TD 2

Exercice 2.8 P 71

Soit X : "La demande quotidienne"

x	-1	0	1	2
$p(x)$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{2}{5}$	$\frac{3}{10}$

a) $E(X) = \sum_{i=1}^{4} x_i \cdot p(x_i)$

$$= (-1)\times\frac{1}{5} + 0\times\frac{1}{10} + 1\times\frac{2}{5} + 2\times\frac{3}{10}$$

$$= \frac{8}{10} = \frac{4}{5}$$

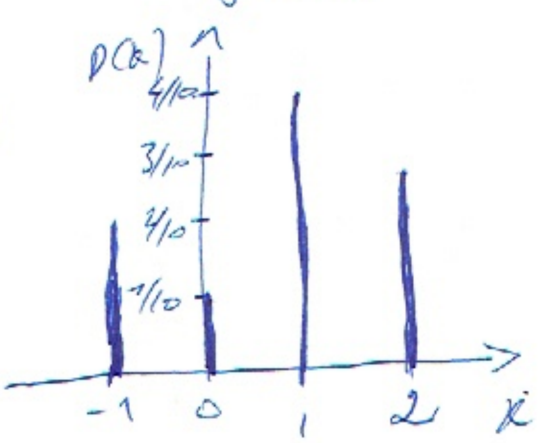
b) $E(X^2) = \frac{9}{5}$

$$V(X) = E(X^2) - E(X)^2$$

$$= \frac{9}{5} - \left(\frac{4}{5}\right)^2 = \frac{9}{5} - \frac{16}{25} = \frac{29}{25}$$

$$\sigma = \sqrt{V(X)} = \sqrt{\frac{29}{25}}$$

c)



d) si $x < -1$ $\quad F(x) = P(X \le x) = 0$

si $-1 \le x < 0$ $\quad F(x) = P(X \le x) = P(X=-1) = \frac{1}{5}$

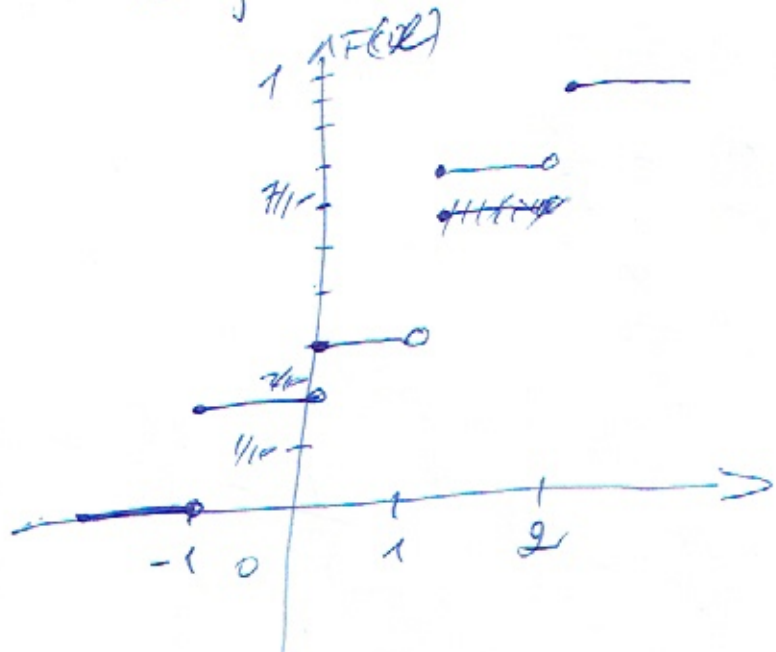
si $0 \le x < 1$ $\quad F(x) = P(X \le x) = P(X=-1) + P(X=0) = \frac{1}{5} + \frac{1}{10} = \frac{3}{10}$

si $1 \le x < 2$ $\quad F(x) = P(X \le x) = P(X=-1) + P(X=0) + P(X=1)$

$$= \frac{1}{5} + \frac{1}{10} + \frac{2}{5} = \frac{7}{10}$$

si $x \ge 2$ $\quad F(x) = P(X \le x) = P(X=-1) + P(X=0) + P(X=1) + P(X=2)$

$$= \frac{1}{5} + \frac{1}{10} + \frac{2}{5} + \frac{3}{10} = 1$$



Exercice 2.11 p72

$$f(x) = \begin{cases} kx & \text{si } 0 \le x < 2 \\ k(4-x) & \text{si } 2 \le x \le 4 \\ 0 & \text{sinon} \end{cases}$$

a) f étant une fonction de densité

donc $\int_{-\infty}^{+\infty} f(x)dx = 1$

$$\int_{-\infty}^{0} f(x)dx + \int_{0}^{2} f(x)dx + \int_{2}^{4} f(x)dx + \int_{4}^{+\infty} f(x)dx = 1$$

$$\int_{-\infty}^{0} 0\,dx + \int_{0}^{2} kx\,dx + \int_{2}^{4} k(4-x)dx + \int_{4}^{+\infty} 0\,dx = 1$$

