西安电子科技大学
XIDIAN UNIVERSITY

B39HF High Frequency Circuits

Lecture 6

The Lossless Transmission Line: General Considerations

Outline

- Fundamental Parameters of Transmission Line
- Voltage Reflection Coefficient
- Standing Waves

Fundamental Parameters

Fundamental Parameters on the Transmission Line

- Propagation constant γ
- Characteristic impedance Z_0
- Guide wavelength λ
- Phase velocity u_p

Fundamental Parameters

Using the lossless line expression for β, we obtain the following expressions for the guide wavelength λ and the phase velocity u_p:

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{L'C'}}$$

$$u_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{L'C'}}$$

According to $L'C' = \mu\varepsilon$, β and u_p can be rewritten as

$$\beta = \omega\sqrt{\mu\varepsilon} \quad (\mathrm{rad/m})$$

$$u_p = \frac{1}{\sqrt{\mu\varepsilon}} \quad (\mathrm{m/s})$$

Where μ and ε are, respectively, the magnetic permeability and electrical permittivity of the insulating material separating the conductors.

Fundamental Parameters

Table 2-2 Characteristic parameters of transmission lines.

	Propagation Constant $\gamma = \alpha + j\beta$	Phase Velocity u_p	Characteristic Impedance Z_0
General case	$\gamma = \sqrt{(R' + j\omega L')(G' + j\omega C')}$	$u_p = \omega/\beta$	$Z_0 = \sqrt{\dfrac{(R' + j\omega L')}{(G' + j\omega C')}}$
Lossless ($R' = G' = 0$)	$\alpha = 0,\ \beta = \omega\sqrt{\epsilon_r}/c$	$u_p = c/\sqrt{\epsilon_r}$	$Z_0 = \sqrt{L'/C'}$
Lossless coaxial	$\alpha = 0,\ \beta = \omega\sqrt{\epsilon_r}/c$	$u_p = c/\sqrt{\epsilon_r}$	$Z_0 = \left(60/\sqrt{\epsilon_r}\right)\ln(b/a)$
Lossless two-wire	$\alpha = 0,\ \beta = \omega\sqrt{\epsilon_r}/c$	$u_p = c/\sqrt{\epsilon_r}$	$Z_0 = \left(120/\sqrt{\epsilon_r}\right) \cdot \ln[(D/d) + \sqrt{(D/d)^2 - 1}]$ $Z_0 \approx \left(120/\sqrt{\epsilon_r}\right)\ln(2D/d)$, if $D \gg d$
Lossless parallel-plate	$\alpha = 0,\ \beta = \omega\sqrt{\epsilon_r}/c$	$u_p = c/\sqrt{\epsilon_r}$	$Z_0 = \left(120\pi/\sqrt{\epsilon_r}\right)(h/w)$

Notes: (1) $\mu = \mu_0$, $\epsilon = \epsilon_r\epsilon_0$, $c = 1/\sqrt{\mu_0\epsilon_0}$, and $\sqrt{\mu_0/\epsilon_0} \approx (120\pi)\ \Omega$, where ϵ_r is the relative permittivity of insulating material. (2) For coaxial line, a and b are radii of inner and outer conductors. (3) For two-wire line, d = wire diameter and D = separation between wire centers. (4) For parallel-plate line, w = width of plate and h = separation between the plates.

Table 2-2 provides a list of the expressions for γ, Z_0 ,and u_p for the general case of several types of lossless lines. The expressions for the lossless lines are based on the equations for L' and C' given in Table 2-1.

2.6.1 Voltage Reflection Coefficient

Voltage Reflection Coefficient

With $\gamma = j\beta$ for the lossless line, the total voltage and current become

$$\tilde{V}(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}$$

$$\tilde{I}(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{j\beta z}$$

two unknowns, V_0^+ and V_0^-.

incident wave: an exponential factor of the form $e^{-j\beta z}$ associated with a wave traveling in the positive z direction, from the source (sending end) to the load (receiving end). with V_0^+ as its voltage amplitude.

reflected wave: the term containing $V_0^- e^{j\beta z}$ represents a with voltage amplitude V_0^-, traveling along the negative z direction, from the load to the source.

Voltage Reflection Coefficient

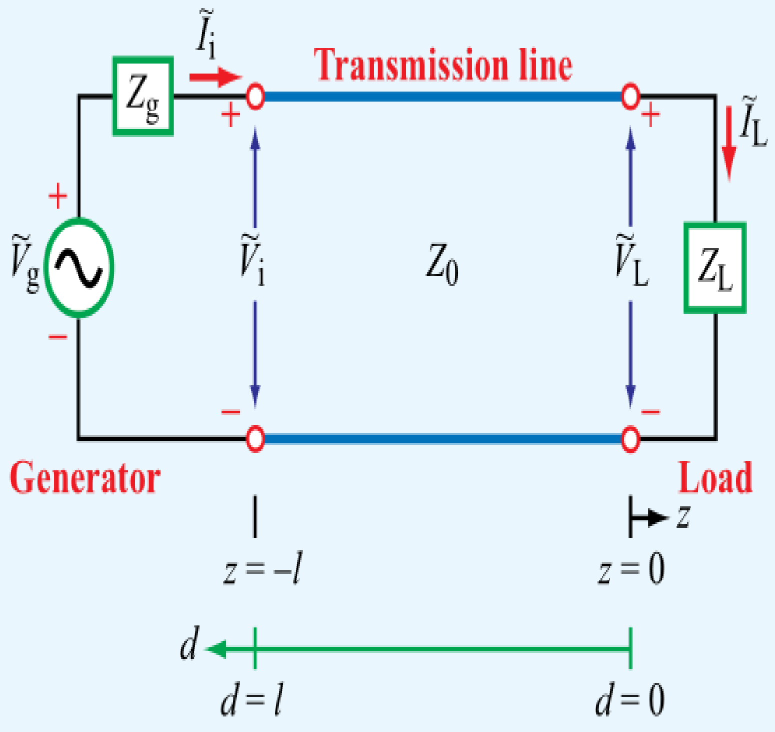


Figure 2-12 Transmission line of length l connected on one end to a generator circuit and on the other end to a load Z_L. The load is located at $z = 0$ and the generator terminals are at $z = -l$. Coordinate d is defined as $d = -z$.

- To determine V_0^+ and V_0^-, we need to consider the lossless transmission line in a complete circuit, including a generator circuit at its input terminals and a load at its output terminals, as shown in Fig. 2-12.
- The line, of length l, is terminated in an arbitrary *load impedance* Z_L.
- For ease, the reference of the spatial coordinate z is chosen such that $z = 0$ corresponds to the location of the load.

Voltage Reflection Coefficient

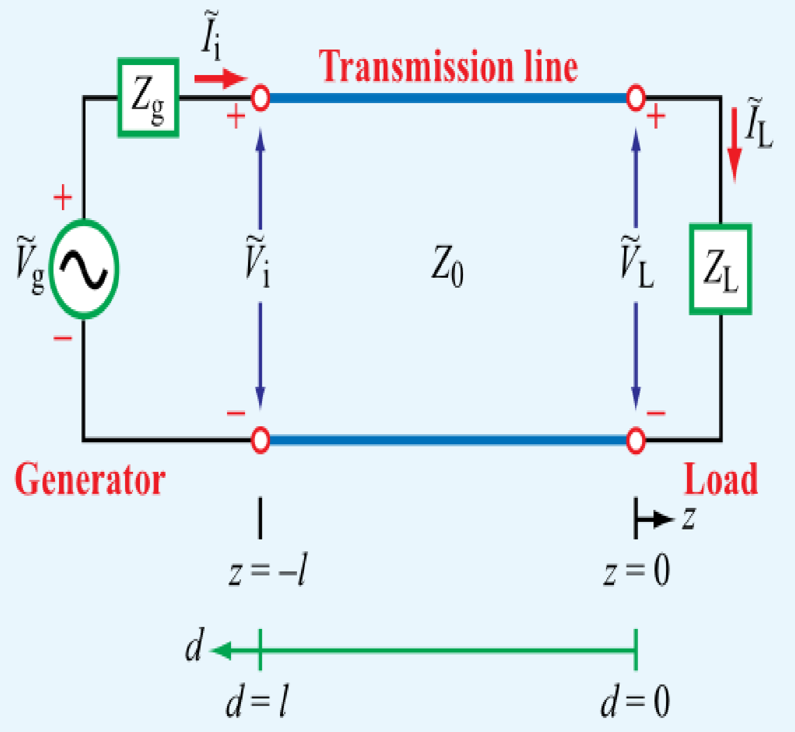


Figure 2-12 Transmission line of length l connected on one end to a generator circuit and on the other end to a load Z_L. The load is located at $z = 0$ and the generator terminals are at $z = -l$. Coordinate d is defined as $d = -z$.

- At the source, i.e., $z = -l$, the line is connected to a sinusoidal voltage source with phasor voltage $\tilde{V}_g$ and internal impedance Z_g.
- Since z points from the generator to the load, positive z corresponds to location beyond the load, therefore irrelevant to our circuit.
- We find it more convenient to work with a spatial dimension that also starts at the load, but whose direction is opposite of z, i.e., the *distance from the load* d and define it as $d = -z$

Voltage Reflection Coefficient

The phasor voltage across the load, $\tilde{V}_{\mathrm{L}}$,and the phasor current through it, $\tilde{I}_{\mathrm{L}}$, are related by the load impedance Z_{L} as

$$Z_{\mathrm{L}} = \frac{\tilde{V}_{\mathrm{L}}}{\tilde{I}_{\mathrm{L}}}$$

The voltage $\tilde{V}_{\mathrm{L}}$ is the total voltage on the line $\tilde{V}(z)$, and $\tilde{I}_{\mathrm{L}}$ is the total current $\tilde{I}(z)$, both evaluated at $z = 0$:

$$\tilde{V}_{\mathrm{L}} = \tilde{V}(z = 0) = V_0^+ + V_0^-$$

$$\tilde{I}_{\mathrm{L}} = \tilde{I}(z = 0) = \frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0}$$

Substitute these expressions into Z_{L}, we obtain

$$Z_{\mathrm{L}} = \left(\frac{V_0^+ + V_0^-}{V_0^+ - V_0^-}\right) Z_0$$

Voltage Reflection Coefficient

Solving for V_0^- gives

$$V_0^- = \left(\frac{Z_L - Z_0}{Z_L + Z_0}\right) V_0^+$$

► The ratio of the amplitudes of the reflected and incident voltage waves at the load is known as the *voltage reflection coefficient* Γ.

$$\Gamma = \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$= \frac{Z_L/Z_0 - 1}{Z_L/Z_0 + 1}$$

$$= \frac{z_L - 1}{z_L - 1} \quad \text{(dimensionless)}$$

where $z_L = Z_L/Z_0$ is the *normalized load impedance*.

Voltage Reflection Coefficient

In many transmission-line problems, we can streamline the necessary computation by normalizing all impedances in the circuit to the characteristic impedance Z_0. *Normalized impedances are denoted by lowercase letters*.

According to the relationship between voltage amplitude and current amplitude, we can obtained that

$$\frac{I_0^-}{I_0^+} = -\frac{V_0^-}{V_0^+} = -\Gamma$$

► We note that whereas the ratio of the voltage amplitudes is equal to Γ, the ratio of the current amplitudes is equal to $-\Gamma$.

Voltage Reflection Coefficient

The reflection coefficient Γ is governed by a single parameter, the normalized load impedance z_{L}. As indicated by $Z_0 = \sqrt{L'/C'}$, Z_0 of a lossless line is a real number.

However, Z_{L} is in general a complex quantity, as in the case of a series RL circuit, for example, for which $Z_{\mathrm{L}} = R + j\omega L$. Hence, in general Γ also is complex and given by

$$\Gamma = |\Gamma| e^{j\theta_{\mathrm{r}}}$$

where $|\Gamma|$ is the magnitude of Γ and θ_{r} is its phase angle. Note that $|\Gamma| \leq 1$.

► A load is said to be *matched* to a transmission line if $Z_{\mathrm{L}} = Z_0$ because then there will be no reflection by the load ($\Gamma = 0$ and $V_0^- = 0$).

Voltage Reflection Coefficient

On the other hand, when the load is an open circuit ($Z_L = \infty$), $\Gamma = 1$ and $V_0^- = V_0^+$, and when it is a short circuit ($Z_L = 0$), $\Gamma = -1$ and $V_0^- = -V_0^+$ (Table 2-3).

Table 2-3 **Magnitude and phase of the reflection coefficient for various types of load. The normalized load impedance $z_L = Z_L/Z_0 = (R + jX)/Z_0 = r + jx$, where $r = R/Z_0$ and $x = X/Z_0$ are the real and imaginary parts of z_L, respectively.**

Reflection Coefficient $\Gamma = |\Gamma|e^{j\theta_r}$

Load	$\lvert\Gamma\rvert$	θ_r
Z_0 — $Z_L = (r + jx)Z_0$	$\left[\dfrac{(r-1)^2 + x^2}{(r+1)^2 + x^2}\right]^{1/2}$	$\tan^{-1}\left(\dfrac{x}{r-1}\right) - \tan^{-1}\left(\dfrac{x}{r+1}\right)$
Z_0 — Z_0	0 (no reflection)	irrelevant
Z_0 — (short)	1	$\pm 180°$ (phase opposition)
Z_0 — (open)	1	0 (in-phase)
Z_0 — $jX = j\omega L$	1	$\pm 180° - 2\tan^{-1} x$
Z_0 — $jX = \dfrac{-j}{\omega C}$	1	$\pm 180° + 2\tan^{-1} x$

2.6.2 Standing Waves

Using the relation $V_0^- = \Gamma V_0^+$ in $\tilde{V}(z)$ and $\tilde{I}(z)$ yields

$$\tilde{V}(z) = V_0^+\left(e^{-j\beta z} + \Gamma e^{j\beta z}\right)$$

$$\tilde{I}(z) = \frac{V_0^+}{Z_0}\left(e^{-j\beta z} - \Gamma e^{j\beta z}\right)$$

These expressions now contain only one unknown, yet to be determined,, V_0^+. Before we proceed to solve for V_0^+, however, let us examine the physical meaning underlying these expressions. We begin by deriving an expression for $\left|\tilde{V}(z)\right|$, the magnitude of $\tilde{V}(z)$.

Standing Waves

Upon using $\Gamma = |\Gamma| e^{j\theta_r}$ in the above expressions and applying the relation $|\tilde{V}(z)| = [\tilde{V}(z)\tilde{V}^*(z)]^{1/2}$, where $\tilde{V}^*(z)$ is the complex conjugate of $\tilde{V}(z)$, we have

$$|\tilde{V}(z)| = \left\{\left[V_0^+\left(e^{-j\beta z} + |\Gamma| e^{j\theta_r} e^{j\beta z}\right)\right]\left[(V_0^+)^*\left(e^{j\beta z} + |\Gamma| e^{-j\theta_r} e^{-j\beta z}\right)\right]\right\}^{1/2}$$

$$= |V_0^+|\left[1 + |\Gamma|^2 + |\Gamma|\left(e^{j(2\beta z+\theta_r)} + e^{-j(2\beta z+\theta_r)}\right)\right]^{1/2}$$

$$e^{jx} + e^{-jx} = 2\cos x$$

$$= |V_0^+|[1 + |\Gamma|^2 + 2|\Gamma|\cos(2\beta z + \theta_r)]^{1/2}$$

To express the magnitude of $\tilde{V}$ as a function of d instead of z, we replace z with $-d$ on the right-hand side of $|\tilde{V}(z)|$:

$$|\tilde{V}(d)| = |V_0^+|[1 + |\Gamma|^2 + 2|\Gamma|\cos(2\beta d - \theta_r)]^{1/2}$$

By applying the same steps to $\tilde{I}(z)$,a similar expression can be derived for $|\tilde{I}(d)|$:

$$|\tilde{I}(d)| = \frac{|V_0^+|}{Z_0}[1 + |\Gamma|^2 - 2|\Gamma|\cos(2\beta d - \theta_r)]^{1/2}$$

Standing Waves

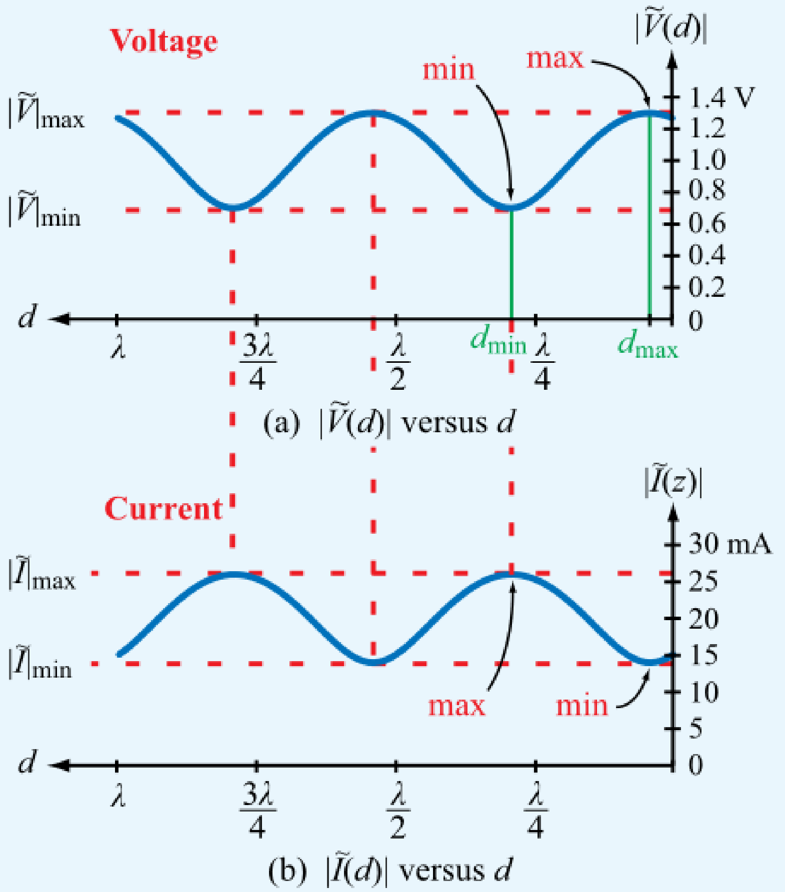


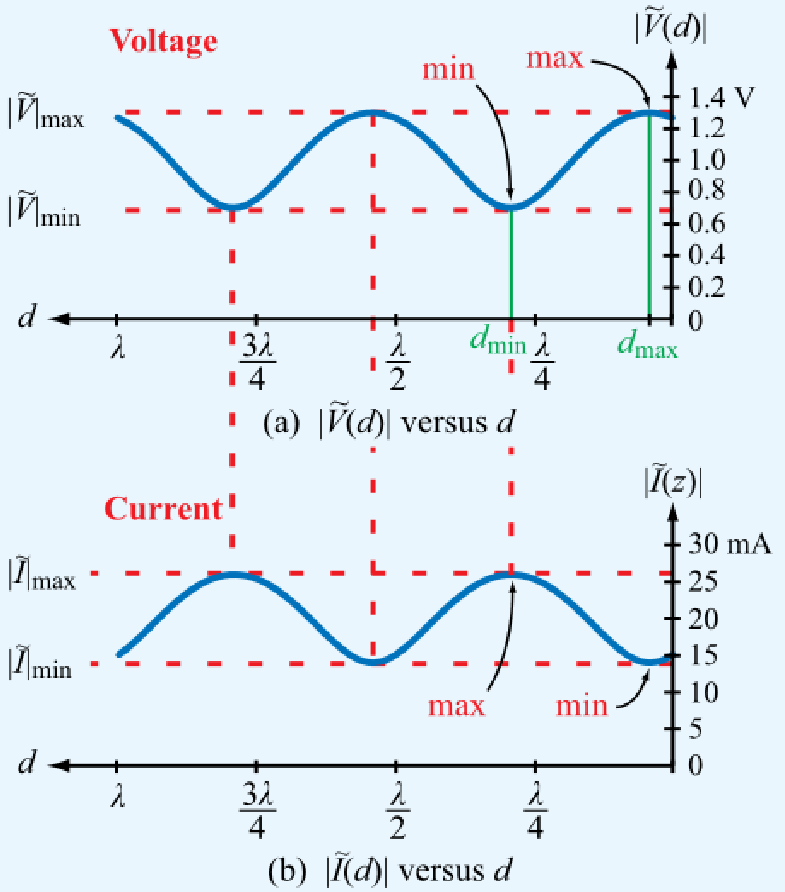
Figure 2-14 Standing-wave pattern for (a) $|\tilde{V}(d)|$ and (b) $|\tilde{I}(d)|$ for a lossless transmission line of characteristic impedance $Z_0 = 50\ \Omega$, terminated in a load with a reflection coefficient $\Gamma = 0.3e^{j30°}$. The magnitude of the incident wave $|V_0^+| = 1$ V. The standing-wave ratio is $S = |\tilde{V}|_{max}/|\tilde{V}|_{min} = 1.3/0.7 = 1.86$.

$$|\tilde{V}(d)| = |V_0^+|[1 + |\Gamma|^2 + 2|\Gamma|\cos(2\beta d - \theta_r)]^{1/2}$$

$$|\tilde{I}(d)| = \frac{|V_0^+|}{Z_0}[1 + |\Gamma|^2 - 2|\Gamma|\cos(2\beta d - \theta_r)]^{1/2}$$

The variations of $|\tilde{V}(d)|$ and $|\tilde{I}(d)|$ as a function of d and the position on the line relative to the load (at $d = 0$), are illustrated in Fig. 2-14, for a line with $|V_0^+| = 1$ V, $|\Gamma| = 0.3$, $\theta_r = 30°$, and $Z_0 = 50\ \Omega$. The sinusoidal patterns are called *standing waves* and are caused by the *interference* of the two traveling waves.

