Coordination in Social Networks

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Motivation

- A collective action may fails due to incomplete information.
  - A joint project, an investment, etc.
- The situation could be severe. Eg. in pro-democracy movement
  1. Showing discontent is threatened by eavesdropping, suppression, etc.
  2. No (fair) voting system, No (fair) mass media, No (uncensored) discussion forum, etc.

Motivation

However, history tells us:

- An event may trigger later events.
  - Consecutive uprisings in East Germany 1989-1990.
- Social networks play important role
  - Ex., Gangster networks (1911 Revolution); Church networks (1989 Berlin Uprising), etc.

Motivation

Question

- *If rational rebels know that a "tiny" event can trigger later events, how do they conduct a decisive collective action within their social network?*

Model

1. No cheap talk. Communication is taking actions.
2. Communication is facing expected cost.
3. Players communicate repeatedly in a network.

Motivation

Looking for

- An equilibrium, where the ex-post efficient outcome played repeatedly after a finite time T in the path when δ is high enough.

Related Literature

- Public good provision.
    - One strand: [Chwe 2000], [Lohmann, 1993,1994], [Bolton and Harris, 1999], [Bramoullé and Kranton, 2007]
    - **This paper adds network-monitoring**
- Social learning.
    - One strand: [Goyal, 2012], [Acemoglu et al., 2011], [Chatterjee and Dutta, 2011].
    - **This paper considers farsighted-learning in the game**
- Repeated game.
    - One strand: [Laclau, 2012], [Wolitzky, 2013], [Wolitzky, 2014]
    - **This paper consider incomplete information and imperfect monitoring**
    - One strand: [Fudenberg and Yamamoto, 2010] [Fudenberg and Yamamoto, 2011] [Wiseman, 2012] [Yamamoto 2014]
    - **This paper consider n-person game without full-rank conditions on public or private signals generated by single-period actions.**

Model

Network

- Let $N = \{1, \ldots, n\}$ be the set of players.
- G_i is a subset of N, where $i \in G_i$
- G_i is i's neighborhood.
- $G = \{G_i\}_i$ is the network.

Definition

1. G is *fixed* if G is not random.
2. G is *finite* if N is finite.
3. G is *undirected* if $j \in G_i \Rightarrow i \in G_j$.
4. A *path* from i to j, $i \neq j$ in an undirected G is

   $$\{i, l_1, \ldots, l_q, j\}$$

   such that $l_1 \in G_i, \ldots, l_q \in G_j$ and $i, l_1, \ldots, l_q, l$ are all distinct.
5. G is *connected*: An undirected G is connected $\Leftrightarrow \forall\ i, j, i \neq j$ there is a path from i to j.
6. G is *acyclic*: An undirected G is acyclic $\Leftrightarrow$ the path from i to j, for $i \neq j$, is unique.

Model

Static k-threshold game [Chwe 2000]

- θ_i: i's type
- $\theta_i \in \Theta_i = \{Rebel, Inert\}$
- $\Theta = \times_{i \in N} \Theta_i$
- $\theta \in \Theta$
- $A_{Rebel_i} = \{\textbf{revolt}, \textbf{stay}\}$; $A_{Inert_i} = \{\textbf{stay}\}$
- $1 \leq k \leq n$
- Static game payoff for player i: $u_{\theta_i}(a_{\theta_i}, a_{-\theta_i})$

$$u_{Inert_i}(a_{Inert_i}, a_{-\theta_i}) = 1 \quad \text{if } a_{Inert_i} = \textbf{stay}$$

$$\begin{aligned}
u_{Rebel_i}(a_{Rebel_i}, a_{-\theta_i}) &= 1 && \text{if } a_{Rebel_i} = \textbf{revolt} \text{ and } \#\{j : a_{\theta_j} = \textbf{revolt}\} \geq k \\
u_{Rebel_i}(a_{Rebel_i}, a_{-\theta_i}) &= -1 && \text{if } a_{Rebel_i} = \textbf{revolt} \text{ and } \#\{j : a_{\theta_j} = \textbf{revolt}\} < k \\
u_{Rebel_i}(a_{Rebel_i}, a_{-\theta_i}) &= 0 && \text{if } a_{Rebel_i} = \textbf{stay}
\end{aligned}$$

- **stay** is a safe arm; **revolt** is a risky arm.

