Central Limit Theorem and Confidence Intervals

B39AX — Fall 2023

Heriot-Watt University

Properties of normal RVs

- Normality is preserved under linear transformations:

$$X \sim \mathcal{N}\big(\mu,\, \sigma^2\big) \qquad \Longrightarrow \qquad a\,X + b \;\sim\; \mathcal{N}\Big(a\,\mu + b,\, a^2\sigma^2\Big)$$

- Sum of independent normal RVs is normal:

$$\begin{cases} X \sim \mathcal{N}\big(\mu_X,\, \sigma_X^2\big) \\ Y \sim \mathcal{N}\big(\mu_Y,\, \sigma_Y^2\big) \\ X \text{ and } Y\text{: independent} \end{cases} \qquad \Longrightarrow \qquad X + Y \;\sim\; \mathcal{N}\Big(\mu_X + \mu_Y,\, \sigma_X^2 + \sigma_Y^2\Big)$$

Sums of normal RVs

Let X_1, X_2, ..., X_n be *independent and identically distributed (i.i.d.)* normal RVs:

$$X_i \overset{\text{i.i.d.}}{\sim} \mathcal{N}\big(\mu,\, \sigma^2\big)\,, \qquad i = 1, \ldots, n$$

Then,

$$S_n = X_1 + X_2 + \cdots + X_n \sim \mathcal{N}\big(n\mu\,,\, n\sigma^2\big)$$

and

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \sim \mathcal{N}(0,\, 1)$$

This holds (asymptotically) even when the X_i's are not normal !

Central Limit Theorem

Theorem (Central Limit Theorem)

Let $X_1, X_2, \ldots, X_n$ be a sequence of i.i.d. RVs with common mean $\mu = \mathbb{E}[X_i]$ and common variance $\sigma^2 = \mathsf{Var}(X_i)$, for $i = 1, \ldots, n$. Then,

$$Z_n := \frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} = \frac{\frac{X_1+X_2+\cdots+X_n}{n} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

converges in distribution to a standard normal RV $\mathcal{N}(0,\, 1)$, i.e.,

$$\lim_{n\to\infty} \mathbb{P}\big(Z_n \le z\big) = \Phi(z)\,, \qquad \forall z\,.$$

Central Limit Theorem

- CLT is surprisingly general !
- Requires only *independence*, *same distribution*, and *finite* μ *and* σ^2
- Convergence can be slow (Berry-Esseen theorem): for some $c > 0$,

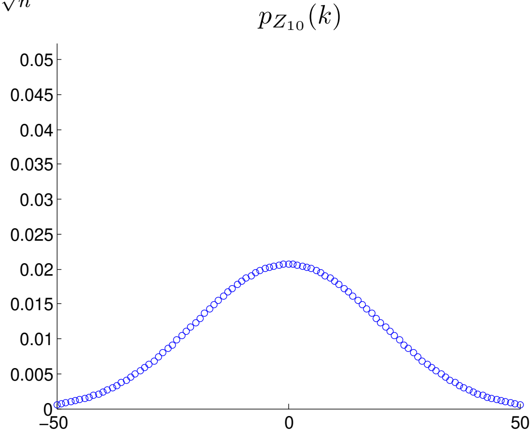
$$\sup_z \Big|\mathbb{P}(Z_n \le z) - \Phi(z)\Big| \le \frac{c}{\sqrt{n}}$$

- Approximation accuracy varies with the distribution:
  - $n \ge 8$ gives a good approximation for continuous uniform RVs
  - $n \ge 30$ for other RVs in this course (except Cauchy RVs)
- Proof of CLT is beyond the scope of this course

Example

X_i : uniformly distributed on $[-10,\, 10] \cap \mathbb{N}$, $i = 1, \ldots, n$ (discrete)

