# Further topics on multivariate continuous distributions

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LEADERS IN IDEAS AND SOLUTIONS

# Plan

- Change of variable – 1D case
- Joint distributions
- Covariance and correlation
- Multivariate Gaussian distribution

# Bivariate distributions

- Joint PDF: $f_{X,Y}(x,y)$ defined on $A$ (e.g., $A = \mathbb{R}^2$)

$$\mathbb{p}_{X,Y}((x,y) \in B) = \iint_{(x,y)\in B} f_{X,Y}(x,y)\, dx\, dy$$

$$f_{X,Y}(x,y) \geq 0, \forall (x,y)$$

$$\iint_{(x,y)\in A} f_{X,Y}(x,y)\, dx\, dy = 1$$

# Bivariate distributions

- Joint CDF: $F_{X,Y}(x, y)$ (defined on $A = \mathbb{R}^2$)

$$F_{X,Y}(x_0, y_0) = \int_{-\infty}^{y_0} \int_{-\infty}^{x_0} f_{X,Y}(x, y)\, dx\, dy$$

$f_{X,Y}(x, y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x\, \partial y}$, (assuming $F_{X,Y}(x, y)$ is twice differentiable)

# Example (I)

- Uniform distribution (rectangle)

$$A = [0,1] \times [1,4]$$

$$f(x,y) = \begin{cases} c, & if\ (x,y) \in A \\ 0, & else. \end{cases}$$

$$1 = \iint_{(x,y)\in A} f_{X,Y}(x,y)\,dx\,dy = \int_0^1 \int_1^4 c\,dy\,dx = 3c$$

- $1/c = 3$ : area of $A$

# Example (II)

- Uniform distribution (disk)

$$A = \{(x, y) | x^2 + y^2 < 9\}$$

$$f(x, y) = \begin{cases} c, & if\ (x, y) \in A \\ 0, & else. \end{cases}$$

$$1 = \iint_{(x,y)\in A} f_{X,Y}(x, y)\, dx\, dy = 9\pi c$$

- $\frac{1}{c} = 9\pi = 3^2\pi$ : area of $A$

# Bayes rule

- Discrete case: $(X, Y)$ are (vectors of) discrete RVs

$$\mathbb{p}(\mathrm{X} = \mathrm{x} | \mathrm{Y} = \mathrm{y}) = \frac{\mathbb{p}(X = x, Y = y)}{\mathbb{p}(Y = y)}$$

- Continuous case: $(\mathrm{X}, \mathrm{Y})$ are (vectors of) continuous RVs

$$f_{X,Y}(x, y) = f_{X|Y}(x|y) f_Y(y)$$

# Marginal distributions

$$f_X(x) = \int_y f_{X,Y}(x,y)\,dy$$

Using Bayes rule: $f_{X,Y}(x,y) = f_{X|Y}(x|y)f_Y(y)$

$$f_X(x) = \int_y f_{X|Y}(x|y)f_Y(y)\,dy$$

# Conditional distributions

- $f_{X|Y}(x|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)}$

- $\mathbb{p}(X \in B_X \,|Y = y) = \int_{B_X} f_{X|Y}(x|y)\, dx$

# Example

- Uniform distribution

$$A = \{(x, y) | x^2 + y^2 < 9\}$$

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{9\pi}, & if\ (x, y) \in A \\ 0, & else. \end{cases}$$

What is $f_X(x)$?

# Example

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dy = \ \ldots$$

- Solution:

$$f_X(x) = \frac{2}{9\pi}\sqrt{9 - x^2}, if\ |x| < 3$$

# Example

- What is $f_{X|Y}(x|y)$?
- $f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{1/(\pi R^2)}{\frac{2}{R^2\pi}\sqrt{R^2-y^2}} = \frac{1}{2\sqrt{R^2-y^2}},$

$if\ x^2 + y^2 \leq R^2$

# Verification using Matlab

- See demo2.m