Standing Waves



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$$|\tilde{V}(d)| = |V_0^+|[1 + |\Gamma|^2 + 2|\Gamma|\cos(2\beta d - \theta_r)]^{1/2}$$

$$|\tilde{I}(d)| = \frac{|V_0^+|}{Z_0}[1 + |\Gamma|^2 - 2|\Gamma|\cos(2\beta d - \theta_r)]^{1/2}$$

The *maximum value* of the *standing-wave pattern* of $|\tilde{V}(d)|$ corresponds to the position on the line at which the incident and reflected waves are *in-phase* $[2\beta d - \theta_r = 2n\pi$ in the expression of $|\tilde{V}(d)|]$ and therefore add constructively to give a value equal to $(1 + |\Gamma|)|V_0^+| = 1.3$ V.

Standing Waves

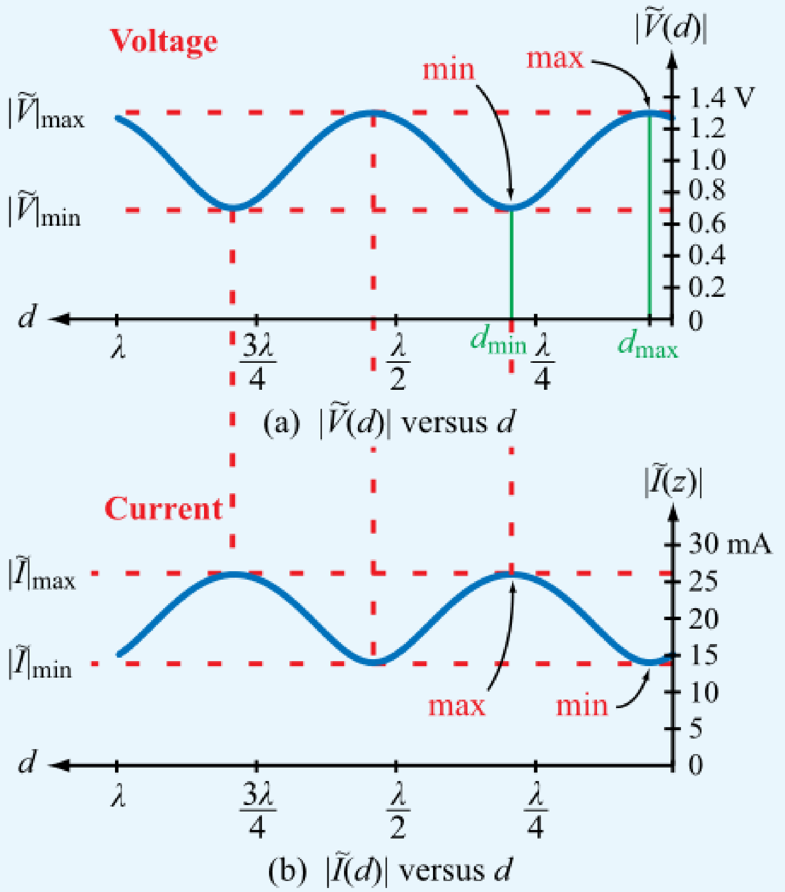

Figure 2-14 Standing-wave pattern for (a) $|\tilde{V}(d)|$ and (b) $|\tilde{I}(d)|$ for a lossless transmission line of characteristic impedance $Z_0 = 50\ \Omega$, terminated in a load with a reflection coefficient $\Gamma = 0.3e^{j30^\circ}$. The magnitude of the incident wave $|V_0^+| = 1$ V. The standing-wave ratio is $S = |\tilde{V}|_{max}/|\tilde{V}|_{min} = 1.3/0.7 = 1.86$.

$$|\tilde{V}(d)| = |V_0^+|[1 + |\Gamma|^2 + 2|\Gamma|\cos(2\beta d - \theta_r)]^{1/2}$$

$$|\tilde{I}(d)| = \frac{|V_0^+|}{Z_0}[1 + |\Gamma|^2 - 2|\Gamma|\cos(2\beta d - \theta_r)]^{1/2}$$

The *minimum value* of $|\tilde{V}(d)|$ occurs when the two waves interfere destructively, which occurs when the incident and reflected waves are in *phase-opposition* $[\,2\beta d - \theta_r = (2n+1)\pi\,]$. In this case, $|\tilde{V}(d)| = (1 - |\Gamma|)|V_0^+| = 0.7$ V.

Standing Waves

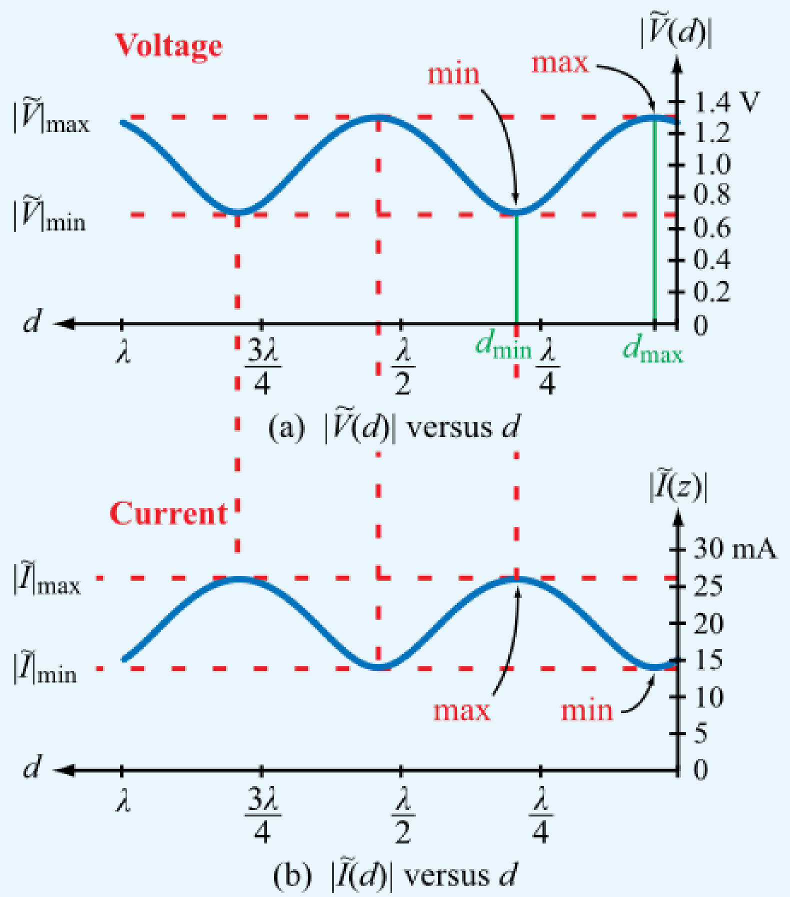

Figure 2-14 Standing-wave pattern for (a) $|\tilde{V}(d)|$ and (b) $|\tilde{I}(d)|$ for a lossless transmission line of characteristic impedance $Z_0 = 50\ \Omega$, terminated in a load with a reflection coefficient $\Gamma = 0.3e^{j30^\circ}$. The magnitude of the incident wave $|V_0^+| = 1$ V. The standing-wave ratio is $S = |\tilde{V}|_{max}/|\tilde{V}|_{min} = 1.3/0.7 = 1.86$.

► Whereas the repetition period is λ for the incident and reflected waves considered individually, the repetition period of the standing-wave pattern is $\lambda/2$. The standing-wave pattern describes the spatial variation of the magnitude of $|\tilde{V}(d)|$ as a function of d.

► If one were to observe the variation of the instantaneous voltage as a function of time at location $d = d_{\text{max}}$ in Fig. 2-14, that variation would be as $\cos\omega t$ and would have an amplitude equal to $1.3\ \text{V}$ [i.e., $v(t)$ would oscillate between $-1.3\ \text{V}$ and $+1.3\ \text{V}$].

Standing Waves

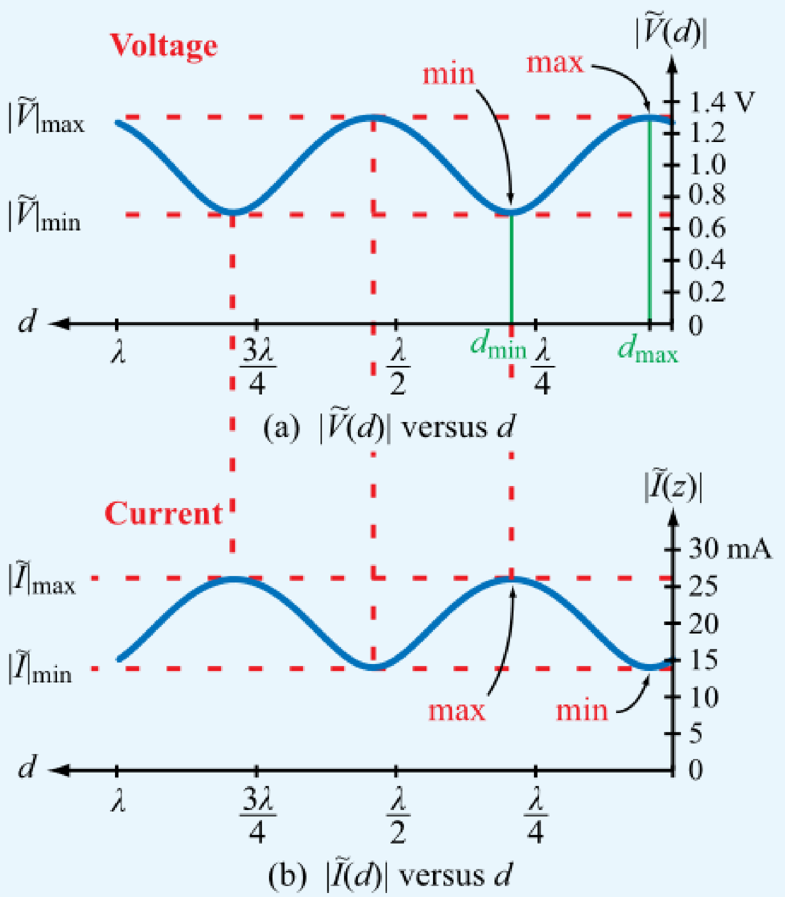

Figure 2-14 Standing-wave pattern for (a) $|\tilde{V}(d)|$ and (b) $|\tilde{I}(d)|$ for a lossless transmission line of characteristic impedance $Z_0 = 50\ \Omega$, terminated in a load with a reflection coefficient $\Gamma = 0.3e^{j30°}$. The magnitude of the incident wave $|V_0^+| = 1$ V. The standing-wave ratio is $S = |\tilde{V}|_{max}/|\tilde{V}|_{min} = 1.3/0.7 = 1.86$.

Similarly, the instantaneous voltage $v(d,t)$ at any location d will be sinusoidal with amplitude equal to $|\tilde{V}(d)|$ at that d.

$$|\tilde{V}(d)| = |V_0^+|[1 + |\Gamma|^2 + 2|\Gamma|\cos(2\beta d - \theta_r)]^{1/2}$$

$$|\tilde{I}(d)| = \frac{|V_0^+|}{Z_0}[1 + |\Gamma|^2 - 2|\Gamma|\cos(2\beta d - \theta_r)]^{1/2}$$

Close inspection of the voltage and current standing-wave patterns shown in Fig. 2-14 reveals that the two patterns are in phase opposition (when one is at a maximum, the other is at a minimum, and vice versa).

Standing Waves

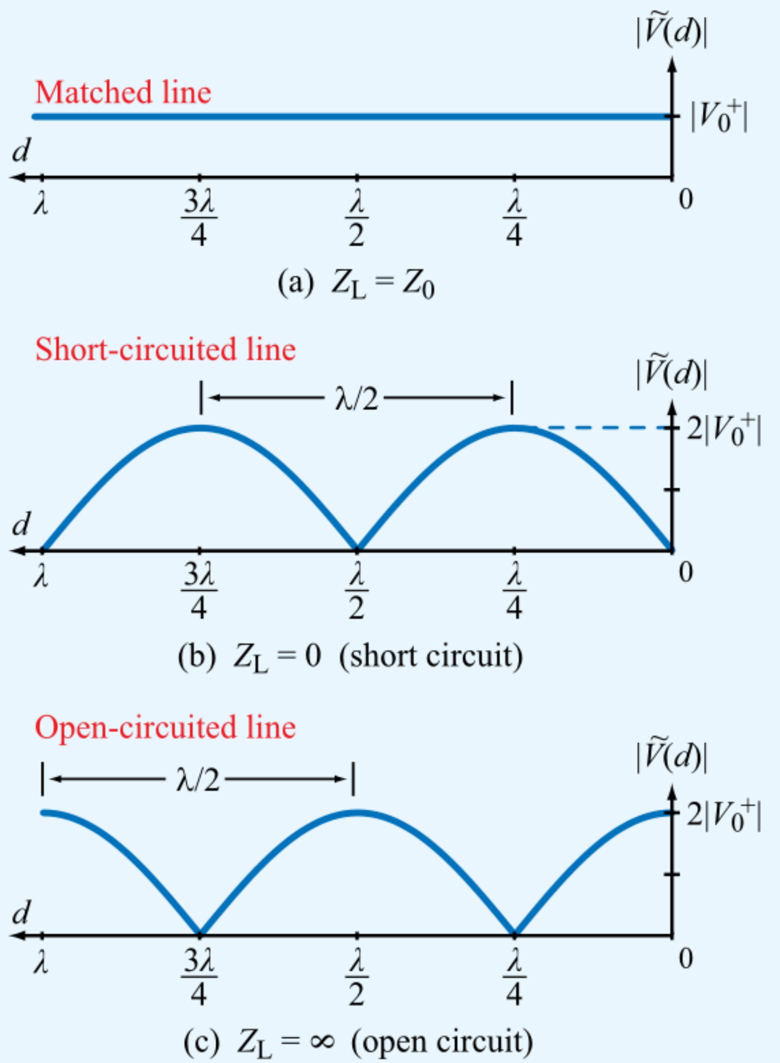

Figure 2-15 Voltage standing-wave patterns for (a) a matched load, (b) a short-circuited line, and (c) an open-circuited line.

$$|\tilde{V}(d)| = |V_0^+|[1 + |\Gamma|^2 + 2|\Gamma|\cos(2\beta d - \theta_r)]^{1/2}$$

$$|\tilde{I}(d)| = \frac{|V_0^+|}{Z_0}[1 + |\Gamma|^2 - 2|\Gamma|\cos(2\beta d - \theta_r)]^{1/2}$$

The standing-wave patterns shown in Fig. 2-14 are for $\Gamma = 0.3e^{j30°}$. The peak-to-peak variation of the pattern ($|\tilde{V}|_{\min}$ to $|\tilde{V}|_{\max}$) depends on $|\Gamma|$, which in general can vary between 0 and 1. For the special case of a matched line with $Z_L = Z_0$, we have $|\Gamma| = 0$ and $|\tilde{V}(d)| = |V_0^+|$ for all values of d, as shown in Fig. 2-15(a).

Standing Waves

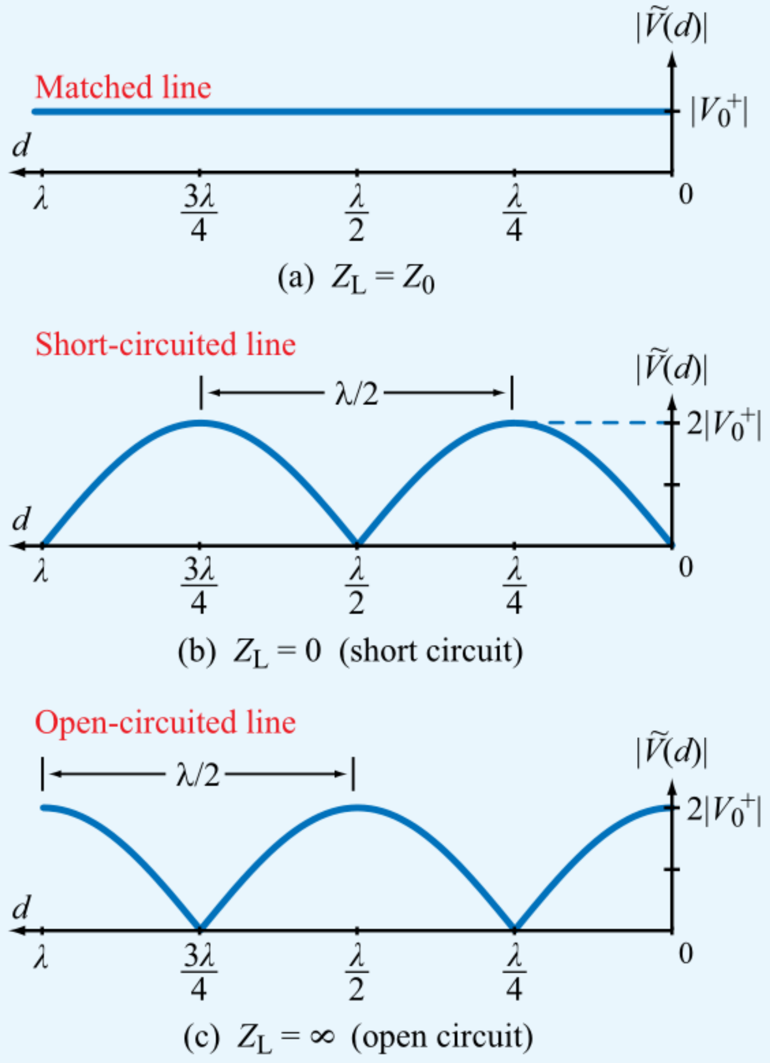

Figure 2-15 Voltage standing-wave patterns for (a) a matched load, (b) a short-circuited line, and (c) an open-circuited line.

$$|\tilde{V}(d)| = |V_0^+|[1 + |\Gamma|^2 + 2|\Gamma|\cos(2\beta d - \theta_r)]^{1/2}$$

$$|\tilde{I}(d)| = \frac{|V_0^+|}{Z_0}[1 + |\Gamma|^2 - 2|\Gamma|\cos(2\beta d - \theta_r)]^{1/2}$$

► With no reflected wave present, there are no interference and no standing waves.

