i} G_j^{t-1},\ I_i^t \equiv \bigcup_{j\in G_i} I_j^{t-1},$$

$$I_{ij}^t \equiv I_i^{t-1} \cap I_j^{t-1} \text{ if } j \in G_i.$$

$$R^t \equiv \{i \in R^{t-1} : \nexists j \in G_i \text{ such that } I_i^t \subseteq G_j^t\}.$$

According to the above definition, the peripheral Rebels in the configuration in Figure 2 are active in 0-block (in R^0) but not active in 1-block (not in R^1), while the central players are active in both 0-block and 1-block. It can be shown that $R^t \subseteq R^{t-1}$ as follows.

Lemma 4.1. *If the θ has strong connectedness, then*

$$R^t \subseteq R^{t-1}$$

for all $t \geq 1$.

Is it enough to let only active Rebels share information about θ while θ can be revealed eventually? The answer is affirmative by Theorem 3 below, given that the network is acyclic and if the θ has strong connectedness.

Theorem 3. *If the network is acyclic and if the θ has strong connectedness, then*

$$[R](\theta) \neq \emptyset \Rightarrow \text{ there exists } t \geq 0 \text{ and } i \in R^t \text{ such that } I_i^{t+1} = [R](\theta).$$

The strategy on the equilibrium path will specify how Rebels act according to the information hierarchy. R^t will be those Rebels who are active in terms of sharing information in the periods for reporting in the t-block.

4.3.2 The equilibrium path in the periods for reporting

If there is no further mention, all the description in this section is for the APEX equilibrium path *before* the terminal period T^θ. For conciseness, let us shorten "periods for reporting in t-block" by O^t, denote $|\mathrm{O}^t|$ as the length of O^t, and shorten **revolt** and **stay** to **r** and **s** receptively henceforth.

$|\mathrm{O}^t|$ is independent from t and determined only the set of players. Firstly, assign each player i a distinguishing prime number x_i starting from 3 (by exploiting the common knowledge of the network). Then let $|\mathrm{O}^t| = x_1 \otimes x_2 \otimes ... \otimes x_n$, where $\otimes$ is the usual multiplication operator. The sequence of actions in O^t is with length $|\mathrm{O}^t|$ and would take one of the forms specified in the right column in Table 2. There, if $I \subseteq N$, then $x_I \equiv \otimes_{i \in I} x_i$. The abbreviations for these sequences are listed in the left column. Since these sequences in the periods for reporting are meant to share information about θ, the terms "playing the sequence" and "reporting the information" will be alternatively used.

It is worth noting that this sequence constructed by prime numbers brings two benefits. Firstly, since the multiplication of distinguishing prime numbers can be uniquely factorized,

Table 2: The notations for the sequences of actions in O^t on the path

Notations		The sequences of actions
$\langle I \rangle$	$\equiv$	$\langle \mathbf{s}, ..., \mathbf{s}, \underbrace{\mathbf{r}, \mathbf{s}, ..., \mathbf{s}}_{x_I} \rangle$
$\langle 1 \rangle$	$\equiv$	$\langle \mathbf{s}, ..., \mathbf{s}, \mathbf{r} \rangle$
$\langle$**all stay**$\rangle$	$\equiv$	$\langle \mathbf{s}, ..., \mathbf{s}, \mathbf{s} \rangle$

the Rebels can utilize such sequence to precisely report players' indexes. Secondly, the undiscounted expected payoff of playing $\langle I \rangle$ for some $I \subseteq N$ is always equal to -1, and therefore it is relatively easy to calculate. This is because, at any period in O^t, there is at most one player would play $\mathbf{r}$ by the property of prime number multiplication.[13]

The sequence $\langle I \rangle$ is used for reporting $I \neq \emptyset$ when I is a set of Rebels; The sequence $\langle$**all stay**$\rangle$ is for the inactive Rebels to report nothing; the sequence $\langle 1 \rangle$ is intentionally created to deal with the free-rider problem. To see how $\langle 1 \rangle$ works, let us formally define the *pivotal Rebel* and the *free-rider problem.*

Definition 4.3 (Pivotal Rebels in O^t)**.** *A Rebel p is pivotal in O^t if $p \in R^t$, $I_p^t \subset [R](\theta)$, $\#I_p^t < k$, and p is certain that he will learn the relevant information in the end of O^t, given that each $i \in R^t$ reports $\langle I_i^t \rangle$.*

From the definition, a pivotal Rebel p in O^t is the one who can learn the relevant information if all of his active Rebel neighbors truthfully report their information about θ to him. He can be further classified into two kinds. The first kind is the one who can learn the true state, while the second is the one who can learn the relevant information only.

For instances, when $k = 6$ and in O^1, in the configuration in Figure 2, only Rebels 4 and 5 are pivotal, and they are of the first kind; in the configuration in Figure 6, only Rebel 5 is pivotal, and he is of the first kind; in the configuration in Figure 7, only Rebel 4 pivotal, and he is of the second kind.

[13]This statement holds given that there is no player who plays $\langle 1 \rangle$.

Figure 6: A configuration of the state and the network in which player 1,3,5,4,7,8 are Rebels while players 2,4,9 are Inerts.

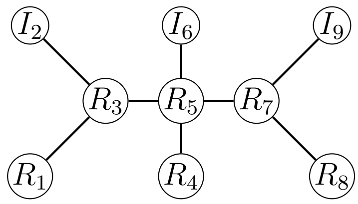
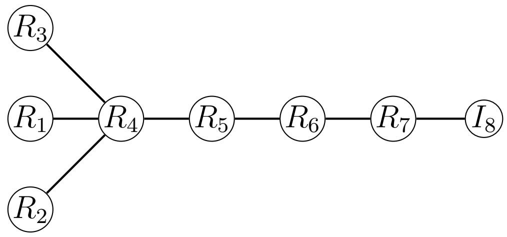


Figure 7: A configuration of the state and the network in which player 1,2,3,4,5,6,7 are Rebels while player 8 is an Inert.



For conciseness, let us call p of the first kind by *θ-pivotal*. For the second kind, if the network is acyclic and the prior has full support on strong connectedness, p is the second kind in O^t only if $I_p^t = k-1$. Since so, let us call the one with $I_p^t = k-1$ by *$k-1$-pivotal.*[14]

Below is the free-rider problem in O^t defined.

Definition 4.4. *There is a free-rider problem in* O^t *if there are multiple θ-pivotal Rebels in* O^t.

The following lemma is crucial.

Lemma 4.2. *If the network is acyclic and if π has full support on strong connectedness, there are at most two θ-pivotal Rebels in the t-block. Moreover, if there are two of them, they are neighbors.*[15]

[14] To show that a pivotal Rebel is the second kind in O^t only if $I_p^t = k-1$, one can follow the same argument in Lemma 4.1 and Theorem 3.

[15] As a remark, Lemma 4.2 is not true when the network is cyclic. To see this, consider a 4-player circle when $\theta = (R, R, R, R)$.

