Coordination in Social Networks: Communication by Actions

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Abstract

1 Introduction

2 Model

There is a set of players $N = \{1, ..., n\}$. They constitute a network $G = (V, E)$ so that the vertices are players ($V = N$) and an edge is a pair of them (E is a subset of the set containing all two-element subsets of N). Throughout this paper, G is assumed to be finite, commonly known, fixed, undirected, and connected.[1]

Time is discrete and denoted by $S = \{0, 1, ...\}$ with index s. Each player could be either type R or type I assigned by the nature at $s = 0$ according to a common prior π; R or I represents a Rebel or an Inert respectively. Call $\theta \in \Theta \equiv \{R, I\}^n$ a state of nature. At each $s \geq 1$, players play a normal form game, the *k-threshold game*, infinitely repeated played with common discounted factor $\delta \in (0, 1)$. In the k-threshold game, $A_R = \{\mathbf{revolt}, \mathbf{stay}\}$

[1] A path in G from i to j is a finite sequence $(l_1, l_2, ..., l_L)$ without repetition so that $l_1 = i$, $l_L = j$, and $\{l_q, l_{q+1}\} \in E$ for all $1 \leq q < L$. G is fixed if G is not random, and G is undirected if there is no order relation over each edge. G is connected if, for all $i, j \in N$, $i \neq j$, there is a path from i to j.

is the set of actions for R and $A_I = \{\textbf{stay}\}$ is that for I. Denote by $\#X$ the cardinality of an set X. A Rebel i's stage-game payoff function is defined as below, while an Inert's stage-game payoff is equal to 1 no matter how other players play.

$$\begin{aligned} u_R(a_i, a_{-i}) &= 1 && \text{if } a_i = \textbf{revolt} \text{ and } \#\{j : a_j = \textbf{revolt}\} \geq k \\ u_R(a_i, a_{-i}) &= -1 && \text{if } a_i = \textbf{revolt} \text{ and } \#\{j : a_j = \textbf{revolt}\} < k \\ u_R(a_i, a_{-i}) &= 0 && \text{if } a_i = \textbf{stay} \end{aligned} .$$

Let $[R](\theta)$ be the set of Rebels given θ and the notion *relevant information* indicate whether or not $\#[R](\theta) \geq k$. Note that the ex-post efficient outcome in the stage game is that every Rebel plays **revolt** whenever $\#[R](\theta) \geq k$, and plays **stay** otherwise.[2]

During the game, every player can observe his and his neighbors' types and his and their histories of actions, but no more. A history of actions played by i from period one to period $s \geq 1$ is denoted by $h_i^s \in H_i^s \equiv \times_{\varsigma=1}^{s} A_{\theta_i}$. Let $G_i \equiv \{j : \{i,j\} \in E\}$ be i's neighbors. Denote $\theta_{G_i} \in \Theta_{G_i} \equiv \{R, I\}^{G_i}$ as the type profile of i's neighbors. Let $h_i^0 = \emptyset$, and denote $h_{G_i}^s \in H_{G_i}^s \equiv \times_{j \in G_i} \times_{\varsigma=1}^{s} H_j^{\varsigma}$ as a history of actions played by i's neighbors from period one to period $s \geq 1$. The information set of i about θ at every period is the cylinder $p(\theta) = \{\theta_{G_i}\} \times \{R, I\}^{N \setminus G_i}$, and the information set about histories of action from period one to period $s \geq 1$ is $\{h_{G_i}^s\} \times H_{N \setminus G_i}^s$. A player i's pure behavior strategy τ_i is a measurable function with respect to i's information partition if τ_i maps $\{\theta_{G_i}\} \times \{R, I\}^{N \setminus G_i} \times \{h_{G_i}^s\} \times H_{N \setminus G_i}^s$ to a single action in his action set for every $s \in \{1, 2, ...\}$ and every $\theta \in \Theta$. I assume that payoffs are hidden to emphasize that observing neighbors' actions are the only channel to infer other players' types and actions.[3]

Likewise, define $H^s \equiv \times_{j \in N} H_j^s$ as the set of histories of actions from period one to period $s \geq 1$ and $H \equiv \bigcup_{\varsigma=0}^{\infty} H^{\varsigma}$ as the collection of histories of actions. By abusing the notation a bit, let $h(\tau, \theta) \in H$ denote the realized history of actions generated by strategy profile $\tau = (\tau_1, \tau_2, ..., \tau_n)$ given θ. Designate $\alpha_{G_i}^{\pi,\tau}(\theta, h^s | \theta_{G_i}, h_{G_i}^s)$ as the conditional distribution over $\Theta \times H^s$ induced by π and τ conditional on i's information up to period $s \geq 1$. The belief

[2] Moreover, at every θ and every k, the ex-post efficient outcome is unique and gives the maximum as well as the same payoff to every Rebel.

[3] Such restriction will be relaxed in the Section 5.

of i over Θ induced by π and τ up to period $s \geq 1$ is defined by

$$\beta_{G_i}^{\pi,\tau}(\theta|h_{G_i}^s) \equiv \sum_{h^s \in H^s} \alpha_{G_i}^{\pi,\tau}(\theta, h^s|\theta_{G_i}, h_{G_i}^s).$$

The equilibrium concept is the weak sequential equilibrium.[4] My objective is looking for the existence of *approaching ex-post efficient equilibrium* or *APEX equilibrium*, which is defined below.

Definition 2.1 (APEX strategy)**.** *A behavior strategy τ is APEX if for all θ, there is a terminal period $T^\theta < \infty$ such that the actions in h_θ^τ after T^θ repeats the static ex-post Pareto efficient outcome.*

Definition 2.2 (APEX equilibrium)**.** *An equilibrium (τ^*, α^*) is APEX if τ^* is APEX.*

In an APEX strategy, all Rebels will play **revolt** forever after some period only if $\#[R](\theta) \geq k$; otherwise, Rebels will play **stay** forever after some period. It is as if the Rebels will learn the relevant information in the equilibrium because they will play the ex-post efficient outcome after a certain point of time and keep on doing so. Notice that, in an APEX equilibrium, it is not only as if the Rebels will learn the relevant information: they must learn that. Lemma 2.1 articulates this fact.

Lemma 2.1 (Learning in the APEX equilibrium)**.** *If the assessment (τ^*, μ^*) is an APEX equilibrium, then for all $\theta \in \Theta$, there is a finite time T_i^θ for every Rebel i so that*

$$\sum_{\theta \in \{\theta : [R](\theta) \geq k\}} \beta_{G_i}^{\pi,\tau^*}(\theta|h_{G_i}^s) = \text{ either } 1 \text{ or } 0$$

whenever $s \geq T_i^\theta$.

[4]A weak sequential equilibrium is an assessment $\{\tau^*, \alpha^*\}$, where α^* is a collection of distributions over players' information sets with the property that, for all $i \in N$ and for all $s = 1, 2, ...,$ $\alpha_{G_i}^*(\theta, h^s|\theta_{G_i}, h_{G_i}^s) = \alpha_{G_i}^{\pi,\tau^*}(\theta, h^s|\theta_{G_i}, h_{G_i}^s)$ whenever the information set is reached with positive probability given τ^*. Moreover, for all $i \in N$ and for all $s = 1, 2, ...,$ τ_i^* maximizes i's continuation expected payoff of

$$E_G^\delta(u_{\theta_i}(\tau_i, \tau_{-i}^*)|\alpha_{G_i}^{\pi,\tau_i,\tau_{-i}^*}(\theta, h^s|\theta_{G_i}, h_{G_i}^s))$$

conditional on θ_{G_i} and $h_{G_i}^s$ for all $h_{G_i}^s \in H_{G_i}^s$.

Definition 2.3 (Learning the relevant information). *A Rebel i learns the relevant information at period ς according to strategy τ if $\sum_{\theta \in \{\theta : [R](\theta) \geq k\}} \beta_{G_i}^{\pi,\tau}(\theta | h_{G_i}^s) =$ either 1 or 0 whenever $s \geq \varsigma$.*

It is clear that an APEX equilibrium exists when $k = 1$. As for other cases, let us start with the case of $k = n$ and then continue on to the case of $1 < k < n$. The proof is by construction. In the case of $k = n$, the constructed APEX equilibrium is intuitive and satisfies a stronger equilibrium concept. My main result tackles the case of $1 < k < n$. In such case, my constructed APEX equilibrium is not trivial and can only works for acyclic networks. Section 5.2 discusses why my constructed equilibrium is intractable in cyclic networks.

3 Equilibrium: APEX for $k = n$

In this section, my objective is to show the existence of APEX equilibrium for the case of $k = n$. In this case, notice that a Rebel can get a better payoff from playing **revolt** than from **stay** *only if* all players are Rebels. Two consequences follow. Firstly, if a Rebel has an Inert neighbor, this Rebel will always play **revolt** in the equilibrium. Secondly, at any period $s \geq 1$, it is credible for every Rebel to punish a deviation by playing **stay** forever *if* there is another one who also plays **stay** forever, independently from the belief held by the punisher. These two features constitute an APEX equilibrium and further transform itself to a sequential equilibrium.

Theorem 1 (APEX equilibrium for the case of $k = n$). *For any n-person repeated k-Threshold game with parameter $k = n$ played in a network, there is a δ^* such that a sequential APEX equilibrium exists whenever $\delta > \delta^*$.*

Image that there are an Inert somewhere as well a Rebel i somewhere. Since the network is connected, there is a path connecting these two players. Along with this path, consider the "closest" Inert from Rebel i; this is an Inert who can be reached by the least number of consecutive edges from i. Note that this Inert's Rebel neighbors will play **stay** forever

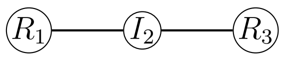

Figure 1: The state and the network in which the APEX equilibrium does not exist when $k = 2$.

since $k = n$. Consider a strategy for Rebels on this path: a Rebel will play **stay** only after observing his neighbor plays **stay**. On this path and according to this strategy, Rebel i will know the existence of such Inert eventually since the network is finite. This contagion argument suggests the following APEX strategy. Every Rebel plays **revolt** initially except for he has an Inert neighbor. Each of them will continuously play **revolt** but switch to **stay** instantly if he observes any of his neighbor plays **stay**. Upon observing a n consecutive **revolt**, a Rebels knows that no Inert exists; otherwise, he knows some Inert exists. The above strategy is an APEX strategy if all Rebels play ex-post efficient outcome after peiord n. To extend it to be an APEX equilibrium, let the deviant play **stay** forever and the punisher who detects it also play **stay** forever. This out-of-path strategy is credible for both the deviant and the punisher, independent from the belief held by the punisher, and hence it is also sequential rational.[5]

4 Equilibrium: APEX for $1 < k < n$

In this section, my objective is to show the existence of APEX equilibrium for the case of $1 < k < n$. In contrast to the case of $k = n$, a Rebel still has the incentive to play **revolt** even if he has an Inert neighbor. This opens a possibility for the non-existence of APEX equilibrium. Example 1 below demonstrates it.

Example 1. Suppose that $k = 2$ and $\theta = (R, I, R)$. The state and the network is represented in Figure 1. Rebel 1 never learns the type of player 3 since Inert 2 cannot reveal it. Therefore no APEX equilibrium exists in this scenario.

