Notations

- $p_{I_i^{t-1}}$: the multiplication of the prime-indexing of members in I_i^{t-1}.
- $\langle I_i^{t-1}\rangle$: reporting $p_{I_i^{t-1}}$ in a sub-block as

$$\langle .,.,.,\underbrace{h,.,.,.}_{p_{I_i^{t-1}}}\rangle$$

, where the . means l.

- $\langle l\rangle$: reporting all ls in a sub-block as

$$\langle .,.,.,\rangle$$

, where the . means l.

- $\langle 2\rangle$: reporting 2 in a sub-block as

$$\langle .,.,.,.,\underbrace{h,.}_{2}\rangle$$

- $\langle 1\rangle$: reporting 1 in a sub-block as

$$\langle .,.,.,.,.,\underbrace{h}_{1}\rangle$$

- $\langle \mathbf{1}_i\rangle$: reporting i's own index in a sub-block as

$$\langle .,.,.,\underbrace{h,.,.,.}_{p_i}\rangle$$

0.1 Belief updating

Notations

- $\tau_i \in T_i$: $T_i = \{H, L\}$, and τ_i is i's type.
- $\tau = \times_i \tau_i$, $T = \times_i T_i$.
- $h^m_{N_i}$: the history of actions made by N_i.
- $T_{h_i^m}$: the all possible states after $h^m_{N_i}$.

Beliefs

1. For node i:

$$\beta_i(\tau|h^m_{N_i}) = \frac{\beta_i(\tau|h^{m-1}_{N_i})\mathbb{I}(\tau \in T_{h^{m-1}_{N_i}})}{\sum_{\tau\in T} \beta_i(\tau|h^{m-1}_{N_i})\mathbb{I}(\tau \in T_{h^{m-1}_{N_i}})} \tag{1}$$

2. **Assumption (Outside of equilibrium)**
   If $\nexists \tau \in T_{h^{m-1}_{N_i}}$, then

$$\beta_i(|(N \setminus I^0_i) \cap [H]| = 0|h^{m'}_{N_i}) = 1 \ , m' \geq m-1 \tag{2}$$

$$\beta_i(T|h^{m'}_{N_i}) = \beta_i(T|h^{m'}_{N_i}) \ , m' \geq m-1 \tag{3}$$

   The Equation 2 says that i will believe that all the nodes other than N_i will be L-types[1]. The Equation 3 says that i will ignore the actions of all his neighbours afterwards.

   From Equation 2 and Equation 3, this assumption will highly reduce the complication in constructing outside-of-equilibrium. We first provide a simple lemma.

   Lemma 0.1.1. *If i has detected a deviation, and if* $|I^0_i| < s$, *then i will play l forever.*

   Proof. By Equation 2 and Equation 3, and l is the dominate strategy if $|[H]| < s$. □

0.1.1 Equilibrium path

Notations

- $|RP^t|$: the length of reporting period in t-block.
- At period m, where $0 \leq m \leq |RP^t|$, denote

$$O^{m,t}_i \subseteq R^{t-1} \cap (N_i \setminus i)$$

  be the set of i's neighbours who has played $\langle I^{t-1} \rangle$ before m in the reporting period.

[1] This outside-of-equilibrium belief can be relaxed as

$$\beta_i(|(Tr_{ij} \setminus j) \cap [H]| = 0|h^{m'}_{N_i}) = 1 \ , m' \geq m-1$$

. I.e. i will consider all the relevant information came from j are wrong, but still consider those information came from the neighbours other than j are still convincing. This relaxation, however, will complicate the construction of outside-of-equilibrium, since i may have reported the relevant information came from j to his neighbours, and therefore he may need to consider the belief formed by his neighbours, his neighbours' neighbours', and so on. We may need another period to let the nodes can "confirm" the provided relevant information, and so that the nodes' beliefs can be "adjusted".

- Let
$$I_i^{m,t} \equiv (\bigcup_{k \in O_i^{m,t}} I_k^{t-1}) \cup I_i^{t-1}$$
be the updated relevant information gathered by i at period m in the reporting period of t-block. Note that $I_i^{0,t} = I_i^{t-1}$ and $I_i^{|RP^t|,t} = I_i^t$.
- Let
$$N_i^{m,t} \equiv (\bigcup_{k \in O_i^{m,t}} N_k^{t-1}) \cup N_i^{t-1}$$
be the updated neighbourhood which contains $I_i^{m,t}$
- Let
$$Ex_{I_i^{m,t}} \equiv \{l \notin N_i^{m,t} | \exists l' \in I_i^{m,t} \text{ such that there exists a } (l,l')\text{-path}\}$$
be all the possible H-nodes outside of $N_i^{m,t}$ given $I_i^{m,t}$.
- Let
$$Tr_{I_i^{m,t} j} \equiv Tr_{ij} \cap (Ex_{I_i^{m,t}} \cup I_i^{m,t})$$
be the set of possible H-nodes in the (ij)-tree given $I_i^{m,t}$.
- Denote the set
$$C^t = \{i \in R^t : \nexists j \in R^{t-1} \cap \bar{G}_i \text{ such that } \exists l_1, l_2 \in Tr_{ij}[[l_1 \in N_j^{t-1} \backslash I_i^{t-1}] \wedge [k_2 \in \bar{G}_{l_1}]]]\}$$
, then we can show

