Coordination in Social Networks: Communication by Actions

Chun-Ting Chen

July 30, 2019

Abstract

This paper studies a collective action problem in a setting of discounted repeated coordination games in which players know their neighbors' inclination to participate as well as monitor their neighbors' past actions. I define *strong connectedness* to characterize those states in which, for every two players who incline to participate, there is a path consisting of players with the same inclination to connect them. Given that the networks are fixed, finite, connected, commonly known, undirected and without cycles, I show that if the priors have full support on the strong connectedness states, there is a (weak) sequential equilibrium in which the ex-post efficient outcome repeats after a finite time T in the path when discount factor is sufficiently high. This equilibrium is constructive and does not depend on public or private signals other than players' actions.

1 Introduction

This paper studies collective actions in a setting of discounted repeated coordination games, where information and monitoring structures are modeled as networks. Players are uncertain about the states of nature but can observe their neighbors' actions. I would like to explore what kinds of networks can induce players to solve the underlying uncertainty in order to coordinate with the ex-post efficient outcome. Though the motive of this study is to understand the dynamic of social movements, a general interest centers on the collective action behaviors within social structures.

Consider pro-democracy movements. Strong discontents overthrowing a regime may exist, but it is difficult to organized around these discontents because information about the existence of such discontents is not always transparent. For instance, in East Germany, the

government had control over the electoral system and the mass media, and the eavesdropping by secret agents had impeded people from showing their discontents. As [Karl-Dieter and Christiane, 1993] or [Chwe, 2000] have suggested, such discontents may be revealed only to someone whom you trust or have an intimate relationship with, but are hardly revealed publicly. This lack of common knowledge about the existence of strong discontent may impede people from conducting a one-shot uprising due to the fear of possible failure (e.g., [Chwe, 2000] in proposing a static model to characterize the networks that provide common knowledge about peoples' discontents). However, an event may trigger a later event (e.g., [Lohmann, 2011] in using informational cascade model to explain consecutive demonstrations in East Germany 1989-1991). When rebels are aware of the scale to transmit relevant information about the level of collective discontent through their actions, they might be willing to act although it might be at risk of facing failure. I view such risky actions as a part of an equilibrium strategy and the entire movement as a learning process.

Inspired by [Chwe, 2000], I model such dynamic collective action in the following way. Players repeatedly play a *k-Threshold game* with a parameter k in a network. There are two types of players located in the network, one we called them *Rebel* and one we called them *Inert*. Players' types and their actions can be observed only by their neighbors. A Rebel has two actions, which are **revolt** or **stay**, while an Inert has only one action, which is **stay**. A Rebel will get payoff as 1 if he chooses **revolt** and more than k players choose **revolt**; he will get payoff as -1 if he chooses **revolt** and less than k players choose **revolt**; he will get payoff as 0 if he chooses **stay**. An Inert will get payoff as 1 if he chooses **stay**.

Since a Rebel may not know how many Rebels exist in this world, Rebels' payoff structure captures the idea that **stay** is a safe arm and **revolt** is a risky arm. Given a common prior π over players' types, players play this k-Threshold game infinitely repeatedly with a common discount factor δ. Cheap talk is not allowed, no outside mechanism serves as an information exchange device.

Rebels then communicate with each other by playing actions. For different k and different network structures, I am looking for a sequential equilibrium which has the property of *approaching ex-post efficient (APEX* henceforth) to investigate the information sharing behavior in the networks. An equilibrium is APEX if and only if the tails of actions in the equilibrium path repeats the static ex-post efficient outcome after a finite time T. This refinement serves to check if players have already learned the relevant information in the equilibrium path. If there are at least k Rebels in this society, then *all* Rebels should **revolt** after T as if they have known that more than k Rebels exist; otherwise, *all* Rebels should **stay** after T. The Rebels' incentives to communicate are affected by their positions in networks since networks are structuring the information and monitoring structure.

In order to get a quick intuition about Rebel's learning process in the proposed framework, consider the k-Threshold game with $k = n$ and assume that payoff is hidden. When $k = n$, a Rebel can get positive payoff only if all the players are Rebels. Given that the networks are fixed, finite, connected, commonly known, and undirected (*networks* henceforth), an APEX sequential equilibrium can be constructed by a contagion-like argument. This argument is to treat **stay** as the message of "there is an Inert out there"; and treat **revolt** as the message of "there could be no Inert out there ". If a Rebel has an Inert neighbor, then he plays **stay** forever. If he has no Inert neighbors, then he plays **revolt** until he observes that some of his neighbors play **stay**, and then he shifts to play **stay** forever. Since the networks are finite, within finite periods, a Rebel will learn that there is an Inert out there if some neighbors have played **stay** and learn that there is no an Inert out there otherwise.

The non-trivial cases appear when $k < n$. The $k = n$ case is easier because the underlying relevant information is to tell "Is there an Inert out there?". I can construct equilibrium when $k = n$ by using single-period binary actions, $\{\textbf{stay}, \textbf{revolt}\}$, to separate the states into two parts, "no Inerts" or "some Inerts". In other words, these single-period actions can generate distinguishable distribution of signals to inform players in telling the true states of nature[1]. However, when $k < n$, the relevant information is to tell "Are there at least k Rebels out there?", and thus these binary actions have to carry more information to reveal the states. As I will show later, several sequences of actions will be used to transmit Rebels' private informations and to control Rebels' beliefs in equilibrium. In the equilibrium path, two kinds of sequence will be used. The first kind, *reporting messages*, is to report their private information about the states of nature; the second one, *coordination messages*, is to inform Rebels about whether some other Rebels have known the relevant information. Specifically, in the equilibrium path, Rebels will play the coordination message to inform other Rebels whenever they have known the relevant information, and those other Rebels will play the same message again to inform other Rebels. The coordination message means to serve as a short-cut to track individuals' higher-order beliefs about "Have some Rebels known the relevant information?", "Have some Rebels known some Rebels have known the relevant information?", etc.

Note that communication is costly in the sense that playing **revolt** is risky. Due to being discounting, Rebels always seek the opportunity to manipulate their messages to save their costs in the time horizontal line[2]. A free-rider problem may occur when reporting information incurs costs. I give an example here to illustrate this issue. Consider a situation

---

[1]e.g., [Fudenberg and Yamamoto, 2010] or [Fudenberg and Yamamoto, 2011].

[2]Indeed, allowing cheap talk or using limit-of-mean preference (e.g., [Renault and Tomala, 1998]) will solve this coordination problem.

where two nearby Rebels exchange information [3]. Suppose that these two Rebels can learn the true state after acquiring information from each other's truthful reporting. Furthermore, we suppose that each of them can freely initiate the coordination after exchanging information. In this instance, truthful reporting is not the best response because a player can wait given that the other will report truthfully. The intuition behind the above scenario is to see the future coordination as a public good. This public good can only be made by Rebels' truthful reporting, which incurs some costs.

The main result will show that this coordination problem can be solved in the acyclic networks . Here, I define a *path* in G is a sequence consisting of nodes without repetition in which a node is a neighbor of a previous node. Then I define an undirected network acyclic G by defining a network in which the path between different nodes is unique. After I define *strong connectedness* as the property that there is always a path consisting of Rebels to connect any pairs of Rebels, the main result shows:

Result 1. (Main Result) *For n-person repeated k-Threshold game with parameter $1 \leq k \leq n$ played in any acyclic network, if π has full support on the strong connectedness, then , there is a δ^* such that a (weak) APEX sequential equilibrium exists whenever $\delta > \delta^*$.*

Here, the assumption that π has full support on strong connectedness means that π assigns positive probability on same states if and only if those states exhibit strong connectedness. This assumption is to make sure that the underlying game is not reduced to an incomplete information game which is without communication. To see this, recall that an Inert always plays **stay**. Rebels cannot communicate with some Rebels by their actions if an Inert happens to separate them. For instance, in a wheel network, an incomplete game without communication is that the central player is an Inert while the peripheral players are all Rebels. It is impossible to find an APEX equilibrium in this instance unless $k = 1$.

The off-path belief serves as a grim trigger as follows. Whenever a Rebel detects a deviation, he believes that all other players outside his neighborhood are Inerts. Thus, if there are less than k Rebels in his neighborhood, he will play **stay** forever. With this off-path belief and the constructed equilibrium strategies, the belief system satisfies *updating consistency*([Perea, 2002]), while it may not satisfy full consistency ([Kreps and Wilson, 1982]). [4]

---

[3]Example 3.3

[4] Updating consistency requires that, for every player, for every player's strategies, for every information sets s^1, s^2 where s^2 follows s^1, if s^2 happens with positive probability given s^1 and given players' strategies contingent on s^1, then the belief over s^2 should satisfy Bayesian updating conditional on the belief over s^1 and players' strategies contingent on s^1. In other words, the updating consistency require that players hold beliefs in every information sets and hold updated beliefs that follows previous beliefs. This requirement

The establishment of an equilibrium construction starts from building a communication protocol. By exploiting the assumption of finite and commonly known network, I assign each node a distinct prime number. Then I let reporting messages carry the information about the multiplication of nodes' prime numbers. Since the multiplication of prime numbers can be defactorized uniquely, the reporting messages thus carry the information about those nodes' locations in a network. Next, I let two phases, reporting period and coordination period, occur in turns in the time horizon, where the reporting (resp. coordination) messages are played in the reporting (resp. coordination) period. In the coordination period, whenever a Rebel tells the relevant information, such Rebel inform his nearby Rebels by sending coordination messages. Those nearby Rebels then continue to inform their nearby Rebels by sending coordination messages, etc. Then, after the coordination period, if a Rebel has received a coordination message, he is certain that all Rebels have commonly known that they can tell the relevant information.

I call a complete two-phases, starting from a reporting period and ending with a following coordination period, a *block*. In a block, I control the inter-temporal incentives in playing between reporting and coordination messages as follows. First, I let both of the coordination messages, one of them can initiate the coordination to **revolt** and another one can initiate the coordination to **stay**, incur no expected cost. Second, I let Rebels play **revolt** after a block only if they have observed the coordination message to **revolt** and observed some reporting messages which incur some expected costs. However, the continuation behavior after observing the coordination message to **stay** is not contingent on any reporting message. When a Rebel looks forward future coordination to **revolt**, he may have the incentive to take a risk to influence Rebels' future behavior forwardly; otherwise, he just plays **stay**. Next, in the equilibrium path, I make sure that Rebels will play ex-post efficient outcome repeatedly right after a block if some Rebels have initiated the coordination in that block. I will argue that only those Rebels who have been able to tell the relevant information after reporting period have the incentive to initiate the coordination. This is because they do not need further evidence to prove that whether the revolution will be successful. This argument is to show that a Rebel other than them will not take advantage to send that free coordination message to initiate the coordination. This is because players cannot update further information if all of their neighbors continue to play the same actions in the future. When δ is high enough, he will not initiate the coordination to impede his own learning process to achieve the ex-post efficient outcome.

I then characterize Rebels' incentive in taking risks and control how much **revolt** they

---

imposes restrictions on off-path beliefs that induce sequential rationality, although it is weaker than full consistency in the sense that full consistency implies updating consistency.

should play to sustain an APEX equilibrium. In the equilibrium path, a Rebel iteratively updates his relevant information given other Rebels' **revolt-stay** finite sequence in reporting their information about the state, and a Rebel takes risks only if his current relevant information has not been acquired by other Rebels. In the equilibrium path, a Rebel thus believes that "more other Rebels are out there" if and only if his nearby Rebels take risks to report their existence. Some specified forms of reporting messages are introduced, and the off-path belief is to enforce Rebels not to play differently from them.

The key step here is to construct a special reporting message which incurs the least expected cost in taking risks, and this message should be considered as a part of the equilibrium path. I denote this special reporting message as $\langle 1 \rangle$. To see its importance, consider the concept of pivotal Rebel. Here, a pivotal Rebel is defined as the Rebel who is sure that he can know the relevant information right after a reporting period given that other Rebels will report their information truthfully. Now suppose playing $\langle 1 \rangle$ is not considered as a part of the equilibrium path, and suppose a Rebel finds himself as a pivotal Rebel during a reporting period while he has not yet reported anything in that period. It could be possible for him to find a profitable deviation by taking less risks (i.e. by playing less **revolt**), which can not be detected by at least k Rebels although some Rebels can indeed detect such deviation. Since those Rebels who detected such deviation will play **stay** forever by the off-path belief, and this pivotal Rebel can initiate the coordination to **revolt** by convincing other Rebels to play **revolt**, the APEX fails. To solve this problem, I introduce message $\langle 1 \rangle$ to let pivotal Rebels identify themselves, while I let coordination messages to **revolt** or to **stay** have to be initiated when $\langle 1 \rangle$ has been played in the equilibrium path to prevent non-pivotal Rebels from mimicking pivotal Rebels.

The major difficulties remaining to solved are the situations where there are multiple pivotal players nearby each other. In such phenomenon, the APEX may fail since playing $\langle 1 \rangle$ does not answer "how many Rebels has a pivotal Rebel known?" although it does address "a pivotal Rebel exists". The assumption of acyclic networks is crucial of solving these problems. If the networks are acyclic, I will show it later that there are only two kinds of pivotal Rebels. One kind is that they have known there are at least $k-1$ Rebels. The other kind is that they will know the true state given other Rebel neighbors' truthful reporting. I call the latter case a free-rider problem. If the networks are acyclic, Lemma 3.1 will show that the free-rider problems only happen between two nearby pivotal Rebels in only one block in the equilibrium path. Further, these two nearby Rebels will know that this free-rider problem will occur before the game entering into this block. The consequence of Lemma 3.1 is that, before the game entering into this block, I can let arbitrary one of them report the information about the state and let the other one play $\langle 1 \rangle$ dependent on

their indexed prime numbers. [5]

This paper contributes to several fields of economics.

First, the future coordination can be viewed as a public good among all Rebels. A strand of public good literature, such as [Lohmann, 1994], is to view information as a public good while generating information is costly[6]. This paper models costly information generation, while adding another aspect, network-monitoring, to investigate a collective action behavior.

Second, this paper is also related to the literature of social learning[7]. Several papers have considered social learning in networks [8]. In this literature, when players are myopic, the information flows could be very complicated because the information they sent can in turns affect their future behaviors. For instance, in [Gale and Kariv, 2003], even for 3-

[5] Such scenario substantially differs from the cyclic counterpart. The free-rider problem becomes intractable in cyclic networks. Let's consider the following example, in which there are 5 players in a cyclic network. Player i is a Rebel and labeled as RB if $i \in \{1, 2, 3, 4, \}$; is an Inert if $i \in \{5\}$.
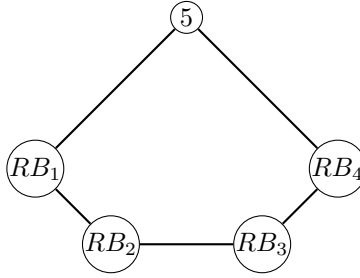
Let's suppose $k = 4$ and assume that, at a certain period t, players know only their immediate neighbors' types. Let's further simplify the scenario and suppose that any Rebel player can exchange information with his neighbors by talking that incurs a fixed cost c at period $t + 1$, as well as freely initiate coordination at $t + 2$. In such a scenario, for *any* set of players, the answer to "Who are the pivotal players before entering $t + 1$?" is not certain at t. Take the set of players $\{1, 2\}$ as an example, and notice that, from the perspective of player 2, the type of player 5 could be Inert. Therefore, player 2 does not know whether player 1 is a pivotal player though player 2 himself is one. (In this case, player 1 is indeed not a pivotal player.) Similarly, player 2 does not know whether player 3 is a pivotal player even player 3 is indeed a pivotal player. The arbitrary selection of free riders, say by choosing player 1, might then fail to reach coordination at $t + 2$.

