Coordination in Social Networks: Communication by Actions

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Abstract

1 Introduction

2 Model

Given a finite set X, denote $\#X$ as its cardinality .

There is a set of players $N = \{1, 2, ..., n\}$. They constitute a network $G = (V, E)$ so that the vertices are players ($V = N$) and an edge is a pair of them (E is a subset of the set containing all two-element subsets of N). Throughout this paper, G is assumed to be finite, commonly known, fixed, undirected, and connected.[1]

Time is discrete with index $s \in \{0, 1, ...\}$. At $s = 0$, the nature chooses a state $\theta \in \Theta = \{R, I\}^n$ once and for all according to a common prior π. Let us interpret R and I as Rebel and Inert respectively. After the nature moves, players play a normal form game, the *k-threshold game*, infinitely repeated played with common discounted factor δ. In the

[1] A path in G from i to j is a finite sequence $(l_1, l_2, ..., l_L)$ without repetition such that $l_1 = i$, $l_L = j$, and $\{l_q, l_{q+1}\} \in E$ for all $1 \leq q < L$. G is fixed if G is not random, and G is undirected if, for all i, j, if $j \in G_i$ then $i \in G_j$. G is connected if, for all $i, j \in N$, $i \neq j$, there is a path from i to j.

k-threshold game, $A_R = \{\textbf{revolt}, \textbf{stay}\}$ is the set of actions for R while $A_I = \{\textbf{stay}\}$ is that for I. A Rebel's static payoff function is defined as follows.

- $u_R(a_R, a_{-i}) = 1$ if $a_R = \textbf{revolt}$ and $\#\{j : a_j = \textbf{revolt}\} \geq k$
- $u_R(a_R, a_{-i}) = -1$ if $a_R = \textbf{revolt}$ and $\#\{j : a_j = \textbf{revolt}\} < k$
- $u_R(a_R, a_{-i}) = 0$ if $a_R = \textbf{stay}$

. An Inert's static payoff is equal to 1 no matter how other players play.

For convenience, let $[R](\theta)$ be the set of Rebels given θ and the notion *relevant information* refer to the information about whether or not $\#[R](\theta) \geq k$.

During the game is played, any player, say i, can observe information only from himself and from his direct neighbors $G_i = \{j | \{i, j\} \in E\}$. These information includes his and his neighbors' types ($\theta_{G_i} \in \Theta_{G_i} = \{R, I\}^{G_i}$) and his and their histories of actions up to period s ($h^s_{G_i} \in H^s_{G_i} \equiv \times^s_{t=1}(\times_{j \in G_i} H^t_j)$). I assume that payoffs are hidden to emphasize that observing neighbors' actions are the only channel to infer other players' types and actions.[2] To be precise, when θ is realized at $s = 0$, i's information set about θ is $P_i(\theta) \equiv \{\theta_{G_i}\} \times \{R, I\}^{N \setminus G_i}$. For the information sets about players' actions, the sets of histories of actions are set to be empty at $s = 0$. At $s > 0$, a history of actions played by i is $h^s_i \in H^s_i \equiv \emptyset \times A^s_i$ while a history of actions played by all players is $h^s \in H^s \equiv \times^s_{t=1}(\times_{j \in N} H^t_j)$. i's information set about other players' histories of actions up to $s > 0$ is $\{h_{G_i}\} \times H^s_{N \setminus G_i}$. A player i's pure behavior strategy τ_i is a measurable function with respect his information partition if it maps $P_i(\theta) \times \{h_{G_i}\} \times H^s_{N \setminus G_i}$ to a single action in his action set for every s and for every θ.

By abusing the notation a bit, let h^τ_θ denote the realized sequence of actions generated by $\tau = (\tau_1, \tau_2, ..., \tau_n)$ given θ. Define $\mu^{\pi,\tau}_{G_i}(\theta, h^s | \theta_{G_i}, h^s_{G_i})$ as the conditional distribution over $\Theta \times H^s$ conditional on i's information up to s, which is induced by π and τ. i's belief over θ up to s is then $\beta^{\pi,\tau}_{G_i}(\theta | \theta_{G_i}, h^s_{G_i}) \equiv \sum_{h^s \in H^s} \alpha^{\pi,\tau}_{G_i}(\theta, h^s | \theta_{G_i}, h^s_{G_i})$.

[2] Such restriction will be relaxed in the Section **??**.

The equilibrium concept is the week sequential equilibrium.[3] I am looking for the existence of approaching ex-post efficient equilibrium (*APEX equilibrium henceforth*), which is formally defined below.

Definition 2.1 (APEX strategy)**.** *A behavior strategy τ is APEX if, for all θ, there is a terminal period $T^\theta < \infty$ such that the actions in h_θ^τ after T^θ repeats the static ex-post Pareto efficient outcome.*

Definition 2.2 (APEX equilibrium)**.** *An equilibrium (τ^*, α^*) is APEX if τ^* is APEX.*

In an APEX strategy, all Rebels will play **revolt** forever after some period if there are more than k Rebels; Rebels will play **stay** forever after some period otherwise. It is as if Rebels will learn the relevant information in the equilibrium since they will play the ex-post efficient outcome eventually and keep doing so. Several properties are noted. In the stage game, the ex-post efficient outcome for each k is unique, which gives the highest as well as the same payoff to every Rebel, and every player play **stay** forever is always an equilibrium. In an APEX equilibrium, it is not only as if Rebels will learn the relevant information but they must learn that by the following lemma.