$Z_n = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$



Normal approximation based on the CLT

Let $X_1, X_2, \ldots, X_n$ be i.i.d. RVs with common mean μ and variance σ^2

$S_n := X_1 + \cdots + X_n \qquad (n \geq 30)$

Then, $\mathbb{P}\big(S_n \leq s\big)$ can be approximated by treating S_n as a normal RV

Procedure:

- Calculate the mean $n\mu$ and the variance $n\sigma^2$ of S_n
- Calculate the normalized value $z = \dfrac{s - n\mu}{\sigma\sqrt{n}}$
- Use the approximation

$$\mathbb{P}\big(S_n \leq s\big) \simeq \Phi(z)\,,$$

where $\Phi(z)$ is available from standard normal CDF tables

Exercise

We load on a plane 100 packages whose weights are independent uniform RVs distributed between 5 and $50\,$kg. What is the probability that the total weight will exceed $3000\,$kg?

Ans: $\simeq 2.74\%$

Parameter estimation

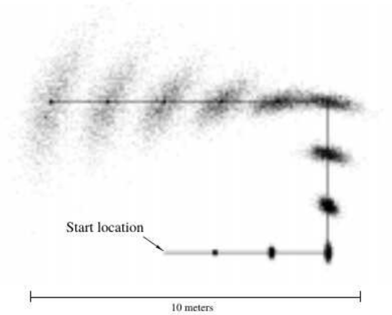
Most distributions have parameters, e.g.,

$$\text{Ber}(p) \qquad \mathcal{N}(\mu,\, \sigma^2) \qquad \text{Bin}(n,\, p) \qquad \text{Poisson}(\lambda) \qquad \text{Exp}(\lambda)$$

How do we estimate a parameter θ from data realizations, i.e., samples from the distribution?

Example

Estimate the position of a robot along time[*,†]



[*] Fox et al., "Monte Carlo Localization: Efficient Position Estimation for Mobile Robots," AAAI, 1999.

[†] Wang et al., "Real-time 3D Human Tracking For Mobile Robots with Multisensors," arXiv:1703.04877v1, 2017.

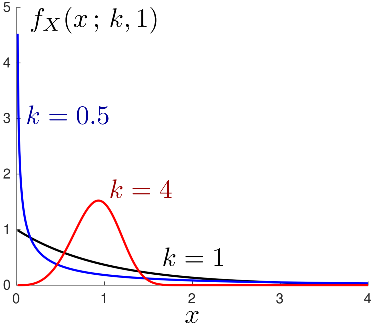
Example: Weibull Distribution and Survival Analysis

X has *Weibull distribution* w/ parameters $k > 0$ (shape), $\lambda > 0$ (scale) if

$$f_X(x\,;\,k,\lambda) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k}\,, \qquad \text{for } x \geq 0$$

Survival function: $S(t) = \mathbb{P}(X > t)$

- Electrical components lifetimes
- Cancer survival rates
- Life expectancies
- Manufacturing delivery times



How to compute parameters k and λ from observations?

Parameter estimation

X: RV whose distribution depends on θ, the parameter to estimate.

Estimator/Statistic: An *estimator* or *statistic* $\widehat{\Theta}_n$ of θ is a function of n *observations* $X_1, \ldots, X_n$ of X, i.e., for some function g,

$$\widehat{\Theta}_n = g\big(X_1, \ldots, X_n\big) \, .$$

When we observe *realizations* of the observations,

$$\widehat{\theta}_n = g\big(x_1, \ldots, x_n\big) \, .$$

Remarks

- The observations are usually independent copies of X
- $\widehat{\Theta}_n$ is a RV, because it is a function of RVs

Example

Let X represent the outcome of a biased coin (or elections): $X \sim \text{Ber}(p)$

$$\mathbb{P}\big(X = 1\big) = p \qquad \mathbb{P}\big(X = 0\big) = 1 - p$$

How do we estimate $\theta = p$ from realizations of n independent tosses?