However, if we can cut the edge between player 4 and player 5 such that the network becomes a tree, player 2 knows that he is the only pivotal player.

[6] For instance, [Lohmann, 1993][Lohmann, 1994] consider that individuals generate information by their actions, where the aggregate outcomes of actions is public. [Bolton and Harris, 1999] consider team experiment in infinite time horizon where the outcomes of experiments are public signals. [Bramoullé and Kranton, 2007] view information as a public good and consider public good provision in networks.

[7] Reviews can be seen in [Bikhchandani et al., 1998] [Cao and Hirshleifer, 2001].

[8] [Goyal, 2012] gives the reviews. Recent papers, e.g., [Acemoglu et al., 2011][Chatterjee and Dutta, 2011], also discuss this topic

person connected undirected networks, the complete network and incomplete network will give different convergence results which highly depend on individuals' initial private signals and their allocations in a network. In [Golub and Jackson, 2010], instead of using Bayesian learning, they use a naive learning protocol to tackle this social learning problem. I consider the social learning in networks as a learning-in-game procedure, where individuals can put more weights on the future learning results. My result gives a hint that the shape of network (without cycle) did not matter too much if players are far-sighted.

Third, a growing literature considers the game played in networks where various games played in various networks with various definitions[9]. Only avfew papers in this literature discuss the repeated game. In complete information game. In [Laclau, 2012], she proves a folk theorem where players play the game locally. In [Wolitzky, 2013] [Wolitzky, 2014], he considers network-like monitoring where a prisoner dilemma game is played globally. My paper is the first paper to consider the incomplete information game repeatedly played in a network.

My paper is also related to the literature in folk theorems in discounted repeated games with incomplete information. In this literature, they consider more general games than the games adopted here. [Fudenberg and Yamamoto, 2010] [Fudenberg and Yamamoto, 2011] [Wiseman, 2012] considering n-person game with public signals jointly generated by the states and actions; [Yamamoto, 2014] considering 2-person game with private signals jointly generated by the states and actions. There, the full-rank conditions are imposed to let single-period actions generate informative signals to separate the states. Here, I consider n-person game without signals and thus the single-period full-rank conditions are not imposed before solving the equilibrium. And my result shows that acyclic networks are sufficient to sustain the ex-post efficiency when discount factor is sufficiently high.

The paper is organized as the followings. I introduce the model in Section 2 . I discusses the equilibrium construction and shows the main result in Section 3.1 and Section 3.2 respectively. Some variations of my model will be discussed in its subsection 3.8. The conclusion is made within Section 4. All the missing proofs can be found in Appendix A.

2   Model

There is a set of players $N = \{1, 2, ..., n\}$. They constitute a network $G = (V, E)$ so that the vertices are players ($V = N$) and an edge is a pair of them (E is a subset of the set containing all two-element subsets of N). Throughout this paper, G is assumed to be finite,

---

[9][Jackson, 2008][Goyal, 2012] gives the reviews.

commonly known, fixed, undirected, and connected[10].

Time is discrete with index $s \in \{0, 1, ...\}$. At $s = 0$, the nature chooses a state $\theta \in \Theta = \{R, I\}^n$ once and for all according to a common prior π. Let us interpret R and I as Rebel and Inert respectively. After the nature moves, players play a normal form game, the *k-threshold game*, infinitely repeated played with common discounted factor δ. In the k-threshold game, $A_R = \{\textbf{revolt}, \textbf{stay}\}$ is the set of actions for R, while $A_I = \{\textbf{stay}\}$ is that for I. A Rebel's static payoff function is defined as follows.

- $u_R(a_R, a_{-i}) = 1$ if $a_R = \textbf{revolt}$ and $\#\{j : a_j = \textbf{revolt}\} \geq k$

- $u_R(a_R, a_{-i}) = -1$ if $a_R = \textbf{revolt}$ and $\#\{j : a_j = \textbf{revolt}\} < k$

- $u_R(a_R, a_{-i}) = 0$ if $a_R = \textbf{stay}$

. An Inert's static payoff is equal to 1 no matter how other players play.

During the game is played, any player, say i, can observe information only from his direct neighbors $G_i = \{j | \{i, j\} \in E\}$. These information includes his neighbors' types ($\theta_{G_i} \in \Theta_{G_i} = \{R, I\}^{G_i}$) and their histories of actions up to period s ($h_{G_i}^s \in H_{G_i}^s \equiv \times_{t=1}^s (\times_{j \in G_i} H_j^t)$). I assume that payoffs are hidden to emphasize that observing neighbors' actions are the only channel to infer other players' types and actions[11]. To be precise, when θ is realized at $s = 0$, i's information set about θ is $P_i(\theta) \equiv \{\theta_{G_i}\} \times \{R, I\}^{N \backslash G_i}$. For the information sets about players' actions, the sets of histories of actions are set to be empty at $s = 0$. At $s > 0$, a history of actions played by i is $h_i^s \in H_i^s \equiv \emptyset \times A_i^s$, while a history of actions played by all players is $h^s \in H^s \equiv \times_{t=1}^s (\times_{j \in N} H_j^t)$. i's information set about other players' histories of actions up to $s > 0$ is $\{h_{G_i}\} \times H_{N \backslash G_i}^s$. A player i's pure behavior strategy τ_i is a measurable function with respect his information partition if it maps $P_i(\theta) \times \{h_{G_i}\} \times H_{N \backslash G_i}^s$ to a single action in his action set for every s and for every θ. For convenience, let $[R](\theta)$ be the set of Rebels given θ.

By abusing the notation a bit, let h_θ^τ denote the realized sequence of actions generated by $\tau = (\tau_1, \tau_2, ..., \tau_n)$ given θ. Define $\mu_{G_i}^{\pi, \tau}(\theta, h^s | \theta_{G_i}, h_{G_i}^s)$ as the conditional distribution over $\Theta \times H^s$ conditional on i's information up to s, which is induced by π and τ. i's belief over θ up to s is then $\beta_{G_i}^{\pi, \tau}(\theta | \theta_{G_i}, h_{G_i}^s) \equiv \sum_{h^s \in H^s} \alpha_{G_i}^{\pi, \tau}(\theta, h^s | \theta_{G_i}, h_{G_i}^s)$.

The equilibrium concept is the week sequential equilibrium [12]. I am looking for the

---

[10]A path in G from i to j is a finite sequence $(l_1, l_2, ..., l_L)$ without repetition such that $l_1 = i$, $l_L = j$, and $\{l_q, l_{q+1}\} \in E$ for all $1 \leq q < L$. G is fixed if G is not random, and G is undirected if, for all i, j, if $j \in G_i$ then $i \in G_j$. G is connected if, for all $i, j \in N$, $i \neq j$, there is a path from i to j.

[11]Such restriction will be relaxed in the Section 3.8

[12]A weak sequential equilibrium is an assessment $\{\tau^*, \mu^*\}$, where μ^* is a collection of distributions over players' information sets with the property that, for all i, for all s, $\mu_{G_i}^*(\theta, h^s | \theta_{G_i}, h_{G_i}^s) = \mu_{G_i}^{\pi, \tau^*}(\theta, h^s | \theta_{G_i}, h_{G_i}^s)$

existence of approaching efficient equilibrium (*APEX equilibrium*), which is formally defined below.

Definition 2.1 (APEX strategy). *A behavior strategy τ is APEX if, for all θ, there is a finite time T^θ such that the actions in h^τ_θ after T^θ repeats the static ex-post Pareto efficient outcome.*

Definition 2.2 (APEX equilibrium). *An equilibrium (τ^*, α^*) is APEX if τ^* is APEX.*

In an APEX strategy, all Rebels will play **revolt** forever after some period if there are more than k Rebels; Rebels will play **stay** forever after some period otherwise. It is as if Rebels will learn the relevant information in the equilibrium since they will play the expost efficient outcome eventually and keep doing so. Several properties are noted. In the stage game, the expost outcome for each k is unique, and every player play **stay** forever is always an equilibrium. In an APEX equilibrium, it is not only as if Rebels will learn the relevant information but they must learn that by the following lemma.

Lemma 2.1. *If (τ^*, μ^*) is an APEX equilibrium, then for all $\theta \in \Theta$, there is a finite time T^θ_i for every Rebel i such that $\sum_{\theta \in \{\theta | [R](\theta) \geq k\}} \beta^{\pi, \tau^*}_{G_i}(\theta | h^s_{G_i}) = $ either 1 or 0 whenever $s \geq T^\theta_i$.*

3 Approaching expost efficient equilibrium

It is clear that an APEX equilibrium exists when $k = 1$. For the other cases, let us start with the case of $k = n$ and then continue to the case of $1 < k < n$.

3.1 The case of $k = n$

In the case of $k = n$, a Rebel can get better payoff from playing **revolt** than that from **stay** *only if* all players are Rebels. Two consequences follows. First, if a Rebel has an Inert neighbor, this Rebel always plays **revolt** in the equilibrium. Secondly, at any period, playing **stay** forever afterwards as a punishment for a deviation is optimal if there is another player who also plays **stay** forever afterwards, independently from the belief held by the punisher. These two features constitute an APEX equilibrium and further turn it to be a sequential equilibrium.

---

whenever the information set is reached with positive probability given τ^*. Moreover, for all i, for all s, τ^*_i maximize i's continuation expected payoff conditional on θ_{G_i} and $h^s_{G_i}$ of

$$E^\delta_G(u_{\theta_i}(\tau_i, \tau^*_{-i}) | \alpha^{\pi, \tau_i, \tau^*_{-i}}_{G_i}(\theta, h^s | \theta_{G_i}, h^s_{G_i}))$$

for all $h^s_{G_i}$.

Theorem 1 (APEX equilibrium for the case of $k = n$). *For any n-person repeated k-Threshold game with parameter $k = n$ played in a network, there is a δ^* such that a sequential APEX equilibrium exists whenever $\delta > \delta^*$.*

Proof. Omitted. □

The proof idea is a contagion argument. Suppose a Rebels play **revolt** at any period except for: (1) he has an Inert neighbor, or (2) he has observed his Rebel neighbor played **stay** once. Since the network is finite (n players) and connected, a Rebel is certain that there is an Inert somewhere if he has seen his neighbor has played **stay**; otherwise, he continues to believe that all platers are Rebels. Observing n consecutive **revolt** will imply that no Inert exist. Clearly, the above strategy is an APEX strategy and therefore ready for the equilibrium path for an APEX equilibrium. For any deviation from the above strategy, construct the out-of-path strategy as to play **stay** forever for both of the deviant and that Rebel (the punisher) who detects that. This out-of-path strategy is optimal for both the deviant and the punisher independent from the belief held by the punisher and hence is also sequential rational.

3.2   The case of $1 < k < n$

In contrast to the case of $k = n$, a Rebel still has incentive to play **revolt** even if he has an Inert neighbor. This allows the non-existence of APEX equilibrium. Let us consider Example 3.1 below.

Example 3.1. Suppose that $k = 2$ and $\theta = (R, I, R)$. The state and the network is represented as follows.

$$\boxed{R_1}\!\!-\!\!\!-\!\!\!-\!\!\boxed{I_2}\!\!-\!\!\!-\!\!\!-\!\!\boxed{R_3}$$
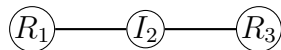
Rebel 1 never learn θ_3 since Inert 2 cannot reveal information about θ_3. The APEX equilibrium does not exist in this scenario.

The following condition, *strong connectedness*, excludes the non-existence of APEX equilibrium.

Definition 3.1 (Strong connectedness). *Given G, a state θ has strong connectedness if, for every pair of Rebels, there is a path consisting of Rebels to connect them*[13].

---

[13]In the language of graph theory, this definition is equivalent to say: given G, θ has strong connectedness if the induced graph by $[R](\theta)$ is connected.

Definition 3.2 (Full support on strong connectedness)**.** *Given G, π has full support on strong connectedness if*

$$\pi(\theta) > 0 \Leftrightarrow \theta \text{ has strong connectedness}$$

The main result of this paper is stated below.

Theorem 2 (APEX equilibrium for the case of $1 < k < n$)**.** *For any n-person repeated k-Threshold game with parameter $1 < k < n$ played in networks, if networks are acyclic[14] and if π has full support on strong connectedness, then there is a δ^* such that an APEX equilibrium exists whenever $\delta > \delta^*$.*

The proof for Theorem 2 is constructive. I show the proof idea throughout Section to Section, while leaving the proof in Appendix. Constructing APEX equilibrium in this case involves difficulties that do not occur in the case of $k = n$. In the case of $k = n$, T^θ can be determined independently from θ by setting $T^\theta = n$[15], but it is not obvious how to obtain T^θ before an equilibrium has been constructed now. Moreover, the free-rider problem might exist in the current case (as demonstrated in Introduction), but this problem never occur in the proposed APEX equilibrium in Theorem 1. As for the punishment scheme, playing **stay** forever is not anymore effective since a deviation might only seen by parts of players (network monitoring), and thus group punishment is hard to be coordinated to execute.

To get better exposition of the proof idea behind Theorem 2, I allow players to endow a talking technology so that they can cheap talk, talk with fixed cost, or talk with a cost function before they play actions until Section. Before Section I also introduce a game, *T-round talking game*, to be an auxiliary scenario that is simpler but mimics relevant features in the original game introduced in Section to shed light on the equilibrium construction. The equilibrium construction for T-round talking game will be studied by means of examples and intentionally serve to demonstrate the free-rider problem. I then argue the equilibrium construction in the original game is an analogue to the one in the T-round talking game. Roughly speaking, players can talk to exchange information about θ for T-round in the T-round talking game. They then play a one-shot k-threshold game at round $T + 1$. Note that in the original game, in an APEX equilibrium path, players would stop update their belief after some finite time and keep playing the same action in the k-threshold game, the game form of the T-round talking game mimics the structure of the APEX equilibrium path in the original game game. We consider in order the case of cheap talk, talking with a fixed cost, and then talking with cost function. We then extend the T-round talking game to allow that T can be endogenously determined in the equilibrium.

---

[14]A network is acyclic if the path from i to j for all $i \neq j$ is unique.

[15]Readers might refer to the proof for Theorem 1 shows

3.3 T-round talking game

The network, the set of states, and the set of players follow exactly the same definitions defined in Section 2. In the T-round talking game, each player endows a *talking technology*. A talking technology for player i is a pair of (W, M_i), where $W = \{\mathbf{s}, \mathbf{r}\}^L$, $L \in \mathbb{N}$ and $M = \times_{t=1}^{T} M_i^t$ is defined recursively by

$$M_i^1 = \{f | f : \Theta_{G_i} \to W\} \cup \{\emptyset\}$$

$$\text{for } 2 \leq t \leq T \ , \ M_i^t = \{f | f : \prod_{j \in G_i} M_j^{t-1} \to W\} \cup \{\emptyset\}.$$

Let us interpret W as the set of sentence and M_i as i's grammar. The meaning of "i talks to his neighbors at round t" is equivalent to "i chooses an element $f \in M_i^t$ to get an element $m \in M$ according to f. Moreover, m can be observed by all of i's neighbors". The time line is as follows.

1. Nature choose θ according to π.

2. Types are then fixed over time.

3. At the first T rounds, players talk to their neighbors.

4. At $T + 1$ round, players play a one-shot k-Threshold game.

5. Game ends.

I begin to find the APEX equilibrium with various k and T in the following examples.

