Coordination in Social Networks

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- How to **"solve"** the problem of **collective action** in the presence of incomplete information?
  - Example of collective action
    - Revolution
    - Raising fund for start-ups

MOTIVATION

- How to **“solve”** the problem of **collective action** in the presence of incomplete information?
  - Example of collective action
    - Revolution
    - Raising fund for start-ups
  - This presentation will be in terms of Revolution.

BACKGROUND

- **Collective action is not static**
  - Protest leads revolution. (East Germany 1989-1990).
- **Information is transmitted within social networks**:
  - Church networks (1989 Berlin Uprising).

- **Dynamics of collective action on networks.**

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- **How people obtain sufficient information over time to coordinate their actions.**

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- Pay-off contingent on global type distribution.

WHAT THIS PAPER DOES?

- Players linked in a fixed and exogenous network.
- Players of two types (**Rebel**,**Inert**). They can **observe own/neighbor's type**.
- Type-contingent action.
- Pay-off contingent on global type distribution.
- Players choose simultaneously and repeatedly. They can **observe own/neighbor's actions**.

Look for

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Result

- Such equilibrium can be constructed under some assumptions.

- Public good provision.

Related Literature

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RELATED LITERATURE

- Public good provision.
  - One strand: [Chwe 2000], [Lohmann, 1993,1994], etc
  - **This paper adds network-monitoring**
- Repeated game in networks.
  - **This paper consider incomplete information and imperfect monitoring**

Network

- Let $N = \{1, ..., n\}$ be the set of players.
- G_i is i's neighborhood, G_i is a subset of N and $i \in G_i$.
- $G = \{G_i\}_i$ is the network.

ASSUMPTION

G is fixed (not random), finite, connected, commonly known, and undirected.

Static k-threshold game [Chwe 2000]

- $1 \leq k \leq n$
- $\theta_i \in \Theta_i = \{Rebel, Inert\}$: i's type
- $\Theta = \times_{i \in N} \Theta_i$; $\theta \in \Theta$

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- $1 \leq k \leq n$
- $\theta_i \in \Theta_i = \{Rebel, Inert\}$: i's type
- $\Theta = \times_{i \in N} \Theta_i$; $\theta \in \Theta$
- $A_{Rebel_i} = \{\textbf{revolt}, \textbf{stay}\}$; $A_{Inert_i} = \{\textbf{stay}\}$

Static k-threshold game [Chwe 2000], **In this presentation,** Or

- Static game payoff for player i: $u_{\theta_i}(a_{\theta_i}, a_{-\theta_i})$

$$u_{Inert_i}(a_{Inert_i}, a_{-\theta_i}) = 1 \quad \text{if } a_{Inert_i} = \textbf{stay}$$

Model

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$$u_{Rebel_i}(a_{Rebel_i}, a_{-\theta_i}) = -1 \quad \text{if } a_{Rebel_i} = \textbf{revolt} \text{ and } \#\{j : a_{\theta_j} = \textbf{revolt}\} < k$$

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- **stay** is a safe arm; **revolt** is a risky arm.

Model

Static k-threshold game [Chwe 2000], **In this presentation,** Or

- Static game payoff for player i: $u_{\theta_i}(a_{\theta_i}, a_{-\theta_i})$

$$
\begin{aligned}
u_{Inert_i}(a_{Inert_i}, a_{-\theta_i}) &= 1 \quad \text{if } a_{Inert_i} = \textbf{stay} \\
\\
u_{Rebel_i}(a_{Rebel_i}, a_{-\theta_i}) &= 1 \quad \text{if } a_{Rebel_i} = \textbf{revolt} \text{ and } \#\{j : a_{\theta_j} = \textbf{revolt}\} \geq k \\
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\\
u_{Rebel_i}(a_{Rebel_i}, a_{-\theta_i}) &= 0 \quad \text{if } a_{Rebel_i} = \textbf{stay}
\end{aligned}
$$

- **stay** is a safe arm; **revolt** is a risky arm.
- Ex-post (Pareto) efficient outcome:
  - Inerts play **stay**.
  - If there are more than k Rebels, all Rebels play **revolt**.
  - Otherwise, all Rebels play **stay**.

Time line

- Time is infinite, discrete.
- Nature choose θ at 0 period according to π.
- Players play the static *k*-threshold game infinitely repeatedly.

