HERIOT WATT UNIVERSITY

Time Zone 2

SCHOOL OF ENGINEERING & PHYSICAL SCIENCES

Electrical, Electronic and Computer Engineer

B39AX

Engineering Mathematics & Statistics

Semester 1 – 2019/20

December 2019

Duration: 2 hours

Instructions to Candidates:

Answer THREE Questions from FOUR

(A breakdown of marks is provided as a guide within each question)

It is important to explain your working fully as credit is given for method as well as for numerical accuracy.

Additional Information:

Exam Data Booklet - supplied
Formula Sheet – attached
Normal Distribution Table – Attached
Student t-Table – Attached

Formulae provided:

Poisson probability:

$$P(x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

Bayes:

$$\Pr(B \mid Y_1) = \frac{\Pr(B)\ \Pr(Y_1 \mid B)}{\Pr(B)\Pr(Y_1 \mid B) + \Pr(B^C)\Pr(Y_1 \mid B^C)}$$

1. (a) Consider the following events in an experiment of tossing a coin twice:

A: at least one head is obtained.

B: two tails are obtained.

Are events A and B independent? Justify. → Not independent

(5 marks)

(b) The transmission on a model of a specific car has a warranty of 40,000 miles. It is known that the lifetime of such a transmission has a normal distribution with mean 72,000 miles and a standard deviation 12,000 miles. What is the probability:

(i) that the transmission will fail before the end of the warranty period?

$p = 0{,}0038$ (5 marks)

(ii) That the transmission will be good for more than 100,000 miles?

$p = 0{,}0099$ (5 marks)

(c) A geneticist is interested in the proportion of African males who have a certain minor blood type disorder. In a random sample of 100 African males, 24 are found to be afflicted. Compute a 99% confidence interval for the proportion of African males who have this blood disorder.

We can be 99% confident that between 12.98% and 35.02% of African males have blood disorder. (10 marks)

2. (a) A washing machine in a Laundromat breaks down an average of three times in a month. Using the Poisson probability distribution formula, calculate the probability that this machine will have:

(i) exactly two breakdowns per month.

$p = 0,2240$

(2 marks)

(ii) at most one breakdown per month.

$p = 0,1992$

(3 marks)

(iii) no breakdowns in 2 months.

$p = 0,0025$

(2 marks)

(b) The processors of X BBQ Sauce indicate on the label that the bottle contains 16 ounces of sauce. The standard deviation of the process is 0.5 ounces. A sample of 36 bottles from last hour's production revealed a mean weight of 16.12 ounces per bottle.

At the 0.05 significance level, can we say that the mean amount per bottle is different from 16 ounces, as claimed by the company?

(9 marks)

With $H_0 : \mu = 16$,
we cannot reject H_0

/Cont…

/Cont…2.

(c) Given the following statistics, what is the probability that a woman has cancer if she has a positive mammogram result?

(i) One percent of women have breast cancer.

~~(3 marks)~~

(ii) Ninety percent of women who have breast cancer test positive on mammograms.

~~(3 marks)~~

(iii) Eight percent of women will have false positives.

~~(3 marks)~~

→ $p = 0{,}1 = 10\,\%$

3. (a) Let X be a continuous random variable, following a uniform distribution in the interval [0,2]. Let Y be a second continuous random variable, following a uniform distribution in the interval [-1,1]. We assume that X and Y are independent.

(i) Compute the mean and variance of X and Y.

$E[x] = 1$ $\quad Var(x) = 1/3$

$E[y] = 0$ $\quad Var(x) = 1/3$

(4 marks)

(ii) Compute the coefficient of correlation between X and the random variable R defined as $R = Y - 3$.

(2 marks)

$c = 0$

/Cont…

/Cont…3(a)

(iii) What is the domain of definition of the random variable?

$$Z = X + Y?$$

(2 marks)

$z \in [-1;3]$

(iv) Compute the probability density function of Z.

$f_z(z) = \begin{cases} \frac{1}{4}[\min(2, z+1) - \max(0, z-1)] & \text{if } z \in [-1;3] \\ 0 & \text{else} \end{cases}$ (4 marks)

(b) Consider the estimation of an unknown continuous parameter x associated with continuous observed data $\boldsymbol{y} = \{y_1, \ldots, y_N\}$ in a Bayesian framework.

(i) What is the name given to the statistical model $f(\boldsymbol{y}|x)$ which relates the observed data to the unknown model parameters of interest?

→ The Likelihood (3 marks)

(ii) What is the name given to the statistical model which incorporates additional information about the unknown parameters? Note that this model does not depend on the data.

→ prior model or prior distribution (3 marks)

(iii) Write the analytical expression (as the solution of an optimization problem, which depends on $f(\boldsymbol{y}|x)$ and $f(x)$) of the maximum a posteriori (MAP) estimator of x.

→ see lecture slides (3 marks)

(iv) Write the analytical expression (as an integral) of the minimum mean square error (MMSE) estimator of x.