Model

Repeated k-threshold game

- Time is infinite, discrete.
- Players located in G at -1 period.
- Nature choose θ at 0 period according to π.
- Players play the static k-threshold game infinitely repeatedly.

Assumption

- *Players know their neighbors' types.*
- *Players perfectly observe their neighbors' actions.*
- *G is FFCCU (fixed, finite, connected, commonly known, undirected)*
- *Payoff is hidden (or noisy, or observable).*
- *π has full support*
- *Common δ.*

Model

Notations:

- $[Rebels](\theta) = \{j : \theta_j = Rebel\}$ for all $\theta \in \Theta$.
- τ: a strategy profile
- $h^m_{G_i}$: the history i can observe up to period m
- $\beta^{\pi,\tau}_i(\theta | h^m_{G_i})$: i's belief for a θ at period m.

APEX

Definition

A sequential equilibrium is *approaching ex-post efficient* (*APEX*) $\Leftrightarrow$

$$\forall\,\theta \text{ there is a finite time } T^{\theta}$$

such that ex-post efficient outcome repeats after T^{θ} in the path.

Lemma

If a sequential equilibrium τ^* is APEX $\Rightarrow$

$$\forall\,\theta\ \forall\, i, \textit{ there is a finite time } T_i^{\theta}$$

such that $\sum_{\theta:\#[Rebels](\theta)\geq k} \beta_{G_i}^{\pi,\tau^*}(\theta|h_{G_i}^{s}) = 1$ *or* $= 0$ *if* $s \geq T_i^{\theta}$.

Leading Example

An Apex Equilibrium for $k = n = 3$ in

- At 1st period
  - Rebel 2 chooses **revolt** if he observes $\theta = (Rebel, Rebel, Rebel)$; Otherwise, chooses **stay** forever.
  - Rebel 1 (or Rebel 3) choose **stay**.
- After 1st period
  - If Rebel 2 chooses **revolt** in the last period, then Rebel 1 (or Rebel 3) chooses **revolt** forever;
  - If Rebel 2 chooses **stay** in the last period, then Rebel 1 (or Rebel 3) chooses **stay** forever.
- Any deviation $\Rightarrow$
  - Choosing **stay** forever.

Goal

Goal
Can we generalize the above result for all FFCCU networks?

Results

- $k = n$: we can.
- $k < n$: with additional assumption,
  - acyclic networks: we can .
  - all networks: open question.

Result: $k = n$

Theorem

In *any FFCCU network,* ***if*** *the prior has full support,* ***then*** *for repeated $k = n$ Threshold game,* ***there is*** *a δ such that a sequential equilibrium which is APEX* ***exists****.*

Proof:

1. Some Inerts neighbors $\Rightarrow$ play **stay** forever.
2. No Inert neighbor $\Rightarrow$ play **revolt** until **stay** is observed, and then play **stay** forever.
3. Any deviation $\Rightarrow$ play **stay** forever.
4. Since networks are FFCCU, there is a finite time T^{θ} such that ex-post efficient outcome repeats afterwards.

Result: $k = n$

Comments:

1. **stay** means "some Inerts are out there."
2. **revolt** means "some Inerts may not be there."
3. Any deviation $\Rightarrow$ punished by shifting to **stay** forever by single player
   - Group punishment is not necessary.

Result and Conjecture: $k < n$

Definition

Strong connectedness$\Leftrightarrow$ for every pair of Rebels, there is a path consisting of Rebels to connect them.

Definition

Full support on strong connectedness$\Leftrightarrow$

$$\pi(\theta) > 0 \text{ if and only if } \theta \text{ has strong connectedness.}$$

Result and Conjecture: $k < n$

Theorem

In *any acyclic FFCCU network,* ***if*** *π has full support on strong connectedness,* ***then*** *for repeated $1 \leq k \leq n$ Threshold game,* ***there is*** *a δ such that a weak sequential equilibrium which is APEX* ***exists****.*

Conjecture

In *any FFCCU network, ...[same as above]...*

Outline

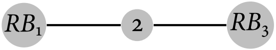
1. The role of Strong Connectedness
2. Communication by actions
3. Communication in the equilibrium
    1. Communication protocol
    2. Reporting and coordination messages in the protocol
    3. Information hierarchy in communication
    4. In-the-path belief updating
    5. Off-path belief
    6. Sketch of proof

The role of Strong Connectedness

The role of Strong Connectedness:

- Otherwise, APEX is impossible for some k.