The following assumption on the prior—*full support on strong connectedness*—excludes

[5] This sequential rationality is in the sense of [Kreps and Wilson, 1982].

the possibility of nonexistence of APEX equilibrium. To this end, I begin with the definition of *strong connectedness*.

Definition 4.1 (Strong connectedness)**.** *Given G, a state θ has strong connectedness if, for every two Rebels, there is a path consisting of Rebels to connect them.*

In the language of graph theory, the following definition is equivalent: given G, θ has strong connectedness if the induced graph by $[R](\theta)$ is connected.

Definition 4.2 (Full support on strong connectedness)**.** *Given G, π has full support on strong connectedness if*

$$\pi(\theta) > 0 \Leftrightarrow \ \theta \text{ has strong connectedness}$$

As a remark, the definition of the full support on strong connectedness is stronger than common knowledge about that every state has strong connectedness. This marginal requirement is subtle and is more convenient in constructing equilibrium.[6]

I am ready to state the main characterization of this paper:

Theorem 2 (APEX equilibrium for the case of $1 < k < n$)**.** *For any n-person repeated k-Threshold game with parameter $1 < k < n$ played in networks, if networks are acyclic and if π has full support on strong connectedness, then there is a δ^* such that an APEX equilibrium exists whenever $\delta > \delta^*$.*[7]

Constructing an APEX equilibrium in this case is convoluted. I illustrate the proof idea throughout this paper while leaving the formal proof in Appendix. Moreover, since the case of $k = 2$ is trivial given that θ has strong connectedness, I focus on $2 < k < n$ cases.[8]

[6] The main result only requires a weaker version: $\pi(\theta) > 0 \Rightarrow \ \theta$ has strong connectedness . However, working on this weaker version is at the expense of much tedious proof. Throughout this paper, I will stick to the original definition.

[7] A network is acyclic if the path from i to j for all $i \neq j$ is unique.

[8] Suppose $[R](\theta) \geq k = 2$, by the full support on strong connectedness, each Rebel have a Rebel neighbor. The following strategy is an APEX strategy. A Rebel plays **revolt** forever from period one if he has a Rebel neighbor; otherwise, he plays **stay** forever from period one. It can be extended to an APEX equilibrium by letting the out-of-path belief be: assigning probability one on the event that all non-neighbor players are Inerts.

To begin, I consider a specific APEX strategy to be the framework in constructing APEX equilibirum; incentive compatibility is not incorporated at this moment. I first study several strategies that lead at least one Rebel to learn the relevant information.

Let us construct a set W that consists of sequences of actions, in which all sequences have equal length, so that there is a one-to-one mapping between Θ and W. This W exists because the network and the states are finite. For instance, the length of each sequence in W is n. Given θ, the i-th component in the corresponding w_θ is **revolt** if i is a Rebel and is **stay** otherwise. Take another example, which is used in the constructed APEX equilibrium for Thereon 2, the length of each sequence in W is the multiplication of a series of prime numbers. In this series, each prime number is distinct and assigned to distinct player. Denote x_i as the prime number assigned to i. The length of a sequence is therefore $\bigotimes_{i\in N} x_i = x_1 \otimes ... \otimes x_n$, where $\otimes$ is the usual multiplication operator. A θ has $[R](\theta)$ Rebels, and the corresponding w_θ is crafted to be:

$$\overbrace{(\textbf{stay}, ..., \textbf{stay}, \underbrace{\textbf{revolt}, \textbf{stay}, ..., \textbf{stay}}_{\bigotimes_{i\in[R](\theta)} x_i})}^{\bigotimes_{i\in N} x_i}.$$

There is a one-to-one mapping between Θ and W since a multiplication of prime numbers can be uniquely factorized. The above observation is organized as follows.

Proposition 4.1. *There is a set $W \subseteq \{\textbf{revolt}, \textbf{stay}\}^L$, where $L \in \mathbb{N}$, so that there is a bijective mapping $f : \Theta \to W$.*

Fix W and then partition the time by $\{\{0\}, \{1, ..., s_1\}, \{s_1{+}1, ..., s_2\}, ..., \{s_{t-1}{+}1, ..., s_t\}, ...\}$, where $t = 1, 2, ...$ and $s_0 = 0$, so that the length of $\{s_{t-1} + 1, ..., s_t\}$ is equal to the length of $w \in W$ for each t. Call $\{s_{t-1} + 1, ..., s_t\}$ the t-block. Let $w_I \in W$ represents the state in which I is the set of Rebels. Note that this w_I is unique given I since the state space for each player is binary.

Given θ, denote the set I_i as i's Rebel neighbors. If i is a Rebel, let $I_i^1 = I_i$, and let $I_i^t = \bigcup_{j\in G_i} I_j^{t-1}$ for $t \geq 2$. If j is an Inert, let $I_j^t = \emptyset$ for $t \geq 1$. In short, I_i^t is the set of Rebels who can be reached from Rebel i by a path consisting of all Rebels; the length of this path is at most t.

The phrase, "i learn θ", indicates there is a period $s \geq 1$ so that i assigns probability one to the event $\{\theta\}$ by observing histories of actions. The following proposition is immediately obtained.

Proposition 4.2. *If θ has strong connectedness, then there is a strategy so that there exists a Rebel who can learn θ.*

Proof. The strategy is as follows.

Strategy 4.2: At each t-block, each Rebel i plays $w_{I_i^t}$

By Bayesian rule, after t-block, Rebel i assigns probability one to the event,

$$\{\theta : \theta_j = R \text{ and } j \in I_i^{t+1}\}.$$

To conclude the proof, the remaining is to show there exists a t so that $I_i^{t+1} = [R](\theta)$. By definition, I_i^t is the set of Rebels who can be reached by at most t consecutive edges from Rebel i, in each of which the endpoints are Rebels. Since θ has strong connectedness, there exists a t_i so that $I_i^{t_i} = [R](\theta)$. What follows is i learn θ at t_i. □

Remark. The strong connectedness assumption in Proposition 4.2 is indispensable as Example 1 demonstrates. Proposition 4.2 is essentially an if-and-only-if result.

The above Strategy 4.2 would be troublesome if incentive compatibility is under consideration, This is because tracing the expected payoff of every player in the network is a giant task. To reduce the complexity, I identify a smaller set of Rebels, *active Rebels*, who are crucial in the information sharing process and thus needed to be traced. First define G_i^t for each t: if i is a Rebel, let $G_i^1 = G_i$, and let $G_i^t = \bigcup_{j \in G_i} G_j^{t-1}$ for $t \geq 2$; if j is an Inert, let $G_j^t = \emptyset$ for $t \geq 1$. In short, G_i^t is the set of players who can be reached from Rebel i by a path consisting of all Rebels; the length of this path is at most t. Then define the active Rebels at t-block as follows.

Definition 4.3 (Active Rebel at t-block)**.** *Set $R^0 = [R](\theta)$. The set of active Rebels at t-block is*

$$R^t \equiv \{i \in R^{t-1} : \nexists j \in G_i \text{ such that } I_i^t \subseteq G_j^t\}.$$

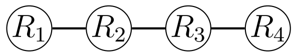

Figure 2: A configuration of the state and the network in which players 1, 2, 3, 4 are Rebels.

In short, an active Rebel in the t-block is an Rebel whose information about θ, I_i^t, is not contained in *any* Rebel's same information. For instance, in the configuration in Figure 2, $R^0 = \{1, 2, 3, 4\}$, $R^1 = \{2, 3\}$, and $R^2 = R^3 = ... = \emptyset$. Furthermore, the active Rebels have to be also the active ones in the previous block; they are fewer and fewer as t goes by.

Lemma 4.1. *If the θ has strong connectedness, then $R^t \subseteq R^{t-1}$ for all $t \geq 1$.*

It is sufficient reveal the relevant information by letting only active Rebels share information about θ given that the network is acyclic and θ has strong connectedness. Theorem 3 articulates this.

Theorem 3. *If the network is acyclic and if the θ has strong connectedness, then there is a strategy so that there exists a R^t Rebel who can learn θ at $t+1$-block.*

The following strategy is for Theorem 3.

Strategy 3: At each t-block, each active Rebel i at t-block plays $w_{I_i^t}$.

Comparing to Strategy 4.2, only fewer Rebels play actions to share information. Theorem 3 is equivalent to the following statement: if the network is acyclic and if the θ has strong connectedness, then there exists $t \geq 0$ and $i \in R^t$ so that $I_i^{t+1} = [R](\theta)$.

Remark. Theorem 3 is not true if the network is cyclic. Take the configuration in Figure3 as an example. There, $R^0 = \{1, 2, 3, 4, 5, 6\}$, $R^1 = \{2, 3\}$, and $R^2 = R^3 = ... = \emptyset$. $I_i^2 \neq [R](\theta)$ for all $i \in R^1$, $I_4^3 = I_5^3 = [R](\theta)$, but neither Rebel 4 or 5 is in R^2.

Neither Proposition 4.2 nor Theorem 3 affirm that all Rebels can learn the relevant information. Next, I create an APEX strategy by modifying Strategy 3. Fix W again, but then partition the time slightly differently from the above mentioned. Partition the time by two alternating phases: *coordination phase* and *reporting phase*, and the time is

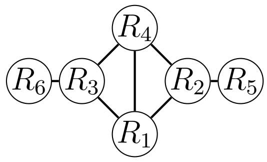

Figure 3: A configuration of the state and the network in which players 1, 2, 3, 4, 5, 6 are Rebels.

$$0 < \underbrace{\text{(coordination phase)} < \text{(reporting phase)}}_{\text{1-block}} < \underbrace{\text{(coordination phase)} < \text{(reporting phase)}}_{\text{2-block}} < ...$$

Figure 4: The partition of the time in the repeated k-threshold game. $<$ is the linear order relation over the time.

starting from coordination phase. The t-th completion of two consecutive phases is called t-block. Figure 4 draws this partition. The length of reporting phase is equal to the length of $w \in W$, and the length of coordination phase is $2n$. The usage of reporting phase is information sharing, and the usage of coordination phase is to coordinate when the ex-post efficient outcome will be played.[9]

Proposition 4.3. *If the network is acyclic and if the θ has strong connectedness, then there is an APEX strategy.*

The following is an APEX strategy for this proposition. Suppose T^θ has been arrived, Rebels play the ex-post efficient outcome. Suppose T^θ has not yet arrived. In the reporting phase, a Rebel follows Strategy 3. In the coordination phase, as follows, the strategy is a contagion process.

1. if a Rebel has been certain $\#[R](\theta) < k$, he plays sequence of actions $(\textbf{stay}, \textbf{stay})$ continuously starting right after he was certain that, and play **stay** forever after this

[9] More precisely, partition the time by $\{\{0\}, \{1, ..., s_{l_1}, s_{l_1}+1, ...s_1\}, \{s_1+1, ..., s_{l_2}, s_{l_2}+1, ..., s_2\}, ..., \{s_{t-1}+1, ..., s_{l_t}, s_{l_t}+1, ..., s_t\}, ...\}$, where $t = 1, 2, ...$ and $s_0 = 0$, so that the length of $\{s_{l_t}+1, ..., s_t\}$ is equal to the length of $w \in W$ for each t, while the length of $\{s_{t-1}+1, ..., s_{l_t}\}$ is $2n$. Call $\{s_{t-1}+1, ..., s_t\}$ the t-block. call $\{s_{l_t}+1, ..., s_t\}$ the *reporting phase* at t-block and call $\{s_{t-1}+1, ..., s_{l_t}\}$ the *coordination phase* at t-block.

phase; T^θ is the period right after this phase.

