**Engineering Mathematics and Statistics (B39AX) Fall 2023**

**Tutorial 8**

**Problem A.**

Consider a set of observations which are independent and identically distributed according to a binomial distribution with known number of trials and whose probability of success is unknown.

1. Compute the maximum likelihood estimator (MLE) of , computed from the observations gathered in .

The pdf of the binomial distribution is given by

Since the observations are independent, we have

The MLE estimator of is obtained by maximizing or equivalently .

To find the maximum of , we compute its derivative with respect to , and look for values of for which the derivative is 0.

The MLE of is thus given by

With .

Adopting a Bayesian approach, is assigned a beta prior distribution whose probability density function is given by

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where are fixed (known) parameters, larger than 1 and is the gamma function. Since , they are omitted in all the pdfs for brevity.

1. Using , compute the mean of the prior distribution .

Note that .

Since is a pdf, we have

Thus,

Finally, we have .

1. Compute the maximum a posteriori (MAP) estimator of .

Using the Bayes’ rule, the posterior distribution is proportional to . The the MAP estimator of can be obtained by maximizing .

We compute its derivative and find the value(s) of for which the derivative is 0.

Which leads to

1. Using the prior distribution defined in 2), show that the distribution is the probability density function of a beta distribution and compute its parameters .

As mentioned above , i.e.,

Where the missing scaling factor does not depend on . This pdf has the same shape as the prior pdf, which corresponds to a beta distribution. Consequently, by identifying the exponents of and , we find

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1. Using the result obtain in 2), compute the posterior mean of , i.e., .

In 2), we showed that the mean of a beta distribution with parameters is . Consequently, the posterior mean (or MMSE estimator of ) is given by

**Problem B.**

Consider a set of observations which are independent and identically distributed according to a Poisson distribution with unknown mean .

1. Compute the maximum likelihood estimator (MLE) of , computed from the observations gathered in .

The pdf of the Poisson distribution is given by

Since the observations are independent, we have

The MLE estimator of is obtained by maximizing or equivalently .

To find the maximum of , we compute its derivative with respect to , and look for values of for which the derivative is 0.

The MLE of is thus given by

Adopting a Bayesian approach, is assigned a gamma prior distribution whose probability density function is given by

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where are fixed (known) parameters, such that and . Note that the gamma distribution is a generalisation of the exponential distribution: the gamma distribution with reduces to an exponential distribution with mean .

1. Using , compute the mean of the prior distribution

Since is a pdf, we have

Thus,

Finally, we have .

1. Compute the maximum a posteriori (MAP) estimator of .

Using the Bayes’ rule, the posterior distribution is proportional to . The MAP estimator of can be obtained by maximizing .

We compute its derivative and find the value(s) of for which the derivative is 0.

Which leads to

1. Using the prior distribution defined in 2), show that the distribution is the probability density function of a gamma distribution and compute its parameters .

As mentioned above , i.e.,

Where the missing scaling factor does not depend on . This pdf has the same shape as the prior pdf, which corresponds to a gamma distribution. Consequently, by identifying the exponents of and , we find

1. Using the result obtain in 2), compute the posterior mean of , i.e., .

In 2), we showed that the mean of a gamma distribution with parameters is . Consequently, the posterior mean (or MMSE estimator of ) is given by