Lemma 2.1 (Learning in the APEX equilibrum)**.** *If (τ^*, μ^*) is an APEX equilibrium, then for all $\theta \in \Theta$, there is a finite time T_i^θ for every Rebel i such that $\sum_{\theta \in \{\theta : [R](\theta) \geq k\}} \beta_{G_i}^{\pi,\tau^*}(\theta | h_{G_i}^s) =$ either 1 or 0 whenever $s \geq T_i^\theta$.*

Proof. In Appendix. □

Definition 2.3 (Learning the relevant information)**.** *A Revel i learns the relevant information at period ξ if $\sum_{\theta \in \{\theta : [R](\theta) \geq k\}} \beta_{G_i}^{\pi,\tau^*}(\theta | h_{G_i}^s) =$ either 1 or 0 whenever $s \geq \xi$.*

[3] A weak sequential equilibrium is an assessment $\{\tau^*, \mu^*\}$, where μ^* is a collection of distributions over players' information sets with the property that, for all i, for all s, $\mu_{G_i}^*(\theta, h^s | \theta_{G_i}, h_{G_i}^s) = \mu_{G_i}^{\pi,\tau^*}(\theta, h^s | \theta_{G_i}, h_{G_i}^s)$ whenever the information set is reached with positive probability given τ^*. Moreover, for all i, for all s, τ_i^* maximize i's continuation expected payoff conditional on θ_{G_i} and $h_{G_i}^s$ of

$$E_G^\delta(u_{\theta_i}(\tau_i, \tau_{-i}^*) | \alpha_{G_i}^{\pi,\tau_i,\tau_{-i}^*}(\theta, h^s | \theta_{G_i}, h_{G_i}^s))$$

for all $h_{G_i}^s$.

It is clear that an APEX equilibrium exists when $k = 1$. For other cases, let us start with the case of $k = n$ and then continue to the case of $1 < k < n$.

3 APEX equilibrium for $k = n$

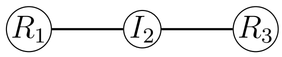
In the case of $k = n$, a Rebel can get better payoff from playing **revolt** than that from **stay** *only if* all players are Rebels. Two consequences follows. First, if a Rebel has an Inert neighbor, this Rebel always plays **revolt** in the equilibrium. Secondly, at any period, playing **stay** forever afterwards as a punishment for a deviation is optimal if there is another player who also plays **stay** forever afterwards, independently from the belief held by the punisher. These two features constitute an APEX equilibrium and further turn it to be a sequential equilibrium.

Theorem 1 (APEX equilibrium for the case of $k = n$)**.** *For any n-person repeated k-Threshold game with parameter $k = n$ played in a network, there is a δ^* such that a sequential APEX equilibrium exists whenever $\delta > \delta^*$.*

Proof. In Appendix. □

The proof is a contagion argument. Suppose a Rebels play **revolt** at any period except for: (1) he has an Inert neighbor, or (2) he has observed his Rebel neighbor played **stay** once. Since the network is finite and connected, a Rebel is certain that there is an Inert somewhere if he has seen his neighbor has played **stay**; otherwise, he continues to believe that all platers are Rebels. Observing n consecutive **revolt** will imply that no Inert exist. The above strategy is an APEX strategy and therefore ready for the equilibrium path for an APEX equilibrium. For any deviation from the above strategy, construct the off-path strategy as to play **stay** forever for both of the deviant and that Rebel (the punisher) who detects that. This off-path strategy is optimal for both the deviant and the punisher, independent from the belief held by the punisher, and hence is also sequential rational.

Figure 1: The state and the network in which the APEX equilibrium does not exist when $k = 2$.



4 APEX equilibrium for $1 < k < n$

In contrast to the case of $k = n$, a Rebel still has incentive to play **revolt** even if he has an Inert neighbor. This opens a possibility of non-existence of APEX equilibrium. Let us consider Example 1 below.

Example 1. Suppose that $k = 2$ and $\theta = (R, I, R)$. The state and the network is represented in Figure 1. Rebel 1 never learn θ_3 since Inert 2 cannot reveal information about θ_3. The APEX equilibrium does not exist in this scenario.

The following condition that works on the prior, *full support on strong connectedness* excludes the possibility of non-existence of APEX equilibrium. To this end, I begin with the definition of *strong connectedness*.

Definition 4.1 (Strong connectedness)**.** *Given G, a state θ has strong connectedness if, for every pair of Rebels, there is a path consisting of Rebels to connect them.*

In the language of graph theory, this definition is equivalent to: given G, θ has strong connectedness if the induced graph by $[R](\theta)$ is connected.

Definition 4.2 (Full support on strong connectedness)**.** *Given G, π has full support on strong connectedness if*

$$\pi(\theta) > 0 \Leftrightarrow \ \theta \textit{ has strong connectedness}$$

This condition emphasizes that only the state that has strong connectedness can happen. As a remark, the definition of the full support on strong connectedness is stronger than common knowledge about the states have strong connectedness. This marginal requirement is subtle and is shown to be more convenient in equilibrium construction.[4]

I am ready to state the main characterization of this paper:

Theorem 2 (APEX equilibrium for the case of $1 < k < n$)**.** *For any n-person repeated k-Threshold game with parameter $1 < k < n$ played in networks, if networks are acyclic and if π has full support on strong connectedness, then there is a δ^* such that an APEX equilibrium exists whenever $\delta > \delta^*$.*[5]

Constructing APEX equilibrium in this case is more convoluted than that in the case of $k = n$. I illustrate the proof idea throughout this paper till Section, while leaving the formal proof in Appendix. In the case of $k = n$, T^θ can be determined independently from θ by setting $T^\theta = n$, but it is not obvious how to obtain T^θ before an equilibrium has been constructed now.[6] Moreover, the free-rider problem might exist in the current case (as demonstrated in Introduction), but this problem never occur in the proposed APEX equilibrium for Theorem 1. As for the punishment scheme, playing **stay** forever is not anymore effective since a deviation might only seen by parts of players (network monitoring), and thus group punishment is hard to be coordinated to execute.

