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B39HF High Frequency Circuits

Lecture 13
Microwave Network Analysis

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Introduction

- When circuits operating at low frequencies, for which the circuit dimensions are small relative to the wavelength, whose dimensions are small enough such that there is negligible phase delay from one point to another.
- This leads to a *quasi-static* type of solution of Maxwell's equations and to the well-known Kirchhoff voltage and current laws and impedance concepts of circuit theory.
- The purpose of this chapter is to show how basic circuit and network concepts can be extended to handle many microwave analysis and design problems of practical interest.

Why we introduce the *network analysis* techniques?

Solving Maxwell's equations gives us much more information about the particular problem under consideration than we really want .

However, usually we are only interested in the voltage or current at a set of terminals, the power flow through a device, or some other type of "terminal" quantity.

Why we introduce the *network analysis* techniques?

Another reason for using circuit or network analysis is that it is then very easy to modify the original problem, or combine several elements together and find the response, without having to reanalyze in detail the behavior of each element in combination with its neighbors.

Outline

- Impedance and admittance matrices
- The scattering matrix
- The transmission (ABCD) matrix

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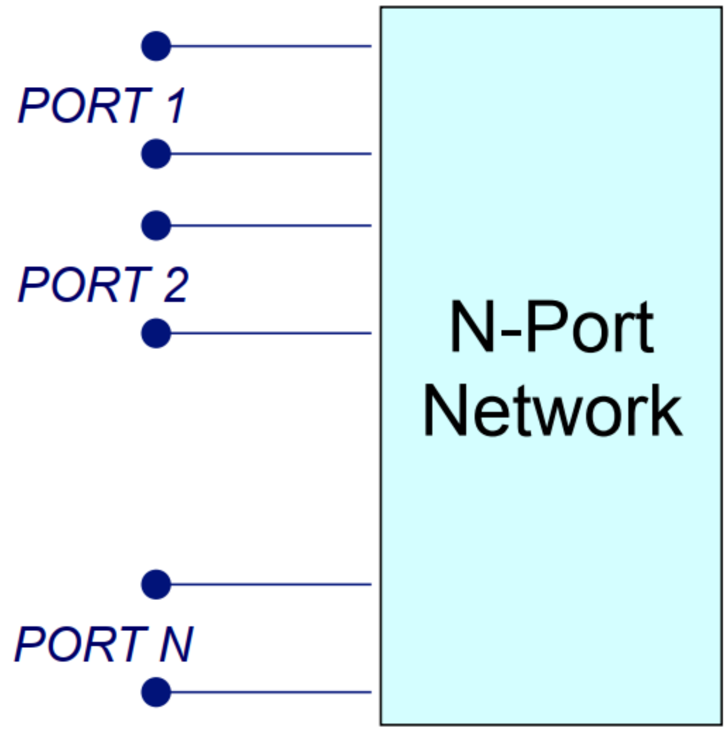
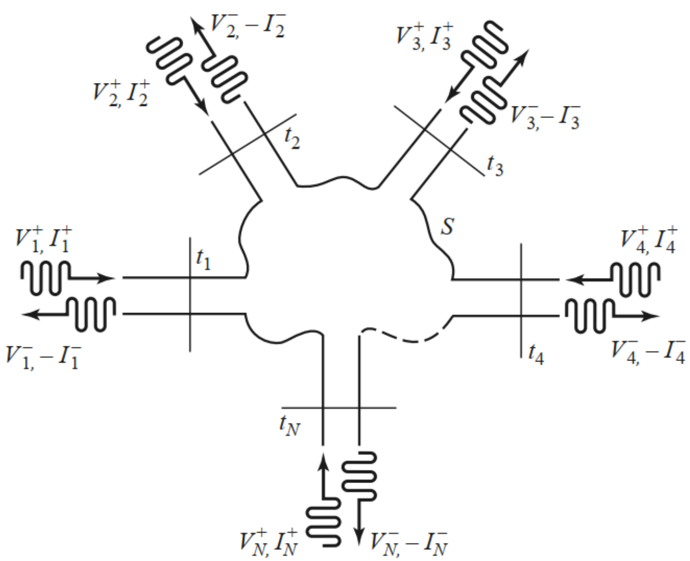
Impedance and Admittance Matrices

Port:

- *Port* was introduced by H. A. Wheeler in the 1950s to replace the less descriptive and more cumbersome phrase "two-terminal pair".
- All circuit networks have *ports*.
- *Ports* allow energy into and out of a network.
- *A port* has a signal line and a reference line.
- The minimum for a network is *one port*, e.g. a resistor.
- A common setup up is a *2-port network*, e.g. one input and one output.

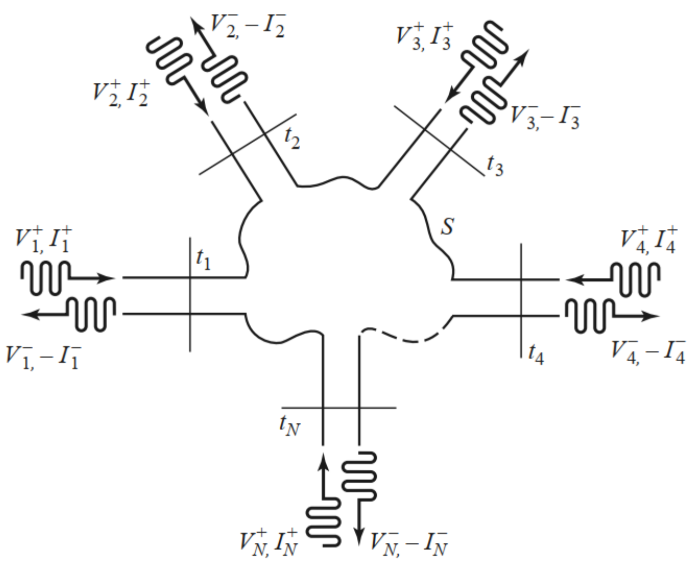


Impedance and Admittance Matrices



- Consider an arbitrary N-port microwave network, as shown in the figure left. The ports may be any type of transmission line or transmission line equivalent of a single propagating waveguide mode.
- At a specific point on the nth port, a terminal plane, t_n, is defined along with equivalent voltages and currents for the incident (V_n^+, I_n^+) and reflected (V_n^-, I_n^-) waves.
- The terminal planes are important in providing a phase reference for the voltage and current phasors.

Impedance and Admittance Matrices



- Now, at the nth terminal plane, the total voltage and current are given by:

$$V_n = V_n^+ + V_n^-$$

$$I_n = I_n^+ - I_n^-$$

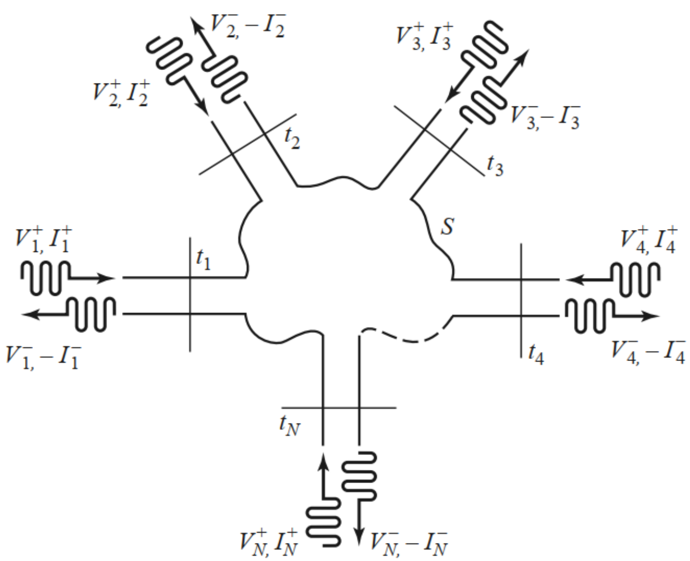
- The impedance matrix $[Z]$ of the microwave network then relates these voltages and currents:

$$\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & \cdots & Z_{1N} \\ Z_{21} & & & \vdots \\ \vdots & & & \vdots \\ Z_{N1} & \cdots & \cdots & Z_{NN} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{bmatrix}$$

- Or in matrix form as

$$[\mathbf{V}] = [\mathbf{Z}][\mathbf{I}]$$

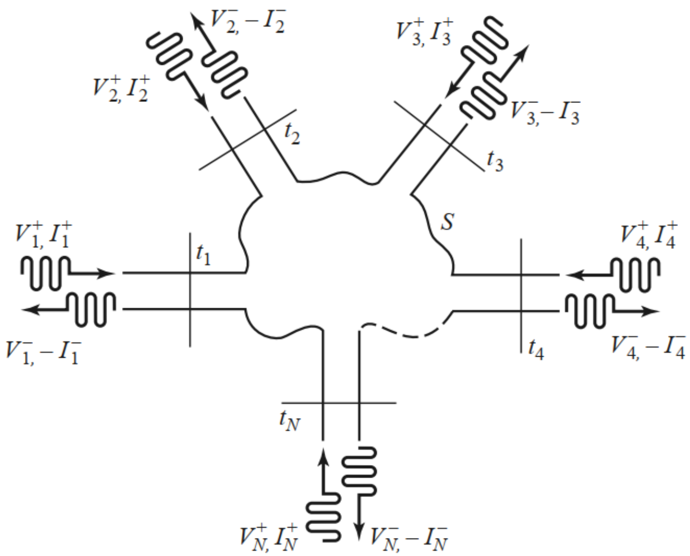
Impedance and Admittance Matrices



$$Z_{ij} = \left.\frac{V_i}{I_j}\right|_{I_k=0,(k\neq j)}$$

- Z_{ij} can be found by driving port j with the *current* I_j, *open-circuiting* all other port (so $I_k = 0$ for $k \neq j$), and measuring the *open-circuit voltage* at port i.
- Z_{ii} is the *input impedance* seen looking into port i when all other ports are *open-circuited*, and Z_{ij} is the *transfer impedance* between ports i and j when all other ports are *open-circuited*.

Impedance and Admittance Matrices



- Similarly, we can define an admittance matrix $[Y]$ as

$$\begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} & \cdots & Y_{1N} \\ Y_{21} & & & \vdots \\ \vdots & & & \vdots \\ Y_{N1} & \cdots & \cdots & Y_{NN} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{bmatrix}$$

- Or in matrix form as

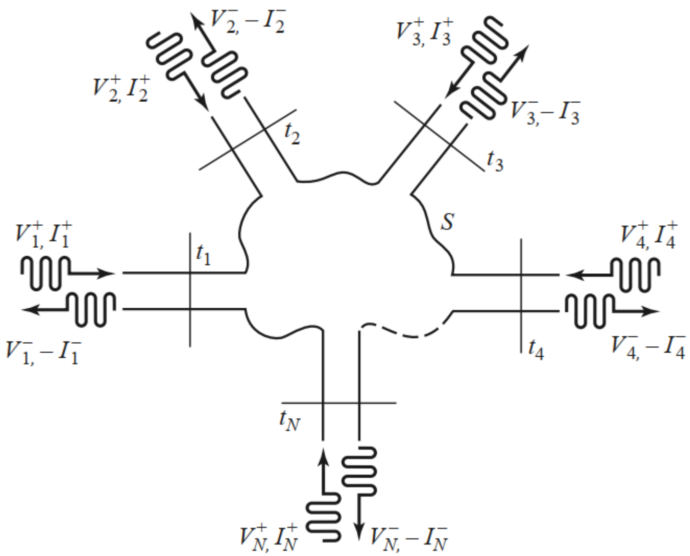
$$[\mathbf{I}] = [\mathbf{Y}][\mathbf{V}]$$

- Of course, the $[Z]$ and $[Y]$ matrices are the inverses of each other:

$$[\mathbf{Y}] = [\mathbf{Z}]^{-1}$$

Impedance and Admittance Matrices

$$Y_{ij} = \left.\frac{I_i}{V_j}\right|_{V_k=0,(k\neq j)}$$



- Y_{ij} can be found by driving port j with the *voltage* V_j, *short-circuiting* all other port (so $V_k = 0$ for $k \neq j$), and measuring the *short-circuit current* at port i.
- Y_{ii} is the *input admittance* seen looking into port i when all other ports are *short-circuited*, and Y_{ij} is the *transfer admittance* between ports i and j when all other ports are *short-circuited*.

Impedance and Admittance Matrices

- In general, each Z_{ij} or Y_{ij} element may be complex. For an arbitrary *N*-port network, the impedance and admittance matrices are $N \times N$ in size, so there are $2N^2$ independent quantities or degrees of freedom.

Impedance and Admittance Matrices

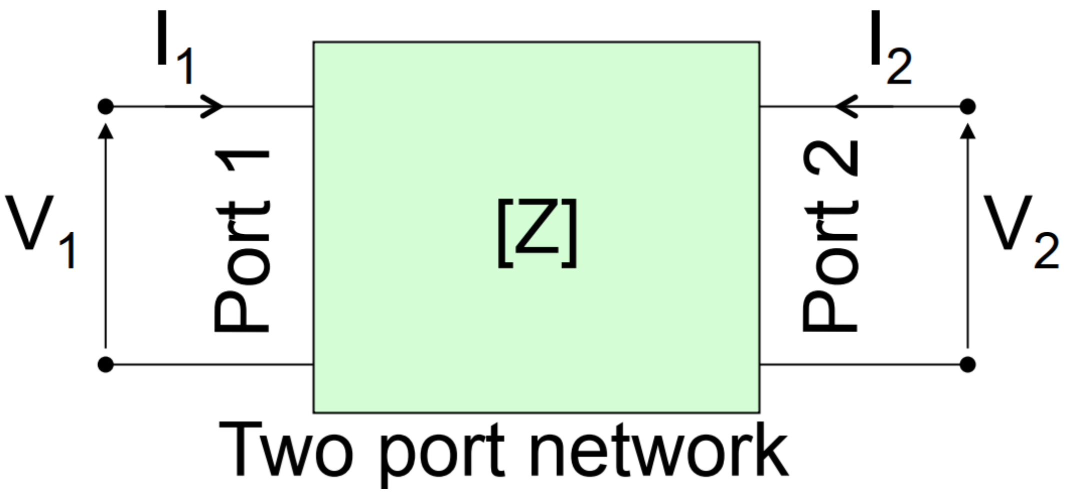
- If the network is reciprocal (not containing any active devices or nonreciprocal media, such as ferrites or plasmas), then the impedance and admittance matrices are symmetric, so that $Z_{ij} = Z_{ji}$, and $Y_{ij} = Y_{ji}$.
- If the network is lossless, then all the impedance Z_{ij} or admittance Y_{ij} elements are purely imaginary.
- Either of these special cases serves to reduce the number of independent quantities or degrees of freedom that an *N*-port network may have.

Impedance and Admittance Matrices

- Specifically, take a two-port network as an example to explain in detail.

Impedance and Admittance Matrices

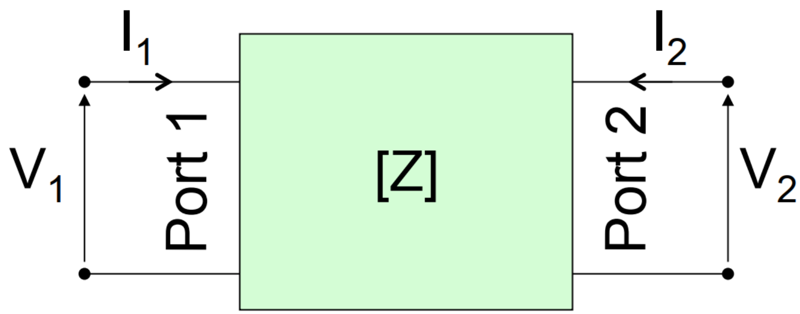
Impedance of two-port network



$$V_1 = Z_{11}I_1 + Z_{12}I_2$$
$$V_2 = Z_{21}I_1 + Z_{22}I_2$$

- The reference direction of voltage and current of the impedance matric is shown in the figure.
- Currents are probed into the circuit.
- The voltages at the ports are measured.
- The solution of the matrix yields the Z-parameters.

Impedance of two-port network



Two port network

$$V_1 = Z_{11}I_1 + Z_{12}I_2$$
$$V_2 = Z_{21}I_1 + Z_{22}I_2$$

$$Z_{11} = \frac{V_1}{I_1}\Big|_{I_2=0}$$ **Input impedance** at port 1, when port 2 is open-circuited

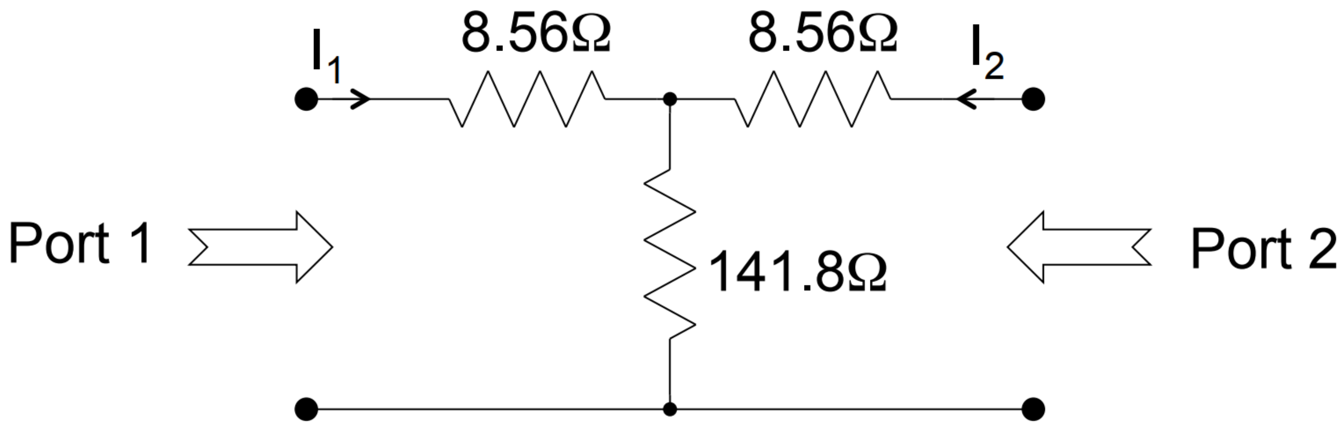
$$Z_{12} = \frac{V_1}{I_2}\Big|_{I_1=0}$$ **Transfer impedance** from port 1 to port 2, when port 1 is open-circuited

$$Z_{21} = \frac{V_2}{I_1}\Big|_{I_2=0}$$ **Transfer impedance** from port 2 to port 1, when port 2 is open-circuited

$$Z_{22} = \frac{V_2}{I_2}\Big|_{I_1=0}$$ **Input impedance** at port 2, when port 1 is open-circuited

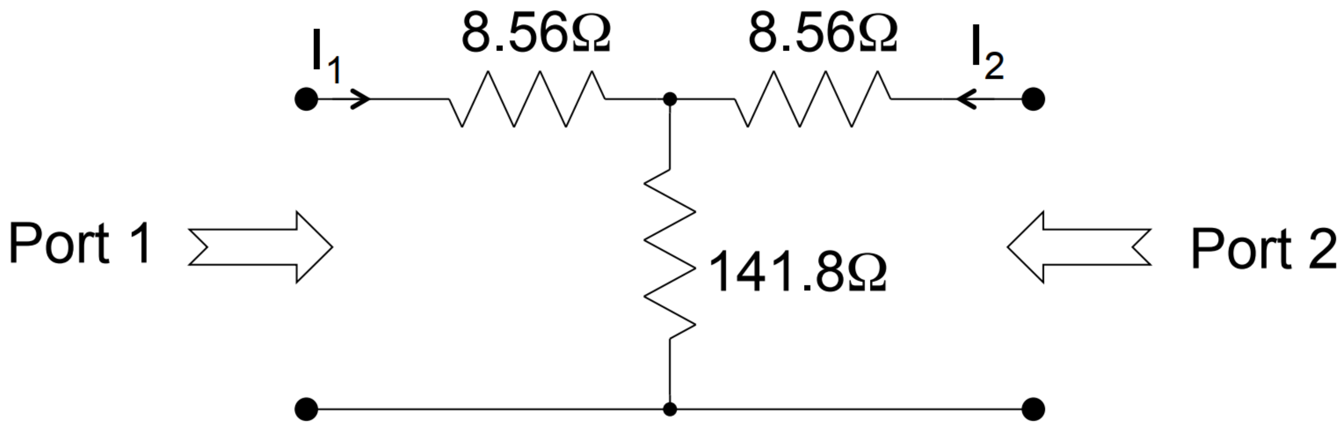
Impedance of two-port network

- Find the Z-parameters matrix of the two-port T-network shown in below.