ASSUMPTION

- *Players know their neighbors' types.*
- *Players perfectly observe their neighbors' actions.*
- π *has full support*
- *Common* δ*.*

Notations:

- $[Rebels](\theta) = \{j : \theta_j = Rebel\}$ for all $\theta \in \Theta$.
- θ_{G_i}: i's private information about the state. ($\theta_{G_i} \in \Theta_{G_i} = \prod_{j \in G_i} \Theta_j$)
- $h^m_{G_i}$: the history observed by i up to period m. ($h^m_{G_i} \in H^m_{G_i} = \prod_{s=1}^m \prod_{j \in G_i} A_{\theta_j}$)
- $h \in H = \prod_{s=1}^{\infty} \prod_{j \in N} A_{\theta_j}$: a infinite sequence of players' actions
- $\tau_i : \Theta_{G_i} \times \bigcup_1^{\infty} H^m_{G_i} \to A_{\theta_i}$, i's strategy.
- $\tau = (\tau_1, ..., \tau_i, ..., \tau_n)$: a strategy profile
- $\beta_i^{\pi,\tau}(\theta | h^m_{G_i})$: i's belief for a θ at period m given τ.

Notations:

- h_θ^τ : a realized h generated by τ given θ.
- Call h_θ^τ a τ_θ-path.
- Call $\{\tau_\theta\}_{\theta\in\Theta}$ the τ-path

DEFINITION

The τ-path is **approaching ex-post efficient** (*APEX*) $\Leftrightarrow$

$$\forall\theta, \text{ there is a finite time } T^\theta$$

such that the actions after T^θ in τ_θ repeats the static ex-post efficient outcome.

Definition

$h^m_{G_i}$ is **reached by τ-path**

$$\Leftrightarrow$$

$\exists\theta$ such that $h^m_{G_i}$ is in τ_θ-path.

Lemma

If the τ-path is APEX $\Rightarrow$

$$\forall\theta\ \forall i, \text{ there is a finite time } T_i^\theta$$

such that $\sum_{\theta:\#[Rebels](\theta)\geq k} \beta^{\pi,\tau}_{G_i}(\theta|h^s_{G_i}) = 1$ or $= 0$ if $s \geq T_i^\theta$ and if $h^s_{G_i}$ reached by τ-path.

DEFINITION (APEX)

A sequential equilibrium (τ^*, β^*) is APEX $\Leftrightarrow$ τ^*-path is APEX, and β^* is the belief system consistent with τ^*.

LEADING EXAMPLE

If pay-off is observable, an Apex Equilibrium for $k = n = 3$ in

- At 1st period

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  - If the pay-off is observed as 1, choose **revolt** afterwards.

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    - All Rebels choose **revolt**.
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    - If the pay-off is observed as 1, choose **revolt** afterwards.
    - Otherwise, choose **stay** afterwards.

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- Any deviation $\Rightarrow$

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  - Choosing **stay** forever.

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  - Rebel 2 chooses **revolt** if he observes $\theta = (\textit{Rebel}, \textit{Rebel}, \textit{Rebel})$; Otherwise, chooses **stay** forever.

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If pay-off is hidden, an Apex Equilibrium for $k = n = 3$ in

- At 1st period
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- After 1st period
  - If Rebel 2 chooses **revolt** in the last period, then Rebel 1 (or Rebel 3) chooses **revolt** forever;

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Goal

Can we generalize the above result?

Assumption

Payoff is hidden (or noisy).

Results

- $k = n$: we can.
- $k < n$: with additional assumptions,
  - acyclic networks: we can .
  - all networks: open question.

$k = n$: Result

Theorem ($k = n$)

***In** any network, **if** the prior has full support, **then** for repeated $k = n$ Threshold game, **there is** a δ such that a sequential equilibrium which is APEX **exists**.*

Proof:

1. Some Inerts neighbors $\Rightarrow$ play **stay** forever.
2. No Inert neighbor $\Rightarrow$ play **revolt** until **stay** is observed, and then play **stay** forever.
3. Any deviation $\Rightarrow$ play **stay** forever.
4. There is a finite time T^{θ} such that ex-post efficient outcome repeats afterwards.

$k = n$: RESULT

Comments for $k = n$:

1. **stay** means "some Inerts are out there."
2. **revolt** means "some Inerts may not be there."
3. Any deviation $\Rightarrow$ punished by shifting to **stay** forever by single player
    - Group punishment is not necessary.

Since a Inert always play **stay**, define

Definition

Strong connectedness $\Leftrightarrow$ for every pair of Rebels, there is a path consisting of Rebels to connect them.

Definition

Full support on strong connectedness $\Leftrightarrow$

$$\pi(\theta) > 0 \text{ if and only if } \theta \text{ has strong connectedness.}$$

to not reduce the game to incomp. info. game without communication.