1. Let $k = 2$.
2. Let

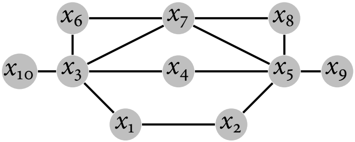


3. Inert 2 block the information transmission.
4. This is an incomplete information game without communication.

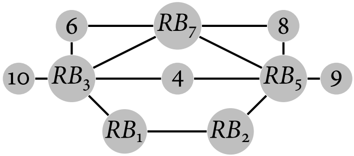
Communication by actions

Communication by binary actions

1. Indexing each node i as a distinct prime number x_i. For instance,



2. Then, If



Rebel 3 report $x_1 \times x_7 \times x_3$ to Rebel 1 by sending a finite sequence

$$\textbf{stay}, \ldots, \textbf{stay}, \underbrace{\textbf{revolt}, \textbf{stay}, \ldots, \textbf{stay}}_{x_1 \times x_7 \times x_3}$$

Communication protocol

Communication phase
Characterize the time horizontal line as

$$\underbrace{\langle\text{coordination period}\rangle}_{0-block}\underbrace{\langle\text{reporting period}\rangle\langle\text{coordination period}\rangle}_{1-block}\ldots$$

- Reporting period: talking about θ
  - Cheap talking: θ will be revealed.
- Why do I need coordination period ?

Communication protocol

Coordination period

Why do I need coordination period ?

- Ans: Since higher-order belief is hard to track.
  - APEX: to find T^θ for all θ.
  - When is T^θ?.
- Sol: Let CD be long enough in a block

$$\overbrace{\langle \ldots \rangle}^{\text{RP}} \overbrace{\langle \langle \cdot \rangle \langle \cdot \rangle \ldots \langle \cdot \rangle \rangle}^{\text{CD}}$$

Communication protocol

Coordination period

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$$\overbrace{\langle \ldots \rangle}^{RP} \overbrace{\langle \langle \cdot \rangle \langle \cdot \rangle \ldots \langle \cdot \rangle \rangle}^{CD}$$

  - If a Rebel knows the relevant info. after RP,

Communication protocol

Coordination period

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- Ans: Since higher-order belief is hard to track.
  - APEX: to find T^{θ} for all θ.
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$$\overbrace{\langle \ldots \rangle}^{RP}\ \overbrace{\langle \langle \cdot \rangle \langle \cdot \rangle \ldots \langle \cdot \rangle \rangle}^{CD}$$

  - If a Rebel knows the relevant info. after RP,$\Rightarrow$ sending messages to let neighbors know that

Communication protocol

Coordination period

Why do I need coordination period ?

- Ans: Since higher-order belief is hard to track.
    - APEX: to find T^{θ} for all θ.
    - When is T^{θ}?.
- Sol: Let CD be long enough in a block

$$\underbrace{\langle \ldots \rangle}_{RP} \underbrace{\langle \langle \cdot \rangle \langle \cdot \rangle \ldots \langle \cdot \rangle \rangle}_{CD}$$

    - If a Rebel knows the relevant info. after RP $\Rightarrow$ sending messages to let neighbors know that $\Rightarrow$ neighbors send msg. to let their neighbors know $\Rightarrow$....$\Rightarrow$ all the Rebels commonly know that after this block.

Communication protocol

After coordination period

- Either stopping or continuing communication
  1. Stopping: if relevant info. is revealed $\Rightarrow$ messages will be sent $\Rightarrow$ all Rebels play the ex-post eff. outcome afterward.
  2. Continuing: otherwise, go to the next block.

Observation

This protocol will let Rebels either stop or continue updating their information about θ after a block.

$\Rightarrow$ a protocol-grim-trigger.