Lemma 0.1.2. *If the network is a tree and the state has Strong Connectivity, then*

1. $0 \leq \#C \leq 2$.
2. *Moreover, suppose there are two nodes in C, then they are the others' neighbour.*

Proof. In appendix. □

Lemma 0.1.3. *If the network is a tree and the state has Strong Connectivity, then*
$$i \in C \Rightarrow \textit{there is no node outside of } \bigcup_{k \in N_i^{t-1}} G_k$$

Proof. In appendix. □

- For convenience, we also denote

$$\langle \textbf{runs [POST-CHECK,1(or 2)]} \rangle$$

be the strategy such that

1. At $0 < m \leq |RP^t| - |\langle 1 \text{ or } 2 \rangle|$, play
$$l$$
2. At $m = |RP^t| - |\langle 1 \text{ or } 2 \rangle| + 1$, runs **POST-CHECK**

Equilibrium path

$i \notin R^t$

- **WHILE LOOP**
  - At $m \geq 0$, if $\#Ex_{I_i^{m,t}} \cup I_i^{m,t} < k$, report $\langle$**stay**$\rangle$ and then play **stay** forever.
  - Otherwise, **runs POST-CHECK**

$i \in R^t$

- **WHILE LOOP**
  - At $m \geq 0$, if $\#Ex_{I_i^{m,t}} \cup I_i^{m,t} < k$, report $\langle$**stay**$\rangle$ and then play **stay** forever.
  - Otherwise, **runs MAIN**
- **MAIN**

  At $m \geq 0$,

  1. At $m = 0$ and if $\#I_i^{t-1} = \#I_i^{0,t} = k-1$, then **runs POST-CHECK**
  2. At $m = 0$ and if $i \in R^t$ and

     $$\nexists j \in R^{t-1}\bar{G}_i \text{ such that } \exists l_1, l_2 \in Tr_{ij}[[l_1 \in N_j^{t-1} \backslash I_i^{t-1}] \wedge [l_2 \in \bar{G}_{l_1}]]]$$

     , then runs **CHECK.**0. Otherwise, recall **MAIN**
  3. At $0 \leq m \leq |RP^t| - |\langle I_i^{t-1} \rangle|$, play

     stay
  4. At $m = |RP^t| - |\langle I_i^{t-1} \rangle| + 1$, then
     (a) if $O_i^{m,t} = \emptyset$, then report

     $$\langle I_i^{t-1} \rangle$$

     (b) if $O_i^{m,t} \neq \emptyset$, then **runs CHECK.k**

- **CHECK.**0

  At $m = 0$, if $i \in C$, i.e. if $i \in R^t$ and

  $$\nexists j \in R^{t-1} \cap \bar{G}_i \text{ such that } [\exists l_1, l_2 \in Tr_{ij}[[l_1 \in N_j^{t-1} \setminus I_i^{t-1}] \wedge [l_2 \in \bar{G}_{l_1}]]]$$

  , then

  1. If $\#C = 1$, then **runs POST-CHECK**
  2. If $\#C = 2$, then denote $i_1, i_2 \in C$ such that $I_{i_1}^{t-2} < I_{i_2}^{t-2}$, and then
     - if $i = i_1$, then **runs POST-CHECK**

- if $i = i_2$, then report

$$\langle I_i^{t-1} \rangle$$

- **CHECK.m**

  At $m > 0$, if $O_i^{m,t} \neq \emptyset$, then there are two cases,

  1. Case 1: If $i \in R^t$ and

     $$\exists j \in O_i^m \text{ such that } \exists l_1, l_2 \in Tr_{I_i^{m,t} j}[[l_1 \in I_j^{t-1} \backslash I_i^{t-1}] \wedge [l_2 \in \bar{G}_{l_1}]]]$$

     , then report

     $$\langle I_i^{t-1} \rangle$$

  2. Case 2: If $i \in R^t$ and

     $$\nexists j \in O_i^m \text{ such that } \exists l_1, l_2 \in Tr_{I_i^{m,t} j}[[l_1 \in I_j^{t-1} \backslash I_i^{t-1}] \wedge [l_2 \in \bar{G}_{l_1}]]]$$