Theorem ($k \leq n$)

In *any acyclic network,* ***if*** π *has full support on strong connectedness,* ***then*** *for repeated* $1 \leq k \leq n$ *Threshold game,* ***there is*** *a* δ *such that a weak sequential equilibrium which is APEX* ***exists****.*

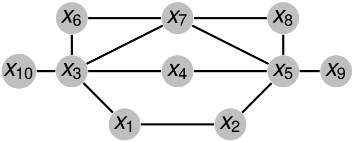
Conjecture ($k \leq n$)

In *any cyclic network, ...[same as above]...*

Outline

1. Communication by actions
2. Communication in the equilibrium
    1. Communication protocol
    2. Reporting and coordination messages in the protocol
    3. Information hierarchy in communication
    4. In-the-path belief updating
    5. Off-path belief
    6. Sketch of proof

1. Indexing each node i as a distinct prime number x_i. For instance,



COMMUNICATION BY ACTIONS

COMMUNICATION BY BINARY ACTIONS

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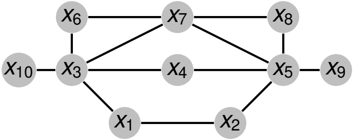
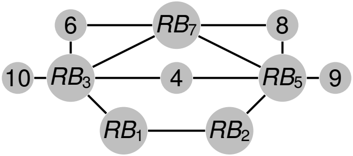


2. Then, If



Rebel 3 report $x_1 \times x_7 \times x_3$ to Rebel 1 by sending a finite sequence

$$\textbf{stay}, \ldots, \textbf{stay}, \underbrace{\textbf{revolt}, \textbf{stay}, \ldots, \textbf{stay}}_{x_1 \times x_7 \times x_3}$$

Two phases, RP and CD, alternate in time horizontal line

$$\underbrace{\langle\text{coordination period}\rangle}_{0-block}\;\underbrace{\langle\text{reporting period}\rangle\langle\text{coordination period}\rangle}_{1-block}\ldots$$

- Reporting period (RP): talking about θ
  - Cheap talking: θ will be revealed.

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- Reporting period (RP): talking about θ
  - Cheap talking: θ will be revealed.
- Why do I need coordination period (CD) ?

Why do I need coordination period ?

- Ans: Since higher-order belief is hard to track.
  - APEX: to find T^θ for all θ.
  - When is T^θ?.

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- Sol: Let *CD* be long enough

$$\overbrace{\langle \dots \rangle}^{RP}\ \overbrace{\langle \langle \cdot \rangle \langle \cdot \rangle \dots \langle \cdot \rangle \rangle}^{CD}$$

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  - If a Rebel i knows relevant info, $\Rightarrow$ i sends msg to inform G_i $\Rightarrow$ $j \in G_i$ sends msg. to inform G_j $\Rightarrow$....all Rebels are informed

In coordination period,

- At least "three" messages to coordinate Rebels
    1. to **revolt**
    2. to **stay**
    3. to continue to next block
- Create these distinguishable messages by binary actions

COORDINATION PERIOD AND MESSAGES

- CD^t: the CD in t-block

$$\overbrace{\langle \underbrace{\langle\cdot\rangle \cdots \langle\cdot\rangle}_{\text{1st division}} \rangle \langle \underbrace{\langle\cdot\rangle \cdots \langle\cdot\rangle}_{\text{2nd division}} \rangle}^{CD^t}$$

- $CD^t_{p,q}$: the p sub-block in q division.

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- $CD^t_{p,q}$: the p sub-block in q division.
- $\langle CD^t_{p,q}\rangle$: the messages in $CD^t_{p,q}$ are distinguishable

$$\begin{array}{ll} \langle \textbf{stay} \rangle & \mathbf{s}, \ldots, \mathbf{s}, \mathbf{s}, \mathbf{s}, \ldots, \mathbf{s} \\ \langle x_i \rangle & \mathbf{s}, \ldots, \mathbf{s}, \underbrace{\mathbf{r}, \mathbf{s}, \ldots, \mathbf{s}}_{x_i} \end{array}$$

- CD^t: the CD in t-block

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- 1st division: sending **message to stay**; otherwise **continue**
- 2nd division: sending **message to revolt**; otherwise **continue**

Coordination period and messages

1st division in CD

Message to **stay**:

- Whenever a Rebel i knows $\#[Rebels](\theta) < k$, he plays **stay** afterward.

$$\overbrace{\langle \underbrace{\langle \textbf{stay} \rangle \ldots}_{\text{1st division}} \rangle \langle \quad \underbrace{\langle \ldots \rangle}_{\text{2nd division}} \quad \rangle}^{CD^t}$$

Message to **stay**:

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- ... then nearby Rebel j plays **stay** afterward

$$\overbrace{\langle \underbrace{\langle \cdot \rangle \langle \textbf{stay} \rangle \ldots}_{\text{1st division}} \rangle \langle \underbrace{\langle \ldots \rangle}_{\text{2nd division}} \quad \rangle}^{CD^t}$$

Coordination period and messages

1st division in CD

Otherwise

- ,

$$\overbrace{\langle \underbrace{\langle x_i\rangle\langle x_i\rangle\ldots\langle\cdot\rangle}_{\text{1st division}}\rangle\langle\underbrace{\langle\cdot\rangle\cdots\langle\cdot\rangle}_{\text{2nd division}}\rangle}^{CD^t}$$

- **Message to revolt**: Whenever a Rebel i know $\#[Rebels](\theta) \geq k$, he play

$$\overbrace{\langle\underbrace{\langle\cdot\rangle \cdots \langle\cdot\rangle}_{\text{1st division}}\rangle\langle\underbrace{\langle\textbf{stay}\rangle\langle\cdot\rangle \cdots \langle\cdot\rangle}_{\text{2nd division}}\rangle}^{CD^t}$$

  in the first sub-block.

- Otherwise ,

$$\overbrace{\langle\underbrace{\langle\cdot\rangle \cdots \langle\cdot\rangle}_{\text{1st division}}\rangle\langle\underbrace{\langle x_i\rangle\langle\cdot\rangle \cdots \langle\cdot\rangle}_{\text{2nd division}}\rangle}^{CD^t}$$

- Message to **revolt**:... then nearby Rebel j play $\langle x_j \rangle$ to inform nearby Rebels, etc

$$\overbrace{\langle \underbrace{\langle\cdot\rangle \cdots \langle\cdot\rangle}_{\text{1st division}} \rangle \langle \underbrace{\langle\cdot\rangle \langle x_j \rangle \cdots \langle\cdot\rangle}_{\text{2nd division}} \rangle}^{CD^t}$$

- Otherwise ,

$$\overbrace{\langle \underbrace{\langle\cdot\rangle \cdots \langle\cdot\rangle}_{\text{1st division}} \rangle \langle \underbrace{\langle\cdot\rangle \langle \textbf{stay} \rangle \cdots \langle\cdot\rangle}_{\text{2nd division}} \rangle}^{CD^t}$$

After coordination period,

- Either stopping or continuing communication
  1. Stopping: if relevant info. is revealed $\Rightarrow$ messages will be sent $\Rightarrow$ all Rebels play the ex-post eff. outcome afterward.
  2. Continuing: otherwise, go to the next block.

OBSERVATION

Either stopping or continuing belief updating.

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  1. Stopping: if relevant info. is revealed $\Rightarrow$ messages will be sent $\Rightarrow$ all Rebels play the ex-post eff. outcome afterward.
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Observation

Either stopping or continuing belief updating.

- "a grim-trigger" (protocol-grim-trigger)

Lemma

Before a Rebel knows $\#[\textit{Rebels}](\theta) < k$ or $\#[\textit{Rebels}](\theta) \geq k$, he will not send ***Message to stay*** *or* ***Message to revolt*** *if δ is high enough.*

1. If he send, then information updating stops (a grim-trigger).

Lemma

*Before a Rebel knows $\#[Rebels](\theta) < k$ or $\#[Rebels](\theta) \geq k$, he will not send **Message to stay** or **Message to revolt** if δ is high enough.*

1. If he send, then information updating stops (a grim-trigger).
2. If he does not send, he can learn the relevant information.

FROM CD TO PR

- No expected cost to send **Message to stay** or **Message to revolt**
- The player who knows the relevant info. is willing to send messages.

FROM CD TO PR

- No expected cost to send **Message to stay** or **Message to revolt**
- The player who knows the relevant info. is willing to send messages.

- However, sending message to reveal information in RP is costly.
- A free rider problem in PR may occur.

From CD to PR

1. $k = 5$
2. Only one block (RP and then CD).
3. No expected cost in CD.

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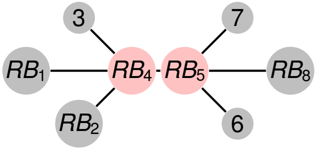

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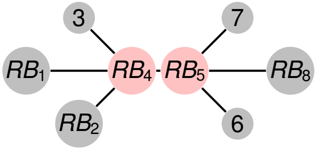

Why? By backward induction,

1. No expected cost to send **Message to stay** or **Message to revolt** in CD.
2. If RB_5 report truthfully, RB_4 can wait for that.
3. If RB_4 report truthfully, RB_5 can wait for that.