► The other case $|\Gamma| = 1$, corresponds to when the load is a short circuit ($\Gamma = -1$) or an open circuit ($\Gamma = 1$). The standing-wave patterns for those two cases are shown in Figs. 2-15(b) and (c); both exhibit maxima of $2|V_0^+|$ and minima equal to zero, but the two patterns are spatially shifted relative to each other by a distance of $\lambda/4$.

Standing Waves

A purely reactive load (capacitor or inductor) also satisfies the condition $|\Gamma| = 1$, but θ_r is generally neither zero nor 180° (Table 2-3). Exercise 2.9 examines the standing-wave pattern for a lossless line terminated in an inductor.

Table 2-3 Magnitude and phase of the reflection coefficient for various types of load. The normalized load impedance $z_L = Z_L/Z_0 = (R + jX)/Z_0 = r + jx$, where $r = R/Z_0$ and $x = X/Z_0$ are the real and imaginary parts of z_L, respectively.

Reflection Coefficient $\Gamma = |\Gamma| e^{j\theta_r}$

Load	$\|\Gamma\|$	θ_r
Z_0 $Z_L = (r + jx)Z_0$	$\left[\frac{(r-1)^2 + x^2}{(r+1)^2 + x^2}\right]^{1/2}$	$\tan^{-1}\left(\frac{x}{r-1}\right) - \tan^{-1}\left(\frac{x}{r+1}\right)$
Z_0 Z_0	0 (no reflection)	irrelevant
Z_0 (short)	1	$\pm 180°$ (phase opposition)
Z_0 (open)	1	0 (in-phase)
Z_0 $jX = j\omega L$	1	$\pm 180° - 2\tan^{-1} x$
Z_0 $jX = \frac{-j}{\omega C}$	1	$\pm 180° + 2\tan^{-1} x$

Standing Waves

Now let us examine the maximum and minimum values of the voltage magnitude. From $|\tilde{V}(d)| = |V_0^+|[1 + |\Gamma|^2 + 2|\Gamma|\cos(2\beta d - \theta_r)]^{1/2}$, $|\tilde{V}(d)|$ is a maximum when the argument of the cosine function is equal to zero or a multiple of 2π. Let us denote $d_{\max}$ as the distance from the load at which $|\tilde{V}(d)|$ is a maximum. It then follows that

$$\left|\tilde{V}(d)\right| = \left|\tilde{V}\right|_{\max} = |V_0^+|[1 + |\Gamma|]$$

when $2\beta d_{\max} - \theta_r = 2n\pi$, with $n = 0$ or a positive integer.

Solving $2\beta d_{\max} - \theta_r = 2n\pi$ for $d_{\max}$, we have

$$d_{\max} = \frac{\theta_r + 2n\pi}{2\beta} = \frac{\theta_r \lambda}{4\pi} + \frac{n\lambda}{2}, \qquad \begin{cases} n = 1,2,\ldots & \text{if } \theta_r < 0 \\ n = 0,1,2,\ldots & \text{if } \theta_r \geq 0 \end{cases}$$

where we have used $\beta = 2\pi/\lambda$.

Standing Waves

The phase angle of the voltage reflection coefficient, θ_{r}, is bounded between $-\pi$ and π radians. If $\theta_{\mathrm{r}} \geq 0$, the *first voltage maximum* occurs at $d_{\max} = \theta_{\mathrm{r}}\lambda/4\pi$, corresponding to $n = 0$, but if $\theta_{\mathrm{r}} < 0$, the first physically meaningful maximum occurs at $d_{\max} = (\theta_{\mathrm{r}}\lambda/4\pi) + \lambda/2$, corresponding to $n = 1$. Negative values of $d_{\max}$ correspond to locations past the end of the line and therefore have no physical significance.

Similarly, the minima of $|\tilde{V}(d)|$ occur at distances $d_{\min}$ for which the argument of the cosine function in $|\tilde{V}(d)| = |V_0^+|[1 + |\Gamma|^2 + 2|\Gamma|\cos(2\beta d - \theta_{\mathrm{r}})]^{1/2}$ is equal to $(2n + 1)\pi$, which gives the result

$$|\tilde{V}|_{\min} = |V_0^+|[1 - |\Gamma|]$$

when $(2\beta d_{\min} - \theta_{\mathrm{r}}) = (2n + 1)\pi$, with $-\pi \leq \theta_{\mathrm{r}} \leq \pi$.

Standing Waves

The first minimum corresponds to $n = 0$. The spacing between a maximum $d_{\max}$ and the adjacent minimum $d_{\min}$ is $\lambda/4$. Hence, the *first minimum* occurs at

$$\begin{cases} d_{\max} + \lambda/4, & \text{if } d_{\max} < \lambda/4 \\ d_{\max} - \lambda/4, & \text{if } d_{\max} \geq \lambda/4 \end{cases}$$

► The locations on the line corresponding to voltage maxima correspond to current minima, and vice versa.

The ratio of $|\tilde{V}|_{\max}$ to $|\tilde{V}|_{\min}$ is called the *voltage standing-wave ratio* S, which is given by

$$S = \frac{|\tilde{V}|_{\max}}{|\tilde{V}|_{\min}} = \frac{1 + |\Gamma|}{1 - |\Gamma|} \quad \text{(dimensionless)}$$

Standing Waves

voltage standing-wave ratio S

$$S = \frac{|\tilde{V}|_{\max}}{|\tilde{V}|_{\min}} = \frac{1 + |\Gamma|}{1 - |\Gamma|} \quad \text{(dimensionless)}$$

This quantity, which often is referred to by its acronym, VSWR, or the shorter acronym *SWR*, provides a measure of the mismatch between the load and the transmission line; for a matched load with $\Gamma = 0$, we get $S = 1$, and for a line with $|\Gamma| = 1$, $S = \infty$.

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B39HF High Frequency Circuits

Lecture 7
Wave Impedance of the Lossless Line

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The standing-wave patterns indicate that on a mismatched line the voltage and current magnitudes are oscillatory with position along the line and in phase opposition with each other. Hence, the voltage to current ratio, called the *wave impedance* $Z(d)$, must vary with position also. Using $\tilde{V}(z) = V_0^+\left(e^{-j\beta z} + \Gamma e^{j\beta z}\right)$ and $\tilde{I}(z) = \frac{V_0^+}{Z_0}\left(e^{-j\beta z} - \Gamma e^{j\beta z}\right)$ with $z = -d$,

$$Z(d) = \frac{\tilde{V}(d)}{\tilde{I}(d)}$$

$$= \frac{V_0^+\left[e^{j\beta d} + \Gamma e^{-j\beta d}\right]}{V_0^+\left[e^{j\beta d} - \Gamma e^{-j\beta d}\right]} Z_0$$

$$= Z_0\left[\frac{1 + \Gamma e^{-j2\beta d}}{1 - \Gamma e^{-j2\beta d}}\right] = Z_0\left[\frac{1 + \Gamma_d}{1 - \Gamma_d}\right] \quad \Omega$$

$$Z(d) = \frac{\tilde{V}(d)}{\tilde{I}(d)} = Z_0 \left[\frac{1+\Gamma_d}{1-\Gamma_d}\right] \quad \Omega$$

where we define

$$\Gamma_d = \Gamma e^{-j2\beta d} = |\Gamma| e^{j\theta_{\mathrm{r}}} e^{-j2\beta d} = |\Gamma| e^{j(\theta_{\mathrm{r}}-2\beta d)}$$

as the *phase-shifted voltage reflection coefficient*, meaning that Γ_d has the same magnitude as Γ, but its phase is shifted by $2\beta d$ relative to that of Γ.

$Z(d)$ is the ratio of the total voltage (incident- and reflected-wave voltages) to the total current at any point d on the line, in contrast with the characteristic impedance of the line Z_0, which relates the voltage and current of each of the two waves individually ($Z_0 = V_0^+/I_0^+ = -V_0^-/I_0^-$).

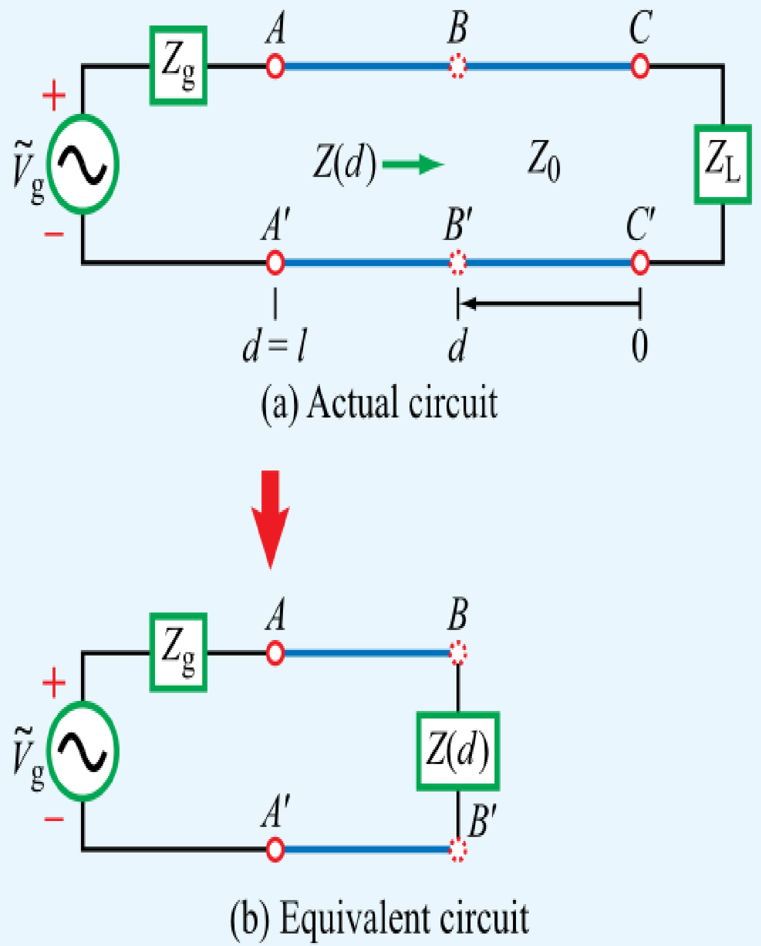

Figure 2-17 The segment to the right of terminals BB' can be replaced with a discrete impedance equal to the wave impedance $Z(d)$.

In the circuit of Fig. 2-17(a), at terminals BB' at an arbitrary location d on the line, $Z(d)$ is the wave impedance of the line when "looking" to the right (i.e., towards the load). Application of the equivalence principle allows us to replace the segment to the right of terminals BB' with a lumped impedance of value $Z(d)$, as depicted in Fig. 2-17(b). From the standpoint of the input circuit to the left of terminals BB', the two circuit configurations are electrically identical.

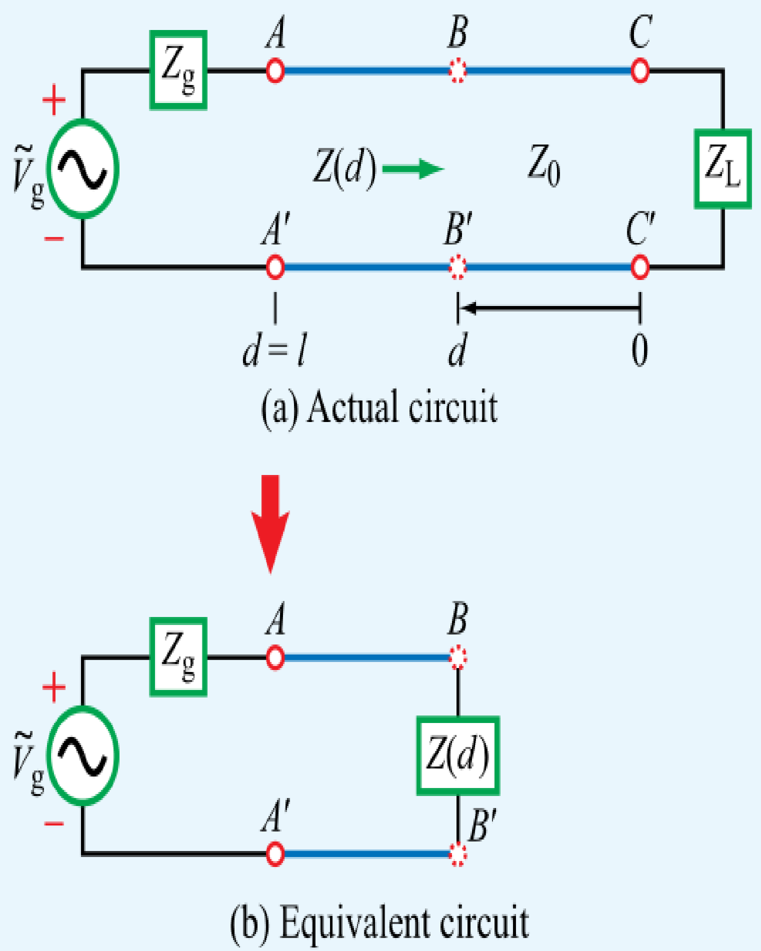

Figure 2-17 The segment to the right of terminals BB' can be replaced with a discrete impedance equal to the wave impedance $Z(d)$.

Of particular interest in many transmission-line problems is the input impedance at the source end of the line, at $d = l$, which is given by

$$Z_{\text{in}} = Z(l) = Z_0 \left[\frac{1+\Gamma_l}{1-\Gamma_l}\right]$$

with

$$\Gamma_l = \Gamma e^{-j2\beta l} = |\Gamma| e^{j(\theta_r - 2\beta l)}$$

By replacing Γ with

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{z_L - 1}{z_L + 1}$$

and using the relations

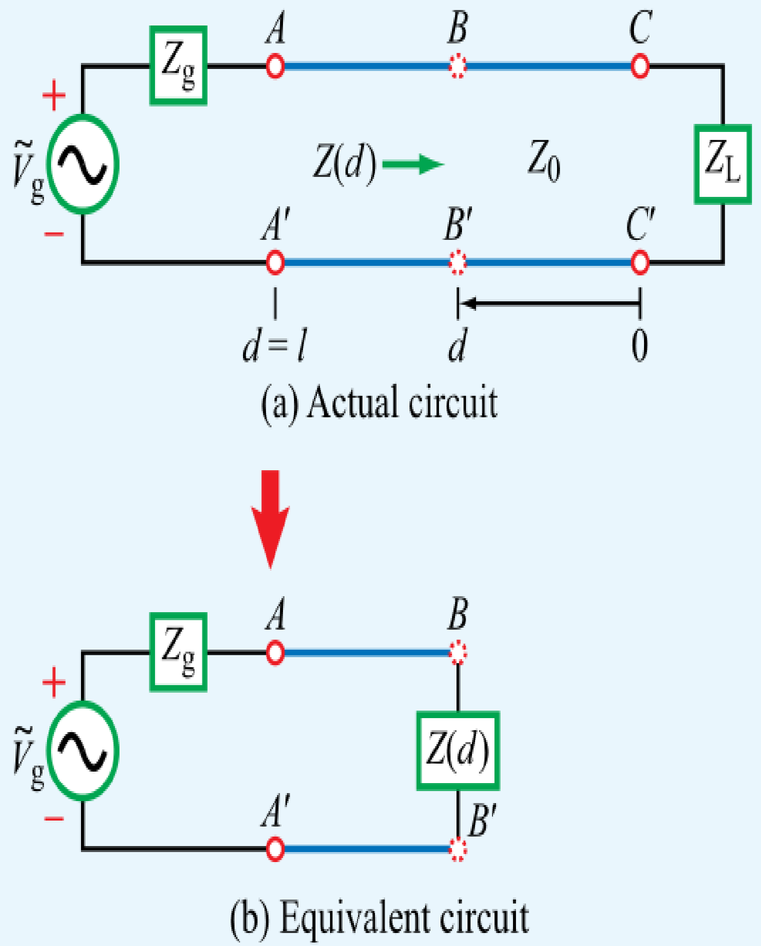

Figure 2-17 The segment to the right of terminals BB' can be replaced with a discrete impedance equal to the wave impedance $Z(d)$.

$$e^{j\beta l} = \cos\beta l + j\sin\beta l$$

$$e^{-j\beta l} = \cos\beta l - j\sin\beta l$$

The expression of Z_{in} can be written in terms of z_{L} as

$$Z_{\text{in}} = Z_0\left(\frac{z_{\text{L}}\cos\beta l + j\sin\beta l}{\cos\beta l + jz_{\text{L}}\sin\beta l}\right)$$

$$= Z_0\left(\frac{z_{\text{L}} + j\tan\beta l}{1 + jz_{\text{L}}\tan\beta l}\right)$$

$$Z(d) = Z_0\frac{\left[e^{j\beta d} + \Gamma e^{-j\beta d}\right]}{\left[e^{j\beta d} - \Gamma e^{-j\beta d}\right]} \qquad d = l$$

$$\Gamma = \frac{Z_{\text{L}} - Z_0}{Z_{\text{L}} + Z_0} = \frac{z_{\text{L}} - 1}{z_{\text{L}} + 1}$$

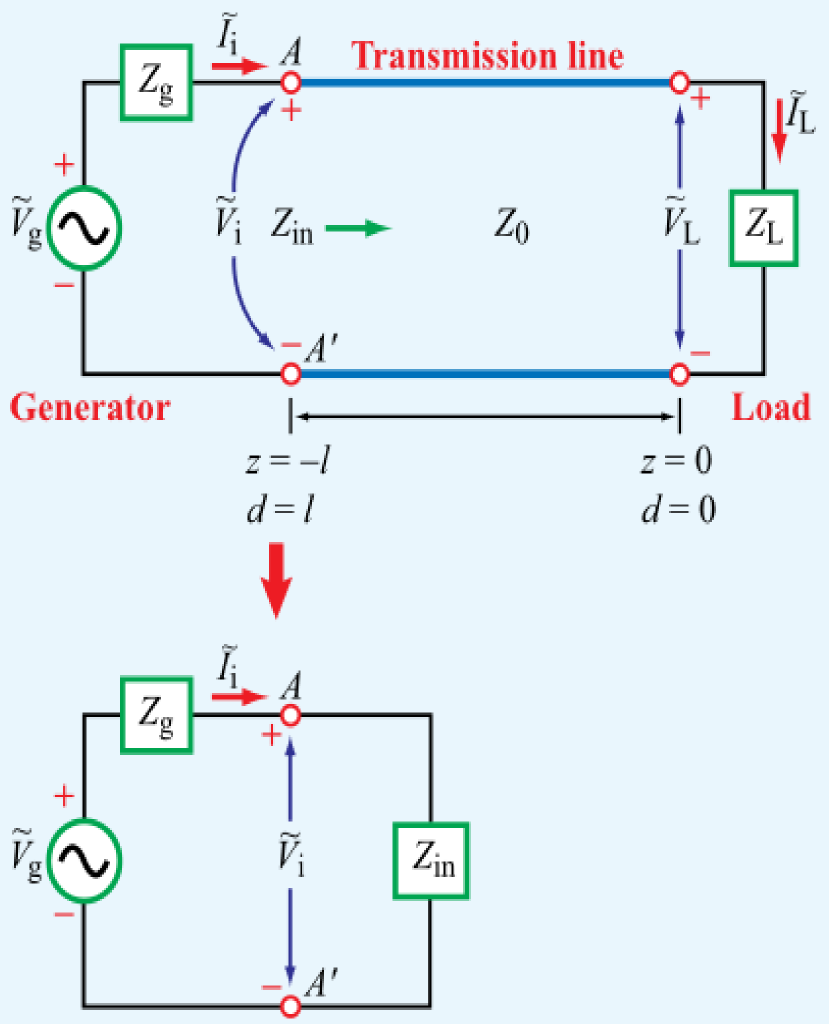

Figure 2-18 At the generator end, the terminated transmission line can be replaced with the input impedance of the line Z_{in}.