Impedance of two-port network

- Answer:

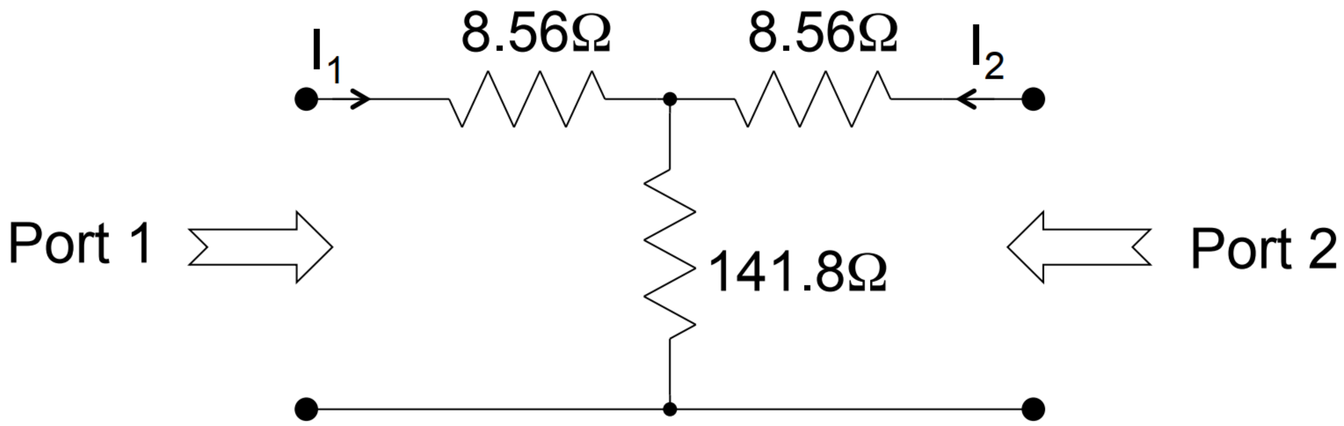


Z_{11} can be found as the input impedance of port 1 when port 2 is open-circuited:

$$Z_{11} = \frac{V_1}{I_1}\bigg|_{I_2=0} = 8.56\Omega + 141.8\Omega = 150.36\Omega$$

Impedance of two-port network

- Answer:

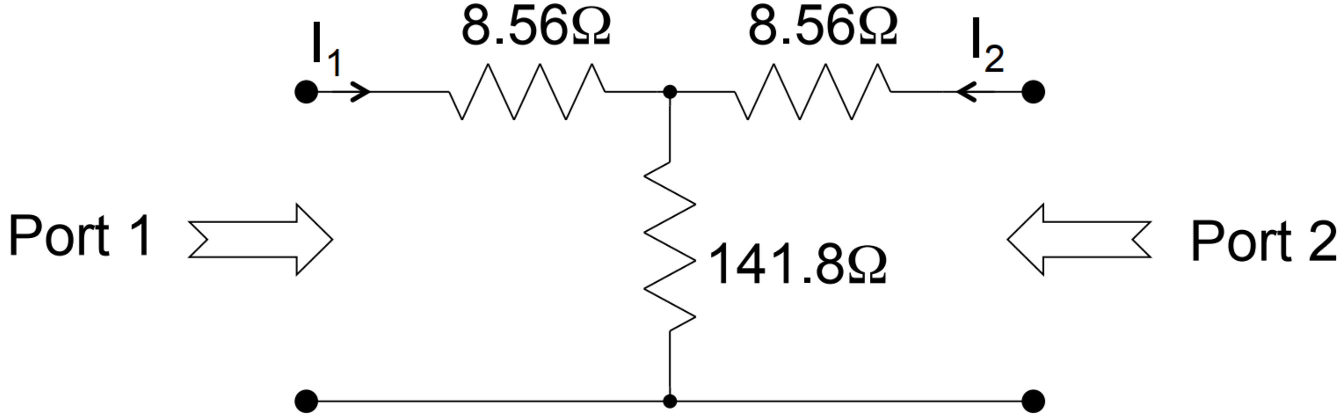


The transfer impedance Z_{12} can be found measuring the open-circuit voltage at port 1 when a current I_2 is applied at port 2. By voltage division,

$$Z_{12} = \frac{V_1}{I_2}\bigg|_{I_1=0} = \frac{V_2}{I_2}\frac{141.8}{8.56+141.8} = 141.8\Omega$$

Impedance of two-port network

- Answer:



According to the symmetry of the T-network, indicating that the circuit is reciprocal, hence:

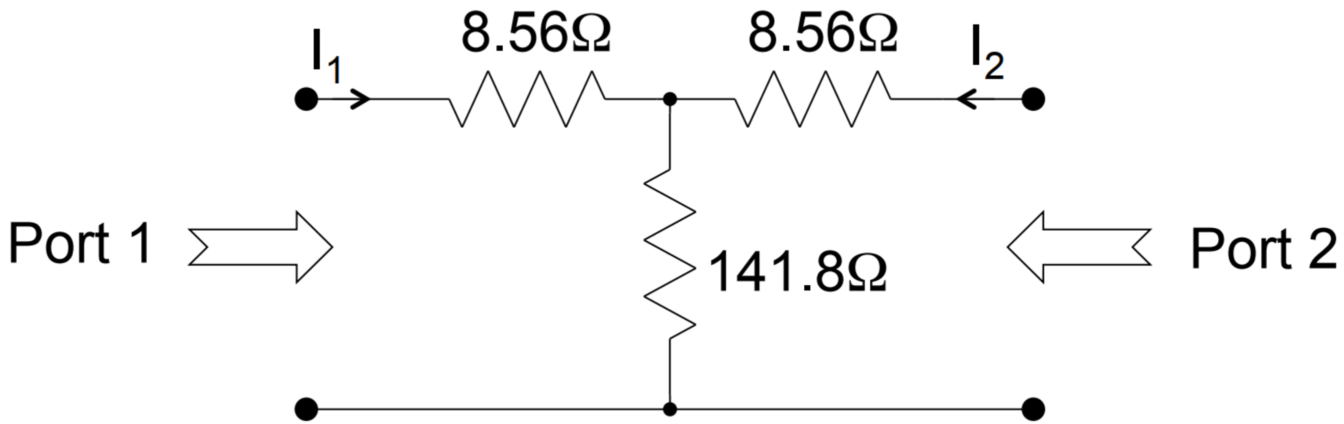
$$Z_{21} = Z_{12}$$

Z_{22} can be found as the input impedance of port 2 when port 1 is open-circuited:

$$Z_{22} = \frac{V_2}{I_2}\bigg|_{I_1=0} = 8.56\Omega + 141.8\Omega = 150.36\Omega$$

Impedance of two-port network

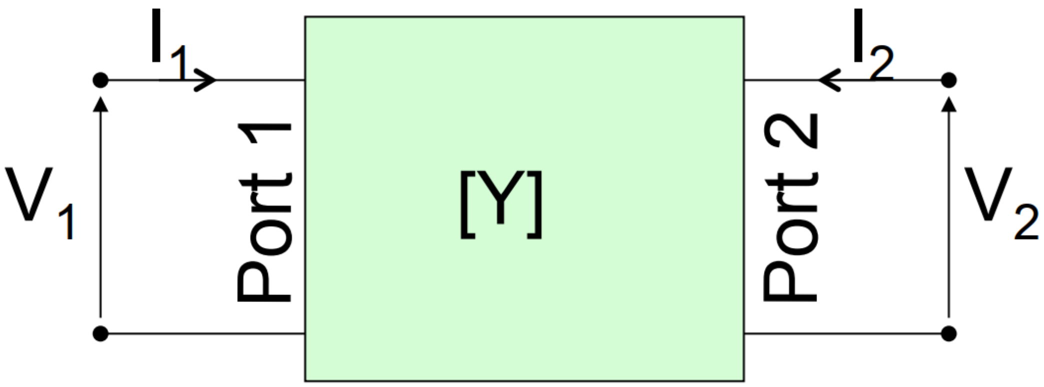
- Answer:



Then the Z-parameters matrix of this two-port network is:

$$\begin{bmatrix} Z_{11} & Z_{21} \\ Z_{12} & Z_{22} \end{bmatrix} = \begin{bmatrix} 150.36 & 141.8 \\ 141.8 & 150.36 \end{bmatrix}$$

Admittance of two-port network

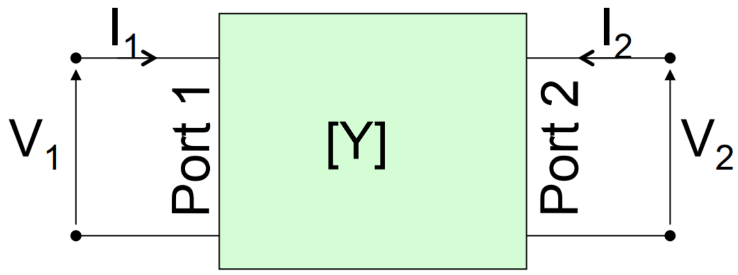


Two port network

$$I_1 = Y_{11}V_1 + Y_{12}V_2$$
$$I_2 = Y_{21}V_1 + Y_{22}V_2$$

- The reference direction of voltage and current of the impedance matric is shown in the figure.
- Voltages are set at the ports.
- The currents at the ports are measured.
- The solution of the matrix yields the Y-parameters.

Admittance of two-port network



Two port network

$$I_1 = Y_{11}V_1 + Y_{12}V_2$$
$$I_2 = Y_{21}V_1 + Y_{22}V_2$$

$Y_{11} = \left.\dfrac{I_1}{V_1}\right|_{V_2=0}$ **Input admittance** at port 1, when port 2 is short-circuited

$Y_{12} = \left.\dfrac{I_1}{V_2}\right|_{V_1=0}$ **Transfer admittance** from port 1 to port 2, when port 1 is short-circuited

$Y_{21} = \left.\dfrac{I_2}{V_1}\right|_{V_2=0}$ **Transfer admittance** from port 2 to port 1, when port 2 is short-circuited

$Y_{22} = \left.\dfrac{I_2}{V_2}\right|_{V_1=0}$ **Input admittance** at port 2, when port 1 is short-circuited

[Z] and [Y] measurement challenges

- Placing o/c and s/c conditions at the ports is problematic.
  - Lead inductance and capacitance.
  - Tuning stubs could compensate but have to be tuned for each measurement frequency.
  - Stubs can make measurements invalid by causing an oscillation condition when device under test is connected.
  - **Process is tedious**. Disconnecting-reconnecting.
- Standing waves.
  - Terminal voltages and current vary in magnitude along the length of the line.

Impedance and Admittance Matrices

- Equivalent voltages and currents, and the related impedance admittance matrices, become somewhat of an abstraction when dealing with high-frequency networks.
- A representation more in accord with direct measurements, and with the ideas of incident, reflected, and transmitted waves, is given by the scattering matrix.

The Scattering Matrix

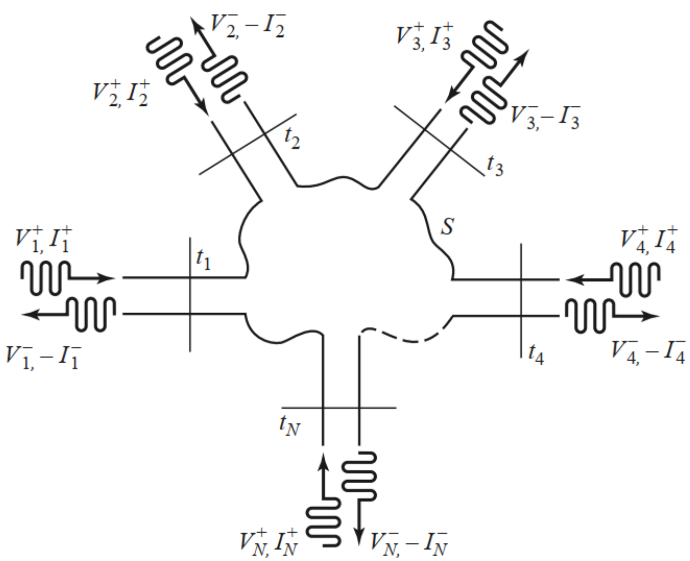
The Scattering Matrix

- Like the impedance and admittance matrix for an *N*-port network, the scattering matrix also provides a complete description of the network as seen at its *N* ports.
- Unlike the impedance and admittance matrices which relate the total voltages and currents at the ports, the scattering matrix relates the voltage waves incident on the ports to those reflected from the ports.
- The network analysis techniques and vector network analyzer can be applied when using scattering matrix.

Advantages of using scattering parameters

- Trouble-free measurement principle at high frequencies.
- Measurement based on voltage travelling waves.
- Matched condition required at the ports.
- No o/c or s/c conditions which may cause devices to oscillate or be damaged.
- Scattering parameters can be cascaded to predict system performance.
- $[\mathbf{S}]$ parameters can be converted to $[\mathbf{Z}]$, $[\mathbf{Y}]$ parameters if desired.
- Measurement similar to gain, loss, reflection coefficient…

The Scattering Matrix



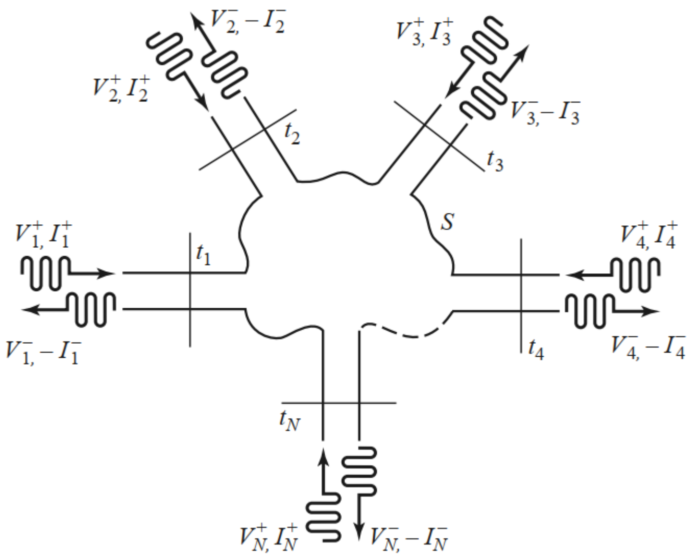
- Consider the N-port microwave network, as shown in the figure left. Where V_n^+ is the amplitude of the voltage wave incident on port n and V_n^- is the amplitude of the voltage wave reflected from port n.
- The scattering matrix, or $[\mathbf{S}]$ matrix, is defined in relation to these incident and reflected voltage waves as:

$$\begin{bmatrix} V_1^- \\ V_2^- \\ \vdots \\ V_N^- \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1N} \\ S_{21} & & & \vdots \\ \vdots & & & \vdots \\ S_{N1} & \cdots & \cdots & S_{NN} \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \\ \vdots \\ V_N^+ \end{bmatrix}$$

- Or in matrix form as

$$\left[\mathbf{V}^-\right] = [\mathbf{S}]\left[\mathbf{V}^+\right]$$

The Scattering Matrix



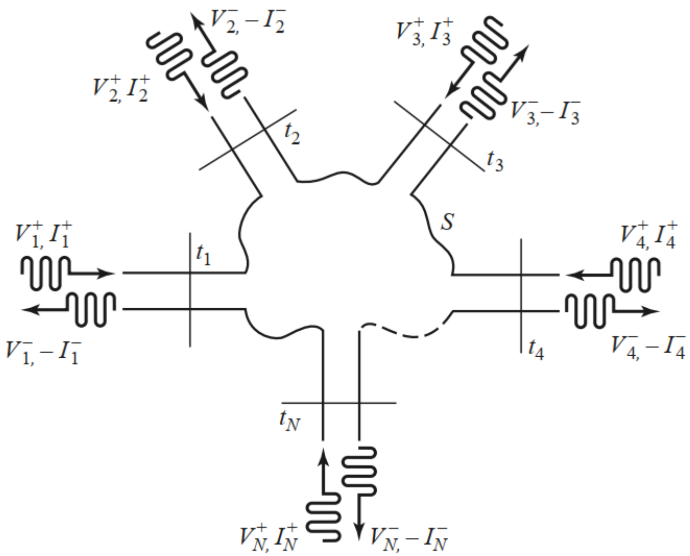
- A specific element of the scattering matrix can be determined as

$$S_{ij} = \left.\frac{V_i^-}{V_j^+}\right|_{V_k^+=0,(k\neq j)}$$

- Which indicates that S_{ij} is found by driving port j with an incident wave of voltage V_j^+ and measuring the reflected wave amplitude V_i^- coming out of port i. The incident wave on all ports expect the jth port are set to zero, which means that all ports should be terminated in matched loads to avoid reflections.

