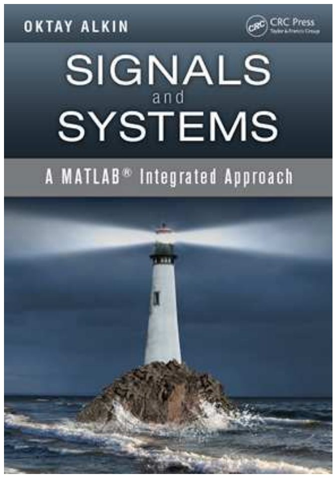

Chapter 1

Signal Representation and Modeling

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CRC Press
Taylor & Francis Group

Objectives

- Understand the concept of a *signal* and how to work with mathematical models of signals.
- Discuss fundamental signal types and signal operations used in the study of signals and systems.
- Experiment with methods of simulating continuous and discrete-time signals with MATLAB.
- Learn various ways of classifying signals and discuss symmetry properties.
- Explore characteristics of sinusoidal signals. Learn *phasor* representation of sinusoidal signals, and how phasors help with analysis.
- Understand the decomposition of signals using unit-impulse functions of appropriate type.
- Learn energy and power definitions.

Mathematical modeling

Mathematical models for signals

The mathematical model for a signal is in the form of a formula, function, algorithm or a graph that approximately describes the time variations of the physical signal.

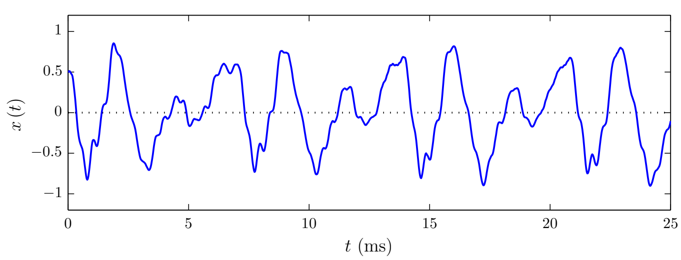
Goals:

1. Understand the characteristics of the signal in terms of its behavior in time and in terms of the frequencies it contains (signal analysis).
2. Develop methods of creating signals with desired characteristics (signal synthesis).
3. Understand how a system responds to a signal and why (system analysis).
4. Develop methods of constructing a system that responds to a signal in some prescribed way (system synthesis).

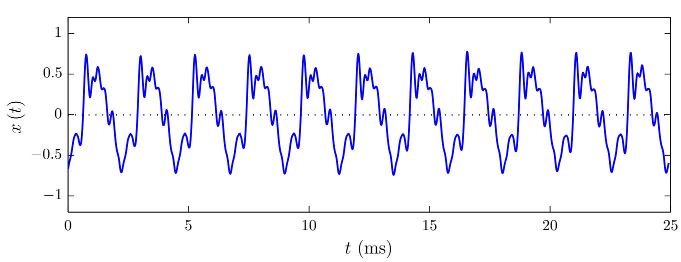
MATLAB Exercise 1.1

Continuous-time signals

A segment from the vowel "o" of the word "hello"



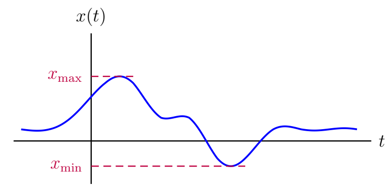
A segment from the sound of a violin

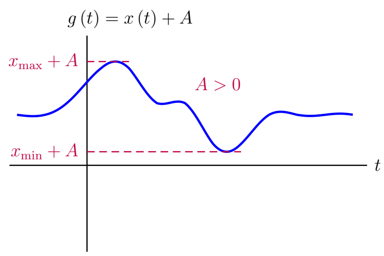


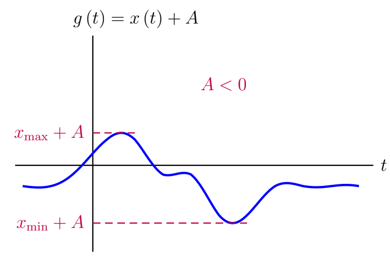
Signal operations

Addition of a constant offset

$$g(t) = x(t) + A$$



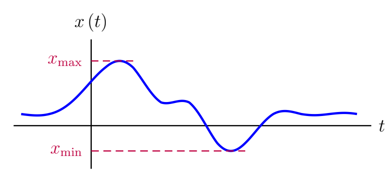


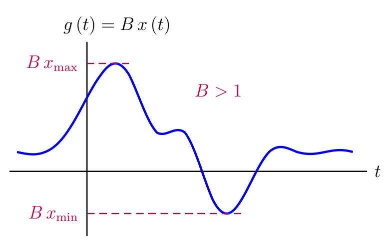


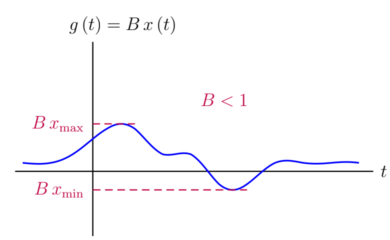
Signal operations (continued)

Multiplication by a constant gain factor

$$g(t) = B\,x(t)$$



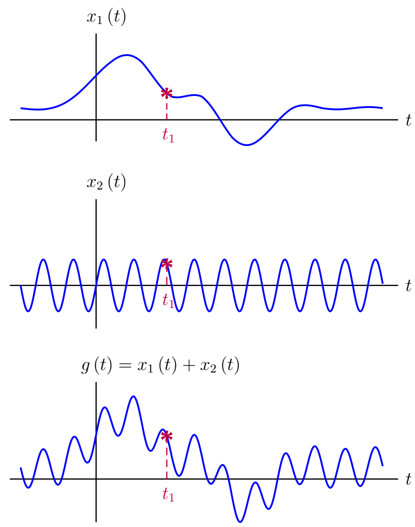




Signal operations (continued)