- Let $X_1, \ldots, X_n$ represent n observations (tosses) of X
- An estimator of θ is the ***sample mean*** $\widehat{\Theta}_n = \overline{X}_n := \dfrac{X_1 + \cdots + X_n}{n}$
- It is "reasonable" estimator, because

$$\mathbb{E}\big[\widehat{\Theta}_n\big] = \frac{\mathbb{E}\big[X_1 + \cdots + X_n\big]}{n} = \frac{\mathbb{E}\big[X_1\big] + \cdots + \mathbb{E}\big[X_n\big]}{n} = p = \theta$$

Properties of estimators

Let $\widehat{\Theta}_n$ be an estimator of θ $\left[\text{i.e., } \widehat{\Theta}_n \text{ is a function of observations } X_1,\right.$ $\left.\ldots, X_n, \text{ whose distribution depends on } \theta\right]$

Unbiased estimator: An estimator is *unbiased* if $\mathbb{E}\big[\widehat{\Theta}_n\big] = \theta$, for all θ

Asymptotically unbiased estimator: An estimator is *asymptotically unbiased* if

$$\lim_{n\to\infty} \mathbb{E}\big[\widehat{\Theta}_n\big] = \theta$$

Mean squared error and the bias-variance tradeoff

A more robust quality measure of an estimator is

Mean squared error:

$$\mathsf{MSE} = \mathbb{E}\Big[\big(\widehat{\Theta}_n - \theta\big)^2\Big]$$

MSE captures both the bias and the variance of the estimator:

$$\mathsf{MSE} = \Big(\underbrace{\mathbb{E}\big[\widehat{\Theta}_n\big] - \theta}_{\text{bias}}\Big)^2 + \mathsf{Var}\big(\widehat{\Theta}_n\big)$$

This is called *bias-variance decomposition* (proof at end of slides)

Example: Estimation of the mean of a RV

$X_1, \ldots, X_n$: independent observations of RV X (unknown distribution)

Consider the following estimators of the mean of X, $\theta = \mathbb{E}[X]$:

- $\widehat{\Theta}_n^{(1)} = \overline{X}_n := \dfrac{X_1 + \cdots + X_n}{n}$ (sample mean)
- $\widehat{\Theta}_n^{(2)} = 0$

Are these estimators biased? What is their MSE?

Example: Estimation of the mean of a RV

We had seen that $\widehat{\Theta}_n^{(1)}$ is unbiased: $\mathbb{E}\left[\widehat{\Theta}_n^{(1)}\right] = \theta$

For MSE, use bias-variance decomposition and independence of X_i's:

$$\begin{aligned}
\mathsf{MSE}^{(1)} &= \Big(\underbrace{\mathbb{E}\big[\widehat{\Theta}_n^{(1)}\big] - \theta}_{=0}\Big)^2 + \mathsf{Var}\Big(\widehat{\Theta}_n^{(1)}\Big) \\
&= \mathbb{E}\left[\Big(\frac{X_1 + \cdots + X_n}{n} - \theta\Big)^2\right] \\
&= \frac{1}{n^2}\underbrace{\mathbb{E}\Big[\big(X_1 + \cdots + X_n - n\theta\big)^2\Big]}_{\mathsf{Var}(X_1 + \cdots + X_n)} \\
&= \frac{1}{n^2}\Big(\mathsf{Var}(X_1) + \cdots + \mathsf{Var}(X_n)\Big) \\
&= \frac{\mathsf{Var}(X_1)}{n} \quad \overset{n\to\infty}{\longrightarrow} \quad 0
\end{aligned}$$

Example: Estimation of the mean of a RV

For $\widehat{\Theta}_n^{(2)} = 0$, we use again the bias-variance decomposition:

$$\begin{aligned}
\mathsf{MSE}^{(2)} &= \Big(\mathbb{E}\big[\widehat{\Theta}_n^{(2)}\big] - \theta\Big)^2 + \mathsf{Var}\big(\widehat{\Theta}_n^{(2)}\big) \\
&= \Big(\mathbb{E}\big[0\big] - \theta\Big)^2 + \mathsf{Var}\big(0\big) \\
&= \theta^2
\end{aligned}$$

Conclusion: $\widehat{\Theta}_n^{(1)}$ is a better estimator than $\widehat{\Theta}_n^{(2)}$, because it is unbiased and its MSE decreases with n.