The Scattering Matrix

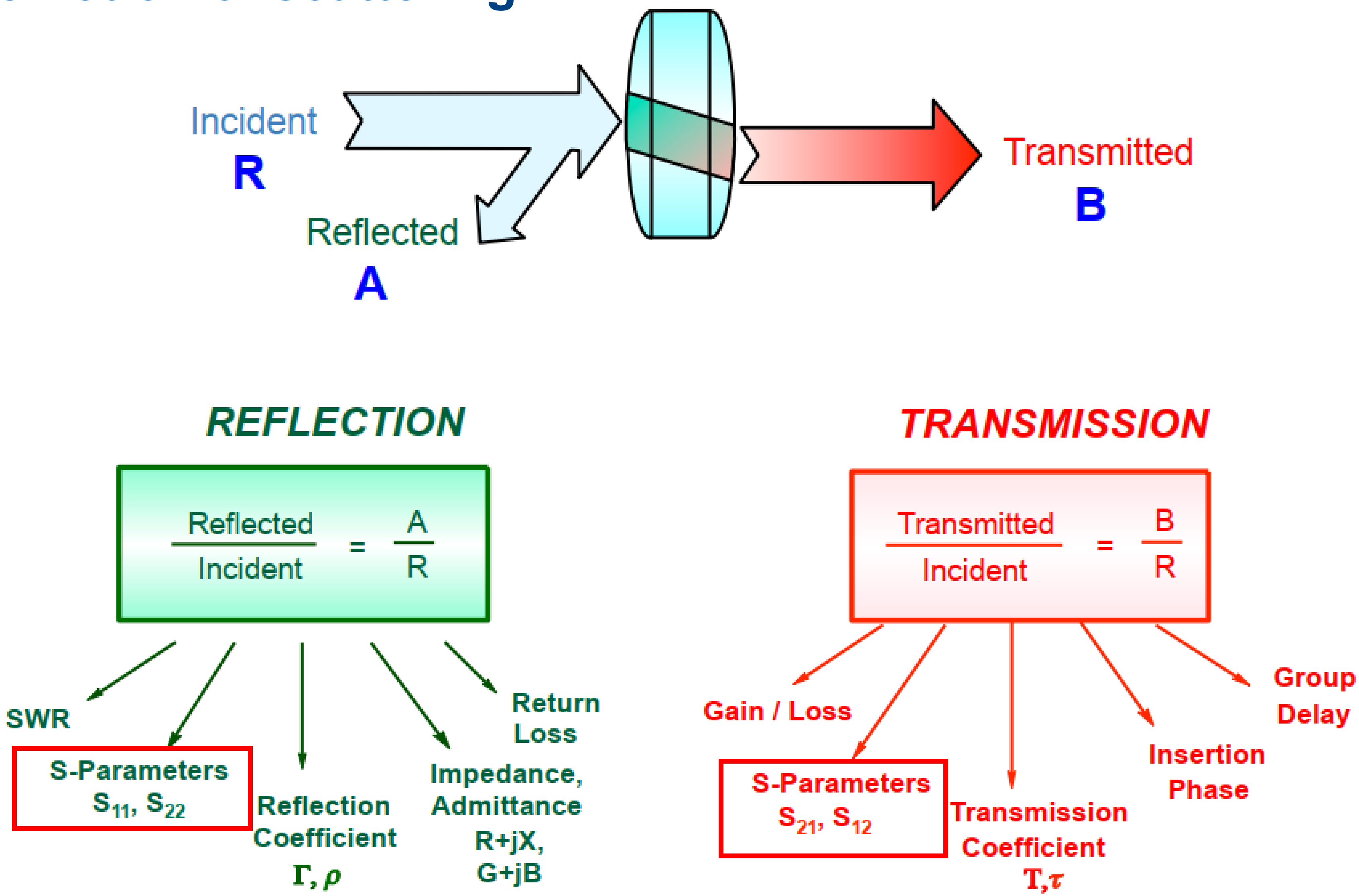
$$S_{ii} = \left. \frac{V_i^-}{V_i^+} \right|_{V_k^+=0,(k\neq i)}$$



- Thus, S_{ii} is the reflection coefficient seen looking into port i when all other ports are terminated in matched loads, and S_{ij} is the transmission coefficient from port j to port i when all other ports are terminated in matched loads.

The Scattering Matrix

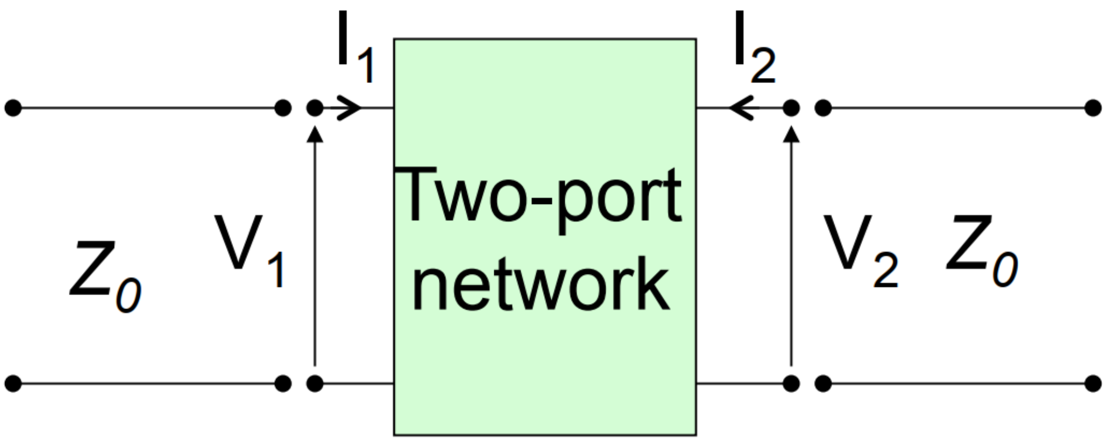
The notion of scattering



The Scattering Matrix

- Here, we also take a two-port network as an example to explain in detail.

The Scattering Matrix



- V_1=voltage at port 1;
- I_1=current at port 1;
- V_2=voltage at port 2;
- I_2=current at port 2;

$$\tilde{V}_i(z) = V_{0i}^{+} e^{-j\beta z} + V_{0i}^{-} e^{j\beta z} = V_i^{+} + V_i^{-}$$

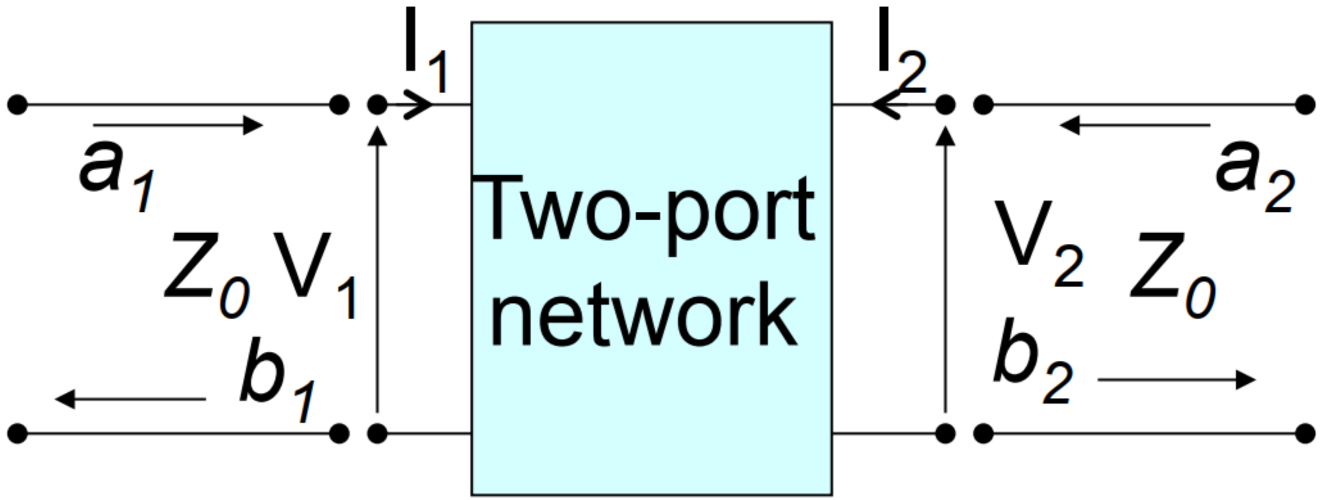
$$\tilde{I}_i(z) = I_{0i}^{+} e^{-j\beta z} + I_{0i}^{-} e^{j\beta z} = \frac{1}{Z_0}\left(V_{0i}^{+} e^{-j\beta z} - V_{0i}^{-} e^{j\beta z}\right) = I_i^{+} + I_i^{-}$$

➢ Therefore, the incident and reflected voltage can be expressed by $\tilde{V}_i$ and $\tilde{I}_i$ as:

$$V_i^{+} = \frac{1}{2}\left(\tilde{V}_i(z) + Z_0 \tilde{I}_i(z)\right) = V_{0i}^{+} e^{-j\beta z}$$

$$V_i^{-} = \frac{1}{2}\left(\tilde{V}_i(z) - Z_0 \tilde{I}_i(z)\right) = V_{0i}^{-} e^{j\beta z}$$

The Scattering Matrix



$$V_i^+ = \frac{1}{2}\left(\tilde{V}_i(z) + Z_0\tilde{I}_i(z)\right) = V_{0i}^+ e^{-j\beta z}$$

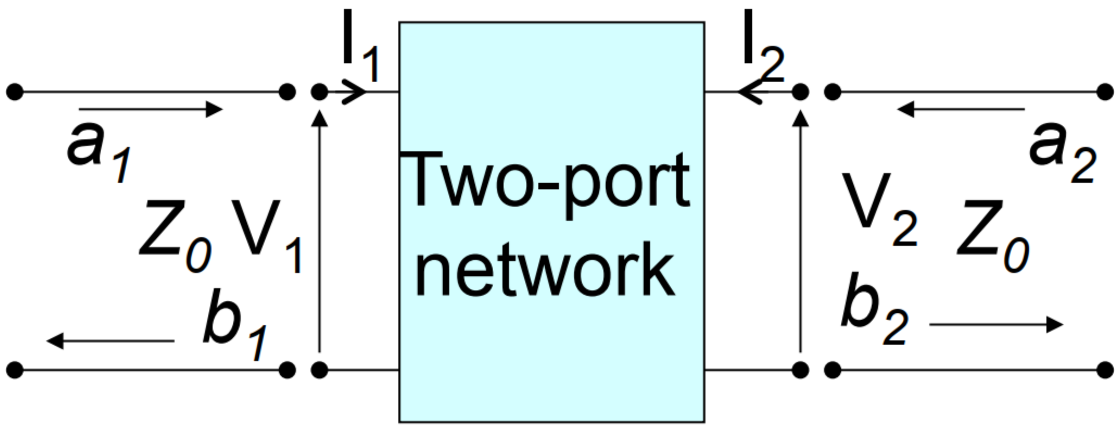
$$V_i^- = \frac{1}{2}\left(\tilde{V}_i(z) - Z_0\tilde{I}_i(z)\right) = V_{0i}^- e^{j\beta z}$$

- Define:

$$a_i = \frac{V_i^+}{\sqrt{Z_0}} = \frac{V_{0i}^+ e^{-j\beta z}}{\sqrt{Z_0}} = \frac{1}{2}\left[\frac{\tilde{V}_i(z)}{\sqrt{Z_0}} + \sqrt{Z_0}\tilde{I}_i(z)\right]$$

$$b_i = \frac{V_i^-}{\sqrt{Z_0}} = \frac{V_{0i}^- e^{j\beta z}}{\sqrt{Z_0}} = \frac{1}{2}\left[\frac{\tilde{V}_i(z)}{\sqrt{Z_0}} - \sqrt{Z_0}\tilde{I}_i(z)\right]$$

The Scattering Matrix



$$a_i = \frac{V_i^+}{\sqrt{Z_0}} = \frac{V_{0i}^+ e^{-j\beta z}}{\sqrt{Z_0}}$$

$$b_i = \frac{V_i^-}{\sqrt{Z_0}} = \frac{V_{0i}^- e^{j\beta z}}{\sqrt{Z_0}}$$

- Average power associated with incident wave at the i-th port:

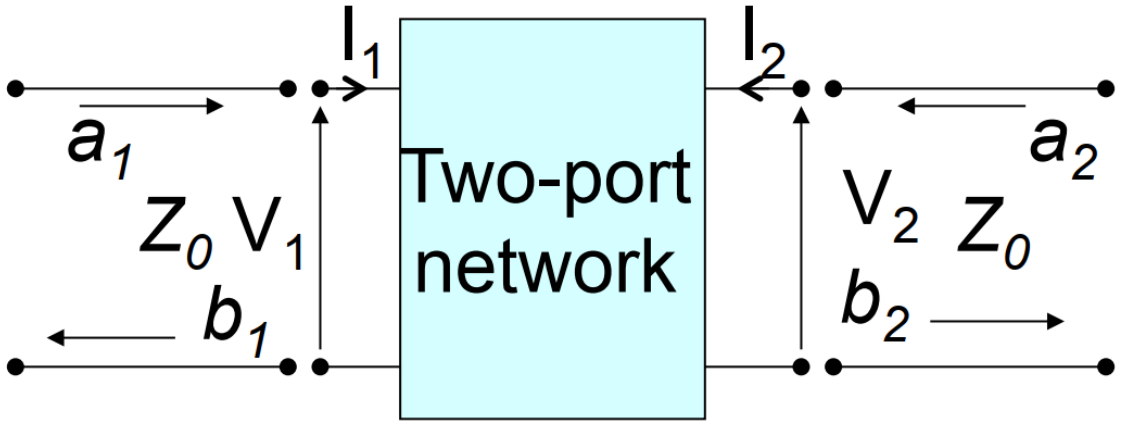
$$P_i^+ = \frac{1}{2}\mathrm{Re}\left(V_i^+\left(I_i^+\right)^*\right) = \frac{1}{2}\mathrm{Re}\left(a_i\sqrt{Z_0}\left(\frac{1}{Z_0}a_i\sqrt{Z_0}\right)^*\right) = \frac{\left|a_i\right|^2}{2}$$

- Average power associated with reflected wave at the i-th port:

$$P_i^- = \frac{1}{2}\mathrm{Re}\left(V_i^-\left(I_i^-\right)^*\right) = \frac{1}{2}\mathrm{Re}\left(b_i\sqrt{Z_0}\left(\frac{1}{Z_0}b_i\sqrt{Z_0}\right)^*\right) = \frac{\left|b_i\right|^2}{2}$$

- a_i and b_i have unit of $\mathrm{Watt}^{1/2}$

The Scattering Matrix



- Four unknowns, V_1, V_2, I_1 and I_2 can be replaced by a_1, a_2, b_1 and b_2:

$$b_1 = S_{11}a_1 + S_{12}a_2$$

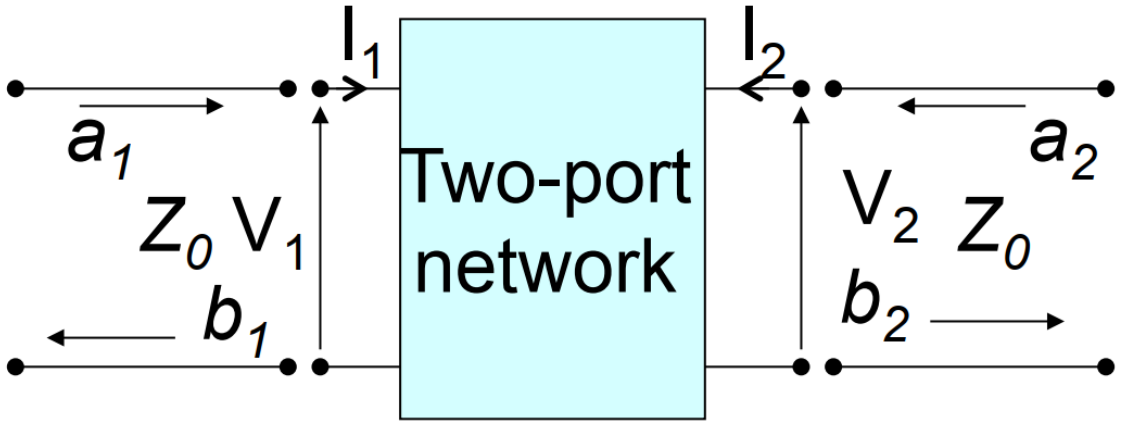
$$b_2 = S_{21}a_1 + S_{22}a_2$$

- S_{ij} are called the scattering parameters or S-parameters

$$a_i = \frac{V_i^+}{\sqrt{Z_0}} = \frac{V_{0i}^+ e^{-j\beta z}}{\sqrt{Z_0}} = \frac{1}{2}\left[\frac{\tilde{V}_i(z)}{\sqrt{Z_0}} + \sqrt{Z_0}\tilde{I}_i(z)\right]$$

$$b_i = \frac{V_i^-}{\sqrt{Z_0}} = \frac{V_{0i}^- e^{j\beta z}}{\sqrt{Z_0}} = \frac{1}{2}\left[\frac{\tilde{V}_i(z)}{\sqrt{Z_0}} - \sqrt{Z_0}\tilde{I}_i(z)\right]$$

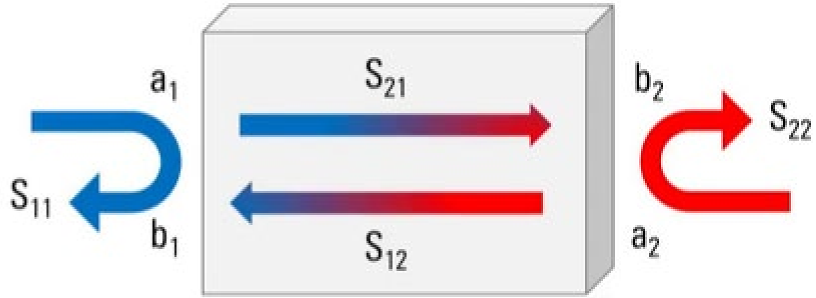
The Scattering Matrix



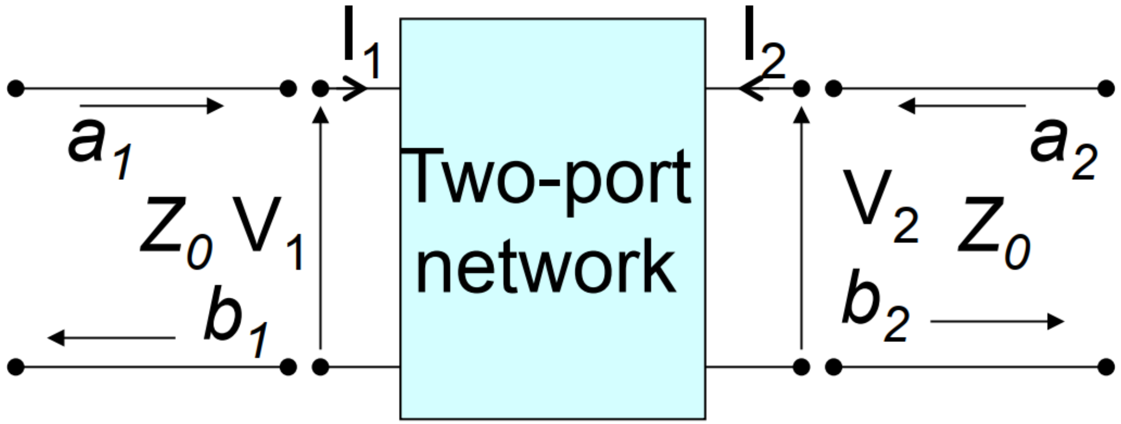
- Assumed the load, Z_L, at port 2 is matched to Z_0. The power delivered to the load is fully absorbed by the load, therefor a_2 is zero:

$$S_{11} = \frac{b_1}{a_1}\bigg|_{a_2=0} = \frac{V_1^-}{V_1^+} = \Gamma_1$$

- S_{11} is the *input reflection coefficient* when port 2 is matched.



