

Introduction to Electric and Magnetic Fields B38EM

Tutorial Week #3 – Solutions

$$\epsilon_0 = 8.85 \times 10^{-12} \text{ Fm}^{-1}, \quad q_{e^-} = 1.6 \times 10^{-19} \text{ C}$$

1. A charge of $Q_1 = 3 \times 10^{-4} \text{ C}$ is located at $R_1 (1,2,3)$ and a charge of $Q_2 = -10^{-4} \text{ C}$ is located at $R_2 (2,0,5)$ in a vacuum. Calculate the force exerted on Q_2 by Q_1 ?

Solution:

Force exerted on charge Q_2 by charge Q_1 :
$$F_{21} = \frac{Q_2 Q_1}{4\pi\epsilon_0} \frac{R_2 - R_1}{|R_2 - R_1|^3}$$

Find the position vectors:

$$\begin{aligned} R_1 &= \hat{x} + 2\hat{y} + 3\hat{z} \\ R_2 &= 2\hat{x} + 5\hat{z} \end{aligned}$$

The vector drawn between Q_1 and Q_2 :

$$R_2 - R_1 = (2\hat{x} + 5\hat{z}) - (\hat{x} + 2\hat{y} + 3\hat{z}) = \hat{x} - 2\hat{y} + 2\hat{z}$$

For a three-dimensional vector $a = (a_1, a_2, a_3)$, the formula for its magnitude is:

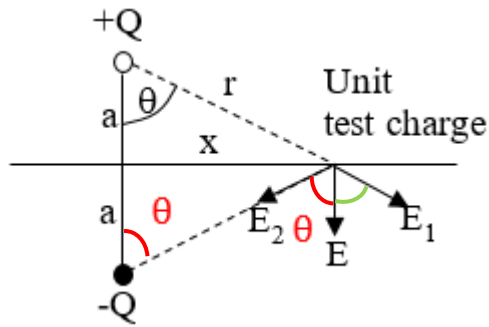
$$\|a\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

$$\begin{aligned} F_{21} &= \frac{-10^{-4} * 3 * 10^{-4}}{4 * \pi * 8.85 * 10^{-12}} \frac{\hat{x} - 2\hat{y} + 2\hat{z}}{\left(\sqrt{(-1)^2 + 2^2 + (-2)^2}\right)^3} = -\frac{269.89}{27} (\hat{x} - 2\hat{y} + 2\hat{z}) \\ &= -(\hat{x} - 2\hat{y} + 2\hat{z}) 9.99 \text{ (N)} \end{aligned}$$

2. Two equal but opposite charges Q are separated by a distance $2a$. Use Coulomb's law and the principle of superposition to find expressions for the electric field \mathbf{E} and electric potential V along the line through the midpoint between the charges and normal to the axis of the charges.

Solution:

The horizontal contributions of the electric field produced by the two charges will cancel each other as the charges are of opposite polarity and of same magnitude. It means that the field has only a vertical component.



$$\text{Triangle: } \cos \theta = \frac{a}{r}, r^2 = a^2 + x^2$$

$$|E_1| = |E_2| = \frac{Q}{4\pi\epsilon_0 r^2}$$

$$\cos \theta = \frac{|E|}{|E_2|} \Rightarrow |E| = |E_2| \cos \theta$$

$$\cos \theta = \frac{|E|}{|E_1|} \Rightarrow |E| = |E_1| \cos \theta$$

$$|E| = E_{1y} + E_{2y} = |E_1| \cos \theta + |E_2| \cos \theta$$

$$|E| = \frac{2Q}{4\pi\epsilon_0(a^2 + x^2)} \frac{a}{\sqrt{a^2 + x^2}}$$

$$E(x) = \frac{Qa}{2\pi\epsilon_0(a^2 + x^2)^{\frac{3}{2}}}$$

In the case of same polarity of charges and equal magnitude, the situation will be the opposite ie no vertical contribution of the total electrical field but a horizontal component.

3. A charge of 3 nC is located at the origin, and another charge of +5 nC is located at 0.3 m along the positive x-axis. Determine the position where the electric field is zero. (remark: nC means nanoCoulomb, which is 10^{-9} C.)

Solution:

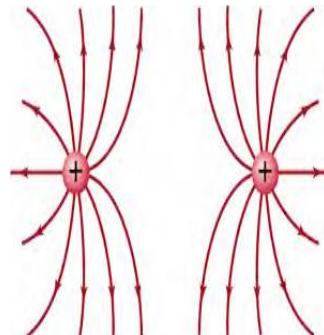


Figure 1 Electrostatic distributions of double charge

It follows from symmetry considerations that \mathbf{E} is zero at some point situated between the charges on the x-axis.

