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Engineering mathematics and statistics - Part II

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B39AX – Fall 2023
Heriot-Watt University

LEADERS IN IDEAS AND SOLUTIONS

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Plan of the second part of the course

- Multivariate distributions (week 7)
- Bayesian estimation (week 8-9)
- Introduction to information theory (week 9)
- Source coding (week 10)
- Channel coding (week 11)

- Assessments: 1 online test (15%) + Final exam (70% covering the whole course)

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Further topics on multivariate continuous distributions

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Plan

- Change of variable – 1D case
- Joint distributions
- Covariance and correlation
- Multivariate Gaussian distribution

Univariate distributions

- Probability density function (PDF): $f_X(x)$
  - Defined on A (e.g., $A = \mathbb{R}$)
  - For $B \subset A$

$$\mathbb{p}(x \in B) = \int_{x \in B} f_X(x)\, dx$$

- Cumulative distribution function (CDF): $F_X(x)$

$$F_X(x_0) = \int_{-\infty}^{x_0} f_X(x)\, dx$$

For $A = \mathbb{R}$

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Univariate distributions

- Expected value

$$\mathbb{E}[X] = \int_{x \in A} x \, f_X(x) \, dx$$

- Variance

$$\mathbb{E}\left[X^2\right] = \int_{x \in A} x^2 \, f_X(x) \, dx$$

- $\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

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Change of variable (1D)

- Assume we have a continuous RV X which follows $X \sim f_X(x)$ defined on $\mathbb{R}$
- Let $g(\cdot)\colon \mathbb{R} \to \mathbb{R}$ be a monotonic function
- What is the distribution of $\mathrm{Y} = g(X)$?

$$f_Y(y) = f_X\left(g^{-1}(y)\right)\left|\frac{d}{dy}(g^{-1}(y))\right|$$

Example (I)

- $X \sim \mathcal{N}(x; 0,1)$ (standard normal distribution)
- $Y = g(X) = aX + b$, with (a, b) scalars $(a \neq 0)$

$$f_Y(y) = f_X\left(g^{-1}(y)\right)\left|\frac{d}{dy}\left(g^{-1}(y)\right)\right|$$

$$X = g^{-1}(Y) = \frac{Y-b}{a}, \frac{d}{dy}\left(g^{-1}(y)\right) = \frac{1}{a}$$

$$f_Y(y) = \frac{f_X\left(\frac{y-b}{a}\right)}{|a|}$$

Example (I)

$$f_Y(y) = \frac{f_X\left(\frac{y-b}{a}\right)}{|a|}$$

- $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$
- $f_Y(y) = \frac{1}{|a|\sqrt{2\pi}} e^{-\frac{\left(\frac{y-b}{a}\right)^2}{2}} = \frac{1}{\sqrt{2\pi a^2}} e^{-\frac{(y-b)^2}{2a^2}}$
- $Y \sim \mathcal{N}(y; b, a^2)$

Example (II)

- Uniform distribution
- $X \sim U_{[0,1]}(x)$
- What is the distribution of $Y = X^2$ and $Z = \sqrt{X}$?

- Solution:
- $f_Y(y) = \frac{1}{2\sqrt{y}}$ for $y \in [0,1]$
- $f_Z(z) = 2z$ for $z \in [0,1]$

Verification using Matlab

- See demo1.m