To get better exposition of the proof idea behind Theorem 2, until Section, I allow players to endow a writing technology so that they can write without cost (cheap talk), write with a fixed cost, or write with a cost function before they play actions. Before Section, I introduce a game, *T-round writing game*, to be an auxiliary scenario that is simpler but mimics relevant features in the original game to shed light on the equilibrium construction. The equilibrium construction for T-round writing game will be studied by means of examples and intentionally serve to demonstrate the free-rider problem. I then argue the equilibrium construction in the original game is an analogue to the one in the T-round writing game. Roughly speaking, in the T-round writing game, players can write

[4]The main result only requires a weaker version: $\pi(\theta) > 0 \Rightarrow$ θ has strong connectedness . However, working on this weaker version is at the expense of almost the same but much tedious proof. Throughout this paper, I stick on the original definition.

[5]A network is acyclic if the path from i to j for all $i \neq j$ is unique.

[6]Readers might refer to the proof of Theorem 1.

to exchange information about θ for T-round. They then play a one-shot k-threshold game at round $T+1$. Note that in an APEX equilibrium path in the original game, players would stop update their belief after some finite time and keep playing the same action in the k-threshold game. The game form of the T-round writing game mimics the structure of the APEX equilibrium path in the original game game. I consider in order the case of writing without cost, writing with a fixed cost, and then writing with cost function. I then modify the T-round writing game to allow that T can be endogenously determined in the equilibrium, which is further analogous to the original game.

4.1 Deterministic T-round writing game

The network, the set of states, and the set of players follow exactly the same definitions defined in Section 2. In the deterministic T-round writing game, each player endows a *writing technology.* A writing technology for player i is a pair of (W, M_i), in which $W = \{\mathbf{r}, \mathbf{s}\}^L$, $L \in \mathbb{N}$, and $M = \times_{t=1}^{T} M_i^t$ is recursively defined.

$$M_i^1 = \{f | f : \Theta_{G_i} \to W\} \cup \{\emptyset\}$$

$$\text{for } 2 \leq t \leq T \text{ , } M_i^t = \{f | f : \times_{j \in G_i} M_j^{t-1} \to W\} \cup \{\emptyset\}.$$

W is interpreted as the set of sentence composed by letters $\mathbf{r}$ or $\mathbf{s}$ with length L while M_i can be understood as i's grammar. The $\emptyset$ is interpreted as keeping silent. The meaning of "i writes to his neighbors at round t" is equivalent to "i chooses an element $f \in M_i^t$ to get an element $w \in W$ according to f. Moreover, m can be observed by all i's neighbors". A sentence combined by $w, w^{'} \in W$ is denoted as $w \oplus w^{'}$ with the property that $(w \oplus w^{'})_l = \mathbf{r}$ if and only if $w_l = \mathbf{r}$ or $w_l^{'} = \mathbf{r}$ for all $l \in \{1, 2, ..., L\}$.

The time line for the T-round writing game is as follows.

1. Nature choose θ according to π.

2. Types are then fixed over time.

3. At $t = 1, ..., T$ round, players write to their neighbors.

Figure 2: A configuration of the state and the network in which player 1,2,4,5,6,8 are a Rebel while player 3,7 are Inerts.

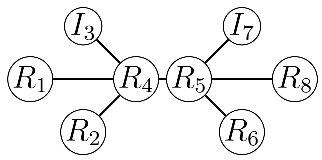
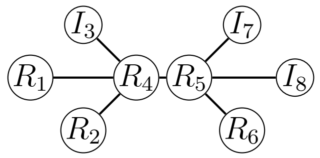


Figure 3: A configuration of the state and the network in which player 1,2,4,5,6 are a Rebel while player 3,7,8 are Inerts.



4. At $T+1$ round, players play a one-shot k-Threshold game.

5. Game ends.

There is no discounting. A Rebel's payoff is the summation of his stage payoff across stages; an Inert's payoff is set to be 1. The equilibrium concept is weak sequential equilibrium. The definition of APEX strategy is adapted as the strategy that induces ex-post outcome in the k-threshold game at $T+1$ round and the definition of APEX equilibrium is adapted accordingly. In the following examples, let us focus on the configuration represented in Figure 2 and Figure 3 with $n = 8$ and $L = 8$. Note that the difference in player 8's type is the only difference between these two configuration. There are specific k and T in each example, and I characterize an APEX equilibrium for each one.

Example 2 (Deterministic T-round writing without cost (cheap talk))**.** Let $k = 6$ and $T = 2$. Assume that writing is costless. Let us consider the following strategy ϕ on its path. Suppose the state and the network are represented in Figure 2. At $t = 1$, ϕ specifies that the peripheral Rebels 1,2,6,8 keep silent; the central player Rebel 4 writes

$(\mathbf{r},\mathbf{r},\mathbf{s},\mathbf{r},\mathbf{r},\mathbf{s},\mathbf{s},\mathbf{s})$; the central player Rebel 5 writes $(\mathbf{s},\mathbf{s},\mathbf{s},\mathbf{r},\mathbf{r},\mathbf{r},\mathbf{s},\mathbf{r})$.[7] On the path of ϕ, Rebel 4's sentence thus reveals that players 1,2,4,5 are Rebels while players 3 is an Inert by writing $\mathbf{r}$ in the i-th component if player i is a Rebel and writing $\mathbf{s}$ in the j-th component if player j's type is Inert or unknown to Rebel 4. Rebel 5's sentence functions the same. Note that the common knowledge of the network structure contributes to the ability in revealing players' types. At $t = 2$, ϕ specifies that the peripheral Rebels 1,2,6,8 still keep silent; Rebel 4 writes $(\mathbf{r},\mathbf{r},\mathbf{s},\mathbf{r},\mathbf{r},\mathbf{r},\mathbf{s},\mathbf{r})$; Rebel 5 writes $(\mathbf{r},\mathbf{r},\mathbf{s},\mathbf{r},\mathbf{r},\mathbf{r},\mathbf{s},\mathbf{r})$. This is to say Rebel 4 and 5 exchange information at $t = 1$ and then coordinate to announce a combined sentence at $t = 2 = T$. At $t = 3 = T + 1$, all Rebels knows that the number of Rebels (by counting $\mathbf{r}$ in Rebel 4 or 5's combined sentence) is greater than or equal to $k = 6$. This leads all Rebels to play the ex-post efficient outcome **revolt** in the k-threshold game.