3.3.1 T-round talking game with cheap talk

Example 3.2. In this example, let us consider $k = 6$, $T = 2$. The state and the network are represented as follows.
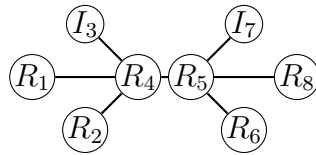
3.3.2 T-round talking game with fixed cost

3.3.3 T-round talking game with cost function

3.4 dispensability of talking technology

The information hierarchy is defined on a network G right after nature chooses a state. I will use the term "node i" instead of "player i" in this step.

I define information hierarchy by defining $\{N_i^{-1}, N_i^0, N_i^1...\}$ and $\{I_i^{-1}, I_i^0, I_i^1...\}$ for each $i \in N$, and then define $\{\leq^0, \leq^1, \leq^2\}$ and $\{R^0, R^1, R^2...\}$ for each iteration in $(0, 1, 2, ...)$. I also use the term "blocks" to represent the "iterations".

Given θ, the information hierarchy is defined as follows.

- **0-block** Denote

$$
\begin{aligned}
N_i^{-1} &\equiv i \\
I_i^{-1} &\equiv i
\end{aligned}
$$

  Then define R^0 as

$$R^0 \equiv [Rebels](\theta) \tag{1}$$

- **1-block** Denote

$$
\begin{aligned}
N_i^0 &\equiv G_i \\
I_i^0 &\equiv G_i \cap R^0
\end{aligned}
$$

  Define the set $\leq^0$ by defining

$$i \in \leq^0 \Leftrightarrow \exists j \in \bar{G}_i [I_i^0 \subseteq N_j^0 \cap R^0] \tag{2}$$

  Then define R^1 as

$$R^1 \equiv \{i \in R^0 | i \notin \leq^0\} \tag{3}$$

- $t+1$**-block,** $t \geq 1$ Denote

$$
\begin{aligned}
N_i^t &\equiv \bigcup_{k \in I_i^{t-1}} G_k \\
I_i^t &\equiv \bigcup_{k \in G_i \cap R^t} I_k^{t-1}
\end{aligned}
$$

  Define the set $\leq^t$ by defining

$$i \in \leq^t \Leftrightarrow \exists j \in \bar{G}_i [I_i^t \subseteq N_j^t \cap R^0] \tag{4}$$

  Then define R^{t+1} as

$$R^{t+1} \equiv \{i \in R^t | i \notin \leq^t\} \tag{5}$$

In other words, $i \in R^t$ if and only if *(1)* i is a Rebel and *(2)* at t-block, there is a $j \in I_i^{t-1}$ who is a Rebel and whose existence is informed to i, but none of i's neighbors has been informed that. Intuitively, the R^t nodes then have more incentives to report their information to others.

From the above definition, Theorem 3 states that it is sufficient to only let R^t nodes to report their information if the underlying network is acyclic.

Theorem 3. *If the network is acyclic and if the state has strong connectedness, then*

$$R^0 \neq \emptyset \Rightarrow \exists t \geq 0 [\exists i \in R^t [I_i^t = R^0]]$$

3.5 Step 2: Equilibrium strategies in the path

In this step, each player in G is indexed with a distinguished prime number to indicate his "location". Such indexation is starting from 3. To be more precise, I index each player i as x_i, where $x_i \geq 3$ is a prime number. Since the multiplication of distinguish prime numbers can be uniquely factorized as those numbers, I then use this property to let Rebels be able to report both the amount and the locations of their Rebel neighbors by reporting the multiplication of their Rebel neighbors' prime numbers.

Denote $\langle \rangle$ as a form of sequence. Denote $|\langle \rangle|$ as the length of a form of finite sequence. Denote $\bar{N} \subset N$ as an non-empty subset of N. The notations for the forms of sequences are shown in Table 1.

The $\langle \rangle$ and the $|\langle \rangle|$ will jointly determine the sequences of actions in the equilibrium path. For example, if a sequence takes the form $\langle 1 \rangle$ and its length $|\langle 1 \rangle| = 3$, then this sequence is $\langle \mathbf{s}, \mathbf{s}, \mathbf{r} \rangle$. Note that the length of a sequence is counted from its end.

In the equilibrium path, two phases, *reporting period* and *coordination period*, occur in turns finitely in the following way,

$$\underbrace{< \text{coordination period} >}_{0-block} \underbrace{< \text{reporting period} >< \text{coordination period} >}_{1-block} ...$$

I.e. after nature chooses a state, all the Rebels start with 0-block, then enter to 1-block,...,and so on. 0-block has only one period, coordination period. The t-blocks, $t \geq 1$ has two periods, reporting period and coordination period, where reporting period occurs first and then coordination period follows. The length of each phase in each block is finite but endogenous.

If a sequence of actions is meant to be played in the reporting period (*resp.* the coordination period), I called it a *reporting message* (*resp. coordination message*). In reporting

Table 1: Notations

$\bar{N}$	$\equiv$	a non-empty subset of N
x_i	$\equiv$	player i's prime-number index
$X_{\bar{N}}$	$\equiv$	$\prod_{j \in \bar{N}} x_j$
$\mathbf{s}$	$\equiv$	$\mathbf{stay}$
$\mathbf{r}$	$\equiv$	$\mathbf{revolt}$
$\langle \mathbf{stay} \rangle$	$\equiv$	$\langle \mathbf{s}, ..., \mathbf{s} \rangle$
$\langle \mathbf{revolt} \rangle$	$\equiv$	$\langle \mathbf{r}, ..., \mathbf{r} \rangle$
$\langle \bar{N} \rangle$	$\equiv$	$\langle \mathbf{s}, ..., \mathbf{s}, \underbrace{\mathbf{r}, \mathbf{s}, ..., \mathbf{s}}_{X_{\bar{N}}} \rangle$
$\langle 1 \rangle$	$\equiv$	$\langle \mathbf{s}, ..., \mathbf{s}, \underbrace{\mathbf{r}}_{1} \rangle$
$\langle x_i \rangle$	$\equiv$	$\langle \mathbf{s}, ..., \mathbf{s}, \underbrace{\mathbf{r}, \mathbf{s}, ..., \mathbf{s}}_{x_i} \rangle$

period in each t-block ($t \geq 1$), Rebels play the sequences defined in Table 2. In coordination period in each t-block ($t \geq 0$), Rebels play the sequences defined in Table 4. After the coordination period in each t-block ($t \geq 0$), players either *(1)* start to repeatedly play some certain actions, or *(2)* enter to the reporting period in $t+1$-block.

I start to give the details of reporting messages and coordination messages.

3.5.1 Reporting messages in reporting period

The reporting period in the t-block is denoted as RP^t. Denote $|RP^t|$ as the total number of periods in RP^t. In the equilibrium path, the sequence of actions played in the RP^t is with length $|RP^t|$ and has to follow one kind of the forms listed Table 2. Any other sequence will be considered as a deviation.

Table 3 shows that the belief formed by Rebel j after j observe his neighbor i's reporting messages in the equilibrium path.

According to Table 3, a Rebel can tell who are R^t after reporting period. The consequence is that, if a Rebel j has observed that all of his neighbors play $\langle \mathbf{stay} \rangle$, then he is sure that $\#[Rebels](\theta) < k$ if $\#I_j^{t-1} < k$.[16]

---

[16]Remind that R^t are those Rebels who have been informed the existence of some Rebels, but any of

Table 2: Reporting messages

Reporting Messages
$\langle \mathbf{stay} \rangle$
$\langle I_i^{t-1} \rangle$
$\langle 1 \rangle$

Table 3: j's belief updating after observing i's reporting messages in the reporting period in t-block

i plays	the event that $j \in \bar{G}_i$ believes with probability one
$\langle \mathbf{stay} \rangle$	$i \notin R^t$
$\langle I_i^{t-1} \rangle$	$i \in R^t$ and $l \in [Rebel](\theta)$ if $l \in I_i^{t-1}$
$\langle 1 \rangle$	$i \in R^t$ and i has known $\#[Rebels](\theta) \geq k - 1$

The important feature here is that a player may play a special sequence—$\langle 1 \rangle$—in the equilibrium path. This special sequence serve as a signal to indicate a *pivotal player* in RP^t—a player who is certain that he will learn the the relevant information given others' truthful reporting right after RP^t. I elaborate this issue as follows.

Pivotal players in RP^t

Definition 3.3 (Pivotal players in RP^t)**.** *Let $\theta \in \Theta$ be given. A player is pivotal in RP^t if he is certain that he can tell whether or not $\#[Rebels](\theta) < k$ given others' truthful reporting right after RP^t.*

There are two kinds of pivotal players in RP^t. The first kind are those players who are certain that they can learn the true state. On the contrary, the second kind of pivotal players can only learn the relevant information.

For the first kind of pivotal players, we may consider the following two examples.

Example 3.3. free-rider Problem

Let $k = 5$ and assume that there are message $\langle M_4 \rangle, \langle M_5 \rangle$ for Rebel 4, 5. To simply the analysis, let's assume that the game is played from 1-block (by discarding the strategies in 0-block and staring the game from the reporting period). Further, assume that Rebels will

---

their neighbors has not been informed that.

play **revolt** forever after observing that $\langle M_4 \rangle$ or $\langle M_5 \rangle$ is played once by Rebel 4 or 5 right after reporting period; otherwise they will play **stay** forever. Let G be the following.
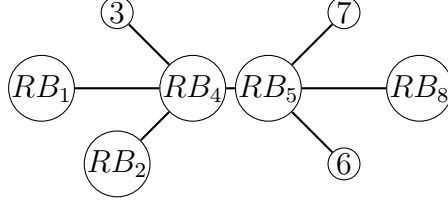
Note that Rebel 4 and Rebel 5 are R^1 members. Let $\langle\rangle_4$ and $\langle\rangle_5$ be the sequences of actions they may use to report the number of Rebel neighbors. If Rebel 5 report truthfully, then Rebel 4 will not report truthfully by arranging the timings in which he plays **revolt**. Since Rebel 4 can use $\langle M_4 \rangle$ to initialize the coordination, such deviation is profitable. Same situation happens for Rebel 5, and then Rebel 4 and Rebel 5 will not report truthfully.

In the above example, two sources constitutes the free-rider problem. One source is that there are coordination messages which will initiate coordination regardless how the reporting messages are played. The other source is that Rebel 4 and Rebel 5 are the pivotal players who can learn the true state. To see the later source more clearly, we may consider the following Example 3.4.

Example 3.4. Pivotal player: Case 1 Let $k = 6$ and suppose that there are message $\langle M_3 \rangle, \langle M_5 \rangle, \langle M_7 \rangle$ for Rebel 3,5,7 to initiate a coordination. Let the game be played from 1-block as Example 3.3. Further, let's suppose that Rebels will play **revolt** forever after observing that $\langle M_3 \rangle$, $\langle M_5 \rangle$, or $\langle M_7 \rangle$ is played once in two periods right after this reporting period; otherwise they will play **stay** forever. Let G be the following.
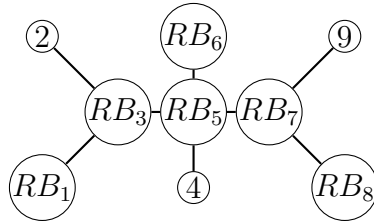
Note that Rebel 3, 5, 7 are R^1 members. In contrast to Example 3.3, although Rebel 3, 7 can use their coordination messages to initiate coordination, they still have incentives to report truthfully. This is because they are not pivotal. Since the coordination to **revolt** has to be initiated immediately after this reporting period, they have incentives to report truthfully to Rebel 5.

Rebel 5, however, has no incentive to report truthfully since he is a pivotal player who can learn the true state.

From the discussions in Example 3.3 and Example 3.4, a way to deal with the free-rider problem is to assign one of those pivotal players who constitute this problem as a *free rider*. If the network is acyclic, Lemma 3.1 shows that the free-rider problem can be identified before the game enter to RP^t. To be more precise, first define TR_{ij} to be a tree rooted in i and spanning in $j \in \bar{G}_i$ as follows.

Definition 3.4. $TR_{ij} \equiv \{l \in N : \text{there is a unique path } \{l, ..., j, i\} \text{ from } l \text{ to } i \text{ through } j\}$

Then, define C^t to be a set containing R^t nodes such that, for all $i \in C^t$, there are no possible Rebel nodes connecting to i by a path consisting of three or more nodes

$$C^t = \{i \in R^t : \nexists j \in R^{t-1} \cap \bar{G}_i [\exists l, l' \in TR_{ij}[l \in N_j^{t-1} \setminus I_i^{t-1} \text{ and } l' \in \bar{G}_l]]\}$$

In words, C^1 is the set of those nodes who can learn the true state right after RP^t. In Example 3.3, for instance, Rebel 4 and Rebel 5 are C^1 nodes. In Example 3.4, Rebel 5 is a C^1 node.

The following two lemmas are useful to identify the free-rider problem in RP^t before the game enters to RP^t.

Lemma 3.1. [17] *If the network is acyclic, and if the state has strong connectedness, then for each t-block,*

$$0 \leq |C^t| \leq 2$$

. *Moreover, suppose there are two nodes in C^t, then they are each other's neighbor.*

Lemma 3.2. *If the network is acyclic, and if the state has strong connectedness, then for each t-block*

$$i \in C^t \Rightarrow \text{there is no possible Rebel node outside of } \bigcup_{k \in N_i^{t-1}} G_k$$

By Lemma 3.1, C^t nodes are each other's neighbor (if there are two C^t nodes). By Lemma 3.2, it is straightforward to show that C^t nodes can identify themselves.[18] In order to solve the free-rider problem in which there are multiple C^t nodes in RP^t, we can let the node in C^t who has smallest prime index to be the free rider, and then solve the problem.

For the second kind of pivotal players, who can only learn the relevant information right after RP^t, we may consider the following example.

---

[17]Generally, this property does not hold if a network is cyclic.

[18]To be more precise, they identify themselves by the following procedure. First, a C^t node, j, assumes that one of his R^{t-1} neighbor, i, is a R^t-node. Second, he check if i is in C^t by checking the definition of C^t. Third, if i is identified as a C^t node, i is the only C^t node other than j (by Lemma 3.1). Finally, j assumes that i will do the same procedure to identify him. Since j himself is a C^t-node, i must be able to identify him if i is a C^t-node, and thus both C^t-nodes i and j can identify each other.

Example 3.5. Pivotal player: Case 2 Let $k = 6$. Again, assume that there are coordination message $\langle M \rangle$s for Rebels. Let the game be played from 1-block as Example 3.3. Let's assume that Rebels will play **revolt** forever after observing that $\langle M \rangle$ is played once in four periods right after reporting period; otherwise they will play **stay** forever.

Let G be the following.
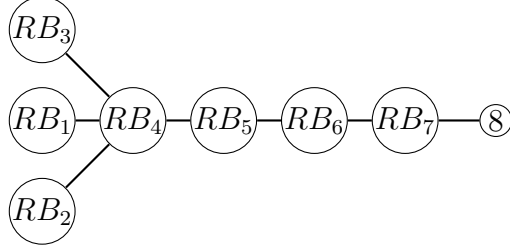
In this case, there is no Rebel in C^1. Rebel 4, nevertheless, will deviate from reporting $\langle I_4^0 \rangle$. Note that Rebel 4 has already known there are 5 Rebels in this world, and therefore he knows that the coordination to **revolt** can be initiated by him if he is informed the existence of one more Rebels. Moreover, if no more Rebels exist, coordination to **stay** can be also initiated by him. Since he can use the message $\langle M \rangle$ to initiate the coordination, his deviation is profitable given others' truthful reporting.
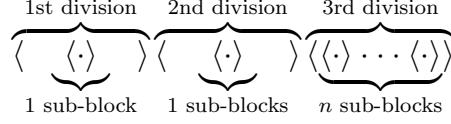
In a summary, the sequence $\langle 1 \rangle$ is introduced in order to let the pivotal players "identify" themselves. In the Appendix A.1, i can show that the pivotal players in RP^t are those Rebels who are either C^t nodes or have already known there are $k-1$ Rebels in the equilibrium path. The equilibrium construction will enforce the pivotal players to play $\langle 1 \rangle$ in the equilibrium path, and therefore the belief updating after observing $\langle 1 \rangle$ is as Table 3 shows.

