**Engineering Mathematics and Statistics (B39AX)**
**Fall 2023**

## Tutorial 3

**Problem A.** A factory produces resistors of $1000\,\Omega$, and the quality control department specifies a tolerance of 10%. Assume that the resistance of each resistor is modeled as a Gaussian random variable with mean $\mu = 1000\,\Omega$ and standard deviation $\sigma = 40\,\Omega$.

(a) What fraction of resistors do you expect to be rejected?

Let $X$ be the random variable representing the resistance of a resistor. Then $X \sim \mathcal{N}(\mu, \sigma^2)$. The quality control department rejects $X$ if it falls outside $[0.9\mu, 1.1\mu] = [900, 1100]$. Therefore,

$$
\begin{aligned}
\mathbb{P}(\text{rejection}) &= \mathbb{P}(X < 900 \cup X > 1100) = \mathbb{P}(X < 900) + \mathbb{P}(X > 1100) \\
&= \mathbb{P}\Big(\frac{X-\mu}{\sigma} < \frac{900-\mu}{\sigma}\Big) + \mathbb{P}\Big(\frac{X-\mu}{\sigma} > \frac{1100-\mu}{\sigma}\Big) \\
&= \mathbb{P}\Big(Z < \frac{900-\mu}{\sigma}\Big) + \mathbb{P}\Big(Z > \frac{1100-\mu}{\sigma}\Big) && \big(Z \sim \mathcal{N}(0,1)\big) \\
&= \Phi\Big(\frac{900-\mu}{\sigma}\Big) + 1 - \Phi\Big(\frac{1100-\mu}{\sigma}\Big) \\
&= \Phi(-2.5) + 1 - \Phi(2.5) \\
&= [1 - \Phi(2.5)] + 1 - \Phi(2.5) && \big(\text{by symmetry}\big) \\
&= 2 - 2\Phi(2.5) \\
&\simeq 2 - 2 \cdot 0.9938 && \big(\text{from the table}\big) \\
&= 0.0124\,,
\end{aligned}
$$

that is, 1.24% of the resistors.

(b) Suppose that a faulty machine produces resistors with $\mu = 1030\,\Omega$, but with the previous standard deviation. What is now the fraction to be rejected?

The rejection interval is the same, so the first four steps of the previous question are the same. We have

$$
\begin{aligned}
\mathbb{P}(\text{rejection}) &= \Phi\Big(\frac{900-\mu}{\sigma}\Big) + 1 - \Phi\Big(\frac{1100-\mu}{\sigma}\Big) \\
&= \Phi(-3.25) + 1 - \Phi(1.75) \\
&= 2 - \Phi(3.25) - \Phi(1.75) \\
&\simeq 2 - 0.9994 - 0.9599 && \big(\text{from the table}\big) \\
&= 0.0407
\end{aligned}
$$

that is, 4.07% of the resistors.

**Problem B.** The lifetime of a high performance electrical motor is expressed in weeks as a Rayleigh random variable, which has the following pdf

$$f_X(x) = \begin{cases} \dfrac{x}{\beta} \exp\Big(-\dfrac{x^2}{2\beta}\Big) & x \geq 0 \\ 0 & x < 0\,, \end{cases}$$

with $\beta = 200$. What is the probability that the motor will fail within the first week?

$$\mathbb{P}(X \leq 1) = \int_0^1 \frac{x}{\beta} \exp\Big(-\frac{x^2}{2\beta}\Big)\, dx = \Big[-\exp\Big(-\frac{x^2}{2\beta}\Big)\Big]_0^1 = -\exp\Big(-\frac{1}{400}\Big) + 1 \simeq 0.00245\,.$$

The probability is 0.0025, or 0.25%.

**Problem C.** A binary message is transmitted as a signal $s$, which is either $-1$ or $+1$. The communication channel corrupts the transmission with additive Gaussian noise with mean $\mu = 0$ and variance $\sigma^2 = 4$. The receiver concludes that the signal $-1$ (or $+1$) was transmitted if the value received is $< 0$ (or $> 0$, respectively). If the probability of transmitting a $+1$ is 0.7, and the probability of transmitting a $-1$ is 0.3, what is the probability of error?

Let $S$ be the random variable representing the transmitted signal. We have

$$\mathbb{P}(S = 1) = 0.7\,, \qquad \mathbb{P}(S = -1) = 0.3\,,$$

Let $Z$ represent additive Gaussian noise, that is, $Z \sim \mathcal{N}(0, \sigma^2)$. And let $Y = S + Z$ represent the received signal. We have

$$\begin{aligned} Y \,|\, (S = 1) &\sim \mathcal{N}(1, \sigma^2) \\ Y \,|\, (S = -1) &\sim \mathcal{N}(-1, \sigma^2) \end{aligned}$$

The receiver makes an error when $Y < 0$ if $S = 1$, or when $Y > 0$ if $S = -1$. Therefore,

$$\begin{aligned} \mathbb{P}(\text{error}) &= \mathbb{P}(Y < 0 \cap S = 1) + \mathbb{P}(Y > 0 \cap S = -1) \\ &= \mathbb{P}(Y < 0 \,|\, S = 1) \cdot \mathbb{P}(S = 1) + \mathbb{P}(Y > 0 \,|\, S = -1) \cdot \mathbb{P}(S = -1) \\ &= 0.7 \cdot \mathbb{P}(Y < 0 \,|\, S = 1) + 0.3 \cdot \mathbb{P}(Y > 0 \,|\, S = -1) \\ &= 0.7 \cdot \mathbb{P}\Big(\frac{Y-1}{\sigma} < \frac{-1}{\sigma} \,\Big|\, S = 1\Big) + 0.3 \cdot \mathbb{P}\Big(\frac{Y+1}{\sigma} > \frac{1}{\sigma} \,\Big|\, S = -1\Big) \\ &= 0.7 \cdot \Phi(-0.5) + 0.3 \cdot [1 - \Phi(0.5)] \\ &= 0.7 \cdot [1 - \Phi(0.5)] + 0.3 \cdot [1 - \Phi(0.5)] \\ &= 1 - \Phi(0.5) \\ &= 1 - 0.6915 \\ &= 0.3085\,. \end{aligned}$$