From the standpoint of the generator circuit, the transmission line can be replaced with an impedance Z_{in}, as shown in Fig. 2-18. The phasor voltage across Z_{in} is given by

$$\tilde{V}_i = \tilde{I}_i Z_{in} = \frac{\tilde{V}_g Z_{in}}{Z_g + Z_{in}}$$

Simultaneously, from the standpoint of the transmission line, the voltage across it at the input of the line is given by $\tilde{V}(z) = V_0^+\left(e^{-j\beta z} + \Gamma e^{j\beta z}\right)$ with $z = -l$:

$$\tilde{V}_i = \tilde{V}(-l) = V_0^+\left(e^{j\beta l} + \Gamma e^{-j\beta l}\right)$$

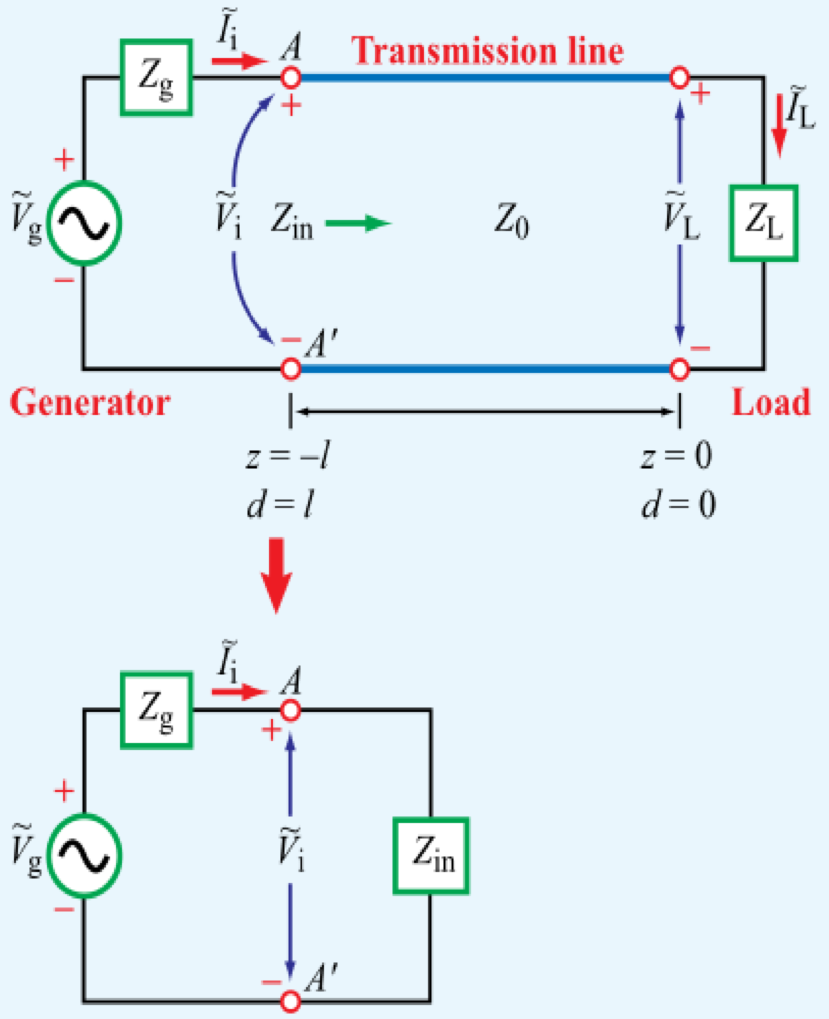

Figure 2-18 At the generator end, the terminated transmission line can be replaced with the input impedance of the line Z_{in}.

Combine the above two expressions about $\tilde{V}_i$ and then solving for V_0^+ leads to

$$V_0^+ = \left(\frac{\tilde{V}_g Z_{in}}{Z_g + Z_{in}}\right)\left(\frac{1}{e^{j\beta l} + \Gamma e^{-j\beta l}}\right)$$

This completes the solution of the transmission-line wave equations, given by

$\frac{d^2\tilde{V}(z)}{dz^2} - \gamma^2\tilde{V}(z) = 0$ and $\frac{d^2\tilde{I}(z)}{dz^2} - \gamma^2\tilde{I}(z) = 0$,

for the special case of a lossless transmission line.

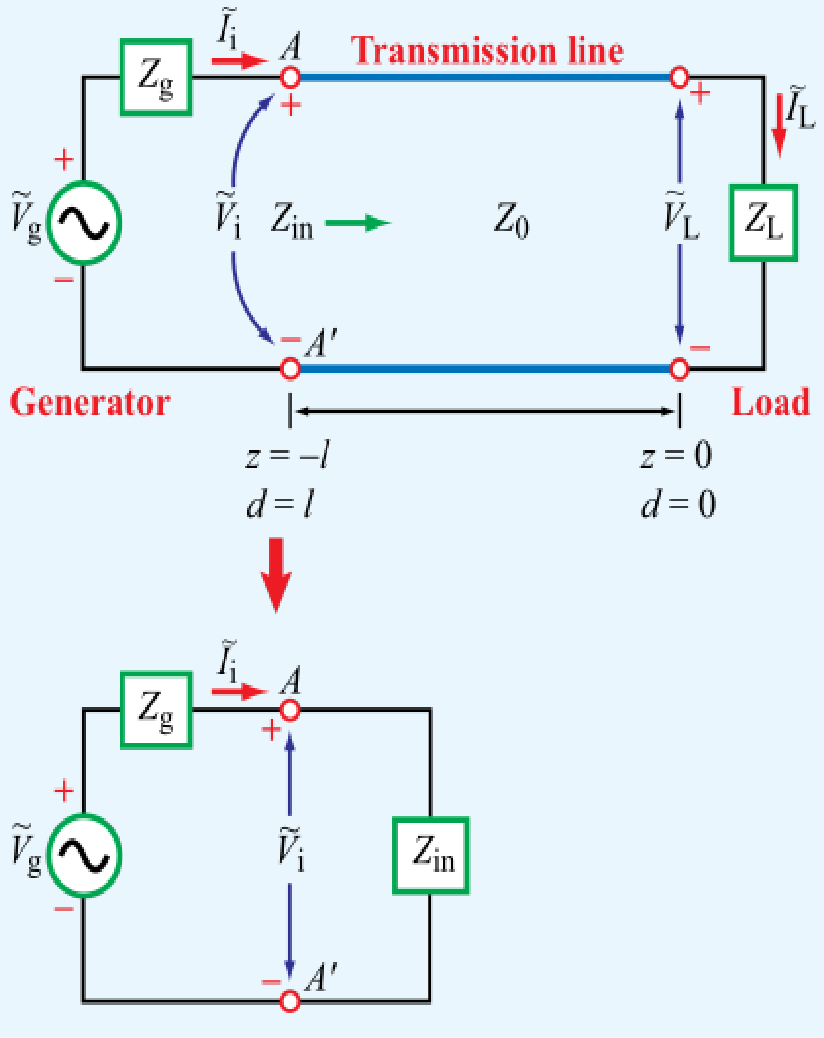

Figure 2-18 At the generator end, the terminated transmission line can be replaced with the input impedance of the line Z_{in}.

We started out with the general solutions given by $\tilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$ and $\tilde{I}(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}$, which included four unknown amplitudes, V_0^+, V_0^-, I_0^+, and I_0^-. We then determined that $Z_0 = V_0^+/I_0^+ = -V_0^-/I_0^-$, thereby reducing the unknowns to the two voltage amplitudes only. Upon applying the boundary condition at the load, we were able to relate V_0^- to V_0^+ through Γ, and, finally, by applying the boundary condition at the source, we obtained an expression for V_0^+.

Example 2-7: Complete Solution for $v(z,t)$ and $i(z,t)$

Q: A 1.05 GHz generator circuit with series impedance $Z_{\mathrm{g}} = 10\ \Omega$ and voltage source given by

$$v_{\mathrm{g}}(t) = 10\sin(\omega t + 30^\circ) \quad (\mathrm{V})$$

is connected to a load $Z_{\mathrm{L}} = (100 + j50)\,\Omega$ through a $50\,\Omega$, $67\ \mathrm{cm}$ long lossless transmission line. The phase velocity of the line is $0.7c$, where c is the velocity of light in a vacuum. Find $v(z,t)$ and $i(z,t)$ on the line.

Solution: From the relationship $u_p = \lambda f$, we find the wavelength

$$\lambda = \frac{u_p}{f} = \frac{0.7 \times 3 \times 10^8}{1.05 \times 10^9} = 0.2 \text{ m}$$

and

$$\beta l = \frac{2\pi}{\lambda} l = \frac{2\pi}{0.2} \times 0.67 = 6.7\pi = 0.7\pi = 126°$$

where we have subtracted multiples of 2π. The voltage reflection coefficient at the load is

$$\Gamma = \frac{Z_\mathrm{L} - Z_0}{Z_\mathrm{L} + Z_0} = \frac{(100 + j50) - 50}{(100 + j50) + 50} = 0.45e^{j26.6°}$$

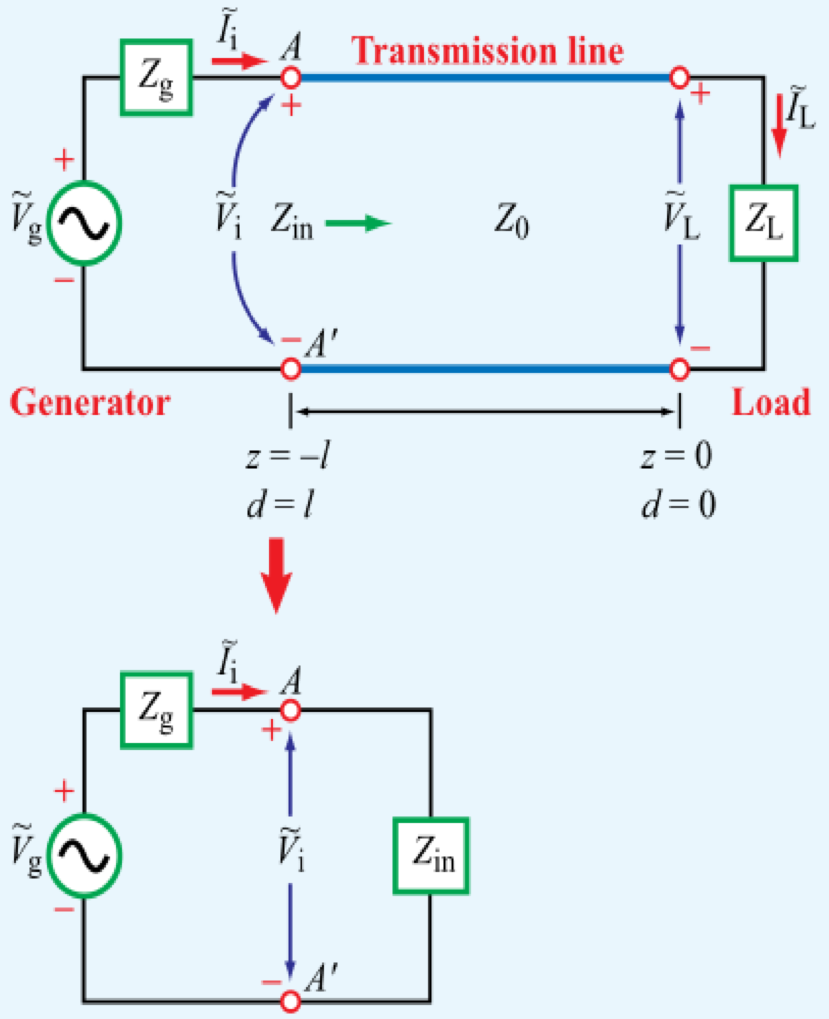

Figure 2-18 At the generator end, the terminated transmission line can be replaced with the input impedance of the line Z_{in}.

With reference to Fig. 2-18, the input impedance of the line, given by $Z_{in} = Z(l) = Z_0\left[\frac{1+\Gamma_l}{1-\Gamma_l}\right]$, is

$$Z_{in} = Z_0\left(\frac{1+\Gamma_l}{1-\Gamma_l}\right)$$

$$= Z_0\left(\frac{1+\Gamma e^{-j2\beta l}}{1-\Gamma e^{-j2\beta l}}\right)$$

$$= 50\left(\frac{1+0.45e^{j26.6^\circ}e^{-j252^\circ}}{1-0.45e^{j26.6^\circ}e^{-j252^\circ}}\right)$$

$$= (21.9 + j17.4)\ \Omega$$

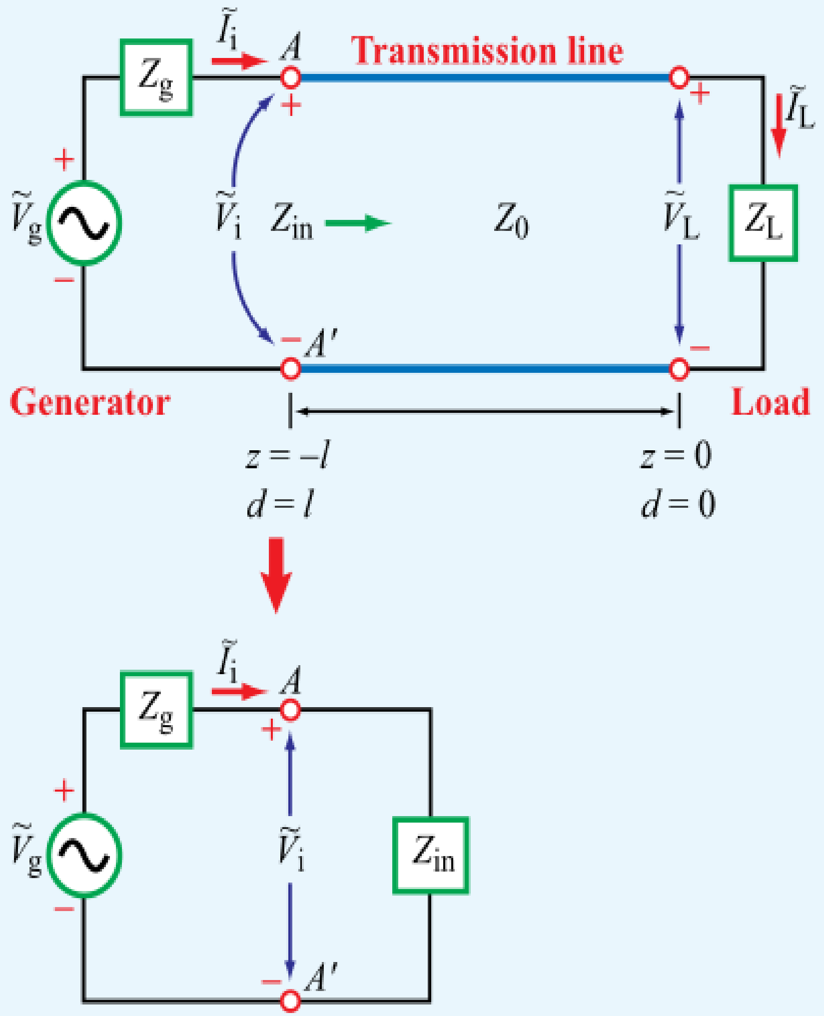

Figure 2-18 At the generator end, the terminated transmission line can be replaced with the input impedance of the line Z_{in}.

Rewriting the expression for the generator voltage with the cosine reference, we have

$$\begin{aligned} v_g(t) &= 10\sin(\omega t + 30°) \\ &= 10\cos(90° - \omega t - 30°) \\ &= 10\cos(\omega t - 60°) \\ &= \Re e\left[10e^{-j60°}e^{j\omega t}\right] \\ &= \Re e\left[\widetilde{V}_g e^{j\omega t}\right] \quad \text{(V)} \end{aligned}$$

Hence, the phasor voltage $\widetilde{V}_g$ is given by

$$\widetilde{V}_g = 10e^{-j60°} = 10\angle -60° \quad \text{(V)}$$

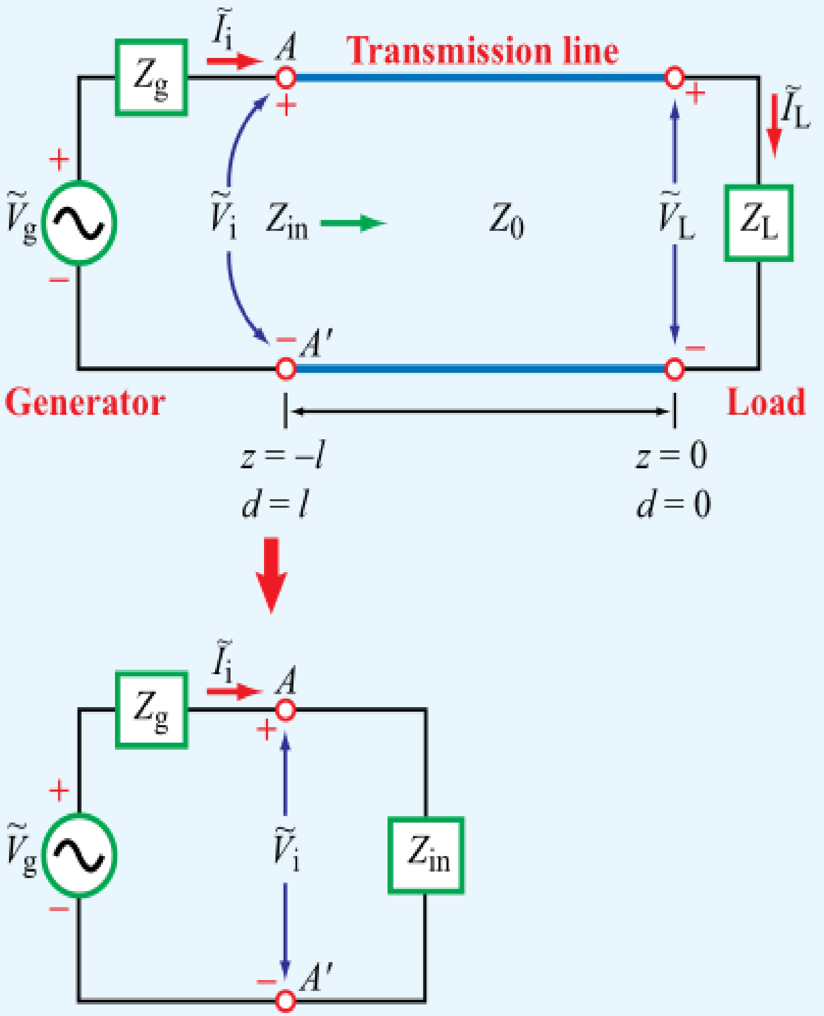

Figure 2-18 At the generator end, the terminated transmission line can be replaced with the input impedance of the line Z_{in}.