Adding two signals

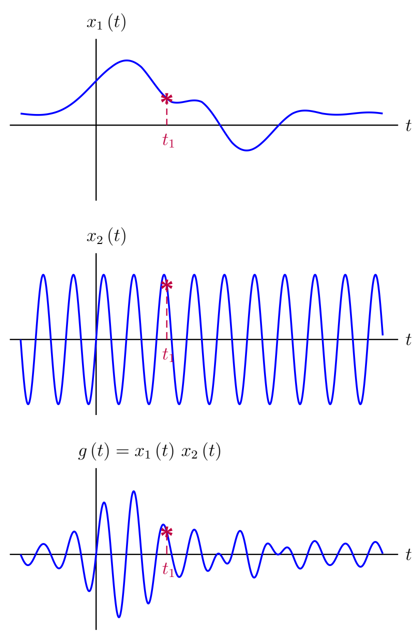
$$g(t) = x_1(t) + x_2(t)$$



Signal operations (continued)

Multiplying two signals

$$g(t) = x_1(t)\; x_2(t)$$



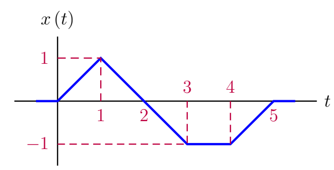
Example 1.1

Constant offset and gain

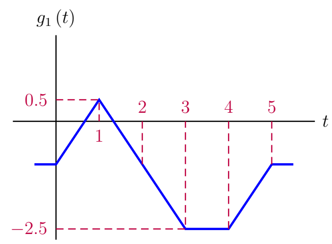
Consider the signal shown. Sketch the signals

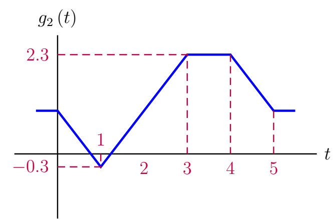
a. $g_1(t) = 1.5\,x(t) - 1$

b. $g_2(t) = -1.3\,x(t) + 1$



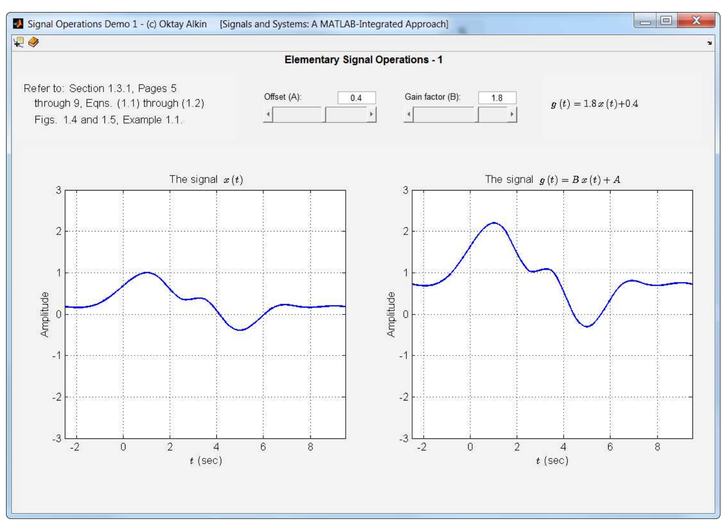
Solution:





Interactive demo: sop_demo1

Experiment by varying parameters A and B.



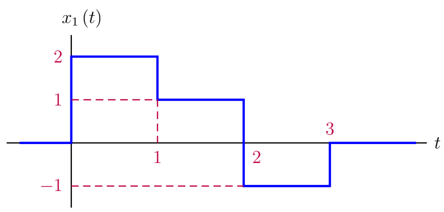
Example 1.2

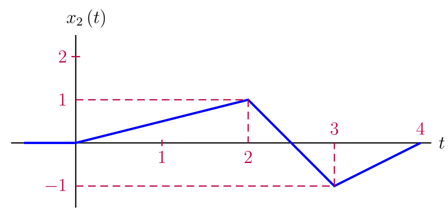
Arithmetic operations with continuous-time signals

Given the signals $x_1(t)$ and $x_2(t)$, sketch the signals

a. $g_1(t) = x_1(t) + x_2(t)$

b. $g_2(t) = x_1(t)\, x_2(t)$





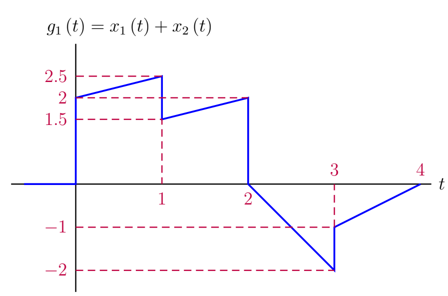
Example 1.2 (continued)

Solution - Part(a):

$$x_1(t) = \begin{cases} 2\,, & 0 < t < 1 \\ 1\,, & 1 < t < 2 \\ -1\,, & 2 < t < 3 \\ 0\,, & \text{otherwise} \end{cases}$$

$$x_2(t) = \begin{cases} \frac{1}{2}t\,, & 0 < t < 2 \\ -2t+5\,, & 2 < t < 3 \\ t-4\,, & 3 < t < 4 \\ 0\,, & \text{otherwise} \end{cases}$$

$$g_1(t) = \begin{cases} \frac{1}{2}t+2\,, & 0 < t < 1 \\ \frac{1}{2}t+1\,, & 1 < t < 2 \\ -2t+4\,, & 2 < t < 3 \\ t-4\,, & 3 < t < 4 \\ 0\,, & \text{otherwise} \end{cases}$$



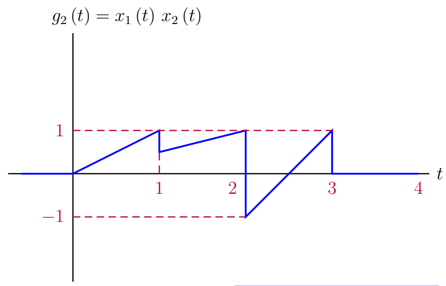
Example 1.2 (continued)