Suppose the configuration is that in Figure 3. At $t = 1$, similarly, ϕ specifies that Rebels 1,2,6,8 keep silent; Rebel 4 writes $(\mathbf{r},\mathbf{r},\mathbf{s},\mathbf{r},\mathbf{r},\mathbf{s},\mathbf{s},\mathbf{s})$; Rebel 5 writes $(\mathbf{s},\mathbf{s},\mathbf{s},\mathbf{r},\mathbf{r},\mathbf{r},\mathbf{s},\mathbf{s})$. At $t = 2$, however, ϕ specifies that the peripheral Rebels 1,2,6,8 keep silent; Rebel 4 keeps silent; Rebel 5 keeps silent as well. On the path, keeping silent by Rebel 4 (or 5) reflects that Rebel 4 (or 5) knows that the total number of Rebels is less than $k = 6$. At $t = 3$, all Rebels know this relevant information and play the ex-post efficient outcome **stay**.

Hence ϕ is a candidate for an APEX equilibrium path. Let ν be the belief system and the in-path belief of ν be the belief induced by ϕ. For the assessment off the path, the off-path strategy of ϕ could be made as follows. If a Rebel make a detectable deviation detected by some others, then the Rebels who detect that deviation keeps silent until $t = T$ and then play **stay** at $t = T + 1$.[8] The off-path belief of ν is the belief that all players who are not neighbors are all Inerts. Since writing is costless, and any deviation by Rebel 4 or 5 would strictly decrease the possibility to achieve ex-post efficient outcome, the assessment (ϕ, ν) constitutes an APEX equilibrium.

Example 3 (Deterministic T-round writing with a fixed cost)**.** Let $k = 6$ and $T = 2$. Suppose that writing incurs a fixed cost $\epsilon > 0$ while keeping silent does not. Let us consider

[7]The notion of "peripheral" and "center" will be formalized in Section

[8]For instance, a wrong sentence that is not according to any grammar, deviating from the in-path ϕ, etc

the assessment (ϕ, ν) in the above example. Since any deviation by Rebel 4 or 5 would strictly decrease the possibility to achieve the ex-post efficient outcome while the ex-post efficient outcome will give the highest expected payoff for every Rebel at $t = 3 = T + 1$ if the relevant information can be revealed then, if ϵ is sufficiently small, (ϕ, ν) also constitutes an APEX equilibrium.

Example 4 (Deterministic T-round writing with cost function—free-rider problem)**.** Let $k = 6$ and $T = 2$. Suppose that keeping silent incurs no cost, but writing incurs a cost $\epsilon > 0$ that is strictly decreasing with the number of $\mathbf{r}$ in a sentence. This is to say writing $(\mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r})$ incurs the least cost while writing $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{s})$ incurs the largest.

If so, that assessment (ϕ, ν) in the previous two examples will no longer be an APEX equilibrium. To see this, first note that the sentence of either Rebel 4 or 5's truthfully reveals their information at $t = 1$ on the path of ϕ. Since that, Rebel 4 will know the relevant information after $t = 1$ (by common knowledge of the network structure) even if he deviates to writing the sentence that indicates that all his neighbors are Rebels.[9][10] Rebel 5 is in the same situation as Rebel 4 and therefore also write the sentence that indicates that all his neighbors are Rebels. However, these sentences are uninformative. It turns out that both of them will deviate, and neither of them can know relevant information after $t = 1$.

Example 5 (Deterministic T-round writing with cost function—solving free-rider problem)**.** The free-rider problem occurs in the previous example can be solved. The solution is to extend the game to more rounds and exploit the assumption of common knowledge about the network. More precisely, let $k = 6$ and $T = 3$. First note that if the previous strategy ϕ is extended to the current case so that Rebel 4 and 5 simultaneously truthfully write their information at either $t = 1$ or $t = 2$, ϕ is not an equilibrium path by the same free-rider argument as the above.

Consider a strategy ϕ' and focus on the interaction between Rebel 4 and 5. On the path

[9] This sentence is $(\mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s}, \mathbf{s})$, which incurs less cost than the truthfully reporting sentence $(\mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s}, \mathbf{s})$.

[10] If he keeps silent, then this benavior wil be considered as a deviation, and therefore he will never get the highest payoff of 1. Hence, he will avoid doing so.

of $\phi^{'}$, at $t = 1$, $\phi^{'}$ specifies that the Rebel with lowest index between Rebel 4 and Rebel 5 is the "free rider", while the other Rebel write his information truthfully. This is to say Rebel 4 will be the free rider—who writes the least-cost sentence; Rebel 5 writes $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r})$ in the configuration of Figure 2 and writes $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s})$ in the configuration of Figure 3. At $t = 2$, Rebel 5 keeps silent; Rebel 4 writes the least-cost sentence if Rebel 5 writes $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{r})$ at $t = 1$ but keeps silent if Rebel 5 writes $(\mathbf{s}, \mathbf{s}, \mathbf{s}, \mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{s}, \mathbf{s})$ then. Thus, Rebel 4' behavior at $t = 2$ reveals the relevant information. At $t = 3 = T$, Rebel 4 keeps silent; Rebel 5 writes the least-cost sentence if Rebel 4 writes the least-cost sentence at $t = 2$ and keeps silent if Rebel 4 keeps silent then. It is straightforward to check that at $t = 4 = T + 1$, all Rebels know the relevant information and play the ex-post efficient outcome accordingly. To construct an APEX equilibrium from $\phi^{'}$, recall (ϕ, ν) and let the in-path belief of $\nu^{'}$ be induced by $\phi^{'}$. The out-of-path strategy follow that in ϕ, and the out-of-path belief follow that of ν.