Estimators of the variance

Let $X_1, \ldots, X_n$ be independent copies (observations) of X.

We want to estimate the variance of X, $\text{Var}(X) = \mathbb{E}\big[(X - \mathbb{E}[X])^2\big]$.

Case 1: mean $\mu = \mathbb{E}[X]$ is *known*. Then,

$$\hat{S}_n^2 = \frac{1}{n} \sum_{i=1}^{n} \big(X_i - \mu\big)^2 \quad \text{is unbiased.}$$

Case 2: mean $\mu = \mathbb{E}[X]$ is *unknown*.* Then,

$$\hat{S}_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} \big(X_i - \overline{X}_n\big)^2 \quad \text{is unbiased.} \qquad \Big(\overline{X}_n\text{: sample mean}\Big)$$

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*The alternative estimator $\overline{S}_n^2 = \frac{1}{n} \sum_{i=1}^{n} \big(X_i - \overline{X}_n\big)^2$ is also a valid estimator, but it is biased (however asymptotically unbiased). All proofs in a separate document.

Exercise

Let X be a RV distributed as $\mathcal{N}(\mu, \sigma^2)$ with unknown μ and σ^2.

We observed the following independent realizations of X:

$$-2.2\,,\ 1.3\,,\ 0.9\,,\ 1.4\,,\ -0.3\,,\ 4.9$$

Compute an unbiased estimate for μ and for σ^2.

Ans: $\widehat{\mu} = 1.0$ and $\widehat{\sigma}^2 = 5.48$ (the biased formula for σ^2 would give 4.57)

t-Student distribution

Let $X_1, \ldots, X_n$ be i.i.d. normal RVs w/ mean μ and variance σ^2

We saw that

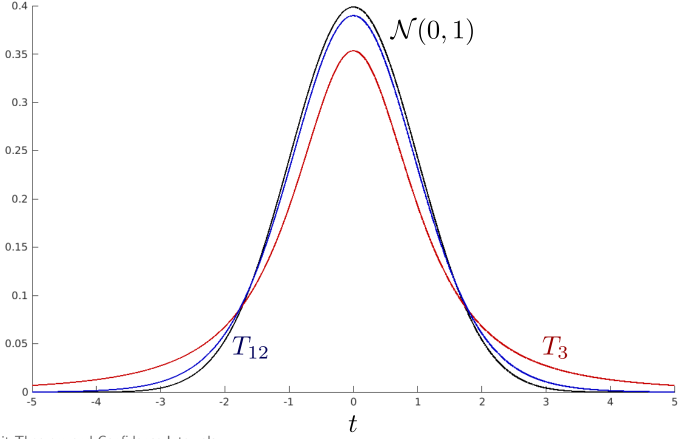
$$Z = \frac{\frac{X_1 + \cdots + X_n}{n} - \mu}{\sigma/\sqrt{n}} = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim \mathcal{N}(0, 1)$$

What if σ^2 is unknown? If we replace it by $\widehat{S}_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(X_i - \overline{X}_n\right)^2$,

$$\text{is} \quad T_n := \frac{\overline{X}_n - \mu}{\widehat{S}_n/\sqrt{n}} \quad \text{normal?}$$

t-Student distribution

$$T_n = \frac{\overline{X}_n - \mu}{\widehat{S}_n/\sqrt{n}}$$ has t-Student distribution with $n-1$ degrees of freedom



t-Student distribution

T_n has a complicated PDF, but its values are tabulated

Usage: Use the t-Student distribution when

- X_i's are normal (or approximately normal)
- σ^2 is unknown
- n is small (e.g., $n < 50$) [otherwise, $\widehat{S}_n^2 \simeq \sigma^2$]

Confidence intervals

X: RV whose distribution depends on parameter θ

Example: $\theta =$ daily electricity consumption in a given household

$X_1, \ldots, X_n$: independent observations of X

We can obtain unbiased estimators $\Theta = g(X_1, \ldots, X_n)$ of θ, for some function g

But how good is an estimator? Why not an interval?

