HERIOT WATT UNIVERSITY

Introduction to channel coding

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B39AX – Fall 2023
Heriot-Watt University

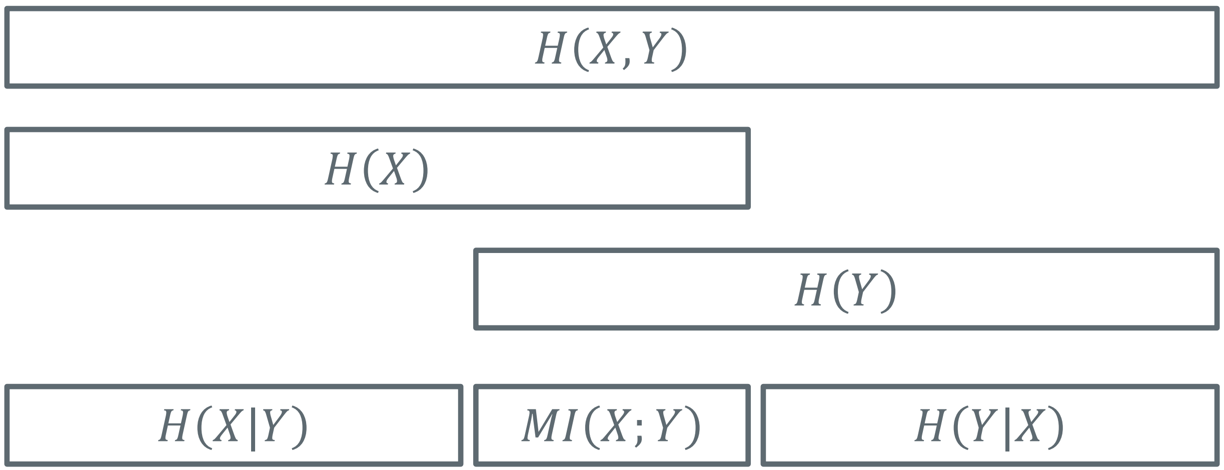
LEADERS IN IDEAS AND SOLUTIONS

Plan

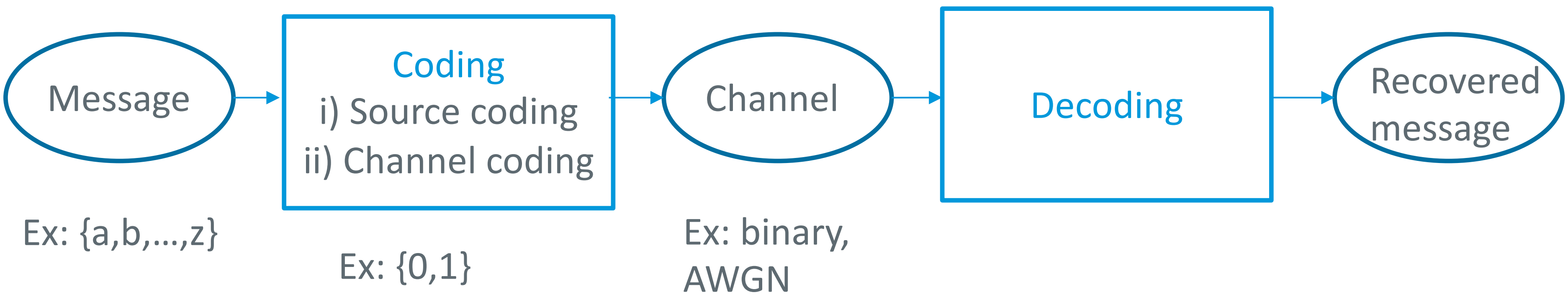
- Joint/conditional entropy
- Mutual information
- Examples of channels
- Channel capacity

- Based on the book: **“Information Theory**, **Inference**, and **Learning Algorithms”**. David J.C. MacKay (Chap. 8-9)

Entropy and mutual information



Typical communication system



- We have seen how to code/compress the message (source) to be sent
- But performance also depend on the channel properties…
- Robust transmission requires channel modelling…

Types of channel

- With memory (not covered in this course)
  - Ex: Markovian structure, multi-path
- Memoryless
  - Current output depends only on current input