Notably,

Lemma 4.3. *If the network is acyclic and if π has full support on strong connectedness, when there are two θ-pivotal Rebels p and $p^{'}$ in the t-block, then they commonly know that they are θ-pivotal Rebels at the beginning of t-block.*

By Lemma 4.3, θ-pivotal Rebels in O^t can identify themselves at the beginning of O^t. It is fundamental, because, on the APEX equilibrium path, if the free-rider problem will occur in O^t, the strategy will specify that the lowest indexed θ-pivotal Rebel p in the free-rider problem plays $\langle 1 \rangle$, while the other one $p^{'}$ plays $\langle I^t_{p^{'}} \rangle$, beforehand. I.e. this knowledge is encoded in the belief system of an APEX equilibrium.

The assumption of acyclic network in Lemma 4.3 is indispensable. In Section 5.1, there is a configuration in a cycle, such that there is no common knowledge of the free-rider problem if players still follow the equilibrium path working for Theorem 2.

Overall, the sequences played in O^t on the path are listed in Table 3.

Table 3: The sequences of actions played in O^t on the path

Rebel i	i plays
$i \notin R^t$	$\langle$**all stay**$\rangle$
$i \in R^t$ but i is not pivotal	$\langle I^t_i \rangle$
i is $k-1$-pivotal	$\langle 1 \rangle$
i is θ-pivotal but not in the free-rider problem	$\langle 1 \rangle$
i is in the free-rider problem with the lowest index	$\langle 1 \rangle$
i is in the free-rider problem without the lowest index	$\langle I^t_i \rangle$

4.3.3 The equilibrium path in the periods for coordination

In this section, I discuss the sequences of actions in the periods for coordination on the path. Let us shorten "periods for coordination in t-block" by K^t. If there is no further mention, all the description in this section is for the APEX equilibrium path *before* the terminal period T^θ.

The main feature in the periods for coordination is, whenever a Rebel i has been thought to be not active starting at some t-block (i.e. $i \notin R^t$ for some $t \in \mathbb{N}$), there is no strategy for i to convince all the Rebels that $\#[R](\theta) \geq k$ even though i might want to propagandize it.

The structure in the periods for coordination is much intrigued as its counterpart Γ' in the indeterministic T-round writing game. In K, the periods are further partitioned by *divisions* and *sub-blocks*. I depict that below, where (K) represents a certain range of periods for coordination.

In K^0,

$$\overbrace{\underbrace{(\mathrm{K})}_{\text{one sub-block}}}^{\text{1-division}}\overbrace{\underbrace{(\mathrm{K})}_{\text{one sub-block}}}^{\text{2-division}}\overbrace{\underbrace{(\mathrm{K})\cdots(\mathrm{K})}_{n\text{ sub-blocks}}}^{\text{3-division}}$$

In K^t, $t > 0$,

$$\overbrace{\underbrace{(\mathrm{K})\cdots(\mathrm{K})}_{n\text{ sub-blocks}}}^{\text{1-division}}\overbrace{\underbrace{(\mathrm{K})\cdots(\mathrm{K})}_{t+1\text{ sub-blocks}}}^{\text{2-division}}\overbrace{\underbrace{(\mathrm{K})\cdots(\mathrm{K})}_{n\text{ sub-blocks}}}^{\text{3-division}}$$

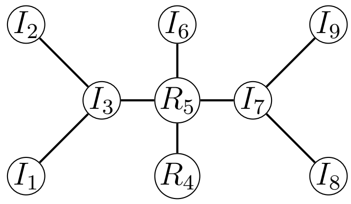
For convenience, in the t-block, denote $\mathrm{K}^t_{u,v}$ as the v-th sub-block in u-division; denote $|\mathrm{K}^t_{u,v}|$ as the length of $\mathrm{K}^t_{u,v}$. Similarly, denote K^t_u as the u-division; denote $|\mathrm{K}^t_u|$ as the length of K^t_u. Let us shorten **revolt** and **stay** to **r** and **s** receptively henceforth. On the path, the length of $|\mathrm{K}^t_{u,v}|$ is determined. For all $v \in \{1, ..., n\}$, $|\mathrm{K}^t_{u,v}| = n$ for $u = 1, 2$, and $|\mathrm{K}^t_{u,v}| = 1$ for $u = 3$. The notations for the sequences of actions on the path are shown in Table 4 shows.[16]

Table 4: The notations for the sequences of actions in K^t_u for $u = 1, 2$, on the path

Notations		The sequences of actions
$\langle i \rangle$	$\equiv$	$\langle \mathbf{s}, ..., \mathbf{s}, \underbrace{\mathbf{r}, \mathbf{s}, ..., \mathbf{s}}_{i} \rangle$
$\langle$**all stay**$\rangle$	$\equiv$	$\langle \mathbf{s}, ..., \mathbf{s}, \mathbf{s} \rangle$

[16]Because, in the 3-division, the length of the sequence of actions is 1, i.e. playing an action, I dispense notations in the 3-division for conciseness.

Figure 8: A configuration of the state and the network in which player 4, 5 are Rebels while players 1,2,3,6,7,8,9 are Inerts.



The equilibrium behavior on the path in K^0 Since the 0-block has a simpler structure, I begin with depicting the equilibrium path in K^0, which are shown in Table 5, Table 6, and Table 7. The description for a Rebel i is whether or not i has learnt the relevant information. Notice that the Rebel i might learn the relevant information by observing his neighbors' behavior, if a Rebel i is certain that $\#[R](\theta) < k$, by the full support on strong connectedness, it must be the case that all Rebels are i's neighbors. All Rebels will be also certian about that immediately after K^0_1. Figure 8 illustrates this scenario when $3 \leq k < n$, in which Rebel 5 is certain $\#[R](\theta) < k$ in K^0_1, and then Rebel 4 is certain that as well immediately after K^0_1.

The intriguing part might be "how a Rebel i initiates the common knowledge about $\#[R](\theta) \geq k$." i does so by *first play* $\langle i \rangle$ *in* K^0_1 *and then play* $\langle$**all stay**$\rangle$ *in* K^0_2. His behavior is then distinguishable from Rebels of other kinds. His neighbors will know $\#[R](\theta) \geq k$ immediately after K^0_2, and then all Rebels will know that by playing $\mathbf{r}$ contagiously in K^0_3.

i will not deviate to play $\langle$**all stay**$\rangle$ even though it might be undetectable. This is by the assumption of acyclic network. If i does so, i will be considered as an inactive Rebel afterwards by all of his neighbors. His behavior will no longer be able to update the belief held by his neighbors whose information hierarchy rank is no less than the one he pretends to be. More precisely, if he is a $R^{\dot{t}}$ Rebel but he pretends not to be, he cannot influence the belief updating of the Rebels who are in R^{t-1}, $t \geq \dot{t}$.[17] He then faces a positive probability that not enough Rebels can be informed of $\#[R](\theta) \geq k$. If this event happens, he will only

[17]This argument is due to Lemma 4.1 and the belief updating on the path described in Table 8, Table 9, Table 10, Table 11, and Table 12.

get zero payoff. However, he can surely get the maximum payoff of 1 afterwards and forever after K_2^0. Sufficiently high discount factor will impede this deviation. In essence, one major part of proofs for Theorem 2 (in the appendix) follows the same argument.

