西安电子科技大学
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B39HF High Frequency Circuits

Lecture 9
Power Flow on a Lossless Transmission Line

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Our discussion thus far has focused on the voltage and current attributes of waves propagating on a transmission line. Now we examine the flow of power carried by the incident and reflected waves. We begin by reintroducing $\tilde{V}(z) = V_0^+\left(e^{-j\beta z} + \Gamma e^{j\beta z}\right)$ and $\tilde{I}(z) = \frac{V_0^+}{Z_0}\left(e^{-j\beta z} - \Gamma e^{j\beta z}\right)$ with $z = -d$:

$$\tilde{V}(d) = V_0^+\left(e^{j\beta d} + \Gamma e^{-j\beta d}\right)$$

$$\tilde{I}(d) = \frac{V_0^+}{Z_0}\left(e^{j\beta d} - \Gamma e^{-j\beta d}\right)$$

In these expressions, the first terms represent the incident-wave voltage and current, and the terms involving Γ represent the reflected-wave voltage and current.

The time-domain expressions for the voltage and current at location d from the load are obtained by transforming above expression to the time domain:

$$v(d,t) = \Re e\left[\tilde{V} e^{j\omega t}\right]$$

$$= \Re e\left[|V_0^+| e^{j\phi^+}\left(e^{j\omega d} + |\Gamma| e^{j\theta_\mathrm{r}} e^{-j\beta d}\right) e^{j\omega t}\right]$$

$$= |V_0^+|[\cos(\omega t + \beta d + \phi^+) + |\Gamma| \cos(\omega t - \beta d + \phi^+ + \theta_\mathrm{r})]$$

$$i(d,t) = \frac{|V_0^+|}{Z_0}[\cos(\omega t + \beta d + \phi^+) - |\Gamma| \cos(\omega t - \beta d + \phi^+ + \theta_\mathrm{r})]$$

where we used the relations $V_0^+ = |V_0^+| e^{j\phi^+}$ and $\Gamma = |\Gamma| e^{j\theta_\mathrm{r}}$.

2.9.1 Instantaneous Power

The instantaneous power carried by the transmission line is equal to the product of $v(d,t)$ and $i(d,t)$:

$$P(d,t) = v(d,t)\, i(d,t)$$

$$= |V_0^+|[\cos(\omega t + \beta d + \phi^+) + |\Gamma|\cos(\omega t - \beta d + \phi^+ + \theta_r)]$$

$$\times \frac{|V_0^+|}{Z_0}[\cos(\omega t + \beta d + \phi^+) - |\Gamma|\cos(\omega t - \beta d + \phi^+ + \theta_r)]$$

$$= \frac{|V_0^+|^2}{Z_0}[\cos^2(\omega t + \beta d + \phi^+) - |\Gamma|^2\cos^2(\omega t - \beta d + \phi^+ + \theta_r)] \quad \text{(W)}$$

Per our earlier discussion in connection with $y(x,t) = A\cos(\omega t + \beta x)$, if the signs preceding ωt and βd in the argument of the cosine term are both positive or both negative, then the cosine term represents a wave traveling in the negative d direction. Since d points from the load to the generator, the first term in $P(d,t) = \frac{|V_0^+|^2}{Z_0}[\cos^2(\omega t + \beta d + \phi^+) - |\Gamma|^2\cos^2(\omega t - \beta d + \phi^+ + \theta_r)]$ represents the *instantaneous incident power* traveling towards the load. This is the power that would be delivered to the load in the absence of wave reflection (when $\Gamma = 0$). Because βd is preceded by a minus sign in the argument of the cosine of the second term in $P(d,t) = \frac{|V_0^+|^2}{Z_0}[\cos^2(\omega t + \beta d + \phi^+) - |\Gamma|^2\cos^2(\omega t - \beta d + \phi^+ + \theta_r)]$, that term represents the *instantaneous reflected power* traveling in the $+d$ direction, away from the load.