- RP^t: the reporting period at t block

$$\overbrace{\langle\langle\cdot\rangle\rangle}^{RP^t}$$

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$$\overbrace{\langle\langle\cdot\rangle\rangle}^{RP^t}$$

- $\langle RP^t \rangle$: the reporting message

Burning moneys	$\neg\langle$**stay**$\rangle$	**s**, ..., **s**, **r**, **s**, ..., **s**
Not burning money	$\langle$**stay**$\rangle$	**s**, ..., **s**, **s**, **s**, ..., **s**

- RP^t: the reporting period at t block

$$\overbrace{\langle\langle\cdot\rangle\rangle}^{RP^t}$$

- $\langle RP^t\rangle$: the reporting message

Burning moneys	$\neg\langle\textbf{stay}\rangle$	**s**, ..., **s**, **r**, **s**, ..., **s**
Not burning money	$\langle\textbf{stay}\rangle$	**s**, ..., **s**, **s**, **s**, ..., **s**

1. Burning moneys+**message to revolt**:
   - coordination to **revolt**
2. Otherwise,
   - no coordination

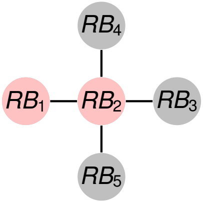
How much money should a Rebel burn?

- Information Hierarchy characterize that.

Information Hierarchy

- Characterizing Rebels' incentives in money burning. ▸ other reason

Ex: $k = 4$,



Information Hierarchy

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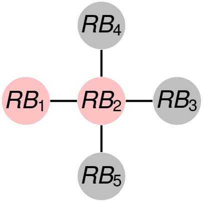
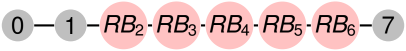


- **Rebel 1 has less incentive**: Rebel 1's information can be reported by Rebel 2.

Information Hierarchy

Information Hierarchy



1. At 0-block, let $R^0 = \{2, 3, 4, 5, 6\}$

Information Hierarchy

Information Hierarchy

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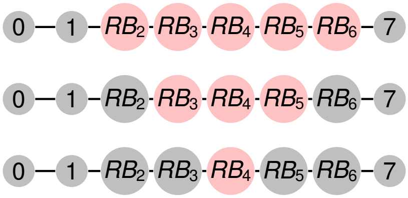
2. At 1-block, let $R^0 = \{\quad 3, 4, 5 \quad\}$

Information Hierarchy



1. At 0-block, let $R^0 = \{2, 3, 4, 5, 6\}$
2. At 1-block, let $R^0 = \{\quad 3, 4, 5 \quad\}$
3. At 2-block, let $R^0 = \{\qquad 4 \qquad\}$

▸ details

Theorem

Given θ, if

1. *the network is acyclic*
2. *the state has strong connectedness*

$\Rightarrow \exists t^{\theta}$ *and* $\exists i \in R^{t^{\theta}}$ *such that* $I_i^{t^{\theta}} \supset [Rebels](\theta)$.

Thus, ideally, APEX can be attained by

- At t block

R^t Rebels	play	$\langle I_i^{t-1} \rangle$	$\mathbf{s}, ..., \mathbf{s}, \overbrace{\mathbf{r}, \mathbf{s}, ..., \mathbf{s}}^{\prod_{j \in I_i^{t-1}} x_j}$
non-R^t Rebels	play	$\langle \mathbf{stay} \rangle$	$\mathbf{s}, ..., \mathbf{s}, \mathbf{s}, \mathbf{s}, ..., \mathbf{s}$

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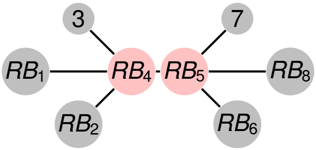
non-R^t Rebels	play	$\langle \mathbf{stay} \rangle$	$\mathbf{s}, ..., \mathbf{s}, \mathbf{s}, \mathbf{s}, ..., \mathbf{s}$

- However, "Pivotal Rebels" will deviate.

DEFINITION (PIVOTAL PLAYER IN RP^t)

$i \in R^t$ and i **will** know relevant info **before** I_i^{t-1} is reported **given** others' truthful reporting.

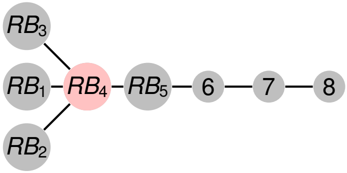
Ex. $k = 5$,



1. Rebel 4 and Rebel 5 are pivotal (**Free Rider problem**)
2. They will manipulate their reporting to save costs.
   - By reporting some other number.