Table 5: The sequences of actions played in K_1^0 on the path

Rebel i	i plays
i is certain $\#[R](\theta) < k$	$\langle$**all stay**$\rangle$
$i \notin R^1$ and is uncertain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$
$i \in R^1$ and is uncertain $\#[R](\theta) \geq k$	$\langle i \rangle$
i is certain $\#[R](\theta) \geq k$	$\langle i \rangle$

Table 6: The sequences of actions played in K_2^0 on the path

Rebel i	i plays
i is certain $\#[R](\theta) < k$	$\langle$**all stay**$\rangle$
$i \notin R^1$ and is uncertain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$
$i \in R^1$ and is uncertain $\#[R](\theta) \geq k$	$\langle i \rangle$
i is certain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$

Table 7: The sequences of actions played in K_3^0 on the path

Rebel i	i plays
i is certain $\#[R](\theta) < k$	$\langle \mathbf{s} \rangle$
$i \notin R^1$ and is uncertain $\#[R](\theta) \geq k$	$\langle \mathbf{s} \rangle$
$i \in R^1$ and is uncertain $\#[R](\theta) \geq k$	$\langle \mathbf{s} \rangle$
i is certain $\#[R](\theta) \geq k$	$\langle \mathbf{r} \rangle$

As a complementary, it is useful to list Rebels' updated beliefs consistent with the equilibrium path after K_1^0 and K_2^0 in Table 8 and Table 9. Players' information filtrations evolves from Table 8 to Table 9.

Table 8: The belief of $j \in G_i$ after observing i's previous actions immediately after K_1^0

i plays	The event $j \in G_i$ assigns with probability one
In K_1^0	
$\langle$**all stay**$\rangle$	$i \notin R^1$ if $j \in R^1$
$\langle$**all stay**$\rangle$	$\#[Rebels](\theta) < k$ if $j \notin R^1$
$\langle i \rangle$	$i \in R^1$ or $\#[Rebels](\theta) \geq k$

Table 9: The belief of $j \in G_i$ after observing i's previous actions immediately after K_2^0

i plays		The event $j \in G_i$ assigns with probability one
In K_1^0	In K_2^0	
$\langle$**all stay**$\rangle$	$\langle$**all stay**$\rangle$	$i \notin R^1$ if $j \in R^1$
$\langle$**all stay**$\rangle$	$\langle$**all stay**$\rangle$	$\#[Rebels](\theta) < k$ if $j \notin R^1$
$\langle i \rangle$	$\langle$**all stay**$\rangle$	$\#[Rebels](\theta) \geq k$
$\langle i \rangle$	$\langle i \rangle$	$i \in R^1$

The equilibrium behavior on the path in K^t for $t \geq 1$ Next, I describe the equilibrium behavior on the path in K^t when $t \geq 1$. Players' belief over states will now be contingent on others' bahavior in O^t. After all, Rebels share information in O^t. As an analogue to the grammar interpretation in the deterministic (or indeterministic) T-round writing game, I first illustrate how players update their belief after observing the equilibrium strategy on the path. After that, the in-path strategy consistent with players' belief is introduced. The belief updating in K^t is shown in Table 10, Table 11, and Table 12. Players' information filtrations evolves from Table 10, to Table 11, and to Table 12.

Table 10: The belief of $j \in G_i$ after observing i's previous actions immediately after O^t

i plays	The event $j \in G_i$ assigns with probability one
In O^t	
$\langle$**all stay**$\rangle$	$i \notin R^t$ and $I_{ji}^{t+1} = I_j^t$
$\langle I_i^t \rangle$	$i \in R^t$ and $I_{ji}^{t+1} = I_j^t \cap I_i^t$
$\langle 1 \rangle$	i is pivotal

Table 11: The belief of $j \in G_i$ after observing i's previous actions immediately after K_1^t contingent on O^t

i plays		The event $j \in G_i$ assigns with probability one
In O^t	In $\mathrm{K}_{1,1}^t$	
$\langle$**all stay**$\rangle$	$\langle i \rangle$	$i \notin R^t$ and $I_{ji}^{t+1} = I_j^t$
$\langle I_i^t \rangle$	$\langle$**all stay**$\rangle$	$\#[Rebels](\theta) < k$
$\langle I_i^t \rangle$	$\langle i \rangle$	$i \in R^t$ and $I_{ji}^{t+1} = I_j^t \cap I_i^t$, or $\#[Rebels](\theta) \geq k$
$\langle 1 \rangle$	$\langle$**all stay**$\rangle$	$\#[Rebels](\theta) < k$
$\langle 1 \rangle$	$\langle i \rangle$	$\#[Rebels](\theta) \geq k$

Table 12: The belief of $j \in G_i$ after observing i's previous actions immediately after K_2^0 contingent on O^t and K_2^0

i plays			The event $j \in G_i$ assigns with probability one
In O^t	In $\mathrm{K}_{1,1}^t$	In $\mathrm{K}_{2,1}^t$	
$\langle$**all stay**$\rangle$	$\langle i \rangle$	$\langle$**all stay**$\rangle$	$i \notin R^t$ and $I_{ji}^{t+1} = I_j^t$
$\langle I_i^t \rangle$	$\langle$**all stay**$\rangle$	$\langle$**all stay**$\rangle$	$\#[Rebels](\theta) < k$
$\langle I_i^t \rangle$	$\langle i \rangle$	$\langle$**all stay**$\rangle$	$\#[Rebels](\theta) \geq k$
$\langle I_i^t \rangle$	$\langle i \rangle$	$\langle i \rangle$	$i \in R^t$ and $I_{ji}^{t+1} = I_j^t \cap I_i^t$
$\langle 1 \rangle$	$\langle$**stay**$\rangle$	$\langle$**all stay**$\rangle$	$\#[Rebels](\theta) < k$
$\langle 1 \rangle$	$\langle i \rangle$	$\langle$**all stay**$\rangle$	$\#[Rebels](\theta) \geq k$

The delicate part in K^t is how a pivotal Rebel p propagandizes the relevant information. p does so by playing $\langle$**all stay**$\rangle$ in $\mathrm{K}^t_{1,1}$ in the case of $\#[R](\theta) < k$, while playing $\langle 1 \rangle$ in the case of $\#[R](\theta) \geq k$. Notice that playing $\langle$**all stay**$\rangle$ in $\mathrm{K}^t_{1,1}$ by p will immediately inform p's neighbors that $\#[R](\theta) < k$. On the contrary, playing $\langle p \rangle$ in $\mathrm{K}^t_{1,1}$ by p has not yet revealed $\#[R](\theta) \geq k$ since $\langle 1 \rangle$ can be also played by other non-pivotal Rebels.