Accordingly, we label these two power components

$$P^{\mathrm{i}}(d,t) = \frac{|V_0^+|^2}{Z_0}\cos^2(\omega t + \beta d + \phi^+) \quad (\mathrm{W})$$

$$P^{\mathrm{r}}(d,t) = -|\Gamma|^2 \frac{|V_0^+|^2}{Z_0}\cos^2(\omega t - \beta d + \phi^+ + \theta_{\mathrm{r}}) \quad (\mathrm{W})$$

Using the trigonometric identity

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

the above expressions can be rewritten as

$$P^{\mathrm{i}}(d,t) = \frac{|V_0^+|^2}{2Z_0}[1 + cos(2\omega t + 2\beta d + 2\phi^+)]$$

$$P^{\mathrm{r}}(d,t) = -|\Gamma|^2 \frac{|V_0^+|^2}{2Z_0}[1 + cos(2\omega t - 2\beta d + 2\phi^+ + 2\theta_{\mathrm{r}})]$$

We note that in each case, the instantaneous power consists of a dc (non–time-varying) term and an ac term that oscillates at an angular frequency of 2ω.

- The power oscillates at twice the rate of the voltage or current.

2.9.2 Time-Average Power

From a practical standpoint, we usually are more interested in the time-average power flowing along the transmission line, $P_{\mathrm{av}}(d)$, than in the instantaneous power $P(d,t)$. To compute $P_{\mathrm{av}}(d)$, we can use a time-domain approach or a computationally simpler phasor-domain approach. For completeness, we consider both.

Time-domain approach

The time-average power is equal to the instantaneous power averaged over one time period $T = \frac{1}{f} = 2\pi/\omega$. For the incident wave, its time-average power is

$$P_{\mathrm{av}}^{\mathrm{i}}(d) = \frac{1}{T}\int_0^T P^{\mathrm{i}}(d,t)dt = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} P^{\mathrm{i}}(d,t)dt$$

Upon inserting $P^{\mathrm{i}}(d,t)=\frac{|V_0^+|^2}{2Z_0}[1+cos(2\omega t+2\beta d+2\phi^+)]$ into $P_{\mathrm{av}}^{\mathrm{i}}(d)=\frac{1}{T}\int_0^T P^{\mathrm{i}}\,(d,t)dt=\frac{\omega}{2\pi}\int_0^{2\pi/\omega} P^{\mathrm{i}}\,(d,t)dt$ and performing the integration, we obtain

$$P_{\mathrm{av}}^{\mathrm{i}}=\frac{|V_0^+|^2}{2Z_0}\qquad \text{(W)}$$

which is identical with the dc term of $P^{\mathrm{i}}(d,t)$ given by $P^{\mathrm{i}}(d,t)=\frac{|V_0^+|^2}{2Z_0}[1+cos(2\omega t+2\beta d+2\phi^+)]$. A similar treatment for the reflected wave gives

$$P_{\mathrm{av}}^{\mathrm{r}}=-|\Gamma|^2\frac{|V_0^+|^2}{2Z_0}=-|\Gamma|^2P_{\mathrm{av}}^{\mathrm{i}}$$

- The average reflected power is equal to the average incident power, diminished by a multiplicative factor of $|\Gamma|^2$.

Note that the expressions for $P_{\mathrm{av}}^{\mathrm{i}}$ and $P_{\mathrm{av}}^{\mathrm{r}}$ are independent of d, which means that the time-average powers carried by the incident and reflected waves do not change as they travel along the transmission line. This is as expected, because the transmission line is lossless.