▸ Go to discussion

Ex. $k = 6$,



1. Rebel 4 is pivotal (given Rebel 5's reporting)
2. He will manipulate his reporting to save costs.
    - By reporting some other number.

Definition (Free rider in RP^t)

1. i is pivotal in RP^t
2. i **will** know $\#[Rebels](\theta)$ **before** I_i^{t-1} is reported.

Definition (Free Rider Problem in RP^t)

There are more than 2 free riders in RP^t.

Lemma

If networks are acyclic, then

- *there is a unique PR^t where Free Rider Problem may occur.*
- *there are only two free riders i, j are involved. Moreover $i \in G_j$.*
- *Moreover, before PR^t and after CD^{t-1}, i, j both know that they will be involved*

Thus, before RP^t and after CD^{t-1}, pick one of them as a free rider.

Non-pivotal R^t Rebels	play	$\langle I_i^{t-1} \rangle$	$\mathbf{s}, ..., \mathbf{s}, \overbrace{\mathbf{r}, \mathbf{s}, ..., \mathbf{s}}^{\prod_{j \in I_i^{t-1}} x_j}$
Pivotal R^t Rebels	may play	$\langle 1 \rangle$	$\mathbf{s}, ..., \mathbf{s}, \mathbf{s}, \mathbf{s}, ..., \mathbf{r}$
non-R^t Rebels	play	$\langle$**stay**$\rangle$	$\mathbf{s}, ..., \mathbf{s}, \mathbf{s}, \mathbf{s}, ..., \mathbf{s}$

I.e. Add $\langle 1 \rangle$ into the equilibrium path.

In the equilibrium path,

LEMMA

If networks are acyclic,

$$i \text{ is pivotal but } i \text{ is not free rider}$$

$\Rightarrow$

$$i \text{ knows that } \#[\mathit{Rebels}](\theta) \geq k-1$$

LEMMA

If networks are acyclic,

$$i \text{ play } \langle 1 \rangle$$

$\Leftrightarrow$

$$i \text{ knows that } \#[\mathit{Rebels}](\theta) \geq k-1$$

Consequently, if i play $\langle 1 \rangle$ in the path

i plays	is i a free rider?	$j \in G_i$ plays	i knows
$\langle 1 \rangle$	yes	$\langle \cdot \rangle$	$\#[Rebels](\theta) \geq k$

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$\langle 1 \rangle$	yes	$\langle \cdot \rangle$	$\#[Rebels](\theta) \geq k$
$\langle 1 \rangle$	no	$\langle 1 \rangle$	$\#[Rebels](\theta) \geq k$
$\langle 1 \rangle$	no	$\langle \textbf{stay} \rangle$	$\#[Rebels](\theta) < k$

$\Rightarrow$ i can tell the relevant info. after RP^t.

Consequently, pivotal i has to play **message to revolt** or **message to revolt**

TABLE : Equilibrium path if i played $\langle 1 \rangle$

In RP^t	In $CD^t_{1,1}$	In $CD^t_{1,2}$	After CD^t
i plays	i plays	i plays	
$\langle 1 \rangle$	$\langle$**stay**$\rangle$	$\langle$**stay**$\rangle$	**stay**
$\langle 1 \rangle$	$\langle \mathbf{x}_i \rangle$	$\langle$**stay**$\rangle$	**revolt**

TABLE : Belief updating after CD^t, $t > 0$

In RP^t	In $CD^t_{1,1}$	In $CD^t_{1,2}$	
i plays	i plays	i plays	The events $j \in G_i$ believes with probability one
$\langle I_i^{t-1} \rangle$	$\langle \textbf{stay} \rangle$	$\langle \textbf{stay} \rangle$	$\#[Rebels](\theta) < k$
$\langle I_i^{t-1} \rangle$	$\langle \mathbf{x}_i \rangle$	$\langle \textbf{stay} \rangle$	$\#[Rebels](\theta) \geq k$
$\langle 1 \rangle$	$\langle \textbf{stay} \rangle$	$\langle \textbf{stay} \rangle$	$\#[Rebels](\theta) < k$
$\langle 1 \rangle$	$\langle \mathbf{x}_i \rangle$	$\langle \textbf{stay} \rangle$	$\#[Rebels](\theta) \geq k$

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i plays	i plays	i plays	The events $j \in G_i$ believes with probability one
$\langle \textbf{stay} \rangle$	$\langle \mathbf{x}_i \rangle$	$\langle \textbf{stay} \rangle$	$i \notin R^t$
$\langle I_i^{t-1} \rangle$	$\langle \mathbf{x}_i \rangle$	$\langle \mathbf{x}_i \rangle$	$i \in R^t$