     (a) Case 2.1: If $i \in R^t$ and

     $$\nexists j \in R^{t-1} \cap (G_i \backslash O_i^{m,t}) \text{ such that } [\exists l_1, l_2 \in Tr_{I_i^{m,t} j}[[l_1 \in N_j^{t-1} \backslash I_i^{t-1}] \wedge [l_2 \in \bar{G}_{l_2}]]]$$

     Note: this case is the case when $i \in C$, thus recall Check.0

     (b) Case 2.2: If

     $$\exists j \in R^{t-1} \cap (G_i \backslash O_i^{m,t}) \text{ such that } [\exists l_1, l_2 \in Tr_{I_i^{m,t} j}[[l_1 \in N_j^{t-1} \backslash I_i^{t-1}] \wedge [l_2 \in \bar{G}_{l_2}]]]$$

     - if $\# I_i^{m,t} = k - 1$, then **runs POST-CHECK**
     - if $\# I_i^{m,t} < k - 1$, then report

     $$\langle I_i^{t-1} \rangle$$

- **CHECK.k**

  At $m \geq 1$,

  1. $O_i^{m,t} \neq \emptyset$, and

     $$\# I_i^{m,t} \geq k$$

     , then **runs POST-CHECK**
  2. $O_i^{m,t} \neq \emptyset$, and

     $$\# I_i^{m,t} < k$$

     , then **runs CHECK.m**

- **POST-CHECK**

  1. At $m = |RP^t|$, then

     (a) If $i \in R^t$ and if $|I_i^{m,t}| \geq k - 1$, then play **revolt**

     (b) if $i \notin R^t$, then play **stay**

0.1.2 Equilibrium path

Notations

- Let
$$Ex_{I_i^t} \equiv \{k \notin I_i^t | \exists k^{'} \in I_i^t \setminus I^{t-1} \text{ such that there exists a } (k,k)\text{-path}\}$$
be all the possible H-nodes outside of N_i^t given I_i^t.
- Let
$$Tr_{I_i^t j} \equiv Tr_{ij} \cap (Ex_{I_i^t} \cup I_i^t)$$
be the set of possible H-nodes in the (ij)-tree given $I_i^{m,t}$.

Equilibrium path

- **1st Division**

  In 1st division, for $t \geq 0$ block and for $1 \leq m \leq n$ sub-block,

  - If i has played $\langle 1 \rangle$, then play
  $$\langle \mathbf{1}_i \rangle$$
  - If $\#Ex_{I_i^t} \cup I_i^t < k$, then
  $$\langle \textbf{play } l \textbf{ forever} \rangle$$
  - If $\#Ex_{I_i^t} \cup I_i^t \geq k$, and there are some $j \in \bar{G}_i$ have played $\langle \textbf{stay} \rangle$, then play
  $$\begin{cases} \langle \mathbf{1}_i \rangle \text{ and then runs } \textbf{COORDINATION} & \text{if} \quad \#I_i^0 \geq k \\ \langle \textbf{play } l \textbf{ forever} \rangle & \text{if} \quad \text{otherwise} \end{cases}$$
  - If $\#Ex_{I_i^t} \cup I_i^t \geq k$, and there is no $j \in \bar{G}_i$ has played $\langle \textbf{stay} \rangle$, then play
  $$\langle \mathbf{1}_i \rangle$$

- **2nd Division**

  If there is no $j \in G_i$ such that j has played $\langle \textbf{stay} \rangle$ in the **1st Division** , then run the following automata. Otherwise, $\langle \textbf{play } l \textbf{ forever} \rangle$

  - $i \notin R^t$
    - In the 1-sub-block: play
    $$\langle \textbf{stay} \rangle$$
    - In the $2 \leq m \leq t+1$ sub-blocks:
      1. If $i \in R^{t^{'}}$ for some $t^{'} \geq 0$ and if there is a $j \in R^{t^{'}+1} \cap \bar{G}_i$ has played
         (a) $\langle \textbf{stay} \rangle$ in $m = 1$ sub-block
         (b) or $\langle \mathbf{1}_j \rangle$ in $m \geq 2$ sub-blocks

, then play

$$\langle \mathbf{1}_i \rangle$$

in $m+1$ sub-block.