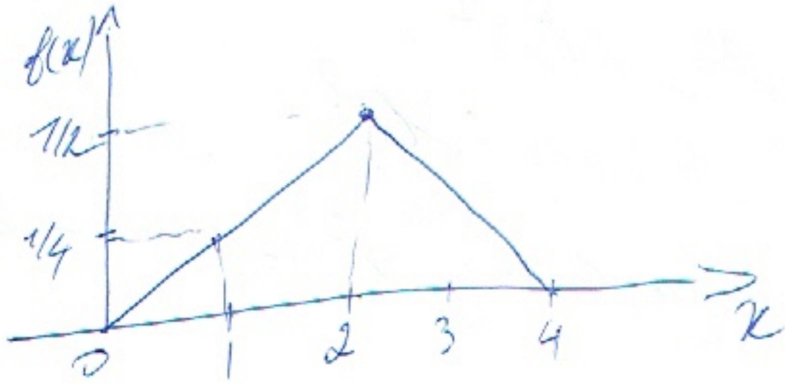
$$k\left[\frac{1}{2}x^2\right]_0^2 + k\left[4x - \frac{1}{2}x^2\right]_2^4 = 1$$

$$k\left(\frac{4}{2} + 16 - 8 - 8 + 2\right) = 1$$

$$4k = 1 \Rightarrow k = \frac{1}{4}$$

$$f(x) = \begin{cases} 1/4\, x & \text{si } 0 \le x < 2 \\ 1 - \frac{1}{4}x & \text{si } 2 \le x \le 4 \\ 0 & \text{sinon} \end{cases}$$

b)



c) $P(X<1 \mid X<2) = \dfrac{P(X<1 \cap X<2)}{P(X<2)}$ $\quad \dfrac{X<1 \cap X<2}{\cancel{\quad}}$

$$= \frac{P(X<1)}{P(X<2)} = \frac{\int_{-\infty}^{1} f(x)dx}{\int_{-\infty}^{2} f(x)dx}$$

$$= \frac{\int_0^1 \frac{1}{4}x\,dx}{\int_0^2 \frac{1}{4}x\,dx} = \frac{[\frac{1}{8}x^2]_0^1}{[\frac{1}{2}x^2]_0^2} = \frac{1/8}{4/8} = \frac{1}{4}$$

$$= \frac{1}{4}$$

d) $\mu = E(X) = \int_{-\infty}^{+\infty} x\, f(x)dx$

$$= \int_0^2 x \cdot \frac{1}{4}x\,dx + \int_2^4 x\cdot(1-\frac{1}{4}x)dx$$

$$= \left[\frac{1}{12}x^3\right]_0^2 + \left[\frac{1}{2}x^2 - \frac{1}{12}x^3\right]_2^4$$

$$= \frac{8}{12} + \left(\frac{16}{2} - \frac{64}{2} - \frac{4}{2} + \frac{8}{12}\right)$$

$$= \frac{2}{3} + 8 - \frac{16}{3} - 2 + \frac{2}{3} = 2$$

$$\sigma^2 = V(X) = E(X^2) - E(X)^2$$

$$= \int_{-\infty}^{+\infty} x^2 f(x)dx - \mu^2$$

$$= \int_0^2 x^2 \cdot \frac{1}{4}x\,dx + \int_2^4 x^2(1-\frac{1}{4}x)dx - 2^2$$

$$\sigma^2 = \frac{1}{16}x^4\Big|_0^2 + \frac{x^3}{3} - \frac{x^4}{16}\Big|_2^4 - 4$$

$$= \frac{16}{16} + \left(\frac{64}{3} - 16 - \frac{8}{3} + 1\right) - 4 = \frac{2}{3}$$

$$\sigma^2 = 2/3$$

e) si $x < 0$ $F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} 0\,dt = 0$

si $0 \leq x < 2$ $F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_0^x f(t)\,dt$

$$= \int_0^x \frac{1}{4}t\,dt = \frac{1}{8}t^2\Big|_0^x$$

$$= \frac{x^2}{8}$$

si $2 \leq x < 4$ $F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_2^x 1 - \frac{1}{4}t\,dt + \int_0^2 \frac{1}{4}t\,dt$