Let it be the point with coordinates (x,0,0). Then:

$$|E_1| = |E_2| \Rightarrow \frac{q_1}{4\pi\epsilon_0 x^2} = \frac{q_2}{4\pi\epsilon_0 (0.3 - x)^2} \Rightarrow \frac{q_1}{x^2} = \frac{q_2}{(0.3 - x)^2}$$

$$x = 0.131 \text{ and } x = -1.03$$

4. Four positive identical charges of 50nC each are located at $A(1,0,0)$, $B(-1,0,0)$, $C(0,1,0)$, and $D(0,-1,0)$ in free space (the coordinates are in meters). Find the total force on the charge at A

Find position vectors: $R_A = \hat{x}$

$$R_B = -\hat{x}$$

$$R_C = \hat{y}$$

$$R_D = -\hat{y}$$

Find forces:

$$F_{AA} = 0$$

$$F_{AB} = \frac{Q_A Q_B}{4\pi\epsilon_0} \frac{R_A - R_B}{|R_A - R_B|^3} = \frac{Q_A Q_B}{4\pi\epsilon_0} \frac{2\hat{x}}{(\sqrt{(2)^2})^3} = \frac{Q_A Q_B}{4\pi\epsilon_0} \frac{1}{8} 2\hat{x}$$

$$= \frac{Q_A Q_B}{4\pi\epsilon_0} \frac{1}{4} \hat{x}$$

$$F_{AC} = \frac{Q_A Q_C}{4\pi\epsilon_0} \frac{R_A - R_C}{|R_A - R_C|^3} = \frac{Q_A Q_C}{4\pi\epsilon_0} \frac{\hat{x} - \hat{y}}{\left(\sqrt{(-1)^2 + 1^2}\right)^3}$$

$$= \frac{Q_A Q_C}{4\pi\epsilon_0} \frac{1}{\sqrt{2}^3} (\hat{x} - \hat{y})$$

$$= \frac{Q_A Q_C}{4\pi\epsilon_0} \frac{1}{(\sqrt{2})^2 (\sqrt{2})^1} (\hat{x} - \hat{y})$$

$$= \frac{Q_A Q_C}{4\pi\epsilon_0} \frac{1}{2\sqrt{2}} (\hat{x} - \hat{y})$$

$$= \frac{Q_A Q_C}{4\pi\epsilon_0} \frac{1}{2} \frac{\sqrt{2}}{2} (\hat{x} - \hat{y})$$

$$F_{AD} = \frac{Q_A Q_D}{4\pi\epsilon_0} \frac{R_A - R_D}{|R_A - R_D|^3} = \frac{Q_A Q_D}{4\pi\epsilon_0} \frac{\hat{x} + \hat{y}}{(\sqrt{1^2 + 1^2})^3}$$

$$= \frac{Q_A Q_D}{4\pi\epsilon_0} \frac{1}{\sqrt{2}^3} (\hat{x} + \hat{y})$$

$$= \frac{Q_A Q_D}{4\pi\epsilon_0} \frac{1}{2} \frac{\sqrt{2}}{2} (\hat{x} + \hat{y})$$

Total Force:

$$F = \frac{(50 \times 10^{-9})^2}{4\pi\epsilon_0} \left(\frac{1}{4} + \frac{1}{2} \frac{\sqrt{2}}{2} + \frac{1}{2} \frac{\sqrt{2}}{2} \right) \hat{x} = 21.5 \hat{x} \mu N$$

5. A charged circular annulus is defined by two circles of centre O, and radii a and R ($a < R$), of surface charge density σ .
- Calculate the E-field generated by these charges at a point M, of height z , situated directly above the point O.
 - What becomes the expression of the field when a tends to zero. Draw the curve $E(z)$ as a function of z .
 - When a is different from zero, what becomes the expression of the field when R increases to infinity. Draw the curve $E(z)$ as a function of z .
 - What is the expression of the field when a tends to zero and R goes to infinity. Draw the curve $E(z)$ as a function of z .

Solution:

For symmetry, the field at M is parallel to axis Oz . Consider a small annulus of thickness dr of radius r and centre O . A point P belongs to this annulus. The field created by this circular elemental annulus, along the Oz axis, is given by:

$$dE = \frac{1}{4\pi\epsilon_0} \cdot \frac{\sigma 2\pi r dr}{MP^2} \cdot \cos \alpha$$

But

$$\cos \alpha = \frac{z}{\sqrt{z^2 + r^2}}, \quad MP = \sqrt{z^2 + r^2}$$

Therefore

$$E = \int_R^a dE = \frac{\sigma}{2\epsilon_0} \left[\frac{z}{\sqrt{z^2 + a^2}} - \frac{z}{\sqrt{z^2 + R^2}} \right]$$

a. First case. If a tends to zero, then:

$$E = \frac{\sigma}{2\epsilon_0} \left[\frac{z}{|z|} - \frac{z}{\sqrt{z^2 + R^2}} \right]$$

b. If a is different from zero and R tends to infinity:

$$E = \frac{\sigma}{2\epsilon_0} \left[\frac{z}{\sqrt{z^2 + a^2}} \right]$$

c. If a is zero and R tends to infinity

$$E = \frac{\sigma}{2\epsilon_0} \left[\frac{z}{|z|} \right]$$