2. if a Rebel has observed the sequence of actions $(\textbf{stay}, \textbf{stay})$, he plays $(\textbf{stay}, \textbf{stay})$ continuously starting right after he observed that, and plays **stay** forever after this phase; T^θ is the period right after this phase.

3. if a Rebel has learnt $\#[R](\theta) \geq k$, he plays sequence of actions $(\textbf{revolt}, \textbf{revolt})$ continuously starting right after he learned that, and play **revolt** forever after this phase; T^θ is the period right after this phase.

4. if a Rebel has observed the sequence of actions $(\textbf{revolt}, \textbf{revolt})$, he plays $(\textbf{revolt}, \textbf{revolt})$ continuously starting right after he observed that, and plays **revolt** forever after this phase; T^θ is the period right after this phase.

5. if a Rebel is uncertain $\#[R](\theta) \geq k$, he plays sequence of actions $(\textbf{revolt}, \textbf{stay})$ continuously.

The idea is simple. Rebels share information in reporting phase. If a Rebel has learnt the relevant information, he disseminates it to all Rebels contagiously in coordination phase; otherwise, he continues to reporting phase.

If incentive compatibility is under consideration, this logic, however, brings a free-rider problem. Suppose that there are two Rebels who share information to each other in a reporting phase, and each of them is certain that he will learn the relevant information if the other one shares *truthful* information to him. Due to sharing information incurs positive or negative payoff, they will not truthfully share their information. This is because each of them will choose his most profitable way of sharing information without impeding learning the relevant information provided that the other one share the truthful information. The free-rider problem turns out to be the main challenge in the construction of an APEX equilibrium. The proof solves it by arguing that if the network is acyclic, the free-rider problem only occurs between two Rebel neigbors who *commonly know it*, while this argument does not hold for cyclic network.[10] With the help from this argument, the constructed equilib-

[10] Section 5.2 provides an example that the free-rider problem is not commonly known between the Rebels who involve.

rium solves the free-rider problem by arbitrarily assigning one of them to be the free rider, who can choose his most profitable way in sharing information, while letting the other one share truthful information.

I then delver into substantial details in constructing an APEX equilibrium. The framework is adapted from Proposition 4.3, and the time is again partitioned by two alternating phases as shown in Figure4. The out-of-path belief is simple, which serves as a grim trigger. Whenever Rebel i detects a deviation at period ς, he forms the following belief:

$$\sum_{\theta \in \{\theta : \theta_j = I, j \notin G_i\}} \beta^{\pi,\tau}_{G_i}(\theta | h^s_{G_i}) = 1, \text{ for all } s \geq \varsigma. \tag{1}$$

Thus, if $\#I^{\varsigma}_i < k$, he will play **stay** forever after he detects a deviation. This out-of-path belief thus serves as a grim trigger.

4.1 The equilibrium path in the reporting phase

If there is no further mention, all the description in this section is for the APEX equilibrium path *before* the terminal period T^{θ}. Let us shorten "reporting phase in t-block" by $\mathcal{O}^t$, denote $|\mathcal{O}^t|$ as the length of $\mathcal{O}^t$, and shorten **revolt** and **stay** to **r** and **s** receptively.

$|\mathcal{O}^t|$ is independent from t and determines only the set of players. Firstly, assign each player i a distinguishing prime number x_i starting from 3. Then let $|\mathcal{O}^t| = \bigotimes_{i \in N} x_i = x_1 \otimes x_2 \otimes ... \otimes x_n$, where $\otimes$ is the usual multiplication operator. The sequence of actions in $\mathcal{O}^t$ is with length $|\mathcal{O}^t|$ and would take one of the forms specified in the right column in Table 1. The abbreviations of these sequences are listed in the left column. Since these sequences in the reporting phase are meant to share information about θ, the terms "playing the sequence" and "reporting the information" are interchangeable and will be alternatively used.

The sequences $\langle I \rangle$ or $\langle 1 \rangle$ are meant to distinguished themselves from $\langle$**all stay**$\rangle$. The sequence $\langle$**all stay**$\rangle$ is for the inactive Rebels at t-block to report nothing. The sequence $\langle I \rangle$ is used for reporting I by active Rebel at t-block when I is a set of Rebels.

It is worth noting that this sequence constructed by prime numbers brings two benefits. Firstly, since the multiplication of distinguishing prime numbers can be uniquely factorized,

Table 1: The notations for the sequences of actions in $\mathcal{O}^t$ on the path

Notations		The sequences of actions
$\langle I \rangle$	$\equiv$	$(\overbrace{\mathbf{s}, ..., \mathbf{s}, \underbrace{\mathbf{r}, \mathbf{s}, ..., \mathbf{s}}_{\bigotimes_{j\in I} x_j}}^{\bigotimes_{i\in N} x_i})$
$\langle 1 \rangle$	$\equiv$	$(\overbrace{\mathbf{s}, ..., \mathbf{s}, \mathbf{r}}^{\bigotimes_{i\in N} x_i})$
$\langle$**all stay**$\rangle$	$\equiv$	$(\overbrace{\mathbf{s}, ..., \mathbf{s}, \mathbf{s}}^{\bigotimes_{i\in N} x_i})$

the Rebels can utilize such sequence to precisely report players' identities. Secondly, the un-discounted expected payoff of playing $\langle I \rangle$ for some $I \subseteq N$ is always equal to -1, and therefore it is relatively easy to calculate. This is because, at any period in $\mathcal{O}^t$, only active Rebel at t-block will report $\langle I \rangle$. Since an active t-block Rebel's I is different from any other active t-block Rebel's same information, at most one Rebel would play $\mathbf{r}$ at any period in $\mathcal{O}^t$ by the property of prime number multiplication.[11]

The sequence $\langle 1 \rangle$ is intentionally crafted to tackle the free-rider problem. To see how $\langle 1 \rangle$ works, let us formally define the *pivotal Rebel* and the *free-rider problem.*

Definition 4.4 (Pivotal Rebels in $\mathcal{O}^t$)**.** *A Rebel p is pivotal in $\mathcal{O}^t$ if p is active at t-block, p has not learnt relevant information, and p is certain that he will learn the relevant information in the end of $\mathcal{O}^t$, given that each $i \in R^t$ reports $\langle I_i^t \rangle$.*

From the definition, a pivotal Rebel in $\mathcal{O}^t$ is one who can learn the relevant information if all of his active Rebel neighbors truthfully report their information about θ to him. The pivotal Rebels can be further classified into two kinds: ones who can learn the true state, and ones who learn only the relevant information. When $k = 6$, in the configuration in Figure **??**, only Rebels 4 and 5 are pivotal, and they are of the first kind; in the configuration in Figure 5, only Rebel 5 is pivotal, and he is of the first kind; in the configuration in Figure 6, only Rebel 4 pivotal, and he is of the second kind.

[11] This statement holds if there is no Rebel who plays $\langle 1 \rangle$.

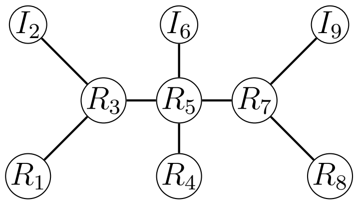

Figure 5: A configuration of the state and the network in which player 1, 3, 4, 5, 7, 8 are Rebels while players 2, 4, 9 are Inerts.

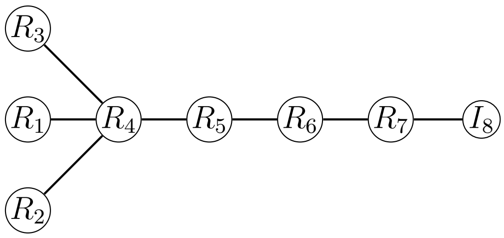

Figure 6: A configuration of the state and the network in which player 1, 2, 3, 4, 5, 6, 7 are Rebels while player 8 is an Inert.

Call p of the first kind by *θ-pivotal*. For the second kind, if the network is acyclic and if the prior has full support on strong connectedness, p is the second kind in $\mathcal{O}^t$ only if $I_p^t = k-1$. Call the one with $I_p^t = k-1$ by $k-1$-*pivotal* Rebel.[12]

Below is the defined free-rider problem in $\mathcal{O}^t$.

Definition 4.5. *A free-rider problem exists in $\mathcal{O}^t$ if there are multiple θ-pivotal Rebels in $\mathcal{O}^t$.*

The following lemma is crucial.

Lemma 4.2. *If the network is acyclic and if π has full support on strong connectedness, there are at most two θ-pivotal Rebels in the t-block. Moreover, they are neighbors.*[13]

Notably,

[12]To show that a pivotal Rebel is the second kind in $\mathcal{O}^t$ only if $I_p^t = k-1$, one can follow the same argument in Lemma 4.1 and Theorem 3.

[13]As a remark, Lemma 4.2 is not true when the network is cyclic. To see this, consider a 4-player circle when $\theta = (R, R, R, R)$.

Lemma 4.3. *If the network is acyclic and if π has full support on strong connectedness, when there are two θ-pivotal Rebels $p, p^{'}$ in the t-block, then they commonly know that they are θ-pivotal Rebels at the beginning of t-block.*

By Lemma 4.3, θ-pivotal Rebels in $\mathcal{O}^t$ can identify themselves at the beginning of $\mathcal{O}^t$. This importance cannot be further emphasized. If the free-rider problem occurs in $\mathcal{O}^t$, the strategy can specify that the lowest indexed θ-pivotal Rebel p in the free-rider problem will play $\langle 1 \rangle$, while the other one $p^{'}$ will play $\langle I^t_{p^{'}} \rangle$ *beforehand.* In short, this knowledge is encoded in the belief system of an APEX equilibrium.

Remark. It is worth noting that the assumption of acyclic network in Lemma 4.3 is indispensable. Lemma 4.3 does not hold if the network is cyclic as Section 5.2 demonstrates it.

Now I can list the sequences played in $\mathcal{O}^t$ on the path in Table 2.

Table 2: The sequences of actions played in $\mathcal{O}^t$ on the path

Rebel i	i plays
$i \notin R^t$	$\langle$**all stay**$\rangle$
$i \in R^t$ but i is not pivotal	$\langle I^t_i \rangle$
i is $k-1$-pivotal	$\langle 1 \rangle$
i is θ-pivotal but not in the free-rider problem	$\langle 1 \rangle$
i is in the free-rider problem with the lowest index	$\langle 1 \rangle$
i is in the free-rider problem without the lowest index	$\langle I^t_i \rangle$

4.2 The equilibrium path in the coordination phase

In this section, I discuss the sequences of actions in the coordination phase on the path. The term "coordination phase in t-block" is shorten by $\mathcal{C}^t$. If there is no further mention, all the description in this section is for the APEX equilibrium path *before* the terminal period T^θ.