In $\mathrm{K}^t_{2,1}$, however, p reveals $\#[R](\theta) \geq k$ by playing $\langle$**all stay**$\rangle$, a costless sequence of actions. It might not seem intuitive at first, but it effectively prevents a potential free-rider problem: there are two pivotal Rebels, say p and $p^{'}$, who have already known $\#[R](\theta) \geq k$ in K^t. If initiating the common knowledge about $\#[R](\theta) \geq k$ incurs negative payoff, p or $p^{'}$ will have incentive (again!) to let the other one initiate it. Playing $\langle$**all stay**$\rangle$ in $\mathrm{K}^t_{2,1}$ proudly becomes the initiation sequence by its cheapness.

The remaining argument is why other non-pivotal Rebels, say i, do not mimic pivotal Rebels' behavior to play $\langle 1 \rangle$ in O^t. i is unwilling to do so because, based on the belief updating on the path, if i plays $\langle 1 \rangle$, all Rebels will learn the relevant information immediately after K^t_2. It implies that the beginning of $t+1$-block is the terminal period. He will not learn the relevant information after that since the belief updating is also terminated. However, he is still uncertain whether or not he can learn the relevant information in O^t since he is not pivotal. Because the ex-post efficient outcome gives him the maximum payoff at every θ, and he will learn the relevant information eventually on the equilibrium path, he prefers not to deviate given that the discount factor is high enough.[18] The proof of Theorem 2 heavily follows the arguments of this kind.

As a complementary, I depict equilibrium strategy consistent with players' belief in Table 13, Table 14, Table 15, and Table 16.

[18]Since R^t Rebels share information on the equilibrium path, by Theorem 3, the belief updating in Table 10, Table 11, and Table 12, and the in-path behavior in Table 13, Table 14, Table 15, and Table 16, the relevant information is learnt by every Rebel eventually on the path.

Table 13: The sequences of actions played in $\mathrm{K}^t_{1,v}$ for $t \geq 1$ and for $v = 1, 2, ..., n$ on the path

Rebel i	i plays
i is certain $\#[R](\theta) < k$	$\langle$**all stay**$\rangle$
$i \notin R^t$ and is uncertain $\#[R](\theta) \geq k$	$\langle i \rangle$
$i \in R^t$ and is uncertain $\#[R](\theta) \geq k$	$\langle i \rangle$
i is certain $\#[R](\theta) \geq k$	$\langle i \rangle$

Table 14: The sequences of actions played in $\mathrm{K}^t_{2,v}$ for $t \geq 1$ for $v = 1$ on the path

Rebel i	i plays
i is certain that $\#[R](\theta) < k$	$\langle$**all stay**$\rangle$
$i \notin R^t$ and is uncertain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$
$i \in R^t$ and is uncertain $\#[R](\theta) \geq k$	$\langle i \rangle$
i is certain that $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$

Table 15: The sequences of actions played in $\mathrm{K}^t_{2,v}$ for $t \geq 1$ for $v = 2, ..., t+1$ on the path

Rebel i	i plays
i is certain that $\#[R](\theta) < k$	$\langle$**all stay**$\rangle$
$i \notin R^t$ and is uncertain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$
$i \in R^t$ and is uncertain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$
i is certain that $\#[R](\theta) \geq k$	$\langle i \rangle$

Table 16: The sequences of actions played in K_3^t for $t \geq 1$ on the path

Rebel i	i plays
i is certain that $\#[R](\theta) < k$	**s**
$i \notin R^1$ and is uncertain $\#[R](\theta) \geq k$	**s**
$i \in R^1$ and is uncertain $\#[R](\theta) \geq k$	**s**
i is certain that $\#[R](\theta) \geq k$	**r**

4.3.4 Out-of-path belief and learning on the path

In this section, I demonstrate examples to show how Rebels learn on the equilibrium path and illustrate the out-of-path belief.

Whenever Rebel i detects a deviation at period s, he forms the following belief:

$$\sum_{\theta \in \{\theta : \theta_j = I, j \notin G_i\}} \beta_{G_i}^{\pi,\tau}(\theta | h_{G_i}^{\dot{s}}) = 1, \text{ for all } \dot{s} \geq s. \tag{1}$$

Thus, if $\#I_i^0 < k$, he will play **stay** forever after he detects a deviation. This out-of-path belief serves as a grim trigger.

Let us take the configuration in Figure 2 (as Example 5 does) as given and consider the example below.

Example 6. Let $k = 6$ and the configuration be that in Figure 2. Players have their prime indexes by $(x_1, x_2, ..., x_8) = (3, 5, 7, 11, 13, 17, 19, 23)$.

At K_1^0, Rebels 1, 2, 6, and 8 play $\langle$**all stay**$\rangle$; Rebels 4 and 5 play $\langle 4 \rangle$ and $\langle 5 \rangle$ respectively.

At K_2^0, Rebels 1, 2, 6, and 8 play $\langle$**all stay**$\rangle$; Rebels 4 and 5 play $\langle 4 \rangle$ and $\langle 5 \rangle$ respectively.

At $\mathrm{K}_{3,v}^0$, for $v = 1, .., n$, all Rebels play **s**.

Immediately after K_3^0, all Rebel are uncertain about $\#[R](\theta) \geq k$.

At O^1, $|\mathrm{O}^1| = 111546435$. Rebels 1, 2, 6, and 8 play

$$\langle \textbf{all stay} \rangle = \langle \overbrace{\mathbf{s}, ..., \mathbf{s}}^{111546435} \rangle;$$

Rebels 4 plays

$$\langle 1 \rangle = \langle \overbrace{\mathbf{s}, ..., \mathbf{s}, \mathbf{r}}^{111546435} \rangle;$$

Rebels 5 plays

$$\langle \{4, 5, 6, 8\} \rangle = \langle \overbrace{\mathbf{s}, ..., \mathbf{s}, \underbrace{\mathbf{r}, \mathbf{s}, ..., \mathbf{s}}_{55913}}^{111546435} \rangle.$$

Immediately after O^1, Rebel 4 is certain $\#[R](\theta) \geq k$; Rebels 1,2,5,6,8 are uncertain about that.

At $\mathrm{K}_{1,v}^1$ for $v = 1, ..., n$, a Rebel i plays $\langle i \rangle$. Rebel 1,2,4,5 are certain $\#[R](\theta) \geq k$, while the others are uncertain about it.

At $K^1_{2,1}$, Rebels 6 and 8 play $\langle 6\rangle$ and $\langle 8\rangle$ respectively; Rebels 1,2,4,5 play $\langle$**all stay**$\rangle$. Immediately after $K^1_{2,1}$, Rebels 6 and 8 also know $\#[R](\theta) \geq k$.