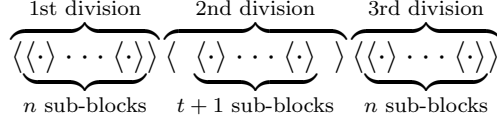### 3.5.2 Coordination messages in coordination period

The ignorance of previous reporting messages after observing a coordination message $\langle M \rangle$ may incur untruthful reporting as the above Example 3.3, 3.4, and 3.5 show. The sequence $\langle 1 \rangle$ is introduced in order to tackle this issue. However, one may have observed that the concatenation of these two messages, $\langle 1 \rangle \langle M \rangle$, is another coordination message itself. To be more precise, $\langle \langle \mathbf{s}, \mathbf{r} \rangle \langle M \rangle \rangle$ is another coordination message by truncating previous actions in $\langle 1 \rangle$ and concatenating the remaining actions to $\langle M \rangle$. If players' continuation behaviors after they observe $\langle \langle \mathbf{s}, \mathbf{r} \rangle \langle M \rangle \rangle$ are independent from the previous reporting messages, then the issue of untruthful reporting is not solved.

In this section, I discuss the coordination messages. And in the equilibrium construction, I let players' belief updating be contingent not only on the coordination messages but also on the previous reporting messages in the equilibrium path.

I depict the "structure" in the coordination period. There are three divisions in coordination period and there are several sub-blocks in each division. In $t = 0$ block, the structure is

$$\overbrace{\langle\ \underbrace{\langle\cdot\rangle}_{\text{1 sub-block}}\ \rangle}^{\text{1st division}}\overbrace{\langle\ \underbrace{\langle\cdot\rangle}_{\text{1 sub-blocks}}\ \rangle}^{\text{2nd division}}\overbrace{\langle\underbrace{\langle\cdot\rangle\cdots\langle\cdot\rangle}_{n\ \text{sub-blocks}}\rangle}^{\text{3rd division}}$$

; in $t > 0$ blocks, the structure is

$$\overbrace{\langle\underbrace{\langle\cdot\rangle\cdots\langle\cdot\rangle}_{n\ \text{sub-blocks}}\rangle}^{\text{1st division}}\overbrace{\langle\ \underbrace{\langle\cdot\rangle\cdots\langle\cdot\rangle}_{t+1\ \text{sub-blocks}}\ \rangle}^{\text{2nd division}}\overbrace{\langle\underbrace{\langle\cdot\rangle\cdots\langle\cdot\rangle}_{n\ \text{sub-blocks}}\rangle}^{\text{3rd division}}$$

, where $n = \#N$.

In the t-block, denote $CD_{m,q}^t$ as the m sub-block in q division and denote $|CD_{m,q}^t|$ as the total number of periods in $CD_{m,q}^t$. The sequence of actions in equilibrium path takes the following forms of sequences with length $|CD_{m,q}^t|$ as Table 4 shows.

Table 4: Coordination messages

Coordination messages
$\langle x_i \rangle$
$\langle \textbf{stay} \rangle$
r
s

Since the 0-block has simpler structure, I will focus on introducing players' behaviors in the coordination period in $t > 0$ block in the following paragraphs, while the Appendix A.1 shows the equilibrium path in $t = 0$ block[19].

The equilibrium path in $(CD_{1,1}^t, ..., CD_{n,1}^t)$**,** $t > 0$

Table 5 shows the belief updating formed by a Rebel j after he observes i's behavior after $CD_{1,1}^t$ in the equilibrium path. According to Table 5, Rebel j can tell whether or not $\#[Rebels](\theta) < k$ after $CD_{1,1}^t$ for some $t > 0$-block. As Table 6 and Table 7 show, in order to transmit the information about whether or not they have learn $\#[Rebels](\theta) < k$, Rebels will play $\langle x_i \rangle$ unless they observe $\langle \textbf{stay} \rangle$ in $CD_{2,1}^t$ to $CD_{n,1}^t$. The information about $\#[Rebels](\theta) < k$ will then be transmitted across all players after $CD_{n,1}^t$, .

---

[19]By equilibrium construction, all Rebels will play **revolt** after 0-block in the equilibrium path if there is a Rebel who has at least k Rebels neighbors.

Note that a Rebel j will play **stay** forever if he believes that $\#[Rebels](\theta) < k$ is with probability one. Thus, the sequence $\langle \mathbf{stay} \rangle$ played in $CD_{1,1}^t$ to $CD_{n,1}^t$ is interpreted as the coordination message to initiate the coordination to **stay**.

Table 5: j's belief updating after $CD_{1,1}^t$ by observing i's previous actions ($t > 0$)

In RP^t	In $CD_{1,1}^t$	
i plays	i plays	The events that $j \in \bar{G}_i$ believe with probability one
$\langle \mathbf{stay} \rangle$	$\langle x_i \rangle$	$i \notin R^t$
$\langle I_i^{t-1} \rangle$	$\langle \mathbf{stay} \rangle$	$\#[Rebels](\theta) < k$
$\langle I_i^{t-1} \rangle$	$\langle x_i \rangle$	$i \in R^t$
$\langle 1 \rangle$	$\langle \mathbf{stay} \rangle$	$\#[Rebels](\theta) < k$
$\langle 1 \rangle$	$\langle x_i \rangle$	$\#[Rebels](\theta) \geq k$

Table 6: In-path strategies in RP^t, $CD_{1,1}^t$, and $CD_{2,1}^t$ ($t > 0$)

In RP^t	In $CD_{1,1}^t$	In $CD_{2,1}^t$
i plays	i plays	$j \in \bar{G}_i$ plays
$\langle \mathbf{stay} \rangle$	$\langle x_i \rangle$	$\langle x_i \rangle$
$\langle I_i^{t-1} \rangle$	$\langle \mathbf{stay} \rangle$	$\langle \mathbf{stay} \rangle$
$\langle I_i^{t-1} \rangle$	$\langle x_i \rangle$	$\langle x_i \rangle$
$\langle 1 \rangle$	$\langle \mathbf{stay} \rangle$	$\langle \mathbf{stay} \rangle$
$\langle 1 \rangle$	$\langle x_i \rangle$	$\langle x_i \rangle$

The equilibrium path in $(CD_{1,2}^t, ..., CD_{t+1,2}^t)$, given $t > 0$

After $CD_{n,1}^t$ and in $CD_{1,2}^t$, Rebels start to check if the coordination to **revolt** can be initiated. The coordination message to initiate the coordination to **revolt** is $\langle \mathbf{stay} \rangle$ as Table 8 shows. The key feature here is that $\langle \mathbf{stay} \rangle$ is a coordination message to initiate the coordination to **revolt** *only if* $\langle I_i^{t-1} \rangle$ or $\langle 1 \rangle$ has been played in RP^t[20]. Although this sequence incurs no expected cost, initiating the coordination to **revolt** by this sequence is "not free" since it requires the Rebels to play **revolt** in the previous reporting period in

---

[20]Although $\langle \mathbf{stay} \rangle$ is also the coordination message to initiate the coordination to **stay** in $CD_{1,1}^t$ to $CD_{n,1}^t$, Rebels are not confused about it.

Table 7: In-path strategies in $CD_{m,1}^t$, where $m \geq 2$ $(t > 0)$

In $CD_{m,1}^t$, $m \geq 2$	In $CD_{m+1,1}^t$, $m \geq 2$
i plays	$j \in \bar{G}_i$ plays
$\langle x_i \rangle$	$\langle x_i \rangle$
$\langle \textbf{stay} \rangle$	$\langle \textbf{stay} \rangle$

order to initiate the coordination to **revolt**. Since the highest continuation payoff contingent on $\theta \in \{\theta : \#[Rebels](\theta) \geq k\}$ is the coordination to **revolt**, players then have incentive to play **revolt** in the previous reporting period.

After $CD_{1,2}^t$, and from $CD_{2,2}^t$ to $CD_{t+1,2}^t$, Rebels start to transmit the information about whether or not they have learn that $\#[Rebels](\theta) \geq k$. According to Table 9 and Table 10, they will play $\langle \textbf{stay} \rangle$ unless they observe someone play $\langle x_i \rangle$. After $CD_{t+1,1}^t$, the information about $\#[Rebels](\theta) \geq k$ will be transmitted across at least k Rebels.

Table 8: j's belief updating after $CD_{1,2}^t$ by observing i's previous actions $(t > 0)$

In RP^t	In $CD_{1,1}^t$	In $CD_{1,2}^t$	
i plays	i plays	i plays	The events j believe with probability one
$\langle \textbf{stay} \rangle$	$\langle x_i \rangle$	$\langle \textbf{stay} \rangle$	$i \notin R^t$
$\langle I_i^{t-1} \rangle$	$\langle \textbf{stay} \rangle$	$\langle \textbf{stay} \rangle$	$\#[Rebels](\theta) < k$
$\langle I_i^{t-1} \rangle$	$\langle x_i \rangle$	$\langle \textbf{stay} \rangle$	$\#[Rebels](\theta) \geq k$
$\langle I_i^{t-1} \rangle$	$\langle x_i \rangle$	$\langle x_i \rangle$	$i \in R^t$
$\langle 1 \rangle$	$\langle \textbf{stay} \rangle$	$\langle \textbf{stay} \rangle$	$\#[Rebels](\theta) < k$
$\langle 1 \rangle$	$\langle x_i \rangle$	$\langle \textbf{stay} \rangle$	$\#[Rebels](\theta) \geq k$

The equilibrium path in $(CD_{1,3}^t, ..., CD_{n,3}^t)$, given $t > 0$

The game finally enters to $CD_{1,3}^t$. In this period, those k Rebels who have learn that $\#[Rebels](\theta) \geq k$ will start to play **revolt** forever. Furthermore, this is the first period in which a Rebel may get positive expected payoff by playing **revolt** (in the equilibrium path). From $CD_{2,3}^t$ to $CD_{n,3}^t$, other Rebels start to transmit this information to all Rebels in order to coordinate to **revolt**. Table 12 shows players' behavior from $CD_{2,3}^t$ to $CD_{n,3}^t$.

Table 9: In-path strategies in RP^t, $CD_{1,1}^t$, $CD_{1,2}^t$, and $CD_{2,2}^t$ $(t > 0)$

In RP^t	In $CD_{1,1}^t$	In $CD_{1,2}^t$	In $CD_{2,2}^t$
i plays	i plays	i plays	$j \in \bar{G}_i$ plays
$\langle \textbf{stay} \rangle$	$\langle x_i \rangle$	$\langle \textbf{stay} \rangle$	$\langle \textbf{stay} \rangle$
$\langle I_i^{t-1} \rangle$	$\langle \textbf{stay} \rangle$	$\langle \textbf{stay} \rangle$	$\langle \textbf{stay} \rangle$
$\langle I_i^{t-1} \rangle$	$\langle x_i \rangle$	$\langle \textbf{stay} \rangle$	$\langle x_i \rangle$
$\langle I_i^{t-1} \rangle$	$\langle x_i \rangle$	$\langle x_i \rangle$	$\langle \textbf{stay} \rangle$
$\langle 1 \rangle$	$\langle \textbf{stay} \rangle$	$\langle \textbf{stay} \rangle$	$\langle \textbf{stay} \rangle$
$\langle 1 \rangle$	$\langle x_i \rangle$	$\langle \textbf{stay} \rangle$	$\langle x_i \rangle$

Table 10: In-path strategies in $CD_{m,2}^t$, where $m \geq 2$ $(t > 0)$

In $CD_{m,2}^t$, $m \geq 2$	In $CD_{m+1,2}^t$, $m \geq 2$
i plays	$j \in \bar{G}_i$ plays
$\langle x_i \rangle$	$\langle x_i \rangle$
$\langle \textbf{stay} \rangle$	$\langle \textbf{stay} \rangle$

3.6   Step 3: Off-path Belief

Whenever Rebel i detects a deviation at period s, he forms the following belief:

$$\sum_{\theta \in \{\theta : \theta_j = Inert, j \notin G_i\}} \beta_{G_i}^{\pi,\tau}(\theta | h_{G_i}^{s'}) = 1, \text{ for all } s' \geq s \tag{6}$$

. Thus, if $\#I_i^0 < k$, he will play **stay** forever. This off-path belief serves as a grim trigger.

3.7   Sketch of the proof for Theorem 2

I have listed Rebels' behavior and their belief updating in my constructed equilibrium path in Table 3, Table 5, and Table 8. The following lemma shows that such equilibrium path is APEX.

Lemma 3.3. *For any n-person repeated k-Threshold game with parameter $k \leq n$ played in an acyclic network, if π has full support on strong connectedness, then there exists an equilibrium path that is APEX.*

Table 11: In-path strategies in $CD_{1,3}^t$, $t > 0$

In $CD_{m,2}^t$, $1 \leq m \leq t+1$	In $CD_{1,3}^t$
i has played	$j \in \bar{G}_i$ plays
$\langle x_i \rangle$	**r**
Otherwise	**s**

Table 12: In-path strategies after $CD_{m,3}^t$, where $m \geq 2$, $t > 0$

In $CD_{m,3}^t$, $m \geq 2$	In $CD_{m+1,3}^t$, $m \geq 2$
i plays	$j \in \bar{G}_i$ plays
r	**r**
s	**s**

I sketech the proof for Theorem 2 as follows. First, I use off-path belief to prevent players from making detectable deviations, such as deviating from playing the specified forms of sequences that are listed in Table 2 or 4. Then I argue that any undetectable deviation made by a Rebel before he learns the relevant information, $\#[Rebels](\theta) \geq k$ or $\#[Rebels](\theta) < k$, will reduce his own expected continuation payoff. To see this, we may consider the case in which a Rebel wants to mimic pivotal plays' behaviors by sending $\langle 1 \rangle$ in a reporting period. According to Table 5 and Table 8, his neighbors' continuation playing after observing $\langle 1 \rangle$ is to play **stay** forever or to play **revolt** forever. But then all of his neighbors will repeat the same action afterward, and hence he cannot get more information to tell whether or not $\#[Rebels](\theta) < k$. When δ is high enough, he can get better continuation payoff by staying in the path in which he can learn the relevant information (by Lemma 2.1). To be more precise, by staying in the path, his static payoff eventually achieves the maximum static payoff as 1 when $\#[Rebels](\theta) \geq k$, and achieves the maximum static payoff as 0 when $\#[Rebels](\theta) < k$ [21].