2. Otherwise, play

$$\langle \mathbf{stay} \rangle$$

in current sub-block

- $i \in R^t$
  - In the 1-sub-block:
    1. If i has played $\langle 1 \rangle$, then play

$$\langle \mathbf{stay} \rangle$$

    2. If i has not played $\langle 1 \rangle$ and if there is a $j \in \bar{G}_i$ has played $\langle 1 \rangle$, then play

$$\langle \mathbf{stay} \rangle$$

    3. If i has not played $\langle 1 \rangle$ and if there is no $j \in \bar{G}_i$ has played $\langle 1 \rangle$, then
       - If $\# I_i^{|RP^t|,t} \geq k$, then play

$$\langle \mathbf{stay} \rangle$$

       - If $\# I_i^{|RP^t|,t} < k$, then play

$$\langle \mathbf{1}_i \rangle$$

  - In the $m \geq 2$-sub-block:
    1. If $i \in R^{t'}$ for some $t' \geq 0$ and if there is a $j \in R^{t'} \cap \bar{G}_i$ has played
       (a) $\langle \mathbf{stay} \rangle$ in $m = 1$ sub-block, or
       (b) $\langle \mathbf{1}_j \rangle$ in $m \geq 2$ sub-blocks

       , then play

$$\langle \mathbf{1}_i \rangle$$

       in $m+1$ sub-block.
    2. Otherwise, play

$$\langle \mathbf{stay} \rangle$$

       in current sub-block.

- **3rd Division**

1. **INITIATING**
   If i has observed $j \in N_i \setminus i$ has played,
   (a) $\langle \mathbf{stay} \rangle$ in 1-sub-block in pre-coordination period or
   (b) $\langle \mathbf{1}_j \rangle$ in $m \geq 2$ sub-blocks in pre-coordination period or
   (c) $\langle h \rangle$ in the **3rd Division**

   , then play

$$\langle \textbf{play } h \textbf{ forever} \rangle$$

2. **NOT INITIATING**
   Otherwise, play **stay** in current period.

0.1.3 Belief in the equilibrium

- If the history $h_{N_i}^{m'}$ is to let i play

  $\langle$**play** h **forever**$\rangle$

  , then i's belief is as

$$\beta_i(|[H]| \geq s | h_{N_i}^{m'}) = 1 \text{ for all } m' \geq m+1 \tag{4}$$

$$\beta_i(T | h_{N_i}^{m'}) = \beta_j(T | h_{N_i}^{m'}) \text{ for all } m' \geq m+1 \tag{5}$$

- If the history $h_{N_i}^{m'}$ is to let i play

  $\langle$**play** l **forever**$\rangle$

  , then i's belief is as

$$\beta_i(|[H]| \geq s | h_{N_i}^{m'}) = 0 \text{ for all } m' \geq m+1 \tag{6}$$

$$\beta_i(T | h_{N_i}^{m'}) = \beta_j(T | h_{N_i}^{m'}) \text{ for all } m' \geq m+1 \tag{7}$$

- Otherwise, form the belief as Equation 1.

0.1.4 Outside of equilibrium

- Deviation by $j \in N_i \setminus i$ and detection by i:

  For convenience, we call the sequences as "$\langle$coordination messages$\rangle$" if the sequences satisfy

  1. $\langle$coordination message$\rangle - 1$: $\langle l \rangle$ in 1-sub-block in pre-coordination period
  2. $\langle$coordination message$\rangle - 2$: or $\langle \mathbf{1}_j \rangle$ in $m \geq 2$ sub-blocks in pre-coordination period
  3. $\langle$coordination messags$\rangle - 3$: or $\langle h \rangle$ in the post-coordination period

  First note that i can distinguish j's information hierarchy by observing if j has reported $\langle l \rangle$ in the reporting period. We then organize the situations where i can detect j's deviation

  1. If j's information hierarchy is strictly lower than i[2], but j play $\langle$coordination message$\rangle - 2$ or $\langle$coordination message$\rangle - 3$ before i.
  2. If i has reported $\langle$coordination messages$\rangle$, but j did not follow to play $\langle$coordination messages$\rangle$.
  3. $j \in R^t$ played $\langle$coordination messages$\rangle$, but it is impossible to find the source to let j play that in the equilibrium path[3].
  4. If j's reporting is other than $\{\langle l \rangle, \langle \mathbf{1}_j \rangle\}$ in failure-checking period or pre-coordination period.

[2] More specifically, j played $\langle l \rangle$ in reporting period.

[3] More specifically, the possibility of $\exists k \in Tr_{I_i^t j}[|I_k^t| \geq s]$ is zero.

5. $|Ex_{I_i^t} \cup I_i^t| < s$, but j report $\langle \mathbf{1}_j \rangle$ in failure-checking period.
6. $|Ex_{I_i^t} \cup I_i^t| \geq s$, but j report $\langle l \rangle$ in failure-checking period.
7. In the case of **[outside of equilibrium]**

- The outside of equilibrium belief for i is as Equation 3 and Equation 2. The strategy for i is then

$$\begin{cases} \text{runs } \textbf{COORDINATION} & \text{if} \quad |I_i^0| \geq s \\ \langle l \rangle \textbf{ and then } \langle \textbf{ play } l \textbf{ forever } \rangle & \text{if} \quad \text{otherwise} \end{cases}$$