→ see lecture slides (4 marks)

4. (a) Consider the discrete random variable (or source) X defined on $\mathcal{A}_X = \{a, b, c, d, e\}$ with probabilities:

$$\mathbb{p}(X = a) = \frac{3}{10}, \mathbb{p}(X = b) = \frac{1-p}{10}, \mathbb{p}(X = c) = \frac{p}{10},$$

$$\mathbb{p}(X = d) = 1/5, \mathbb{p}(X = e) = 4/10,$$

(i) Compute the information content of the outcome $X = d$. (2 marks)

$I(x=d) = 2.3219$ b.

(ii) Compute the entropy $H(X)$ for $p = 0,75$. (3 marks)

$H(X) = 1.9276$ b/symbol.

(iii) Compute the value of p for which $H(X)$ is maximized? (2 marks)

$p = 0,5$

(iv) For $p = 0.5$ and assuming that the source X has a rate of 1000 symbol/s, what is the information rate of the source? (3 marks)

$R = 1946,4$ b/s

(b) Consider the discrete random variable (or source) X defined on $\mathcal{A}_X = \{1,2,3,4,5,6,7\}$ with probabilities:

$$\begin{cases} \mathbb{p}(X = 1) = 1/7 \\ \mathbb{p}(X = 2) = 1/21 \\ \mathbb{p}(X = 3) = 3/14 \\ \mathbb{p}(X = 4) = 1/14 \\ \mathbb{p}(X = 5) = 2/21 \\ \mathbb{p}(X = 6) = 6/21 \\ \mathbb{p}(X = 7) = 1/7 \end{cases}.$$

(i) Compute the entropy $H(X)$. (2 marks)

$H(X) = 2.5989$ b/symbol.

/Cont…

/Cont…4(b)

(ii) Use Huffman coding to devise a code for the source X.

One solution : 1 -> 000 4 -> 00001
2 -> 100 5 -> 101
3 -> 11 6 -> 01
7 -> 001

(3 marks)

(iii) What is the expected length of the code obtained?

$L(X, C) = 2.6429$ b

(3 marks)

(iv) Is the code obtained via Huffman coding a prefix code?

yes.

(2 marks)

(c) Consider a binary source X defined on $\mathcal{A}_X = \{0,1\}$, with $\mathbb{p}(X = 0) = p$. Consider also the Z channel, with input X, output Y and characterized by

$$\begin{cases} \mathbb{p}(Y = 0|X = 0) = 1 \\ \mathbb{p}(Y = 1|X = 0) = 0 \\ \mathbb{p}(Y = 1|X = 1) = 1 - f \\ \mathbb{p}(Y = 1|X = 1) = f \end{cases}$$

(i) Let $p = 0.6$ and $f = 0.01$. Are X and Y independent variables?

No

(1 mark)

(ii) Compute the joint entropy $H(X, Y)$ for $p = 0.6$ and $f = 0.01$.

$H(X, Y) = 1.0033$ b/symbol.

(2 marks)

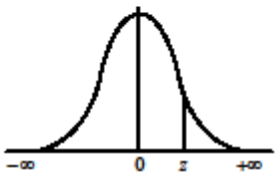
(iii) Compute the mutual information $I(X; Y)$ for $p = 0.6$ and $f = 0.01$.

(2 marks)

$H(X/Y) = 0,0347$

END OF PAPER

NORMAL DISTRIBUTION TABLE



	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

TABLE of CRITICAL VALUES for STUDENT'S *t* DISTRIBUTIONS

Column headings denote probabilities (α) ***above*** tabulated values.