3.8 Discussion

3.8.1 Variation: payoff as signals

The hidden payoff assumption can be relaxed without change the main result. One may consider a situation in which the static payoff not only depends on players' joint efforts but

---

[21]Claim 7 in the Appendix A.1 shows this argument

also on some random shocks, says the weather.[22]. To be more precise, there is a public signal $y \in \{y_1, y_2\}$ generated by Rebels' actions. Let Rebel i's payoff function be $u_{Rebel}(a_{Rebel_i}, y)$, and let $u_{Rebel}(\textbf{stay}, y_1) = u_{Rebel}(\textbf{stay}, y_2) = u_0$. The distribution of y_1 and y_2 is

$$
\begin{aligned}
p_{1s} &= \Pr(y = y_1 | \#\textbf{revolt} \geq k) \\
p_{1f} &= \Pr(y = y_1 | \#\textbf{revolt} < k) \\
p_{2s} &= \Pr(y = y_2 | \#\textbf{revolt} \geq k) \\
p_{2f} &= \Pr(y = y_2 | \#\textbf{revolt} < k)
\end{aligned}
$$

such that

$$
p_{1s} u_{Rebel}(\textbf{revolt}, y_1) + p_{2s} u_{Rebel}(\textbf{revolt}, y_2) > u_0 > p_{1f} u_{Rebel}(\textbf{revolt}, y_1) + p_{2f} u_{Rebel}(\textbf{revolt}, y_2)
\tag{7}
$$

and

$$
1 > p_{1s} > 0, 1 > p_{2s} > 0, p_{1f} = 1 - p_{1s}, p_{2f} = 1 - p_{2s}
\tag{8}
$$

Equation 7 is a generalization of the k-threshold game. Equation 8 is a full support assumption on signal y.

If Equation 8 holds, we can construct exactly the same equilibrium strategy by ignoring the noisy signal y. To see this, we can check the equilibrium path constructed in the previous sections. According to Table 2 and Table 4, given a period s before some Rebels play $\langle 1 \rangle$, there is at most one Rebel who plays action **revolt**. Since that, the signal y is not relevant before some Rebels play $\langle 1 \rangle$. We then check if a Rebel wants to play $\langle 1 \rangle$ in order to get additional information coming from y. However, playing $\langle 1 \rangle$ will initiate either the coordination to **stay** or the coordination to **revolt** as Table 5 and Table 8 shows. After a coordination is initiated, a Rebel can not get additional information to learn the relevant information. This is because the signal y is noisy, and Rebels' actions will repeat. By the same argument in Claim 7 (in the Appendix A.1), a Rebel is better to stay in the equilibrium path.

If Equation 8 does not hold, says $p_{1s} = p_{2f} = 1$, then the signal y is not noisy, and therefore the strategies constructed in the previous section is no longer an equilibrium. However, another APEX equilibrium can be constructed by letting all Rebels play **revolt** in the first period, and then keep playing **revolt** or **stay** contingent on the signals $y = y_1$ or $y = y_2$.

---

[22]e.g.,[Shadmehr and Bernhardt, 2011]

3.8.2   Variation: Rebels with different levels of efforts

We may also consider a model in which players contribute different levels of efforts to a collective action. Let the set of states of nature be $\hat{\Theta} = \Theta \times \Xi$, where $\Theta = \{Rebel, Inert\}^n$ and $\Xi = \{1, 2, ..., k\}^n$. Let $\hat{\theta} = (\theta, e)$ be a state of nature. After $\hat{\theta}$ is realized, a player i will hold an endowment e_i, where $e_i \in \{1, 2, ..., k\}$. The payoff structure is modified as the following.

1. $u_{Rebel_i}(a_{Rebel_i}, a_{-\theta_i}) = b_i$ if $a_{Rebel_i} = \textbf{revolt}$ and $\sum_{j:a_{\theta_j}=\textbf{revolt}} e_j \geq k$

2. $u_{Rebel_i}(a_{Rebel_i}, a_{-\theta_i}) = -e_i$ if $a_{Rebel_i} = \textbf{revolt}$ and $\sum_{j:a_{\theta_j}=\textbf{revolt}} e_j < k$

3. $u_{Rebel_i}(a_{Rebel_i}, a_{-\theta_i}) = 0$ if $a_{Rebel_i} = \textbf{stay}$

4. $u_{Inert_i}(a_{Inert_i}, a_{-\theta_i}) = 1$ if $a_{Inert_i} = \textbf{stay}$

After $\hat{\theta}$ is realized, players repeatedly play the above game in a network G. To see that the strategies constructed in previous section is still an equilibrium, we can transform $(G, \hat{\Theta})$ to $(G', \hat{\Theta}')$, where each player i is attached with $\#e_i$ different players in G', and $\hat{\Theta}' = \Theta \times \{1\}^n$.

### 3.8.3   Variation: networks with cycles

The prime indexing can deal with potential free problems when players play the repeated k-threshold game in a cyclic network. We may consider the following example.

Example 3.6. Let $k = 6$ and let θ and G be the following. In this network, Rebel 3 and Rebel 4 have the same information $I_3^1 = I_4^1$. If there is no punishment, Rebel 3 (or Rebel 4) may shirk and deviate from truthfully reporting $\langle I^3 \rangle$ (or $\langle I^4 \rangle$) at a reporting period if Rebel 4 (or Rebel 3) can reports truthfully. But this kind of deviation can be detected by Rebel 5 (or Rebel 2) since I_3^1 should be equal to I_4^1.
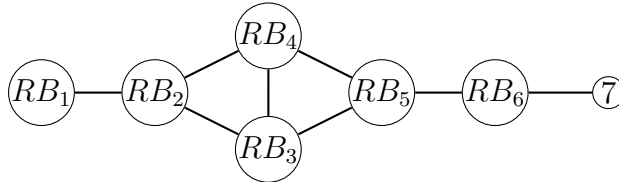

However, there is another free-rider problem that is hard to solve. Remind that, when the network is acrylic, the free-rider problem is solved by selecting a player as a free rider before the game enters to the reporting period. In cyclic network, we may need more elaborations to select a free rider to solve the free-rider problem. Let's consider the Example 3.7.

Example 3.7. Let $k = 6$. Suppose the network and θ is as follows.
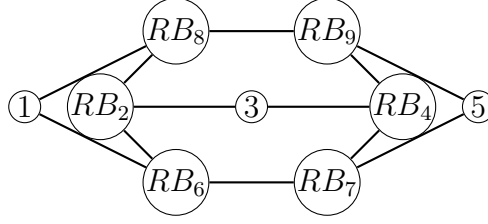
Let's assume that players follow the equilibrium path constructed in the previous section, and assume that they are in the end of 1-block. In such case, Rebel 2 has been informed that $I_2^1 = \{RB_2, RB_6, RB_8, RB_9, RB_7\}$, Rebel 4 has been informed that $I_4^1 = \{RB_4, RB_7, RB_9, RB_8, RB_6\}$, and so on. One more round of reporting period will let Rebels 2,6,7,4,9,8 know the true state θ, and therefore Rebels 2,6,7,4,9,8 are all pivotal players who constitute a free-rider problem (as Example 3.3). We may select a player to be a free rider, says Rebel 4 is selected, before the game enters to the reporting period in 2-block. However, this selection is ex-post. From the point of players' view, the state could be θ' as the following.
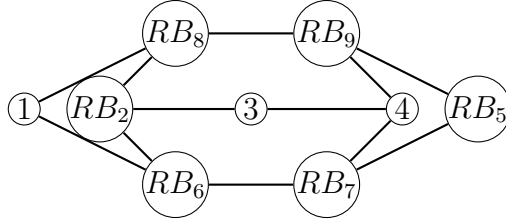
In θ', player 4 is an Inert and therefore not a pivotal Rebel, and hence he can not be selected as a free rider. We then need another rule in order to select a free rider during the game is played in the reporting period in the 2-block.

As Example 3.3 or Example 3.7 show, the free-rider problems occur if a free rider has not yet been selected before the game enter to a reporting period. In cyclic networks, this problem become more difficult to solve. Unfortunately, the solution is still infeasible in this paper.

I leave a conjecture in this paper and end this section.

Conjecture 3.1. *For any n-person repeated k-Threshold game with parameter $k < n$ played in any network, if π has full support on strong connectedness, then there exists a δ^* such that an APEX equilibrium exists whenever $\delta > \delta^*$.*

4 Conclusion

I model a coordination game and illustrate the learning processes generated by strategies in a sequential equilibrium and answer the question proposed in the beginning: what kind of networks can conduct coordination in a collective action game with information barrier. In the equilibrium, players transmit the relevant information by encoding such information by their actions in the time horizontal line. Since there is an expected cost in coding information, the potential free-rider problems may occur to impede the learning process. When the networks are without cycle, players can always learn the underlying relevant information and conduct the coordination only by their actions. However, what kinds of equilibrium strategies can constitute a learning process to learn the relevant information in cyclic networks still remains to be answered.

The construction of communication protocol exploits the assumption of finite type space and the finite threshold. Since the relevant information has been parameterized as a threshold, players can acquire this information by jointly incrementally reporting their own private information. The major punishment to keep players staying in the equilibrium path is then the joint shifting to play same actions as the stopping to update their information. The threshold model seems a general model in proofing that a communication protocol not only leads a learning process but also constitutes an equilibrium to reveal the relevant information in finite time.

Existing literatures in political science and sociology have recognized the importance of social network in influencing individual's behavior in participating social movements, e.g., [Passy, 2003][McAdam, 2003][Siegel, 2009]. This paper views networks as routes for communication where rational individuals initially have local information, and they can influence nearby individuals by taking actions. Such influence may take long time to travel across individuals, and the whole process incurs inefficient outcomes in many periods. A characterization in the speed of information transmission across a network is not answered here, although it is an important topic in order to give more attentions in investigating the most efficient way to let the information be spread . This question would remain for the future research.

References

D. Acemoglu, M. A. Dahleh, I. Lobel, and A. Ozdaglar. Bayesian Learning in Social Networks. *The Review of Economic Studies*, 78(4):1201–1236, 2011.

S. Bikhchandani, D. Hirshleifer, and I. Welch. Learning from the Behavior of Others: Conformity, Fads, and Informational Cascades, 1998.

P. Bolton and C. Harris. Strategic Experimentation. *Econometrica*, 67(2):349–374, 1999.

Y. Bramoullé and R. Kranton. Public Goods in Networks. *Journal of Economic Theory*, 135(1):478–494, 2007.

H. H. Cao and D. A. Hirshleifer. Conversation, Observational Learning, and Informational Cascades. *SSRN Electronic Journal*, Nov. 2001.

K. Chatterjee and B. Dutta. Credibility and Strategic Learning in Networks, Jan. 2011.

M. S.-Y. Chwe. Communication and Coordination in Social Networks. *Review of Economic Studies*, 67(1):1–16, 2000.

D. Fudenberg and Y. Yamamoto. Repeated Games Where the Payoffs and Monitoring Structure Are Unknown. *Econometrica*, 78(5):1673–1710, 2010.

D. Fudenberg and Y. Yamamoto. Learning from Private Information in Noisy Repeated Games. *Journal of Economic Theory*, 146(5):1733–1769, 2011.

D. Gale and S. Kariv. Bayesian Learning in Social Networks. *Games and Economic Behavior*, 45(2):329–346, 2003.

B. Golub and M. O. Jackson. Naïve Learning in Social Networks and the Wisdom of Crowds. *American Economic Journal: Microeconomics*, 2(1):112–149, 2010.

S. Goyal. *Connections: An Introduction to the Economics of Networks*. Princeton University Press, 2012.

M. O. Jackson. *Social and Economic Networks*. Princeton University Press, 2008.

O. Karl-Dieter and G. Christiane. Dissident Groups, Personal Networks, and Spontaneous Cooperation: The East German Revolution of 1989. *American Sociological Review*, 58 (5):659–680, 1993.

D. M. Kreps and R. Wilson. Sequential Equilibria. *Econometrica*, 50(4):863–94, July 1982.

M. Laclau. A Folk Theorem for Repeated Games Played on a Network. *Games and Economic Behavior*, 76(2):711–737, 2012.

S. Lohmann. A Signaling Model of Informative and Manipulative Political Action. *American Political Science Review*, 87(2):319–333, 1993.

S. Lohmann. Information Aggregation Through Costly Political Action. *The American Economic Review*, 84(3):518–530, 1994.

S. Lohmann. The Dynamics of Informational Cascades: The Monday Demonstrations in Leipzig, East Germany, 1989–91. *World Politics*, 47(01):42–101, June 2011.

D. McAdam. Beyond Structural Analysis: Toward a More Dynamic Understanding of Social Movements. In *Social Movements and Networks*. Oxford University Press, Oxford, 2003.

F. Passy. Social Networks Matter. But How? In *Social Movements and Networks*, pages 21–48. Oxford U Press, 2003.

A. Perea. A Note on The One-deviation Property in Extensive Form Games. *Games and Economic Behavior*, 40(2):322 – 338, 2002.

J. m. Renault and T. Tomala. Repeated Proximity Games. *International Journal of Game Theory*, 27(4):539–559, 1998.

M. Shadmehr and D. Bernhardt. Collective Action with Uncertain Payoffs: Coordination, Public Signals, and Punishment Dilemmas. *American Political Science Review*, 105(04): 829–851, Nov. 2011.

D. A. Siegel. Social Networks and Collective Action. *American Journal of Political Science*, 53(1):122–138, 2009.

T. Wiseman. A Partial Folk Theorem for Games with Private Learning. *Theoretical Economics*, 7(2):217–239, 2012.

A. Wolitzky. Cooperation with Network Monitoring. *Review of Economic Studies*, 80(1): 395–427, 2013.

A. Wolitzky. Communication with Tokens in Repeated Games on Networks. *Theoretical Economics*, Jan. 2014.

Y. Yamamoto. Individual Learning and Cooperation in Noisy Repeated Games. *Review of Economic Studies*, 81(1):473–500, 2014.

A   Appendix

proof for Lemma 2.1

Proof. The proof is done by contradiction. Suppose Rebels' strategies constitute an APEX. By definition of APEX, there is a time T^θ when actions start to repeat at state θ. Let $T = \max_{\theta \in \Theta} T^\theta$. Pick that time $T_i = T + 1$ and suppose the consequence did not not holds so that $0 < \sum_{\theta:\#[Rebels](\theta) \geq k} \beta_{G_i}^{\pi,\tau^*}(\theta|h_{G_i}^s) < 1$ for some $s \geq T_i$. Then this Rebel puts some positive weights on some $\theta \in \{\theta : \#[Rebels](\theta) < k\}$ and puts some positive weights on $\theta \in \{\theta : \#[Rebels](\theta) \geq k\}$ at that time s. Note this Rebel i has already known θ_j if $j \in G_i$, and therefore Rebel i put some positive weights on $\theta \in \{\theta : \#[Rebels](\theta) < k, \theta_l = Rebel, l \notin G_i\}$ and $\theta \in \{\theta : \#[Rebels](\theta) < k, \theta_l = Inert, l \notin G_i\}$. Since actions start to repeat at T, all i's neighbors will play the same actions as the actions at time T, but then Rebel i can not update information from his neighborhood by Bayesian rule. Suppose i's continuation strategy is to play **revolt** repeatedly, then this is not ex-post efficient if $\#[Rebels](\theta) < k$. Suppose i's continuation strategy is to play **stay** repeatedly, then this is not ex-post efficient if $\#[Rebels](\theta) \geq k$ $\qquad \square$

proof for Theorem 3

This proof follows three useful claims, Claim 1, Claim 2 and Claim 3. First note that I_i^t and N_i^t, $t \geq 1$ can be expressed as

$$I_i^t = \bigcup_{k_0 \in G_i \cap R^t} \bigcup_{k_1 \in G_{k_0} \cap R^{t-1}} \cdots \bigcup_{k_{t-1} \in G_{k_{t-2}} \cap R^1} G_{k_{t-1}} \cap R^0 \tag{9}$$

, while H_i^t can be expressed as

$$N_i^t = \bigcup_{k_0 \in G_i \cap R^{t-1}} \bigcup_{k_1 \in G_{k_0} \cap R^{t-2}} \cdots \bigcup_{k_{t-2} \in G_{k_{t-3}} \cap R^0} G_{k_{t-2}} \tag{10}$$

Claim 1. $I_i^t \subset N_i^t$ *for* $t \geq 1$

Proof. $I_0^t \subset N_0^t$ by definition. Since $R^t \subset R^{t-1}$ for $t \geq 1$, $I_i^t \subset N_i^t$ for $t \geq 1$ by comparing Equation 9 and Equation 10. $\qquad \square$

Claim 2. *If the network is without cycle, then for each $t \geq 1$ block, we have $i \in R^t \Leftrightarrow i \in R^{t-1}$ and $\exists k_1, k_2 \in R^{t-1} \cap \bar{G}_i$, where $k_1 \neq k_2$.*

Proof. The proof is done by induction. We first show that the statement is true for $t = 1$.
   Base: $i \in R^1 \Leftrightarrow [i \in R^0] \wedge [\exists k_1, k_2 \in (R^0 \cap \bar{G}_i)]$.
   $\Rightarrow$: Since $i \in R^1$, then $i \in R^0$ and then $I_i^0 \nsubseteq N_j^0$ for all $j \in \bar{G}_i$ by definition. Since $I_i^0 = R^0 \cap G_i$, then $\forall j \in \bar{G}_i [\exists k \in (R^0 \cap \bar{G}_i)[k \notin N_j^0]]$. Since the $j \in \bar{G}_i$ is arbitrary, we then have a pair of $k_1, k_2 \in (R^0 \cap \bar{G}_i)$ such that $k_1 \notin N_{k_2}^0$ and $k_2 \notin N_{k_1}^0$.