At $K^1_{2,2}$, all Rebels play $\langle$**all stay**$\rangle$.

At $K^1_{2,v}$ for $v = 1, ..., n$, all Rebels play $\mathbf{r}$.

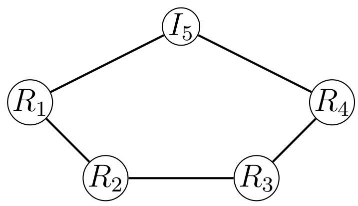
After $K^1_{2,n}$, all Rebels play $\mathbf{r}$ forever.

5 Discussion

5.1 Cyclic networks

Scenarios in cyclic networks substantially differ from the acyclic counterpart. The free-rider problem becomes intractable in cyclic networks, at least in the way of my equilibrium construction. Let us consider the configuration in Figure 9.

Figure 9: A configuration of the state and the network in which player 1,2,3,4 are Rebels while player 5 is an Inert.



Suppose $k = 4$ and the period s at the beginning of 1-block is not terminal. By the definition of pivotal Rebel in Section 4.3.2, Rebel 2 and 3 are θ-pivotal. From the perspective of Rebel 2's view, the type of player 5 could be Inert. Therefore, Rebel 2 does not know whether or not Rebel 1 is pivotal. Similarly, Rebel 2 does not know whether or not Rebel 3 is pivotal, *even though* player 3 is indeed θ-pivotal. Therefore there is no common knowledge of the free-rider problem at period s.

However, suppose we cut the edge between player 4 and 5, Rebel 2 knows that he is the only θ-pivotal Rebel.

6 Conclusion

References

A Appendix

A.1 The APEX equilibrium for Theorem 2

A.1.1 Equilibrium path

By definition of information hierarchy,

$$\begin{aligned} G_i^t &= \bigcup_{k_1 \in G_i} \bigcup_{k_2 \in G_{k_0}} \cdots \bigcup_{k_t \in G_{k_{t-1}}} G_{k_t} \\ &= \{j \in N : \text{ there is a path } (i, k_1 ... k_l, j) \text{ such that } l \leq t-1 \text{ and } \theta_i = \theta_{k_1} = ... = \theta_{k_l} = R\}. \end{aligned}$$

Further define

Definition A.1 (Extended tree by G_i^t)**.**

$$\begin{aligned} X_i^t \equiv \{j \in N : \\ &\textit{there is a path } (i, k_1 ... k_l, j_1, ..., j_m, j) \textit{ such that } l \leq t-1 \textit{ and } \theta_i = \theta_{k_1} = ... = \theta_{k_l} = \theta_{j_1} = R\}. \end{aligned}$$

X_i^t represents the tree extended by G_i^t. By assuming the network is acyclic, this is equivalent to the set of all possible Rebels in G.

The equilibrium will be represented as a *finite register machine* (or a *finite automata*).

Definition A.2 (Finite register machine)**.** *A finite register machine for i consists of finite registers Σ. A register is a tuple*

$$(\Omega, \times_{G_i} A_R, f, \lambda),$$

in which Ω are sets of events induced by H_i. $\times_{G_i} A_R$ is the sets of input. $f : \Omega \to A_R$ assigns an action to each event. $\lambda : \Omega \times Y \to \Sigma$ is the transition function. There is a set of initial registers. If i's information is included in the intersection of some events $\dot{\omega}$ and $\ddot{\omega}$, while $\dot{\omega} \subseteq \ddot{\omega}$ ($\dot{\omega}$ could be equal to $\ddot{\omega}$), i's action to be played is specified by $f(\ddot{\omega})$.

Note that what i's register machine does is to specify i's action according his information at a certain period, not including a characterization of i's information. i's information up to period s has been characterized by $P_i(\theta) \times \{h^s_{G_i}\} \times H^s_{N \setminus G_i}$ as that in Section 2.

Definition A.3 (m-register in t-block)**.** *A m-register in a (sub)block or a division is a register such that m is the m-th period in that (sub)block or division.*

To shorten the notation, denote $m - \Gamma$ as the m-register in the (sub)block Γ.

Definition A.4 (Terminal register)**.** *A register is terminal if its every $f \in F$ is a constant and the image of $\lambda \in \Lambda$ is a singleton containing itself.*

Though players act as if acting a whole sequence, they in fact act period by period. For convenience, for any finite sequence of action $\langle\rangle$, denote $\langle\rangle_m$ as the m-th (counting from the beginning) component in $\langle\rangle$, and denote $\langle\rangle(m)$ as the prefix of $\langle\rangle$ with length m. Let us also shorten action **revolt** to be **r** and **stay** to be **s**.

Equilibrium path in K^0

Table 17: The m-register in K_1^0 on the path, where $1 \leq m \leq |\mathrm{K}_1^0| - 1$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, y)$
$\#X_i^t < k$	$\langle \textbf{all stay} \rangle_m$		terminal $\mathbf{s}$
$i \notin R^1$	$\langle \textbf{all stay} \rangle_m$		$m + 1 \dashv \mathrm{K}_1^0$
$i \in R^1$	$\langle i \rangle_m$		$m + 1 - \mathrm{K}_1^0$
$I_i^1 \geq k$	$\langle i \rangle_m$		$m + 1 - \mathrm{K}_1^0$

Table 18: The m-register in K_1^0 on the path, where $m = |\mathrm{K}_1^0|$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, y)$
$\#X_i^t < k$		$\langle \textbf{all stay} \rangle_m$		terminal $\mathbf{s}$
$i \notin R^1$	all j play $\langle \textbf{all stay} \rangle (m-1)$	$\langle \textbf{all stay} \rangle_m$	all j play $\mathbf{s}$	terminal $\mathbf{s}$
$i \notin R^1$	$\exists j$ plays $\langle j \rangle (m-1)$	$\langle \textbf{all stay} \rangle_m$	such j plays $\langle j \rangle_m$	$1 - \mathrm{K}_2^0$
$i \in R^1$		$\langle i \rangle_m$		$1 - \mathrm{K}_2^0$
$I_i^1 \geq k$		$\langle i \rangle_m$		$1 - \mathrm{K}_2^0$

Table 19: The m-register in K_2^0 on the path, where $1 \leq m \leq |\mathrm{K}_2^0| - 1$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, y)$
$i \notin R^1$	$\langle \textbf{all stay} \rangle_m$		$m + 1 - \mathrm{K}_2^0$
$i \in R^1$	$\langle i \rangle_m$		$m + 1 - \mathrm{K}_2^0$
$I_i^1 \geq k$	$\langle \textbf{all stay} \rangle_m$		$m + 1 - \mathrm{K}_2^0$