Solution - Part(b):

$$x_1(t) = \begin{cases} 2\,, & 0 < t < 1 \\ 1\,, & 1 < t < 2 \\ -1\,, & 2 < t < 3 \\ 0\,, & \text{otherwise} \end{cases}$$

$$x_2(t) = \begin{cases} \frac{1}{2}t\,, & 0 < t < 2 \\ -2t+5\,, & 2 < t < 3 \\ t-4\,, & 3 < t < 4 \\ 0\,, & \text{otherwise} \end{cases}$$

$$g_2(t) = \begin{cases} t\,, & 0 < t < 1 \\ \frac{1}{2}t\,, & 1 < t < 2 \\ 2t-5\,, & 2 < t < 3 \\ 0\,, & \text{otherwise} \end{cases}$$

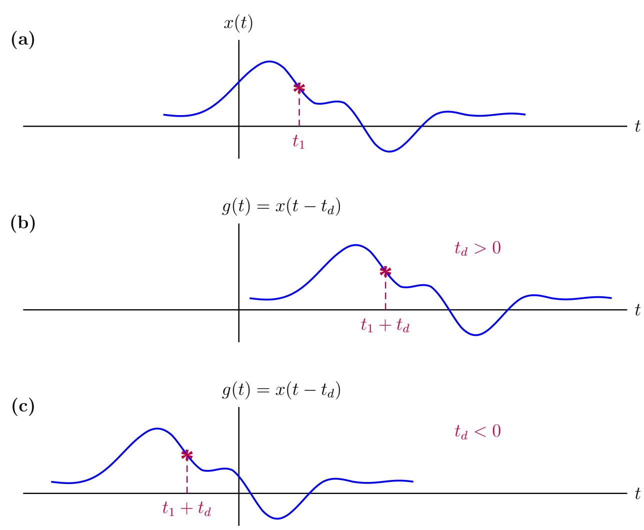


MATLAB Exercise 1.2

Signal operations (continued)

Time shifting

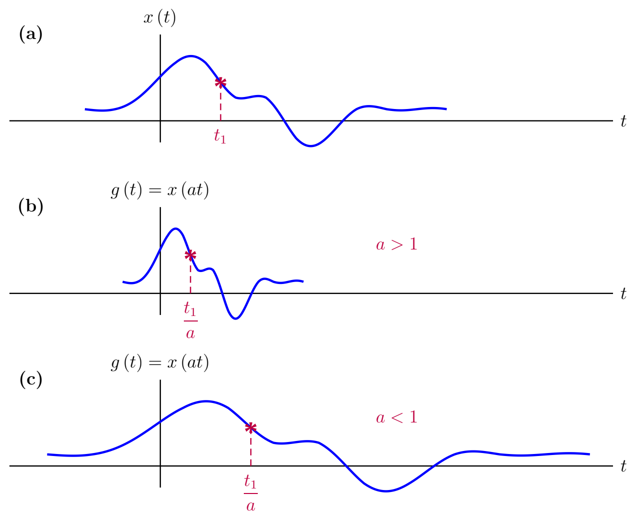
$$g(t) = x(t - t_d)$$



Signal operations (continued)

Time scaling

$$g(t) = x(at)$$

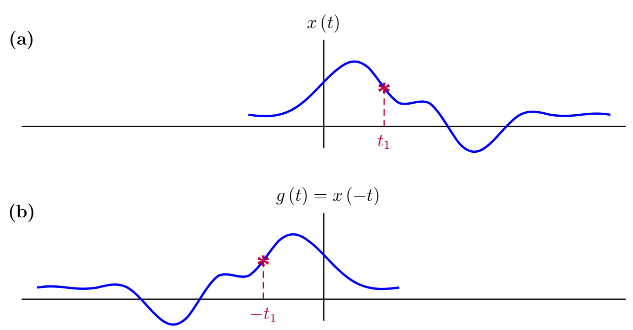


Chapter 1
Continuous-Time Signals
Signal operations

Signal operations (continued)

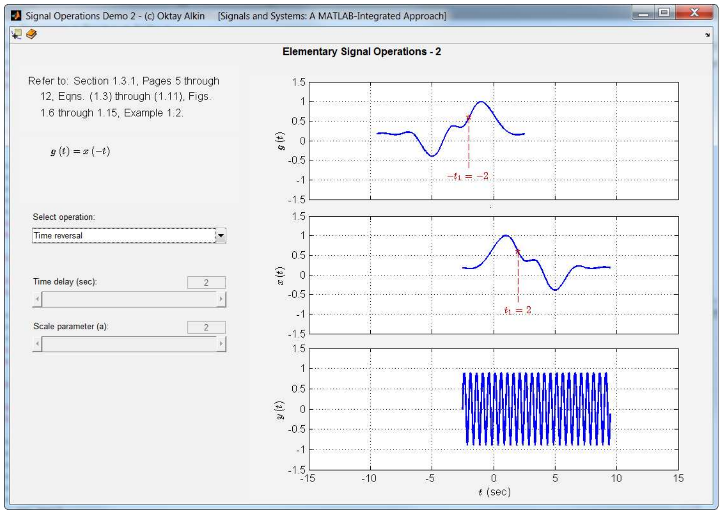
Time reversal

$$g(t) = x(-t)$$



Interactive demo: sop_demo2

Experiment by varying applicable parameter values.



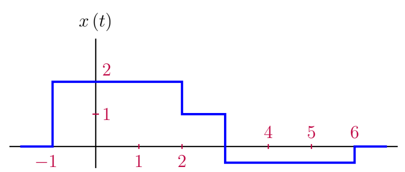
Example 1.3

Basic operations for continuous-time signals

Consider the signal $x(t)$ shown. Sketch the following signals:

a. $g(t) = x(2t - 5)$,

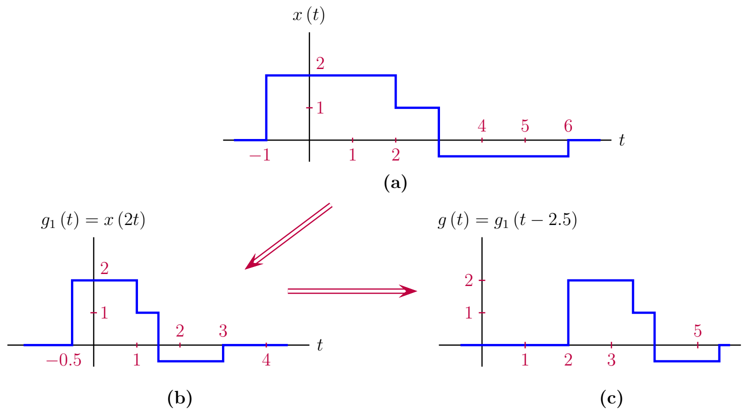
b. $h(t) = x(-4t + 2)$.