To explicitly depict the structure in the coordination phase is tiresome, but it is actually a simple contagion scenario: Rebels jointly decide to terminate or continue their information sharing during this phase. For that, a coordination phase is partitioned into three *divisions*. In the first division, if there is a Rebel has learnt that $\#[R](\theta) < k$, all Rebels will play **stay** forever after this division; otherwise, they continue to the next one. In the second one, if there is a Rebel has learnt that $\#[R](\theta) \geq k$, a certain portion of Rebels will play **revolt** forever after this division; otherwise, they continue to the next one. In the third one, if there is a Rebel has learnt that $\#[R](\theta) \geq k$ in previous divisions, all Rebel will play **revolt** forever after this division; otherwise, they continue to the next phase—a reporting phase.

To fulfil the above contagion argument, create a set of sequences of actions to be played in the equilibrium path so Rebels' belief is updated by observing them. To illustrate these sequence and the equilibrium path, partition a division into *sub-blocks*. I depict the whole partition in the coordination phase below, where $\square$ represents a sub-block in coordination phase.

In $\mathcal{C}^1$,

$$\overbrace{\underbrace{\square}_{\text{one sub-block}}}^{\text{1-division}}\overbrace{\underbrace{\square}_{\text{one sub-block}}}^{\text{2-division}}\overbrace{\underbrace{\square\cdots\square}_{n\text{ sub-blocks}}}^{\text{3-division}}$$

In $\mathcal{C}^t$, $t \geq 2$,

$$\overbrace{\underbrace{\square\cdots\square}_{n\text{ sub-blocks}}}^{\text{1-division}}\overbrace{\underbrace{\square\cdots\square}_{t+1\text{ sub-blocks}}}^{\text{2-division}}\overbrace{\underbrace{\square\cdots\square}_{n\text{ sub-blocks}}}^{\text{3-division}}$$

In the t-block, denote $\mathcal{C}^t_u$ as the u-division and $|\mathcal{C}^t_u|$ as the length of $\mathcal{C}^t_u$. Likewise, denote $\mathcal{C}^t_{u,v}$ as the v-th sub-block in u-division and $|\mathcal{C}^t_{u,v}|$ as the length of $\mathcal{C}^t_{u,v}$. Let us shorten **revolt** and **stay** to **r** and **s** receptively. For $v = 1, ..., n$, let $|\mathcal{C}^t_{u,v}| = n$ if $u = 1, 2$ and $|\mathcal{C}^t_{u,v}| = 1$ if $u = 3$. The notations for the sequences of actions on the path are shown in Table 3.[14]

[14]Because, in the 3-division, the length of the sequence of actions is 1, i.e. playing an action, I dispense notations in the 3-division for conciseness.

Table 3: The notations for the sequences of actions in $\mathcal{C}^t_{u,v}$ for $u = 1, 2$, $v = 1, ..., n$, on the path

Notations		The sequences of actions
$\langle i \rangle$	$\equiv$	$(\overbrace{\mathbf{s}, ..., \mathbf{s}, \underbrace{\mathbf{r}, \mathbf{s}, ..., \mathbf{s}}_{i}}^{n})$
$\langle$**all stay**$\rangle$	$\equiv$	$(\overbrace{\mathbf{s}, ..., \mathbf{s}, \mathbf{s}}^{n})$

4.2.1 The equilibrium behavior on the path in $\mathcal{C}^1$

Since the 1-block has a simpler structure, I begin with depicting the equilibrium path in $\mathcal{C}^1$, which is shown in Table 4.

There is a non-trivial construction in the equilibrium path: "How Rebel i initiates the common knowledge about $\#[R](\theta) \geq k$." Rebel i does so by playing $\langle i \rangle$ in $\mathcal{C}^1_{1,1}$ and then play $\langle$**all stay**$\rangle$ in $\mathcal{C}^1_{2,1}$. His behavior is thus distinguishable from other kinds of Rebels. His neighbors will know $\#[R](\theta) \geq k$ right after $\mathcal{C}^1_{2,1}$, and then all Rebels will know that by playing $\mathbf{r}$ contagiously in $\mathcal{C}^1_3$.

Rebel i will not deviate to play $\langle$**all stay**$\rangle$ even though it might be undetectable. This is because the network is acyclic. If i does so, i will be considered as an inactive Rebel at 1-block afterwards by all of his neighbors. This implies each of i's behavior will be certain that there is no Rebel behind i. i then faces the possibility that no Rebels can learn $\#[R](\theta) \geq k$ even if $\#[R](\theta) \geq k$. If this happens, i will only get zero payoff. However, i can surely get payoff of 1 forever after $\mathcal{C}^1_{2,1}$. Sufficiently high discount factor will deter this deviation.

Rebels' belief updating after $\mathcal{C}^1_1$ and $\mathcal{C}^1_2$ on the path are listed in Table 5. The evolution of information filtrations can be tracked through this table.

Table 4: The sequences of actions played in $\mathcal{C}^1$ on the path

The sequences of actions played in $\mathcal{C}^1_{1,1}$ on the path

Rebel i	i plays
i is certain $\#[R](\theta) < k$	$\langle$**all stay**$\rangle$
$i \notin R^1$ and is uncertain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$
$i \in R^1$ and is uncertain $\#[R](\theta) \geq k$	$\langle i \rangle$
i is certain $\#[R](\theta) \geq k$	$\langle i \rangle$

The sequences of actions played in $\mathcal{C}^1_{2,1}$ on the path

Rebel i	i plays
i is certain $\#[R](\theta) < k$	$\langle$**all stay**$\rangle$
$i \notin R^1$ and is uncertain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$
$i \in R^1$ and is uncertain $\#[R](\theta) \geq k$	$\langle i \rangle$
i is certain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$

The sequences of actions played in $\mathcal{C}^1_{3_v}$, where $v = 1, ..., n$, on the path

Rebel i	i plays
i is certain $\#[R](\theta) < k$	**s**
$i \notin R^1$ and is uncertain $\#[R](\theta) \geq k$	**s**
$i \in R^1$ and is uncertain $\#[R](\theta) \geq k$	**s**
i is certain $\#[R](\theta) \geq k$	**r**

Table 5: On the path, in $\mathcal{C}^1$, the belief of i's neighbor j after observing i's previous actions.

i plays	The event on which j assigns probability one right after $\mathcal{C}_1^1$
In $\mathcal{C}_1^1$	
$\langle$**all stay**$\rangle$	$i \notin R^1$ if $j \in R^1$
$\langle$**all stay**$\rangle$	$\#[R](\theta) < k$ if $j \notin R^1$
$\langle i \rangle$	$i \in R^1$ or $\#[R](\theta) \geq k$

i plays		The event on which j assigns probability one right after $\mathcal{C}_2^1$
In $\mathcal{C}_1^1$	In $\mathcal{C}_2^1$	
$\langle$**all stay**$\rangle$	$\langle$**all stay**$\rangle$	$i \notin R^1$ if $j \in R^1$
$\langle$**all stay**$\rangle$	$\langle$**all stay**$\rangle$	$\#[R](\theta) < k$ if $j \notin R^1$
$\langle i \rangle$	$\langle$**all stay**$\rangle$	$\#[R](\theta) \geq k$
$\langle i \rangle$	$\langle i \rangle$	$i \in R^1$

4.2.2 The equilibrium behavior on the path in $\mathcal{C}^t$ for $t \geq 2$

Next, I describe the equilibrium behavior on the path in $\mathcal{C}^t$ whenever $t \geq 2$. Players' belief over states will now be contingent on the behavior in $\mathcal{O}^t$ since Rebels share information in $\mathcal{O}^t$. I illustrate players' belief updating in Table 6. The evolution of information filtrations can be tracked in this table. After that, the in-path strategy contingent on players' belief is introduced in Table 7. Let us denote $I_{ij}^{t+1} = I_i^t \cap I_j^t$.

The delicate part in $\mathcal{C}^t$ is how a pivotal Rebel p propagandizes the relevant information. p does so by playing $\langle$**all stay**$\rangle$ in $\mathcal{C}_{1,1}^t$ in the case of $\#[R](\theta) < k$, while playing $\langle p \rangle$ in the case of $\#[R](\theta) \geq k$. Notice that playing $\langle$**all stay**$\rangle$ in $\mathcal{C}_{1,1}^t$ by p will immediately inform p's neighbors that $\#[R](\theta) < k$. On the contrary, playing sequence other than $\langle$**all stay**$\rangle$ has not yet revealed $\#[R](\theta) \geq k$ since that might be played by other non-pivotal Rebels.

In $\mathcal{C}_{2,1}^t$, p reveals $\#[R](\theta) \geq k$ by playing $\langle$**all stay**$\rangle$, which is a costless sequence of actions. It might not seem intuitive at first sight, but it effectively prevents a potential

Table 6: On the path, in $\mathcal{C}^t$, the belief of i's neighbor j after observing i's previous actions.

i plays	The event on which j assigns probability one right after $\mathcal{O}^t$
In $\mathcal{O}^t$	
$\langle$**all stay**$\rangle$	$i \notin R^t$ and $I_{ji}^{t+1} = I_j^t$
$\langle I_i^t \rangle$	$i \in R^t$ and $I_{ji}^{t+1} = I_j^t \cap I_i^t$
$\langle 1 \rangle$	i is pivotal

i plays		The event on which j assigns probability one right after $\mathcal{C}_{1,1}^t$
In $\mathcal{O}^t$	In $\mathcal{C}_{1,1}^t$	
$\langle$**all stay**$\rangle$	$\langle i \rangle$	$i \notin R^t$ and $I_{ji}^{t+1} = I_j^t$
$\langle I_i^t \rangle$	$\langle$**all stay**$\rangle$	$\#[R](\theta) < k$
$\langle I_i^t \rangle$	$\langle i \rangle$	$(\#[R](\theta) \geq k)$ or $(i \in R^t$ and $I_{ji}^{t+1} = I_j^t \cap I_i^t)$
$\langle 1 \rangle$	$\langle$**all stay**$\rangle$	$\#[R](\theta) < k$
$\langle 1 \rangle$	$\langle i \rangle$	$\#[R](\theta) \geq k$

i plays			The event on which j assigns probability one right after $\mathcal{C}_{2,1}^t$
In $\mathcal{O}^t$	In $\mathcal{C}_{1,1}^t$	In $\mathcal{C}_{2,1}^t$	
$\langle$**all stay**$\rangle$	$\langle i \rangle$	$\langle$**all stay**$\rangle$	$i \notin R^t$ and $I_{ji}^{t+1} = I_j^t$
$\langle I_i^t \rangle$	$\langle$**all stay**$\rangle$	$\langle$**all stay**$\rangle$	$\#[R](\theta) < k$
$\langle I_i^t \rangle$	$\langle i \rangle$	$\langle$**all stay**$\rangle$	$\#[R](\theta) \geq k$
$\langle I_i^t \rangle$	$\langle i \rangle$	$\langle i \rangle$	$i \in R^t$ and $I_{ji}^{t+1} = I_j^t \cap I_i^t$
$\langle 1 \rangle$	$\langle$**stay**$\rangle$	$\langle$**all stay**$\rangle$	$\#[R](\theta) < k$
$\langle 1 \rangle$	$\langle i \rangle$	$\langle$**all stay**$\rangle$	$\#[R](\theta) \geq k$

Table 7: The sequences of actions played in $\mathcal{C}^t$, $t \geq 2$ on the path

The sequences of actions played in $\mathcal{C}^t_{1,v}$ for $t \geq 2$ and for $v = 1, 2, ..., n$ on the path