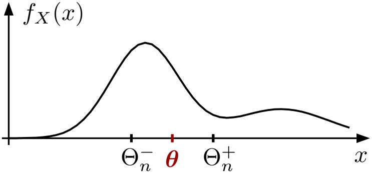
Confidence intervals

Confidence interval

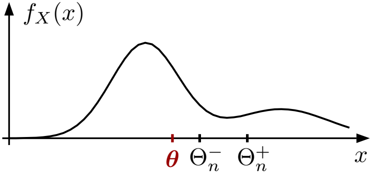
Fix $0 < \alpha < 1$ (typically small, e.g., 0.1, 0.05, 0.01).

A *confidence interval with confidence level* $1 - \alpha$ for a parameter θ is an interval $\left[\Theta_n^-, \Theta_n^+\right]$, where $\Theta_n^- \leq \Theta_n^+$ are RVs such that, for all θ,

$$\mathbb{P}\left(\Theta_n^- \leq \theta \leq \Theta_n^+\right) \geq 1 - \alpha$$



Success (prob $\geq 1 - \alpha$)



Failure (prob $\leq \alpha$)

Example

- $X \sim \mathcal{N}(\mu\,,\,\sigma^2)$ *with unknown μ, but known σ^2*
- Estimator of μ: $\widehat{\Theta}_n = \overline{X}_n := \dfrac{X_1 + \cdots + X_n}{n}$ $\big(X_i$: i.i.d. copy of $X\big)$

How to derive a confidence interval for $\theta = \mu$ with $\alpha = 0.05$?

- Use the fact that $\widehat{\Theta}_n \sim \mathcal{N}\Big(\mu\,,\,\dfrac{\sigma^2}{n}\Big)$ and thus $Z = \dfrac{\widehat{\Theta}_n - \mu}{\sigma/\sqrt{n}} \sim \mathcal{N}(0,\,1)$
- Therefore,

$$\begin{aligned}
\mathbb{P}\big(-c \le Z \le c\big) &= \mathbb{P}\left(-c \le \frac{\widehat{\Theta}_n - \mu}{\sigma/\sqrt{n}} \le c\right) \\
&= \mathbb{P}\left(\widehat{\Theta}_n - c\frac{\sigma}{\sqrt{n}} \le \mu \le \widehat{\Theta}_n + c\frac{\sigma}{\sqrt{n}}\right) \\
&\ge 1 - \alpha
\end{aligned}$$

Example

Parameter c is determined from the normal table:

$\mathbb{P}(-c \leq Z \leq c) \geq 1 - \alpha \quad \Longleftrightarrow \quad \Phi(c) \geq 1 - \frac{\alpha}{2} = 0.975,$

which gives $c = 1.96$. The confidence interval for μ is then

$$\left[\widehat{\Theta}_n - 1.96\frac{\sigma}{\sqrt{n}}\ ,\ \widehat{\Theta}_n + 1.96\frac{\sigma}{\sqrt{n}}\right].$$

Remarks:

- Once realizations $x_1, \ldots, x_n$ are available, we replace $\widehat{\theta}_n = \frac{1}{n}\sum_{i=1}^{n} x_i$
- **Wrong interpretation:** "The probability of $\mu \in$ CI is 95%"
- **Correct interpretation:** "If we repeatedly build CIs from i.i.d. samples, 95% of them will contain $\theta = \mu$"
- Building CIs requires knowing the distribution of $\widehat{\Theta}_n$

Exercise

We conduct a poll to estimate the fraction of the population that supports a given candidate for office. Out of the 1200 people that were interviewed, 684 support the candidate (57%). Build a 95% confidence interval for the fraction of the population that supports the candidate.

