The second-order geometry of short-and-sparse blind deconvolution

Yenson Lau

March 22, 2019

Abstract

These notes describe a set of experiments for interrogating the curvature of nonconvex objective functions for short-and-sparse blind deconvolution (SaS-BD) over the course of optimization.

1 Introduction

In SaS-BD, we observe a length-m signal

$$\mathbf{y} \;=\; \iota\mathbf{a}_0 \,\circledast\, \mathbf{x}_0 \,+\, \mathbf{w},$$

the cyclic convolution between a *short* kernel $\mathbf{a}_0 \in \mathbb{S}^{p_0-1}$, zero-padded to length m, and a *sparse* activation map $\mathbf{x}_0 \in \mathbb{R}^m$, with possible additive noise $\mathbf{w}$. Our goal is to recover $\mathbf{a}_0$ and $\mathbf{x}_0$ up to some scaling or cyclic shift, as

$$\iota\mathbf{a}_0 \,\circledast\, \mathbf{x}_0 \;\equiv\; \alpha\, s_\ell[\iota\mathbf{a}_0] \,\circledast\, \alpha^{-1} s_{-\ell}[\mathbf{x}_0]$$

for any $\alpha \in \{\mathbb{R} \setminus 0\}$ and $\ell \in \mathbb{Z}$. We refer to these as the scaling and shift symmetries respectively.

A natural approach is to begin by formulating SaS-BD as a nonconvex optimization problem,

$$\min_{\mathbf{a}\in\mathbb{S}^{p-1},\mathbf{x}} \;\Big[\; \Psi_\lambda(\mathbf{a},\mathbf{x}) \;\doteq\; \psi(\mathbf{a},\mathbf{x};\mathbf{y}) \;+\; \lambda\rho(\mathbf{x}) \;\Big], \tag{1.1}$$

which minimizes a reconstruction error between $\mathbf{y}$ and $\mathbf{a} \circledast \mathbf{x}$, plus a sparse penalty ρ on $\mathbf{x}$. For example, combining a squared error loss with an ℓ_1-norm penalty yields the *bilinear-lasso* (BL) formulation

$$\Psi_\lambda(\mathbf{a},\mathbf{x}) \;=\; \tfrac{1}{2}\left\|\mathbf{a} \circledast \mathbf{x} - \mathbf{y}\right\|_2^2 \;+\; \lambda \left\|\mathbf{x}\right\|_1. \tag{BL}$$

A simplified problem, known as the *dropped-quadratic* formulation (DQ), can be obtained when the circulant matrix of $\mathbf{a}$, $\mathbf{C}_\mathbf{a}$, satisfies $\mathbf{C}_\mathbf{a}^T\mathbf{C}_\mathbf{a} \simeq \mathbf{I}$ over the course of optimization,

$$\begin{aligned}\Psi_\lambda(\mathbf{a},\mathbf{x}) \;&=\; \tfrac{1}{2}\left\|\mathbf{x}\right\|_2^2 \;-\; \langle \mathbf{a} \circledast \mathbf{x}, \mathbf{y}\rangle \;+\; \tfrac{1}{2}\left\|\mathbf{y}\right\|_2^2 + \lambda\rho(\mathbf{x}), & \text{(DQ)}\\ \rho(\mathbf{x}) \;&=\; \textstyle\sum_i \left(x_i^2 + \delta^2\right)^{\frac{1}{2}}. & \text{(1.2)}\end{aligned}$$

1.1 Regional landscape geometry

When $\mathbf{x}_0$ is generic, the objective function becomes regular on $\mathbf{a}$ after marginalization on $\mathbf{x}$,

$$(1.1) \;\equiv\; \min_{\mathbf{a}\in\mathbb{S}^{p-1}} \left[\, \varphi_\lambda(\mathbf{a}) \;\doteq\; \min_{\mathbf{x}} \;\Psi_\lambda(\mathbf{a},\mathbf{x}) \,\right]. \tag{1.3}$$

This phenomenon appears to be *largely independent* of the specific formulation used for SaS-BD, but finds a relatively simple expression in the (DQ) formulation, for which is summarized as follows [KLZW19]:

1.2 Sparsity-coherence tradeoff

2 Experimental outline

References

[KLZW19] Han-Wen Kuo, Yenson Lau, Yuqian Zhang, and John Wright. Geometry and symmetry in short-and-sparse deconvolution. *arXiv preprint arXiv:1901.00256*, 63(7):4497–4520, 2019.