Table 20: The m-register in K_2^0 on the path, where $m = |\mathrm{K}_2^0|$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, y)$
$i \notin R^1$	$\forall j$ play $\langle j \rangle (m-1)$	$\langle$**all stay**$\rangle_m$	$\forall j$ play $\langle j \rangle_m$	$1 - \mathrm{K}_3^0$
$i \notin R^1$	$\exists j$ plays $\langle$**all stay**$\rangle(m-1)$	$\langle$**all stay**$\rangle_m$	such j plays $\langle$**all stay**$\rangle_m$	terminal **r**
$i \in R^1$	$\forall j$ play $\langle j \rangle (m-1)$	$\langle i \rangle_m$	$\forall j$ play $\langle j \rangle_m$	$1 - \mathrm{K}_3^0$
$i \in R^1$	$\exists j$ plays $\langle$**all stay**$\rangle(m-1)$	$\langle i \rangle_m$	such j plays $\langle$**all stay**$\rangle_m$	terminal **r**
$I_i^1 \geq k$		$\langle i \rangle_m$		terminal **r**

Table 21: The m-register in K_3^0 on the path, where $1 \leq m \leq |\mathrm{K}_3^0| - 1$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, y)$
$i \notin R^1$	**s**	$\forall j$ play **s**	$m + 1 - \mathrm{K}_3^0$
$i \notin R^1$	**s**	$\exists j$ play **r**	terminal **r**
$i \in R^1$	**s**	$\forall j$ play **s**	$m + 1 - \mathrm{K}_3^0$
$i \in R^1$	**s**	$\exists j$ play **r**	terminal **r**

Table 22: The m-register in K_3^0 on the path, where $m = |\mathrm{K}_3^0|$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, y)$
$i \notin R^1$	**s**	$\forall j$ play **s**	$1 - \mathrm{O}^1$
$i \notin R^1$	**s**	$\exists j$ play **r**	terminal **r**
$i \in R^1$	**s**	$\forall j$ play **s**	$1 - \mathrm{O}^1$
$i \in R^1$	**s**	$\exists j$ play **r**	terminal **r**

Equilibrium path in O^t Since players in fact act period by period, some notations are introduced here for convenience. Let $m_j = |\mathrm{O}^t| - x_{I_j^t}$ be the period in which j report I_j^t (i.e. the period where $\mathbf{r}$ occurs in $\langle I_j^t \rangle$). Define $I_i^{t+1}(m)$ to br the unions of I_i^t and all the I_j^t with $j \in G_i$ and $m_j \leq m$. This is i's information about Rebels up to the first m periods in O^t. Similarly, define i's extended neighbors $G_i^{t+1}(m)$ to be the unions of G_i^t and all the I_j^t with $j \in G_i$ and $m_j \leq m$. Then define $X_i^{t+1}(m)$ to be the extended tree from $G_i^t(m)$ in the same way as Definition A.1.

Table 23: The m-register in O^t on the path, where $1 \leq m < |\mathrm{O}^t|$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, y)$
$i \notin R^t$		$\langle$**all stay**$\rangle_m$		$m + 1 - \mathrm{O}^t$
$i \in R^t$	$I_i^{t+1}(m-1) < k$, $X_i^{t+1}(m) \geq k$	$\langle I_i^t \rangle_m$		$m + 1 - \mathrm{O}^t$
$i \in R^t$	$I_i^{t+1}(m-1) < k$, $X_i^{t+1}(m-1) < k$			terminal $\mathbf{s}$
$i \in R^t$	$I_i^{t+1}(m-1) \geq k-1$	$\langle 1 \rangle_m$		$m + 1 - \mathrm{O}^t$
i is the free rider		$\langle 1 \rangle_m$		$m + 1 - \mathrm{O}^t$
i is the free rider	$X_i^{t+1}(m-1) < k$			terminal $\mathbf{s}$
i is pivotal		$\langle 1 \rangle_m$		$m + 1 - \mathrm{O}^t$
i is pivotal	$X_i^{t+1}(m-1) < k$			terminal $\mathbf{s}$

Table 24: The m-register in O^t on the path, where $m = |\mathrm{O}^t|$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, y)$
$i \notin R^t$		$\langle \textbf{all stay} \rangle_m$		$1 - \mathrm{K}^t_{1,1}$
$i \in R^t$	$I^{t+1}_i(m-1) < k$, $X^{t+1}_i(m) \geq k$	$\langle I^t_i \rangle_m$		$1 - \mathrm{K}^t_{1,1}$
$i \in R^t$	$I^{t+1}_i(m-1) < k$, $X^{t+1}_i(m-1) < k$			terminal **s**
$i \in R^t$	$I^{t+1}_i(m-1) \geq k-1$	$\langle 1 \rangle_m$		$1 - \mathrm{K}^t_{1,1}$
i is the free rider		$\langle 1 \rangle_m$		$1 - \mathrm{O}^t$
i is the free rider	$X^{t+1}_i(m-1) < k$			terminal **s**
i is pivotal		$\langle 1 \rangle_m$		$1 - \mathrm{K}^t_{1,1}$
i is pivotal	$X^{t+1}_i(m-1) < k$			terminal **s**

Equilibrium path in K^t for $t \geq 1$

Table 25: The m-register in $\mathrm{K}^t_{1,v}$ on the path, where $1 \leq m < |\mathrm{K}^t_{1,v}|$, $v = 1, ..., n$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, y)$
$i \notin R^t$	$\langle i \rangle_m$	$\exists j = m$ such that $a_j = \mathbf{s}$	terminal $\mathbf{s}$
$i \notin R^t$	$\langle i \rangle_m$	$\forall j$ such that $a_j = \langle j \rangle_m$	$m + 1 - \mathrm{K}^t_{1,v}$
$i \in R^t$	$\langle 1 \rangle_m$	$\exists j = m$ such that $a_j = \mathbf{s}$	terminal $\mathbf{s}$
$i \in R^t$	$\langle 1 \rangle_m$	$\forall j$ such that $a_j = \langle j \rangle_m$	$m + 1 - \mathrm{K}^t_{1,v}$

Table 26: The m-register in $\mathrm{K}^t_{1,v}$ on the path, where $m = |\mathrm{K}^t_{1,v}|$, $v = 1, ..., n$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, y)$
$i \notin R^t$	$\langle i \rangle_m$	$\exists j = m$ such that $a_j = \mathbf{s}$	terminal $\mathbf{s}$
$i \notin R^t$	$\langle i \rangle_m$	$\forall j$ such that $a_j = \langle j \rangle_m$	$1 - \mathrm{K}^t_{1,v+1}$
$i \in R^t$	$\langle 1 \rangle_m$	$\exists j = m$ such that $a_j = \mathbf{s}$	terminal $\mathbf{s}$
$i \in R^t$	$\langle 1 \rangle_m$	$\forall j$ such that $a_j = \langle j \rangle_m$	$1 - \mathrm{K}^t_{1,v+1}$