Rebel i	i plays
i is certain $\#[R](\theta) < k$	$\langle$**all stay**$\rangle$
$i \notin R^t$ and is uncertain $\#[R](\theta) \geq k$	$\langle i \rangle$
$i \in R^t$ and is uncertain $\#[R](\theta) \geq k$	$\langle i \rangle$
i is certain $\#[R](\theta) \geq k$	$\langle i \rangle$

The sequences of actions played in $\mathcal{C}^t_{2,v}$ for $t \geq 2$ for $v = 1$ on the path

Rebel i	i plays
i is certain that $\#[R](\theta) < k$	$\langle$**all stay**$\rangle$
$i \notin R^t$ and is uncertain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$
$i \in R^t$ and is uncertain $\#[R](\theta) \geq k$	$\langle i \rangle$
i is certain that $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$

The sequences of actions played in $\mathcal{C}^t_{2,v}$ for $t \geq 2$ for $v = 2, ..., t+1$ on the path

Rebel i	i plays
i is certain that $\#[R](\theta) < k$	$\langle$**all stay**$\rangle$
$i \notin R^t$ and is uncertain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$
$i \in R^t$ and is uncertain $\#[R](\theta) \geq k$	$\langle$**all stay**$\rangle$
i is certain that $\#[R](\theta) \geq k$	$\langle i \rangle$

The sequences of actions played in $\mathcal{C}^t_3$ for $t \geq 2$ on the path

Rebel i	i plays
i is certain that $\#[R](\theta) < k$	**s**
$i \notin R^1$ and is uncertain $\#[R](\theta) \geq k$	**s**
$i \in R^1$ and is uncertain $\#[R](\theta) \geq k$	**s**
i is certain that $\#[R](\theta) \geq k$	**r**

free-rider problem: there are two pivotal Rebels, say p and $p^{'}$, who have already known $\#[R](\theta) \geq k$ in $\mathcal{C}^t$. If initiating the common knowledge about $\#[R](\theta) \geq k$ incurs negative payoff, p or $p^{'}$ will have the incentive (again!) to let the other one initiate it. Playing $\langle$**all stay**$\rangle$ in $\mathcal{C}^t_{2,1}$ proudly becomes the initiation sequence by its cheapness.

The remaining argument is why other non-pivotal Rebels, say i, do not mimic the pivotal Rebels' behavior to play $\langle 1 \rangle$ in $\mathcal{O}^t$. If i plays $\langle 1 \rangle$, all Rebels will learn the relevant information right after $\mathcal{C}^t_2$ based on the belief updating on the path. It implies that the beginning of $t+1$-block is the terminal period. He will not learn the relevant information after that because the belief updating will be also terminated. However, he is still uncertain whether he can learn the relevant information in $\mathcal{O}^t$ since he is not pivotal. Since that the ex-post efficient outcome gives him the maximum payoff at every θ, and that he will learn the relevant information eventually on the equilibrium path, he prefers not to deviate given that the discount factor is high enough.[15]

5 Discussion

5.1 Payoff as signals

The hidden payoff assumption can be relaxed without changing the main result. One may consider a situation in which the stage payoff depends not only on players' joint efforts but also on a random shock, say the weather. To fix the idea, there is a public signal $y \in \{r, s\}$ generated according to the action profile. Let a Rebel's payoff function be $u_R(a_R, y)$ such

[15]Since R^t Rebels share information on the equilibrium path, by Theorem 3, the belief updating in Table 5, Table 6, and the in-path behavior in Table 7, the relevant information is learnt by every Rebel eventually on the path.

that $u_R(\mathbf{stay}, r) = u_R(\mathbf{stay}, s) = u_0$. y is drawn from the distribution of

$$\begin{aligned}
p_{rr} &= \Pr(y = r | \#\{j : a_j \mathbf{revolt}\} \geq k) \\
p_{sr} = 1 - p_{rr} &= \Pr(y = s | \#\{j : a_j \mathbf{revolt}\} \geq k) \\
p_{ss} &= \Pr(y = s | \#\{j : a_j \mathbf{revolt}\} < k) \\
p_{rs} = 1 - p_{ss} &= \Pr(y = r | \#\{j : a_j \mathbf{revolt}\} < k)
\end{aligned}$$

such that

$$p_{rr} u_R(\mathbf{revolt}, r) + p_{sr} u_R(\mathbf{revolt}, s) > u_0 > p_{rr} u_R(\mathbf{revolt}, r) + p_{ss} u_R(\mathbf{revolt}, s),$$

and

$$0 \leq p_{rs} \leq 1, 0 \leq p_{ss} \leq 1.$$

The APEX equilibrium constructed for Theorem 2 is still a one in this scenario. Note that in that APEX equilibrium path, at most one **revolt** can occur at every period before some Rebel plays $\langle 1 \rangle$. This implies that the signal y is completely uninformative before some Rebel plays $\langle 1 \rangle$. If a Rebel i deviates to play $\langle 1 \rangle$ in $\mathcal{O}^t$ at some t in the hope gathering information from y, he will not learn the relevant information after $\mathcal{O}^t$ since the terminal period will come right after t-block. He will, however, learn the relevant information and play the ex-post efficient outcome if he is on the path, and hence he will not deviate.

5.2 Cyclic networks

Scenarios in cyclic networks substantially differ from the acyclic counterpart. The free-rider problem could become intractable in cyclic networks. Let us consider the configuration in Figure 7, and suppose $k = 4$.

In Figure 7, Rebels 2 and 3 are θ-pivotal by definition. From the perspective of Rebel 2, the type of player 5 could be Inert. Therefore, Rebel 2 does not know that Rebel 1 is pivotal. Similarly, Rebel 2 does not know that Rebel 3 is pivotal, *even though* player 3 is indeed θ-pivotal. Therefore there is no common knowledge of the free-rider problem at the beginning of 1-block.

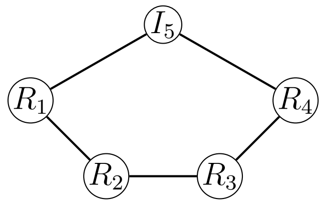

Figure 7: A configuration of the state and the network in which player 1, 2, 3, 4 are Rebels while player 5 is an Inert.

However, the common knowledge of engaging in a free-rider problem is restored when we cut the edge between player 4 and 5; Rebel 2 knows that he is the only θ-pivotal Rebel.

I leave a conjecture in this paper and end this section.

Conjecture 5.1. *For any n-person repeated k-Threshold game with parameter $k < n$ played in any network, if π has full support on strong connectedness, then there exists a $\delta^* \in (0,1)$ such that an APEX equilibrium exists whenever $\delta > \delta^*$.*

6 Conclusion

I model a coordination game and illustrate the learning processes generated by strategies in a (weak) sequential equilibrium to answer the question proposed in the beginning: what kind of networks can conduct coordination in a collective action with information barrier. In the equilibrium, players transmit the relevant information by encoding such information by their actions as time goes by. Since there might be an negative expected payoff in coding information, the potential free-rider problems might occur to impede the learning process. My result show that if the network is acyclic, players can always learn the underlying relevant information and conduct the coordination only by actions. In cyclic networks, however, what kinds of equilibrium strategies can lead to learning the relevant information still remains to be answered.

The construction of the communication protocol by actions exploits the assumption of the common knowledge of the network and the finite type space. Since the relevant information has been parametrized as a threshold in the stage game, players can acquire this information by jointly incrementally reporting their own private information period by period. The major punishment to deter deviation is then the joint shifting to play that same action as the stopping to update information. The threshold game thus seems a potential model in proofing that a communication protocol by actions not only leads a learning process but also constitutes an equilibrium to reveal the relevant information in finite time.

Existing literatures in political science and sociology have recognized the importance of social network in influencing individual's behavior in participating social movements ([Passy, 2003][McAdam, 2003][Siegel, 2009]). This paper views networks as routes for communication in which rational individuals initially have local information but they can influence nearby individuals by taking actions. Such influence may take long time to travel across individuals and the whole process incurs inefficient outcomes in many periods. A characterization in the speed of information transmission across a network is not answered here, although it is an important topic in investigating the most efficient way to let the information be spread. This question would remain for the future research.

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A Appendix

A.1 The APEX equilibrium for Theorem 2

A.1.1 Equilibrium path

By definition of information hierarchy,

$$\begin{aligned} I_i^t &= \bigcup_{k_1 \in G_i} \bigcup_{k_2 \in G_{k_0}} \cdots \bigcup_{k_t \in G_{k_{t-1}}} I_{k_t}^1 \\ &= \{j \in [R](\theta) : \exists \text{ a path } (i, k_1 \ldots k_l, j) \text{ s.t. } 0 \le l \le t-1 \text{ and } \theta_i = \theta_{k_1} = \ldots = \theta_{k_l} = R\} \end{aligned}$$

Let us define several notions.

Definition A.1 (Extended tree by I_i^t)**.**

$$\begin{aligned} X_i^t \;\equiv\; & \{j \in N : \\ & \exists \textit{ a path } (i, k_1 \ldots k_l, k_{l+1}) \textit{ s.t. } k_{l+1} = j,\ l \ge t-1,\ \{i, k_1, \ldots, k_t\} \subset I_i^t\} \cup I_i^t \end{aligned}$$

X_i^t is the set of all possible Rebels in G given information I_i^t.

Definition A.2 (The tree rooted in i and spanning in the direction toward j)**.**

$$TR_{ij} \equiv \{v \in N : \textit{there is a path from } i \textit{ to } v \textit{ through } j,\ j \in G_i\} \cup \{i, j\}$$

Definition A.3 (Extended vertices outside I_i^t in TR_{ij})**.**

$$Y_{ij}^t \equiv TR_{ij} \cap (X_i^t \setminus I_i^t)$$

Definition A.4 (i's capable neighbors by I_i^t)**.**

$$D_i^t \equiv \{j \in G_i : Y_{ij}^t \neq \emptyset\}$$

Definition A.5 (Finite register machine)**.** *A finite register machine for i consists of finite registers Σ. A register is a tuple*

$$(\Omega, \times_{G_i} A_R, f, \lambda),$$

in which Ω are sets of events induced by H_i. $\times_{G_i} A_R$ is the sets of input. $f : \Omega \to A_R$ assigns an action to each event. $\lambda : \Omega \times \times_{G_i} A_R \to \Sigma$ is the transition function. There is a set of initial registers.

i's register specifies i's action according his information at a certain period but does not characterize i's information transition. The register machine here is more like the *switch function* instead of the finite automata. The information of i up to period s is $P_i(\theta) \times \{h^s_{G_i}\} \times H^s_{N \setminus G_i}$ characterized in Section 2.

Definition A.6 (m-register in t-block)**.** *A m-register in a (sub)block or a division is the register in the m-th period in that (sub)block or division.*

To shorten the notation, denote $m \dashv \Gamma$ as the m-register in the (sub)block or division Γ.

Definition A.7 (Terminal **r**)**.** *The terminal **r** is a register such that the image of f is $\{$**revolt**$\}$ and the image of λ is a singleton containing itself.*

Definition A.8 (Terminal **s**)**.** *The terminal **s** is a register such that the image of f is $\{$**stay**$\}$ and the image of λ is a singleton containing itself.*

The equilibrium will be represented as a finite register machine. Moreover, though players act as if acting a whole sequence, they in fact act period by period. For convenience, for any finite sequence of action $\langle\rangle$, denote $\langle\rangle_m$ as the m-th (counting from the beginning) component in $\langle\rangle$, and denote $\langle\rangle(m)$ as the prefix of $\langle\rangle$ with length m. Let us also shorten action **revolt** to be **r** and **stay** to be **s**.