The Scattering Matrix



- According to the definition of S_{11} , we can define the *input impedance* at port 1, which is the ratio of the voltage and current at port 1 and also related to the reflection coefficient (S_{11}) at port 1.

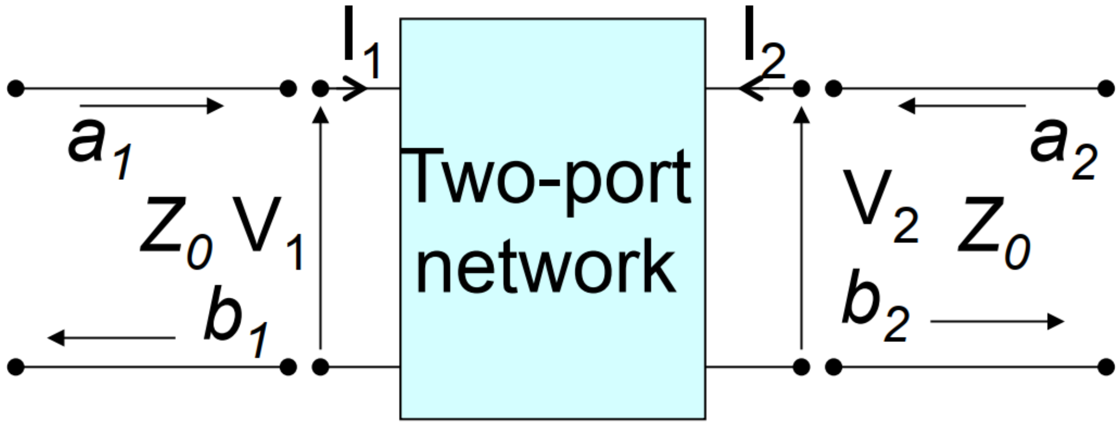
$$Z_1^{in} = \frac{V_1}{I_1}$$

$$S_{11} = \frac{Z_1^{in} - Z_0}{Z_1^{in} + Z_0}$$

- Then the input impedance Z_1^{in} expressed by S_{11} is:

$$Z_1^{in} = Z_0 \frac{1 + S_{11}}{1 - S_{11}} (\Omega)$$

The Scattering Matrix



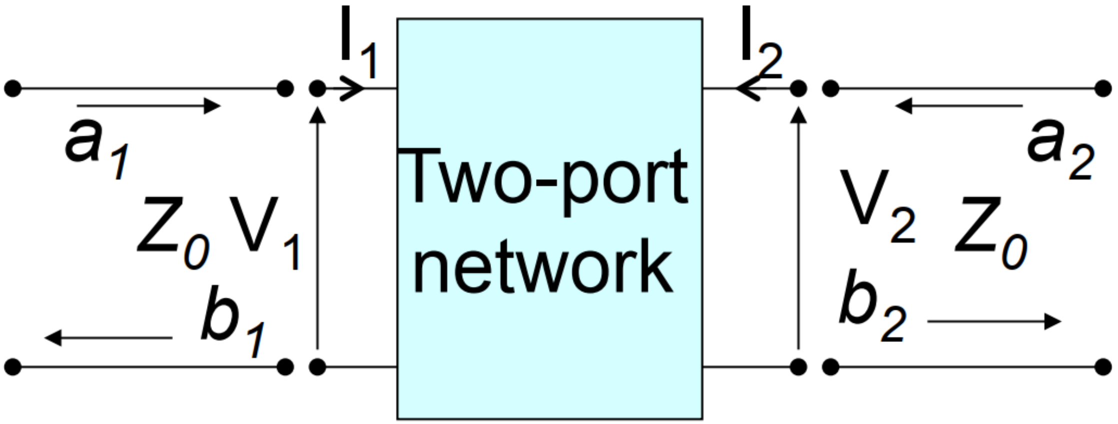
- S_{11} is the ratio of amplitude of reflected voltage to that of incident voltage, according to the relationship between voltage and power, we can define the *power ratio*, which is the ratio of the power reflected at the port 1 load and that incident into port 1 of the network:

$$|S_{11}|^2 = \frac{\text{power reflected at the port 1}}{\text{power incident into port 1 of the network}}$$

- *Return loss* is defined as:

$$R.L. = -10\log_{10}|S_{11}|^2\,(dB)$$

The Scattering Matrix



- S_{21} indicate the *transmission* effect from port 1 to port 2， defined as the ratio of the amplitude of transmitted voltage to the port 2 to that of incident voltage into port 1, then we can define the *power ratio*:

$$\left|S_{21}\right|^2 = \frac{\text{power transmitted to port 2}}{\text{power incident into port 1 of the network}}$$

- *Insertion Loss* is defined as:

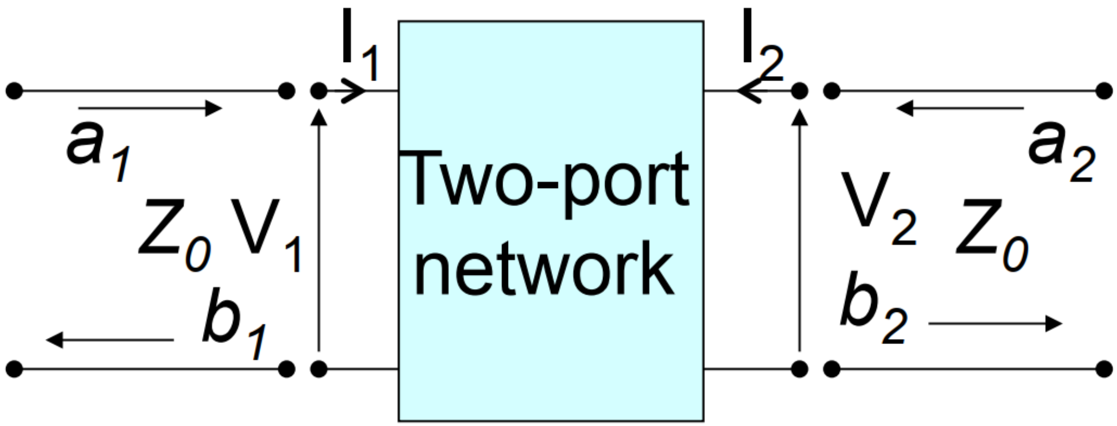
$$I.L. = -10\log_{10}\left|S_{21}\right|^2 (dB)$$

- *Forward Power Gain* is defined as:

$$Gain = 10\log_{10}\left|S_{21}\right|^2 (dB)$$

$$S_{21} = \left.\frac{b_2}{a_1}\right|_{a_2=0}$$

The Scattering Matrix



$$b_1 = S_{11}a_1 + S_{12}a_2$$
$$b_2 = S_{21}a_1 + S_{22}a_2$$

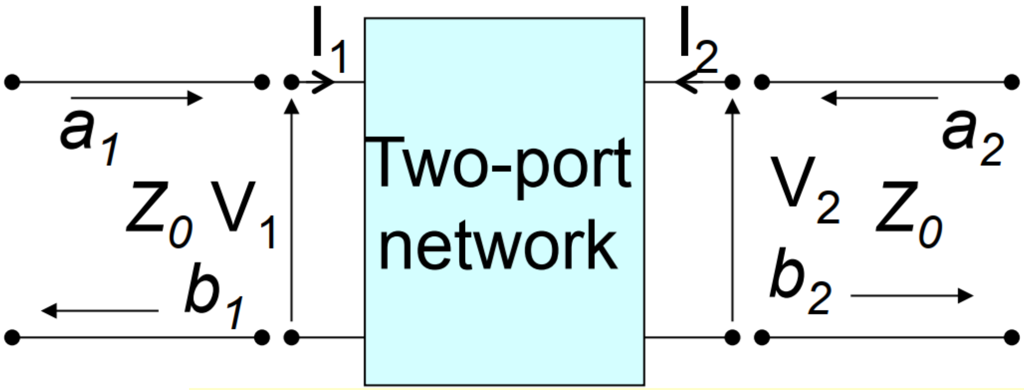
$S_{11} = \left.\dfrac{b_1}{a_1}\right|_{a_2=0}$ **Input reflection coefficient,** when port 2 is matched

$S_{12} = \left.\dfrac{b_1}{a_2}\right|_{a_1=0}$ **Reverse transmission gain** when port 1 is matched

$S_{21} = \left.\dfrac{b_2}{a_1}\right|_{a_2=0}$ **Forward transmission gain** when port 2 is matched

$S_{22} = \left.\dfrac{b_2}{a_2}\right|_{a_1=0}$ **Output reflection coefficient** when port 1 is matched

The Scattering Matrix



$$b_1 = S_{11}a_1 + S_{12}a_2$$
$$b_2 = S_{21}a_1 + S_{22}a_2$$

$$|S_{11}|^2 = \frac{\text{power reflected at port 1}}{\text{power incident into port 1 of the network}}$$

$$|S_{12}|^2 = \frac{\text{power transmitted to the port 1}}{\text{power incident into port 2 of the network}}$$

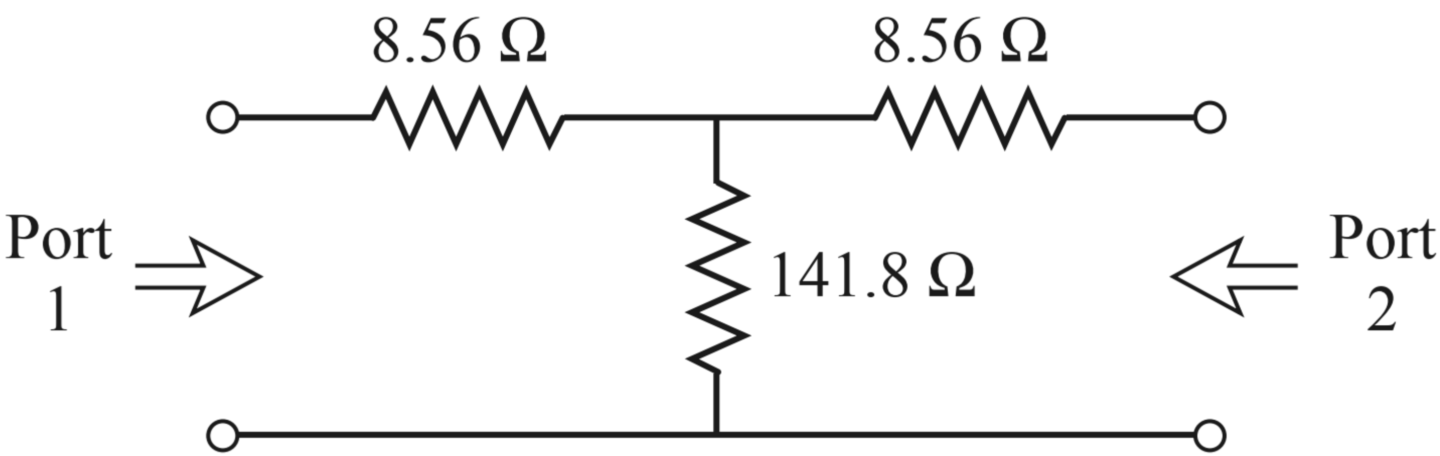
$$|S_{21}|^2 = \frac{\text{power transmitted to the port 2}}{\text{power incident into port 1 of the network}}$$

$$|S_{22}|^2 = \frac{\text{power reflected at port 2}}{\text{power incident into port 2 of the network}}$$

- these are true only for the case of equal impedance for input and output port

Evaluation of scattering parameters

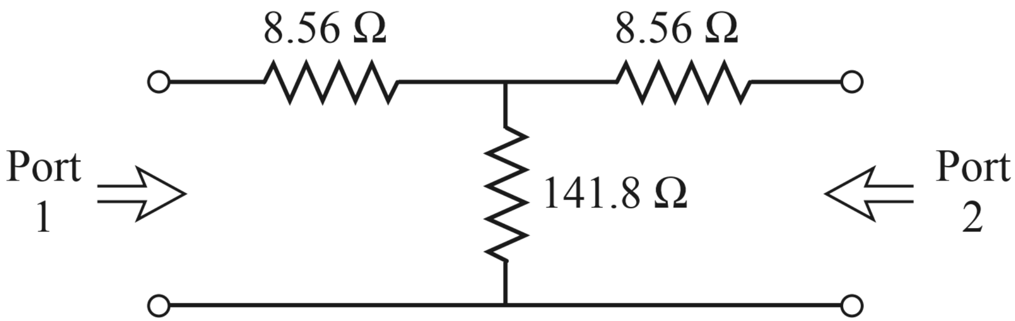
Question: Find the scattering parameters of the 3 dB attenuator circuit shown in below.



The Scattering Matrix

Evaluation of scattering parameters

Solution:



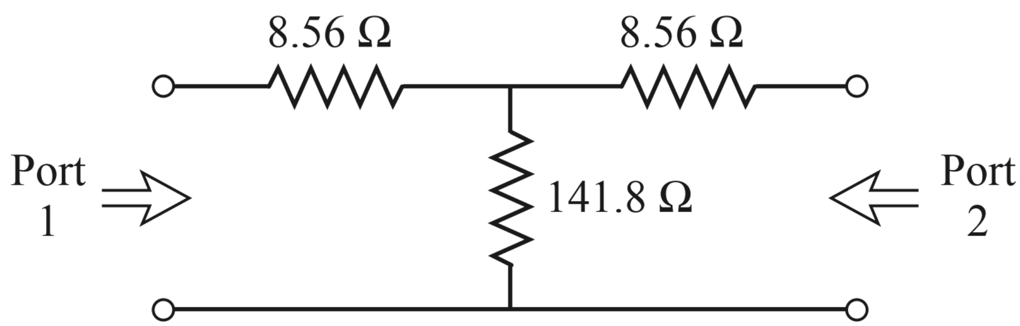
$$S_{11} = \frac{b_1}{a_1}\Big|_{a_2=0} = \frac{V_1^-}{V_1^+} = \Gamma_1$$

As we already know, S_{11} can be determined as the reflection coefficient seen at port 1 when port 2 is terminated in a matched load ($Z_0 = 50\Omega$):

$$S_{11} = \frac{V_1^-}{V_1^+}\Big|_{V_2^+=0} = \Gamma^{(1)}\Big|_{V_2^+=0} = \frac{Z_{in}^{(1)} - Z_0}{Z_{in}^{(1)} + Z_0}\Big|_{Z_0 \text{ on port } 2}$$

Evaluation of scattering parameters

Solution:



$$Z_{in}^{(1)} = 8.56 + \frac{141.8(8.56+50)}{141.8+(8.56+50)} = 50\Omega$$

Hence S_{11} can be calculated:

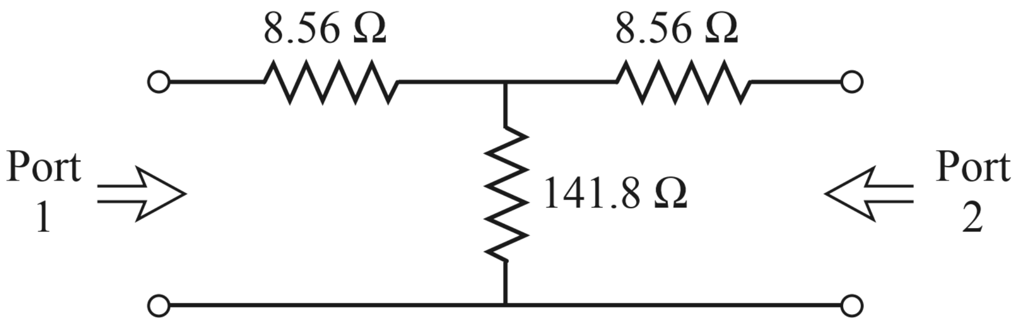
$$S_{11} = \frac{Z_{in}^{(1)} - Z_0}{Z_{in}^{(1)} + Z_0} = 0$$

According to the symmetry of the circuit, it could be obtained directly that :

$$S_{22} = 0$$

Evaluation of scattering parameters

Solution:

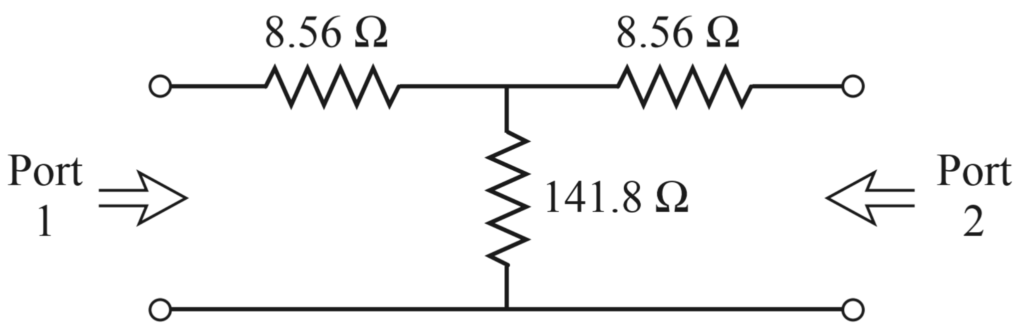


Then the transmission parameter S_{21} can be found by applying an incident wave at port 1, V_1^+, and measuring the outcoming wave at port 2, V_2^-. This is equivalent to the transmission coefficient from port 1 to port 2:

$$S_{21} = \frac{V_2^-}{V_1^+}\bigg|_{V_2^+=0}$$

Evaluation of scattering parameters

Solution:



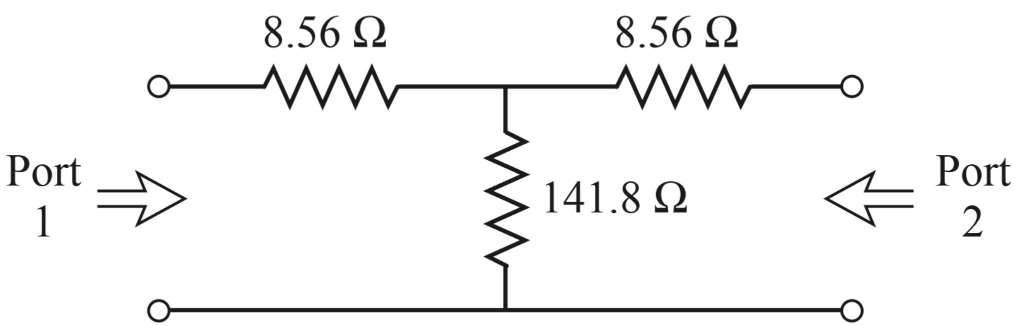
From the fact that $S_{11} = S_{22} = 0$, we know that $V_1^- = 0$ when port 2 is terminated in $Z_0 = 50\Omega$, and that $V_2^+ = 0$. In this case we have that $V_1^+ = V_1$ and $V_2^- = V_2$. By applying a voltage V_1 at port 1 and using voltage division twice we find $V_2^- = V_2$ as the voltage across the 50Ω load resistor at port 2:

$$V_2^- = V_2 = V_1\left(\frac{41.44}{41.44+8.56}\right)\left(\frac{50}{50+8.56}\right) = 0.707V_1$$

The Scattering Matrix

Evaluation of scattering parameters

Solution:



$$V_2^- = V_2 = V_1\left(\frac{41.44}{41.44+8.56}\right)\left(\frac{50}{50+8.56}\right) = 0.707V_1$$

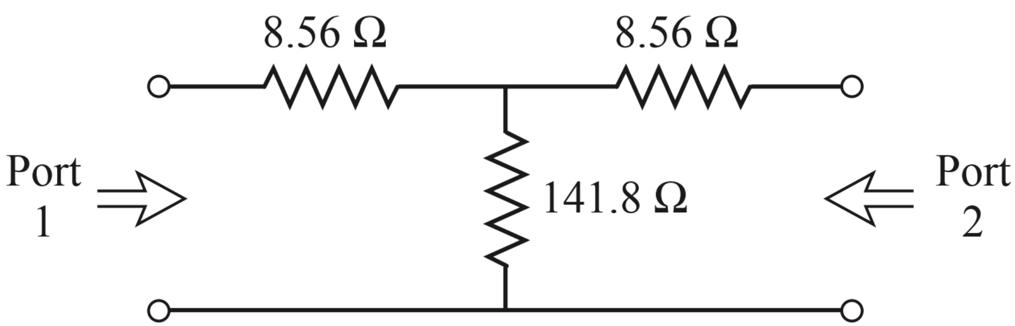
Where the resistance value 41.44Ω is from the 141.8Ω resistor in parallel with the sum of the 50Ω load and the 8.56Ω resistor

$$141.8\Omega // (8.56\Omega + 50\Omega) = \frac{141.8\Omega(8.56\Omega+50\Omega)}{141.8\Omega+(8.56\Omega+50\Omega)} = 41.44\Omega$$

The Scattering Matrix

Evaluation of scattering parameters

Solution:



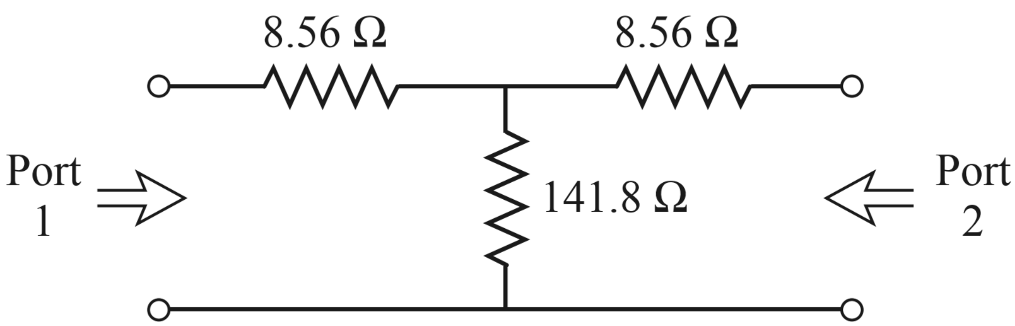
Thus, S_{21} can be obtained, and according to the symmetry of the circuit, S_{12} could also be obtained.

$$S_{21} = S_{12} = 0.707$$

$$[\mathbf{S}] = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0.707 \\ 0.707 & 0 \end{bmatrix}$$

Evaluation of scattering parameters

Solution:

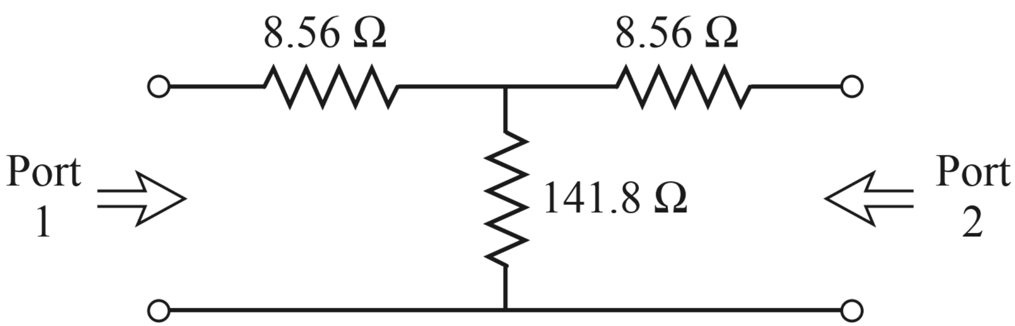


Let us briefly analyze the application of this circuit. Analyze the relationship between the output and input of this circuit, what can we find?

- Tips: The ratio of the voltage at the output port to the voltage at the input port, the ratio of the power at the output port to that at the input port.

Evaluation of scattering parameters

Solution:



If the input power is $|V_1^+|^2/2Z_0$, then according to the S-parameter S_{21}, it could be obtained that the output voltage is $V_2^- = S_{21}V_1^+$, and the output power is $|V_2^-|^2/2Z_0 = |S_{21}V_1^+|^2/2Z_0 = |V_1^+|^2/4Z_0$.

Notice that the output power is one-half of the input power, which is $\log_{10}\frac{1}{2} = -3dB$, hence this circuit is called 3 dB attenuator circuit.

The Scattering Matrix

- So far, we have discussed three types of matrix charactering the microwave circuits, which sets up the relationship matrix between reference quantities from different perspectives.
- So what kind of transformation relationship does these three network matrices have?

The relationship between the [S] and [Z] matrices

► The scattering matrix [**S**] can be determined from the [**Z**] (or [**Y**]) matrix and vice versa.

First, wassume that the characteristic impedance, Z_{0n}, of all the ports are identical. (This restriction will be removed when we discuss generalized scattering parameters.) Then, for convenience, we can set $Z_{0n}=1$. Then the total voltage and current at the nth port can be written as

$$V_n = V_n^+ + V_n^-$$

$$I_n = I_n^+ - I_n^- = V_n^+ - V_n^-$$

Using the definition of [**Z**], ***i.e.***, [**V**] = [**Z**][**I**], the equation above can be rewritten as:

$$[\mathbf{Z}][\mathbf{I}] = [\mathbf{Z}]\left[\mathbf{V}^+\right] - [\mathbf{Z}]\left[\mathbf{V}^-\right] = [\mathbf{V}] = \left[\mathbf{V}^+\right] + \left[\mathbf{V}^-\right]$$

The relationship between the [S] and [Z] matrices

► The scattering matrix [**S**] can be determined from the [**Z**] (or [**Y**]) matrix and vice versa.

$$[\mathbf{Z}][\mathbf{I}]=[\mathbf{Z}]\left[\mathbf{V}^{+}\right]-[\mathbf{Z}]\left[\mathbf{V}^{-}\right]=[\mathbf{V}]=\left[\mathbf{V}^{+}\right]+\left[\mathbf{V}^{-}\right]$$

which can be rewritten as:

$$([\mathbf{Z}]+[\mathbf{U}])\left[\mathbf{V}^{-}\right]=([\mathbf{Z}]-[\mathbf{U}])\left[\mathbf{V}^{+}\right]$$

where [**U**] is an identity matrix defined as:

$$[\mathbf{U}]=\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \\ 0 & & \cdots & 1 \end{bmatrix}$$

The Scattering Matrix

The relationship between the [S] and [Z] matrices

► The scattering matrix [**S**] can be determined from the [**Z**] (or [**Y**]) matrix and vice versa.

$$([\mathbf{Z}]+[\mathbf{U}])\left[\mathbf{V}^-\right]=([\mathbf{Z}]-[\mathbf{U}])\left[\mathbf{V}^+\right]$$

By applying transformation, the scattering matrix can be derived as,

$$\left[\mathbf{V}^-\right]=([\mathbf{Z}]+[\mathbf{U}])^{-1}([\mathbf{Z}]-[\mathbf{U}])\left[\mathbf{V}^+\right]$$

$$\left[\mathbf{V}^-\right]=[\mathbf{S}]\left[\mathbf{V}^+\right]$$

Hence it could be obtained the relationship between [**S**] and [**Z**] :

$$[\mathbf{S}]=([\mathbf{Z}]+[\mathbf{U}])^{-1}([\mathbf{Z}]-[\mathbf{U}])$$

The relationship between the $[S]$ and $[Z]$ matrices

► The scattering matrix $[\mathbf{S}]$ can be determined from the $[\mathbf{Z}]$ (or $[\mathbf{Y}]$) matrix and vice versa.

Given scattering matrix in terms of impedance matrix, for one-port network , the relationship between $[\mathbf{S}]$ and $[\mathbf{Z}]$ is simplified as :

$$S_{11} = \frac{z_{11} - 1}{z_{11} + 1}$$

in agreement with the result for the reflection coefficient seen looking into a load with a normalized input impedance of z_{11}.

The Scattering Matrix

The relationship between the [S] and [Z] matrices

► The scattering matrix [**S**] can be determined from the [**Z**] (or [**Y**]) matrix and vice versa.

To find [**Z**] in terms of [**S**], rewrite the expression:

$$[\mathbf{S}] = ([\mathbf{Z}] + [\mathbf{U}])^{-1}([\mathbf{Z}] - [\mathbf{U}])$$

as:

$$[\mathbf{Z}][\mathbf{S}] + [\mathbf{U}][\mathbf{S}] = [\mathbf{Z}] - [\mathbf{U}]$$

solve [**Z**] :

$$[\mathbf{Z}] = ([\mathbf{U}] + [\mathbf{S}])([\mathbf{U}] - [\mathbf{S}])^{-1}$$

The Scattering Matrix

- As we have discussed in previous section, we will discuss the characteristics of the scattering matrix of reciprocal network and lossless network.

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Reciprocal Networks and Lossless Networks

Reciprocal Network and Lossless Network

- As we discussed in the section about the impedance and admittance matrices, we already know that the impedance and admittance matrices are symmetric for reciprocal networks, and are purely imaginary for lossless network.
- Of course the scattering matrices for these particular types of networks also have special properties.
- The conclusion is that (1) the scattering matrix for a reciprocal network is symmetric, and that (2) the scattering matrix for a lossless network is unitary.
- Next, we will conduct a detailed derivation and discussion.

Reciprocal Network

$$V_n = V_n^+ + V_n^-$$

$$I_n = I_n^+ - I_n^- = V_n^+ - V_n^-$$

- By adding two equations above, we can obtain

$$V_n^+ = \frac{1}{2}\left(V_n + I_n\right)$$

- or

$$\left[\mathbf{V}^+\right] = \frac{1}{2}\left([\mathbf{Z}] + [\mathbf{U}]\right)[\mathbf{I}]$$

- By subtracting two equations above, we can obtain

$$V_n^- = \frac{1}{2}\left(V_n - I_n\right)$$

- or

$$\left[\mathbf{V}^-\right] = \frac{1}{2}\left([\mathbf{Z}] - [\mathbf{U}]\right)[\mathbf{I}]$$

Reciprocal Network

$$\left[\mathbf{V}^-\right]=\frac{1}{2}([\mathbf{Z}]-[\mathbf{U}])[\mathbf{I}] \qquad \left[\mathbf{V}^+\right]=\frac{1}{2}([\mathbf{Z}]+[\mathbf{U}])[\mathbf{I}]$$

- Eliminating $[\mathbf{I}]$ from the expression of $[\mathbf{V}^+]$ and that of $[\mathbf{V}^-]$ gives

$$\left[\mathbf{V}^-\right]=([\mathbf{Z}]-[\mathbf{U}])([\mathbf{Z}]+[\mathbf{U}])^{-1}\left[\mathbf{V}^+\right]$$

- so that

$$[\mathbf{S}]=([\mathbf{Z}]-[\mathbf{U}])([\mathbf{Z}]+[\mathbf{U}])^{-1}$$

- Taking the transpose of the equation gives

$$[\mathbf{S}]^T=\left\{([\mathbf{Z}]+[\mathbf{U}])^{-1}\right\}^T([\mathbf{Z}]-[\mathbf{U}])^T$$

- Now $[\mathbf{U}]$ is diagonal, so that $[\mathbf{U}]^T=[\mathbf{U}]$, and if the network is reciprocal, $[\mathbf{Z}]$ is symmetric. hence $[\mathbf{Z}]^T=[\mathbf{Z}]$. The above equation then reduces to

$$[\mathbf{S}]^T=([\mathbf{Z}]+[\mathbf{U}])^{-1}([\mathbf{Z}]-[\mathbf{U}])$$

Reciprocal Network

$$[\mathbf{S}]^T = ([\mathbf{Z}]+[\mathbf{U}])^{-1}([\mathbf{Z}]-[\mathbf{U}])$$

- Comparing the expression to the $[\mathbf{S}]$ matrix in terms of $[\mathbf{Z}]$ that is:

$$[\mathbf{S}] = ([\mathbf{Z}]+[\mathbf{U}])^{-1}([\mathbf{Z}]-[\mathbf{U}])$$

- We have thus shown that

$$[\mathbf{S}]^T = [\mathbf{S}]$$

- Hence the scattering matrix is symmetric for reciprocal networks.

The Scattering Matrix

Reciprocal Network

- Then, let us give a brief discussion of the characteristic of the element of the scattering matrix for the reciprocal network.

$$[\mathbf{S}]^T = [\mathbf{S}]$$

- The network has the same transmission characteristic between two ports irrespective of the direction taken;
- For two-port network, when it is reciprocal, then we have $S_{21} = S_{12}$;
- For *n*-port network, when it is reciprocal, then we have $S_{nm} = S_{mn}$ when $n \neq m$;
- Many passive structures are reciprocal;
- Examples of non-reciprocal components are ferrite devices and active devices such as microwave transistors.

Lossless Network

- If the network is lossless, no real power can be delivered to the network. Thus, the relationship of the power is:

Incident Power = Reflected Power + Transmitted Power

- For a *n*-port lossless network, it can be expressed that:

$$\sum_{i=1}^{n} a_i a_i^* = \sum_{i=1}^{n} b_i b_i^*$$

- Express the above summation expression in matrix form:

$$[\mathbf{a}]^T [\mathbf{a}]^* = [\mathbf{b}]^T [\mathbf{b}]^*$$

Lossless Network

- As we already defined of the scattering matrix, the relationship between $[\mathbf{a}]$ and $[\mathbf{b}]$ also satisfy:

$$[\mathbf{b}] = [\mathbf{S}][\mathbf{a}]$$

- Hence $[\mathbf{b}]^T$ and $[\mathbf{b}]^*$ can be written in $[\mathbf{a}]$-related equation :

$$[\mathbf{b}]^T = [\mathbf{a}]^T [\mathbf{S}]^T$$

$$[\mathbf{b}]^* = [\mathbf{S}]^* [\mathbf{a}]^*$$

- Substituting the above two formulas into the lossless network condition

$$[\mathbf{a}]^T [\mathbf{a}]^* = [\mathbf{b}]^T [\mathbf{b}]^*$$

- Then we have:

$$[\mathbf{a}]^T [\mathbf{a}]^* = [\mathbf{a}]^T [\mathbf{S}]^T [\mathbf{S}]^* [\mathbf{a}]^*$$

The Scattering Matrix

Lossless Network

$$[\mathbf{a}]^T[\mathbf{a}]^* = [\mathbf{a}]^T[\mathbf{S}]^T[\mathbf{S}]^*[\mathbf{a}]^*$$

- To hold the equation, the scattering matrix [**S**] must satisfy :

$$[\mathbf{S}]^T[\mathbf{S}]^* = [\mathbf{I}]$$

- or:

$$[\mathbf{S}]^* = \left\{[\mathbf{S}]^T\right\}^{-1} \quad \text{or} \quad [\mathbf{S}][\mathbf{S}]^{*T} = [\mathbf{I}]$$

- A matrix that satisfies the condition of the above equations is called ***Unitary Matrix***.

$$\sum_{k=1}^{N} S_{ki}S_{ki}^* = \delta_{ij}, \text{for all } i, j$$

if $i = j$, $$\sum_{k=1}^{N} S_{ki}S_{ki}^* = 1$$

if $i \neq j$ $$\sum_{k=1}^{N} S_{ki}S_{kj}^* = 0, \text{for } i \neq j$$

Lossless Network

- As for the two-port network, when the network is lossless, its scattering matrix satisfies:

$$\begin{bmatrix} S_{11} & S_{21} \\ S_{12} & S_{22} \end{bmatrix}\begin{bmatrix} S_{11}^* & S_{12}^* \\ S_{21}^* & S_{22}^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

or

$$S_{11}S_{11}^* + S_{21}S_{21}^* = 1 \quad (1)$$
$$S_{12}S_{12}^* + S_{22}S_{22}^* = 1 \quad (2)$$
$$S_{11}S_{12}^* + S_{21}S_{22}^* = 0 \quad (3)$$
$$S_{12}S_{11}^* + S_{22}S_{21}^* = 0 \quad (4)$$

- Equations (1) and (2) represent of conservation of energy principle at port 1 and port 2;
- The other two equations indicates that simultaneously exciting both ports of matched lossless network can result in no output energy from the network.