An observation is worth noting. Why is Rebel 5 willing to concede that Rebel 4 is chosen to be the free rider at $t = 1$? This is because he *knows* that, by common knowledge about the network, he and Rebel 4 are in a free-rider problem. Moreover, again by common knowledge about the network, he knows that Rebel 4 knows this, an so on to infinite inference hierarchy. This is to say, at least in this case, Rebel 4 and 5 commonly known that they are engaged in a free-rider problem due to the common knowledge assumption on network. In Section, this property of common knowledge about engaging in a free-rider problem will be formally characterized. Roughly speaking, this property is not a merely special case. It holds for any acyclic network in the constructed APEX equilibrium in the original game.

4.2 Indeterministic T-round writing game

In this section, the setting is exactly the same as that in the deterministic T-round writing game, except for that players can now jointly decide the round in which they will play the one-shot k-threshold game and then end the game. This is to say, before they play the one-shot k-threshold game, they have to reach an agreement—a common knowledge of a

Figure 4: The linearly ordered labelled rounds in the indeterministic T-round writing game after θ is realized.

$$0_1^{'} < 0_2^{'} < ... < 0_{l_0}^{'} < 1 < 1_1^{'} < 1_2^{'} < ... < 1_{l_1}^{'} < 2 < ...,$$

where $l_0, l_1, ...$ are all finite numbers.

terminal round T. The set of rounds is countably infinite and linearly ordered with generic element t. The writing technology is the same as that in deterministic T-round writing game, except for letting $W = \{\mathbf{r}, \mathbf{s}\}^L \cup \{\mathbf{r}, \mathbf{s}\}^{L^{'}}$ now. In the example below, let $L = 8$ and $L^{'} = 1$.

Conceptually, there could be two kinds of rounds. In the first kind, players write to exchange information about θ (as they do in the deterministic T-round writing game). In the second kind, players write to form the common knowledge of T. Let us partition the set of rounds into two parts, Γ and $\Gamma^{'}$, which represent the first kind and the second kind respectively. The round in Γ is labelled by γ while the round in $\Gamma^{'}$ is labelled by $\gamma_l^{'}$. The rounds is linearly ordered by $<$ (after θ is realized). To be more precise, the rounds is ordered as shown in Figure 4. As an indeterministic T-round writing game is illustrated below, an APEX equilibrium is constructed .

Example 6 (Indeterministic T-round writing with cost function)**.** Let $l_j = 2$ for $j = 0, 1, ...$ Suppose that the setting is exactly the same as that in Example 5, except for that T is not deterministic. Let us consider the path of a strategy ψ. At a round in $\Gamma^{'}$, ψ specifies that, if a Rebel thinks "it is certain that the number of Rebels outnumbers k (i.e. $\#[R](\theta) \geq k$) and the nearest forthcoming round in Γ is the terminal round," he write $(\mathbf{r})$; if a Rebel thinks "it is possible but not certain that $\#[R](\theta) \geq k$," he writes $(\mathbf{s})$; otherwise, he writes $\emptyset$ to show that "it is impossible that $\#[R](\theta) \geq k$ and the nearest forthcoming round in Γ is the terminal round." According to this strategy, $t = 1$ is not terminal if no Rebel has write $(\mathbf{r})$ or $\emptyset$ before. For instance, $t = 1$ is not terminal in the configuration in Figure 2 (or Figure 3). If $t = 1$ is not terminal, at $t = 1$, Rebel 4 and 5 are in a free-rider problem as Example 4 shows. ψ solves it by specifying Rebel 4 is the free rider and Rebel 5 writes his information about θ truthfully (as what ϕ' does in Example 5). At $t = 1_1'$, Rebel 4 knows $\#[R](\theta) \geq k$ in the configuration in Figure 2 and knows $\#[R](\theta) < k$ in the configuration in Figure 3. Therefore he writes $(\mathbf{r})$ and $\emptyset$ respectively for these two configuration; as for other Rebels, they writes $(\mathbf{s})$. After $t = 1_1'$, it is straightforward to check that all the Rebels will learn the relevant information (by seeing the writing of Rebel 4 at $t = 1_1'$) and will terminate their writing at $t = 2$. Therefore, $t = 1$ is the terminal round, and Rebels play a one-shot k-threshold game at $t = 2$.

Denote the belief system as υ. ψ can be made to be an APEX equilibrium strategy in a usual way by setting the in-path belief as that induced by ψ and adopting the out-of-path assessment of (ϕ, ν) in the previous example.

4.3 Dispensability of writing technology

In essence, writing technology is dispensable, and repeated actions are sufficient to serve as a communication protocol to achieve ex-post outcome in an equilibrium. In this section, I draw the analogue between the writing game and the original game in Table 1. More precisely, in the equilibrium construction in the original game, let us partition the periods, and each part in the partition is analogous to a round in the writing game. The length of periods in a part is analogous to the length of sentence. Since actions played in a certain part of periods will incur an expected payoff, it is an analogue that writing is costly at a certain round in the writing game. The disjoint unions of parts of periods also constitute a coarser partition of periods, which is analogous to partitioning the rounds. As an analogue of the partition of $\Gamma \cup \Gamma'$ in the indeterministic T-round writing game above, the analogue of Γ is the set of *periods for reporting* in the original game to emphasize that these periods are for reporting information about θ; Γ' is the set of *periods for coordination* in the original game to emphasize that these periods are for coordinating to play the ex-post efficient outcome. The partition of periods is linearly ordered by $<$ (after θ is realized), and let us define a coarser partition with parts *t-blocks* indexed by $t \in \{0, 1, ...\}$ along with the order of partition of periods as shown in Figure 5. One could see that Figure 4 and Figure 5 are harmoniously analogous to each other.

