Conditional Probability and Independence

B39AX — Fall 2023

Heriot-Watt University

Intuition

Let A and B be two events.

If we know B happened, what does this tell us about A?

Example 1:

$A =$ "I am left-handed"

$B =$ "One of my parents is left-handed"

Example 2:

$A =$ "Today it rains"

$B =$ "Today 270 babies were born"

Intuition

Let $\mathbb{P}(A \mid B)$ be the probability that A occurs ***given that B occurred.***

In the frequentist interpretation, repeat an experiment T times:

$$\mathbb{P}(A \mid B) \simeq \frac{\#\text{ times } A \text{ and } B \text{ occurred}}{\#\text{ times } B \text{ occurred}} = \frac{T \cdot f_T(A \cap B)}{T \cdot f_T(B)} \simeq \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

If B ***does not inform*** about A, then $\mathbb{P}(A \mid B) = \mathbb{P}(A)$, or

$$\boxed{\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)}$$

In this case, we say that A and B are ***independent***.

Definitions

Conditional probability:

Let A and B be events such that $\mathbb{P}(B) > 0$.

The *conditional probability of A given B* is

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \,.$$

Independence:

The events A and B are *independent* if $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$

Exercise

Roll a fair die.

- $A =$ "Get number 3"

  $B =$ "Get an odd number"

  Compute $\mathbb{P}(A \mid B)$ and $\mathbb{P}(B \mid A)$.

- $A =$ "Get an even number"

  $B =$ "Get number 3"

  Are A and B independent?

Exercise

A family has two children. What is the probability that both are boys given that at least one is a boy?

Answer: $1/3$

Properties of independence

- A and B independent implies

  A and B^c independent

  A^c and B independent

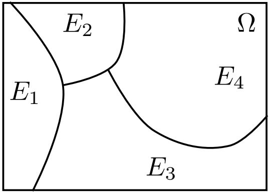
  A^c and B^c independent

- If $\mathbb{P}(B) > 0$ and A and B are independent, then $\mathbb{P}(A \mid B) = \mathbb{P}(A)$

- Let $\mathbb{P}(B) > 0$. Then, $\mathbb{Q}(\cdot) := \mathbb{P}(\cdot \mid B)$ is a probability measure (satisfies the axioms)

Partition

$\{E_i\}_{i=1}^{n}$ is a **partition** of Ω if



- $E_i \neq \emptyset$ for all i
- $E_i \cap E_j = \emptyset$ for all $i \neq j$
- $\bigcup_{i=1}^{n} E_i = \Omega$

Example: A and A^c are a partition of Ω.

Three Important Theorems

Multiplication Rule

Motivation

Suppose we have a random password generator that creates passwords using only lowercase characters $\{a, b, c, \ldots, z\}$ and never repeats the same character in a password.

If we ask it to generate a 3-length password, what is the probability that it is "abc"?

Multiplication Rule

Theorem (Multiplication Rule)

Let A_1, A_2, ..., A_n be events and $\mathbb{P}(A_1 \cap \cdots \cap A_{n-1}) > 0$. Then,

$$\mathbb{P}\big(A_1 \cap \cdots \cap A_n\big) = \mathbb{P}\big(A_1\big) \cdot \mathbb{P}\big(A_2 \,|\, A_1\big) \cdots \mathbb{P}\big(A_n \,|\, A_1 \cap \cdots \cap A_{n-1}\big)$$

Ans (prev prob.): $\frac{1}{26} \cdot \frac{1}{25} \cdot \frac{1}{24}$

Total Probability Theorem

Motivation

Suppose I have two coins:

- Coin 1 is fair: H and T have the same probability
- Coin 2 is biased: H is twice as likely as T

I select one of the coins with equal probability and flip it.

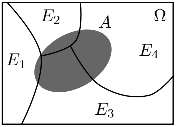
What is the probability of obtaining H?

Total Probability Theorem

Theorem (Total Probability Theorem)

Let $\{E_i\}_{i=1}^n$ be a partition of Ω. For any event A,

$$\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A \mid E_i) \cdot \mathbb{P}(E_i)$$



Ans (prev prob.): $\frac{7}{12}$

Bayes Rule

Motivation

Suppose I have two coins:

- Coin 1 is fair: H and T have the same probability
- Coin 2 is biased: H is twice as likely as T

I select one of the coins with equal probability and flip it.

If I obtained H, what is the probability that I chose coin 2?

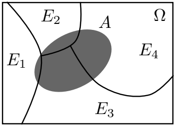
Bayes Rule

Theorem (Bayes Rule)

Let $\{E_i\}_{i=1}^n$ be a partition of Ω. If $\mathbb{P}(A) > 0$, then

$$\mathbb{P}\big(E_i \,|\, A\big) = \frac{\mathbb{P}(A \,|\, E_i) \cdot \mathbb{P}(E_i)}{\sum_{j=1}^n \mathbb{P}(A \,|\, E_j) \cdot \mathbb{P}(E_j)}$$

for any $i = 1, \ldots, n$.



Allows updating "beliefs" when A happens: $\mathbb{P}\big(E_i\big) \to \mathbb{P}\big(E_i \,|\, A\big)$

Ans (prev prob.): $\frac{4}{7}$

Example

Consider a disease and a test for the disease.

- The test has accuracy 95%
- The test gives false positives with probability 5%
- The disease affects 0.1% of the population

I was randomly selected to take the test, which turned out positive.