Equilibrium path

Coordination period and messages

Idea

- At least "three" messages to coordinate Rebels
  1. to **revolt**
  2. to **stay**
  3. to continue to next block
- Create these distinguishable messages by binary actions

Equilibrium path

Coordination period and messages

- CD^t: the CD in t-block

$$\overbrace{\langle\underbrace{\langle\cdot\rangle\cdots\langle\cdot\rangle}_{\text{1st division}}\rangle\langle\underbrace{\langle\cdot\rangle\cdots\langle\cdot\rangle}_{\text{2nd division}}\rangle}^{CD^t}$$

- $CD^t_{p,q}$: the p sub-block in q division.
- $\langle CD^t_{p,q}\rangle$: the messages in $CD^t_{p,q}$ are distinguishable

$$\begin{array}{ll} \langle \textbf{stay} \rangle & \mathbf{s}, \ldots, \mathbf{s}, \mathbf{s}, \mathbf{s}, \ldots, \mathbf{s} \\ \langle x_i \rangle & \mathbf{s}, \ldots, \mathbf{s}, \underbrace{\mathbf{r}, \mathbf{s}, \ldots, \mathbf{s}}_{x_i} \end{array}$$

- 1st division: sending **message to stay**; otherwise **continue**
- 2nd division: sending **message to revolt**; otherwise **continue**

Equilibrium path

Message to stay

- Whenever a Rebel i knows $\#[Rebels](\theta) < k$

$$\overbrace{\langle \underbrace{\langle \textbf{stay} \rangle \langle \cdot \rangle \ldots \langle \cdot \rangle}_{\text{1st division}} \rangle \langle \underbrace{\langle \cdot \rangle \cdots \langle \cdot \rangle}_{\text{2nd division}} \rangle}^{CD^t}$$

- Otherwise ,

$$\overbrace{\langle \underbrace{\langle x_i \rangle \langle \cdot \rangle \ldots \langle \cdot \rangle}_{\text{1st division}} \rangle \langle \underbrace{\langle \cdot \rangle \cdots \langle \cdot \rangle}_{\text{2nd division}} \rangle}^{CD^t}$$

Equilibrium path

Message to stay

- Then nearby Rebel j play **stay** afterward

$$\overbrace{\langle \underbrace{\langle x_i\rangle\langle \textbf{stay}\rangle \ldots \langle\cdot\rangle}_{\text{1st division}} \rangle \langle \underbrace{\langle\cdot\rangle \cdots \langle\cdot\rangle}_{\text{2nd division}} \rangle}^{CD^t}$$

- Otherwise,

$$\overbrace{\langle \underbrace{\langle x_i\rangle\langle x_i\rangle \ldots \langle\cdot\rangle}_{\text{1st division}} \rangle \langle \underbrace{\langle\cdot\rangle \cdots \langle\cdot\rangle}_{\text{2nd division}} \rangle}^{CD^t}$$

Equilibrium path

Message to revolt

- Whenever a Rebel i know $\#[Rebels](\theta) \geq k$

$$\overbrace{\langle \underbrace{\langle\cdot\rangle \cdots \langle\cdot\rangle}_{\text{1st division}} \rangle \langle \underbrace{\langle \textbf{stay} \rangle \langle\cdot\rangle \cdot\cdot \langle\cdot\rangle}_{\text{2nd division}} \rangle}^{CD^t}$$

- Otherwise ,

$$\overbrace{\langle \underbrace{\langle\cdot\rangle \cdots \langle\cdot\rangle}_{\text{1st division}} \rangle \langle \underbrace{\langle x_i \rangle \langle\cdot\rangle \cdot\cdot \langle\cdot\rangle}_{\text{2nd division}} \rangle}^{CD^t}$$

Equilibrium path

Message to revolt

- Then nearby Rebel j play $\langle x_j \rangle$ to inform nearby Rebels, etc

$$\overbrace{\langle \underbrace{\langle \cdot \rangle \cdots \langle \cdot \rangle}_{\text{1st division}} \rangle \langle \underbrace{\langle x_j \rangle \langle x_j \rangle \cdot \cdot \langle \cdot \rangle}_{\text{2nd division}} \rangle}^{CD^t}$$

- Otherwise ,

$$\overbrace{\langle \underbrace{\langle \cdot \rangle \cdots \langle \cdot \rangle}_{\text{1st division}} \rangle \langle \underbrace{\langle x_j \rangle \langle \textbf{stay} \rangle \cdot \cdot \langle \cdot \rangle}_{\text{2nd division}} \rangle}^{CD^t}$$

Equilibrium path

Coordination messages

- No expected cost to send **Message to stay** in $CD^t_{1,1}$ or **Message to revolt** in $CD^t_{1,2}$
  - Both are $\langle \textbf{stay} \rangle$
- In this protocol, if δ is high enough then

Lemma

*Before a Rebel knows $\#[Rebels](\theta) < k$ or $\#[Rebels](\theta) \geq k$, he will not send **Message to stay** or **Message to revolt**.*

1. Ex-post efficient outcome gives the maximum static ex-post payoff.
2. protocol-grim-trigger: information updating stops after $CD^t_{1,1} = \langle \textbf{stay} \rangle$ or $CD^t_{1,2} = \langle \textbf{stay} \rangle$.
3. So, he is better not to send those messages..