Application of $V_0^+ = \left(\frac{\tilde{V}_g Z_{in}}{Z_g + Z_{in}}\right)\left(\frac{1}{e^{j\beta l} + \Gamma e^{-j\beta l}}\right)$

gives

$$V_0^+ = \left(\frac{\tilde{V}_g Z_{in}}{Z_g + Z_{in}}\right)\left(\frac{1}{e^{j\beta l} + \Gamma e^{-j\beta l}}\right)$$

$$= \left[\frac{10e^{-j60^\circ}(21.9 + j17.4)}{10 + 21.9 + j17.4}\right]$$

$$\cdot \left(e^{j126^\circ} + 0.45e^{j26.6^\circ}e^{-j126^\circ}\right)^{-1}$$

$$= 10.2e^{j159^\circ} \ \text{(V)}$$

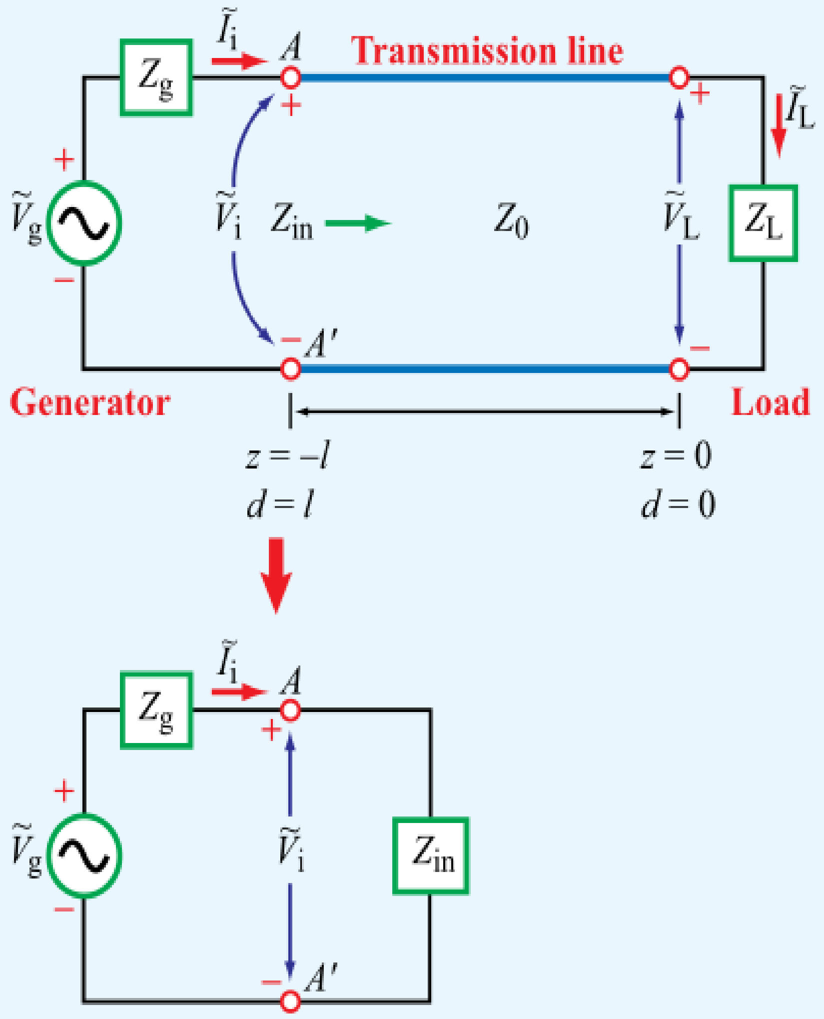

Figure 2-18 At the generator end, the terminated transmission line can be replaced with the input impedance of the line Z_{in}.

Using $\tilde{V}(z) = V_0^+\left(e^{-j\beta z} + \Gamma e^{j\beta z}\right)$ with $z = -d$, the phasor voltage on the line is

$$\tilde{V}(d) = V_0^+\left(e^{j\beta d} + \Gamma e^{-j\beta d}\right)$$

$$= 10.2e^{j159^\circ}\left(e^{j\beta d} + 0.45e^{j26.6^\circ}e^{-j\beta d}\right)$$

and the corresponding instantaneous voltage $v(d,t)$ is

$$v(d,t) = \Re e\left[\tilde{V}(d)e^{j\omega t}\right]$$

$$= 10.2\cos(\omega t + \beta d + 159^\circ)$$

$$+4.55\cos(\omega t - \beta d + 185.6^\circ)\ \ (\text{V})$$

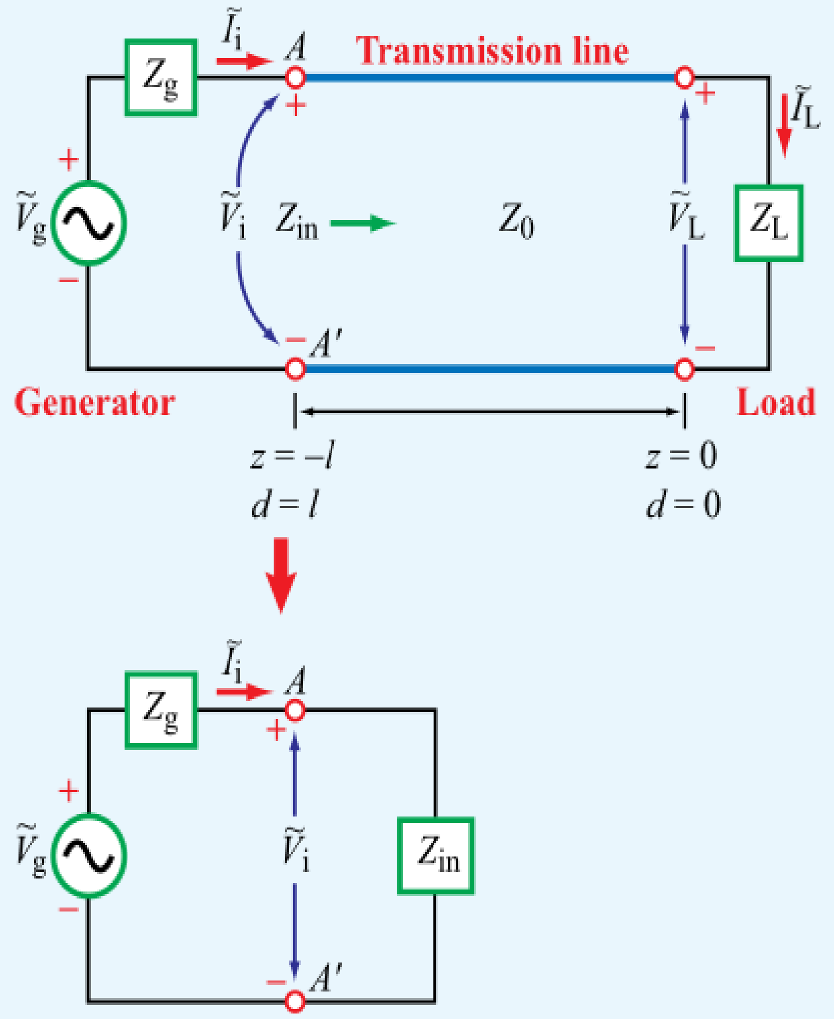

Figure 2-18 At the generator end, the terminated transmission line can be replaced with the input impedance of the line Z_{in}.

Similarly, $\tilde{I}(z) = \frac{V_0^+}{Z_0}\left(e^{-j\beta z} - \Gamma e^{j\beta z}\right)$ leads

to

$$\tilde{I}(d) = 0.2e^{j159°}\left(e^{j\beta d} - 0.45e^{j26.6°}e^{-j\beta d}\right)$$

$$i(d,t) = 0.2\cos(\omega t + \beta d + 159°)$$

$$+4.55\cos(\omega t - \beta d + 185.6°) \ \ \text{(A)}$$

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B39HF High Frequency Circuits

Lecture 8
Special Cases of the Lossless Line

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previous remarks

We often encounter situations involving lossless transmission lines with particular terminations or lines whose lengths lead to particularly useful line properties. We now consider some of these special cases.

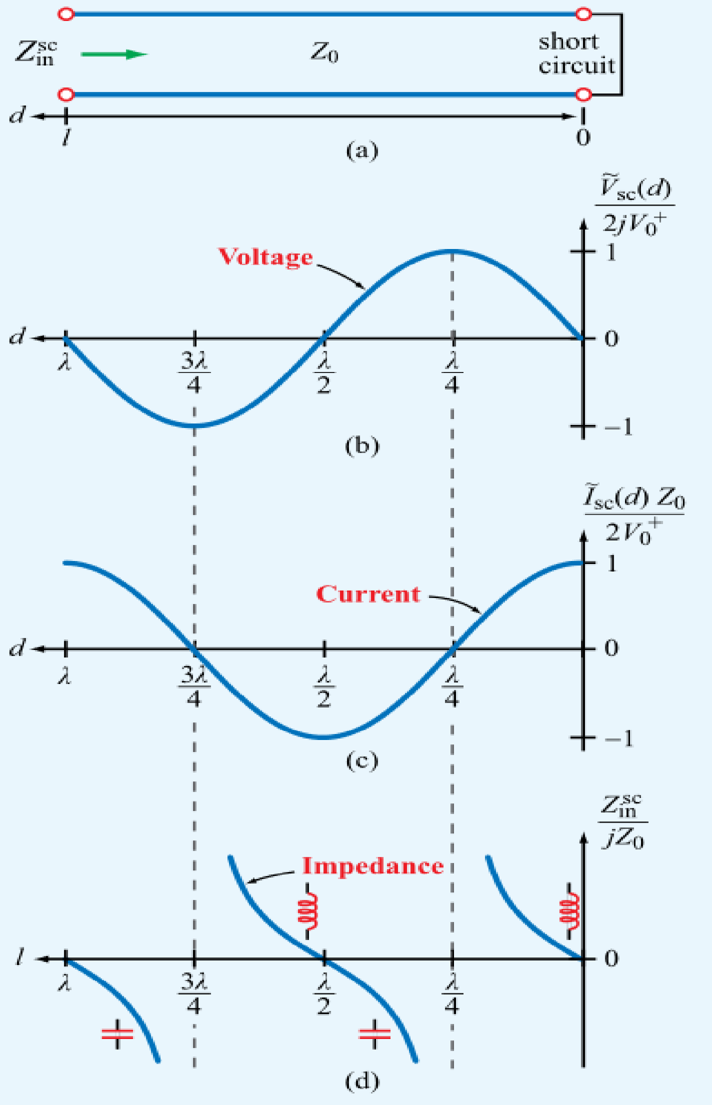


Figure 2-19 Transmission line terminated in a short circuit: (a) schematic representation, (b) normalized voltage on the line, (c) normalized current, and (d) normalized input impedance.

2.8.1 Short-Circuited Line

The transmission line shown in Fig. 2-19(a) is terminated in a short circuit, $Z_L = 0$. Consequently, the voltage reflection coefficient defined by $\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{z_L - 1}{z_L - 1}$ is $\Gamma = -1$, and the voltage standing-wave ratio given by $S = \frac{|\tilde{V}|_{max}}{|\tilde{V}|_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|}$ is $S = \infty$.

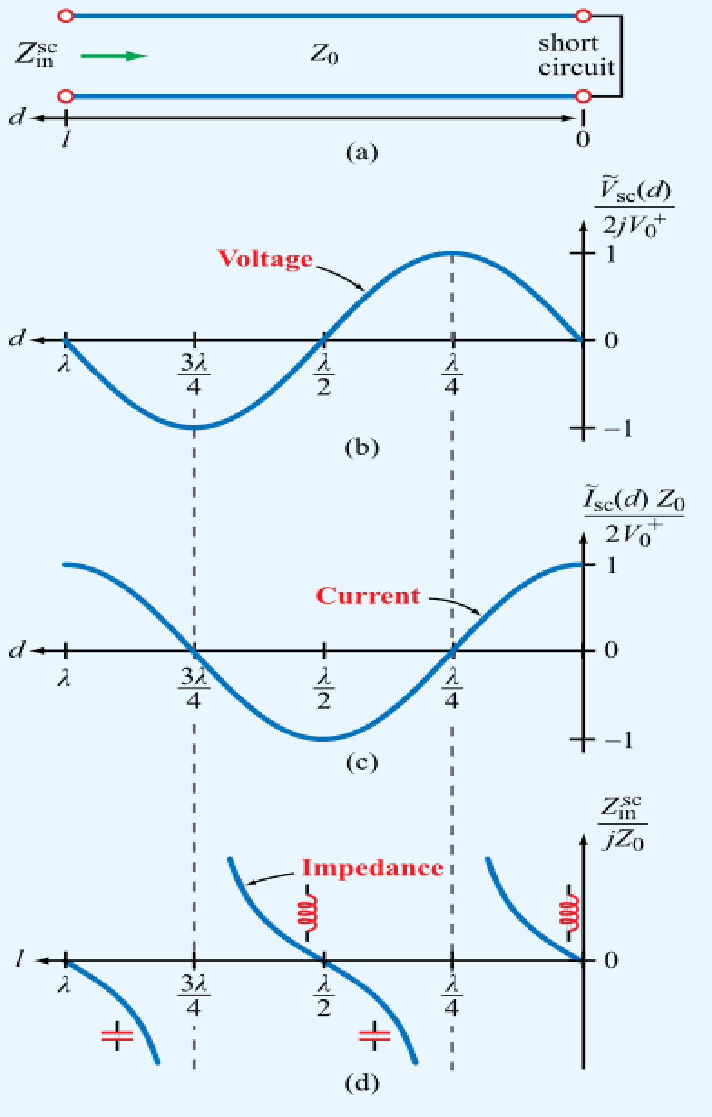


Figure 2-19 Transmission line terminated in a short circuit: (a) schematic representation, (b) normalized voltage on the line, (c) normalized current, and (d) normalized input impedance.

With $z = -d$ and $\Gamma = -1$ in $\tilde{V}(z) = V_0^+\left(e^{-j\beta z} + \Gamma e^{j\beta z}\right)$ and $\tilde{I}(z) = \frac{V_0^+}{Z_0}\left(e^{-j\beta z} - \Gamma e^{j\beta z}\right)$,

and $\Gamma = -1$ in $Z(d) = \frac{\tilde{V}(d)}{\tilde{I}(d)} = Z_0\left[\frac{1+\Gamma_d}{1-\Gamma_d}\right]$, the voltage, current, and wave impedance on a short-circuited lossless transmission line are given by

$$\tilde{V}_{sc}(d) = V_0^+\left[e^{j\beta d} - e^{-j\beta d}\right] = 2jV_0^+\sin\beta d$$

$$\tilde{I}_{sc}(d) = \frac{V_0^+}{Z_0}\left[e^{j\beta d} + e^{-j\beta d}\right] = \frac{2V_0^+}{Z_0}\cos\beta d$$

$$Z_{sc}(d) = \frac{\tilde{V}_{sc}(d)}{\tilde{I}_{sc}(d)} = jZ_0\tan\beta d$$

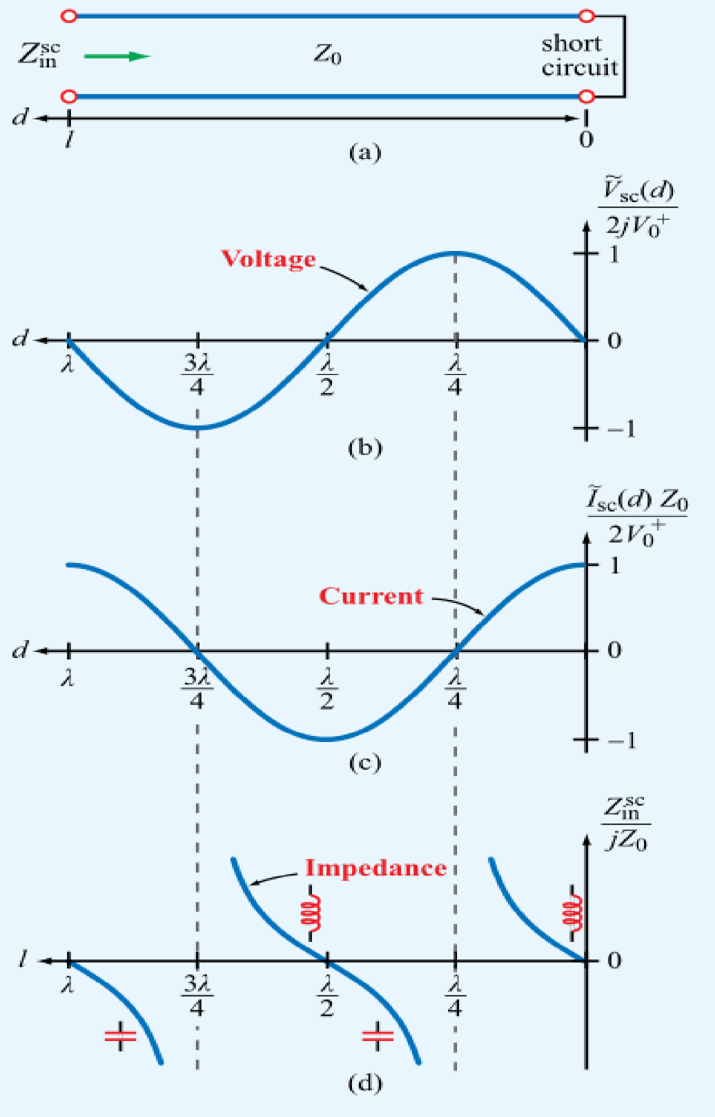


Figure 2-19 Transmission line terminated in a short circuit: (a) schematic representation, (b) normalized voltage on the line, (c) normalized current, and (d) normalized input impedance.

The voltage $\tilde{V}_{sc}(d)$ is zero at the load ($d = 0$), as it should be for a short circuit, and its amplitude varies as $\sin\beta d$. In contrast, the current $\tilde{I}_{sc}(d)$ is a maximum at the load and it varies as $\cos\beta d$. Both quantities are displayed in Fig. 2-19 as a function of d.

Denoting Z_{in}^{sc} as the input impedance of a short-circuited line of length d,

$$Z_{in}^{sc} = \frac{\tilde{V}_{sc}(l)}{\tilde{I}_{sc}(l)} = jZ_0 \tan\beta l$$

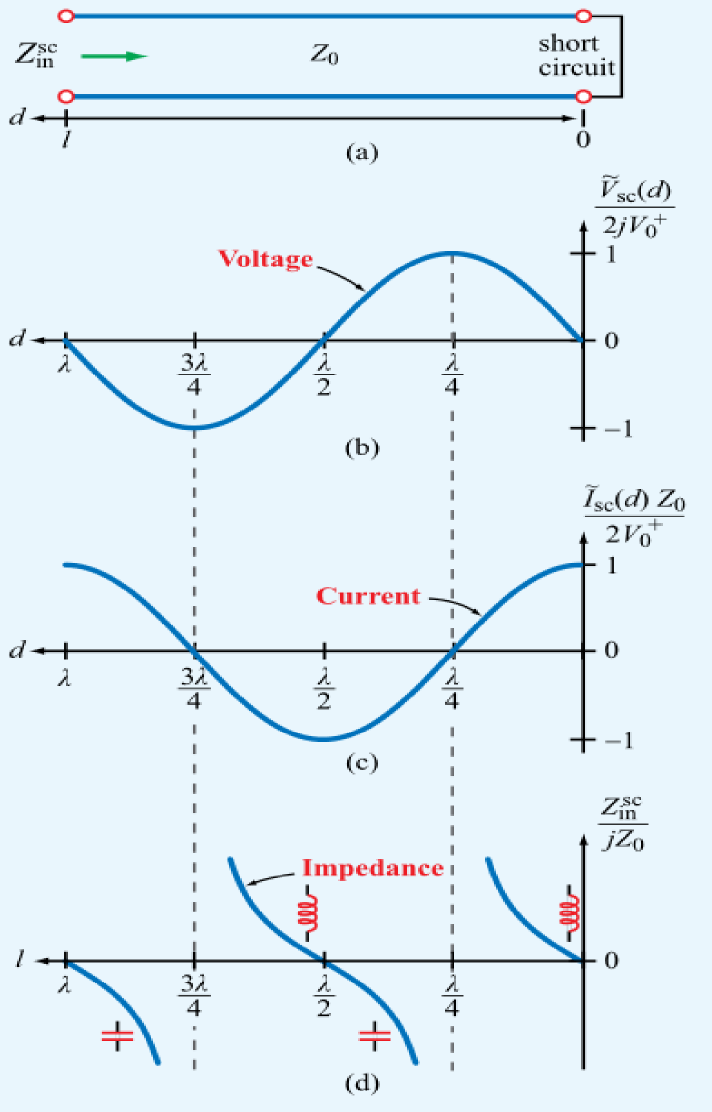


Figure 2-19 Transmission line terminated in a short circuit: (a) schematic representation, (b) normalized voltage on the line, (c) normalized current, and (d) normalized input impedance.