$$= \frac{1}{8}t^2\Big|_0^2 + t - \frac{1}{8}t^2\Big|_2^x = x - \frac{x^2}{8} - 2 + \frac{4}{8} + \frac{4}{8}$$

$$= x - \frac{x^2}{8} - 1 = -1 + x - \frac{x^2}{8}$$

si $x \geq 4$ $F(x) = \int_{-\infty}^{x} f(t)\,dt$

$$= \int_0^2 \frac{1}{4}t\,dt + \int_2^4 1 - \frac{1}{4}t\,dt + \int_4^x 0\,dt$$

$$= \frac{1}{8}t^2\Big|_0^2 + t - \frac{1}{8}t^2\Big|_2^4$$

$$= \frac{4}{8} + 4 - \frac{16}{8} - 2 + \frac{4}{8} = 1$$

$$F(x) = \begin{cases} 0 & \text{si } x < 0 \\ \frac{x^2}{8} & \text{si } 0 \leq x < 2 \\ -1 + x - \frac{x^2}{8} & 2 \leq x < 4 \\ 1 & \text{si } x \geq 4 \end{cases}$$

Exercice 3

$$f_X(x) = \begin{cases} \frac{1}{2\theta} \text{ si } -\theta \leq x \leq \theta \\ 0 \text{ sinon} \end{cases}$$

a) f_X étant une densité donc $\int_{-\infty}^{+\infty} f_X(x)dx = 1$

$$\int_{-\theta}^{\theta} \frac{1}{2\theta} dx = 1$$

$$\frac{x}{2\theta}\Big|_{-\theta}^{\theta} = 1 \Rightarrow \frac{\theta}{2\theta} + \frac{\theta}{2\theta} = 1$$

$$\mu = E(X) = \int_{-\infty}^{+\infty} x f_X(x) dx$$

$$= \int_{-\theta}^{\theta} \frac{x}{2\theta} dx = \frac{x^2}{4\theta}\Big|_{-\theta}^{\theta} = \frac{\theta^2 - \theta^2}{4\theta} = 0$$

$$\sigma^2 = E(X^2) - E(X)^2 = \int_{-\infty}^{+\infty} x^2 f_X(x) dx - \mu^2$$

$$= \int_{-\theta}^{\theta} \frac{x^2}{2\theta} dx - \mu^2$$

$$\frac{x^3}{6\theta}\Big|_{-\theta}^{\theta} - 0 = \frac{\theta^3 + \theta^3}{6\theta} = \frac{\theta^2}{3} = 1^2$$

$$\Rightarrow \theta = \sqrt{3}$$

$$\sigma = \frac{\theta}{\sqrt{3}} = 1$$

b) $\mu = 0$

Exercice 4

soit T : "la durée avant la 1ère panne"

$$f_T(t) = \begin{cases} \frac{1}{4} e^{-t/4} & \text{si } t \geq 0 \\ 0 & \text{sinon} \end{cases}$$

a) $P(T \leq 1) = \int_{-\infty}^{1} f_T(t)dt$

$= \int_0^1 \frac{1}{4} e^{-1/4 t} dt = \left[- e^{-1/4 t} \right]_0^1$

$= 1 - e^{-1/4}$

b) soit Y : le profit réalisé

si $T \leq 1$ $\quad Y = 200 - 200 = 0$

si $T > 1$ $\quad Y = 200$

$P(Y=0) = P(T \leq 1) = 1 - e^{-1/4}$

$P(Y=200) = P(T > 1) = 1 - P(T \leq 1) = e^{-1/4}$

Y	0	200
$P(Y)$	$1 - e^{-1/4}$	$e^{-1/4}$

$E(Y) = 0 \times P(Y=0) + 200 \times P(Y=200) = 0 \times (1 - e^{-1/4}) + 200 \times e^{-1/4}$

$= 200\, e^{-1/4} = 155,76$ \$

c) $F(x) = \int_{-\infty}^{x} f_X(x)\,dx$

$f_X(x) = \begin{cases} \frac{1}{2\sqrt{3}} & \text{si } -\sqrt{3} \le x \le \sqrt{3} \\ 0 & \text{sinon} \end{cases}$

si $x < -\sqrt{3}$

$F(x) = \int_{-\infty}^{x} f_X\,dt = \int_{-\infty}^{x} 0\,dt = 0$

si $-\sqrt{3} \le x < \sqrt{3}$

$F(x) = \int_{-\infty}^{x} f_X(t)\,dt = \int_{-\infty}^{-\sqrt{3}} 0\,dt + \int_{-\sqrt{3}}^{x} f(t)\,dt$