Equilibrium path

Reporting period and messages
Idea

- "Burning moneys" before sending **message to revolt**.
    1. Gives incentives to report θ.
    2. Prevent potential free rider problems.
- Characterizing "how much money a Rebel should burn"
    - Building Information Hierarchy

Equilibrium path

Reporting period and messages

- RP^t: the reporting period at t block

$$\overbrace{\langle\langle\cdot\rangle\rangle}^{RP^t}$$

- $\langle RP^t \rangle$: the reporting message

Burning moneys	$\neg\langle \textbf{stay} \rangle$	**s, ..., s, r, s, ..., s**
Not burning money	$\langle \textbf{stay} \rangle$	**s, ..., s, s, s, ..., s**

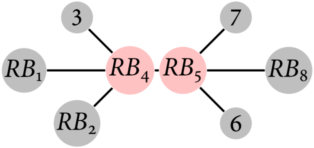
- Burning moneys+**message to revolt**:
    - Rebels believe that $\#[Rebels](\theta) \geq k$
- Not burning moneys+**message to revolt**:
    - Rebels don't believe that $\#[Rebels](\theta) \geq k$

Equilibrium path

An example for free rider problem if no money burning

Assume,

1. Only one block.
2. No expected cost in CD.
3. Obs. $\langle M \rangle$ in $CD \Rightarrow$ play **revolt** forever;
4. Obs. $\neg\langle M \rangle$ in $CD \Rightarrow$ play **stay** forever.
5. $k = 5$
6. Free riders:



7. Rebel 4 will not burn money if Rebel 5 report truthfully, and vise versa.

Observation

1. *Some Rebels* *will* *know* $\#[Rebels](\theta) \geq k$ *or* $\#[Rebels](\theta) < k$ *after RP.*
2. *The "meaning" of* $\langle M \rangle$ *or* $\neg\langle M \rangle$ *should not be free from burning money.*

Equilibrium path

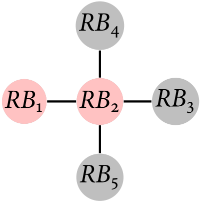
How much money should a Rebel burns?

- Burning money is to convince Rebels to coordination to revolt.
- Information Hierarchy: how much money should be burned?.

Information Hierarchy

Main goal of **Information Hierarchy**

- Characterizing Rebels' incentives in money burning.
- Ex: $k = 4$ and

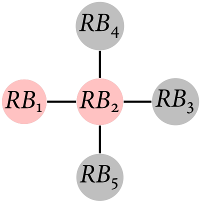


1. Rebel 1's information can be reported by Rebel 2.

Information Hierarchy

Main goal of **Information Hierarchy**

- Easing the punishment scheme when monitoring is imperfect.
- Note that $k < n$, punishment by single player is not enough.
- Ex: $k = 4$ and



1. Rebel 1 can only be monitored by Rebel 2.
2. Suppose Rebel 2,3,4,5 can coordinate at period T and play **revolt** forever.
3. If Rebel 1 did not burn money at period $T-1$, Rebel 2 has no incentive to punish him.

Information Hierarchy

Information Hierarchy

$$0 - 1 - RB_2 - RB_3 - RB_4 - RB_5 - RB_6 - 7$$

At 0-block, let

$$R^0 = [Rebels](\theta)$$

Information Hierarchy

$$0 - 1 - RB_2 - RB_3 - RB_4 - RB_5 - RB_6 - 7$$

At 1-block, first let

$$\begin{aligned} G_i^0 &\equiv G_i \\ I_i^0 &\equiv G_i \cap R^0 \end{aligned}$$

For instance,

$$\begin{aligned} I_2^0 &= \{2, 3\} & G_2^0 &= \{1, 2, 3\} \\ I_3^0 &= \{2, 3, 4\} & G_3^0 &= \{2, 3, 4\} \end{aligned}$$