$\Leftarrow$: Pick $k \in \{k_1, k_2\} \subseteq (R^0 \cap \bar{G}_i)$, and pick an arbitrary $j \in \bar{G}_i \backslash \{k\}$. Note that $k \notin N_j^0$, otherwise there is a cycle from i to i. Hence $[k \in (R^0 \cap \bar{G}_i)] \wedge [k \notin N_j^0]$ and therefore $[k \in I_i^0] \wedge [k \notin N_j^0]$. Then we have $I_i^0 \nsubseteq N_j^0$ for arbitrary $j \in \bar{G}_i$, and thus $i \in R^1$.

Induction hypothesis: the statement is true for $\{1, 2, .., t\}$ where $t \geq 1$.

If the hypothesis is true, then $i \in R^{t+1} \Leftrightarrow [i \in R^t] \wedge [\exists k_1, k_2 \in (R^t \cap \bar{G}_i)]$

$\Rightarrow$: since $i \in R^{t+1}$, then $i \in R^t$ and $I_i^t \nsubseteq N_j^t$ for all $j \in \bar{G}_i$ by definition. Recall that I_i^t can be expressed as Equation 9 and H_i^t can be expressed as Equation 10, then for every $l \in I_i^{t-1}$, we can find a path connecting i to l by the induction hypothesis. If $j \in \bar{G}_i$, then we can find a path connecting j to l by connecting j to i, and then connecting i to l. Thus, if $l \in I_i^{t-1}$ then $l \in N_J^t$, and hence $I_i^{t-1} \subseteq N_j^t$ for all $j \in \bar{G}_i$. Recall that $I_i^t = \bigcup_{k \in N_i \cap R^t} I_k^{t-1}$ and $i \in R^{t+1}$, then we must have $\forall j \in \bar{G}_i [\exists k \in (R^t \cap \bar{G}_i)[I_k^{t-1} \nsubseteq N_j^t]]$, since $I_i^{t-1} \subseteq N_j^t$. Note that such $j \in \bar{G}_i$ is arbitrary, we then have a pair of $k_1, k_2 \in (R^t \cap \bar{G}_i)$ such that $k_1 \notin N_{k_2}^t$ and $k_2 \notin N_{k_1}^t$.

$\Leftarrow$: By the induction hypothesis, we have a chain $k_{1_0}, ..., k_{1_t}, i, k_{2_t}, ..., k_{2_0}$ with $k_{1_0} \in R^0, ..., k_{1_t} \in R^t$, $i \in R^t$, $k_{2_t} \in R^t, ..., k_{1_0} \in R^0$, where $k_{1_t}, k_{2_t} \in (R^t \cap \bar{G}_i)$, $k_{1_0} \in I_{k_{1_t}}^{t-1}$ and $k_{2_0} \in I_{k_{2_t}}^{t-1}$. Note that $k_{1_0} \notin N_j^t$ whenever $j \in \bar{G}_i$, otherwise there is a cycle from i to i since $\{i, k_{2_t}, ..., k_{2_0}\} \in N_j^t$, and hence $[k_{1_0} \in I_{k_{1_t}}^{t-1}] \wedge [k_{1_0} \notin N_j^t]$ for all $j \in \bar{G}_i$. Therefore we have $[I_{k_{1_t}}^{t-1} \in I_i^t] \wedge [I_{k_{1_t}}^{t-1} \nsubseteq N_j^t]$ for all $j \in \bar{G}_i$ since $k_{1_t}, k_{2_t} \in (R^t \cap \bar{G}_i)$ and $[k_{1_0} \in I_{k_{1_t}}^{t-1}] \wedge [k_{1_0} \notin N_j^t]$ for all $j \in \bar{G}_i$. Then we have $I_i^t = \bigcup_{k \in N_i \cap R^t} I_k^{t-1} \nsubseteq N_j^t$ for arbitrary $j \in \bar{G}_i$, and thus $i \in R^{t+1}$.

We can then conclude that the statement is true by induction. $\qquad \square$

Claim 3. *If the network is without cycle and if the state has strong connectedness, then if there is a pair of R^t nodes, then there exists a R^t-path connecting them.*

Proof. The proof is done by induction and by Claim 2. Since the state has strong connectedness, we have a R^0-path connecting each pair of R^0 nodes. Since all pairs of R^0 nodes are connected by a R^0-path, then for all pairs of R^1 nodes must be in some of such paths by Claim 2, and then connected by a R^0-path. But then all the R^0-nodes in such path are all R^1 nodes by Claim 2 again and by $R^t \subseteq R^{t-1}$ for $t \geq 1$ by definition. Thus, for all pairs of R^1 nodes has a R^1-path connecting them. The similar argument holds for $t > 1$, we then get the result.

$\qquad \square$

I begin to prove Theorem 3. I first claim that if $R^t \neq \emptyset$ and if $R^{t+1} = \emptyset$, then $R^0 \subset I_i^t$ whenever $i \in R^t$. Then I claim that if $R^t \neq \emptyset$ then $\#R^{t+1} < \#R^t$. Finally, I iterate R^t with $t \geq 0$ to get the conclusion.

If $R^t \neq \emptyset$ but $R^{t+1} = \emptyset$, I claim that $R^0 \subset I_i^t$ for all $i \in R^t$. The proof is by contradiction. If $R^0 \not\subset I_i^t$, there is a $j \in R^0$ but $j \notin I_i^t$. Since I_i^t can be expressed as Equation 9, there is no such a path $\{i, k_0, k_1, ..., k_{t-1}, j\}$, where $k_0 \in G_i \cap R^t$, $k_1 \in G_{k_0} \cap R^{t-1}, ..., k_{t-1} \in N_{k_{t-2}} \cap R^1$. Since $R^{t+1} = \emptyset$ and therefore $R^{t'} = \emptyset$ if $t' \geq t+1$, and hence there is no such a path containing a node in $R^{t'} = \emptyset$, where $t' \geq t+1$ connecting i to j. But $i \in R^t$ and $i, j \in R^0$, if there is no such a path, then it violate either Claim 3 or Claim 2. Contradiction.

Next I claim that if $R^t \neq \emptyset$ then $\#R^{t+1} < \#R^t$. The proof is the followings. Given a node i in R^t, let $j \in R^t$ (could be i itself) be the node connected with i with the maximum shortest R^t path. This j can be found since $R^t \neq \emptyset$ and the network is finite. Then there is no R^t node in j's neighborhood other than the nodes in this path. Since the network is without cycle, there is at most one R^t node in j's neighborhood. But then $j \notin R^{t+1}$ since it violate Claim 2.

Starting from $R^0 \neq \emptyset$ and iterating R^t with $t \geq 0$, if $R^t \neq \emptyset$ but $R^{t+1} = \emptyset$, then there is some i with $R^0 \subset I_i^t$ as the above paragraph shows; if $R^t \neq \emptyset$ and $R^{t+1} \neq \emptyset$, then we starting from R^{t+1} and iterating R^{t+1} with $t \geq t+1$. Since $\#R^{t+1} < \#R^t$ as the above paragraph shows, there is a time t^* with $R^{t^*} = \emptyset$, then we get the conclusion.

proof for Lemma 3.1

Proof. Denote (i, j)-path as the set of paths from i to j. The proof is by contradiction. Suppose there are three or more R^t-nodes in C^t, then pick any three nodes of them, and denote them as i_1, i_2, i_3. Let's say i_2 is in a (i_1, i_3)-path by strong connectedness, and therefore $i_2 \in TR_{i_1 i_2}$ and $i_3 \in TR_{i_2 i_3}$. First we show that $i_1 \in G_{i_2}$ (or $i_3 \in G_{i_2}$). Suppose $i_1 \notin N_{i_2}$, since $i_1, i_2 \in R^t$, then the (i_1, i_2)-path is a R^t-path by Claim 2. Let this $(i_1 i_2)$-path be $\{i_1, j_1, ..., j_n, i_2\}$. Since $i_1, j_1, ..., j_n, i_2 \in R^t$, we then have $I_{i_1}^{t-1} \not\subseteq N_{j_1}^{t-1}, ..., I_{j_n}^{t-1} \not\subseteq N_{i_2}^{t-1}$ and $I_{j_1}^{t-1} \not\subseteq N_{i_1}^{t-1}, ..., I_{i_2}^{t-1} \not\subseteq N_{j_n}^{t-1}$. Since $I_{i_1}^{t-1} \subseteq N_{i_1}^{t-1}, ..., I_{i_2}^{t-1} \subseteq N_{i_2}^{t-1}$ by Claim 1, we further have $\exists k_1 \in R^0[k_1 \in N_{j_1}^{t-1}\backslash I_{i_1}^{t-1}], ..., \exists k_n \in R^0[k_n \in N_{j_n}^{t-1}\backslash I_{i_2}^{t-1}]$. Since the state has strong connectedness, there is a R^0 path connecting $k_1, ..., k_n$ by Claim 3. But then we have already found k_1, k_2 such that $k_1 \in N_{j_1}^{t-1}\backslash I_{i_1}^{t-1}$ and $k_2 \in \bar{G}_{k_1}$. It is a contradiction that $i_1 \in C$.

Now, i_1, i_2, i_3 will form a R^t-path as $\{i_1, i_2, i_3\}$. With the same argument as the above, we then have $\exists k_1 \in R^0[k_1 \in N_{i_2}^{t-1}\backslash I_{i_1}^{t-1}]$ and $\exists k_2 \in R^0[k_2 \in N_{i_3}^{t-1}\backslash I_{i_2}^{t-1}]$, and thus i_1 is not in C. $\qquad\square$

proof for Lemma 3.2

Proof. The proof is done by contradiction. Since $i \in R^t$, there is a $j \in (R^{t-1} \cap \bar{G}_i)$ by Lemma 2. Note that $N_j^{t-1} \subseteq \bigcup_{k \in N_i^{t-1}} N_k$ since $N_j^{t-1} = \bigcup_{k \in I_j^{t-2}} N_k$, and $I_j^{t-2} \subseteq I_i^{t-1} \subseteq N_i^{t-1}$. If

there is another node outside $\bigcup_{k \in N_i^{t-1}} N_k$ in TR_{ij}, then there must be another node such that there is a path connected to some nodes in N_j^{t-1} since the network is connected. It is a contradiction that $i \in C$.

$\square$

A.1 Equilibrium

A.1.1 Out-off-path belief

If Rebel i detects a deviation at m period, he forms the belief as

$$\beta_i(\{\theta : \theta \in \times_{j \in G_i} \{\theta_j\} \times \{Inert\}^{n-\#G_i}\} | h_{G_i}^{m'}) = 1 \ , \ m' \geq m \tag{11}$$

A.1.2 Equilibrium Path: Notations

- $\langle\rangle$ is a finite sequence.

- $|\langle\rangle|$ is the length of finite sequence $\langle\rangle$.

- $\langle\rangle_r$ is the set of finite sequences in which the action **r** occurs once and only once.

- $PF(\langle\rangle, m)$ is the m-periods prefix of a finite sequence $\langle\rangle$.

- If $\langle\rangle \in \langle\rangle_r$, then let $||\langle\rangle|| = \arg\min\{m \in \{1, ..., |\langle\rangle|\} | PF(\langle\rangle, m) \in \langle\rangle_r\}$

- (i, j)-path is the set of paths from i to j.

- $(RP)^t$ is the period in the end of RP^t.

A.1.3 Equilibrium Path: reporting period

reporting period: notations

- m is a period in reporting period in t block.

- $|RP^t|$ is the total periods in reporting period in t-block

- $O_i^{m,t}$ is the set of i's neighbors js who has played a sequence M such that $M = PF(\langle I_j^{t-1}\rangle, m)$ and $M \in \langle\rangle_r$ at period m.

- $I_i^{m,t} \equiv (\bigcup_{k \in O_i^{m,t}} I_k^{t-1}) \cup I_i^{t-1}$ is the updated relevant information gathered by i at period m. Note that $I_i^{0,t} = I_i^{t-1}$ and $I_i^{|RP^t|,t} = I_i^t$.