Example 1.3 (continued)

Solution - Part(a):

$$g(t) = g_1(t-2.5) = x(2[t-2.5]) = x(2t-5)$$

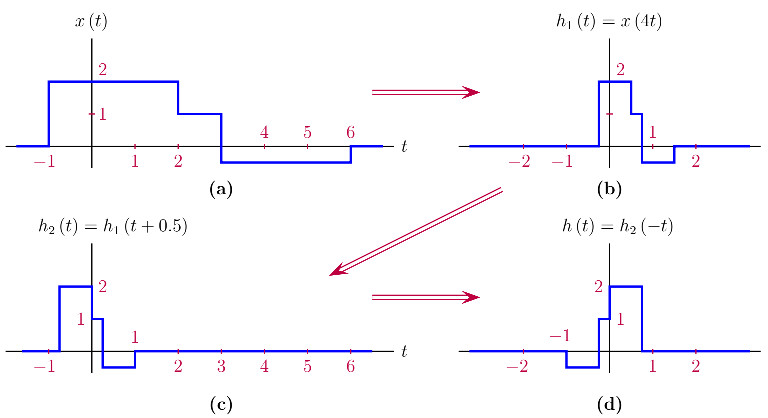


Example 1.3 (continued)

Solution - Part(b):

$$h_2(t) = h_1(t+0.5) = x(4[t+0.5]) = x(4t+2)$$

$$h(t) = h_2(-t) = x(-4t+2)$$



MATLAB Exercise 1.3

Basic building blocks

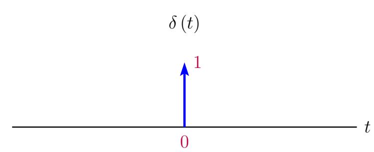
- Unit-impulse function
- Unit-step function
- Unit-pulse function
- Unit-ramp function
- Unit-triangle function
- Sinusoidal signals

Unit-impulse function

Mathematical definition

$$\delta(t) = \begin{cases} 0\,, & \text{if } t \neq 0 \\ \text{undefined}\,, & \text{if } t = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

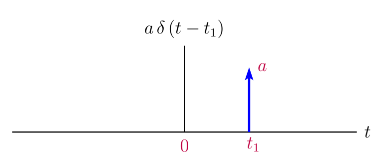


Scaling and time shifting:

$$a\,\delta(t-t_1) = \begin{cases} 0\,, & \text{if } t \neq t_1 \\ \text{undefined}\,, & \text{if } t = t_1 \end{cases}$$

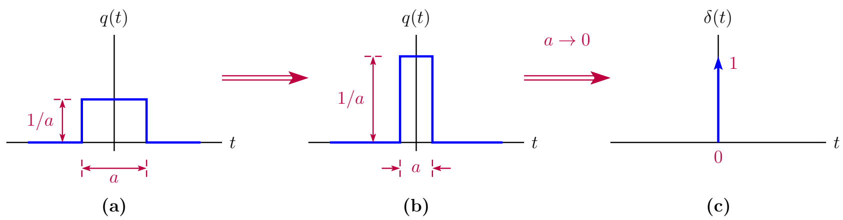
and

$$\int_{-\infty}^{\infty} a\,\delta(t-t_1)\, dt = a$$



Obtaining unit-impulse function from a rectangular pulse

$$\text{Let} \quad q(t) = \begin{cases} \dfrac{1}{a}\,, & |t| < \dfrac{a}{2} \\ 0\,, & |t| > \dfrac{a}{2} \end{cases}$$



$$\delta(t) = \lim_{a \to 0} \left[q(t) \right]$$

Chapter 1
Continuous-Time Signals
Basic building blocks for continuous-time signals

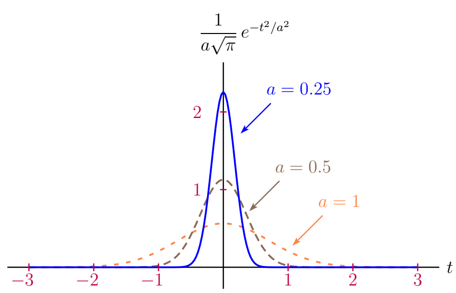
Other functions that can produce a unit-impulse

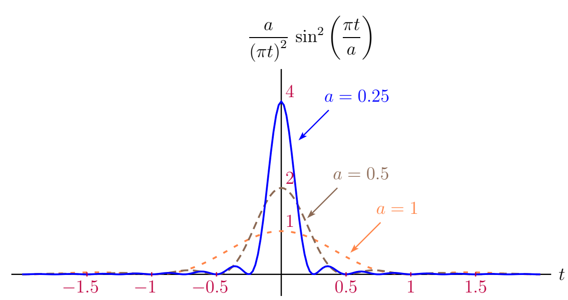
Gaussian function:

$$\delta(t) = \lim_{a \to 0} \left[\frac{1}{a\sqrt{\pi}}\, e^{-t^2/a^2} \right]$$

Squared sinc pulse:

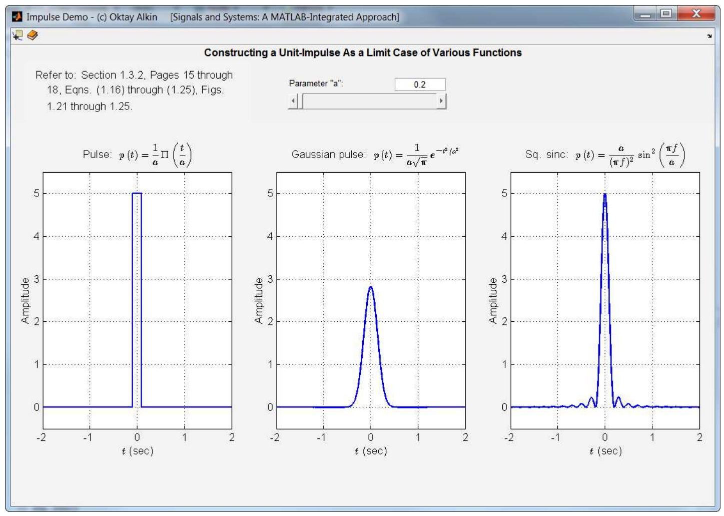
$$\delta(t) = \lim_{a \to 0} \left[\frac{a}{(\pi t)^2}\, \sin^2\left(\frac{\pi t}{a}\right) \right]$$





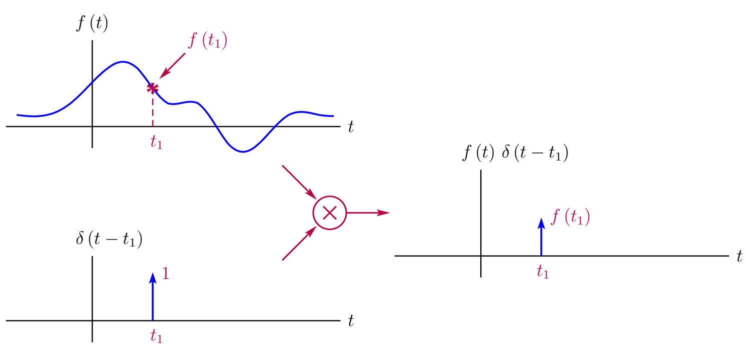
Interactive demo: imp_demo.m

Experiment by changing the parameter a.



Sampling property of the unit-impulse function

$$f(t)\,\delta(t-t_1) = f(t_1)\,\delta(t-t_1)$$



The function $f(t)$ must be continuous at $t = t_1$.

Sifting property of the unit-impulse function

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-t_1)\,dt = f(t_1)$$

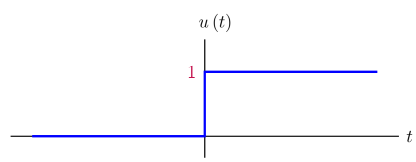
$$\int_{t_1-\Delta t}^{t_1+\Delta t} f(t)\,\delta(t-t_1)\,dt = f(t_1)$$

The function $f(t)$ must be continuous at $t=t_1$. Also, $\Delta t > 0$.

Unit-step function

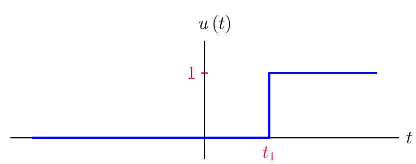
Mathematical definition

$$u(t) = \begin{cases} 1\,, & t>0 \\ 0\,, & t<0 \end{cases}$$



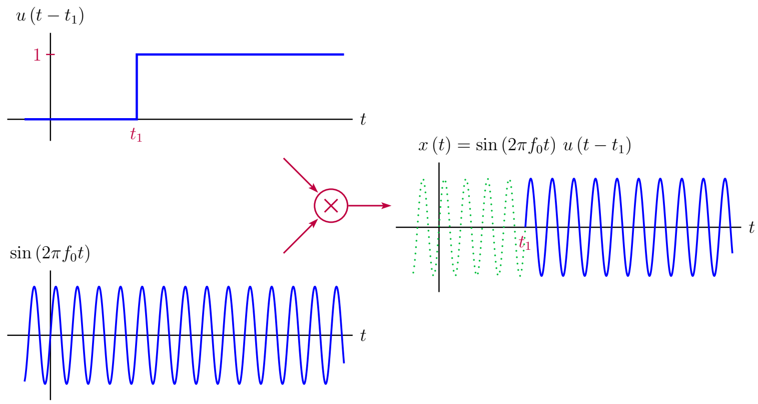
Time shifting the unit-step function:

$$u(t-t_1) = \begin{cases} 1\,, & t>t_1 \\ 0\,, & t<t_1 \end{cases}$$



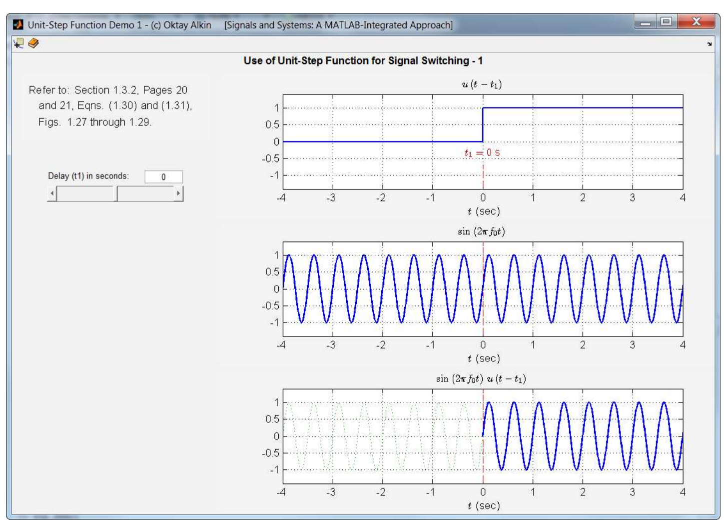
Using the unit-step function to turn a signal on

$$x(t) = \sin(2\pi f_0 t)\; u(t - t_1) = \begin{cases} \sin(2\pi f_0 t)\ , & t > t_1 \\ 0\ , & t < t_1 \end{cases}$$