A plot of $Z_{\text{in}}^{\text{sc}}/jZ_0$ versus l is shown in Fig. 2-19(d). For the short-circuited line, if its length is less than $\lambda/4$, its impedance is equivalent to that of an inductor, and if it is between $\lambda/4$ and $\lambda/2$, it is equivalent to that of a capacitor.

In general, the input impedance Z_{in} of a line terminated in an arbitrary load has a real part, called the *input resistance* R_{in}, and an imaginary part, called the *input reactance* X_{in}:

$$Z_{\text{in}} = R_{\text{in}} + jX_{\text{in}}$$

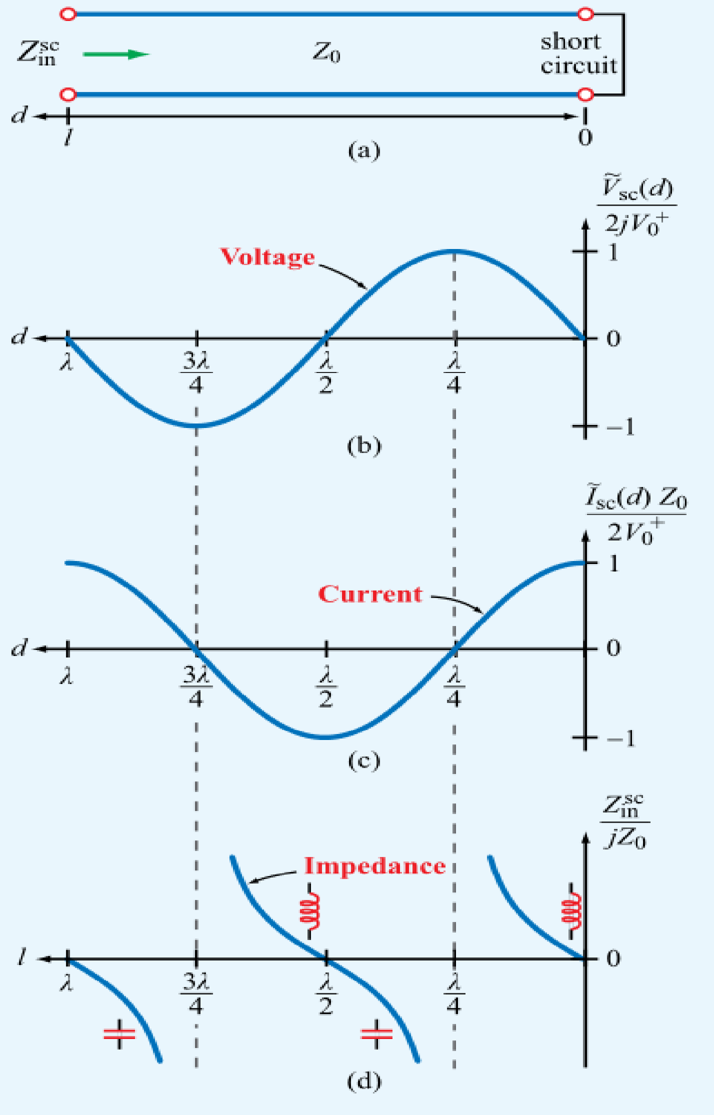


Figure 2-19 Transmission line terminated in a short circuit: (a) schematic representation, (b) normalized voltage on the line, (c) normalized current, and (d) normalized input impedance.

In the case of the short-circuited lossless line, the input impedance is purely reactive ($R_{\text{in}} = 0$). If $\tan\beta l \geq 0$, the line appears inductive to the source, acting like an equivalent inductor L_{eq} whose impedance equals $Z_{\text{in}}^{\text{sc}}$. Thus,

$$j\omega L_{\text{eq}} = jZ_0 \tan\beta l, \qquad \text{if } \tan\beta l \geq 0$$

or

$$L_{\text{eq}} = \frac{Z_0 \tan\beta l}{\omega} \quad (\text{H})$$

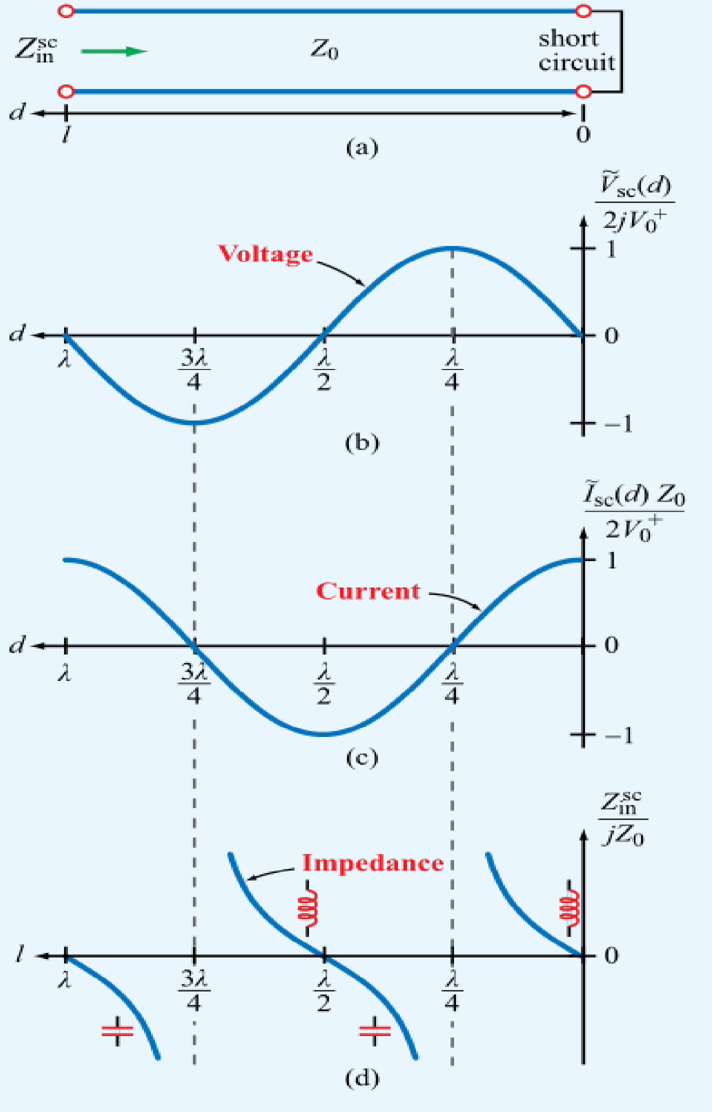


Figure 2-19 Transmission line terminated in a short circuit: (a) schematic representation, (b) normalized voltage on the line, (c) normalized current, and (d) normalized input impedance.

The minimum line length l that would result in an input impedance Z_{in}^{sc} equivalent to that of an inductor with inductance L_{eq} is

$$l = \frac{1}{\beta}\tan^{-1}\left(\frac{\omega L_{\text{eq}}}{Z_0}\right) \qquad \text{(m)}$$

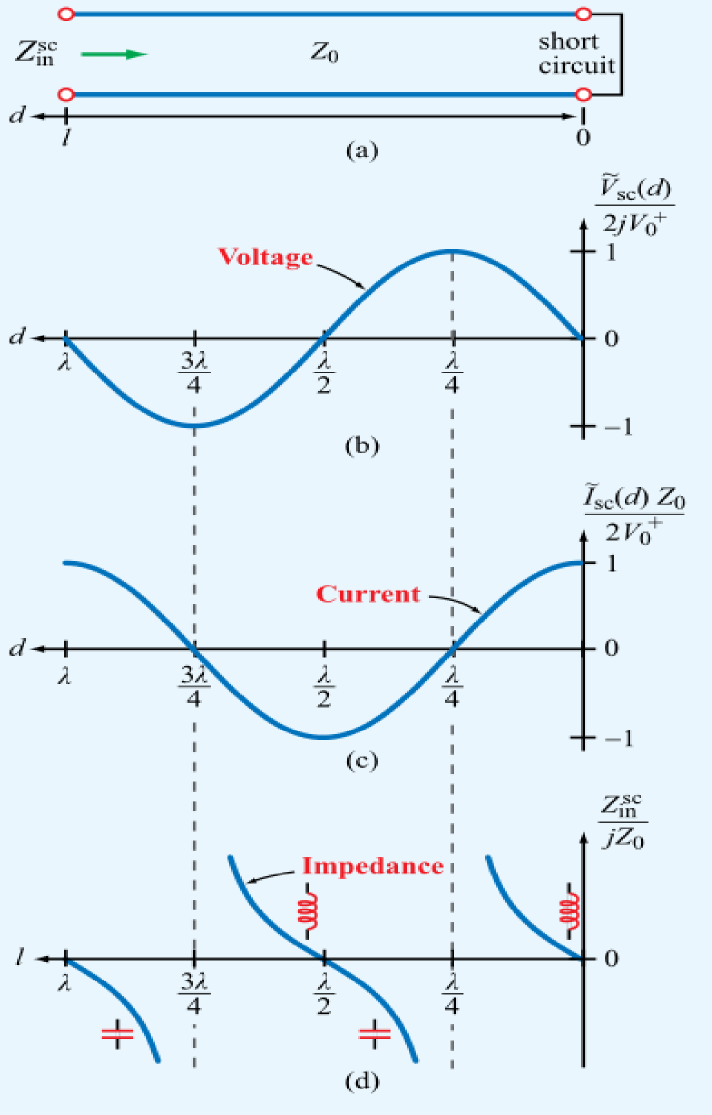


Figure 2-19 Transmission line terminated in a short circuit: (a) schematic representation, (b) normalized voltage on the line, (c) normalized current, and (d) normalized input impedance.

Similarly, if $\tan\beta l \le 0$, the input impedance is capacitive, in which case the line acts like an equivalent capacitor with capacitance C_{eq} such that

$$\frac{1}{j\omega C_{\text{eq}}} = jZ_0\tan\beta l, \qquad \text{if } \tan\beta l \le 0$$

or

$$C_{\text{eq}} = -\frac{1}{Z_0\omega\tan\beta l} \qquad \text{(F)}$$

Since l is a positive number, the shortest length l for which $\tan\beta l \leq 0$ corresponds to the range $\pi/2 \leq \beta l \leq \pi$. Hence, the minimum line length l that would result in an input impedance $Z_{\text{in}}^{\text{sc}}$ equivalent to that of a capacitor of capacitance C_{eq} is

$$l = \frac{1}{\beta}\left[\pi - \tan^{-1}\left(\frac{1}{\omega C_{\text{eq}} Z_0}\right)\right] \qquad \text{(m)}$$

- These results imply that, through proper choice of the length of a short-circuited line, we can make them into equivalent capacitors and inductors of any desired reactance.

Such a practice is indeed common in the design of microwave circuits and high-speed integrated circuits, because making an actual capacitor or inductor often is much more difficult than fabricating a shorted microstrip transmission line on a circuit board.

Example 2-8: Equivalent Reactive Elements

Q: Choose the length of a shorted $50\ \Omega$ lossless transmission line (Fig. 2-20) such that its input impedance at $2.25\ \mathrm{GHz}$ is identical to that of a capacitor with capacitance $C_{\mathrm{eq}} = 4\ \mathrm{pF}$. The wave velocity on the line is $0.75c$.

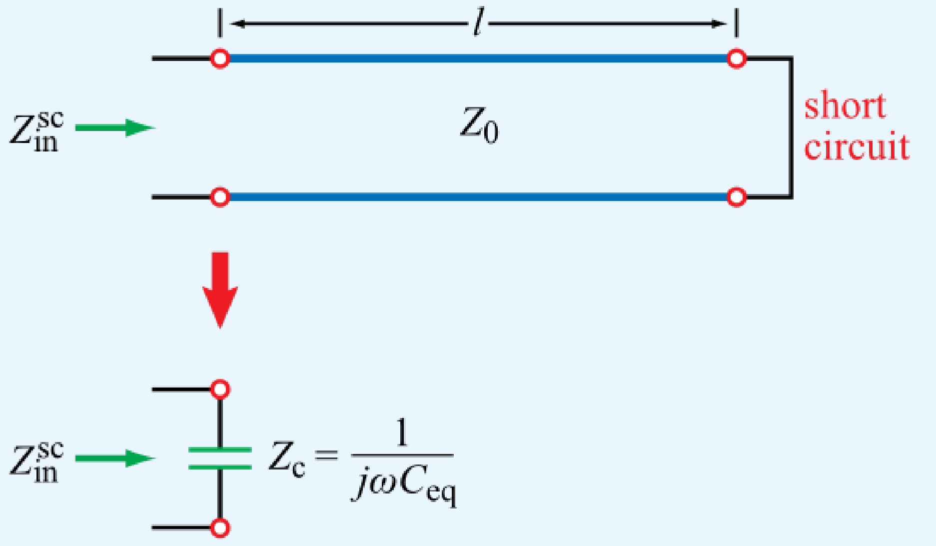


Figure 2-20 Shorted line as equivalent capacitor (Example 2-8).

Solution:

$$u_p = 0.75c = 0.75 \times 3 \times 10^8 = 2.25 \times 10^8 \ \text{m/s}$$

$$Z_0 = 50\ \Omega \qquad f = 2.25\ \text{GHz} = 2.25 \times 10^9\ \text{Hz} \qquad C_{\text{eq}} = 4\ \text{pF} = 4 \times 10^{-12}\ \text{F}$$

The phase constant is

$$\beta = \frac{2\pi}{\lambda} = \frac{2\pi f}{u_p} = \frac{2\pi \times 2.25 \times 10^9}{2.25 \times 10^8} = 62.8 \quad (\text{rad/m})$$

Form $\frac{1}{j\omega C_{\text{eq}}} = jZ_0 \tan\beta l$, it follows that

$$\tan\beta l = -\frac{1}{Z_0 \omega C_{\text{eq}}} = -\frac{1}{50 \times 2\pi \times 2.25 \times 10^9 \times 4 \times 10^{-12}} = -0.354$$

The tangent function is negative when its argument is in the second or fourth quadrants. The solution for the second quadrant is

$$\beta l_1 = 2.8\ \text{rad} \quad or \quad l_1 = \frac{2.8}{\beta} = \frac{2.8}{62.8} = 4.46\ \text{cm}$$

and the solution for the fourth quadrant is

$$\beta l_2 = 5.94\ \text{rad} \quad or \quad l_2 = \frac{5.94}{62.8} = 9.46\ \text{cm}$$

We also could have obtained the value of l_1 by applying $l = \frac{1}{\beta}\left[\pi - \tan^{-1}\left(\frac{1}{\omega C_{\text{eq}} Z_0}\right)\right]$. The length l_2 is greater than l_1 by exactly $\lambda/2$. In fact, any length $l = 4.46\text{cm} + \text{n}\lambda/2$, where n is a positive integer, also is a solution.

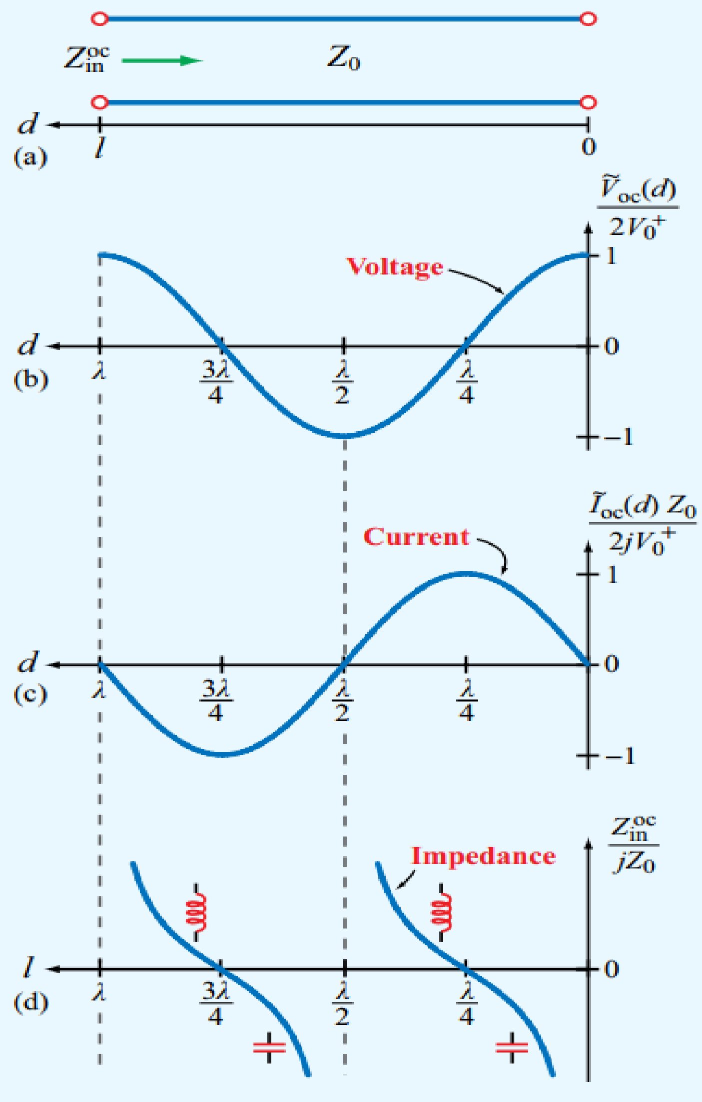


Figure 2-21 Transmission line terminated in an open circuit: (a) schematic representation, (b) normalized voltage on the line, (c) normalized current, and (d) normalized input impedance.

2.8.2 Open-Circuited Line

With $Z_L = \infty$, as illustrated in Fig. 2-21(a), we have $\Gamma = 1$, $S = \infty$, and the voltage, current, and input impedance are given by

$$\tilde{V}_{oc}(d) = V_0^+\left[e^{j\beta d} + e^{-j\beta d}\right] = 2V_0^+ \cos\beta d$$

$$\tilde{I}_{oc}(d) = \frac{V_0^+}{Z_0}\left[e^{j\beta d} - e^{-j\beta d}\right] = \frac{2jV_0^+}{Z_0}\sin\beta d$$

$$Z_{in}^{oc} = \frac{\tilde{V}_{oc}(l)}{\tilde{I}_{oc}(l)} = -jZ_0 \cot\beta d$$

Plots of these quantities are displayed in Fig. 2-21 as a function of d.

2.8.3 Application of Short-Circuit/Open-Circuit Technique

A network analyzer is a radio-frequency (RF) instrument capable of measuring the impedance of any load connected to its input terminal. When used to measure

（1） $Z_{\text{in}}^{\text{sc}}$, in a short circuit, and the input impedance of a lossless line when terminated

（2） $Z_{\text{in}}^{\text{oc}}$, the input impedance of the line when terminated in an open circuit, the combination of the two measurements can be used to determine the characteristic impedance of the line Z_0 and its phase constant β.