$= \int_{-\sqrt{3}}^{x} \frac{1}{2\sqrt{3}}\,dt = \frac{t}{2\sqrt{3}}\Big|_{-\sqrt{3}}^{x} = \frac{x}{2\sqrt{3}} + \frac{1}{2}$

si $x \ge \sqrt{3}$ $\quad F(x) = \int_{-\infty}^{x} f(t)\,dt$

$F(x) = \int_{-\infty}^{-\sqrt{3}} 0\,dt + \int_{-\sqrt{3}}^{\sqrt{3}} \frac{1}{2\sqrt{3}}\,dt + \int_{\sqrt{3}}^{+\infty} 0\,dt$

$= \frac{t}{2\sqrt{3}}\Big|_{-\sqrt{3}}^{\sqrt{3}} = \frac{\sqrt{3}+\sqrt{3}}{2\sqrt{3}} = 1$

$$F(x) = \begin{cases} 0 & \text{si } x < -\sqrt{3} \\ \frac{x}{2\sqrt{3}} + \frac{1}{2} & \text{si } -\sqrt{3} \le x < \sqrt{3} \\ 1 & \text{si } x \ge \sqrt{3} \end{cases}$$

Exercice 5

D: Defectueux

$R_X = \{2,3,4,5,6,7,8,9\}$

a) $P(X=2) = \frac{3}{10} \cdot \frac{2}{9} = \frac{1}{15}$ $\quad DD$

$P(X=3) = \frac{2!}{1!1!} \cdot \frac{7}{10} \cdot \frac{3}{9} \cdot \frac{2}{8} = \frac{7}{60}$ $\quad \bar{D}DD$, $D\bar{D}D$

$P(X=4) = \frac{3!}{2!1!} \frac{7}{10} \frac{6}{9} \frac{3}{8} \frac{2}{7} = \frac{3}{20}$ $\quad \begin{cases} \bar{D}\bar{D}DD \\ \bar{D}D\bar{D}D \\ D\bar{D}\bar{D}D \end{cases}$

$P(X=5) = \frac{4!}{3!1!} \frac{7}{10} \cdot \frac{6}{9} \cdot \frac{5}{8} \cdot \frac{3}{7} \cdot \frac{2}{6} = \frac{1}{6}$

$P(X=6) = \frac{5!}{4!1!} \cdot \frac{7}{10} \cdot \frac{6}{9} \cdot \frac{5}{8} \cdot \frac{4}{7} \cdot \frac{3}{6} \cdot \frac{2}{5} = \frac{1}{6}$

$P(X=7) = \frac{6!}{5!1!} \cdot \frac{7}{10} \cdot \frac{6}{9} \cdot \frac{5}{8} \cdot \frac{4}{7} \cdot \frac{3}{6} \cdot \frac{3}{5} \cdot \frac{2}{4} = \frac{3}{20}$

$P(X=8) = \frac{7!}{6!1!} \cdot \frac{7}{10} \cdot \frac{6}{9} \cdot \frac{5}{8} \cdot \frac{4}{7} \cdot \frac{3}{6} \cdot \frac{2}{5} \cdot \frac{3}{4} \cdot \frac{2}{3} = \frac{7}{60}$

$P(X=9) = \frac{8!}{7!1!} \cdot \frac{7}{10} \cdot \frac{6}{9} \cdot \frac{5}{8} \cdot \frac{4}{7} \cdot \frac{3}{6} \cdot \frac{2}{5} \cdot \frac{1}{4} \cdot \frac{3}{3} \cdot \frac{2}{2} = \frac{1}{15}$

$$F(x) = \begin{cases} 0 & si \ x < 2 \\ 1/15 & si \ 2 \le x < 3 \\ \frac{11}{60} & si \ 3 \le x < 4 \\ 1/3 & si \ 4 \le x < 5 \\ 1/2 & si \ 5 \le x < 6 \\ 2/3 & si \ 6 \le x < 7 \\ 49/60 & si \ 7 \le x < 8 \\ 14/15 & si \ 8 \le x < 9 \\ 1 & si \ x \geq 9 \end{cases}$$

c) $\mu = E(X) = \sum_{i=1}^{8} x_i \, p(x_i) = \frac{11}{2}$

$\sigma^2 = V(X) = \sum_{i=1}^{8} x_i^2 \, p(x_i) - \mu^2$

$= 3,85 \Rightarrow \sigma = 1,96$