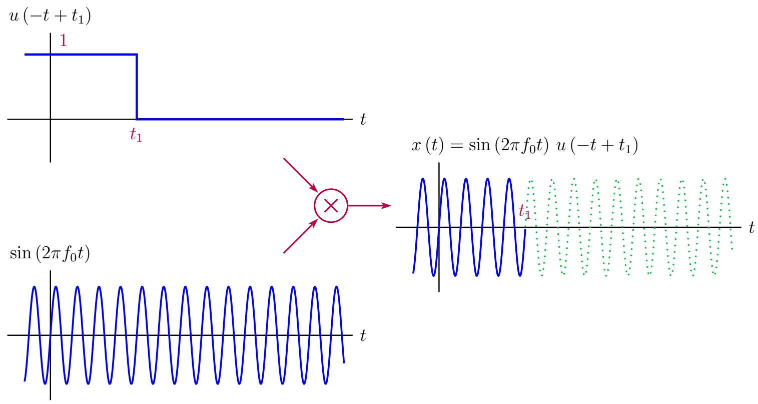
Interactive demo: stp_demo1

Experiment by changing the delay t_1.



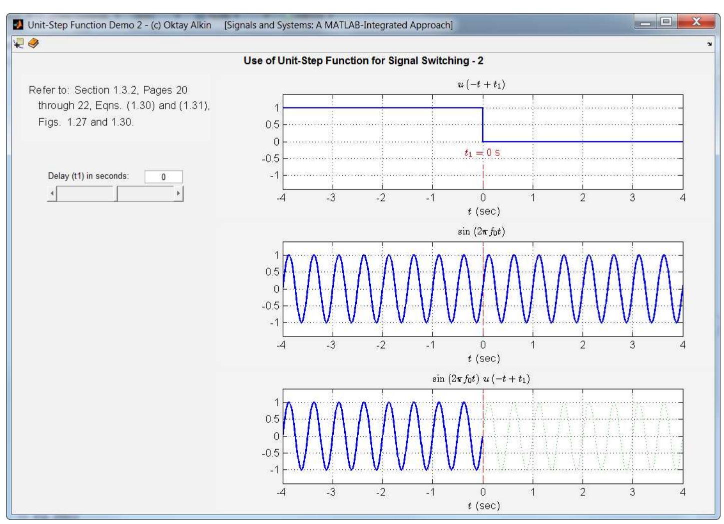
Using the unit-step function to turn a signal off

$$x(t) = \sin(2\pi f_0 t)\; u(-t+t_1) = \begin{cases} \sin(2\pi f_0 t)\;, & t < t_1 \\ 0\;, & t > t_1 \end{cases}$$



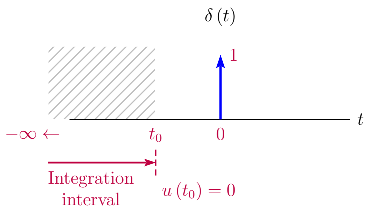
Interactive demo: stp_demo2

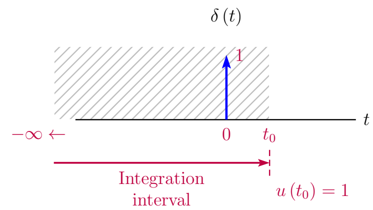
Experiment by changing the delay t_1.

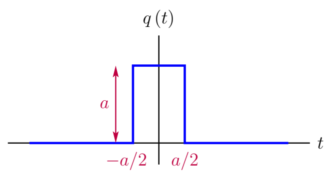


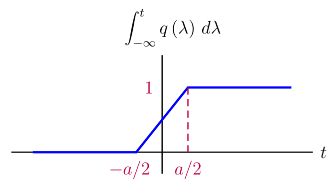
Relationship between unit-step and unit-impulse functions

$$u(t) = \int_{-\infty}^{t} \delta(\lambda)\, d\lambda \quad \Rightarrow \quad \delta(t) = \frac{du}{dt}$$





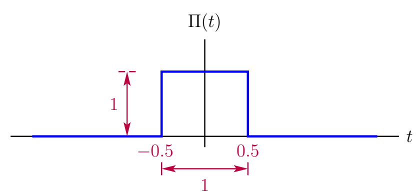




Unit-pulse function

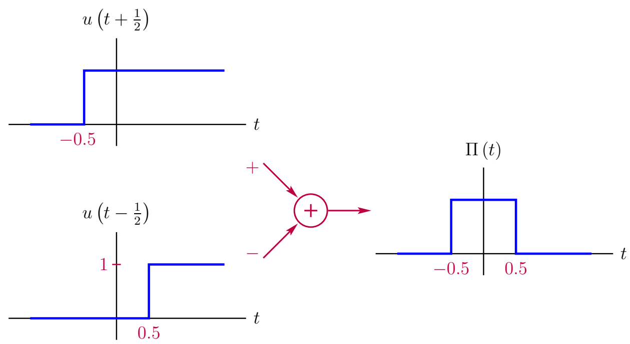
Mathematical definition

$$\Pi(t) = \begin{cases} 1, & |t| < \frac{1}{2} \\ 0, & |t| > \frac{1}{2} \end{cases}$$



Constructing a unit-pulse from unit-step functions

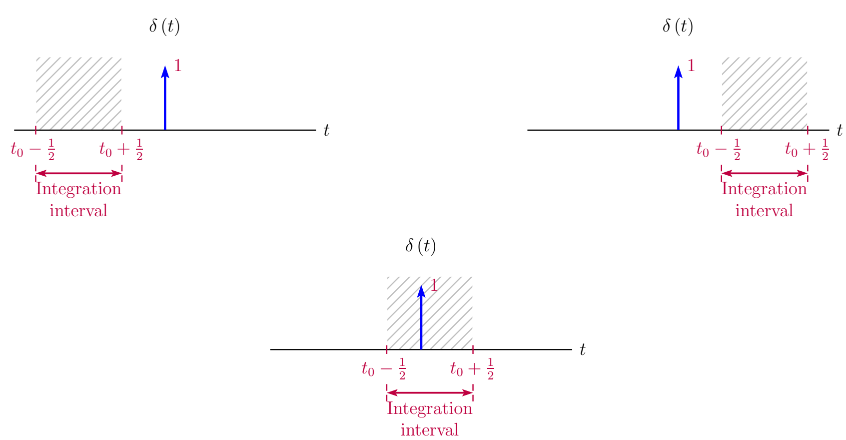
$$\Pi(t) = u\left(t+\tfrac{1}{2}\right) - u\left(t-\tfrac{1}{2}\right)$$