What is the probability of having the disease? *Should I worry?*

80% of people in a survey answered 95% [The Economist, 20 Feb, 1999]

Ans: 1.87%

Proofs of the Theorems

Proof of Multiplication Rule

Apply conditional probability recursively: $\mathbb{P}\big(A \cap B\big) = \mathbb{P}\big(A \,|\, B\big) \cdot \mathbb{P}(B)$

(a formal proof can be done via induction)

$$
\begin{aligned}
\mathbb{P}\big(A_1 \cap A_2\big) &= \mathbb{P}\big(A_2 \,|\, A_1\big) \cdot \mathbb{P}(A_1) \\
\mathbb{P}\big(A_1 \cap A_2 \cap A_3\big) &= \mathbb{P}\big(\big(A_1 \cap A_2\big) \cap A_3\big) \\
&= \mathbb{P}\big(A_3 \,|\, A_1 \cap A_2\big) \cdot \mathbb{P}\big(A_1 \cap A_2\big) \\
&= \mathbb{P}\big(A_3 \,|\, A_1 \cap A_2\big) \cdot \mathbb{P}\big(A_2 \,|\, A_1\big) \cdot \mathbb{P}(A_1) \\
&\vdots \\
\mathbb{P}\big(A_1 \cap \cdots \cap A_n\big) &= \mathbb{P}\big(A_n \,|\, A_1 \cap \cdots \cap A_{n-1}\big) \cdots \mathbb{P}\big(A_2 \,|\, A_1\big) \cdot \mathbb{P}\big(A_1\big)
\end{aligned}
$$

$\square$

Proofs of the Theorems

Proof of Total Probability

For any event A, we have $A = A \cap \Omega$. And because the sets E_1, E_2, . . . , E_n are a partition of Ω,

$$\begin{aligned}
\mathbb{P}(A) &= \mathbb{P}\big(A \,\cap\, \Omega\big) \\
&= \mathbb{P}\Big(A \,\cap\, (E_1 \,\cup\, E_2 \,\cup\, \cdots \,\cup\, E_n)\Big) \\
&= \mathbb{P}\Big(\big(A \cap E_1\big) \,\cup\, \big(A \cap E_2\big) \,\cup\, \cdots \cup \big(A \cap E_n\big)\Big) \\
&= \mathbb{P}\big(A \cap E_1\big) + \mathbb{P}\big(A \cap E_2\big) + \cdots + \mathbb{P}\big(A \cap E_n\big) \\
&= \mathbb{P}\big(A \,|\, E_1\big) \cdot \mathbb{P}\big(E_1\big) + \mathbb{P}\big(A \,|\, E_2\big) \cdot \mathbb{P}\big(E_2\big) + \cdots + \mathbb{P}\big(A \,|\, E_n\big) \cdot \mathbb{P}\big(E_n\big)\,.
\end{aligned}$$

The 4th equality is due to $E_i \cap E_j = \emptyset \implies (A \cap E_i) \cap (A \cap E_j) = \emptyset$.

$\square$

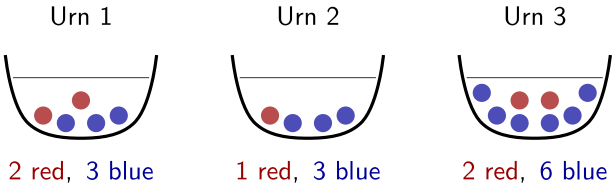
Proofs of the Theorems

Proof of Bayes Rule

$$\begin{aligned}\mathbb{P}(E_i \mid A) &= \frac{\mathbb{P}(A \cap E_i)}{\mathbb{P}(A)} \\ &= \frac{\mathbb{P}(A \mid E_i) \cdot \mathbb{P}(E_i)}{\mathbb{P}(A)} \\ &= \frac{\mathbb{P}(A \mid E_i) \cdot \mathbb{P}(E_i)}{\sum_{j=1}^{n} \mathbb{P}(A \mid E_j) \cdot \mathbb{P}(E_j)}\,,\end{aligned}$$

where the last step used the total probability theorem. $\square$

Exercise



Question 1: I select one of the urns uniformly at random, and draw a ball. What is the probability that it is blue? Ans: $\frac{7}{10}$

Question 2: From urn 3, I draw 3 balls consecutively *without* putting them back in.* What is the probability that all of them are blue? $\frac{5}{14}$

---

*This is called ***sampling without replacement***. If we put the balls back in, it is called ***sampling with replacement.***

Counting

Computing probabilities often requires counting $\#$ of elements in sets

$$\text{Uniform probabilities} \quad \Longrightarrow \quad \mathbb{P}(A) = \frac{\#\text{ of elements in } A}{\#\text{ of elements in } \Omega}$$

Example: $\{$Alistair, Beatrix, Colin, Donald, Eleanor$\}$ are waiting in line.

- How many different configurations can they form?
- Only 3 can be served. How many configurations now? (order matters)
- How many ways of splitting them into 3 served, 2 not served?

Counting

Given a set with n elements,

Permutations: $n\,! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$

k-**permutations:** $A_k^n = \dfrac{n\,!}{(n-k)!} = n \cdot (n-1) \cdots (n-k+1)$

k-**combinations:** $\dbinom{n}{k} = \dfrac{A_k^n}{k\,!} = \dfrac{n\,!}{(n-k)!k!}$

Convention: $0\,! = 1$