Examples of channels

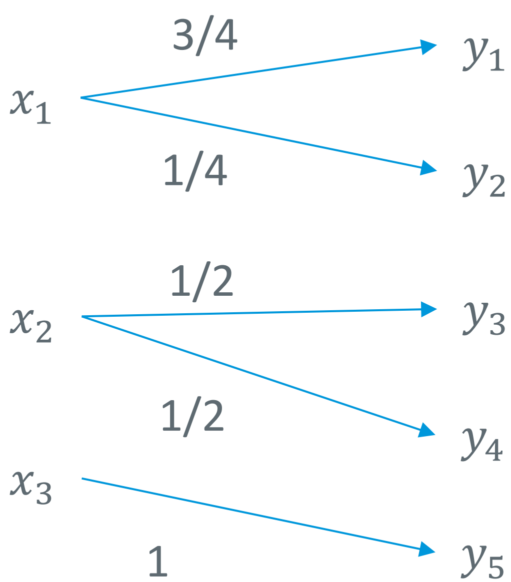
- $\mathcal{A}_X = \{0,1,\dots,N\}$ (discrete source)
- $\mathcal{A}_Y = \{0,1,\dots,N\}$ (discrete)
- Noiseless (deterministic) channel

$$X \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ 1 & \longrightarrow & 1 \\ \dots & & \dots \\ N & \longrightarrow & N \end{array} \quad Y$$

$$\mathbb{p}(Y = j | X = i) = \begin{cases} 1 \; if \; j = i \\ 0 \; if \; j \neq \; i \end{cases}$$

Examples of channels

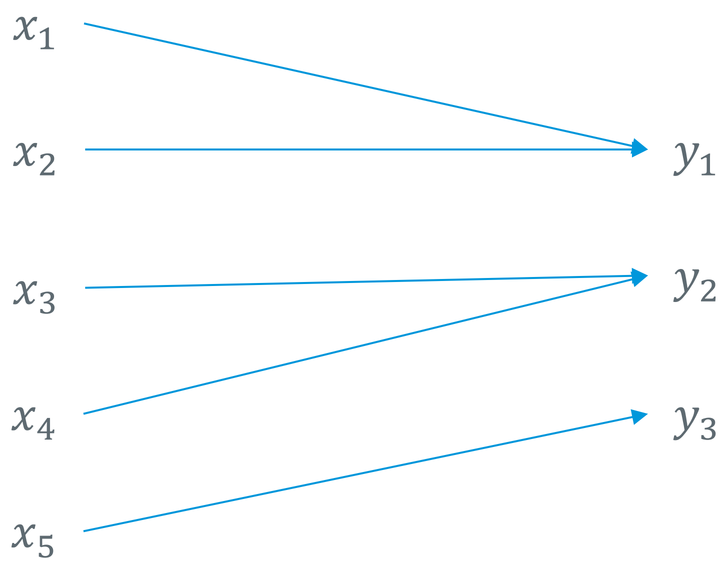
- $\mathcal{A}_X = \{0,1,\ldots,N\}$ (discrete source)
- $\mathcal{A}_Y = \{0,1,\ldots,M\}$ (discrete)
- Lossless channel



$$\sum_{m=1}^{M} \mathbb{p}(y_m|x_n) = 1, \forall n$$

Examples of channels

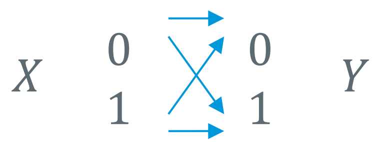
- $\mathcal{A}_X = \{0,1,\dots,N\}$ (discrete source)
- $\mathcal{A}_Y = \{0,1,\dots,M\}$ (discrete) $M \leq N$
- Deterministic channel



$$\sum_{m=1}^{M} \mathbb{p}(y_m | x_n) = 1, \forall n$$

Examples of channels

- $\mathcal{A}_X = \{0,1\}$ (binary source)
- $\mathcal{A}_Y = \{0,1\}$ (binary source)
- Binary symmetric channel (BSC)



$$\mathbb{p}(Y = 0|X = 0) = 1 - f, \mathbb{p}(Y = 1|X = 1) = 1 - f$$
$$\mathbb{p}(Y = 1|X = 0) = f, \mathbb{p}(Y = 0|X = 1) = f$$

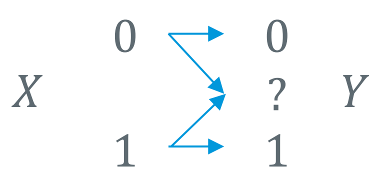
Examples of channels

- $\mathcal{A}_X = \{0,1\}$ (binary source)
- $\mathcal{A}_Y = \{0,1\}$ (binary source)
- Z channel

$$\mathbb{p}(Y=0|X=0)=1, \mathbb{p}(Y=1|X=1)=1-f$$
$$\mathbb{p}(Y=1|X=0)=0, \mathbb{p}(Y=0|X=1)=f$$

Examples of channels

- $\mathcal{A}_X = \{0,1\}$ (binary source)
- $\mathcal{A}_Y = \{0,?,1\}$ (binary source)
- Binary erasure channel



$$\mathbb{p}(Y = 0|X = 0) = 1 - f, \mathbb{p}(Y = 1|X = 1) = 1 - f$$
$$\mathbb{p}(Y =?\,|X = 0) = f, \mathbb{p}(Y =?\,|X = 1) = f$$
$$\mathbb{p}(Y = 1|X = 0) = 0, \mathbb{p}(Y =?\,|X = 1) = 0$$

