Engineering Mathematics and Statistics (B39AX) Fall 2023

Tutorial 8

Problem A.

Consider a set of N observations $y_1, \dots, y_N$ which are independent and identically distributed according to a binomial distribution with known number of trials n and whose probability of success p is unknown.

1) Compute the maximum likelihood estimator (MLE) $\hat{p}_{MLE}$ of p, computed from the N observations gathered in $\boldsymbol{y} = [y_1, \dots, y_N]$.

Adopting a Bayesian approach, p is assigned a beta prior distribution whose probability density function is given by

$$f(p) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1},$$

where (α, β) are fixed (known) parameters, larger than 1 and $\Gamma(\cdot)$ is the gamma function. Since (α, β), they are omitted in all the pdfs for brevity.

2) Using $E_{f(p)}[p] = \int p\, f(p) dp$, compute the mean of the prior distribution $f(p)$.

   Note that $\forall x > 0, \Gamma(x+1) = x\Gamma(x)$.

3) Compute the maximum a posteriori (MAP) estimator of p.
4) Using the prior distribution defined in 2), show that the distribution $f(p|\boldsymbol{y})$ is the probability density function of a beta distribution and compute its parameters $(\alpha_{post}, \beta_{post})$.
5) Using the result obtain in 2), compute the posterior mean of p, i.e., $E_{f(p|\boldsymbol{y})}[p]$.

Problem B.

Consider a set of N observations $y_1, \dots, y_N$ which are independent and identically distributed according to a Poisson distribution with unknown mean λ.

1) Compute the maximum likelihood estimator (MLE) $\hat{\lambda}_{MLE}$ of λ, computed from the N observations gathered in $\boldsymbol{y} = [y_1, \dots, y_N]$.

Adopting a Bayesian approach, λ is assigned a gamma prior distribution whose probability density function is given by

$$f(\lambda) = \frac{1}{\Gamma(k)\theta^k} \lambda^{k-1} e^{-\frac{\lambda}{\theta}},$$

where (k, θ) are fixed (known) parameters, such that $k \geq 1$ and $\theta \geq 0$. Note that the gamma distribution is a generalisation of the exponential distribution: the gamma distribution with $k = 1$ reduces to an exponential distribution with mean θ.

2) Using $E_{f(\lambda)}[\lambda] = \int \lambda\, f(\lambda) d\lambda$, compute the mean of the prior distribution $f(\lambda)$
3) Compute the maximum a posteriori (MAP) estimator of λ.
4) Using the prior distribution defined in 2), show that the distribution $f(\lambda|\boldsymbol{y})$ is the probability density function of a gamma distribution and compute its parameters $(k_{post}, \theta_{post})$.
5) Using the result obtain in 2), compute the posterior mean of λ, i.e., $E_{f(\lambda|\boldsymbol{y})}[\lambda]$.