Information Hierarchy

$$0 - 1 - RB_2 - RB_3 - RB_4 - RB_5 - RB_6 - 7$$

Then define

$$\leq^0$$

by

$$i \in \leq^0 \Leftrightarrow \exists j \in \bar{G}_i (I_i^0 \subseteq G_j^0 \cap R^0)$$

- For instance,

$$2 \in \leq^0, 3 \notin \leq^0$$

- Since

$$\begin{aligned} I_2^0 &= \{2, 3\} & G_2^0 \cap R^0 &= \{2, 3\} \\ I_3^0 &= \{2, 3, 4\} & G_3^0 \cap R^0 &= \{2, 3, 4\} \end{aligned}$$

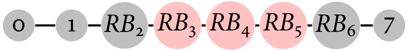
Information Hierarchy

$$0 - 1 - RB_2 - RB_3 - RB_4 - RB_5 - RB_6 - 7$$

At 1-block, let

$$R^1 \equiv \{ i \in R^0 \mid i \notin \leq^0 \} = \{ \quad 3, 4, 5 \quad \}$$

Information Hierarchy



At 2-block, let

$$
\begin{aligned}
G_i^1 &\equiv \bigcup_{k \in I_i^0} G_k \\
I_i^1 &\equiv \bigcup_{k \in G_i \cap R^1} I_k^0
\end{aligned}
$$

For instance,

$$
\begin{aligned}
I_3^1 &= \{2, 3, 4, 5\} \qquad & G_3^1 &= \{1, 2, 3, 4, 5\} \\
I_4^1 &= \{2, 3, 4, 5, 6\} \qquad & G_4^1 &= \{2, 3, 4, 5, 6\}
\end{aligned}
$$

Information Hierarchy

$$0 - 1 - RB_2 - RB_3 - RB_4 - RB_5 - RB_6 - 7$$

Then define

$$\leq^1$$

by

$$i \in \leq^1 \Leftrightarrow \exists j \in \bar{G}_i \left(I_i^1 \subseteq G_j^1 \cap R^0 \right)$$

- For instance,

$$3 \in \leq^1, 4 \notin \leq^0$$

- Since

$$\begin{array}{ll} I_3^1 = \{2,3,4,5\} & G_3^1 \cap R^0 = \{2,3,4,5\} \\ I_4^1 = \{2,3,4,5,6\} & G_4^1 \cap R^0 = \{2,3,4,5,6\} \end{array}$$

Information Hierarchy

$$0 - 1 - RB_2 - RB_3 - RB_4 - RB_5 - RB_6 - 7$$

At 2-block, let

$$R^2 \equiv \{ i \in R^1 | i \notin \leq^1 \} = \{ \quad 4 \quad \}$$

Information Hierarchy

Theorem

Given θ, if

1. *the network is FFCCU and acyclic*
2. *the state has strong connectedness*

$\Rightarrow \exists t^{\theta}$ *and* $\exists i \in R^{t^{\theta}}$ *such that* $I_i^{t^{\theta}} \supset [Rebels](\theta)$.

So, APEX can be attained by

R^t Rebels	play	$\langle I_i^{t-1} \rangle$	$\mathbf{s}, \ldots, \mathbf{s}, \overbrace{\mathbf{r}, \mathbf{s}, \ldots, \mathbf{s}}^{\prod_{j \in I_i^{t-1}} x_j}$
non-R^t Rebels	play	$\langle \mathbf{stay} \rangle$	$\mathbf{s}, \ldots, \mathbf{s}, \mathbf{s}, \mathbf{s}, \ldots, \mathbf{s}$

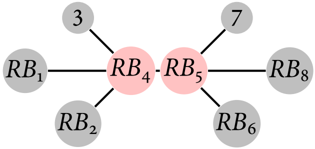
However, "Pivotal Rebels" will deviate.

Pivotal players

Definition

i is pivotal in $RP^t \Leftrightarrow i \in R^t$ and i **will** know $\#[Rebels](\theta) \geq k$ or $\#[Rebels](\theta) < k$ after RP^t **before** I_i^{t-1} is reported.