Initial registers The initial register for each Rebel is $1 \dashv \mathcal{C}^1_1$, which is defined in the next section.

Registers in $\mathcal{C}^1$

Table 8: The $m \dashv \mathcal{C}_1^1$ on the path

$1 \le m \le |\mathcal{C}_1^1| - 1$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$\#X_i^1 < k$	$\langle \textbf{all stay} \rangle_m$		terminal $\mathbf{s}$
$i \notin R^1$, $\#X_i^1 \ge k$, $I_i^1 < k$	$\langle \textbf{all stay} \rangle_m$		$m+1 \dashv \mathcal{C}_1^1$
$i \in R^1$, $\#X_i^1 \ge k$, $I_i^1 < k$	$\langle i \rangle_m$		$m+1 \dashv \mathcal{C}_1^1$
$i \in R^1$, $\#X_i^1 \ge k$, $I_i^1 \ge k$	$\langle i \rangle_m$		$m+1 \dashv \mathcal{C}_1^1$

$m = |\mathcal{C}_1^1|$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$\#X_i^1 < k$		$\langle \textbf{all stay} \rangle_m$		terminal $\mathbf{s}$
$i \notin R^1$, $\#X_i^1 \ge k$, $I_i^1 < k$	all j play $\langle \textbf{all stay} \rangle(m-1)$	$\langle \textbf{all stay} \rangle_m$	all j play $\mathbf{s}$	terminal $\mathbf{s}$
$i \notin R^1$, $\#X_i^1 \ge k$, $I_i^1 < k$	$\exists j$ plays $\langle j \rangle(m-1)$	$\langle \textbf{all stay} \rangle_m$	such j plays $\langle j \rangle_m$	$1 \dashv \mathcal{C}_2^1$
$i \in R^1$, $\#X_i^1 \ge k$, $I_i^1 < k$		$\langle i \rangle_m$		$1 \dashv \mathcal{C}_2^1$
$i \in R^1$, $\#X_i^1 \ge k$, $I_i^1 \ge k$		$\langle i \rangle_m$		$1 \dashv \mathcal{C}_2^1$

Table 9: The $m \dashv \mathcal{C}_2^1$ on the path

$1 \leq m < |\mathcal{C}_2^1|$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$i \notin R^1$		$\langle \textbf{all stay} \rangle_m$		$m+1 \dashv \mathcal{C}_2^1$
$i \in R^1$, $I_i^1 < k$	$\exists j \in G_i$, j plays $< j >_j = s$	$\langle \textbf{all stay} \rangle_m$		$m+1 \dashv \mathcal{C}_2^1$
$i \in R^1$, $I_i^1 < k$	$\forall j \in G_i$, j plays $< j >_j = r$	$\langle i \rangle_m$		$m+1 \dashv \mathcal{C}_2^1$
$i \in R^1$, $I_i^1 \geq k$		$\langle \textbf{all stay} \rangle_m$		$m+1 \dashv \mathcal{C}_2^1$

$m = |\mathcal{C}_2^1|$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$i \notin R^1$	$\forall j \in G_i$, j plays $\langle j \rangle(m-1)$	$\langle \textbf{all stay} \rangle_m$	$\forall j \in G_i$, j plays $\langle j \rangle_m$	$1 \dashv \mathcal{C}_3^1$
$i \notin R^1$	$\exists j \in G_i$, j plays $\langle \textbf{all stay} \rangle(m-1)$	$\langle \textbf{all stay} \rangle_m$	such j plays $\langle \textbf{all stay} \rangle_m$	terminal $\mathbf{r}$
$i \in R^1$, $I_i^1 < k$	$\forall j \in G_i$, j play $\langle j \rangle(m-1)$	$\langle i \rangle_m$	$\forall j \in G_i$, j plays $\langle j \rangle_m$	$1 \dashv \mathcal{C}_3^1$
$i \in R^1$, $I_i^1 < k$	$\exists j \in G_i$, j plays $\langle \textbf{all stay} \rangle(m-1)$	$\langle i \rangle_m$	such j plays $\langle \textbf{all stay} \rangle_m$	terminal $\mathbf{r}$
$i \in R^t$, $I_i^1 \geq k$		$\langle \textbf{all stay} \rangle_m$		terminal $\mathbf{r}$

Table 10: The $m \dashv \mathcal{C}_3^1$ on the path

$1 \leq m < |\mathcal{C}_3^1|$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
	$\mathbf{s}$	$\forall j$ play $\mathbf{s}$	$m+1 \dashv \mathcal{C}_3^1$
	$\mathbf{s}$	$\exists j$ play $\mathbf{r}$	terminal $\mathbf{r}$

$1 \leq m = |\mathcal{C}_3^1|$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
	$\mathbf{s}$	$\forall j$ play $\mathbf{s}$	$1 \dashv \mathcal{O}^1$
	$\mathbf{s}$	$\exists j$ play $\mathbf{r}$	terminal $\mathbf{r}$

Registers in $\mathcal{O}^t$ Let $m_i = |\mathcal{O}^t| - x_{I_i^t}$ be the period in which i report I_i^t. I.e. m_i is the period where $\mathbf{r}$ occurs in $\langle I_i^t \rangle$. Denote $G_i(m) = \{j \in G_i : m_j < m\}$. Define $I_i^{t+1}(m) \equiv I_i^t \cup \bigcup_{j \in G_i(m)} I_j^t$ to be the information of i up to the m-th period in $\mathcal{O}^t$. Define $X_i^{t+1}(m)$ to be the extended tree from $I_i^t(m)$ in the same way as that in Definition A.1, and define $Y_{ij}^t(m)$ and $D_i^t(m)$ accordingly.

Table 11: The $m \dashv \mathcal{O}^t$ on the path, where $1 \leq m < |\mathcal{O}^t|$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$i \notin R^t$		$\langle$**all stay**$\rangle_m$		$m+1 \dashv \mathcal{O}^t$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I_i^{t+1}(m) < k$, $X_i^{t+1}(m) < k$	$\langle$**all stay**$\rangle_m$		terminal **s**
$i \in R^t$, not free rider, not $k-1$-pivotal	$I_i^{t+1}(m) < k$, $X_i^{t+1}(m) \geq k$	$\langle I_i^t \rangle_m$		$m+1 \dashv \mathcal{O}^t$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I_i^{t+1}(m) \geq k$, $X_i^{t+1}(m) \geq k$	$\langle 1 \rangle_m$		$m+1 \dashv \mathcal{O}^t$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I_i^{t+1}(m) \geq k-1$, $X_i^{t+1}(m) \geq k$, $D_i^t = 1$	$\langle 1 \rangle_m$		$m+1 \dashv \mathcal{O}^t$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I_i^{t+1}(m) \geq k-1$, $X_i^{t+1}(m) \geq k$, $D_i^t > 1$	$\langle I_i^t \rangle_m$		$m+1 \dashv \mathcal{O}^t$
$i \in R^t$, the free rider	$X_i^{t+1}(m) \geq k$	$\langle 1 \rangle_m$		$m+1 \dashv \mathcal{O}^t$
$i \in R^t$, the free rider	$X_i^{t+1}(m) < k$	$\langle$**all stay**$\rangle_m$		terminal **s**
$i \in R^t$, i is $k-1$-pivotal	$X_i^{t+1}(m) \geq k$	$\langle 1 \rangle_m$		$m+1 \dashv \mathcal{O}^t$
$i \in R^t$, i is $k-1$-pivotal	$X_i^{t+1}(m) < k$	$\langle$**all stay**$\rangle_m$		terminal **s**

Table 12: The $m \dashv \mathcal{O}^t$ on the path, where $m = |\mathcal{O}^t|$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$i \notin R^t$		$\langle \textbf{all stay} \rangle_m$		$1 \dashv \mathcal{C}^t_{1,1}$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I_i^{t+1}(m) < k$, $X_i^{t+1}(m) < k$	$\langle \textbf{all stay} \rangle_m$		terminal $\mathbf{s}$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I_i^{t+1}(m) < k-1$, $X_i^{t+1}(m) \geq k$	$\langle I_i^t \rangle_m$		$1 \dashv \mathcal{C}^t_{1,1}$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I_i^{t+1}(m) \geq k$, $X_i^{t+1}(m) \geq k$	$\langle 1 \rangle_m$		$1 \dashv \mathcal{C}^t_{1,1}$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I_i^{t+1}(m) \geq k-1$, $X_i^{t+1}(m) \geq k$, $D_i^t = 1$	$\langle 1 \rangle_m$		$1 \dashv \mathcal{C}^t_{1,1}$
$i \in R^t$, not free rider, not $k-1$-pivotal	$I_i^{t+1}(m) \geq k-1$, $X_i^{t+1}(m) \geq k$, $D_i^t > 1$	$\langle I_i^t \rangle_m$		$1 \dashv \mathcal{C}^t_{1,1}$
$i \in R^t$, the free rider	$X_i^{t+1}(m) \geq k$	$\langle 1 \rangle_m$		$1 \dashv \mathcal{C}^t_{1,1}$
$i \in R^t$, the free rider	$X_i^{t+1}(m) < k$	$\langle \textbf{all stay} \rangle_m$		terminal $\mathbf{s}$
$i \in R^t$, i is $k-1$-pivotal	$X_i^{t+1}(m) \geq k$	$\langle 1 \rangle_m$		$1 \dashv \mathcal{C}^t_{1,1}$
$i \in R^t$, i is $k-1$-pivotal	$X_i^{t+1}(m) < k$	$\langle \textbf{all stay} \rangle_m$		terminal $\mathbf{s}$

Registers in $\mathcal{C}^t$ for $t \geq 2$

Table 13: The $m \dashv \mathcal{C}^t_{1,v}$ for $v = 1, ..., n$ on the path

$1 \leq m < |\mathcal{C}^t_{1,v}|$, where $v = 1, ..., n$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$X_i^{t+1} < k$	$\langle \textbf{all stay} \rangle_m$		terminal $\mathbf{s}$
$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\exists j \in G_i, j = m$ such that $a_j = \mathbf{s}$	terminal $\mathbf{s}$
$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\forall j \in G_i$ such that $a_j = \langle j \rangle_m$	$m + 1 \dashv \mathcal{C}^t_{1,v}$

$m = |\mathcal{C}^t_{1,v}|$, where $v = 1, ..., n - 1$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$X_i^{t+1} < k$	$\langle \textbf{all stay} \rangle_m$		terminal $\mathbf{s}$
$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\exists j \in G_i, j = m$ such that $a_j = \mathbf{s}$	terminal $\mathbf{s}$
$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\forall j \in G_i$ such that $a_j = \langle j \rangle_m$	$1 \dashv \mathcal{C}^t_{1,v+1}$

$1 \leq m < |\mathcal{C}^t_{1,n}|$

$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\exists j \in G_i, j = m$ such that $a_j = \mathbf{s}$	terminal $\mathbf{s}$
$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\forall j \in G_i$ such that $a_j = \langle j \rangle_m$	$m + 1 \dashv \mathcal{C}^t_{1,n}$