Constructing a unit-pulse from unit-impulse functions

$$\int_{t-1/2}^{t+1/2} \delta(\lambda)\, d\lambda = \begin{cases} 1\,, & \text{if} \quad t-\frac{1}{2} < 0 \text{ and } t+\frac{1}{2} > 0 \\ 0\,, & \text{otherwise} \end{cases}$$

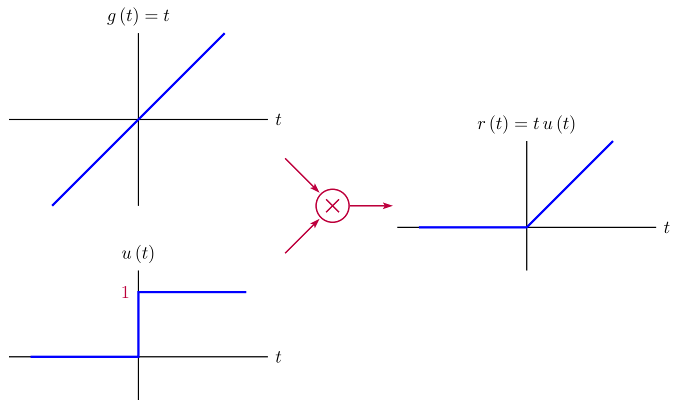
$$= \begin{cases} 1\,, & -\frac{1}{2} < t < \frac{1}{2} \\ 0\,, & \text{otherwise} \end{cases}$$



Unit-ramp function

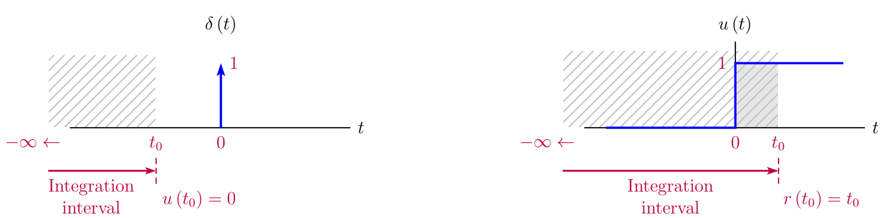
Mathematical definition

$$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases} \qquad \text{or, equivalently} \qquad r(t) = t\,u(t)$$



Constructing a unit-ramp from a unit-step

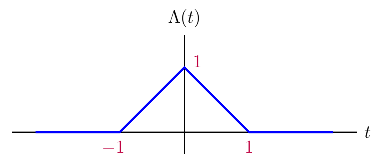
$$r(t) = \int_{-\infty}^{t} u(\lambda)\, d\lambda$$



Unit-triangle function

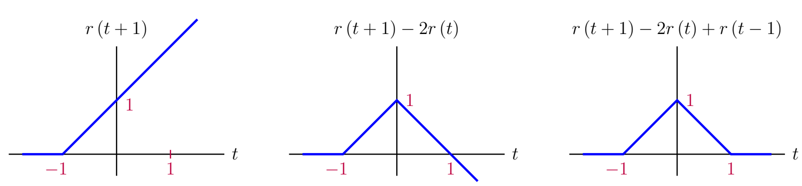
Mathematical definition

$$\Lambda\left(t\right)=\begin{cases} t+1, & -1\le t<0 \\ -t+1, & 0\le t<1 \\ 0, & \text{otherwise}\end{cases}$$



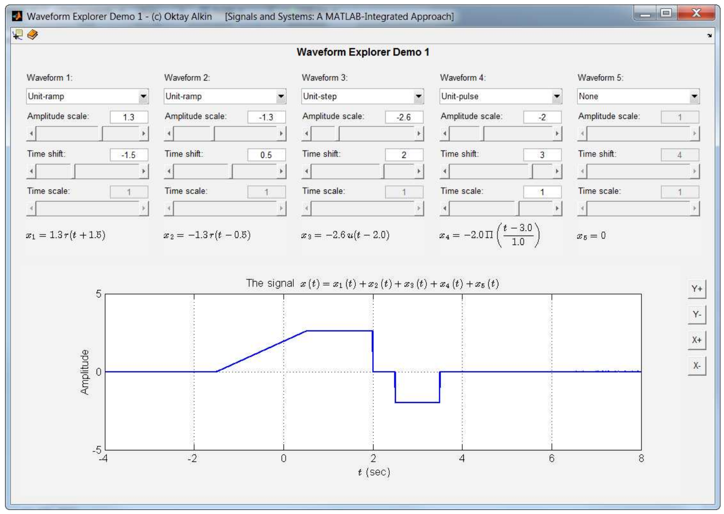
Constructing a unit-triangle using unit-ramp function

$$\Lambda\left(t\right)=r\left(t+1\right)-2\,r\left(t\right)+r\left(t-1\right)$$



Interactive demo: wav_demo1

Construct waveforms using basic building blocks.