- $N_i^{m,t} \equiv (\bigcup_{k \in O_i^{m,t}} N_k^{t-1}) \cup N_i^{t-1}$ is the updated neighborhood which contains $I_i^{m,t}$

- Let
$$Ex_{I_i^{m,t}} \equiv \{l \notin N_i^{m,t} | \exists l' \in I_i^{m,t} \text{ such that there exists a } (l,l')\text{-path}\}$$
be all the possible Rebel nodes outside of $N_i^{m,t}$ given $I_i^{m,t}$

- Let
$$TR_{I_i^{m,t}j} \equiv TR_{ij} \cap (Ex_{I_i^{m,t}} \cup I_i^{m,t})$$
be all the possible Rebel nodes in the TR_{ij} given $I_i^{m,t}$.

reporting period: automata

$i \notin R^t$

- **WHILE LOOP**
  - At $m \geq 0$, if $\#Ex_{I_i^{m,t}} \cup I_i^{m,t} < k$, report $\langle$**stay**$\rangle$ and then play **stay** forever.
  - Otherwise, **runs POST-CHECK**

$i \in R^t$

- **WHILE LOOP**
  - At $m \geq 0$, if $\#Ex_{I_i^{m,t}} \cup I_i^{m,t} < k$, report $\langle$**stay**$\rangle$ and then play **stay** forever.
  - Otherwise, **runs MAIN**

- **MAIN**

  At $m \geq 0$,

  1. At $m = 0$ and if $\#I_i^{t-1} = \#I_i^{0,t} = k - 1$, then **runs POST-CHECK**
  2. At $m = 0$ and if $i \in R^t$ and
     $$\nexists j \in R^{t-1} \cap \bar{G}_i \text{ such that } \exists l_1, l_2 \in TR_{ij}[[l_1 \in N_j^{t-1}\backslash I_i^{t-1}] \wedge [l_2 \in \bar{G}_{l_1}]]]$$
     , then runs **CHECK**.0. Otherwise, recall **MAIN**
  3. At $0 \leq m \leq |RP^t| - ||\langle I_i^{t-1}\rangle||$, play

     $$\text{stay}$$

  4. At $m = |RP^t| - ||\langle I_i^{t-1}\rangle|| + 1$, then

(a) if $O_i^{m,t} = \emptyset$, then report
$$\langle I_i^{t-1} \rangle$$

(b) if $O_i^{m,t} \neq \emptyset$, then **runs CHECK.k**

- **CHECK.0**

At $m = 0$, if $i \in C^t$, i.e. if $i \in R^t$ and

$$\nexists j \in R^{t-1} \cap \bar{G}_i \text{ such that } [\exists l_1, l_2 \in TR_{ij}[[l_1 \in N_j^{t-1} \setminus I_i^{t-1}] \wedge [l_2 \in \bar{G}_{l_1}]]]$$

, then

1. If $\#C^t = 1$, then **runs POST-CHECK**

2. If $\#C^t = 2$, then denote $i_1, i_2 \in C$ such that $I_{i_1}^{t-2} < I_{i_2}^{t-2}$, and then
   - if $i = i_1$, then **runs POST-CHECK**
   - if $i = i_2$, then report
   $$\langle I_i^{t-1} \rangle$$

- **CHECK.m**

At $m > 0$, if $O_i^{m,t} \neq \emptyset$, then there are two cases,

1. Case 1: If $i \in R^t$ and

   $$\exists j \in O_i^m \text{ such that } \exists l_1, l_2 \in TR_{I_i^{m,t}j}[[l_1 \in I_j^{t-1} \setminus I_i^{t-1}] \wedge [l_2 \in \bar{G}_{l_1}]]]$$

   , then report
   $$\langle I_i^{t-1} \rangle$$

2. Case 2: If $i \in R^t$ and

   $$\nexists j \in O_i^m \text{ such that } \exists l_1, l_2 \in TR_{I_i^{m,t}j}[[l_1 \in I_j^{t-1} \setminus I_i^{t-1}] \wedge [l_2 \in \bar{G}_{l_1}]]]$$

   (a) Case 2.1: If $i \in R^t$ and

   $$\nexists j \in R^{t-1} \cap (G_i \setminus O_i^{m,t}) \text{ such that } [\exists l_1, l_2 \in TR_{I_i^{m,t}j}[[l_1 \in N_j^{t-1} \setminus I_i^{t-1}] \wedge [l_2 \in \bar{G}_{l_2}]]]$$

   Note: this case is the case when $i \in C$, thus recall Check.0

   (b) Case 2.2: If $i \in R^t$ and

   $$\exists j \in R^{t-1} \cap (G_i \setminus O_i^{m,t}) \text{ such that } [\exists l_1, l_2 \in TR_{I_i^{m,t}j}[[l_1 \in N_j^{t-1} \setminus I_i^{t-1}] \wedge [l_2 \in \bar{G}_{l_2}]]]$$

      − if $\#I_i^{m,t} = k-1$, then **runs POST-CHECK**

      − if $\#I_i^{m,t} < k-1$, then report

$$\langle I_i^{t-1} \rangle$$

- **CHECK.k**

  At $m \geq 1$,

  1. $O_i^{m,t} \neq \emptyset$, and
  $$\#I_i^{m,t} \geq k$$
  , then **runs POST-CHECK**

  2. $O_i^{m,t} \neq \emptyset$, and
  $$\#I_i^{m,t} < k$$
  , then **runs CHECK.**m

- **POST-CHECK**

  1. At $m = |RP^t|$, then
     (a) If $i \in R^t$ and if $\#I_i^{m,t} \geq k-1$, then play **revolt**
     (b) if $i \notin R^t$, then play **stay**

A.1.4 Equilibrium path: coordination period

coordination period: notations

- m is a sub-block in the coordination period.

- Let
  $$Ex_{I_i^t} \equiv \{l \notin I_i^t | \exists l' \in I_i^t \setminus I^{t-1} \text{ such that there exists a } (l,l')\text{-path}\}$$
  be all the possible Rebel nodes outside of N_i^t given I_i^t.

- Let
  $$TR_{I_i^t j} \equiv TR_{ij} \cap (Ex_{I_i^t} \cup I_i^t)$$
  be the set of possible Rebel nodes in the TR_{ij} given I_i^t.

coordination period: automata

- **1st Division**

  In 1st division, for $t = 0$ block,

  - If $\#Ex_{I_i^t} \cup I_i^t < k$, then play **stay** forever.
  - If $\#Ex_{I_i^t} \cup I_i^t \geq k$, and if $i \notin R^1$, then play

    $$\langle \mathbf{stay} \rangle$$

  - If $\#Ex_{I_i^t} \cup I_i^t \geq k$, and if $i \in R^1$, then play

    $$\langle x_i \rangle$$

  In 1st division, for $t > 0$ block and for $1 \leq m \leq n$ sub-block,

  - If i has played $\langle 1 \rangle$, then play

    $$\langle x_i \rangle$$

  - If $\#Ex_{I_i^t} \cup I_i^t < k$, then play **stay** forever.
  - If $\#Ex_{I_i^t} \cup I_i^t \geq k$, and there are some $j \in \bar{G}_i$ have played $\langle \mathbf{stay} \rangle$, then play **stay** forever.
  - If $\#Ex_{I_i^t} \cup I_i^t \geq k$, and there is no $j \in \bar{G}_i$ has played $\langle \mathbf{stay} \rangle$, then play

    $$\langle x_i \rangle$$

- **2nd Division**

  In $t = 0$ block

  - If $i \notin R^1$, play

    $$\langle \mathbf{stay} \rangle$$

    .

  - If $i \in R^1$, and if $\#I_i^0 \geq k$, play

    $$\langle \mathbf{stay} \rangle$$

    .

  - If $i \in R^1$, if $\#I_i^0 < k$, if $\#Ex_{I_i^t} \cup I_i^t \geq k$ and if some $j \in \bar{G}_i$ have played play $\mathbf{1}_j$ in the 1st division, then play

    $$\langle \mathbf{stay} \rangle$$

    .

– If $i \in R^1$, if $\#I_i^0 < k$, if $\#Ex_{I_i^t} \cup I_i^t \geq k$ and if no $j \in \bar{G}_i$ has played play $\mathbf{1}_j$ in the 1st division, then play **stay** forever.

In $t > 0$ block, if there is no $j \in G_i$ such that j has played $\langle \mathbf{stay} \rangle$ in the **1st Division**, then run the following automata. Otherwise, play **stay** forever.

– $i \notin R^t$
   * In the 1-sub-block: play
   $$\langle \mathbf{stay} \rangle$$
   * In the $2 \leq m \leq t+1$ sub-blocks:
      1. If $i \in R^{t'}$ for some $t' \geq 0$ and if there is a $j \in R^{t'+1} \cap \bar{G}_i$ has played
         (a) $\langle \mathbf{stay} \rangle$ in $m = 1$ sub-block
         (b) or $\langle \mathbf{1}_j \rangle$ in $m \geq 2$ sub-blocks
         , then play
         $$\langle x_i \rangle$$
         in $m + 1$ sub-block.
      2. Otherwise, play
         $$\langle \mathbf{stay} \rangle$$
         in current sub-block
– $i \in R^t$
   * In the 1-sub-block:
      1. If i has played $\langle 1 \rangle$, then play
         $$\langle \mathbf{stay} \rangle$$
      2. If i has not played $\langle 1 \rangle$ and if there is a $j \in \bar{G}_i$ has played $\langle 1 \rangle$, then play
         $$\langle \mathbf{stay} \rangle$$
      3. If i has not played $\langle 1 \rangle$ and if there is no $j \in \bar{G}_i$ has played $\langle 1 \rangle$, then
         · If $\#I_i^{|RP^t|,t} \geq k$, then play
         $$\langle \mathbf{stay} \rangle$$
         · If $\#I_i^{|RP^t|,t} < k$, then play
         $$\langle \mathbf{1}_i \rangle$$
   * In the $m \geq 2$-sub-block:

1. If $i \in R^{t'}$ for some $t' \geq 0$ and if there is a $j \in R^{t'} \cap \bar{G}_i$ has played

   (a) $\langle \mathbf{stay} \rangle$ in $m = 1$ sub-block, or

   (b) $\langle \mathbf{1}_j \rangle$ in $m \geq 2$ sub-blocks

   , then play

   $$\langle x_i \rangle$$

   in $m + 1$ sub-block.

2. Otherwise, play

   $$\langle \mathbf{stay} \rangle$$

   in current sub-block.

- **3rd Division**

  1. **INITIATING**

     If i has observed $j \in \bar{G}_i$ has played

     (a) $\langle \mathbf{stay} \rangle$ in 1-sub-block in **2nd Division** or

     (b) $\langle \mathbf{1}_j \rangle$ in $m \geq 2$ sub-blocks **2nd Division** or

     (c) **s** in the **3rd Division**

     , then play **revolt** forever

  2. **NOT INITIATING**

     Otherwise, play **stay** in current period.

A.1.5 Proof for Theorem 2

The proof is organized as follows. In Claim 4 and Lemma 3.3, I show that a Rebel will learn $\#[Rebels](\theta) \geq k$ or $\#[Rebels](\theta) < k$ in the equilibrium path. Lemma 3.3 also show that the equilibrium path is ex-post efficient. Since that, there is a T such that a Rebel's static payoff after T is 1 if $\#[Rebels](\theta) \geq k$; is 0 if $\#[Rebels](\theta) \geq k$. Such payoff is the maximum static payoff contingent on θ after time T. In Claim 5, I show that if a Rebel makes detectable deviation, then there is a positive probability event E (by the full support assumption) contingent on this deviation such that his expected continuation static payoff is strictly lower than that in equilibrium path after T. Finally, in Claim 6, Claim 7, Claim 8, and Claim 9, I show that if a Rebel makes undetectable deviation, then there is a positive probability event E (by the full support assumption) contingent on this deviation such that his expected continuation static payoff is also strictly lower than that in the equilibrium path after T. Since the static payoff after T is maximum for all θ

in equilibrium path, there is a δ such that a Rebel will not deviate. I then conclude this theorem.

To simplify the notations, if $P(\theta)$ is a property of θ, then I abuse the notations by letting $\beta_{G_i}^{\pi,\tau^*}(P(\theta)|h_{G_i}^m) \equiv \sum_{\theta:P(\theta)} \beta_{G_i}^{\pi,\tau^*}(\theta|h_{G_i}^m)$. I also say "$i$ knows $P(\theta)$" to mean $\beta_{G_i}^{\pi,\tau^*}(P(\theta)|h_{G_i}^m) = 1$.

Claim 4. *Assume that players follow the equilibrium path. Then if $\#Ex_{I_i^{m,t}} \cup I_i^{m,t} \geq k$, where m is a period in RP^t, then if i reports $\langle 1 \rangle$, then Rebels coordinate to **revolt** after t-block, or $\#R^0 < k$.*

Proof. By directly checking the equilibrium path, we have

1. if $\#I_i^{|RP^t|,t} \geq k$, then the coordination can be initiated by such i.

2. if $\#I_i^{|RP^t|,t} = k - 1$, and if there is one more node who reported $\langle 1 \rangle$, then the coordination can be initiated by i.

3. if $\#I_i^{|RP^t|,t} = k - 1$, and if there are no nodes who reported in current reporting period, then $\#I_i^{|RP^t|,t} = \#I_i^t = k - 1$. We now check the conditions guiding i to **POST-CHECK**.

   - If i is coming from the conditions in **MAIN**, it means that there are no further Rebels outside I_i^{t-1}, thus outside $\bigcup_{k \in I_i^{t-1}} G_k$.
   - If i is coming from the conditions in **CHECK.0**, it means that there are no further Rebels outside $\bigcup_{k \in I_i^{t-1}} G_k \cap R^0$, and thus outside $\bigcup_{k \in I_i^{t-1}} G_k$.
   - If i is coming from the conditions in **CHECK.m**, it means that there are no further Rebels outside $\bigcup_{k \in I_i^{t-1}} G_k \cap R^0$, and thus outside $\bigcup_{k \in I_i^{t-1}} G_k$.

   Since $I_i^t = \bigcup_{k \in I_i^{t-1}} G_k \cap R^0 \subset \bigcup_{k \in I_i^{t-1}} G_k$ and $\#I_i^t < k$, and hence $\#R^0 < k$.

$\square$

proof for Lemma 3.3

Proof. We want to show that all Rebels play **revolt** eventually when θ satisfies $\#[Rebels](\theta) \geq k$; all Rebels play **stay** eventually when θ satisfies $\#[Rebels](\theta) < k$.

1. If all Rebels only play $\langle I^{t-1} \rangle$ or $\langle \mathbf{stay} \rangle$ in the reporting period for all $t \geq 1$ block, then, in the equilibrium path, those nodes played $\langle I^{t-1} \rangle$ are R^t-node, and those nodes played $\langle \mathbf{stay} \rangle$ are non-R^t nodes.

If there are some Rebels play $\langle \textbf{stay} \rangle$ in $CD_{1,1}^t$, then all the Rebels play **stay** eventually; If R^t Rebels play $\langle \textbf{stay} \rangle$ in $CD_{1,2}^t$, then all the Rebels will play **revolt** after third division in coordination period in this block. Otherwise, all the Rebels go to the next reporting period.

By Theorem 3, there is a t^* such that there is a R^{t^*} node knows θ, and therefore he knows if θ satisfying $\#[Rebels](\theta) \geq k$ or $\#[Rebels](\theta) < k$. In the equilibrium path, such node that plays $\langle \textbf{stay} \rangle$ is either in $CD_{1,1}^{t^*}$ or in $CD_{1,2}^{t^*}$. Thus, the equilibrium path is APEX.

2. If there are some Rebels play $\langle 1 \rangle$ in reporting period for a $t \geq 1$ block, then by Claim 4, such nodes will knows if θ satisfying $\#[Rebels](\theta) \geq k$ or $\#[Rebels](\theta) < k$ after reporting period in this t-block. Then $\langle \textbf{stay} \rangle$ is either played in the first sub-block in first division or played in the first sub-block in second division in coordination period. Thus, the equilibrium path is APEX.

$\square$

Next, I prepare the claims to show that a Rebel will not deviate. I start with Claim 5 in which the deviation is detectable.

Claim 5. *Assume that player i follows equilibrium path before m period. Denote D be the set of Rebels who detect i's deviation at m period. Then if $\#Ex_{I_i^{m,t}} \cup I_i^{m,t} \geq k$, then if $\#I_i^{m,t} < k$ and if $D \neq \emptyset$, there is a δ such that i will not deviate.*

Proof. Denote D be the set of neighbors who detect i's deviation. Let the events be

$$
\begin{aligned}
E_1 &= \{\theta : \#[Rebels](\theta) < k\} \\
E_2 &= \{\theta : k \leq \#[Rebels](\theta) < k + \#D\} \\
E_3 &= \{\theta : \#[Rebels](\theta) \geq k + \#D\}
\end{aligned}
$$

In the equilibrium path, there are periods t^s (t^f) such that, if θ satisfying $\#[Rebels](\theta) \geq k$ ($\#[Rebels](\theta) < k$) then Rebels play **revolt** (**stay**) forever. If i follows the equilibrium path, the expected static payoff after $\max\{t^s, t^f\}$[23] is

$$
\beta_i(E_2 | h_{N_i}^m) + \beta_i(E_3 | h_{N_i}^m)
$$

If i deviate, the expected static payoff after $\max\{t^s, t^f\}$ is

$$
\beta_i(E_3 | h_{N_i}^m)
$$

---

[23]There is t^s or t^f for each θ. The maximum is among those possible θ.

