Engineering Mathematics and Statistics (B39AX)
Fall 2023

Tutorial 1

Problem A. A die is weighted so that the probability of each face is proportional to the number that it contains. For example, 6 is twice as likely to occur as 3.

(a) Describe the sample space and find the probability of each outcome.

The sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$. Let $p_i := \mathbb{P}(\omega_i)$, for $\omega_i \in \Omega$. We have $p_i = \alpha \cdot i$, for $i = 1, \ldots, 6$. Using the normalization axiom,

$$\sum_{i=1}^{6} p_i = 1 \quad \Longleftrightarrow \quad \alpha \sum_{i=1}^{6} i = 1 \quad \Longleftrightarrow \quad \alpha \frac{1+6}{2} 6 = 1 \quad \Longleftrightarrow \quad \alpha = \frac{1}{21}\,.$$

So the probability of each outcome is $p_i = \frac{i}{21}$, $i = 1, \ldots, 6$.

(b) What is the probability of obtaining an even number? And what is the probability of obtaining a prime number?

The probability of obtaining an even number is

$$\mathbb{P}(\{2,4,6\}) = \mathbb{P}(\{2\}) + \mathbb{P}(\{4\}) + \mathbb{P}(\{6\}) = \frac{2}{21} + \frac{4}{21} + \frac{6}{21} = \frac{12}{21} = \frac{4}{7}\,.$$

The probability of obtaining a prime number is

$$\mathbb{P}(\{2,3,5\}) = \mathbb{P}(\{2\}) + \mathbb{P}(\{3\}) + \mathbb{P}(\{5\}) = \frac{2}{21} + \frac{3}{21} + \frac{5}{21} = \frac{10}{21}\,.$$

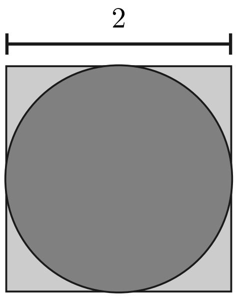
(c) What is the probability of obtaining a number larger than or equal to 3?

$$\mathbb{P}(\{3,4,5,6\}) = \frac{3}{21} + \frac{4}{21} + \frac{5}{21} + \frac{6}{21} = \frac{18}{21} = \frac{6}{7}$$

(d) What is the probability of obtaining 1? Is there an alternative way to obtain this result using the previous answers?

We have $\mathbb{P}(\{1\}) = 1/21$. To obtain this result from the previous answers, let $A = \{2,4,6\}$, $B = \{3,4,5,6\}$, and $C = \{1\}$. We have $C = \Omega \backslash \{A \cup B\}$ and $A \cap B = \{4,6\}$, which has probability $\mathbb{P}(A \cap B) = 10/21$. Therefore,

$$\mathbb{P}(C) = 1 - \mathbb{P}(A \cup B) = 1 - \mathbb{P}(A) - \mathbb{P}(B) + \mathbb{P}(A \cap B) = 1 - \frac{4}{7} - \frac{6}{7} + \frac{10}{21} = 1 - \frac{30}{21} + \frac{10}{21} = \frac{1}{21}\,.$$



Problem B. A square of side 2 has a circle perfectly inscribed, as shown in the figure. We throw a dart, which lands at any point inside the square with equal probability. What is the probability that it lands outside the circle?

Let C be the event “dart lands inside the circle”. Then, we want to compute

$$\begin{aligned}\mathbb{P}(\text{"dart lands outside the circle"}) &= 1 - \mathbb{P}(C)\\ &= 1 - \text{"ratio of the areas"}\\ &= 1 - \frac{\pi}{4}\,.\end{aligned}$$

Problem C. Let A and B be events with probabilities $3/4$ and $1/3$, respectively.

(a) Show that the probability of $A \cap B$ is smaller than or equal to $1/3$. Describe the situation in which the probability is equal to $1/3$.

We have $\mathbb{P}(A) = \frac{3}{4}$ and $\mathbb{P}(B) = \frac{1}{3}$.

Because $A \cap B \subseteq B$ (or through a picture), we have $\mathbb{P}(A \cap B) \leq \mathbb{P}(B) = \frac{1}{3}$.

An alternative is $\frac{1}{3} = \mathbb{P}(B) = \mathbb{P}(A \cap B) + \underbrace{\mathbb{P}(A^c \cap B)}_{\geq 0}$, which implies $\frac{1}{3} \geq \mathbb{P}(A \cap B)$.

And equality occurs when $A \cap B = B$, that is, $B \subseteq A$ (or through a picture).

An alternative answer is $A^c \cap B = \emptyset$.

(b) Show that the probability of $A \cap B$ is larger than or equal to $1/12$. Describe the situation in which the probability is equal to $1/12$.

We have

$$\begin{aligned}1 \geq \mathbb{P}(A \cup B) &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)\\ &= \frac{3}{4} + \frac{1}{3} - \mathbb{P}(A \cap B)\,.\end{aligned}$$

That is,

$$\mathbb{P}(A \cap B) \geq \frac{13}{12} - 1 = \frac{1}{12}\,.$$

Equality occurs when $\mathbb{P}(A \cup B) = 1$, that is, $A \cup B = \Omega$ (or through a picture).

Problem D. Suppose I toss a fair coin three times. In each toss, let H denote heads and T denote tails.

(a) Describe the sample space and determine the size of the set of possible events.

The sample space is

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

And the set of possible events is

$$\mathcal{F} = 2^\Omega = \{\emptyset, \{HHH\}, \ldots, \{TTT\}, \{HHH, HHT\}, \{HHH, HTH\}, \ldots, \{TTH, TTT\}, \\ \ldots \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}\}.$$

It contains $2^{|\Omega|} = 2^8 = 128$ elements.

(b) Let A be the event "obtain exactly two heads." Compute $\mathbb{P}(A)$.

The events from $\mathcal{F}$ that contain exactly two heads are

$$A = \{HHT, HTH, THH\}.$$

Since the coin is fair, each of the eight events in the sample space Ω is equiprobable, and thus

$$\mathbb{P}(A) = \frac{3}{8}.$$

(c) Let B be the event "obtain heads in the first toss." Is B independent from A?

We need to compute $\mathbb{P}(B)$ and $\mathbb{P}(A \cap B)$. We have

$$B = \{HHH, HHT, HTH, HTT\},$$

and thus $\mathbb{P}(B) = 4/8 = 1/2$. We also have

$$A \cap B = \{HHT, HTH\}.$$

And thus $\mathbb{P}(A \cap B) = 2/8 = 1/4$. To determine if A and B are independent, we check if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$:

$$\frac{1}{4} = \mathbb{P}(A \cap B) \neq \mathbb{P}(A) \cdot \mathbb{P}(B) = \frac{3}{8} \cdot \frac{1}{2} = \frac{3}{16}.$$

Thus, A and B are not independent.

Problem E. An urn contains 7 red and 3 white marbles. The marbles are taken out from the urn one at a time. What is the probability that the first two marbles are red and the third one white?

Let S_i denote the event "colour of marble at draw i". We can use the multiplication rule:

$$\begin{aligned}\mathbb{P}(S_1 = R,\, S_2 = R,\, S_3 = W) &= \mathbb{P}(S_1 = R) \cdot \mathbb{P}(S_2 = R \,|\, S_1 = R) \cdot \mathbb{P}(S_3 = W \,|\, S_1 = R,\, S_2 = R) \\ &= \frac{7}{10} \cdot \frac{6}{9} \cdot \frac{3}{8} \\ &= \frac{7}{40}\,.\end{aligned}$$