Figure 5: The linearly ordered partitions of periods in the repeated k-threshold game after θ is realized.

$$\underbrace{\text{(periods for coordination)}}_{\text{0-block}} < \underbrace{\text{(periods for reporting)} < \text{(periods for coordination)}}_{\text{1-block}} < \ldots$$

Table 1: The analogue between indeterministic T-round writing game and repeated k-threshold game

Indeterministic T-round writing game	Repeated k-threshold game
A round	A range of periods
A sentence	A sequence of actions
The length of a sentence in a round	The length of a part of periods
A chosen digit in a sentence	A chosen action
The cost of writing a sentence	The expected payoff in a part of periods
The fixed grammar	The equilibrium path

Note that the notions of *peripheral* and *central* in Example 2 is not yet formalized as well as analogizing to the original game. I generalize these notions in the original game by defining *information hierarchy* among players for each t-block below.

4.3.1 Information hierarchy

The information hierarchy across Rebels at t-block in G is a tuple

$$(\{G_i^t\}_{i \in N}, \{I_i^t\}_{i \in N}, R^t, \theta).$$

G_i^t is meant to represent *the extended neighbors* to represent the following. $j \in G_i^t$ if j can be reached by t consecutive edges from i such that the endpoints of $t-1$ edges are both Rebels but the endpoints of the remaining one are j and a Rebel; I_i^t is interpreted as *the extended Rebel neighbors*—the set of Rebels in G_i^t; R^t is interpreted as *the active Rebels*—those Rebels who are *active* in the sense that their extended Rebel neighbors are not a subset their direct neighbors' extended Rebel neighbors. They are defined recursively:

At $t = 0$,

$$\text{if } \theta_i = I,\ G_i^0 \equiv \emptyset,\ I_i^0 \equiv \emptyset.$$

$$\text{if } \theta_i = R,\ G_i^0 \equiv \{i\},\ I_i^0 \equiv \{i\}.$$

$$R^0 \equiv [R](\theta).$$

At $t = 1$,

$$\text{if } \theta_i = I,\ G_i^1 \equiv \emptyset,\ I_i^1 \equiv \emptyset.$$

$$\text{if } \theta_i = R,\ G_i^1 \equiv G_i,\ I_i^0 \equiv G_i \cap R^0.$$

$$R^1 \equiv \{i \in R^0 : \nexists j \in G_i \cap R^0 \text{ such that } I_i^1 \subseteq G_j^1\}.$$

At $t > 1$,

$$G_i^t \equiv \bigcup_{j \in G_i \cap R^{t-1}} G_j^{t-1},\ I_i^t \equiv \bigcup_{j \in G_i \cap R^{t-1}} I_j^{t-1}, \text{ and } R^t \equiv \{i \in R^{t-1} : \nexists j \in G_i \cap R^0 \text{ such that } I_i^t \subseteq G_j^t\}.$$

In other words, according to the above definition, the peripheral Rebels in the configuration in Figure 2 are active in 0-block (in R^0) but not active in 1-block (not in R^1), while the central players are active in both 0-block and 1-block. It can be shown that $R^t \subset R^{t-1}$ by the following lemma.

Lemma 4.1. *If the network is acyclic and if the θ has strong connectedness, then*

$$R^t \subset R^{t-1}$$

for all $t \geq 1$

Proof. To this end, I first show that if $i \notin R^{t-1}$ then $i \notin R^t$ for all t,and then show that the equality does not hold. Given $t = \tilde{t}$, if $i \notin R^{\tilde{t}}$ then there is a j such that all Rebels can be reached by $\tilde{t}$ consecutive edged from i can be reach by $\tilde{t}$ consecutive edged from j. Then if i can reach more Rebels at any $t > \tilde{t}$ by t consecutive edges (therefore I_i^t is enlarged), all members in I_i^t can be also be reached by t consecutive edges by j. I then show that the equality does not hold. Take an $R^t = R^{t+1}$ for some t. (TBA) □

The question is whether or not it is enough to let only active Rebels exchange information about θ while θ can be revealed eventually? The answer is positive. Theorem 3 below states that it is sufficient to only let Rebels in R^t in the t-block to report their information if the network is acyclic and the state has strong connectedness.

Theorem 3. *If the network is acyclic and if the θ has strong connectedness, then*

$$[R](\theta) \neq \emptyset \Rightarrow \exists t \geq 0, \exists i \in R^t \text{ such that } I_i^t = [R](\theta).$$

Proof. In Appendix. □

4.3.2 The equilibrium path in periods for reporting

If there is no further mentioned, all the description in this section is for the APEX equilibrium path *before* the terminal period T^θ is reached. For conciseness, let us denote RP^t as the periods for reporting in t-block, denote $|RP^t|$ as the length of RP^t, and shorten **revolt** and **stay** to **r** and **s** receptively henceforth.

$|RP^t|$ is independent from t and determined only the set of players. To determine $|RP^t|$, first assign each player i a distinguished prime number x_i starting from 3 (by exploiting the common knowledge about network structure). Then let $|RP^t| = x_1 \otimes x_2 \otimes ... \otimes x_n$, where $\otimes$ is the usual multiplication operator. The sequence of actions in RP^t is with length $|RP^t|$

and would take one of the forms specifies in the right column in Table 2. There, if $I \subseteq N$, then $x_I \equiv \otimes_{i\in I} x_i$. The abbreviations for these sequences are listed in the left column. Since these sequences in the periods for reporting are meant to exchange information about θ, I would alternate using "playing the sequence" with "reporting the information" to mean the same behavior in the periods for reporting.

Table 2: The notations for the sequences of actions in RP^t on the path

Notations		The sequence of actions
$\langle I \rangle$	$\equiv$	$\langle \mathbf{s}, ..., \mathbf{s}, \underbrace{\mathbf{r}, \mathbf{s}, ..., \mathbf{s}}_{X_I} \rangle$
$\langle 1 \rangle$	$\equiv$	$\langle \mathbf{s}, ..., \mathbf{s}, \mathbf{r} \rangle$
$\langle$**all stay**$\rangle$	$\equiv$	$\langle \mathbf{s}, ..., \mathbf{s}, \mathbf{s} \rangle$

It is worth noting that this sequence constructed by prime numbers brings two benefits. First, since the multiplication of distinguish prime numbers can be uniquely factorized, the Rebels can utilize such sequence to precisely report players' indexes. Secondly and ultimately conveniently, the un-discounted expected payoff of playing $\langle I \rangle$ for some $I \subseteq N$ is always equal to -1. This is because, at any period in RP^t, if there is no player playing $\langle 1 \rangle$, there is at most one player would play $\mathbf{r}$ by the property of prime number multiplication.