Therefore there is a loss in expected static payoff of

$$\beta_i(E_2|h_{N_i}^m)$$

Thus, there is a loss in expected continuation payoff contingent on E_2 by

$$\delta^{\max\{t^s, t^f\}} \frac{\beta_i(E_2|h_{N_i}^m)}{1-\delta}$$

Note that $\beta_i(E_2|h_{N_i}^m) > 0$, since $\#Ex_{I_i^{m,t}} \cup I_i^{m,t} \geq k$ and therefore $\beta_i(\#[Rebels](\theta) \geq k|h_{N_i}^m) > 0$ by full support assumption. $\qquad\square$

Next, I prepare the claims to show that a Rebel will not deviate if such deviation is undetectable.

Claim 6. *Assume that player i follows equilibrium path before m period. Then if $\#Ex_{I_i^{m,t}} \cup I_i^{m,t} \geq k$ and m is a period in RP^t, then if $\#I_i^{m,t} < k$, there is a δ such that i will not deviate by reporting $\bar{I}_i^{t-1} \neq I_i^{t-1}$ if such deviation is not detected by i's neighbor.*

Proof. Assume $\bar{I}_i^{t-1} \neq I_i^{t-1}$. Since a detection of deviation has not occur, it must be the case that there is a non-empty set $F = \{j \in \bar{I}_i^{t-1} : \theta_j = Inerts\}$[24].

Let the set

$$E_1 = \{\bar{\theta} : \bar{\theta}_j = Rebel \text{ if } j \in F \text{ and } \bar{\theta}_j = \theta_j \text{ if } j \notin F\}$$

be the set of pseudo events by changing θ_j where $j \in F$. And let

$$E_2 = \{\theta : \theta_j = Inert \text{ if } j \in F \text{ and } \bar{\theta}_j = \theta_j \text{ if } j \notin F\}$$

be the set of true event.

Then consider the event

$$
\begin{aligned}
E &= \{\bar{\theta} \in E_1 : \#[Rebels](\bar{\theta}) \geq k\} \\
&= \{\theta \in E_2 : \#[Rebels](\theta) \geq k - \#F\}
\end{aligned}
$$

Partition E as sub events

$$
\begin{aligned}
E_3 &= \{\theta \in E_2 : \#[Rebels](\theta) \geq k\} \\
E_4 &= \{\theta \in E_2 : k > \#[Rebels](\theta) \geq k - \#F\}
\end{aligned}
$$

By Lemma 3 and by following the strategies in equilibrium path (since i has not been detected), there is a block $\bar{t}^s$ with respect to $\bar{\theta}$ such that if $\bar{\theta} \in E$ then there some $R^{\bar{t}^s}$ Rebel js, says J, will initiate the coordination, and then Rebels play **revolt** forever after $\bar{t}^s$-block. Note that such j is with $\#I_i^{\bar{t}^s} \geq k$ by checking the equilibrium path.

We have several cases:

---

[24]Otherwise, there is a detection of deviation. Recall the definition in information hierarchy: $I_i^{-1} \subset I_i^0 \subset \dots \subset I_i^{t-1}$ for all $i \in R^0$

44

1. Case 1: If $i \in J$, his own initiation will only depends on $\#I_i^{\bar{t}^s}$ by Claim 7 and Claim 8, which is the same as he has reported $\langle I_i^{t-1} \rangle$. He is strictly better off by not deviating since playing $\langle \bar{I}_i^{t-1} \rangle$ is more costly than $\langle \bar{I}_i^{t-1} \rangle$ (since $X_{\bar{I}_i^{t-1}} > X_{I_i^{t-1}}$).

2. Case 2: If there is another j such that $\bar{I}_i^{t-1} \not\subset I_j^{\bar{t}^s}$, then since such j's initiation of coordination dependent on his own information $I_j^{\bar{t}^s}$ by Claim 7 and Claim 8, and i's deviation did not change j's information. It is strictly better by not deviating since playing $\langle \bar{I}_i^{t-1} \rangle$ is more costly than $\langle \bar{I}_i^{t-1} \rangle$ (since $X_{\bar{I}_i^{t-1}} > X_{I_i^{t-1}}$).

3. Case 3: Suppose there is another j such that $\bar{I}_i^{t-1} \subset I_j^{\bar{t}^s}$ and $\#I_i^{\bar{t}^s} \geq k$, then such j will initiate the coordination to **revolt**. If i did not follow j's initiation of coordination, then there is a detection of deviation by checking the equilibrium path. i will not deviate as Claim 5 shows. If i follows, and $\#I_i^{\bar{t}^s} \geq s$, we are in the Case 1. If i follows, but $\#I_i^{\bar{t}^s} < s$, then i's expected static payoff after $\bar{t}^s$ is at most

$$\max\{\beta_i(E_3|h_{N_i}^m) \times 1 + \beta_i(E_4|h_{N_i}^m) \times (-1), 0\}$$

However, if i follows the equilibrium path, there is are t^s, t^f such that the expected static payoff after $\max\{t^s, t^f\}$ is

$$\max\{\beta_i(E_3|h_{N_i}^{m'}), 0\}$$

Thus, there is a loss in expected continuation payoff contingent on E by

$$\delta^{\max\{t^s, t^f\}} \frac{\min\{\beta_i(E_3|h_{G_i}^m), \beta_i(E_4|h_{G_i}^m)\}}{1 - \delta}$$

Note that $\beta_i(E_3|h_{N_i}^m) > 0$ and $\beta_i(E_3|h_{N_i}^m) > 0$, since $\#Ex_{I_i^{m,t}} \cup I_i^{m,t} \geq k$ and $\#I_i^{m,t} < k$, and therefore $1 > \beta_i(\#[Rebels](\theta) \geq k|h_{N_i}^m) > 0$ by full support assumption. $\square$

Claim 7. *Assume that player i follows equilibrium path before m period. Then if $\#Ex_{I_i^{m,t}} \cup I_i^{m,t} \geq k$ and m is a period in RP^t, then if $\#I_i^{m,t} \leq k - 1$, i will not play $\langle 1 \rangle$ given that $i \notin C^t$ or i does not satisfy the conditions to play $\langle 1 \rangle$.*

Proof. Let

$$E' = \{\theta : \#I_i^{RP^t, t} \leq k - 1\}$$

. Note that such event is not empty by checking the timing where i deviated:

1. If i has a neighbor $j \in C^t$, then $j \notin O_i^{RP^t, t}$, and therefore we can construct E' by assuming that all other neighbors (other than i, j and other than $l \in O_i^{m,t}$) are non-R^t.

2. If
$$\exists j \in R^{t-1} \cap \bar{G}_i \text{ such that } \exists k_1, k_2 \in TR_{ij}[[k_1 \in N_j^{t-1}\backslash I_i^{t-1}] \wedge [k_2 \in \bar{G}_{k_2}]]$$
, then just let $E' = \{\theta : N_i^t \cap R^0 \leq k-1\} = \{\theta : I_i^t \leq k-1\}$[25].

Next, let

$$
\begin{aligned}
E_1 &= \{\theta : \#[Reble](\theta) < k\} \cap E' \\
E_2 &= \{\theta : \#[Reble](\theta) \geq k\} \cap E'
\end{aligned}
$$

Note that E_1 and E_2 are not empty. According to the equilibrium path, if i did not follow the conditions to play $\langle 1 \rangle$, it must be the case that there are some nodes outside I_i^t but there is a path consisting of Rebels to connect them. By strong connectedness, E_1 and E_2 are not empty.

Since i deviate to play $\langle 1 \rangle$, his behavior after $CD_{1,1}^t$ will decide the following three cases:

1. If i play $\langle \mathbf{stay} \rangle$ in $CD_{1,1}^t$, then the coordination to $\mathbf{stay}$ starts after $CD_{1,1}^t$.

2. If i play $\langle x_i \rangle$ in $CD_{1,1}^t$, then the coordination to $\mathbf{revolt}$ will be initiate after $CD_{1,2}^t$ if he mimic the behavior of a pivotal player (i.e., by mimicking those players who played $\langle 1 \rangle$ in the equilibrium path).

3. If i play $\langle x_i \rangle$ in $CD_{1,1}^t$, but he did not mimic the behavior of pivotal player, then such deviation will be detected.

Thus, i's expected static payoff after the coordination period in this t-block is at most

$$\max\{\beta_i(E_2|h_{N_i}^m) \times 1 + \beta_i(E_1|h_{N_i}^m) \times (-1), 0\}$$

However, if he stay in the equilibrium, there is a t^s (t^f) such that Rebels play $\mathbf{revolt}$ ($\mathbf{stay}$) contingent on E_2 (E_1), and thus after $t^* = \max\{t^s, t^f\}$ he get the expected payoff as

$$\max\{\beta_i(E_2|h_{N_i}^m) \times 1, 0\}$$

After some calculation, after t^*, there is a loss of

$$\delta^{t^*} \frac{\min\{\beta_i(E_2|h_{G_i}^m), \beta_i(E_1|h_{G_i}^m)\}}{1-\delta}$$

Note that $\beta_i(E_1|h_{N_i}^m) > 0$ and $\beta_i(E_2|h_{N_i}^m) > 0$, by E_1 and E_2 are not empty and by full support assumption.

$\square$

---

[25]note that $I_i^t = I_i^{RP^t, t}$

Claim 8. *Assume that player i follows equilibrium path before $(RP)^t - 1$ period. Then if $\beta_i(\#[Rebels](\theta) \geq k | h_{G_i}^{(RP)^t-1}) > 0$, then if i can report $\langle 1 \rangle$, i will not report $\langle \boldsymbol{stay} \rangle$ when δ is high enough.*

Proof. There are two cases when i can play $\langle 1 \rangle$.

- Case 1: If $\#I_i^{|RP^t|-1,t} \geq k$, let the event E be

$$E = \{\theta : \#[Rebels](\theta) = \#I_i^{|RP^t|,t}\}$$

  That is, the event that no more Rebels outside i's information about Rebels. Contingent on E, there is no more Rebel can initiate the coordination. This is because for all $j \in O_i^{|RP^t|-1,t}$, j is with $\#I_j^{t-1} < k-1$, and for all $j \in \bar{G}_i$ who have not yet reported, $j \notin R^t$ since all the Rebels are in $I_i^{|RP^t|-1,t}$. Since only i can initiate the coordination, if i deviated, compared to equilibrium, there is a loss in expected continuation payoff as

$$\delta^q \frac{\beta_i(E|h_{G_i}^{(RP)^t-1})}{1-\delta}$$

  , where q is a period after t-block.

- Case 2: If $\#I_i^{|RP^t|-1,t} = k - 1$, since $\beta_i(\#[Rebels](\theta) \geq k | h_{G_i}^{|RP^t|}) > 0$, the following event E_1 must have positive probability; otherwise, since no neighbors can report after current period, and thus $\beta_i(\#[Rebels](\theta) \geq k | h_{G_i}^{|RP^t|}) = 0$.

  Let

$$E_1 = \{\theta : \exists j \in \bar{G}_i, j \notin O_i^{|RP^t|-1,t}[\#I_j^{|RP^t|-1,t} \geq k-1]\}$$

  Let sub-events $E_1' \subset E_1$ as

$$E_1' = \{\theta : \text{ exist a unique} j \in \bar{G}_i, j \notin O_i^{|RP^t|-1,t}[\#I_j^{|RP^t|-1,t} \geq k-1]\}$$

  Note that this E_1' can be constructed since the network is tree. If there is θ admits 2 or more js in the definition E_1, these js are not each others' neighbor. Suppose there are two js, says j, j', there must be at least one node in $I_j^{|RP^t|-1,t}$ but outside of $I_{j'}^{|RP^t|-1,t}$. We then pick a j, and suppose those nodes outside $I_j^{|RP^t|-1,t}$ are all Inerts.

  Now, dependent on such j, let

$$E = \{\theta : \#[Rebels](\theta) = \#I_j^{|RP^t|-1,t} \cup I_i^{|RP^t|-1,t}\}$$

  If i report $\langle \boldsymbol{stay} \rangle$, there are following consequences.

- j will believe that $i \notin R^t$, and thus i can not initiate the coordination.

- Such j has $\#I_j^{|RP^t|,t} = \#I_j^t < k$. Since there is no more Rebel outside $I_j^{|RP^t|-1,t} \cup I_i^{|RP^t|-1,t}$ contingent on E, such j will then play stay forever after t-block.

- Without such extra Rebels in $I_j^{|RP^t|,t}$, only $\#I_i^{|RP^t|-1,t} = k - 1$ Rebels may play **revolt**, and therefore there is no coordination to **revolt**

However, if i play $\langle 1 \rangle$, coordination can be initiated by himself in the following coordination period. Thus, there is a loss in expected continuation payoff by

$$\delta^q \frac{\beta_i(E|h_{G_i}^{(RP)^t-1})}{1-\delta}$$

, where q is a period after t-block.

$\square$

Claim 9. *Assume that player i follows equilibrium path before $(RP)^t$. Then, suppose there is no $j \in G_i$ has played $\langle 1 \rangle$ in RP^t, suppose $\#I_i^t < k$, and suppose $\#Ex_{I_i^t} \cup I_i^t \geq k$, then there is δ such that*

- *if i has not observed $\langle \boldsymbol{stay} \rangle$ played by $j \in G_i$ in $CD_{1,2}^t$, or*

- *if i has not observed $\langle \boldsymbol{1}_j \rangle$ played by $j \in G_i$ in $CD_{q,2}^t$, $g \geq 2$*

, then i will not play

- *$\langle \boldsymbol{stay} \rangle$ in $CD_{1,2}^t$ and*

- *$\langle \boldsymbol{1}_j \rangle$ in $CD_{q+1,2}^t$, $g \geq 2$*

Proof. If i deviate, all i's neighbor who did not detect the deviation will play **revolt** after coordination period in this block; if i's deviation is detected by some neighbors, we are in the case of Claim 5 and so that i will not deviate. We then check if i deviate but no neighbor detect it. Let

$$E' = \{\theta : \#I_i^t \leq k - 1\}$$

and let

$$\begin{aligned} E_1 &= \{\theta : \#[Reble](\theta) < k\} \cap E' \\ E_2 &= \{\theta : \#[Reble](\theta) \geq k\} \cap E' \end{aligned}$$

Since $\#I_i^t < k$ and $\#Ex_{I_i^t} \cup I_i^t \geq k$, due to the full support assumption and the equilibrium strategies played by i's neighbors, we have

$$0 < \beta_i(\#[Rebels](\theta) \geq k | h_{G_i}^m) < 1$$

, and thus E_1 and E_2 have positive probability. Since after i's deviation, all the Rebels will play **revolt** after this block, i's expected static payoff after the coordination period in this t-block is at most

$$\max\{\beta_i(E_2 | h_{N_i}^m) \times 1 + \beta_i(E_1 | h_{N_i}^m) \times (-1), 0\}$$

However, if he stay in the equilibrium, there is a t^s (t^f) such that Rebels play **revolt** (**stay**) contingent on E_2 (E_1), and thus after $t^* = \max\{t^s, t^f\}$ he get the expected payoff as

$$\max\{\beta_i(E_2 | h_{N_i}^m) \times 1, 0\}$$

After some calculation, after t^*, there is a loss of

$$\delta^{t^*} \frac{\min\{\beta_i(E_2 | h_{G_i}^m), \beta_i(E_1 | h_{G_i}^m)\}}{1 - \delta}$$

$\square$

After the above claims, we can take a sufficiently high δ to let all the above claims hold. Since a deviation is either detectable or non-detectable, and a deviation happens either in reporting period or coordination period, I conclude that this theorem holds by above claims.