Table 27: The m-register in $\mathrm{K}^t_{1,v}$ on the path, where $m = |\mathrm{K}^t_{1,n}|$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, y)$
$i \notin R^t$	$\langle i \rangle_m$	$\exists j = m$ such that $a_j = \mathbf{s}$	terminal $\mathbf{s}$
$i \notin R^t$	$\langle i \rangle_m$	$\forall j$ such that $a_j = \langle j \rangle_m$	$1 - \mathrm{K}^t_{2,1}$
$i \in R^t$	$\langle 1 \rangle_m$	$\exists j = m$ such that $a_j = \mathbf{s}$	terminal $\mathbf{s}$
$i \in R^t$	$\langle 1 \rangle_m$	$\forall j$ such that $a_j = \langle j \rangle_m$	$1 - \mathrm{K}^t_{2,1}$

Table 28: The m-register in $\mathrm{K}^t_{2,v}$ on the path, where $1 \leq m < |\mathrm{K}^t_{2,v}|$, $v = 1, ..., t+1$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, y)$
$i \notin R^t$	$\exists j,\ \langle j\rangle_j = \mathbf{s}$	$\langle\textbf{all stay}\rangle_m$		$m + 1 - \mathrm{K}^t_{2,v}$
$i \notin R^t$	$\forall j,\ \langle j\rangle_j = \mathbf{r}$	$\langle i\rangle_m$		$m + 1 - \mathrm{K}^t_{2,v}$
$i \in R^t$, $I_i^{t+1} \geq k$		$\langle\textbf{all stay}\rangle_m$		$m + 1 - \mathrm{K}^t_{2,v}$
$i \in R^t$, $I_i^{t+1} < k$	$\exists j,\ \langle j\rangle_j = \mathbf{s}$	$\langle\textbf{all stay}\rangle_m$		$m + 1 - \mathrm{K}^t_{2,v}$
$i \in R^t$, $I_i^{t+1} < k$	$\forall j,\ \langle j\rangle_j = \mathbf{r}$	$\langle i\rangle_m$		$m + 1 - \mathrm{K}^t_{2,v}$

Table 29: The m-register in $\mathrm{K}^t_{2,v}$ on the path, where $m = |\mathrm{K}^t_{2,v}|$, $v = 1, ..., t$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, y)$
$i \notin R^t$	$\exists j,\ \langle j\rangle_j = \mathbf{s}$	$\langle\textbf{all stay}\rangle_m$		$1 - \mathrm{K}^t_{2,v+1}$
$i \notin R^t$	$\forall j,\ \langle j\rangle_j = \mathbf{r}$	$\langle i\rangle_m$		$1 - \mathrm{K}^t_{2,v}$
$i \in R^t$, $I_i^{t+1} \geq k$		$\langle\textbf{all stay}\rangle_m$		$1 - \mathrm{K}^t_{2,v}$
$i \in R^t$, $I_i^{t+1} < k$	$\exists j,\ \langle j\rangle_j = \mathbf{s}$	$\langle\textbf{all stay}\rangle_m$		$1 - \mathrm{K}^t_{2,v+1}$
$i \in R^t$, $I_i^{t+1} < k$	$\forall j,\ \langle j\rangle_j = \mathbf{r}$	$\langle i\rangle_m$		$1 - \mathrm{K}^t_{2,v+1}$

Table 30: The m-register in $\mathrm{K}^t_{2,v}$ on the path, where $m = |\mathrm{K}^t_{2,t+1}|$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, y)$
$i \notin R^t$	$\exists j,\ \langle j\rangle_j = \mathbf{s}$	$\langle\textbf{all stay}\rangle_m$		terminal **r**
$i \notin R^t$	$\forall j,\ \langle j\rangle_j = \mathbf{r}$	$\langle i\rangle_m$		$1 - \mathrm{K}^t_{2,v}$
$i \in R^t$, $I_i^{t+1} \geq k$		$\langle\textbf{all stay}\rangle_m$		terminal **r**
$i \in R^t$, $I_i^{t+1} < k$	$\exists j,\ \langle j\rangle_j = \mathbf{s}$	$\langle\textbf{all stay}\rangle_m$		terminal **r**
$i \in R^t$, $I_i^{t+1} < k$	$\forall j,\ \langle j\rangle_j = \mathbf{r}$	$\langle i\rangle_m$		$1 - \mathrm{K}^t_{2,v+1}$

Table 31: The m-register in K_3^t on the path, where $1 \leq m \leq |\mathrm{K}_3^t| - 1$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, y)$
$i \notin R^t$	**s**	$\forall j$ play **s**	$m + 1 - \mathrm{K}_3^t$
$i \notin R^t$	**s**	$\exists j$ play **r**	terminal **r**
$i \in R^t$	**s**	$\forall j$ play **s**	$m + 1 - \mathrm{K}_3^t$
$i \in R^t$	**s**	$\exists j$ play **r**	terminal **r**

Table 32: The m-register in K_3^t on the path, where $m = |\mathrm{K}_3^t|$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, y)$
$i \notin R^t$	**s**	$\forall j$ play **s**	$1 - \mathrm{O}^{t+1}$
$i \notin R^t$	**s**	$\exists j$ play **r**	terminal **r**
$i \in R^t$	**s**	$\forall j$ play **s**	$1 - \mathrm{O}^{t+1}$
$i \in R^t$	**s**	$\exists j$ play **r**	terminal **r**

A.2 Missing proofs

proof of Lemma 2.1

Proof. The proof is done by contraposition. Suppose Rebels' strategies constitute an APEX equilibrium. By definition of the APEX equilibrium, at every θ, there is a period T^{θ} when all Rebels' actions start to repeat themselves. Let $T = \max_{\theta \in \Theta} T^{\theta}$. For Rebel i, let $T_i = T + 1$, and suppose $0 < \sum_{\theta:\#[R](\theta)\geq k} \beta_{G_i}^{\pi,\tau^*}(\theta|h_{G_i}^s) < 1$ for some $s \geq T_i$. Then this Rebel assigns positive weight at some $\dot{\theta} \in \{\theta : \#[R](\theta) < k\}$ and some positive weight at some $\ddot{\theta} \in \{\theta : \#[R](\theta) \geq k\}$ at period s. Note that i has already known θ_j if $j \in G_i$, and therefore i assigns positive weight at some $\theta' \in \{\theta : \#[R](\theta) < k, \theta_l = R, l \notin G_i\}$ and positive weight at some $\theta'' \in \{\theta : \#[R](\theta) < k, \theta_l = I, l \notin G_i\}$. Since all Rebels' actions start to repeat themselves at period T, i cannot update information afterwards. Suppose i's continuation strategy is to continuously play **revolt**, then this is not ex-post efficient when $\#[R](\theta) < k$; suppose i's continuation strategy is to continuously play **stay**, then this is not ex-post efficient when $\#[R](\theta) \geq k$. □

proof of Theorem 1

Proof. Let τ^* be the following strategy. After the nature moves, a Rebel i plays **revolt** if he has no Inert neighbor; i plays **stay** forever if he has an Inert neighbor. After the first period, if i has not detected a deviation and observes that all his Rebel neighbors play **revolt** continuously previously, he plays **revolt** in the current period; otherwise, he plays **stay** afterwards and forever. If a Rebel j deviates, then j plays **stay** afterwards and forever.