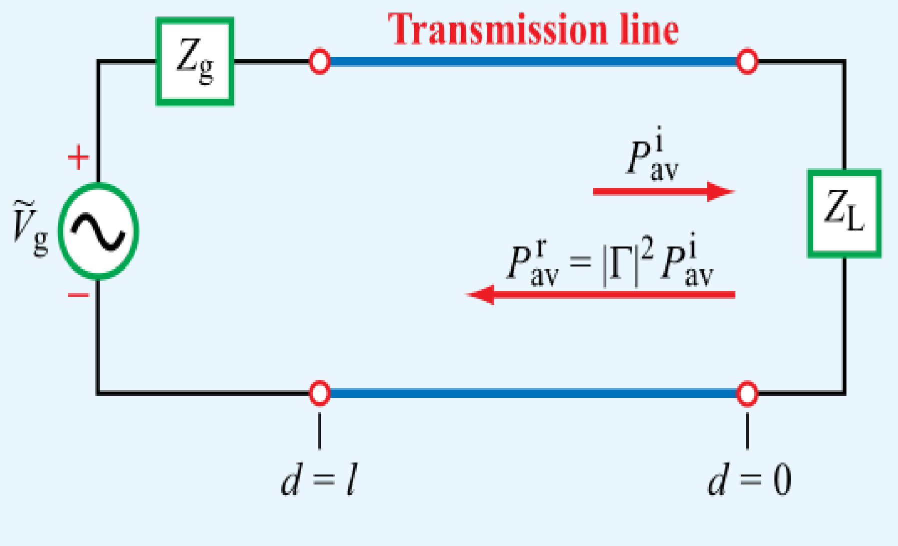

Figure 2-23 The time-average power reflected by a load connected to a lossless transmission line is equal to the incident power multiplied by $|\Gamma|^2$.

The *net average power flowing towards (and then absorbed by) the load* shown in Fig. 2-23 is

$$P_{\mathrm{av}} = P_{\mathrm{av}}^{\mathrm{i}} + P_{\mathrm{av}}^{\mathrm{r}}$$

$$= \frac{|V_0^+|^2}{2Z_0}\left[1 - |\Gamma|^2\right] \quad \text{(W)}$$

Phasor-domain approach

For any propagating wave with voltage and current phasors $\tilde{V}$ and $\tilde{I}$, a useful formula for computing the time-average power is

$$P_{\text{av}} = \frac{1}{2}\Re e\,[\tilde{V}\cdot\tilde{I}^*]$$

where $\tilde{I}^*$ is the complex conjugate of $\tilde{I}$. Application of this formula to $\tilde{V}(d) = V_0^+\left(e^{j\beta d} + \Gamma e^{-j\beta d}\right)$ and $\tilde{I}(d) = \frac{V_0^+}{Z_0}\left(e^{j\beta d} - \Gamma e^{-j\beta d}\right)$ gives

$$P_{\text{av}} = \frac{1}{2}\Re e\left[V_0^+\left(e^{j\beta d} + \Gamma e^{-j\beta d}\right)\cdot\frac{{V_0^+}^*}{Z_0}\left(e^{-j\beta d} - \Gamma^* e^{j\beta d}\right)\right]$$

$$= \frac{1}{2}\Re e\left[\frac{|V_0^+|^2}{Z_0}\left(1 - |\Gamma|^2 + \Gamma e^{-j2\beta d} - \Gamma^* e^{j2\beta d}\right)\right]$$

$$P_{\mathrm{av}} = \frac{1}{2}\Re e\left[V_0^+\left(e^{j\beta d} + \Gamma e^{-j\beta d}\right)\cdot\frac{{V_0^+}^*}{Z_0}\left(e^{-j\beta d} - \Gamma^* e^{j\beta d}\right)\right]$$

$$= \frac{1}{2}\Re e\left[\frac{|V_0^+|^2}{Z_0}\left(1 - |\Gamma|^2 + \Gamma e^{-j2\beta d} - \Gamma^* e^{j2\beta d}\right)\right]$$

$$= \frac{|V_0^+|^2}{2Z_0}\left\{[1 - |\Gamma|^2] + \Re e\left[|\Gamma| e^{-j(2\beta d - \theta_{\mathrm{r}})} - |\Gamma| e^{j(2\beta d - \theta_{\mathrm{r}})}\right]\right\}$$

$$= \frac{|V_0^+|^2}{2Z_0}\{[1 - |\Gamma|^2] + |\Gamma|[\cos(2\beta d - \theta_{\mathrm{r}}) - \cos(2\beta d - \theta_{\mathrm{r}})]\}$$

$$= \frac{|V_0^+|^2}{2Z_0}[1 - |\Gamma|^2]$$

which is identical to $P_{\mathrm{av}} = P_{\mathrm{av}}^{\mathrm{i}} + P_{\mathrm{av}}^{\mathrm{r}} = \frac{|V_0^+|^2}{2Z_0}[1 - |\Gamma|^2]$.

Exercise 2-14:

For a $50\ \Omega$ lossless transmission line terminated in a load impedance $Z_L = (100 + j50)\ \Omega$, determine the fraction of the average incident power reflected by the load.