Use

- CLT to approximate a sum of Bernoulli RVs as a normal RV;
- The unbiased estimate for the variance, and assume it is accurate.

Exercise: solution

Define the RVs

$X =$ "a person chosen from the population supports the candidate."

$X_i =$ "the ith person interviewed supports the candidate," $\quad i = 1, \ldots, n$.

The X_i's are $n = 1200$ independent copies of X.

We have $X \sim \text{Ber}(p)$, and we wish to estimate $\mu := \mathbb{E}[X] = p = \theta$.

We are given a realization of the unbiased estimator of μ:

$$\widehat{\theta}_n = \overline{x}_n = \frac{x_1 + \cdots + x_n}{n} = \frac{684}{1200} = 0.57$$

Exercise: solution

As $n > 30$, CLT approximates $\frac{X_1+\cdots+X_n-n\mu}{\sigma\sqrt{n}}$ by $Z \sim \mathcal{N}(0,\,1)$, and

$$\begin{aligned} 0.95 &= \mathbb{P}\big(-c \le Z \le c\big) \\ &\simeq \mathbb{P}\left(-c \le \frac{X_1+\cdots+X_n-n\mu}{\sigma\sqrt{n}} \le c\right) \\ &= \mathbb{P}\left(\overline{X}_n - c\frac{\sigma}{\sqrt{n}} \;\le\; \mu \;\le\; \overline{X}_n + c\frac{\sigma}{\sqrt{n}}\right) \end{aligned}$$

We know $\overline{x}_n = 0.57$, $n = 1200$, and $c = 1.96$ (normal table). As $n > 50$,

$$\sigma^2 \simeq \widehat{S}_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2 = \frac{1}{1199}\left[684(1-0.57)^2 + 516\cdot 0.57^2\right]$$

$$= 0.245$$

The 95% CI is $\left[0.57 - 1.96\sqrt{\frac{0.245}{1200}}, 0.57 + 1.96\sqrt{\frac{0.245}{1200}}\right] = [0.542,\,0.598]$.

Procedure for computing a CI for the mean of a RV

X: RV whose distribution has mean μ (a.k.a. ***population mean***)

$X_1, \ldots, X_n$: i.i.d. copies of X

Case 1: **variance σ^2 of X is known**

- Use sample mean $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ as the estimator of μ
- If X has normal distribution or if CLT is applicable ($n \geq 30$),

$$\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \sim \mathcal{N}(0\,,\,1)$$

- CI is

$$\left[\overline{X}_n - c\frac{\sigma}{\sqrt{n}}\,,\,\overline{X}_n + c\frac{\sigma}{\sqrt{n}}\right],$$

  where c is such that $\mathbb{P}(-c \leq Z \leq c) \geq 1 - \alpha$.

Procedure for computing a CI for the mean of a RV

X: RV whose distribution has mean μ

$X_1, \ldots, X_n$: i.i.d. copies of X

Case 2: **variance $\boldsymbol{\sigma^2}$ of $\boldsymbol{X}$ is unknown and $\boldsymbol{n \geq 50}$**

- Use sample mean $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ as the estimator of μ
- As n is large, $\widehat{S}_n^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X}_n)^2$ accurately estimates σ^2
- As n is large, CLT is applicable and

$$\frac{\overline{X}_n - \mu}{\widehat{S}_n/\sqrt{n}} \sim \mathcal{N}(0\,,\,1)$$

- CI is

$$\left[\overline{X}_n - c\frac{\widehat{S}_n}{\sqrt{n}}\,,\,\overline{X}_n + c\frac{\widehat{S}_n}{\sqrt{n}}\right],$$

  where c is such that $\mathbb{P}(-c \leq Z \leq c) \geq 1 - \alpha$.