At period s, according to τ^*, if i has not detected a deviation, but he observe his Rebel neighbors plays **stay** in the current period, he forms the belief of

$$\sum_{\theta:\#[R](\theta)\geq k} \beta_{G_i}^{\pi,\tau^*}(\theta|h_{G_i}^s) = 0$$

afterwards and forever. Therefore, he plays **stay** afterwards and forever as his best response.

At period s, if a Rebel detects a deviation, or he has deviated, to play **stay** afterwards and forever is his best response since at least one player will play **stay** afterwards and forever.

Since the network is finite with n vertices, if all players do not deviate, after period n, each Rebel plays **revolt** and gets payoff 1 forever if $\theta \in \{\theta : \#[R](\theta) \geq k\}$; each Rebels plays **stay** and gets payoff 0 forever if $\theta \in \{\theta : \#[R](\theta) < k\}$. However, a Rebel who has deviated surely gets payoff 0 forever after period n. Therefore, there is a $0 < \delta < 1$ large enough to impede Rebels to deviate.

To check if τ^* and $\{\beta_{G_i}^{\pi,\tau^*}(\theta|h_{G_i}^s)\}_{i \in N}$ satisfy full consistency[19], take any $0 < x < 1$ such that Rebels play τ^* with probability $1-x$ and play other behavior strategies with probability x. Clearly, when $x \to 0$, the belief converges to $\{\beta_{G_i}^{\pi,\tau^*}(\theta|h_{G_i}^s)\}_{i \in N}$. Since the out-of-path strategy is the best response for both of the Rebel who detects deviation and the Rebel who makes deviation, for arbitrary beliefs they hold, τ^* is a sequential equilibrium. □

proof of Lemma 4.1

Proof. I show that if $i \notin R^{t-1}$ then $i \notin R^t$ for all t.

By definition,

$$\begin{aligned} G_i^t &= \bigcup_{k_1 \in G_i} \bigcup_{k_2 \in G_{k_0}} \cdots \bigcup_{k_t \in G_{k_{t-1}}} G_{k_t} \\ &= \{j \in N : \text{ there is a path } (i, k_1 ... k_l, j) \text{ such that } l \leq t-1 \text{ and } \theta_i = \theta_{k_1} = ... = \theta_{k_l} = R\}, \end{aligned}$$

while

$$\begin{aligned} I_i^t &= \bigcup_{k_1 \in G_i} \bigcup_{k_2 \in G_{k_0}} \cdots \bigcup_{k_t \in G_{k_{t-1}}} G_{k_t} \cap [R](\theta) \\ &= \{j \in [R](\theta) : \text{ there is a path } (i, k_1 ... k_l, j) \text{ such that } l \leq t-1 \text{ and } \theta_i = \theta_{k_1} = ... = \theta_{k_l} = R\}. \end{aligned}$$

The above equality says that, at $t = \dot{t}$, if $i \notin R^{\dot{t}}$, then there is a j such that the Rebels, who can be reached by $\dot{t}$ consecutive edges from i, can be also reached by $\dot{t}$ consecutive edges from j. Therefore, if there are new Rebels who can be reached from i at any $\ddot{t} > \dot{t}$ by $\ddot{t}$ consecutive edges, those new ones can be also be reached by $\ddot{t}$ consecutive edges by j. Hence, $i \notin R^{\ddot{t}}$. □

proof of Theorem 3

[19]Krep and Wilson (1982)

Proof. First define

Definition A.5 (TR_{ij})**.**

$$TR_{ij} \equiv \{v \in N : \textit{there is a path from } i \textit{ to } v \textit{ through } j\}.$$

Since the θ has strong connectedness and since $[R](\theta) \neq \emptyset$, for each i, there is a minimum t_i such that $I_i^{t_i} = [R](\theta)$. Let $P = \underset{i \in N}{\arg\min}\{t_1, ..., t_n\}$ with generic element p. Therefore $I_p^{t_p} = [R](\theta)$. I show that $p \in R^{t_p - 1}$ to complete the proof. I prove it by contradiction. If $p \notin R^{t_p-1}$, then $I_p^{t_p-1} \subseteq G_j^{t_p-1}$ for some $j \in G_p$. Then, all the Rebels in TR_{jp} are in $G_j^{t_p-1}$, but there exist Rebels in TR_{pj} who are in $G_j^{t_p-1}$ but not in $I_p^{t_p-1}$. This is because the network is acyclic and $I_p^{t_p-1} \subset [R](\theta)$. However, $p \notin P$, since $I_j^{t_j-1} = [R](\theta)$ already. I then conclude that $p \in R^{t_p-1}$.

$\square$

proof of Lemma 4.2

Proof. The proof is by contradiction. Suppose that, at t-block and before T^θ, there are three or more θ-pivotal Rebels. Since θ has strong connectedness, there are three of them, p_1, p_2, p_3, with the property $p_1 \in G_{p_2}$ and $p_2 \in G_{p_3}$.

Since the network is acyclic, $p_1 \notin G_{p_3}$ and $p_3 \notin G_{p_1}$. Since p_1 is θ-pivotal, $I^t \subset [R](\theta)$ and $I_p^{t+1} = [R](\theta)$. It implies that, in $TR_{p_1p_2}$, p_1 can reach all Rebels by $t+1$ edges, but cannot reach all of them by t edges. The same situation applies to p_3. However, it means that p_2 can reach all Rebels in $TR_{p_1p_1}$ by t edges and reach all Rebels in $TR_{p_1p_1}$ by t edges, and hence $I_{p_2}^t = [R](\theta)$. It contradict to the definition of θ-pivotal Rebel.

$\square$

proof of Lemma 4.3

Proof. A θ-pivotal p knows that $p^{'} \in G_i$ if $p^{'}$ is another one. p picks a neighbor p' and checks whether or not $[R](\theta) \subseteq I_p^t \cup I_{p'}^t$ for all possible $I_{p'}^t$. By common knowledge of the network, p knows $G_{p'}^t$. Since p is θ-pivotal, he is certain that all the Rebel in the direction from p toward $p^{'}$ is in $G_{p'}^t$ and hence in $I_{p'}^t$. Then p can check whether or not $[R](\theta) \subseteq I_p^t \cup I_{p'}^t$ for all possible $I^t_{p'}$. If so, then p knows $p^{'}$ is also θ-pivotal by the definition of θ-pivotal. Similarly, a θ-pivotal $p^{'}$ can do the same procedure. Therefore, if there are two θ-pivotal p and $p^{'}$, they commonly know that they are θ-pivotal. They commonly know this at the beginning of t-block since they know I^t_p and $I^t_{p'}$ by the construction of information hierarchy. $\square$

proof of Theorem 2