Indeed, the product of the expressions given by $Z_{\text{in}}^{\text{sc}} = \frac{\tilde{V}_{\text{sc}}(l)}{\tilde{I}_{\text{sc}}(l)} = jZ_0 \tan\beta l$ and $Z_{\text{in}}^{\text{oc}} = \frac{\tilde{V}_{oc}(l)}{\tilde{I}_{oc}(l)} = -jZ_0 \cot \beta l$ gives

$$Z_0 = \sqrt[+]{Z_{\text{in}}^{\text{sc}} Z_{\text{in}}^{\text{oc}}}$$

and the ratio of the same expressions leads to

$$\tan\beta l = \sqrt{\frac{-Z_{\text{in}}^{\text{sc}}}{Z_{\text{in}}^{\text{oc}}}}$$

Because of the π phase ambiguity associated with the tangent function, the length l should be less than or equal to $\lambda/2$ to provide an unambiguous result.

Example 2-9: Measuring Z_0 and β

Q: Find Z_0 and β of a $57\ \mathrm{cm}$ long lossless transmission line whose input impedance was measured as $Z_{\mathrm{in}}^{\mathrm{sc}} = j40.42\ \Omega$ when terminated in a short circuit and as $Z_{\mathrm{in}}^{\mathrm{oc}} = -j121.24\ \Omega$ when terminated in an open circuit. From other measurements, we know that the line is between 3 and 3.25 wavelengths long.

Solution: From $Z_0 = {}^{+}\sqrt{Z_{\mathrm{in}}^{\mathrm{sc}} Z_{\mathrm{in}}^{\mathrm{oc}}}$ and $\tan\beta l = \sqrt{\frac{-Z_{\mathrm{in}}^{\mathrm{sc}}}{Z_{\mathrm{in}}^{\mathrm{oc}}}}$,

$$Z_0 = {}^{+}\sqrt{Z_{\mathrm{in}}^{\mathrm{sc}} Z_{\mathrm{in}}^{\mathrm{oc}}} = \sqrt{(j40.42)(-j121.24)} = 70\ \Omega$$

$$\tan\beta l = \sqrt{\frac{-Z_{\mathrm{in}}^{\mathrm{sc}}}{Z_{\mathrm{in}}^{\mathrm{oc}}}} = \sqrt{\frac{1}{3}}$$

Since l is between 3λ and 3.25λ, $\beta l = (2\pi l/\lambda)$ is between 6π radians and $(13\pi/2)$ radians. This places βl in the first quadrant $(0\ to\ \pi/2)$ radians. Hence, the only acceptable solution for $\tan\beta\ell = \sqrt{1/3}$ is $\beta l = \pi/6$ radians. This value, however, does not include the 2π multiples associated with the integer λ multiples of l. Hence, the true value of βl is

$$\beta l = 6\pi + \frac{\pi}{6} = 19.4 \quad (\mathrm{rad})$$

in which case

$$\beta = \frac{19.4}{0.57} = 34 \quad (\mathrm{rad/m})$$

2.8.4 Lines of Length $l = n\lambda/2$

If $l = n\lambda/2$, where n is an integer,

$$\tan\beta l = \tan[(2\pi/\lambda)(n\lambda/2)] = \tan n\pi = 0$$

Consequently, $Z_{\mathrm{in}} = Z_0\left(\frac{z_{\mathrm{L}}+j\tan\beta l}{1+jz_{\mathrm{L}}\tan\beta l}\right)$ reduces to

$$Z_{\mathrm{in}} = Z_{\mathrm{L}}, \quad \text{for } l = n\lambda/2$$

which means that a half-wavelength line (or any integer multiple of $\lambda/2$) does not modify the load impedance.

2.8.5 Quarter-Wavelength Transformer

Another case of interest is when the length of the line is a quarter-wavelength (or $\lambda/4 + n\lambda/2$, where $n = 0$ or a positive integer), corresponding to $\beta l = (2\pi/\lambda)(\lambda/4) = \pi/2$. From $Z_{\text{in}} = Z_0 \left(\frac{z_{\text{L}} + j\tan \beta l}{1 + jz_{\text{L}}\tan \beta l} \right)$, the input impedance becomes

$$Z_{\text{in}} = \frac{{Z_0}^2}{Z_{\text{L}}}, \quad \text{for} \;\; l = \lambda/4 + n\lambda/2$$

The utility of such a quarter-wave transformer is illustrated by Example 2-10.

Example 2-10: $\lambda/4$ Transformer

Q: A $50\,\Omega$ lossless transmission line is to be matched to a resistive load impedance with $Z_L = 100\,\Omega$ via a quarter-wave section as shown in Fig. 2-22, thereby eliminating reflections along the feedline. Find the required characteristic impedance of the quarter-wave transformer.

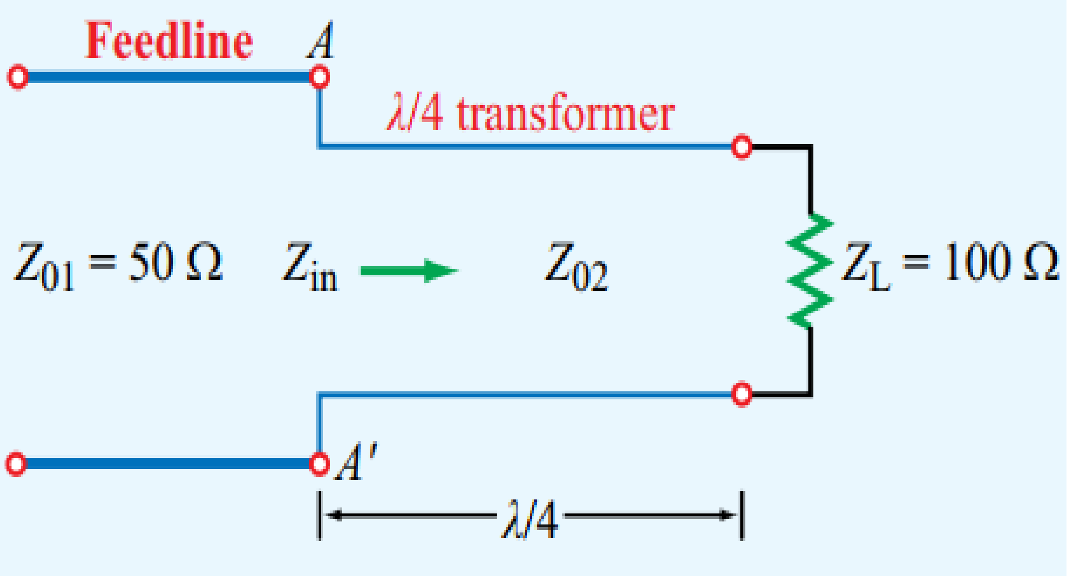


Figure 2-22 Configuration for Example 2-10.

Solution: To eliminate reflections at terminal AA', the input impedance Z_{in} looking into the quarter-wave line should be equal to Z_{01}, the characteristic impedance of the feedline. Thus, $Z_{\mathrm{in}} = 50\ \Omega$. From $Z_{\mathrm{in}} = \frac{{Z_0}^2}{Z_{\mathrm{L}}}$,

$$Z_{\mathrm{in}} = \frac{Z_{02}^2}{Z_{\mathrm{L}}}$$

or
$$Z_{02} = \sqrt{Z_{\mathrm{in}} Z_{\mathrm{L}}} = \sqrt{50 \times 100} = 70.7\ \Omega$$

Whereas this eliminates reflections on the feedline, it does not eliminate them on the $\lambda/4$ line. However, since the lines are lossless, all the power incident on AA' will end up getting transferred into the load Z_{L}.

In this example, Z_{L} is purely resistive. To apply the $\lambda/4$ transformer technique to match a transmission line to a load with a complex impedance, a slightly more elaborate procedure is required (Section 2-11).

2.8.6 Matched Transmission Line: $Z_{\mathrm{L}} = Z_0$

For a matched lossless transmission line with $Z_{\mathrm{L}} = Z_0$, (1) the input impedance $Z_{\mathrm{in}} = Z_0$ for all locations d on the line, (2) $\Gamma = 0$, and (3) all the incident power is delivered to the load, regardless of the line length l. A summary of the properties of standing waves is given in Table 2-4.

Table 2-4 Properties of standing waves on a lossless transmission line.

Voltage maximum **Voltage minimum**	$\lvert\widetilde{V}\rvert_{max} = \lvert V_0^+\rvert[1+\lvert\Gamma\rvert]$ $\lvert\widetilde{V}\rvert_{min} = \lvert V_0^+\rvert[1-\lvert\Gamma\rvert]$
Positions of voltage maxima (also positions of current minima)	$d_{max} = \frac{\theta_r\lambda}{4\pi} + \frac{n\lambda}{2}, \quad n = 0, 1, 2, \ldots$
Position of first maximum (also position of first current minimum)	$d_{max} = \begin{cases} \frac{\theta_r\lambda}{4\pi}, & \text{if } 0 \le \theta_r \le \pi \\ \frac{\theta_r\lambda}{4\pi} + \frac{\lambda}{2}, & \text{if } -\pi \le \theta_r \le 0 \end{cases}$
Positions of voltage minima (also positions of current maxima)	$d_{min} = \frac{\theta_r\lambda}{4\pi} + \frac{(2n+1)\lambda}{4}, \quad n = 0, 1, 2, \ldots$
Position of first minimum (also position of first current maximum)	$d_{min} = \frac{\lambda}{4}\left(1 + \frac{\theta_r}{\pi}\right)$
Input impedance	$Z_{in} = Z_0\left(\frac{z_L + j\tan\beta l}{1 + jz_L\tan\beta l}\right) = Z_0\left(\frac{1+\Gamma_l}{1-\Gamma_l}\right)$
Positions at which Z_{in} is real	at voltage maxima and minima
Z_{in} at voltage maxima	$Z_{in} = Z_0\left(\frac{1+\lvert\Gamma\rvert}{1-\lvert\Gamma\rvert}\right)$
Z_{in} at voltage minima	$Z_{in} = Z_0\left(\frac{1-\lvert\Gamma\rvert}{1+\lvert\Gamma\rvert}\right)$
Z_{in} of short-circuited line	$Z_{in}^{sc} = jZ_0\tan\beta l$
Z_{in} of open-circuited line	$Z_{in}^{oc} = -jZ_0\cot\beta l$
Z_{in} of line of length $l = n\lambda/2$	$Z_{in} = Z_L, \quad n = 0, 1, 2, \ldots$
Z_{in} of line of length $l = \lambda/4 + n\lambda/2$	$Z_{in} = Z_0^2/Z_L, \quad n = 0, 1, 2, \ldots$
Z_{in} of matched line	$Z_{in} = Z_0$
$\lvert V_0^+\rvert$ = amplitude of incident wave; $\Gamma = \lvert\Gamma\rvert e^{j\theta_r}$ with $-\pi < \theta_r < \pi$; θ_r in radians; $\Gamma_l = \Gamma e^{-j2\beta l}$.	

Concept Question

Concept Question 2-10:

What is the difference between the characteristic impedance Z_0 and the input impedance Z_{in}? When are they the same?

The characteristic impedance is the characteristic of the transmission line, the inherent property of the transmission line, and it is related to the shape of the transmission line, material of the transmission line, etc. However, input impedance will be changing with the actual circuit and the load.

When the impedance of the load is matching to the characteristic impedance of the transmission line, they are the same.

$$Z(d) = \frac{\tilde{V}(d)}{\tilde{I}(d)} = \frac{V_0^+\left[e^{j\beta d} + \Gamma e^{-j\beta d}\right]}{V_0^+\left[e^{j\beta d} - \Gamma e^{-j\beta d}\right]} Z_0 = Z_0\left[\frac{1 + \Gamma e^{-j2\beta d}}{1 - \Gamma e^{-j2\beta d}}\right] = Z_0\left[\frac{1 + \Gamma_d}{1 - \Gamma_d}\right] \quad \Omega$$

Concept Question 2-11:

What is a quarter-wave transformer? How can it be used?

The quarter-wave transformer is when the length of the line is a quarter-wavelength (or $\lambda/4 + \mathrm{n}\lambda/2$, where $\mathrm{n} = 0$ or a positive integer), corresponding to $\beta l = (2\pi/\lambda)(\lambda/4) = \pi/2$.

The utility of such a quarter-wave transformer is illustrated by Example 2-10.

$$Z_{\mathrm{in}} = \frac{{Z_0}^2}{Z_{\mathrm{L}}}, \quad \text{for } \; l = \lambda/4 + n\lambda/2$$

Concept Question 2-12:

A lossless transmission line of length l is terminated in a short circuit. If $l < \lambda/4$, is the input impedance inductive or capacitive?

For the short-circuited line, if its length is less than $\lambda/4$, its impedance is equivalent to that of an inductor.

Concept Question 2-13:

What is the input impedance of an infinitely long line?

The input impedance of an infinitely long line is equal to its characteristic Impendence.

Concept Question 2-14:

If the input impedance of a lossless line is inductive when terminated in a short circuit, will it be inductive or capacitive when the line is terminated in an open circuit?

The answer is capacitive.

Exercise 2-12:

A $50\ \Omega$ lossless transmission line uses an insulating material with $\varepsilon_{\mathrm{r}} = 2.25$. When terminated in an open circuit, how long should the line be for its input impedance to be equivalent to a $10\ \mathrm{pF}$ capacitor at $50\ \mathrm{MHz}$?

Answer: The following quantities are given:

$$Z_0 = 50\ \Omega \qquad C = 10\ \mathrm{pF} = 10^{-11}\ \mathrm{F} \qquad f = 50 \times 10^6\ \ \mathrm{Hz}$$

From the relationship $\beta = \omega\sqrt{\varepsilon_{\mathrm{r}}}/c$,

$$\beta = \frac{2\pi f\sqrt{\varepsilon_{\mathrm{r}}}}{c} = \frac{2 \times \pi \times 50 \times 10^6 \times \sqrt{2.25}}{3 \times 10^8} = \frac{\pi}{2}$$

According to $Z_{\text{in}}^{\text{oc}} = -jZ_0 \cot \beta d$, we know $\frac{1}{j\omega C} = -jZ_0 \cot \beta d$, so

$$\cot \beta d = \frac{1}{Z_0 \omega C}$$

$$\cot \frac{\pi}{2} d = \frac{1}{50 \times 2\pi \times 50 \times 10^6 \times 10^{-11}} = \frac{20}{\pi}$$

$$\cot \frac{\pi}{2} d = \frac{1}{\tan \frac{\pi}{2} d} = \frac{20}{\pi}$$

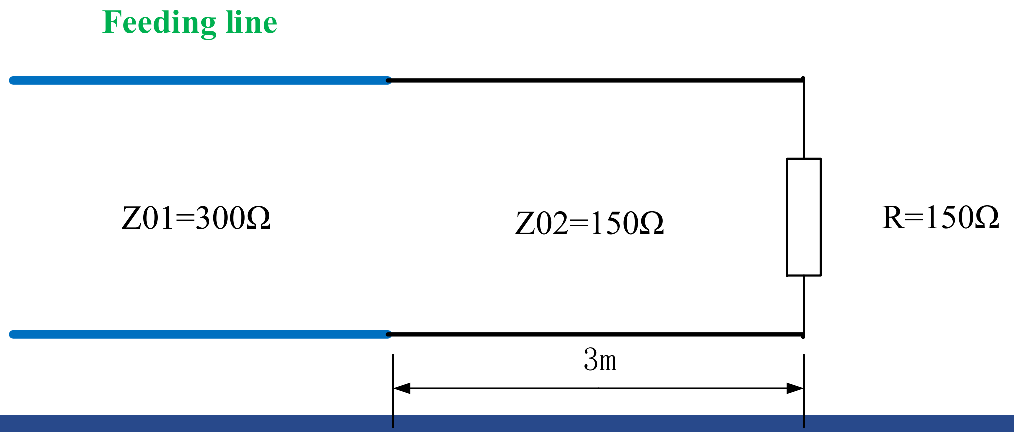
$$\frac{\pi}{2} d = \arctan \frac{\pi}{20} = 0.1558$$

$$d = 0.1558 \times \frac{2}{\pi} = 0.0992 \ \text{m} = 9.92 \ \text{cm}$$

Exercise 2-13:

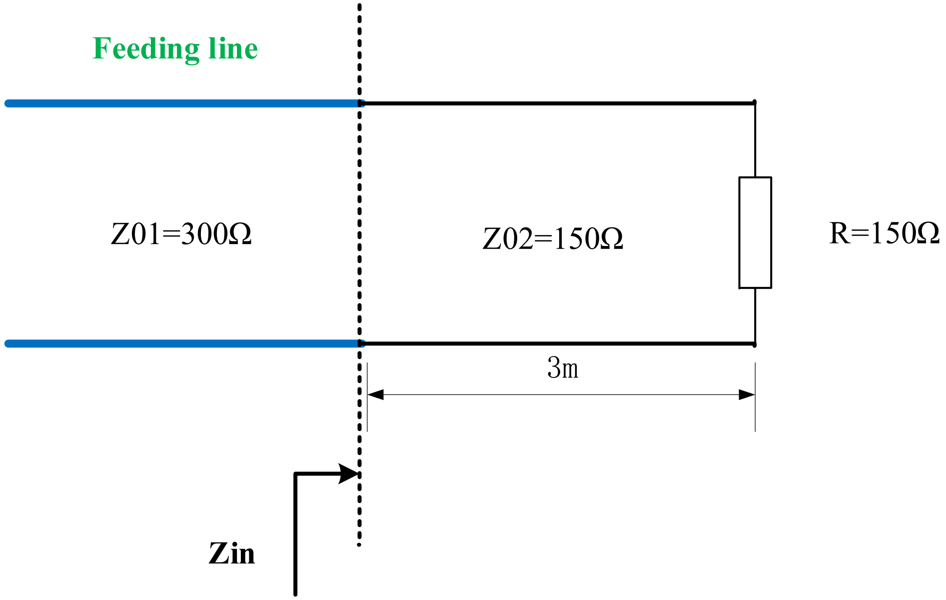
A $300\ \Omega$ feedline is to be connected to a $3\ \mathrm{m}$ long, $150\ \Omega$ line terminated in a $150\ \Omega$ resistor. Both lines are lossless and use air as the insulating material, and the operating frequency is $50\ \mathrm{MHz}$. Determine (a) the input impedance of the $3\ \mathrm{m}$ long line, (b) the voltage standing-wave ratio on the feedline, and (c) the characteristic impedance of a quarter-wave transformer were it to be used between the two lines in order to achieve $S = 1$ on the feedline.

Answer: According to the expression above, the diagram can be obtained as follow



Exercise 2-13:

(a) the input impedance of the 3 m long line,



Answer:

(a) For the right hand of the dash line,

$$\Gamma_1 = \frac{Z_L - Z_{02}}{Z_L + Z_{02}} = 0$$

The operating frequency is 50 MHz, the wavelength can be calculated as

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{5 \times 10^7} = 6\text{m}$$

The length of the transmission line l is 3m,

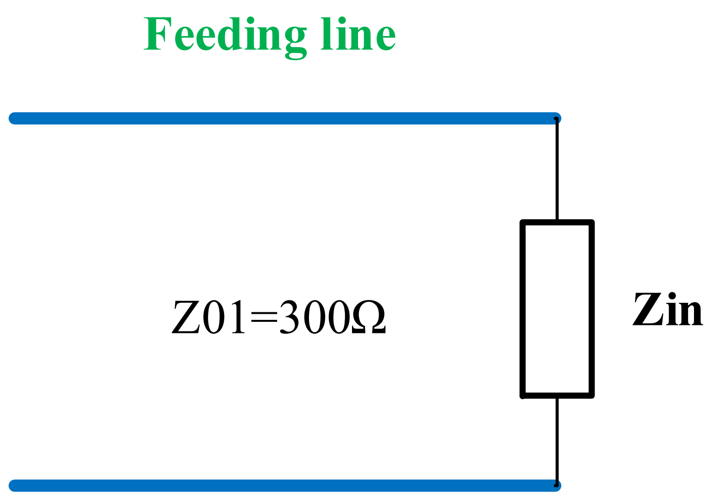
$$l = \frac{\lambda}{2}$$

According to the impedance transformed with $\frac{\lambda}{2}$ as the period, the input impedance $Z_{in} = Z_L = 150\ \Omega$

Exercise 2-13:

(b) the voltage standing-wave ratio on the feedline

Answer: (b) The equivalent circuit diagram is shown on the right.



