Notes about tutorial 8

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An important property used in this tutorial is the so-called conjugacy between distributions

This happens when $f(y/x)$ (seen as a function of x) has the same shape as $f(x)$ (the prior distribution), which results in $f(x/y)$ being in the same family of densities as $f(x)$. It is useful as it simplifies the exploitation of $f(x/y)$.

Some examples:

Likelihood	prior	posterior	
$y/p \sim$ Bernoulli(p)	$p \sim$ Beta	$p/y \sim$ Beta	
$y/p,m \sim$ Binomial(m,p)	"	"	
$y/x \sim \mathcal{N}(x, \sigma^2)$	$x \sim \mathcal{N}(m, s^2)$	$x/y \sim \mathcal{N}(\quad)$	(σ^2 fixed/known)
$y/\sigma^2 \sim \mathcal{N}(x, \sigma^2)$	$\sigma^2 \sim$ Inverse Gamma	$\sigma^2/y \sim$ Inverse Gamma	(x fixed/known)
$y/\lambda \sim$ Poisson(λ)	$\lambda \sim$ Gamma	$\lambda/y \sim$ Gamma	

We do not need to compute $f(x)$ all the time and perfectly to find its mean/variance. or $f(x/y)$

Sometimes, it is sufficient to identify if $f(x)$ belongs to a known family of distributions.

Ex: $f(x) \propto e^{-ax^2+bx}$ with $a>0$ and $x \in \mathbb{R}$

What is $E_{f(x)}[x]$?

Solution 1: We know that $\int f(x)\,dx = 1$

So we can compute $K>0$ such that

$$\int f(x)\,dx = \int \frac{1}{K} e^{-ax^2+bx}\,dx = 1$$

Once we have K we compute

$$E_{f(x)}[x] = \int x f(x)\,dx = \frac{1}{K}\int x e^{-ax^2+b}\,dx$$

This requires computing 2 integrals

Solution 2 (sometimes faster):

$$f(x) \propto e^{g(x)} \leftarrow \text{2nd order polynomial in } x$$

This is very similar to a normal distribution

$$x \sim \mathcal{N}(m, \sigma^2) \to f(x) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{1}{2}} e^{-\frac{(x-m)^2}{2\sigma^2}} \propto e^{-\frac{(x-m)^2}{2\sigma^2}}$$

If we develop $-\frac{(x-m)^2}{2\sigma^2} = -\frac{x^2}{2\sigma^2} + \frac{m}{\sigma^2}x - \frac{m^2}{2\sigma^2}$

$$f(x) \propto e^{-\frac{x^2}{2\sigma^2} + \frac{m}{\sigma^2}x - \frac{m^2}{2\sigma^2}} \leftarrow \text{does not depend on } x$$

$$\propto e^{-\frac{x^2}{2\sigma^2} + \frac{m}{\sigma^2}x}$$

We can identify the terms in x^2 and x

$$a = \frac{1}{2\sigma^2} \text{ and } b = \frac{m}{\sigma^2}$$

$$\Rightarrow \sigma^2 = \frac{1}{2a} \qquad m = b\sigma^2 = \frac{b}{2a}$$

If $f(x) \propto e^{-ax^2+bx}$, then $x \sim \mathcal{N}\left(\frac{b}{2a}, \frac{1}{2a}\right)$.

and $x \in \mathbb{R}$

It follows that $E_{f(x)}[x] = \frac{b}{2a}$.