Engineering Mathematics and Statistics (B39AX)
Fall 2023

Tutorial 2

Problem A. Suppose that six married couples (i.e., 12 people) are in a room, and we select four of them randomly.

(a) What is the probability that we select two married couples?

We can solve the problem in two different ways:

- Using the multiplication rule, we view the selection of people sequentially. Let S_i denote the event "person selected at time i is married to a previously selected person", for $i = 1, 2, 3, 4$. Note that $\mathbb{P}(S_1) = 0$ and $\mathbb{P}(S_1^c) = 1$, so we will ignore the selection of the first person. We have

$$\text{"2 married couples are chosen"} = \left(S_2 \cap S_3^c \cap S_4\right) \cup \left(S_2^c \cap S_3 \cap S_4\right).$$

Therefore,

$$\begin{aligned}
&\mathbb{P}\Big(\left(S_2 \cap S_3^c \cap S_4\right) \cup \left(S_2^c \cap S_3 \cap S_4\right)\Big) \\
&= \mathbb{P}(S_2 \cap S_3^c \cap S_4) + \mathbb{P}(S_2^c \cap S_3 \cap S_4) \\
&= \mathbb{P}(S_2) \cdot \mathbb{P}(S_3^c \,|\, S_2) \cdot \mathbb{P}(S_4 \,|\, S_2 \cap S_3^c) + \mathbb{P}(S_2^c) \cdot \mathbb{P}(S_3 \,|\, S_2^c) \cdot \mathbb{P}(S_4 \,|\, S_2^c \cap S_3) \\
&= \frac{1}{11} \cdot 1 \cdot \frac{1}{9} + \frac{10}{11} \cdot \frac{2}{10} \cdot \frac{1}{9} \\
&= \frac{3}{99} \\
&= \frac{1}{33}.
\end{aligned}$$

- Using counting, we see that the number of ways of selecting 4 people out of 12 is $\binom{12}{4} = 495$, and the number of ways of selecting 2 couples out of 6 is $\binom{6}{2} = 15$. So the probability of choosing 2 married couples is given by the ratio

$$\frac{\binom{6}{2}}{\binom{12}{4}} = \frac{15}{495} = \frac{1}{33}.$$

(b) What is the probability that there is no married couple in the people we choose?

Again, we have two ways of solving the problem:

- Using the multiplication rule, we want to compute the probability of the event $S_2^c \cap S_3^c \cap S_4^c$:

$$\mathbb{P}\left(S_2^c \cap S_3^c \cap S_4^c\right) = \mathbb{P}(S_2^c) \cdot \mathbb{P}\left(S_3^c \,|\, S_2^c\right) \cdot \mathbb{P}\left(S_4^c \,|\, S_2^c \cap S_3^c\right) = \frac{10}{11} \cdot \frac{8}{10} \cdot \frac{6}{9} = \frac{16}{33}.$$

- Using counting, we observe that the 4 unmarried people come from 4 different couples, and there are $\binom{6}{4}$ such choices. But, for each of these configurations, we can select either member of the couple, and there are 2^4 such ways of doing that. Therefore, the probability is

$$\frac{2^4 \cdot \binom{6}{4}}{\binom{12}{4}} = \frac{16 \cdot 15}{495} = \frac{16}{33}.$$

(c) What is the probability that exactly one married couple is chosen?

This can be obtained from the previous answers by noticing that all these events form a partition of the sample space, i.e.,

$$\begin{aligned}\mathbb{P}(\text{“2 married couples are chosen”}) + \mathbb{P}(\text{“1 married couple is chosen”})\\ + \mathbb{P}(\text{“no married couple is chosen”}) = 1\end{aligned}$$

Therefore, $\mathbb{P}(\text{“1 married couple is chosen”}) = 1 - \frac{1}{33} - \frac{16}{33} = \frac{16}{33}$.