Channel capacity

$$MI(X;Y) = H(X) - H(X|Y)$$

- Definition:

$$C_s = \max_{\{\mathbb{p}(X=x_n)\}_n} MI(X;Y)$$

The capacity C_s (b/symbol) can be interpreted as the maximal amount of information that can be transmitted through the channel

Channel capacity per second

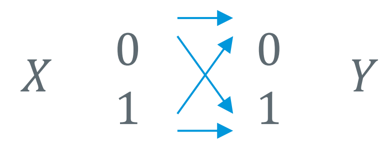
$$C_s = \max_{\{\mathbb{p}(X=x_n)\}_n} MI(X;Y)$$

If r symbols are being transmitted per second, the maximum rate of transmission of information per second is rC_s.

$$C = rC_s \text{ (b/s)}$$

Capacity of the BSC

- $\mathcal{A}_X = \{0,1\}$ (binary source)
- $\mathcal{A}_Y = \{0,1\}$ (binary source)
- Binary symmetric channel (BSC)
- From demo1.m (+ tutorial)



$$C_s = 1 + f log_2(f) + (1 - f)\log_2(1 - f)$$

Example (I)

- BSC: transmitting at the rate of 1000 symbols per second. 1% of received symbols is incorrect. What is the rate of transmission of information rC_s?

Possible first guess, 990 bits per second, is not correct because we don't know where errors occur.

- r=1000, $p = 0.5$ (source assumed equiprobable), $f = 0.01$

Example (I)

- BSC: transmitting at the rate of 1000 symbols per second. 1% of received symbols is incorrect. What is the rate of transmission of information rC_s?
- r=1000, $p = 1/2$

$$C_s = 1 + f log_2(f) + (1 - f)\log_2(1 - f)$$

Example (I)

- BSC: transmitting at the rate of 1000 symbols per second. 1% of received symbols is incorrect. What is the rate of transmission of information rC_s?
- r=1000, $p = 1/2$

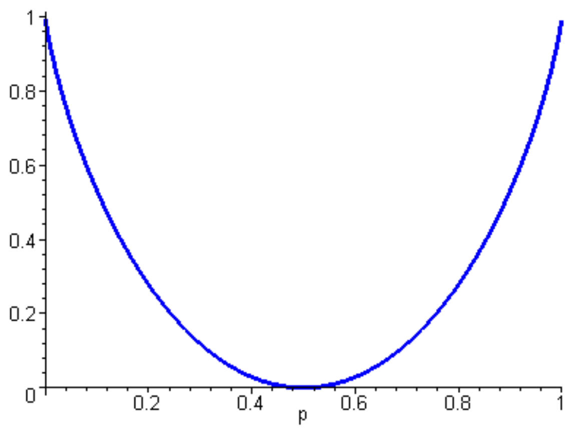
$$C_s = 1 + f\log_2 f + (1-f)\log_2(1-f)$$

$f = 0 \Rightarrow C_s = 1$ (noiseless)

$f = 1 \Rightarrow C_s = 1$ (inverter)

$f = 0.5 \Rightarrow C_s = 0$

$f = 0.01 \Rightarrow C_s = 0.919$



Example (II)

- $\mathcal{A}_X = \{0,1\}$ (binary source)
- $\mathbb{p}(X = 0) = 0.7$, $\mathbb{p}(X = 1) = 0.3$
- Binary symmetric channel ($f = 0.1$)
- Find $\mathbb{p}(Y = 0)$ and $\mathbb{p}(Y = 1)$
- $\mathbb{p}(Y = 0) = 0.66$, $\mathbb{p}(Y = 1) = 0.34$

Existence of errors tends to equalize probabilities because symbols that occur more frequently are transmitted wrongly more often. A noisy channel increases uncertainty ($H(Y) \geq H(X)$).

Useful results

- Here we only considered the transmission of symbols with fixed lengths (often 1) from a memoryless sources through a memoryless channel.
- This can be extended to cases where a variable-length code is used for X (see source coding)
- Key idea: tailor the input (after compression) depending on the channel to maximize the mutual information between the input and output of the channel

Beyond the scope of this course

- Similar results apply to more complex transmission schemes
  - Additive white Gaussian noise (AWGN) channel: Y is continuous
  - Transmission of complex numbers (modulation, see also B39SA)
- We focused on coding. Decoding and error detection/correction is also very important
- For more advanced topics: **"Information Theory**, **Inference**, and **Learning Algorithms"**. David J.C. MacKay

Summary

- Information theory:
  - Useful tools for communications but not only… (data compression, cryptography, quantum computing,…)
  - Basic blocks: source and channel coding
  - Central concept: entropy as a measure of uncertainty + Bayes rule
  - Channel capacity: linked to mutual information