$m = |\mathcal{C}^t_{1,n}|$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\exists j \in G_i, j = m$ such that $a_j = \mathbf{s}$	terminal $\mathbf{s}$
$X_i^{t+1} \geq k$	$\langle i \rangle_m$	$\forall j \in G_i$ such that $a_j = \langle j \rangle_m$	$1 \dashv \mathcal{C}^t_{2,1}$

Table 14: The $m \dashv \mathcal{C}^t_{2,v}$ for $v = 1, ..., t+1$ on the path

$1 \leq m < |\mathcal{C}^t_{2,v}|$, where $v = 1, ..., t+1$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$I_i^{t+1} < k$	$\exists j,\ \langle j \rangle_j = \mathbf{s}$	$\langle \textbf{all stay} \rangle_m$		$m+1 \dashv \mathcal{C}^t_{2,v}$
$I_i^{t+1} < k$	$\forall j,\ \langle j \rangle_j = \mathbf{r}$	$\langle i \rangle_m$		$m+1 \dashv \mathcal{C}^t_{2,v}$
$I_i^{t+1} \geq k$		$\langle \textbf{all stay} \rangle_m$		$m+1 \dashv \mathcal{C}^t_{2,v}$

$m = |\mathcal{C}^t_{2,v}|$, where $v = 1, ..., t$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$I_i^{t+1} < k$	$\exists j \in G_i,\ \langle j \rangle_j = \mathbf{s}$	$\langle \textbf{all stay} \rangle_m$		$1 \dashv \mathcal{C}^t_{2,v+1}$
$I_i^{t+1} < k$	$\forall j \in G_i,\ \langle j \rangle_j = \mathbf{r}$	$\langle i \rangle_m$		$1 \dashv \mathcal{C}^t_{2,v+1}$
$I_i^{t+1} \geq k$		$\langle \textbf{all stay} \rangle_m$		$1 \dashv \mathcal{C}^t_{2,v+1}$

$1 \leq m < |\mathcal{C}^t_{2,t+1}|$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$I_i^{t+1} < k$	$\exists j \in G_i,\ \langle j \rangle_j = \mathbf{s}$	$\langle \textbf{all stay} \rangle_m$		$m+1 \dashv \mathcal{C}^t_{2,t+1}$
$I_i^{t+1} < k$	$\forall j \in G_i,\ \langle j \rangle_j = \mathbf{r}$	$\langle i \rangle_m$		$m+1 \dashv \mathcal{C}^t_{2,t+1}$
$I_i^{t+1} \geq k$		$\langle \textbf{all stay} \rangle_m$		$m+1 \dashv \mathcal{C}^t_{2,t+1}$

$m = |\mathcal{C}^t_{2,t+1}|$

ω_i		$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
$I_i^{t+1} < k$	$\exists j \in G_i,\ \langle j \rangle_j = \mathbf{s}$	$\langle \textbf{all stay} \rangle_m$		terminal **r**
$I_i^{t+1} < k$	$\forall j \in G_i,\ \langle j \rangle_j = \mathbf{r}$	$\langle i \rangle_m$		$1 \dashv \mathcal{C}^t_{3,1}$
$I_i^{t+1} \geq k$		$\langle \textbf{all stay} \rangle_m$		terminal **r**

Table 15: The $m \dashv \mathcal{C}_3^t$ on the path

$1 \leq m < |\mathcal{C}_3^t|$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
	$\mathbf{s}$	$\forall j \in G_i$, j plays $\mathbf{s}$	$m+1 \dashv \mathcal{C}_3^t$
	$\mathbf{s}$	$\exists j \in G_i$, j plays $\mathbf{r}$	terminal $\mathbf{r}$

$m = |\mathcal{C}_3^t|$

ω_i	$f(\omega_i)$	a_{G_i}	$\lambda(\omega_i, a_{G_i})$
	$\mathbf{s}$	$\forall j \in G_i$, j plays $\mathbf{s}$	$1 \dashv \mathcal{O}^{t+1}$
	$\mathbf{s}$	$\exists j \in G_i$, j play $\mathbf{r}$	terminal $\mathbf{r}$

A.2 Missing proofs

Proof of Lemma 2.1

Proof. The proof is done by contraposition. Suppose Rebels' strategies constitute an APEX equilibrium. By definition of the APEX equilibrium, at every θ, there is a period T^{θ} when all Rebels' actions start to repeat themselves. Let $T = \max_{\theta \in \Theta} T^{\theta}$. For Rebel i, let $T_i = T + 1$, and suppose $0 < \sum_{\theta : \#[R](\theta) \geq k} \beta_{G_i}^{\pi,\tau^*}(\theta | h_{G_i}^s) < 1$ for some $s \geq T_i$. Then this Rebel assigns positive weight at some $\theta' \in \{\theta : \#[R](\theta) < k\}$ and some positive weight at some $\theta'' \in \{\theta : \#[R](\theta) \geq k\}$ at period s. Note that i has already known θ_j if $j \in G_i$, and therefore i assigns positive weight at some $\theta' \in \{\theta : \#[R](\theta) < k, \theta_l = R, l \notin G_i\}$ and positive weight at some $\theta'' \in \{\theta : \#[R](\theta) < k, \theta_l = I, l \notin G_i\}$. Since all Rebels' actions start to repeat themselves at period T, i cannot update information afterwards. Suppose i's continuation strategy is to continuously play **revolt**, then this is not ex-post efficient when $\#[R](\theta) < k$; suppose i's continuation strategy is to continuously play **stay**, then this is not ex-post efficient when $\#[R](\theta) \geq k$. $\square$

Proof of Theorem 1

Proof. Let τ^* be the following strategy. After the nature moves, a Rebel i plays **revolt** if he has no Inert neighbor; i plays **stay** forever if he has an Inert neighbor. After the first period, if i has not detected a deviation and observes that all his Rebel neighbors play **revolt** continuously previously, he plays **revolt** in the current period; otherwise, he plays **stay** afterwards and forever. If a Rebel j deviates, then j plays **stay** afterwards and forever.

At period s, according to τ^*, if i has not detected a deviation, but he observe his Rebel neighbors plays **stay** in the current period, he forms the belief of

$$\sum_{\theta : \#[R](\theta) \geq k} \beta_{G_i}^{\pi,\tau^*}(\theta | h_{G_i}^s) = 0$$

afterwards and forever. Therefore, he plays **stay** afterwards and forever as his best response.

At period s, if a Rebel detects a deviation, or he has deviated, to play **stay** afterwards and forever is his best response since at least one player will play **stay** afterwards and forever.

Since the network is finite with n vertices, if all players do not deviate, after period n, each Rebel plays **revolt** and gets payoff 1 forever if $\theta \in \{\theta : \#[R](\theta) \geq k\}$; each Rebels plays **stay** and gets payoff 0 forever if $\theta \in \{\theta : \#[R](\theta) < k\}$. However, a Rebel who has deviated surely gets payoff 0 forever after period n. Therefore, there is a $0 < \delta < 1$ large enough to impede Rebels to deviate.

To check if τ^* and $\{\beta_{G_i}^{\pi,\tau^*}(\theta|h_{G_i}^s)\}_{i\in N}$ satisfy full consistency[16], take any $0 < x < 1$ such that Rebels play τ^* with probability $1-x$ and play other behavior strategies with probability x. Clearly, when $x \to 0$, the belief converges to $\{\beta_{G_i}^{\pi,\tau^*}(\theta|h_{G_i}^s)\}_{i\in N}$. Since the out-of-path strategy is the best response for both of the Rebel who detects deviation and the Rebel who makes deviation, for arbitrary beliefs they hold, τ^* is a sequential equilibrium. □

Proof of Lemma 4.1

Proof. I show that if $i \notin R^{t-1}$ then $i \notin R^t$ for all t.

By definition,

$$\begin{aligned} G_i^t &= \bigcup_{k_1\in G_i} \bigcup_{k_2\in G_{k_0}} \cdots \bigcup_{k_t\in G_{k_{t-1}}} G_{k_t}^1 \\ &= \{j \in N : \exists \text{ a path } (i, k_1 \ldots k_l, j) \text{ such that } l \leq t-1 \text{ and } \theta_i = \theta_{k_1} = \ldots = \theta_{k_l} = R\}, \end{aligned}$$

while

$$\begin{aligned} I_i^t &= \bigcup_{k_1\in G_i} \bigcup_{k_2\in G_{k_0}} \cdots \bigcup_{k_t\in G_{k_{t-1}}} I_{k_t}^1 \\ &= \{j \in [R](\theta) : \exists \text{ a path } (i, k_1 \ldots k_l, j) \text{ such that } l \leq t-1 \text{ and } \theta_i = \theta_{k_1} = \ldots = \theta_{k_l} = R\}. \end{aligned}$$

The above equality says that, at $t = \dot{t}$, if $i \notin R^{\dot{t}}$, then there is a j such that the Rebels, who can be reached by $\dot{t}$ consecutive edges from i, can be also reached by $\dot{t}$ consecutive edges from j. Therefore, if there are new Rebels who can be reached from i at any $\ddot{t} > \dot{t}$

[16]Krep and Wilson (1982)

by $\ddot{t}$ consecutive edges, those new ones can be also be reached by $\ddot{t}$ consecutive edges by j. Hence, $i \notin R^{\ddot{t}}$. $\square$

Proof of Theorem 3

Proof. Since the network is finite, θ has strong connectedness, and $[R](\theta) \neq \emptyset$, there is a minimum t_i such that $I_i^{t_i} = [R](\theta)$ for each i by the definition of I_i^t. Let $P = \arg\min_{i \in N}\{t_1, ..., t_n\}$ with generic element p. Therefore $I_p^{t_p} = [R](\theta)$. I show that $p \in R^{t_p - 1}$ to complete the proof. I prove it by contradiction. If $p \notin R^{t_p-1}$, then $I_p^{t_p-1} \subseteq G_j^{t_p-1}$ for some $j \in G_p$. Then, all the Rebels in TR_{jp} are in $G_j^{t_p-1}$, but there exist Rebels in TR_{pj} who are in $G_j^{t_p-1}$ but not in $I_p^{t_p-1}$. This is because the network is acyclic and $I_p^{t_p-1} \subset [R](\theta)$. But then $p \notin P$ since $I_j^{t_j-1} = [R](\theta)$ already. I then conclude that $p \in R^{t_p-1}$.

$\square$

Proof of Lemma 4.2

Proof. The proof is by contradiction. Suppose that, at t-block and before T^θ, there are three or more θ-pivotal Rebels. Since θ has strong connectedness, there are three of them, p_1, p_2, p_3, with the property $p_1 \in G_{p_2}$ and $p_2 \in G_{p_3}$.

Since the network is acyclic, $p_1 \notin G_{p_3}$ and $p_3 \notin G_{p_1}$. Since p_1 is θ-pivotal, $I^t \subset [R](\theta)$ and $I_p^{t+1} = [R](\theta)$. It implies that, in $TR_{p_1p_2}$, p_1 can reach all Rebels by $t+1$ edges, but cannot reach all of them by t edges. The same situation applies to p_3. However, it means that p_2 can reach all Rebels in $TR_{p_1p_1}$ by t edges and reach all Rebels in $TR_{p_1p_1}$ by t edges, and hence $I_{p_2}^t = [R](\theta)$. It contradict to the definition of θ-pivotal Rebel.