The Scattering Matrix

Matched Network

- As we have mentioned in previous section, S_{ii} is the reflection coefficient seen looking into port i when all other ports are terminated in matched loads.
- Parameters S_{11} and S_{22} represent the reflection coefficients at the ports of a 2-port network under matched load conditions. Therefore:
- For a two-port network, when it is matched indicating that:

$$S_{11} = S_{22} = 0$$

- For a N-port network, when it is matched indicating that:

$$S_{nn} = 0; \text{for} \quad n = 1, 2, \ldots, N$$

- Most microwave components are designed to have a small reflection coefficient. That is necessary in a cascaded system to avoid energy being reflected between circuit-blocks.

Summary of S-matrix for two-port network

General

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

Matched, non-reciprocal

$$\begin{bmatrix} 0 & S_{12} \\ S_{21} & 0 \end{bmatrix}$$

Reciprocal

$$\begin{bmatrix} S_{11} & S_{21} \\ S_{21} & S_{22} \end{bmatrix}$$

Lossless (Unitary Condition)

$$\begin{bmatrix} S_{11} & S_{21} \\ S_{12} & S_{22} \end{bmatrix} \begin{bmatrix} S_{11}^* & S_{12}^* \\ S_{21}^* & S_{22}^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

EXAMPLE Application of Scattering Parameters

Question: A certain two-port network is measured and the following scattering matrix is obtained:

$$[S]=\begin{bmatrix} 0.1\angle 0^\circ & 0.8\angle 90^\circ \\ 0.8\angle 90^\circ & 0.2\angle 0^\circ \end{bmatrix}$$

- Determine whether the network is reciprocal or lossless.
- If a short circuit is placed on port 2, what will be the resulting return loss at port 1?

EXAMPLE Application of Scattering Parameters

Solution:

$$[\mathbf{S}]=\begin{bmatrix} 0.1\angle 0^\circ & 0.8\angle 90^\circ \\ 0.8\angle 90^\circ & 0.2\angle 0^\circ \end{bmatrix}$$

Reciprocal

Obviously, the scattering matrix $[S]$ is symmetry, satisfying the characteristic of the scattering matrix of the reciprocal network

$$[\mathbf{S}]^T=\begin{bmatrix} 0.1\angle 0^\circ & 0.8\angle 90^\circ \\ 0.8\angle 90^\circ & 0.2\angle 0^\circ \end{bmatrix}=[\mathbf{S}]$$

Hence the network is reciprocal.

EXAMPLE Application of Scattering Parameters

Solution:

$$[\mathbf{S}]=\begin{bmatrix} 0.1\angle 0^\circ & 0.8\angle 90^\circ \\ 0.8\angle 90^\circ & 0.2\angle 0^\circ \end{bmatrix}$$

Lossless

To be lossless, the conditions that [$\boldsymbol{S}$] matrix should satisfy :

Matrix form condition: $[\mathbf{S}]^T[\mathbf{S}]^* = [\mathbf{I}]$

Scalar form condition:

$$\begin{cases} \sum_{k=1}^{N} S_{ki}S_{ki}^* = 1 \\ \sum_{k=1}^{N} S_{ki}S_{kj}^* = 0, \text{for} \quad i \neq j \end{cases}$$

EXAMPLE Application of Scattering Parameters

Solution:

$$[S]=\begin{bmatrix} 0.1\angle 0^\circ & 0.8\angle 90^\circ \\ 0.8\angle 90^\circ & 0.2\angle 0^\circ \end{bmatrix}$$

Lossless

For $i=j$=1 and $N=2$, we have

$$\sum_{k=1}^{2} S_{k1}S_{k1}^{*}=S_{11}S_{11}^{*}+S_{21}S_{21}^{*}=\left|S_{11}\right|^{2}+\left|S_{21}\right|^{2}=(0.1)^{2}+(0.8)^{2}=0.65\neq 1$$

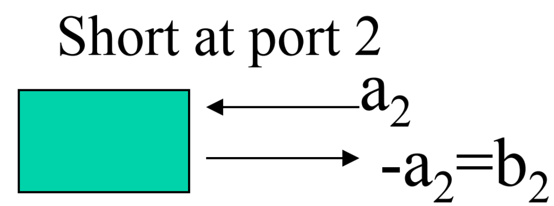
Obviously, the scattering matrix of this two-port network can not satisfy the lossless condition, thus it is not a lossless network.

The Scattering Matrix

EXAMPLE Application of Scattering Parameters

Solution:

$$[S]=\begin{bmatrix} 0.1\angle 0^\circ & 0.8\angle 90^\circ \\ 0.8\angle 90^\circ & 0.2\angle 0^\circ \end{bmatrix}$$



Return loss

Reflected power at port 1 when port 2 is shorted can be calculated as follow. Note that $a_2 = -b_2$ holds when port 2 is short circuited. Express the relationship between two ports by the S-parameter

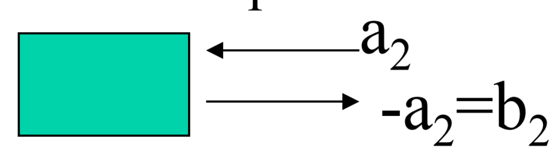
$$b_1 = S_{11}a_1 + S_{12}a_2$$
$$b_2 = S_{21}a_1 + S_{22}a_2$$

EXAMPLE Application of Scattering Parameters

Solution:

$$[S]=\begin{bmatrix} 0.1\angle 0^\circ & 0.8\angle 90^\circ \\ 0.8\angle 90^\circ & 0.2\angle 0^\circ \end{bmatrix}$$

Short at port 2



Return loss

$$b_1 = S_{11}a_1 + S_{12}a_2$$
$$b_2 = S_{21}a_1 + S_{22}a_2$$

Substituting $a_2 = -b_2$ into the equations above, we have

$$b_1 = S_{11}a_1 + S_{12}a_2 = S_{11}a_1 - S_{12}b_2 \quad (1)$$
$$b_2 = S_{21}a_1 + S_{22}a_2 = S_{21}a_1 - S_{22}b_2 \quad (2)$$

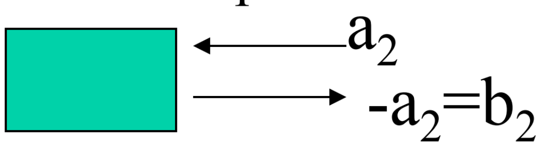
The Scattering Matrix

EXAMPLE Application of Scattering Parameters

Solution:

$$[S]=\begin{bmatrix} 0.1\angle 0^\circ & 0.8\angle 90^\circ \\ 0.8\angle 90^\circ & 0.2\angle 0^\circ \end{bmatrix}$$

Short at port 2



Return loss

$$b_1=S_{11}a_1+S_{12}a_2=S_{11}a_1-S_{12}b_2 \qquad (1)$$

$$b_2=S_{21}a_1+S_{22}a_2=S_{21}a_1-S_{22}b_2 \qquad (2)$$

According to the equation (2), we have the relationship between b_2 and a_1:

$$b_2=\frac{S_{21}}{\left(1+S_{22}\right)}a_1 \qquad (3)$$

EXAMPLE Application of Scattering Parameters

Solution:

$$[\mathbf{S}] = \begin{bmatrix} 0.1\angle 0^\circ & 0.8\angle 90^\circ \\ 0.8\angle 90^\circ & 0.2\angle 0^\circ \end{bmatrix}$$

Return loss

Substituting the result of equation (3) into equation (1), we have

$$\Gamma = \frac{b_1}{a_1} = S_{11} - S_{12}\frac{b_2}{a_1} = S_{11} - S_{12}\frac{S_{21}}{1+S_{22}} = 0.1 - \frac{(j0.8)(j0.8)}{1+0.2} = 0.633$$

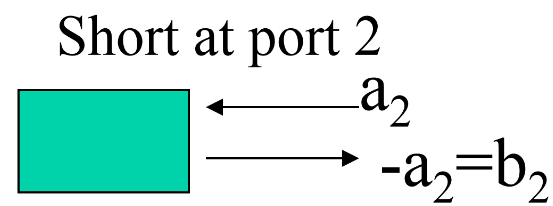
- Notice that S_{11} is the reflection coefficient at port 1 when port 2 is matched.
- While in this question, port 2 is short circuited, the reflection coefficient Γ is not equal to S_{11} .

The Scattering Matrix

EXAMPLE Application of Scattering Parameters

Solution:

$$[S]=\begin{bmatrix} 0.1\angle 0^\circ & 0.8\angle 90^\circ \\ 0.8\angle 90^\circ & 0.2\angle 0^\circ \end{bmatrix}$$



Return loss

According to the definition of the return loss, i.e., the ratio of the reflected power and the incident power , and in this question, based on the relationship between the voltage and the power, the return loss can be calculated as a function of the reflection coefficient Γ.

$$R.L.=-10\log_{10}|\Gamma|^2=-20\log_{10}|\Gamma|=-20\log_{10}(0.633)=3.97(dB)$$

EXAMPLE 4.5 Application of Scattering Parameters

Question: A two-port network is known to have the following scattering matrix.

$$[S]=\begin{bmatrix} 0.15\angle 0^\circ & 0.85\angle -45^\circ \\ 0.85\angle 45^\circ & 0.2\angle 0^\circ \end{bmatrix}$$

- Determine if the network is reciprocal and lossless.
- If port 2 is terminated with a matched load, what is the return loss seen at port 1?
- If port 2 is terminated with a short circuit, what is the return loss at port 1?

EXAMPLE 4.5 Application of Scattering Parameters

Solution:

$$[\mathbf{S}] = \begin{bmatrix} 0.15\angle 0^\circ & 0.85\angle -45^\circ \\ 0.85\angle 45^\circ & 0.2\angle 0^\circ \end{bmatrix}$$

Reciprocal

Because the scattering matrix $[S]$ is not symmetry,

$$[\mathbf{S}]^T = \begin{bmatrix} 0.15\angle 0^\circ & 0.85\angle 45^\circ \\ 0.85\angle -45^\circ & 0.2\angle 0^\circ \end{bmatrix} \neq [\mathbf{S}]$$

according to the condition of the reciprocal network, so conclusion can be drawn that the network is not reciprocal.

EXAMPLE 4.5 Application of Scattering Parameters

Solution:

$$[\mathbf{S}] = \begin{bmatrix} 0.15\angle 0° & 0.85\angle -45° \\ 0.85\angle 45° & 0.2\angle 0° \end{bmatrix}$$

Lossless

For $i = j$=1, and $N = 2$, we have

$$\sum_{k=1}^{2} S_{ki} S_{ki}^{*} = S_{11} S_{11}^{*} + S_{21} S_{21}^{*} = \left|S_{11}\right|^2 + \left|S_{21}\right|^2 = (0.15)^2 + (0.85)^2 = 0.745 \neq 1$$

Obviously, the sum is not equal to 1, this two-port network is not lossless

EXAMPLE 4.5 Application of Scattering Parameters

Solution:

$$[\mathbf{S}]=\begin{bmatrix} 0.15\angle 0^\circ & 0.85\angle -45^\circ \\ 0.85\angle 45^\circ & 0.2\angle 0^\circ \end{bmatrix}$$

Return loss at port 1 when port 2 is terminated with a matched load

When port 2 is terminated with a matched load, then the reflection coefficient seen at port 1 is equal to S_{11}, that is:

$$\Gamma = S_{11} = 0.15$$

So the return loss is:

$$R.L. = -20\log_{10}|\Gamma| = -20\log_{10}(0.15) = 16.5(dB)$$

EXAMPLE 4.5 Application of Scattering Parameters

Solution:

$$[S]=\begin{bmatrix} 0.15\angle 0^\circ & 0.85\angle -45^\circ \\ 0.85\angle 45^\circ & 0.2\angle 0^\circ \end{bmatrix}$$

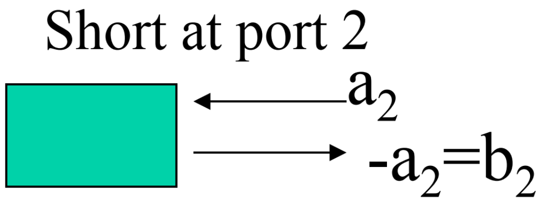
Return loss at port 1 when port 2 is terminated with a short circuit

As we have calculated previously, when port 2 is terminated with a short circuit, then the reflection coefficient at port 1 is no longer equal to S_{11}. We should calculate the reflection coefficient at port 1 to obtain the return loss at port 1 for short-circuit case.

EXAMPLE 4.5 Application of Scattering Parameters

Solution:

$$[S]=\begin{bmatrix} 0.15\angle 0^\circ & 0.85\angle -45^\circ \\ 0.85\angle 45^\circ & 0.2\angle 0^\circ \end{bmatrix}$$



Return loss at port 1 when port 2 is terminated with a short circuit

When port 2 is terminated with a short circuit, we have $a_2=-b_2$. Then as in previous exercise, we have:

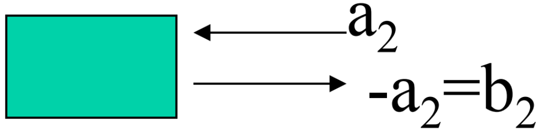
$$b_1 = S_{11}a_1 + S_{12}a_2$$
$$b_2 = S_{21}a_1 + S_{22}a_2$$

EXAMPLE 4.5 Application of Scattering Parameters

Solution:

$$[S]=\begin{bmatrix} 0.15\angle 0^\circ & 0.85\angle -45^\circ \\ 0.85\angle 45^\circ & 0.2\angle 0^\circ \end{bmatrix}$$

Short at port 2



Return loss at port 1 when port 2 is terminated with a short circuit

Substituting $a_2=-b_2$ for the short-circuit at port 2, then we have:

$$b_1=S_{11}a_1+S_{12}a_2=S_{11}a_1-S_{12}b_2 \qquad (1)$$

$$b_2=S_{21}a_1+S_{22}a_2=S_{21}a_1-S_{22}b_2 \qquad (2)$$

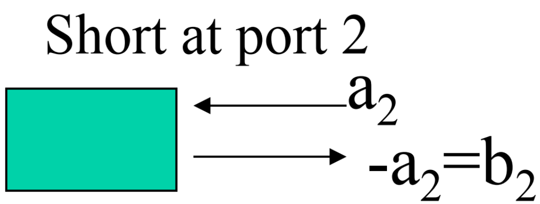
equation (2) gives:

$$b_2=\frac{S_{21}}{(1+S_{22})}a_1 \qquad (3)$$

EXAMPLE 4.5 Application of Scattering Parameters

Solution:

$$[S]=\begin{bmatrix} 0.15\angle 0^\circ & 0.85\angle -45^\circ \\ 0.85\angle 45^\circ & 0.2\angle 0^\circ \end{bmatrix}$$



Return loss at port 1 when port 2 is terminated with a short circuit

Dividing equation (1) by a_1 and substituting the result of equation (3) into equation (1) , we have

$$\Gamma=\frac{b_1}{a_1}=S_{11}-S_{12}\frac{S_{21}}{1+S_{22}}=0.15-\frac{(0.85\angle -45^\circ)(0.85\angle 45^\circ)}{1+0.2}=-0.452$$

So the return loss is:

$$R.L.=-20\log_{10}|\Gamma|=-20\log_{10}(0.452)=6.9(dB)$$

The Scattering Matrix

- An important point to understand scattering parameters is that the reflection coefficient looking into port n is not always equal to S_{nn};
- Only if all other ports are terminated with the matched load, the reflection coefficient at port n equals S_{nn};
- Similarly, the transmission coefficient from port m to port n is not equal to S_{nm};
- Unless all other ports are terminated with the matched load, then the transmission coefficient from port m to port n equals S_{nm}.

The Scattering Matrix

- The scattering parameters of a network reflects properties of only the network itself (network is linear), and are defined under the condition that all ports are matched;
- Changing the terminations or excitations of a network does not change its scattering parameters, but may change the reflection coefficient seen at specific port, or the transmission coefficient between two ports.

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A Shift in Reference Planes

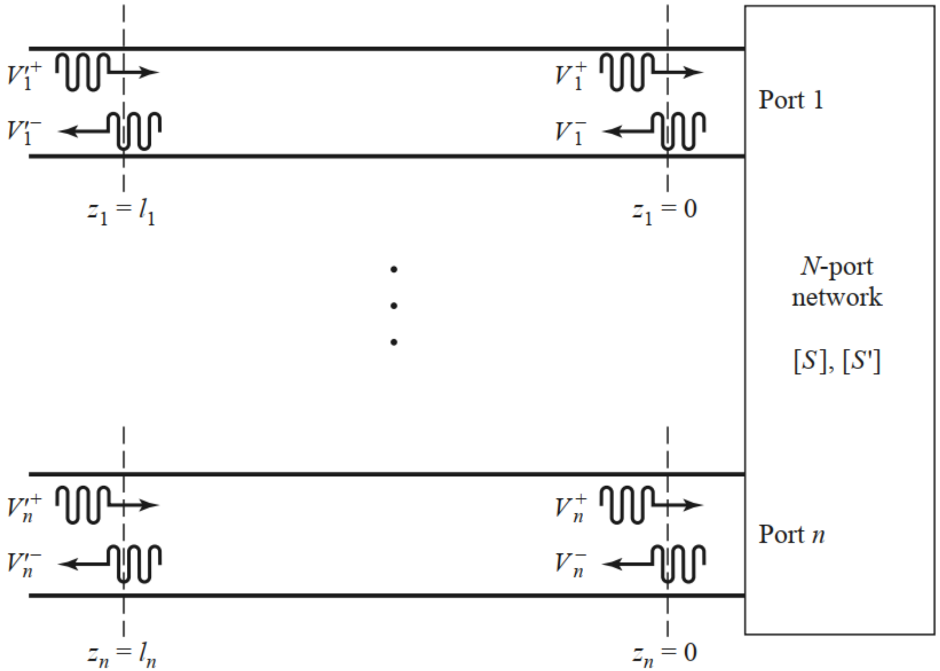
? **Why** we will discuss the impact of the shift in reference planes on S-parameters?

- Because scattering parameters relate amplitudes (magnitude and phase) of traveling waves incident on and reflected from a microwave network, phase reference planes must be specified for each port of the network.