TABLE : Belief updating after CD^t, $t > 0$

In RP^t	In $CD^t_{1,1}$	In $CD^t_{1,2}$	
i plays	i plays	i plays	The events $j \in G_i$ believes with probability one
$\langle \textbf{stay} \rangle$	$\langle \mathbf{x}_i \rangle$	$\langle \textbf{stay} \rangle$	$i \notin R^t$
$\langle I_i^{t-1} \rangle$	$\langle \textbf{stay} \rangle$	$\langle \textbf{stay} \rangle$	$\#[Rebels](\theta) < k$
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$\langle I_i^{t-1} \rangle$	$\langle \mathbf{x}_i \rangle$	$\langle \mathbf{x}_i \rangle$	$i \in R^t$
$\langle 1 \rangle$	$\langle \textbf{stay} \rangle$	$\langle \textbf{stay} \rangle$	$\#[Rebels](\theta) < k$
$\langle 1 \rangle$	$\langle \mathbf{x}_i \rangle$	$\langle \textbf{stay} \rangle$	$\#[Rebels](\theta) \geq k$

Off-path Belief

Whenever i detects a deviation, he believes that

$$\text{for all } j \notin G_i,\ \theta_j \neq \textit{Rebel}$$

1. If he has less than k Rebel-neighbors, he will play **stay** forever.

Off-path Belief

Whenever i detects a deviation, he believes that

$$\text{for all } j \notin G_i,\ \theta_j \neq \textit{Rebel}$$

1. If he has less than k Rebel-neighbors, he will play **stay** forever.
2. This off-path belief then also serve as another "grim trigger" (belief-grim-trigger).

Sketch of proof

1. The equilibrium path is APEX.
2. APEX outcome gives maximum ex-post continuation pay-off after some T.
3. Detectable deviation $\Rightarrow$ belief-grim-trigger. belief-grim-trigger
4. Undetectable deviation $\Rightarrow$ protocol-grim-trigger. protocol-grim-trigger
5. Any deviation will let APEX fail in a positive probability.
6. Sufficiently high δ will impede deviation.

1. From the above steps, an APEX equilibrium for **acyclic** networks is constructed.
   - At most **2** free riders will occur. ▸ example
2. Solving Pivotal-player problem for **cyclic** networks need more elaboration.
   - More than **3** free riders will occur. ▸ example

1. payoff is perfectly observed
   - Play **revolt** in the first period, then the relevant information revealed.
2. payoff is noisy
   - With full support assumption, the existing equilibrium is APEX.
   - Ex.

$$
\begin{aligned}
p_{1s} &= \Pr(y = y_1 | \#\textbf{revolt} \geq k) \\
p_{1f} &= \Pr(y = y_1 | \#\textbf{revolt} < k) \\
p_{2s} &= \Pr(y = y_2 | \#\textbf{revolt} \geq k) \\
p_{2f} &= \Pr(y = y_2 | \#\textbf{revolt} < k)
\end{aligned}
$$

$$1 > p_{1s} > 0, 1 > p_{2s} > 0, p_{1f} = 1 - p_{1s}, p_{2f} = 1 - p_{2s} \tag{1}$$

Further works

1. Cyclic networks.
2. A general model in which players can communicate only by their actions to learn the relevant information in finite time when $\delta < 1$, while the communication protocol itself is an equilibrium.
3. Equilibrium selection.

OR, Static k-threshold game [Chwe 2000]

- Static game payoff for player i: $u_{\theta_i}(a_{\theta_i}, a_{-\theta_i})$

$$u_{Inert_i}(a_{Inert_i}, a_{-\theta_i}) = 1 \quad \text{if } a_{Inert_i} = \textbf{stay}$$

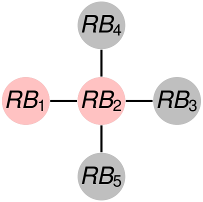
$$\begin{aligned} u_{Rebel_i}(a_{Rebel_i}, a_{-\theta_i}) &= 1 \quad \text{if } a_{Rebel_i} = \textbf{revolt} \text{ and } \#\{j : a_{\theta_j} = \textbf{revolt}\} \geq k \\ u_{Rebel_i}(a_{Rebel_i}, a_{-\theta_i}) &= \text{-1} \quad \text{if } a_{Rebel_i} = \textbf{revolt} \text{ and } \#\{j : a_{\theta_j} = \textbf{revolt}\} < k \end{aligned}$$

$$\begin{aligned} u_{Rebel_i}(a_{Rebel_i}, a_{-\theta_i}) &= 1 \quad \text{if } a_{Rebel_i} = \textbf{stay} \text{ and } \#\{j : a_{\theta_j} = \textbf{revolt}\} \geq k \\ u_{Rebel_i}(a_{Rebel_i}, a_{-\theta_i}) &= 0 \quad \text{if } a_{Rebel_i} = \textbf{stay} \text{ and } \#\{j : a_{\theta_j} = \textbf{revolt}\} < k \end{aligned}$$

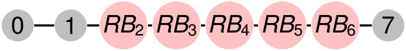
Main goal of **Information Hierarchy**

- Easing the punishment scheme when monitoring is imperfect.