The sequence $\langle 1 \rangle$ is intentionally created to deal with the free-rider problem. To see how it functions, let us formally define the free-rider problem by first defining the *pivotal Rebel* as follows.

Definition 4.3 (Pivotal Rebels in RP^t)**.** *A Rebel is pivotal in RP^t if he is in R^t and certain that he will learn the relevant information in the end of RP^t, given that each Rebel in R^t, say i, reports $\langle I_i^t \rangle$.*

From the definition, a pivotal Rebel p in RP^t is the one who can learn the relevant information if all of his active Rebel neighbors truthfully report their information about θ to him. He can be further classified into tow kinds. The first kind is the one who can learn

Figure 6: A configuration of the state and the network in which player 1,3,5,6,7,8 are Rebels while players 2,4,9 are Inerts.

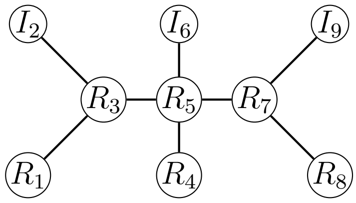
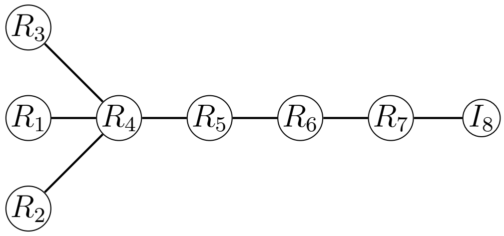


Figure 7: A configuration of the state and the network in which player 1,2,3,4,5,6,7 are Rebels while players 8 is an Inert.



the true state, while the second kind is the one who can learn the relevant information only. In fact, if the network is acyclic and the prior has full support on strong connectedness, it can be shown that p in RP^t is the second kind only if $I_p^t = k-1$. In other words, p is either with $I_p^t = k-1$ or the one who can learn the true state, or both. For conciseness, call p of the first kind by *θ-pivotal*; call the one with $I_p^t = k-1$ by *$k-1$-pivotal*. As instances, when $k=6$ and in RP^1, in the configuration in Figure 2, only Rebels 4 and 5 are pivotal (θ-pivotal); in the configuration in Figure 6, only Rebels 5 is pivotal (θ-pivotal); in the configuration in Figure 7, only Rebels 4 pivotal ($k-1$-pivotal).

Below is the free-rider problem in RP^t defined.

Definition 4.4. *There is a free-rider problem in RP^t if there are multiple θ-pivotal Rebels in RP^t.*

The following lemma says that there are at most two θ-pivotal Rebels in a R^t for all t. Therefore, if there is a free-rider problem, it occurs between two θ-pivotal Rebels. Moreover, if there are two of them, they are neighbors.

Lemma 4.2. *If the network is acyclic and if π has full support on strong connectedness, then for each t-block, there are at most two θ-pivotal Rebels. Moreover, if there are two of them, they are neighbors.*[11]

Proof. In Appendix. □

Especially,

Lemma 4.3. *If the network is acyclic and if π has full support on strong connectedness, then for each t-block, if there are two θ-pivotal Rebels p and $p^{'}$, then they commonly know that they are θ-pivotal Rebels at the beginning of t-block.*

Proof. In Appendix. □

By Lemma 4.3, θ-pivotal Rebels in RP^t can identify themselves at the beginning of RP^t. On the APEX equilibrium path, if the free-rider problem will occur RP^t, the strategy will specify that θ-pivotal Rebel p in RP^t who has the lowest index plays $\langle 1 \rangle$, while the other one $p^{'}$ plays $\langle I^t_{p^{'}} \rangle$.

Overall, the sequences played in RP^t on the path are specified in Table 3.

Table 3: The sequences of actions played in RP^t on the path

Rebel i	i plays
$i \notin R^t$	$\langle$**all stay**$\rangle$
$i \in R^t$ but i is not pivotal	$\langle I^t_i \rangle$
i is $k-1$-pivotal	$\langle 1 \rangle$
i is θ-pivotal but not in the free-rider problem	$\langle 1 \rangle$
i is in the free-rider problem with the lowest index	$\langle 1 \rangle$
i is in the free-rider problem without the lowest index	$\langle I^t_i \rangle$

[11] As a remark, the above lemma is not true when the network is cyclic. To see this, consider a 4-player circle when $\theta = (R, R, R, R)$.

4.3.3 The equilibrium path in periods for coordination

In this section, I discuss the sequences of action in periods of coordination on the path. If there is no further mentioned, all the description in this section is for the APEX equilibrium path *before* the terminal period T^θ is reached. The crucial feature in periods of coordination is that, in short, whenever a Rebel i has been thought to be not active starting at some t-block (i.e. $i \notin R^t$ for some $t \in \mathbb{N}$), there is no strategy for i to convince all the Rebels that $\#[R](\theta) \geq k$ even though i wants to propagandize it.

The structure in the periods for coordination is more intrigued than that in periods for reporting, as that Γ' in indeterministic T-round writing game. In periods of coordination, these periods are further partitioned by *divisions* and *sub-blocks*. I depict that in details below, where (CD) represents a certain range of periods for coordination. Let us first denote CD^t as the periods for coordination in t-block.