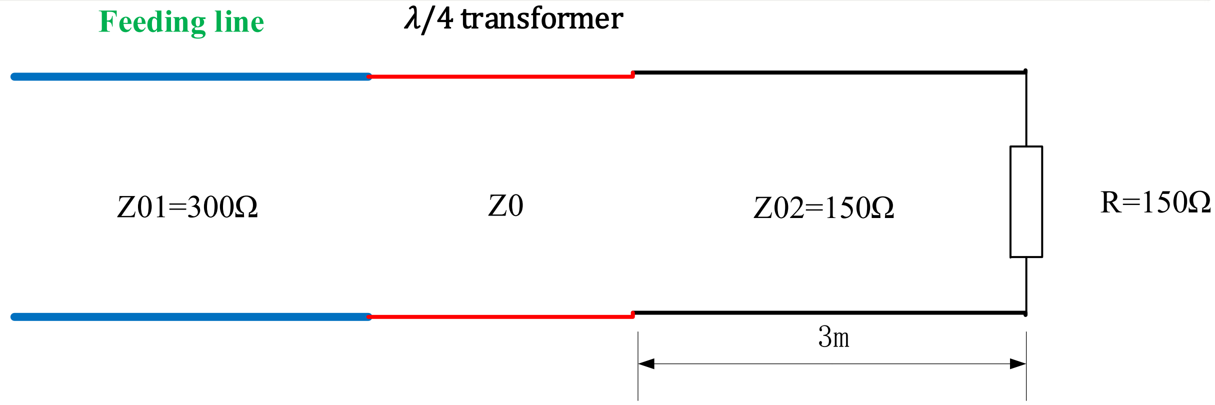
$$\Gamma_2 = \frac{Z_{01} - Z_{\text{in}}}{Z_{01} + Z_{\text{in}}} = \frac{1}{3}$$

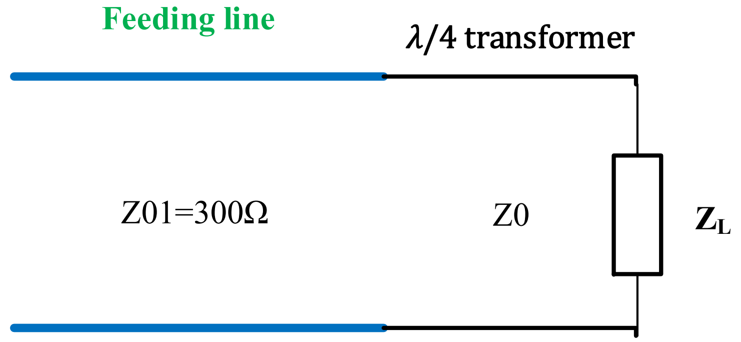
so the voltage standing-wave ratio

$$S = \frac{1 + |\Gamma_2|}{1 - |\Gamma_2|} = 2$$

Exercise 2-13:

(c) the characteristic impedance of a quarter-wave transformer were it to be used between the two lines in order to achieve $S = 1$ on the feedline.





The circuit diagram is shown on the right

According to

$$S = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

when $S = 1$, $|\Gamma| = 0$.

According to the answer of (a), $Z_{\text{in}} = 300\ \Omega$

$$Z_{\text{in}} = \frac{{Z_0}^2}{Z_{\text{L}}}$$

So the characteristic impedance of the quarter-wave transformer is

$$Z_0 = \sqrt{Z_{\text{in}} Z_{\text{L}}} = \sqrt{300 \times 150} = 212.1\ \ \Omega$$

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B39HF High Frequency Circuits

Lecture 9
Power Flow on a Lossless Transmission Line

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Our discussion thus far has focused on the voltage and current attributes of waves propagating on a transmission line. Now we examine the flow of power carried by the incident and reflected waves. We begin by reintroducing $\tilde{V}(z) = V_0^+\left(e^{-j\beta z} + \Gamma e^{j\beta z}\right)$ and $\tilde{I}(z) = \frac{V_0^+}{Z_0}\left(e^{-j\beta z} - \Gamma e^{j\beta z}\right)$ with $z = -d$:

$$\tilde{V}(d) = V_0^+\left(e^{j\beta d} + \Gamma e^{-j\beta d}\right)$$

$$\tilde{I}(d) = \frac{V_0^+}{Z_0}\left(e^{j\beta d} - \Gamma e^{-j\beta d}\right)$$

In these expressions, the first terms represent the incident-wave voltage and current, and the terms involving Γ represent the reflected-wave voltage and current.

The time-domain expressions for the voltage and current at location d from the load are obtained by transforming above expression to the time domain:

$$v(d,t) = \Re e\left[\tilde{V} e^{j\omega t}\right]$$

$$= \Re e\left[|V_0^+| e^{j\phi^+}\left(e^{j\omega d} + |\Gamma| e^{j\theta_{\mathrm{r}}} e^{-j\beta d}\right) e^{j\omega t}\right]$$

$$= |V_0^+|[\cos(\omega t + \beta d + \phi^+) + |\Gamma|\cos(\omega t - \beta d + \phi^+ + \theta_{\mathrm{r}})]$$

$$i(d,t) = \frac{|V_0^+|}{Z_0}[\cos(\omega t + \beta d + \phi^+) - |\Gamma|\cos(\omega t - \beta d + \phi^+ + \theta_{\mathrm{r}})]$$

where we used the relations $V_0^+ = |V_0^+| e^{j\phi^+}$ and $\Gamma = |\Gamma| e^{j\theta_{\mathrm{r}}}$.

2.9.1 Instantaneous Power

The instantaneous power carried by the transmission line is equal to the product of $v(d,t)$ and $i(d,t)$:

$$P(d,t) = v(d,t)\, i(d,t)$$

$$= |V_0^+|[\cos(\omega t + \beta d + \phi^+) + |\Gamma|\cos(\omega t - \beta d + \phi^+ + \theta_r)]$$

$$\times \frac{|V_0^+|}{Z_0}[\cos(\omega t + \beta d + \phi^+) - |\Gamma|\cos(\omega t - \beta d + \phi^+ + \theta_r)]$$

$$= \frac{|V_0^+|^2}{Z_0}[\cos^2(\omega t + \beta d + \phi^+) - |\Gamma|^2\cos^2(\omega t - \beta d + \phi^+ + \theta_r)] \quad \text{(W)}$$

Per our earlier discussion in connection with $y(x,t)=A\cos(\omega t+\beta x)$, if the signs preceding ωt and βd in the argument of the cosine term are both positive or both negative, then the cosine term represents a wave traveling in the negative d direction. Since d points from the load to the generator, the first term in $P(d,t)=\frac{|V_0^+|^2}{Z_0}[\cos^2(\omega t+\beta d+\phi^+)-|\Gamma|^2\cos^2(\omega t-\beta d+\phi^++\theta_\mathrm{r})]$ represents the *instantaneous incident power* traveling towards the load. This is the power that would be delivered to the load in the absence of wave reflection (when $\Gamma=0$). Because βd is preceded by a minus sign in the argument of the cosine of the second term in $P(d,t)=\frac{|V_0^+|^2}{Z_0}[\cos^2(\omega t+\beta d+\phi^+)-|\Gamma|^2\cos^2(\omega t-\beta d+\phi^++\theta_\mathrm{r})]$, that term represents the *instantaneous reflected power* traveling in the $+d$ direction, away from the load.

Accordingly, we label these two power components

$$P^{\mathrm{i}}(d,t) = \frac{|V_0^+|^2}{Z_0}\cos^2(\omega t + \beta d + \phi^+) \quad (\mathrm{W})$$

$$P^{\mathrm{r}}(d,t) = -|\Gamma|^2\frac{|V_0^+|^2}{Z_0}\cos^2(\omega t - \beta d + \phi^+ + \theta_{\mathrm{r}}) \quad (\mathrm{W})$$

Using the trigonometric identity

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

the above expressions can be rewritten as

$$P^{\mathrm{i}}(d,t) = \frac{|V_0^+|^2}{2Z_0}[1 + cos(2\omega t + 2\beta d + 2\phi^+)]$$

$$P^{\mathrm{r}}(d,t) = -|\Gamma|^2\frac{|V_0^+|^2}{2Z_0}[1 + cos(2\omega t - 2\beta d + 2\phi^+ + 2\theta_{\mathrm{r}})]$$

We note that in each case, the instantaneous power consists of a dc (non–time-varying) term and an ac term that oscillates at an angular frequency of 2ω.

- The power oscillates at twice the rate of the voltage or current.

2.9.2 Time-Average Power

From a practical standpoint, we usually are more interested in the time-average power flowing along the transmission line, $P_{\mathrm{av}}(d)$, than in the instantaneous power $P(d,t)$. To compute $P_{\mathrm{av}}(d)$, we can use a time-domain approach or a computationally simpler phasor-domain approach. For completeness, we consider both.

Time-domain approach

The time-average power is equal to the instantaneous power averaged over one time period $T = \frac{1}{f} = 2\pi/\omega$. For the incident wave, its time-average power is

$$P_{\mathrm{av}}^{\mathrm{i}}(d) = \frac{1}{T}\int_0^T P^{\mathrm{i}}(d,t)dt = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} P^{\mathrm{i}}(d,t)dt$$

Upon inserting $P^{\mathrm{i}}(d,t)=\frac{|V_0^+|^2}{2Z_0}[1+cos(2\omega t+2\beta d+2\phi^+)]$ into $P_{\mathrm{av}}^{\mathrm{i}}(d)=\frac{1}{T}\int_0^T P^{\mathrm{i}}\,(d,t)dt=\frac{\omega}{2\pi}\int_0^{2\pi/\omega} P^{\mathrm{i}}\,(d,t)dt$ and performing the integration, we obtain

$$P_{\mathrm{av}}^{\mathrm{i}}=\frac{|V_0^+|^2}{2Z_0} \qquad \text{(W)}$$

which is identical with the dc term of $P^{\mathrm{i}}(d,t)$ given by $P^{\mathrm{i}}(d,t)=\frac{|V_0^+|^2}{2Z_0}[1+cos(2\omega t+2\beta d+2\phi^+)]$. A similar treatment for the reflected wave gives

$$P_{\mathrm{av}}^{\mathrm{r}}=-|\Gamma|^2\frac{|V_0^+|^2}{2Z_0}=-|\Gamma|^2P_{\mathrm{av}}^{\mathrm{i}}$$

- The average reflected power is equal to the average incident power, diminished by a multiplicative factor of $|\Gamma|^2$.

Note that the expressions for $P^{\mathrm{i}}_{\mathrm{av}}$ and $P^{\mathrm{r}}_{\mathrm{av}}$ are independent of d, which means that the time-average powers carried by the incident and reflected waves do not change as they travel along the transmission line. This is as expected, because the transmission line is lossless.

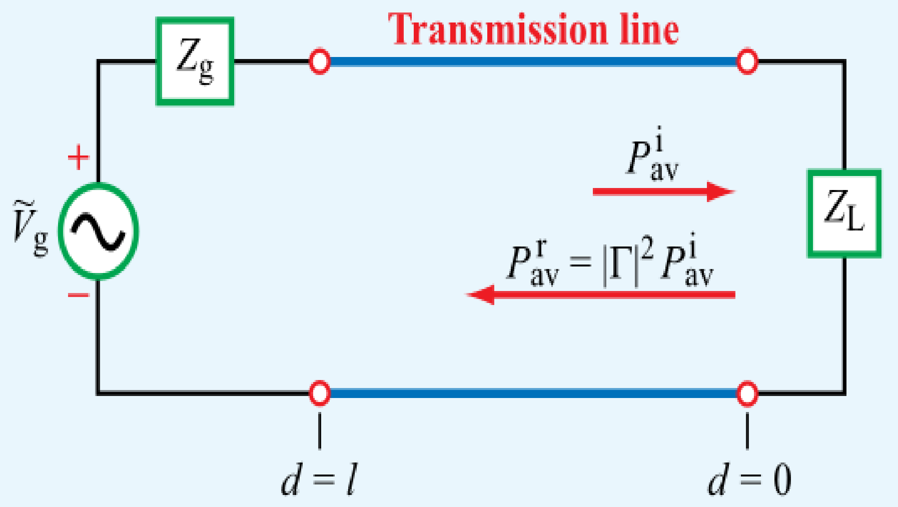

Figure 2-23 The time-average power reflected by a load connected to a lossless transmission line is equal to the incident power multiplied by $|\Gamma|^2$.

The *net average power flowing towards (and then absorbed by) the load* shown in Fig. 2-23 is

$$P_{\mathrm{av}} = P^{\mathrm{i}}_{\mathrm{av}} + P^{\mathrm{r}}_{\mathrm{av}}$$

$$= \frac{|V_0^+|^2}{2Z_0}\left[1 - |\Gamma|^2\right] \quad \text{(W)}$$

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Phasor-domain approach

For any propagating wave with voltage and current phasors $\tilde{V}$ and $\tilde{I}$, a useful formula for computing the time-average power is

$$P_{\text{av}} = \frac{1}{2}\Re e\,[\tilde{V}\cdot\tilde{I}^*]$$

where $\tilde{I}^*$ is the complex conjugate of $\tilde{I}$. Application of this formula to $\tilde{V}(d) = V_0^+\left(e^{j\beta d} + \Gamma e^{-j\beta d}\right)$ and $\tilde{I}(d) = \frac{V_0^+}{Z_0}\left(e^{j\beta d} - \Gamma e^{-j\beta d}\right)$ gives

$$P_{\text{av}} = \frac{1}{2}\Re e\left[V_0^+\left(e^{j\beta d} + \Gamma e^{-j\beta d}\right)\cdot\frac{{V_0^+}^*}{Z_0}\left(e^{-j\beta d} - \Gamma^* e^{j\beta d}\right)\right]$$

$$= \frac{1}{2}\Re e\left[\frac{|V_0^+|^2}{Z_0}\left(1 - |\Gamma|^2 + \Gamma e^{-j2\beta d} - \Gamma^* e^{j2\beta d}\right)\right]$$

$$P_{\text{av}} = \frac{1}{2}\Re e\left[V_0^+\left(e^{j\beta d} + \Gamma e^{-j\beta d}\right)\cdot\frac{V_0^{+*}}{Z_0}\left(e^{-j\beta d} - \Gamma^* e^{j\beta d}\right)\right]$$

$$= \frac{1}{2}\Re e\left[\frac{|V_0^+|^2}{Z_0}\left(1 - |\Gamma|^2 + \Gamma e^{-j2\beta d} - \Gamma^* e^{j2\beta d}\right)\right]$$

$$= \frac{|V_0^+|^2}{2Z_0}\left\{[1 - |\Gamma|^2] + \Re e\left[|\Gamma|e^{-j(2\beta d-\theta_{\text{r}})} - |\Gamma|e^{j(2\beta d-\theta_{\text{r}})}\right]\right\}$$

$$= \frac{|V_0^+|^2}{2Z_0}\{[1 - |\Gamma|^2] + |\Gamma|[\cos(2\beta d - \theta_{\text{r}}) - \cos(2\beta d - \theta_{\text{r}})]\}$$

$$= \frac{|V_0^+|^2}{2Z_0}[1 - |\Gamma|^2]$$

which is identical to $P_{\text{av}} = P_{\text{av}}^{\text{i}} + P_{\text{av}}^{\text{r}} = \frac{|V_0^+|^2}{2Z_0}[1 - |\Gamma|^2]$.

Exercise 2-14:

For a $50\,\Omega$ lossless transmission line terminated in a load impedance $Z_L = (100 + j50)\,\Omega$, determine the fraction of the average incident power reflected by the load.

Answer: The following quantities are given

$$Z_0 = 50\,\Omega \qquad\qquad Z_L = (100 + j50)\,\Omega$$

The voltage reflection coefficient is

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{50 + j50}{150 + j50} = \frac{1 + j1}{3 + j1}$$

$$= \frac{1.414e^{j45^\circ}}{3.162e^{j18.4^\circ}} = 0.45e^{j26.6^\circ}$$

So $|\Gamma| = 0.45$.

According to $P_{\mathrm{av}}^{\mathrm{r}} = -|\Gamma|^2 \frac{|V_0^+|^2}{2Z_0} = -|\Gamma|^2 P_{\mathrm{av}}^{\mathrm{i}}$, we can obtain

$$\frac{|P_{\mathrm{av}}^{\mathrm{r}}|}{|P_{\mathrm{av}}^{\mathrm{i}}|} = |\Gamma|^2 = 0.45^2 = 20\%$$

Exercise 2-15:

For the line of Exercise 2-14, what is the magnitude of the average reflected power if $|V_0^+| = 1\ \mathrm{V}$?

Answer: The following quantities are given

$$Z_0 = 50\ \Omega \qquad |V_0^+| = 1\ \mathrm{V} \qquad |\Gamma| = 0.45$$

According to $P_{\mathrm{av}}^{\mathrm{r}} = -|\Gamma|^2 \frac{|V_0^+|^2}{2Z_0}$,

$$|P_{\mathrm{av}}^{\mathrm{r}}| = |\Gamma|^2 \frac{|V_0^+|^2}{2Z_0}$$

$$= 0.45^2 \cdot \frac{1^2}{2 * 50} = 2\ \ \mathrm{mW}$$

Concept Question

Concept Question 2-15:

According to $P^{\mathrm{r}}(d,t) = -|\Gamma|^2 \frac{|V_0^+|^2}{2Z_0}[1 + cos(2\omega t - 2\beta d + 2\phi^+ + 2\theta_{\mathrm{r}})]$, the instantaneous value of the reflected power depends on the phase of the reflection coefficient θ_{r} , but the average reflected power given by $P_{\mathrm{av}}^{\mathrm{r}} = -|\Gamma|^2 \frac{|V_0^+|^2}{2Z_0} = -|\Gamma|^2 P_{\mathrm{av}}^{\mathrm{i}}$ does not. Explain.

In $P^{\mathrm{r}}(d,t)$, the instantaneous power consists of a dc (non–time-varying) term and an ac term that oscillates at an angular frequency of 2ω. The average reflected power $P_{\mathrm{av}}^{\mathrm{r}}$ just consists of a dc (non–time-varying) term.

Concept Question 2-16:

What is the average power delivered by a lossless transmission line to a reactive load?

$$P_{\mathrm{av}} = P_{\mathrm{av}}^{\mathrm{i}} + P_{\mathrm{av}}^{\mathrm{r}}$$

$$= \frac{|V_0^+|^2}{2Z_0}[1 - |\Gamma|^2] \quad (\mathrm{W})$$

Concept Question 2-17:

What fraction of the incident power is delivered to a matched load?

100%

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Concept Question 2-18:

Verify that $\frac{1}{T}\int_0^T \cos^2(\frac{2\pi t}{T}+\beta d+\phi)\,dt=\frac{1}{2}$ regardless of the values of d and ϕ, so long as neither is a function of t.

$$\frac{1}{T}\int_0^T \cos^2(\frac{2\pi t}{T}+\beta d+\phi)\,dt=\frac{1}{T}\int_0^T \frac{\cos 2\left(\frac{2\pi t}{T}+\beta d+\phi\right)+1}{2}dt$$

$$=\frac{1}{2T}\left[\int_0^T \cos\left(\frac{4\pi t}{T}+2\beta d+2\phi\right)dt+\int_0^T 1\,dt\right]$$

$$=\frac{1}{2T}\left[\left[\frac{\sin\left(\frac{4\pi t}{T}+2\beta d+2\phi\right)}{\frac{4\pi}{T}}\right]_0^T+[t]_0^T\right]=\frac{1}{2T}\cdot T=\frac{1}{2}$$