Ex: $k = 4$,



1. **Rebel 1 can only be monitored by Rebel 2.**
2. Suppose Rebel 2,3,4,5 can coordinate at period T and play **revolt** forever.
3. If Rebel 1 did not burn money at period $T-1$, Rebel 2 has no incentive to punish him.



At 1-block, first let

$$
\begin{aligned}
G_i^0 &\equiv G_i \\
I_i^0 &\equiv G_i \cap R^0
\end{aligned}
$$

For instance,

$$
\begin{aligned}
I_2^0 &= \{2,3\} & G_2^0 &= \{1,2,3\} \\
I_3^0 &= \{2,3,4\} & G_3^0 &= \{2,3,4\}
\end{aligned}
$$

$$0 - 1 - RB_2 \cdot RB_3 \cdot RB_4 \cdot RB_5 \cdot RB_6 - 7$$

Then define

$$\leq^0$$

by

$$i \in \leq^0 \Leftrightarrow \exists j \in \bar{G}_i (I_i^0 \subseteq G_j^0 \cap R^0)$$

- For instance,

$$2 \in \leq^0, 3 \notin \leq^0$$

- Since

$$\begin{array}{ll} I_2^0 = \{2, 3\} & G_2^0 \cap R^0 = \{2, 3\} \\ I_3^0 = \{2, 3, 4\} & G_3^0 \cap R^0 = \{2, 3, 4\} \end{array}$$

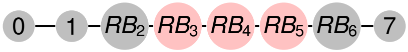
APPENDIX-INFORMATION HIERARCHY

$$0 - 1 - RB_2 \cdot RB_3 \cdot RB_4 \cdot RB_5 \cdot RB_6 - 7$$

At 1-block, let

$$R^1 \equiv \{i \in R^0 | i \notin \leq^0\} = \{ \quad 3, 4, 5 \quad \}$$

Appendix-Information Hierarchy



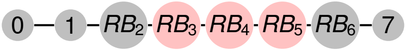
At 2-block, let

$$
\begin{aligned}
G_i^1 &\equiv \bigcup_{k \in I_i^0} G_k \\
I_i^1 &\equiv \bigcup_{k \in G_i \cap R^1} I_k^0
\end{aligned}
$$

For instance,

$$
\begin{aligned}
I_3^1 &= \{2, 3, 4, 5\} \qquad & G_3^1 &= \{1, 2, 3, 4, 5\} \\
I_4^1 &= \{2, 3, 4, 5, 6\} \qquad & G_4^1 &= \{2, 3, 4, 5, 6\}
\end{aligned}
$$

Appendix-Information Hierarchy



Then define

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- For instance,

$$3 \in \leq^1, 4 \notin \leq^0$$

- Since

$$I_3^1 = \{2,3,4,5\} \qquad G_3^1 \cap R^0 = \{2,3,4,5\}$$
$$I_4^1 = \{2,3,4,5,6\} \qquad G_4^1 \cap R^0 = \{2,3,4,5,6\}$$

APPENDIX-INFORMATION HIERARCHY

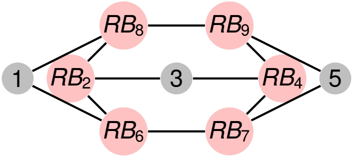
$$0 - 1 - RB_2 - RB_3 - RB_4 - RB_5 - RB_6 - 7$$

At 2-block, let

$$R^2 \equiv \{i \in R^1 | i \notin \leq^1\} = \{ \quad 4 \quad \}$$

Go back to IH

More than 3 free riders will occur **at** a block in cyclic network.



We may pick one of free riders.

Go to discussion

APPENDIX-$\geq$ 3 FREE RIDERS

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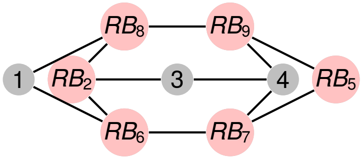

We may pick one of free riders. How to pick?

Go to discussion