$\square$

Proof of Lemma 4.3

Proof. A θ-pivotal p knows that $p^{'} \in G_i$ if $p^{'}$ is another one. p picks a neighbor p' and checks whether or not $[R](\theta) \subseteq I_p^t \cup I_{p'}^t$ for all possible $I_{p'}^t$. By common knowledge of the network, p knows $G_{p'}^t$. Since p is θ-pivotal, he is certain that all the Rebel in the direction from p toward $p^{'}$ is in $G_{p^{'}}^t$ and hence in $I_{p^{'}}^t$. Then p can check whether or not $[R](\theta) \subseteq I_p^t \cup I_{p^{'}}^t$ for all possible $I^t_{p'}$. If so, then p knows $p^{'}$ is also θ-pivotal by the definition of θ-pivotal. Similarly, a θ-pivotal $p^{'}$ can do the same procedure. Therefore, if there are two θ-pivotal p and $p^{'}$, they commonly know that they are θ-pivotal. They commonly know this at the beginning of t-block since they know I^t_p and $I^t_{p'}$ by the construction of information hierarchy. $\square$

Proof of Theorem 2 I begin with the following lemmas stating that all Rebels eventually learn the relevant information on the path.

Lemma A.1. *If the network is acyclic and if the θ has strong connectedness, then the equilibrium path specified in Section A.1.1 is an APEX strategy.*

Proof. Firstly, suppose θ satisfies $\#[R](\theta) < k$. I show that, all Rebels will enter terminal **s** eventually without entering terminal **r**. Let p be the Rebel defined in the proof of Theorem 3 so that $I^{t_p}_p = [R](\theta)$, where p is one of the earliest Rebels who knows $\#[R](\theta) < k$. I claim that $\#X^{t_p}_p < k$ if and only if $I^{t_p}_p = [R](\theta)$. For the only if part, the proof is by way of contradiction. If not, by the full support on strong connectedness, there is a possible Rebel outside $I^{t_p}_p$, and therefore p is uncertain $\#[R](\theta) < k$. For the if part, note that $I^{t_p}_p \subset X^{t_p}_p$ and therefore $\#I^{t_p}_p < \#X^{t_p}_p < k$. I also claim that $\#X^{t_p}_p(m) < k$ if and only $I^{t_p}_p(m) = [R](\theta)$. The proof is exactly the same as the noted above by replacing $I^{t_p}_p$ to $I^{t_p}_p$ and $I^{t_p}_p(m)$ to $I^{t_p}_P(m)$

Referring to Table 8 to Table 12, whenever there is a p so that $\#X^{t_p}_p < k$, p plays **stay** forever. It implies that all Rebels enter terminal **s** right after $\mathcal{C}^t_1, t \geq 0$. Notice that Rebels entering to terminal **r** only after some period after $\mathcal{C}^t_1$ and therefore all Rebels will enter terminal **s** before terminal **r**.

Secondly, suppose θ satisfies $\#[R](\theta) \geq k$. I show that all Rebels will enter terminal **r** eventually. Note first that if there is a Rebel p so that $\#I^1_p \geq k$, all Rebels enter terminal **r** after $\mathcal{C}^1_3$ by referring to Table 8, Table 9, and Table 10. At $t > 0$, if there is a Rebel p who has play $\langle 1 \rangle$ at $\mathcal{O}^t$, by the postulate of $\#[R](\theta) \geq k$, after $\mathcal{C}^t_3$, all Rebels enter terminal **r** according to the equilibrium path specified in Table 11, Table 12, Table 13, Table 14, and Table 15. There must be some Rebel $p \in R^t$ who plays $\langle 1 \rangle$ at $\mathcal{O}^t$ for some t by the same argument in the proof of Theorem 3. $\square$

Due to Lemma A.1, define $T^{\theta}_{\tau^*}$ as the earliest period at which all Rebels play ex-post efficient outcome afterwards according to an APEX equilibrium τ^*. For simplicity, I suppress the notation $\beta^{\pi,\tau}_{G_i}(\theta|h^s_{G_i})$ to $\beta^{\tau}_{G_i}(\theta|h^s_{G_i})$ and the notation $\alpha^{\pi,\tau}_{G_i}(\theta, h^s|\theta_{G_i}, h^s_{G_i})$ to $\alpha^{\tau}_{G_i}(\theta, h^s|h^s_{G_i})$. If $P(\theta)$ is a property of θ, define

$$\beta^{\tau}_{G_i}(P(\theta)|h^s_{G_i}) \equiv \sum_{\theta \in \{\theta : P(\theta)\}} \beta^{\tau}_{G_i}(\theta|h^s_{G_i}).$$

Furthermore, if m, s are periods and $m > s$, denote $h^{m|s}$ as a history in H^m so that $(h^s, h^{m|s}) \in H^m$. Denote $\tau'|^s_{\tau}$ as a strategy following τ til period s.

Claim 1. *Suppose Rebel i follows an APEX equilibrium τ^* til period s. If there is a strategy $(\tau_i, \tau_{-i})|^s_{\tau^*}$ generating a history $h^{m|s}, \infty > m > s$ so that i will be uncertain about the relevant information and stop belief updating after m, then Rebel i will not deviate to $\tau|^s_{\tau^*}$ if $\delta \in (0, 1)$ is sufficiently high.*

Proof. Denote $\beta^{\tau|^s_{\tau^*}}_{G_i}(\theta|h^{m|s}, h^s_{G_i})$ as i's belief about θ at m following $h^{m|s}$ induced by $\tau|^s_{\tau^*}$. By the postulate, $0 < \beta^{\tau|^s_{\tau^*}}_{G_i}(\#[R](\theta)|h^{m|s}, h^s_{G_i}) < 1$. From the perspective that i holds a belief of $\beta^{\tau^*}_{G_i}(\#[R](\theta) \geq k|h^s_{G_i})$ at period s, $h^{m|s}$ can be thought of an imperfect signal at period m to infer whether or not $\#[R](\theta) \geq k$: if $\#[R](\theta) \geq k$, $h^{m|s}$ occurs with probability η and does not occur with probability $1 - \eta$; if $\#[R](\theta) < k$, $h^{m|s}$ occurs with probability μ and does not occur with probability $1 - \mu$ so that $0 \leq \eta \leq 1$, $0 \leq \mu \leq 1$, and $0 < \eta/\mu < \infty$. Denote $M = \max\{m, T^{\theta}_{\tau^*}\}$. Rebel i's maximum expected stage-game payoff starting from M by following $h^{m|s}$ calculated at period s is

$$V = \max\{\eta\beta^{\tau^*}_{G_i}(\#[R](\theta) \geq k|h^s_{G_i}) - \mu\beta^{\tau^*}_{G_i}(\#[R](\theta) < k|h^s_{G_i}), 0\}.$$

The first term $\eta\beta^{\tau^*}_{G_i}(\#[R](\theta) \geq k|h^s_{G_i}) - \mu\beta^{\tau^*}_{G_i}(\#[R](\theta) < k|h^s_{G_i})$ is i's expected stage-game payoff if all Rebels play **revolt** afterwards starting from M. The second term 0 is the one by playing **stay** afterwards. Rebel i's expected stage-game payoff starting from M by following τ^* calculated at period s is

$$\beta^{\tau^*}_{G_i}(\#[R](\theta) \geq k|h^s_{G_i}) > V.$$

The inequality above is due to $0 < \eta < 1, 0 < \mu < 1$. There is a difference in present value of

$$W(\delta) = \frac{\delta^{M-s}(\beta_{G_i}^{\tau^*}(\#[R](\theta) \geq k | h_{G_i}^s) - V)}{1-\delta}.$$

Denote L as the summation of all gains from deviation calculated from period s to period M. L is finite since the stage-game payoff is finite and $M - s$ is finite. Taking sufficiently high $\delta \in (0,1)$ so that $W(\delta) > L$ will deter this deviation. $\square$

Claim 2. *Suppose Rebel i follows an APEX equilibrium τ^* til period s. If i deviates to a strategy $(\tau_i, \tau_{-i})|_{\tau^*}^s$ so that there are $d > 0$ Rebels detects this deviation, then Rebel i will not deviate to $\tau|_{\tau^*}^s$ if $\delta \in (0,1)$ is sufficiently high.*

Proof. First suppose $I_i^s < k$. If $\tau|_{\tau^*}^s$ leads to a strategy so that i is uncertain how many Rebels are, i will not deviate by Claim 1. If $\tau|_{\tau^*}^s$ leads to a strategy so that i is certain that there are k' Rebels, $k \leq k' < k + d$, play some action forever after some period $m_{k'}$, i's stage-game payoff after m_d is at most 0. If Rebel i follows τ^*, i's stage-game payoff is 1 after T^θ. Therefore after $M = \max\{T^\theta, m_d\}$, there is a difference in present value of $1/(1-\delta)$ at M. Since $I_i^s < k$, $0 < \beta_{G_i}^{\tau^*}(\#[R](\theta) \geq k | h_{G_i}^s) < 1$, and therefore $\eta = \beta_{G_i}^{\tau^*}(k \leq \#[R](\theta) < k + d | h_{G_i}^s)$ is positive. Hence there is a difference in present value of $\eta\delta^{M-s}/(1-\delta)$. As for the case $\tau|_{\tau^*}^s$ leads to a strategy so that i is certain that there are k' Rebels, $k' \geq k + d$ or $k' < k$, play **revolt** forever or play some action forever respectively after some period, i's stage-game payoff after $M = \max\{T^\theta, m_d\}$ is the same as i follows τ^*. Denote L as the summation of all gains from deviation calculated from period s to period M. L is finite since the stage-game payoff is finite and $M - s$ is finite. Taking sufficiently high $\delta \in (0,1)$ so that $\eta\delta^{M-s}/(1-\delta)$ will deter this deviation.

Next suppose $I_i^s \geq k$. $\square$

If Rebel i makes detectable deviation. $I_i^t \geq k$. In $\mathcal{O}^t$, playing $\langle I \rangle$ for some $I \subset N$ incurs more negative expected payoff than does $\langle 1 \rangle$. In $\mathcal{C}_1^t$ after $\mathcal{O}^t$, detectable deviation is the sequence other than $\langle i \rangle$ or $\langle$**all stay**$\rangle$. deviation will be detected by all neighbors, all neighbors play **s** contagiously, the expected continuation payoff after some period S is zero. On the path, the one is $1/(1-\delta)$. In K_2^t, detectable deviation is the sequence other than $\langle i \rangle$

or $\langle \textbf{all stay} \rangle$. deviation will be detected by all neighbors, all neighbors play $\mathbf{s}$ contagiously, the expected continuation payoff after some period S is zero. In K_1^0, detectable deviation is the sequence other than $\langle i \rangle$ or $\langle \textbf{all stay} \rangle$. deviation will be detected by all neighbors, all neighbors play $\mathbf{s}$ contagiously, the expected continuation payoff after some period S is zero.

If $I_i^t < k$. suppose there are $m > 0$ Rebel neighbors detects this deviation. The expected continuation payoff is at most $\beta_{G_i}(\#[R](\theta) \geq k + m)|h_{G_i}^s)$