- We now show how scattering parameters are transformed when the reference planes are shifted from their original locations.

The Scattering Matrix

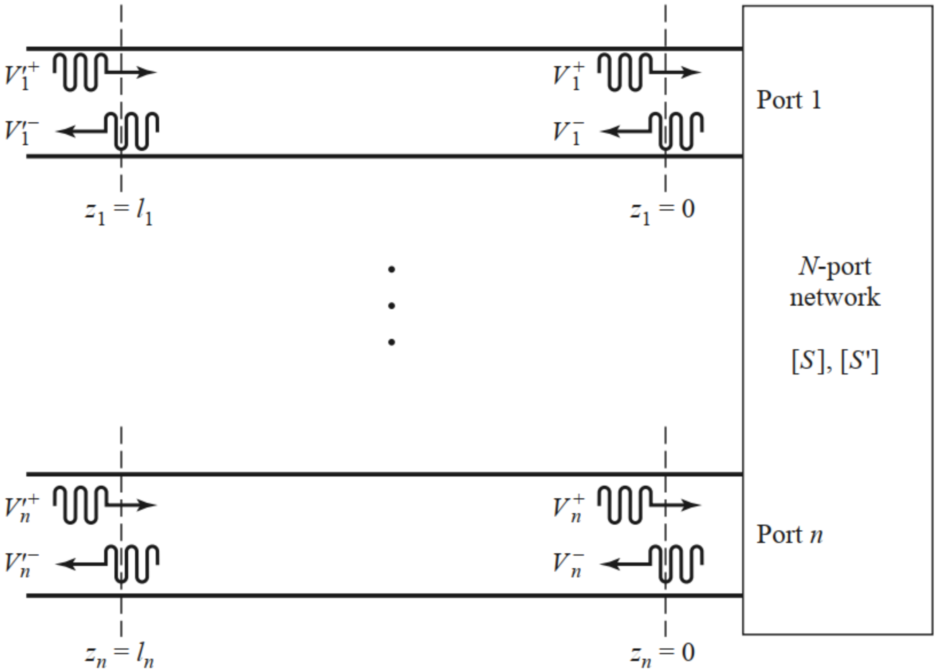
A Shift in Reference Planes



Consider the *N*-port microwave network shown in left, where the original terminal planes are assumed to be at $z_n = 0$ for the nth port, where z_n is an arbitrary coordinate measured along the transmission line feeding the nth port. The scattering matrix for the network with this set of terminal planes is denoted by $[S]$. Then in terms of the incident and reflected port voltages we have

$$\left[\mathbf{V}^-\right] = [\mathbf{S}]\left[\mathbf{V}^+\right] \quad (1)$$

A Shift in Reference Planes



$$\left[\mathbf{V}^-\right] = [\mathbf{S}]\left[\mathbf{V}^+\right] \qquad (1)$$

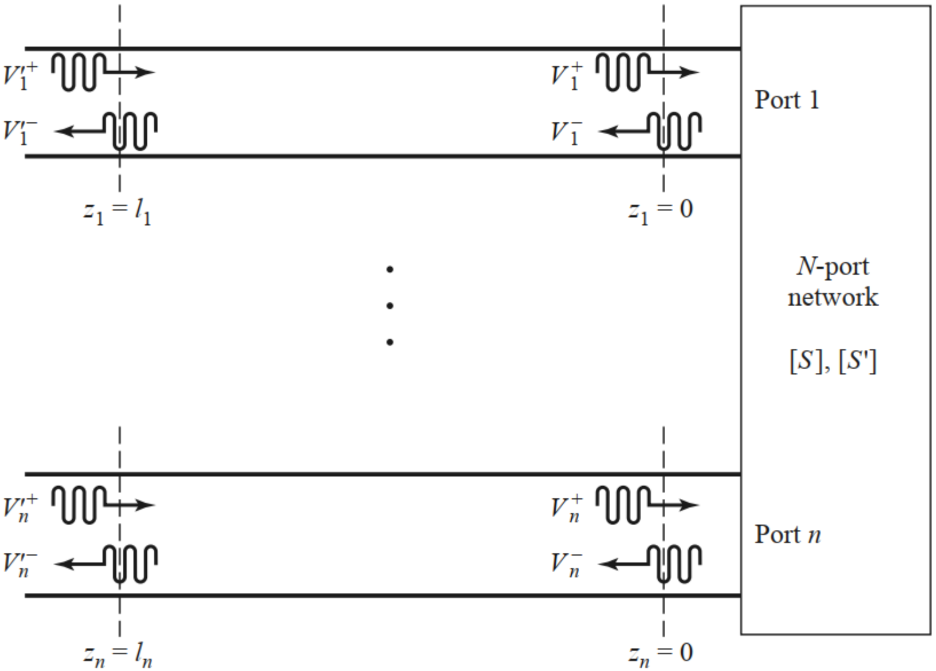
$$\left[\mathbf{V}'^-\right] = [\mathbf{S}']\left[\mathbf{V}'^+\right] \qquad (2)$$

Now consider a new set of reference planes defined at $z_n = l_n$ for the nth port, and let the new scattering matrix be denoted as $[\boldsymbol{S}']$. Then in terms of the incident and reflected port voltages we have that [left equation (2)]

where the unprimed quantities are referenced to the original terminal planes at $z_n = 0$, and the primed quantities are referenced to the new terminal planes at $z_n = l_n$.

The Scattering Matrix

A Shift in Reference Planes



$$[\mathbf{V}^-] = [\mathbf{S}][\mathbf{V}^+] \quad (1)$$

$$[\mathbf{V}'^-] = [\mathbf{S}'][\mathbf{V}'^+] \quad (2)$$

From the theory of traveling waves on lossless transmission lines we can relate the new wave amplitudes to the original ones as

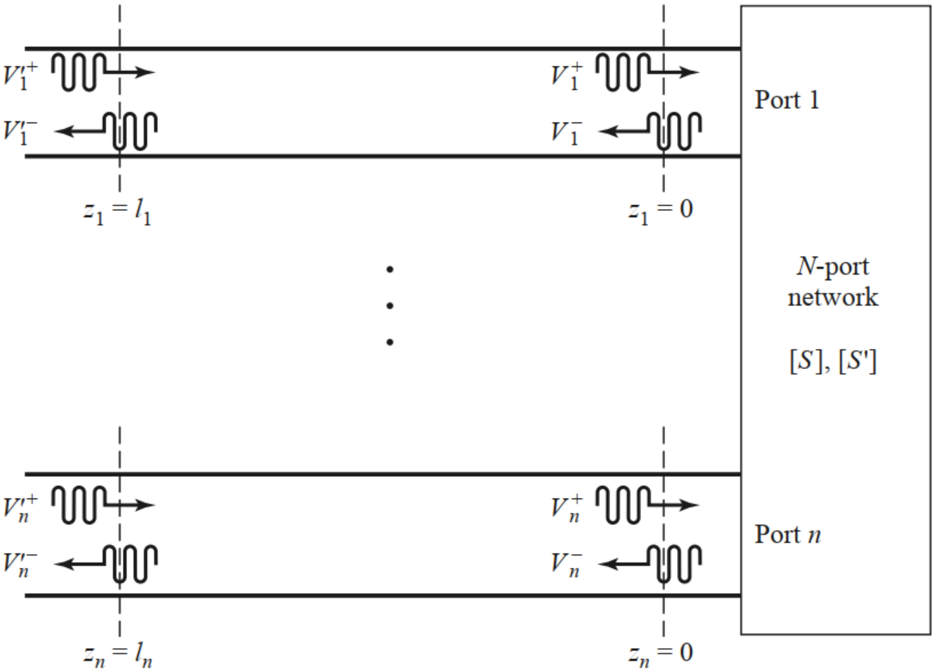
$$V_n^+ = V_n'^+ e^{-j\theta_n}$$

$$V_n'^- = V_n^- e^{-j\theta_n}$$

where $\theta_n = \beta_n l_n$ is the electrical length of the outward shift of the reference plane of port n.

The Scattering Matrix

A Shift in Reference Planes



$$V_n^+ = V_n'^+ e^{-j\theta_n}$$
$$V_n'^- = V_n^- e^{-j\theta_n}$$

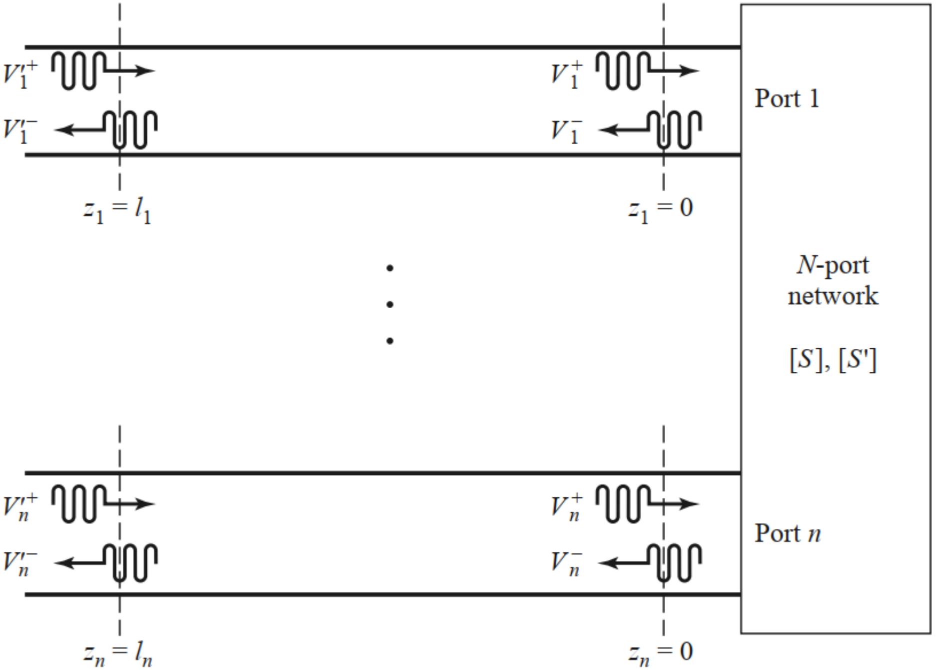
Perform transformations on equations above, we have

$$V_n'^+ = V_n^+ e^{j\theta_n} \quad (3)$$
$$V_n'^- = V_n^- e^{-j\theta_n} \quad (4)$$

$$\left[\mathbf{V}^-\right] = [\mathbf{S}]\left[\mathbf{V}^+\right] \quad (1)$$
$$\left[\mathbf{V}'^-\right] = [\mathbf{S}']\left[\mathbf{V}'^+\right] \quad (2)$$

The Scattering Matrix

A Shift in Reference Planes



$$\left[\mathbf{V}^-\right]=[\mathbf{S}]\left[\mathbf{V}^+\right] \qquad (1)$$

$$\left[\mathbf{V}'^-\right]=[\mathbf{S}']\left[\mathbf{V}'^+\right] \qquad (2)$$

$$V_n'^+ = V_n^+ e^{j\theta_n} \qquad (3)$$

$$V_n'^- = V_n^- e^{-j\theta_n} \qquad (4)$$

Rewriting Equations (3) and (4) in matrix form:

$$\begin{bmatrix} e^{-j\theta_1} & 0 & \cdots & 0 \\ 0 & e^{-j\theta_2} & & \\ \vdots & & \ddots & \\ 0 & & & e^{-j\theta_N} \end{bmatrix}\left[\mathbf{V}'^+\right]=\left[\mathbf{V}^+\right]$$

$$\begin{bmatrix} e^{j\theta_1} & 0 & \cdots & 0 \\ 0 & e^{j\theta_2} & & \\ \vdots & & \ddots & \\ 0 & & & e^{j\theta_N} \end{bmatrix}\left[\mathbf{V}'^-\right]=\left[\mathbf{V}^-\right]$$

A Shift in Reference Planes

Substituting the equations above into equation (1):

$$\begin{bmatrix} e^{j\theta_1} & 0 & \cdots & 0 \\ 0 & e^{j\theta_2} & & \\ \vdots & & \ddots & \\ 0 & & & e^{j\theta_N} \end{bmatrix} \left[\mathbf{V}'^{-}\right] = [\mathbf{S}] \begin{bmatrix} e^{-j\theta_1} & 0 & \cdots & 0 \\ 0 & e^{-j\theta_2} & & \\ \vdots & & \ddots & \\ 0 & & & e^{-j\theta_N} \end{bmatrix} \left[\mathbf{V}'^{+}\right]$$

Multiplying by the inverse of the first matrix on the left yields

$$\left[\mathbf{V}'^{-}\right] = \begin{bmatrix} e^{-j\theta_1} & 0 & \cdots & 0 \\ 0 & e^{-j\theta_2} & & \\ \vdots & & \ddots & \\ 0 & & & e^{-j\theta_N} \end{bmatrix} [\mathbf{S}] \begin{bmatrix} e^{-j\theta_1} & 0 & \cdots & 0 \\ 0 & e^{-j\theta_2} & & \\ \vdots & & \ddots & \\ 0 & & & e^{-j\theta_N} \end{bmatrix} \left[\mathbf{V}'^{+}\right]$$

A Shift in Reference Planes

$$\left[\mathbf{V}'^{-}\right]=\begin{bmatrix} e^{-j\theta_1} & 0 & \cdots & 0 \\ 0 & e^{-j\theta_2} & & \\ \vdots & & \ddots & \\ 0 & & & e^{-j\theta_N} \end{bmatrix}[\mathbf{S}]\begin{bmatrix} e^{-j\theta_1} & 0 & \cdots & 0 \\ 0 & e^{-j\theta_2} & & \\ \vdots & & \ddots & \\ 0 & & & e^{-j\theta_N} \end{bmatrix}\left[\mathbf{V}'^{+}\right]$$

Comparing the equation above with the equation (2) shows that

$$[\mathbf{S}']=\begin{bmatrix} e^{-j\theta_1} & 0 & \cdots & 0 \\ 0 & e^{-j\theta_2} & & \\ \vdots & & \ddots & \\ 0 & & & e^{-j\theta_N} \end{bmatrix}[\mathbf{S}]\begin{bmatrix} e^{-j\theta_1} & 0 & \cdots & 0 \\ 0 & e^{-j\theta_2} & & \\ \vdots & & \ddots & \\ 0 & & & e^{-j\theta_N} \end{bmatrix}$$

A Shift in Reference Planes

$$[\mathbf{S}'] = \begin{bmatrix} e^{-j\theta_1} & 0 & \cdots & 0 \\ 0 & e^{-j\theta_2} & & \\ \vdots & & \ddots & \\ 0 & & & e^{-j\theta_N} \end{bmatrix} [\mathbf{S}] \begin{bmatrix} e^{-j\theta_1} & 0 & \cdots & 0 \\ 0 & e^{-j\theta_2} & & \\ \vdots & & \ddots & \\ 0 & & & e^{-j\theta_N} \end{bmatrix}$$

- Note that

$$S'_{nn} = e^{-2j\theta_n} S_{nn}$$

indicating that the phase of S_{nn} is shifted by twice the electrical length of the shift in terminal plane n because the wave travels twice over this length , transmission and reflection.

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The Transmission (*ABCD*) Matrix

The Transmission *(ABCD)* Matrix

? **What** the shortcomings of the network matrices previously discussed?

► The *Z*, *Y*, and *S* parameter representations can be used to characterize a microwave network with an arbitrary number of ports, but in practice many microwave networks consist of a cascade connection of two or more two-port networks.

► In terms of the cascade-connected networks, the matrices like *Z*-, *Y*-, and *S*-parameter matrices are inconvenient to shown the relationship between the inputs and outputs.

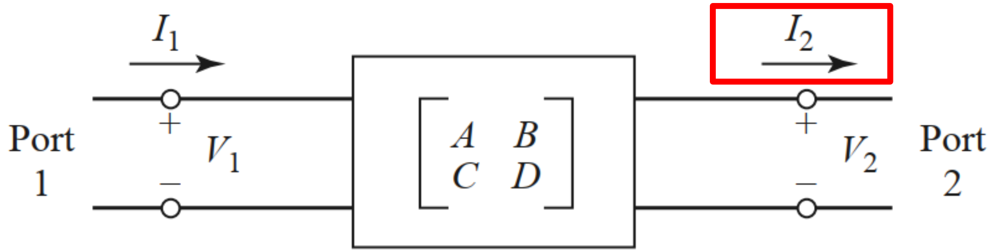
The Transmission *(ABCD)* Matrix

? **Why** introduce the ABCD matrix?

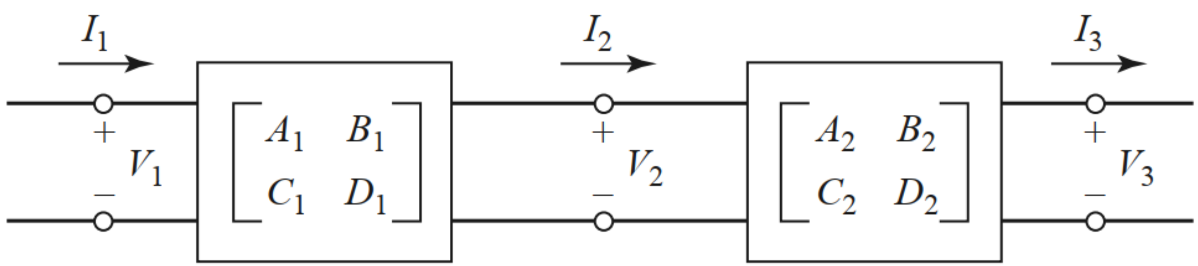
► In cascade case, it is convenient to define a 2×2 ***transmission***, or ***ABCD***, ***matrix***, for each two-port network.

► The *ABCD matrix* of the cascade connection of two or more two-port networks can be easily found by multiplying the *ABCD matrices* of the individual two-ports.