Problem B. Having taught Statistics for many years, I have found that 80% of the students who do the coursework exercises pass the exam, but only 10% of the students who don't do the coursework pass the exam. Every year only 60% of students do the coursework.

(a) What is the percentage of students who pass the exam?

Define the events $P =$ "student passes the exam" and $C =$ "student does the coursework". We are given the following information

$$\begin{aligned}\mathbb{P}(P \,|\, C) &= 0.8\\ \mathbb{P}(P \,|\, C^c) &= 0.1\\ \mathbb{P}(C) &= 0.6\,,\end{aligned}$$

and are asked to compute $\mathbb{P}(P)$. Note that C, C^c is a partition of Ω. So, we can use the total probability theorem:

$$\mathbb{P}(P) = \mathbb{P}(P \cap C) + \mathbb{P}(P \cap C^c) = \underbrace{\mathbb{P}(P|C)}_{0.8} \cdot \underbrace{\mathbb{P}(C)}_{0.6} + \underbrace{\mathbb{P}(P|C^c)}_{0.1} \cdot \underbrace{\mathbb{P}(C^c)}_{0.4} = 0.48 + 0.04 = 0.52\,.$$

Thus, only 52% of students pass the exam.

(b) Of the students who pass the exam, what percentage did the coursework?

We need to compute $\mathbb{P}(C \,|\, P)$:

$$\mathbb{P}(C \,|\, P) = \frac{\mathbb{P}(C \cap P)}{\mathbb{P}(P)} = \frac{\mathbb{P}(P|C) \cdot \mathbb{P}(C)}{\mathbb{P}(P)} = \frac{0.48}{0.52} \simeq 0.92\,.$$

Problem C. An urn contains 5 red and 3 white marbles. A marble is selected at random, discarded, and two marbles of the other colour are then placed in the urn. Next, a second marble is selected from the urn.

(a) What is the probability that the second marble is red?

Define the events

$$\begin{aligned} R_i &= \text{“Red marble is selected at time i,”}\ i = 1, 2 \\ W_i &= \text{“White marble is selected at time i,”}\ i = 1, 2 \end{aligned}$$

We want to compute $\mathbb{P}(R_2)$. To do so, we use the total probability theorem, noticing that R_1 and W_1 form a partition of the sample space Ω. Thus,

$$\begin{aligned} \mathbb{P}(R_2) &= \mathbb{P}(R_2 \cap \Omega) \\ &= \mathbb{P}\Big(R_2 \cap (R_1 \cup W_1)\Big) \\ &= \mathbb{P}\Big((R_2 \cap R_1) \cup (R_2 \cap W_1)\Big) \\ &= \mathbb{P}\Big(R_2 \cap R_1\Big) + \mathbb{P}\Big(R_2 \cap W_1\Big) \\ &= \mathbb{P}(R_1) \cdot \mathbb{P}(R_2 \,|\, R_1) + \mathbb{P}(W_1) \cdot \mathbb{P}(R_2 \,|\, W_1) \\ &= \frac{5}{8} \cdot \frac{4}{9} + \frac{3}{8} \cdot \frac{7}{9} \\ &= \frac{41}{72}\,. \end{aligned}$$

(b) What is the probability that the two selected marbles have the same colour?

The only way for the two marbles to have the same colour is if they are both white or both red, that is $(R_1 \cap R_2) \cup (W_1 \cap W_2)$. Noticing that both events are disjoint, we can just sum their probabilities:

$$\begin{aligned} &\mathbb{P}(R_1 \cap R_2) + \mathbb{P}(W_1 \cap W_2) \\ &= \mathbb{P}(R_1) \cdot \mathbb{P}(R_2 \,|\, R_1) + \mathbb{P}(W_1) \cdot \mathbb{P}(W_2 \,|\, W_1) \\ &= \frac{5}{8} \cdot \frac{4}{9} + \frac{3}{8} \cdot \frac{2}{9} \\ &= \frac{26}{72} \\ &= \frac{13}{36}\,. \end{aligned}$$