1. Ex. $k = 5$
2. Rebel 4 and Rebel 5 are pivotal (**Free Rider problem**)



3. They will manipulate their reporting to save costs.
   - By reporting some other number.

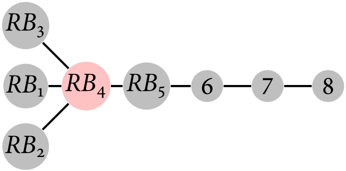
Information Hierarchy

Pivotal players

Definition

i is pivotal in $RP^t \Leftrightarrow i \in R^t$ and i **will** know that $\#[Rebels](\theta) \geq k$ or $\#[Rebels](\theta) < k$ after RP^t **before** I_i^{t-1} is reported.

1. Ex. $k = 6$
2. Rebel 4 is pivotal



3. He will manipulate his reporting to save costs.
   - By reporting some other number.

Information Hierarchy

Solving Pivotal-player problem. Step 1.

Definition

Free Rider Problem A FRP in a t-block is that $\exists i, j \in R^t$, $i \neq j$ such that

1. i, j is pivotal in RP^t
2. i, j **will** know the $\#[Rebels](\theta)$ after RP^t **before** I_i^{t-1} is reported.

Solving Pivotal-player problem. Step 1.

Lemma

If networks are acyclic, then

- *there is a unique block* B^t *where FRP may occur.*
- *there are only two* $i, j \in R^t$ *are involved, and* $i \in G_j$.
- *Moreover, both of* i, j *know that they will be involved before* B^t *and after* B^{t-1}.

Thus, before B^t and after B^{t-1}, pick one of them be pivotal player.

- By their prim number.

Information Hierarchy

Solving Pivotal-player problem. Step 2.

Non-pivotal R^t Rebels	play	$\langle I_i^{t-1} \rangle$	$\mathbf{s}, \ldots, \mathbf{s}, \overbrace{\mathbf{r}, \mathbf{s}, \ldots, \mathbf{s}}^{\prod_{j \in I_i^{t-1}} x_j}$
Pivotal R^t Rebels	may play	$\langle 1 \rangle$	$\mathbf{s}, \ldots, \mathbf{s}, \mathbf{s}, \mathbf{s}, \ldots, \mathbf{r}$
non-R^t Rebels	play	$\langle \mathbf{stay} \rangle$	$\mathbf{s}, \ldots, \mathbf{s}, \mathbf{s}, \mathbf{s}, \ldots, \mathbf{s}$

I.e. Add $\langle 1 \rangle$ into the equilibrium path.

Solving Pivotal-player problem. Step 3.
In the equilibrium path,

Lemma

If networks are acyclic, in RP^t, before i plays I_i^{t-1}

$$i \textit{ knows that } \#[Rebels](\theta) \geq k-1$$

$\Leftrightarrow$

$$i \textit{ is pivotal but } i \textit{ may not know } \#[Rebels](\theta) \textit{ after } RP^t$$

Lemma

If networks are acyclic,

$$i \textit{ knows that } \#[Rebels](\theta) \geq k-1$$

$\Leftrightarrow$

$$i \textit{ play } \langle 1 \rangle$$

Information Hierarchy

Solving Pivotal-player problem. Step 3.
Consequently, in the path,

i has played	i in FRP	$j \in G_i$ play	i knows
$\langle 1 \rangle$	yes	$\langle \cdot \rangle$	$\#[Rebels](\theta) \geq k$
$\langle 1 \rangle$	no	$\langle 1 \rangle$	$\#[Rebels](\theta) \geq k$
$\langle 1 \rangle$	no	$\langle \textbf{stay} \rangle$	$\#[Rebels](\theta) < k$

i has to play **message to revolt** or **message to revolt** if he played $\langle 1 \rangle$

Table : Equilibrium path if i played $\langle 1 \rangle$

In RP^t	In $CD^t_{1,1}$	In $CD^t_{1,2}$	After
i plays	i plays	i plays	
$\langle 1 \rangle$	$\langle \textbf{stay} \rangle$	$\langle \textbf{stay} \rangle$	**coordination to stay**
$\langle 1 \rangle$	$\langle \mathbf{x}_i \rangle$	$\langle \textbf{stay} \rangle$	**coordination to revolt**

