Further topics on multivariate continuous distributions

Dr. Yoann Altmann

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LEADERS IN IDEAS AND SOLUTIONS

Plan

- Change of variable – 1D case
- Joint distributions
- Covariance and correlation
- Multivariate Gaussian distribution

Expectations

$g(\cdot)$: arbitrary function

$$\mathbb{E}[g(X,Y)] = \iint_{(x,y)\in A} g(x,y) f_{X,Y}(x,y)\, dx\, dy$$

Important example:

$$\mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + c$$

where a, b and c are scalars

Expectations using conditional distributions

- Marginal expectation

$$\mathbb{E}[g(X)] = \iint_{(x,y)\in A} g(x) f_{X,Y}(x, y)\, dx\, dy$$

$$\mathbb{E}[g(X)] = \int g(x) \int f_{X,Y}(x, y) dy\, dx = \int g(x) f_X(x) dx$$

- Conditional expectation

$$\mathbb{E}[g(X)|Y = y] = \int g(x) f_{X|Y}(x|y) dx$$

$\mathbb{E}[X] = \int \ \mathbb{E}[X|Y = y]\, f_Y(y)\, dy$ (application of total expectation theorem)

Independent variables

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

This is equivalent to

$$f_{X|Y}(x|y) = f_X(x)$$

"The distribution of $X|Y$ does not depend on Y"

Independent variables

- X and Y jointly continuous random variables

- $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$

- $\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\mathbb{E}[h(Y)]$

- $\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$

Higher dimensions

- X (and Y) can now be vectors of RVs

$$f_X(x) = \int_y f_{X,Y}(x,y)\,dy$$

… becomes a multidimensional integral

$Y = \{U, V\}$

$$f_X(x) = \int_y f_{X,Y}(x,y)\,dy = \int_u \int_v f_{X,U,V}(x,u,v)\,du\,dv$$

- Bayes rule

$$f_{X,Y}(x,y) = f_{X|Y}(x|y)\,f_Y(y)$$

Sum of independent RVs

- Let X and Y be independent, jointly continuous random variables
- What is the distribution of $Z = X + Y$?

$$f_Z(z) = \int f_{Z|X}(z|x) f_X(x) dx = \int f_Y(z-x) f_X(x) dx$$

$$\mathbb{p}(Z < z|X = x) = \mathbb{p}(X + Y < z|X = x)$$
$$\mathbb{p}(x + Y < z|X = x) = \mathbb{p}(Y < z - x)$$

Sum of independent RVs

$$\mathbb{p}(Z < z|X = x) = \int_{-\infty}^{z} f_{Z|X}(z'|x)dz'$$

By differentiating w.r.t. z: $\frac{\partial \mathbb{p}(Z<z|X=x)}{\partial z} = f_{Z|X}(z|x)$

$$\mathbb{p}(Y < z - x) = \int_{-\infty}^{z-x} f_Y(y)dy$$

By differentiating w.r.t. z: $\frac{\partial \mathbb{p}(Y<z-x)}{\partial z} = f_Y(z - x)$

Sum of independent RVs

We obtain

$$f_{Z|X}(z|x) = f_Y(z - x).$$

Using Bayes' rule

$$f_Z(z) = \int f_{Z|X}(z|x) f_X(x) dx = \int f_Y(z - x) f_X(x) dx$$

Example (I)

- Let X and Y be independent, jointly continuous random variables: $X \sim U_{[0,1]}(x)$, $\mathrm{Y} \sim U_{[0,1]}(x)$
- What is the distribution of $Z = X + Y$?

- Solution: $f_Z(z) = \begin{cases} \min\{1, z\} - \max\{0, z-1\}, & if\ z \in [0,2] \\ 0, & otherwise \end{cases}$

Example (II)

- Let X and Y be independent, jointly continuous random variables: $X \sim \mathcal{N}(x; m, s^2)$, $\mathrm{Y} \sim \mathcal{N}(y; 0, \sigma^2)$
- What is the distribution of $Z = X + Y$?

- Solution: $Z \sim \mathcal{N}(y; m, s^2 + \sigma^2)$

⚠️ Here $f_Z(\cdot)$ is in the same family as $f_X(\cdot)$ and $f_Y(\cdot)$ but it is not generally the case!

Verification using Matlab

- See demo3.m