Procedure for computing a CI for the mean of a RV

X: RV whose distribution has mean μ

$X_1, \ldots, X_n$: i.i.d. copies of X

Case 3: **variance $\boldsymbol{\sigma^2}$ of $\boldsymbol{X}$ is unknown and $\boldsymbol{n < 50}$**

- Use sample mean $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ as the estimator of μ
- Estimate σ^2 with $\widehat{S}_n^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X}_n)^2$
- As n is small and we have only an estimation of σ^2, we use

$$T_n := \frac{\overline{X}_n - \mu}{\widehat{S}_n/\sqrt{n}} \;\sim\; t\text{-Student}(n-1)$$

- CI is

$$\left[\overline{X}_n - t\frac{\widehat{S}_n}{\sqrt{n}}\,,\,\overline{X}_n + t\frac{\widehat{S}_n}{\sqrt{n}}\right],$$

where t is such that $\mathbb{P}(-t \leq T_n \leq t) \geq 1 - \alpha$.

Exercise

The weight of an object is measured 8 times using a scale that reports the true weight, plus a random error with zero mean and unknown variance. Assume that the errors in the observations are independent. The following results were obtained:

$$0.5547\,,\ \ 0.5404\,,\ \ 0.6364\,,\ \ 0.6438\,,\ \ 0.4917\,,\ \ 0.5674\,,\ \ 0.5564\,,\ \ 0.6066$$

Note that

$$\overline{x}_n = 0.5747\,, \qquad \widehat{s}_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x}_n)^2 = 2.636 \times 10^{-3}$$

Compute a 95% confidence interval for the mean.

Exercise: solution

As the variance is unknown and n is small ($n \leq 50$),

$$\frac{\overline{X}_n - \mu}{\widehat{S}_n/\sqrt{n}} \;\sim\; t\text{-Student}(7)$$

The CI is then

$$\left[\overline{X}_n - t\frac{\widehat{S}_n}{\sqrt{n}}\,,\, \overline{X}_n + t\frac{\widehat{S}_n}{\sqrt{n}}\right].$$

From the t-Student table, $\mathbb{P}(T_8 \geq t) \geq \frac{\alpha}{2} = 0.025$ implies $t = 2.365$.

Replacing the values, we obtain the CI

$$\left[0.5747 - 2.365\frac{\sqrt{2.636 \times 10^{-3}}}{\sqrt{8}}\,,\, 0.5747 + 2.365\frac{\sqrt{2.636 \times 10^{-3}}}{\sqrt{8}}\right]$$

$$= \left[0.532\,,\, 0.618\right].$$

Proofs

Bias-Variance Decomposition

$$\mathsf{MSE} := \mathbb{E}\left[\left(\widehat{\Theta}_n - \theta\right)^2\right] = \left(\mathbb{E}\left[\widehat{\Theta}_n\right] - \theta\right)^2 + \mathsf{Var}\left(\widehat{\Theta}_n\right)$$

Proof

$$\begin{aligned}
\mathbb{E}\left[\left(\widehat{\Theta}_n - \theta\right)^2\right] &= \mathbb{E}\left[\left(\widehat{\Theta}_n - \mathbb{E}\left[\widehat{\Theta}_n\right] + \mathbb{E}\left[\widehat{\Theta}_n\right] - \theta\right)^2\right] \\
&= \mathbb{E}\left[\left(\widehat{\Theta}_n - \mathbb{E}\left[\widehat{\Theta}_n\right]\right)^2\right] + \underbrace{\mathbb{E}\left[\left(\mathbb{E}\left[\widehat{\Theta}_n\right] - \theta\right)^2\right]}_{\mathsf{Var}(\widehat{\Theta}_n)} \\
&\quad + 2\,\underbrace{\mathbb{E}\left[\left(\widehat{\Theta}_n - \mathbb{E}\left[\widehat{\Theta}_n\right]\right) \cdot \left(\mathbb{E}\left[\widehat{\Theta}_n\right] - \theta\right)\right]}_{=0}.
\end{aligned}$$

The last term is zero because of the linearity of the expected value. $\square$