The Transmission *(ABCD)* Matrix



(a)

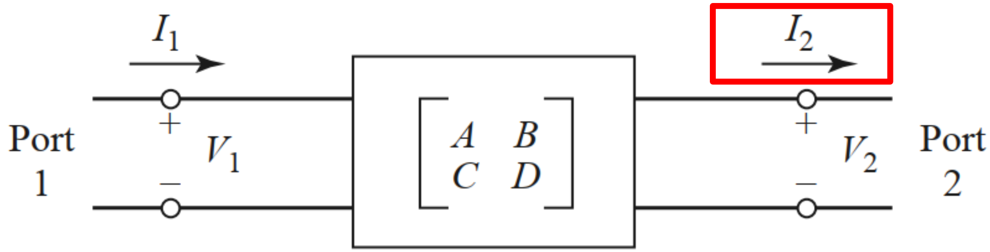


(b)

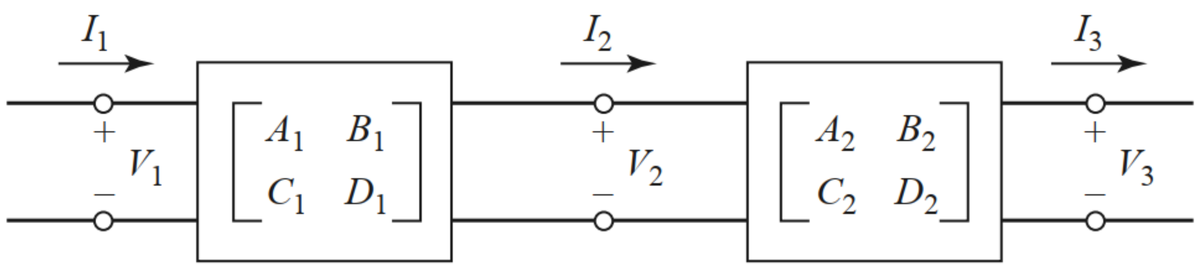
- The ABCD matrix is defined for a two-port network in terms of the total voltages and currents as shown in Figure (a) and the following

$$V_1 = AV_2 + BI_2$$
$$I_1 = CV_2 + DI_2$$

The Transmission *(ABCD)* Matrix



(a)

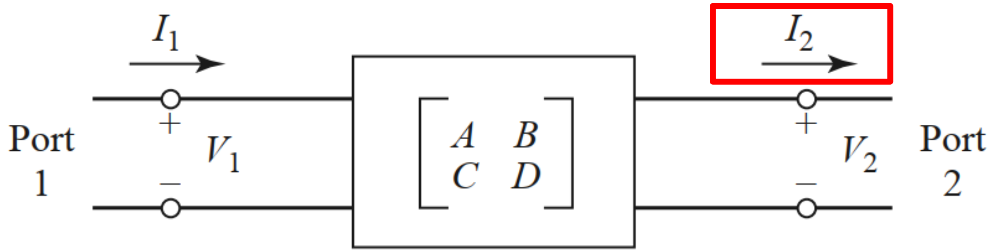


(b)

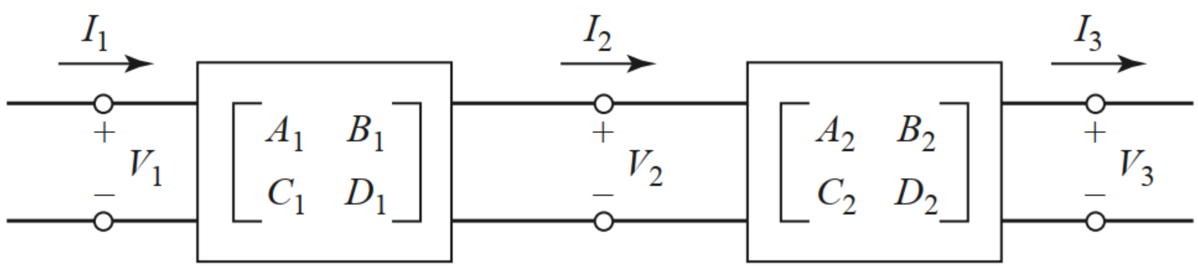
- or in matrix form as [Figure (a)]

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \quad (1)$$

The Transmission *(ABCD)* Matrix



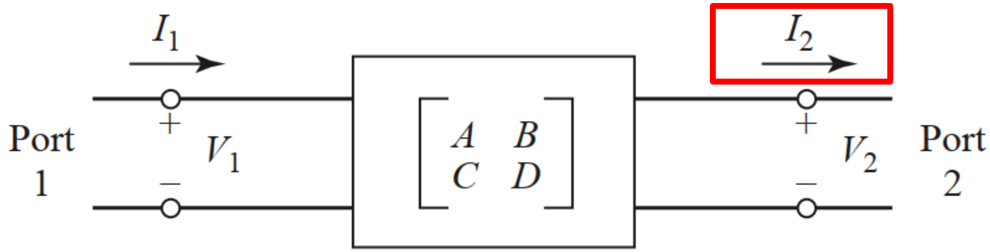
(a)



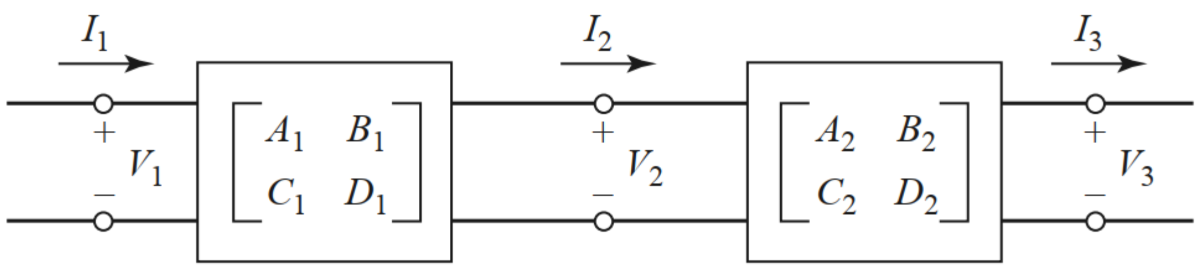
(b)

- It is important to note that a change in the sign convention of I_2 has been made from our previous definitions(I_2 the current flowing *into* port 2).

The Transmission *(ABCD)* Matrix



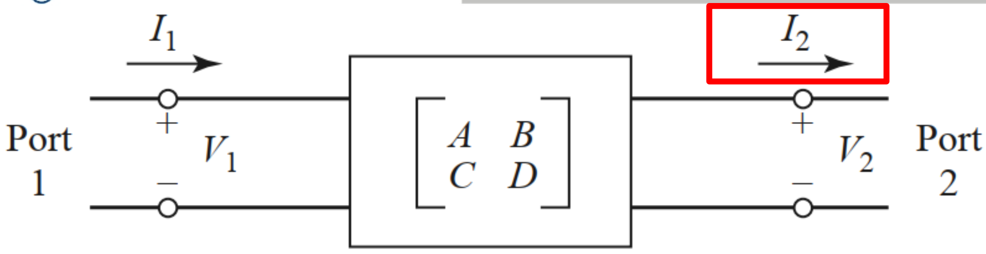
(a)



(b)

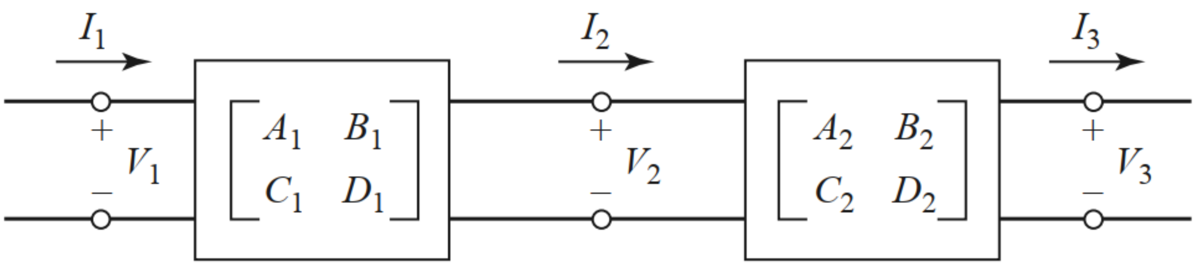
- The convention that I_2 flows ***out*** of port 2 will be used when dealing with *ABCD matrices* so that in a cascade network I_2 will be the same current that flows into the adjacent network, as shown in Figure (b).

The Transmission *(ABCD)* Matrix



(a)

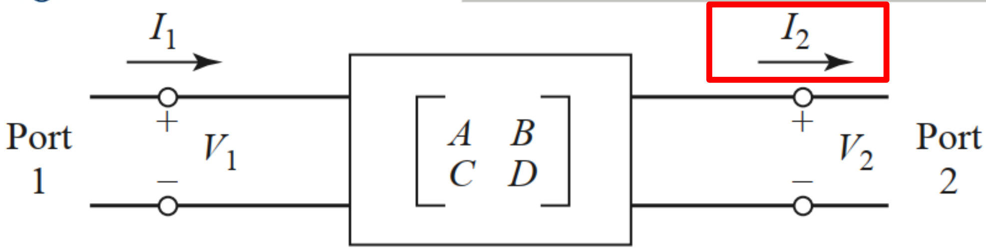
$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \quad (1)$$



(b)

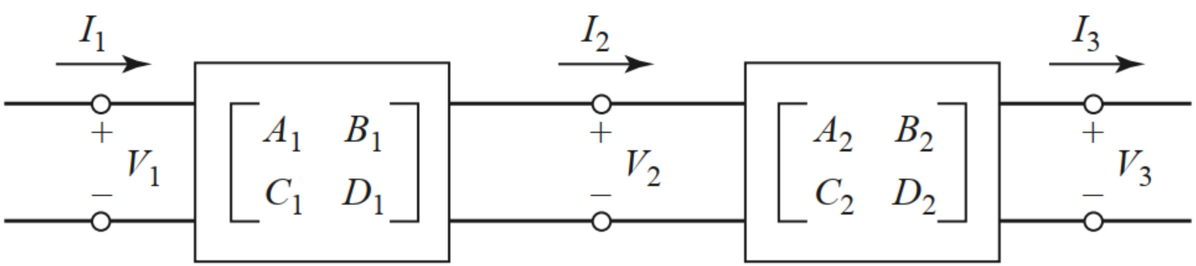
- Then the left-hand side of equation (1) represents the voltage and current at port 1 of the network, while the column on the right-hand side of equation (1) represents the voltage and current at port 2.

The Transmission *(ABCD)* Matrix



(a)

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \quad (1)$$

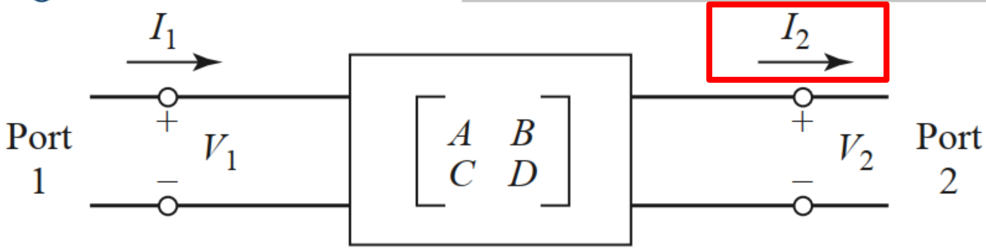


(b)

- In the cascade connection of two-port networks shown in Figure (b) we have that

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \quad (2a) \qquad \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} V_3 \\ I_3 \end{bmatrix} \quad (2b)$$

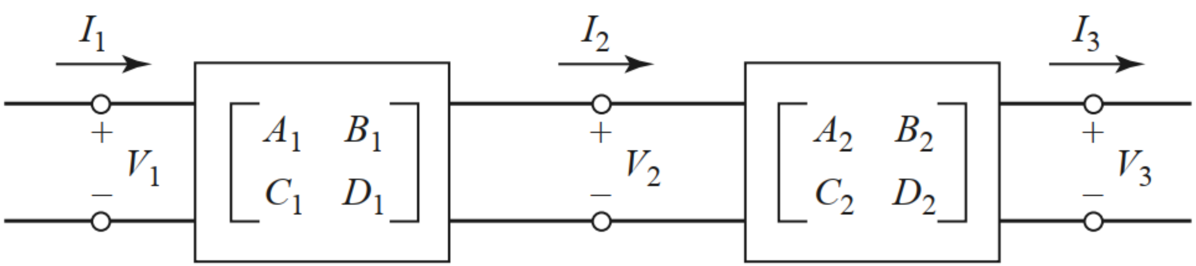
The Transmission *(ABCD)* Matrix



(a)

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \quad (1)$$

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \quad (2a)$$



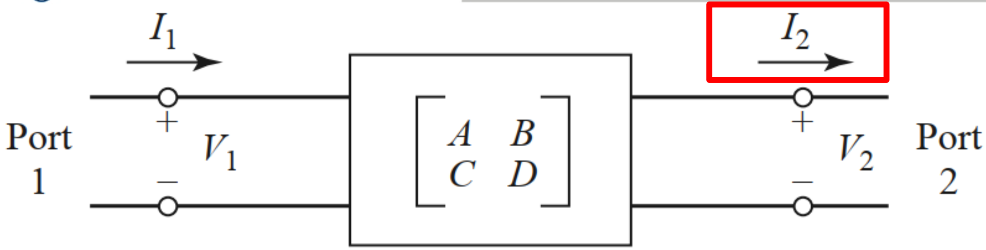
(b)

$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} V_3 \\ I_3 \end{bmatrix} \quad (2b)$$

- Substituting (2b) into (2a) gives

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} V_3 \\ I_3 \end{bmatrix} \quad (3)$$

The Transmission *(ABCD)* Matrix

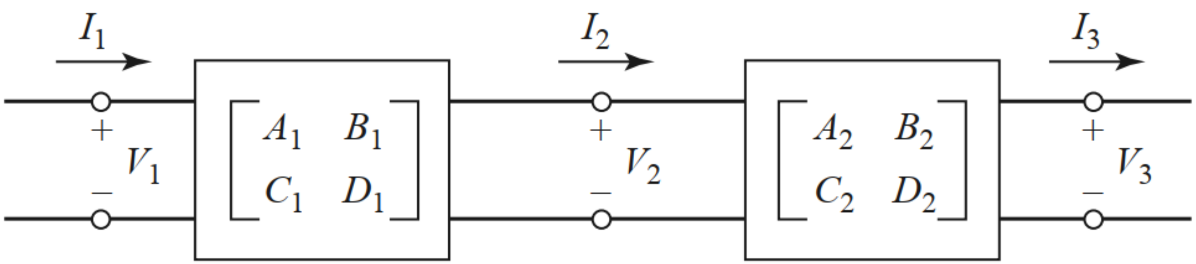


(a)

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \quad (1)$$

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \quad (2a)$$

$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} V_3 \\ I_3 \end{bmatrix} \quad (2b)$$

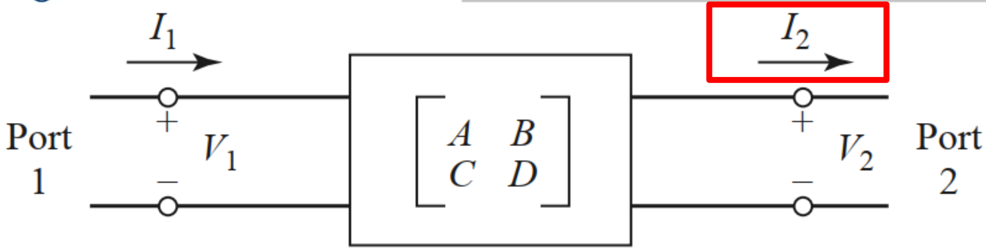


(b)

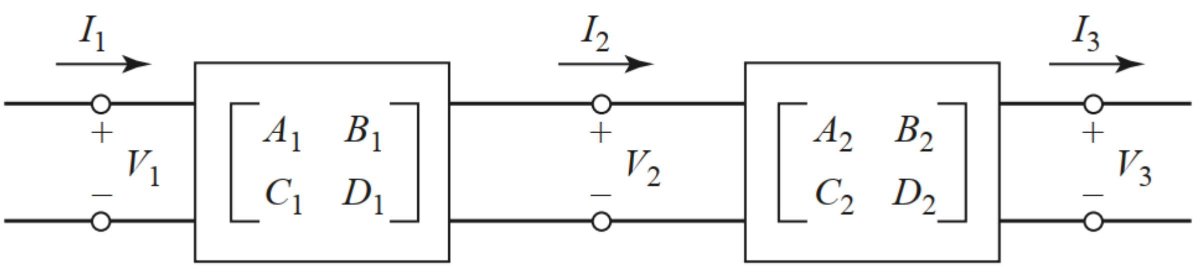
$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} V_3 \\ I_3 \end{bmatrix} \quad (3)$$

- The equation (3) shows that the ABCD matrix of the cascade connection of the two networks is equal to the product of the ABCD matrices representing the individual two-ports.

The Transmission *(ABCD)* Matrix



(a)



(b)

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \quad (1)$$

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \quad (2a)$$

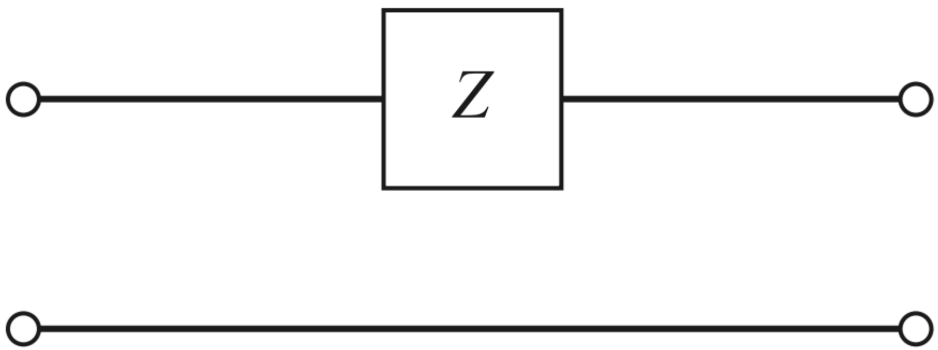
$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} V_3 \\ I_3 \end{bmatrix} \quad (2b)$$

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} V_3 \\ I_3 \end{bmatrix} \quad (3)$$

- Note that the order of multiplication of the matrix must be the same as the order in which the networks are arranged since matrix multiplication is not, in general, commutative.

EXAMPLE 4.6 Evaluation of *ABCD* Parameters

Question: Find the ABCD parameters of a two-port network consisting of a series impedance Z between ports 1 and 2 (the first entry in Table 1).



$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

EXAMPLE 4.6 Evaluation of *ABCD* Parameters

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \quad (1)$$

Solution:

From the defining relations of equation (1), we have that

$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0}$$

which indicates that A is found by applying a voltage V_1 at port 1, and measuring the open-circuit voltage V_2 at port 2. Thus

$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0} = 1$$

$$B = \left.\frac{V_1}{I_2}\right|_{V_2=0} = \frac{V_1}{V_1/Z} = Z,$$

$$C = \left.\frac{I_1}{V_2}\right|_{I_2=0} = 0,$$

$$D = \left.\frac{I_1}{I_2}\right|_{V_2=0} = \frac{I_1}{I_1} = 1,$$

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Thanks