In CD^0,

$$\overbrace{\underbrace{(CD)}_{\text{one sub-block}}}^{\text{1-division}}\overbrace{\underbrace{(CD)}_{\text{one sub-block}}}^{\text{2-division}}\overbrace{\underbrace{(CD)\cdots(CD)}_{n \text{ sub-blocks}}}^{\text{3-division}}$$

In CD^t, $t > 0$,

$$\overbrace{\underbrace{(CD)\cdots(CD)}_{n \text{ sub-blocks}}}^{\text{1-division}}\overbrace{\underbrace{(CD)\cdots(CD)}_{t+1 \text{ sub-blocks}}}^{\text{2-division}}\overbrace{\underbrace{(CD)\cdots(CD)}_{n \text{ sub-blocks}}}^{\text{3-division}}$$

For convenience, in the t-block, denote $CD^t_{u,v}$ as the v-th sub-block in u-division; denote $|CD^t_{u,v}|$ as the length of $CD^t_{u,v}$. Similarly, denote CD^t_u as the u-division; denote $|CD^t_u|$ as the length of CD^t_u. Let us shorten **revolt** and **stay** to **r** and **s** receptively henceforth. On the path, for all $v \in \mathbb{N}$, $|CD^t_{u,v}| = n$ for $u = 1, 2$ and $|CD^t_{u,v}| = 1$ for $u = 3$. The notations for the sequences of actions on the path are shown in Table 4 shows.[12]

Since the 0-block has simpler structure, I begin with depicting the equilibrium path in CD^0 as shown in Table 5, Table 6, and Table 7. The description for a Rebel i there is whether or not i has learnt the relevant information. The Rebel i might learn the relevant

[12] Since, in 3-division, the length of the sequence of actions is one, i.e. playing an action, I dispense notations there for conciseness.

Table 4: The notations for the sequences of actions in $CD^t_{u,v}$ for $u = 1, 2$, on the path

Notations		The sequence of actions
$\langle i \rangle$	$\equiv$	$\langle \mathbf{s}, ..., \mathbf{s}, \underbrace{\mathbf{r}, \mathbf{s}, ..., \mathbf{s}}_{i} \rangle$
$\langle$**all stay**$\rangle$	$\equiv$	$\langle \mathbf{s}, ..., \mathbf{s}, \mathbf{s} \rangle$

information at the beginning of $CD^0_{1,1}$ or later by observing his neighbors' behavior. If a Rebel i is certain that $\#[R](\theta) < k$, by the full support on strong connectedness, it must be the case that all the Rebels are i's neighbors and therefore aware that as well immediately after $CD^0_{1,1}$.

The intriguing and unintuitive part here might be how a Rebel i initiates the common knowledge about $\#[R](\theta) \geq k$. i does so by first play $\langle i \rangle$ in $CD^0_{1,1}$ and then play $\langle$**all stay**$\rangle$ in $CD^0_{2,1}$. His behavior is then distinguishable from Rebels of other kinds, then his neighbor will know $\#[R](\theta) \geq k$ immediately after $CD^0_{2,1}$, and then all the Rebels will know that by playing $\mathbf{r}$ contagiously in CD^0_3. Notice that, by the assumption of acyclic network, i will not deviate to play $\langle$**all stay**$\rangle$ even though it might be undetectable. This is because, if so, i will be considered as an inactive Rebels, as a "dead end", afterwards by *all* of his neighbors. His neighbors' belief about the state will no longer influenced from his behavior. He then faces a positive probability that there are not enough Rebels can be informed of $\#[R](\theta) \geq k$ and thus he will only get zero payoff in that scenario. Taking sufficiently high discount factor impedes his deviation. In essence, all the proofs for the equilibrium behavior on the path follow this argument in Appendix.

It is a useful complementary to list Rebels' updated beliefs consistent with the equilibrium path after CD^0_2, as Table shows.

Table 5: The sequences of actions played in $CD^0_{1,1}$ on the path

Rebel i	i plays
i is certain that $\#[R](\theta) < k$	$\langle$**all stay**$\rangle$
$i \notin R^1$ and is uncertain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$
$i \in R^1$ and is uncertain $\#[R](\theta) \geq k$	$\langle i \rangle$
i is certain that $\#[R](\theta) \geq k$	$\langle i \rangle$

Table 6: The sequences of actions played in $CD^0_{2,1}$ on the path

Rebel i	i plays
i is certain that $\#[R](\theta) < k$	$\langle$**all stay**$\rangle$
$i \notin R^1$ and is uncertain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$
$i \in R^1$ and is uncertain $\#[R](\theta) \geq k$	$\langle i \rangle$
i is certain that $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$

Table 7: The sequences of actions played in CD^0_3 on the path

Rebel i	i plays
i is certain that $\#[R](\theta) < k$	$\langle \mathbf{s} \rangle$
$i \notin R^1$ and is uncertain $\#[R](\theta) \geq k$	$\langle \mathbf{s} \rangle$
$i \in R^1$ and is uncertain $\#[R](\theta) \geq k$	$\langle \mathbf{s} \rangle$
i is certain that $\#[R](\theta) \geq k$	$\langle \mathbf{r} \rangle$

Table 8: The belief of $j \in G_i$ after observing i's previous actions immediately after CD_2^0

In $CD_{1,1}^0$	In $CD_{2,1}^0$	
i plays	i plays	The event $j \in G_i$ assigns with probability one
$\langle$**all stay**$\rangle$	$\langle$**all stay**$\rangle$	$i \notin R^1$ or $\#[Rebels](\theta) < k$
$\langle i \rangle$	$\langle$**all stay**$\rangle$	$\#[Rebels](\theta) \geq k$
$\langle i \rangle$	$\langle i \rangle$	$i \in R^1$

4.3.4 Step 3: Off-path Belief

Whenever Rebel i detects a deviation at period s, he forms the following belief:

$$\sum_{\theta \in \{\theta : \theta_j = Inert, j \notin G_i\}} \beta_{G_i}^{\pi,\tau}(\theta | h_{G_i}^{s'}) = 1, \text{ for all } s' \geq s \tag{1}$$

. Thus, if $\#I_i^0 < k$, he will play **stay** forever. This off-path belief serves as a grim trigger.

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