d.f.	0.40	0.25	0.10	0.05	0.04	0.025	0.02	0.01	0.005	0.0025	0.001	0.0005
1	0.325	1.000	3.078	6.314	7.916	12.706	15.894	31.821	63.656	127.321	318.289	636.578
2	0.289	0.816	1.886	2.920	3.320	4.303	4.849	6.965	9.925	14.089	22.328	31.600
3	0.277	0.765	1.638	2.353	2.605	3.182	3.482	4.541	5.841	7.453	10.214	12.924
4	0.271	0.741	1.533	2.132	2.333	2.776	2.999	3.747	4.604	5.598	7.173	8.610
5	0.267	0.727	1.476	2.015	2.191	2.571	2.757	3.365	4.032	4.773	5.894	6.869
6	0.265	0.718	1.440	1.943	2.104	2.447	2.612	3.143	3.707	4.317	5.208	5.959
7	0.263	0.711	1.415	1.895	2.046	2.365	2.517	2.998	3.499	4.029	4.785	5.408
8	0.262	0.706	1.397	1.860	2.004	2.306	2.449	2.896	3.355	3.833	4.501	5.041
9	0.261	0.703	1.383	1.833	1.973	2.262	2.398	2.821	3.250	3.690	4.297	4.781
10	0.260	0.700	1.372	1.812	1.948	2.228	2.359	2.764	3.169	3.581	4.144	4.587
11	0.260	0.697	1.363	1.796	1.928	2.201	2.328	2.718	3.106	3.497	4.025	4.437
12	0.259	0.695	1.356	1.782	1.912	2.179	2.303	2.681	3.055	3.428	3.930	4.318
13	0.259	0.694	1.350	1.771	1.899	2.160	2.282	2.650	3.012	3.372	3.852	4.221
14	0.258	0.692	1.345	1.761	1.887	2.145	2.264	2.624	2.977	3.326	3.787	4.140
15	0.258	0.691	1.341	1.753	1.878	2.131	2.249	2.602	2.947	3.286	3.733	4.073
16	0.258	0.690	1.337	1.746	1.869	2.120	2.235	2.583	2.921	3.252	3.686	4.015
17	0.257	0.689	1.333	1.740	1.862	2.110	2.224	2.567	2.898	3.222	3.646	3.965
18	0.257	0.688	1.330	1.734	1.855	2.101	2.214	2.552	2.878	3.197	3.610	3.922
19	0.257	0.688	1.328	1.729	1.850	2.093	2.205	2.539	2.861	3.174	3.579	3.883
20	0.257	0.687	1.325	1.725	1.844	2.086	2.197	2.528	2.845	3.153	3.552	3.850
21	0.257	0.686	1.323	1.721	1.840	2.080	2.189	2.518	2.831	3.135	3.527	3.819
22	0.256	0.686	1.321	1.717	1.835	2.074	2.183	2.508	2.819	3.119	3.505	3.792
23	0.256	0.685	1.319	1.714	1.832	2.069	2.177	2.500	2.807	3.104	3.485	3.768
24	0.256	0.685	1.318	1.711	1.828	2.064	2.172	2.492	2.797	3.091	3.467	3.745
25	0.256	0.684	1.316	1.708	1.825	2.060	2.167	2.485	2.787	3.078	3.450	3.725
26	0.256	0.684	1.315	1.706	1.822	2.056	2.162	2.479	2.779	3.067	3.435	3.707
27	0.256	0.684	1.314	1.703	1.819	2.052	2.158	2.473	2.771	3.057	3.421	3.689
28	0.256	0.683	1.313	1.701	1.817	2.048	2.154	2.467	2.763	3.047	3.408	3.674
29	0.256	0.683	1.311	1.699	1.814	2.045	2.150	2.462	2.756	3.038	3.396	3.660
30	0.256	0.683	1.310	1.697	1.812	2.042	2.147	2.457	2.750	3.030	3.385	3.646
31	0.256	0.682	1.309	1.696	1.810	2.040	2.144	2.453	2.744	3.022	3.375	3.633
32	0.255	0.682	1.309	1.694	1.808	2.037	2.141	2.449	2.738	3.015	3.365	3.622
33	0.255	0.682	1.308	1.692	1.806	2.035	2.138	2.445	2.733	3.008	3.356	3.611
34	0.255	0.682	1.307	1.691	1.805	2.032	2.136	2.441	2.728	3.002	3.348	3.601
35	0.255	0.682	1.306	1.690	1.803	2.030	2.133	2.438	2.724	2.996	3.340	3.591
36	0.255	0.681	1.306	1.688	1.802	2.028	2.131	2.434	2.719	2.990	3.333	3.582
37	0.255	0.681	1.305	1.687	1.800	2.026	2.129	2.431	2.715	2.985	3.326	3.574
38	0.255	0.681	1.304	1.686	1.799	2.024	2.127	2.429	2.712	2.980	3.319	3.566
39	0.255	0.681	1.304	1.685	1.798	2.023	2.125	2.426	2.708	2.976	3.313	3.558
40	0.255	0.681	1.303	1.684	1.796	2.021	2.123	2.423	2.704	2.971	3.307	3.551
60	0.254	0.679	1.296	1.671	1.781	2.000	2.099	2.390	2.660	2.915	3.232	3.460
80	0.254	0.678	1.292	1.664	1.773	1.990	2.088	2.374	2.639	2.887	3.195	3.416
100	0.254	0.677	1.290	1.660	1.769	1.984	2.081	2.364	2.626	2.871	3.174	3.390
120	0.254	0.677	1.289	1.658	1.766	1.980	2.076	2.358	2.617	2.860	3.160	3.373
140	0.254	0.676	1.288	1.656	1.763	1.977	2.073	2.353	2.611	2.852	3.149	3.361
160	0.254	0.676	1.287	1.654	1.762	1.975	2.071	2.350	2.607	2.847	3.142	3.352
180	0.254	0.676	1.286	1.653	1.761	1.973	2.069	2.347	2.603	2.842	3.136	3.345
200	0.254	0.676	1.286	1.653	1.760	1.972	2.067	2.345	2.601	2.838	3.131	3.340
250	0.254	0.675	1.285	1.651	1.758	1.969	2.065	2.341	2.596	2.832	3.123	3.330
inf	0.253	0.674	1.282	1.645	1.751	1.960	2.054	2.326	2.576	2.807	3.090	3.290