Answer: The following quantities are given

$$Z_0 = 50\ \Omega \qquad\qquad Z_L = (100 + j50)\ \Omega$$

The voltage reflection coefficient is

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{50 + j50}{150 + j50} = \frac{1 + j1}{3 + j1}$$

$$= \frac{1.414e^{j45^\circ}}{3.162e^{j18.4^\circ}} = 0.45e^{j26.6^\circ}$$

So $|\Gamma| = 0.45$.

According to $P_{\mathrm{av}}^{\mathrm{r}} = -|\Gamma|^2 \frac{\left|V_0^+\right|^2}{2Z_0} = -|\Gamma|^2 P_{\mathrm{av}}^{\mathrm{i}}$, we can obtain

$$\frac{\left|P_{\mathrm{av}}^{\mathrm{r}}\right|}{\left|P_{\mathrm{av}}^{\mathrm{i}}\right|} = |\Gamma|^2 = 0.45^2 = 20\%$$

Exercise 2-15:

For the line of Exercise 2-14, what is the magnitude of the average reflected power if $|V_0^+| = 1\ \mathrm{V}$?

Answer: The following quantities are given

$$Z_0 = 50\ \Omega \qquad |V_0^+| = 1\ \mathrm{V} \qquad |\Gamma| = 0.45$$

According to $P_{\mathrm{av}}^{\mathrm{r}} = -|\Gamma|^2 \frac{|V_0^+|^2}{2Z_0}$,

$$|P_{\mathrm{av}}^{\mathrm{r}}| = |\Gamma|^2 \frac{|V_0^+|^2}{2Z_0}$$

$$= 0.45^2 \cdot \frac{1^2}{2 * 50} = 2\ \ \mathrm{mW}$$

Concept Question

Concept Question 2-15:

According to $P^{\mathrm{r}}(d,t) = -|\Gamma|^2 \frac{|V_0^+|^2}{2Z_0}[1 + cos(2\omega t - 2\beta d + 2\phi^+ + 2\theta_{\mathrm{r}})]$, the instantaneous value of the reflected power depends on the phase of the reflection coefficient θ_{r} , but the average reflected power given by $P_{\mathrm{av}}^{\mathrm{r}} = -|\Gamma|^2 \frac{|V_0^+|^2}{2Z_0} = -|\Gamma|^2 P_{\mathrm{av}}^{\mathrm{i}}$ does not. Explain.

In $P^{\mathrm{r}}(d,t)$, the instantaneous power consists of a dc (non–time-varying) term and an ac term that oscillates at an angular frequency of 2ω. The average reflected power $P_{\mathrm{av}}^{\mathrm{r}}$ just consists of a dc (non–time-varying) term.

Concept Question 2-16:

What is the average power delivered by a lossless transmission line to a reactive load?

$$P_{\mathrm{av}} = P_{\mathrm{av}}^{\mathrm{i}} + P_{\mathrm{av}}^{\mathrm{r}}$$

$$= \frac{|V_0^+|^2}{2Z_0}\left[1 - |\Gamma|^2\right] \quad (\mathrm{W})$$

Concept Question 2-17:

What fraction of the incident power is delivered to a matched load?

100%

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Concept Question 2-18:

Verify that $\frac{1}{T}\int_0^T \cos^2(\frac{2\pi t}{T} + \beta d + \phi)\, dt = \frac{1}{2}$ regardless of the values of d and ϕ, so long as neither is a function of t.

$$\frac{1}{T}\int_0^T \cos^2(\frac{2\pi t}{T} + \beta d + \phi)\, dt = \frac{1}{T}\int_0^T \frac{\cos 2\left(\frac{2\pi t}{T} + \beta d + \phi\right) + 1}{2} dt$$

$$= \frac{1}{2T}\left[\int_0^T \cos\left(\frac{4\pi t}{T} + 2\beta d + 2\phi\right) dt + \int_0^T 1\, dt\right]$$

$$= \frac{1}{2T}\left[\left[\frac{\sin\left(\frac{4\pi t}{T} + 2\beta d + 2\phi\right)}{\frac{4\pi}{T}}\right]_0^T + [t]_0^T\right] = \frac{1}{2T}\cdot T = \frac{1}{2}$$