Beliefs in equilibrium path

In the equilibrium path

Table : In RP^t

R^t	either play	$\langle I_i^{t-1} \rangle$	$\mathbf{s}, \ldots, \mathbf{s}, \overbrace{\mathbf{r}, \mathbf{s}, \ldots, \mathbf{s}}^{\prod_{j \in I_i^{t-1}} x_j}$
R^t	or play	$\langle 1 \rangle$	$\mathbf{s}, \ldots, \mathbf{s}, \mathbf{s}, \mathbf{s}, \ldots, \mathbf{r}$
R^t	play	$\langle \mathbf{stay} \rangle$	$\mathbf{s}, \ldots, \mathbf{s}, \mathbf{s}, \mathbf{s}, \ldots, \mathbf{s}$

Table : Belief updating after CD^t, $t > 0$

In RP^t	In $CD^t_{1,1}$	In $CD^t_{1,2}$	
i plays	i plays	i plays	The events j believe with probability one
$\langle \mathbf{stay} \rangle$	$\langle \mathbf{x}_i \rangle$	$\langle \mathbf{stay} \rangle$	$i \notin R^t$
$\langle I_i^{t-1} \rangle$	$\langle \mathbf{stay} \rangle$	$\langle \mathbf{stay} \rangle$	$\#[Rebels](\theta) < k$
$\langle I_i^{t-1} \rangle$	$\langle \mathbf{x}_i \rangle$	$\langle \mathbf{stay} \rangle$	$\#[Rebels](\theta) \geq k$
$\langle I_i^{t-1} \rangle$	$\langle \mathbf{x}_i \rangle$	$\langle \mathbf{x}_i \rangle$	$i \in R^t$
$\langle 1 \rangle$	$\langle \mathbf{stay} \rangle$	$\langle \mathbf{stay} \rangle$	$\#[Rebels](\theta) < k$
$\langle 1 \rangle$	$\langle \mathbf{x}_i \rangle$	$\langle \mathbf{stay} \rangle$	$\#[Rebels](\theta) \geq k$

Off-path Belief

Whenever i detects a deviation, he believes that

$$\text{for all } j \notin G_i,\ \theta_j \neq \text{Rebel}$$

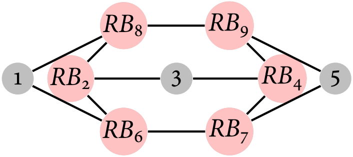
1. If $\#(G_i \cap [Rebels](\theta)) < k$, he will play **stay** forever.
2. This off-path belief then also serve as a grim trigger - belief-grim-trigger.

Sketch of proof

1. The equilibrium path is APEX.
2. If game enters B^t, all Rebels have not know relevant info. before B^t.
3. Detectable deviation $\Rightarrow$ APEX **may** fail by belief-grim-trigger.
4. Undetectable deviation $\Rightarrow$ APEX **may** fail by protocol-grim-trigger
    - pivotal R^t, non-pivotal R^t, non-R^t, will not mimic each other.
5. Ex-post outcome gives maximum ex-post static pay-off.
6. Sufficiently high δ will impede deviation.

Discussion

1. From the above steps, an APEX equilibrium for acyclic networks is constructed.
2. Solving Pivotal-player problem for cyclic networks need more elaboration



Discussion

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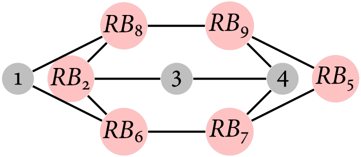

Discussion

1. payoff is perfectly observed
   - Play **revolt** in the first period, then the relevant information revealed.
2. payoff is noisy
   - With full support assumption, the existing equilibrium is APEX.
   - Ex.

$$
\begin{aligned}
p_{1s} &= \Pr(y = y_1 | \#\textbf{revolt} \geq k) \\
p_{1f} &= \Pr(y = y_1 | \#\textbf{revolt} < k) \\
p_{2s} &= \Pr(y = y_2 | \#\textbf{revolt} \geq k) \\
p_{2f} &= \Pr(y = y_2 | \#\textbf{revolt} < k)
\end{aligned}
$$

$$
1 > p_{1s} > 0, 1 > p_{2s} > 0, p_{1f} = 1 - p_{1s}, p_{2f} = 1 - p_{2s} \tag{1}
$$

Further works

1. For the networks with circles, the proof for an APEX equilibrium is still open.
2. There should be a general model in which players can communicate only by their actions to learn the relevant information in finite time when $\delta < 1$, while the communication protocol itself is an equilibrium.
3. Communication in network could serve as a criteria in equilibrium selection.