Sinusoidal signals

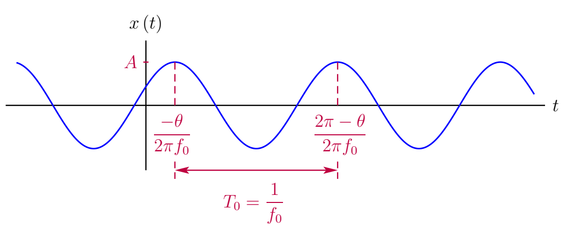
Sinusoidal signal

$$x(t) = A\cos(\omega_0 t + \theta)$$

A : Amplitude

ω_0 : Radian frequency (rad/s)

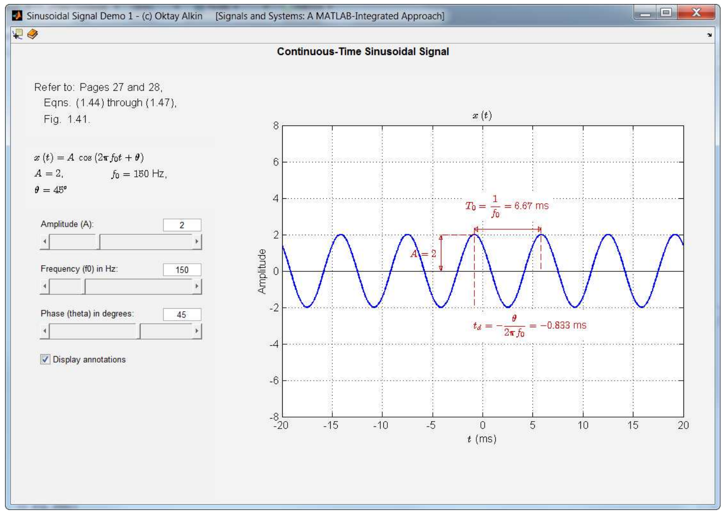
θ : Phase (radians)



▸ MATLAB Exercise 1.5

Interactive demo: sin_demo2

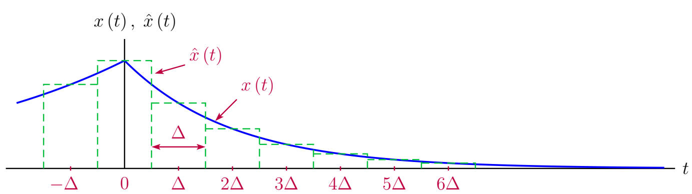
Experiment by varying the amplitude A, the frequency f_0 and the phase θ.



Impulse decomposition for continuous-time signals

Rough approximation to the signal $x(t)$:

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x(n\Delta)\ \Pi\left(\frac{t-n\Delta}{\Delta}\right)$$

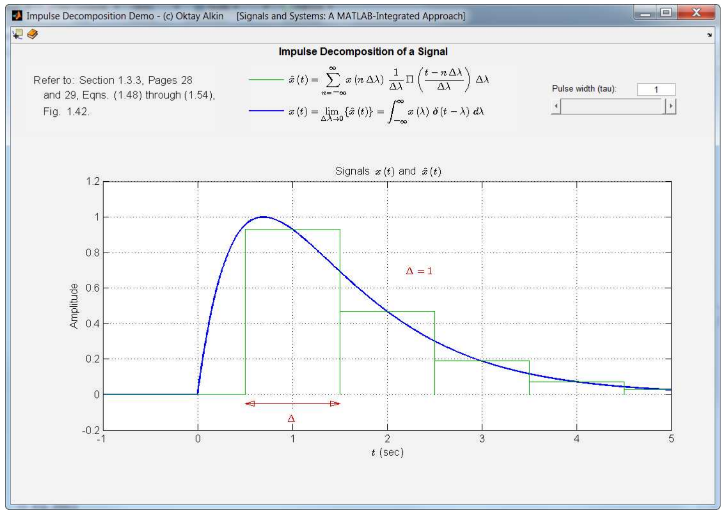


Take the limit as $\Delta \to 0$:

$$\begin{aligned} x(t) &= \lim_{\Delta\to 0}\left[\hat{x}(t)\right] \\ &= \int_{-\infty}^{\infty} x(\lambda)\ \delta(t-\lambda)\ d\lambda \end{aligned}$$

Interactive demo: id_demo

Experiment by varying the parameter Δ.



Real vs. complex signals

Complex signal in Cartesian form

$$x\,(t) = x_r\,(t) + j\,x_i\,(t)$$

Complex signal in polar form

$$x\,(t) = |x\,(t)|\;e^{j\measuredangle x(t)}$$

$$|x(t)| = \left[x_r^2(t) + x_i^2(t)\right]^{1/2}$$

$$\measuredangle x(t) = \tan^{-1}\left[\frac{x_i(t)}{x_r(t)}\right]$$

$$x_r(t) = |x(t)|\,\cos\left(\measuredangle x(t)\right)$$

$$x_i(t) = |x(t)|\,\sin\left(\measuredangle x(t)\right)$$

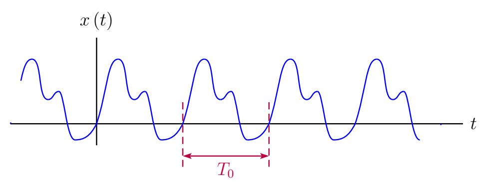
Periodic signals

Definition

A signal is said to be *periodic* if it satisfies

$$x\left(t+T_0\right)=x\left(t\right)$$

at all time instants t, and for a specific value of $T_0 \neq 0$.



If a signal is periodic with period T_0, then it is also periodic with periods of $2T_0, 3T_0, \ldots, kT_0, \ldots$ where k is any integer.

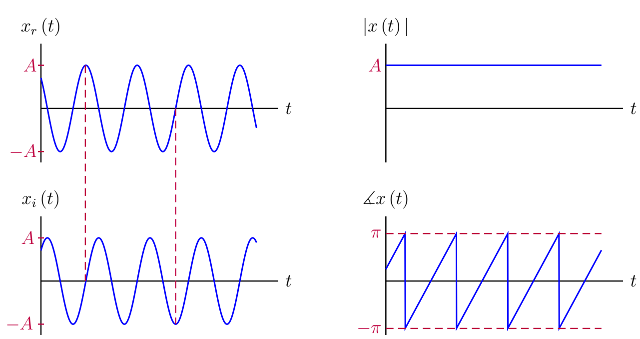
Example 1.4

Working with a complex periodic signal

Consider a signal defined by

$$\begin{aligned} x\left(t\right) =& x_r\left(t\right)+j\,x_i\left(t\right) \\ =& A\,\cos\left(2\pi f_0 t+\theta\right)+j\,A\,\sin\left(2\pi f_0 t+\theta\right) \end{aligned}$$

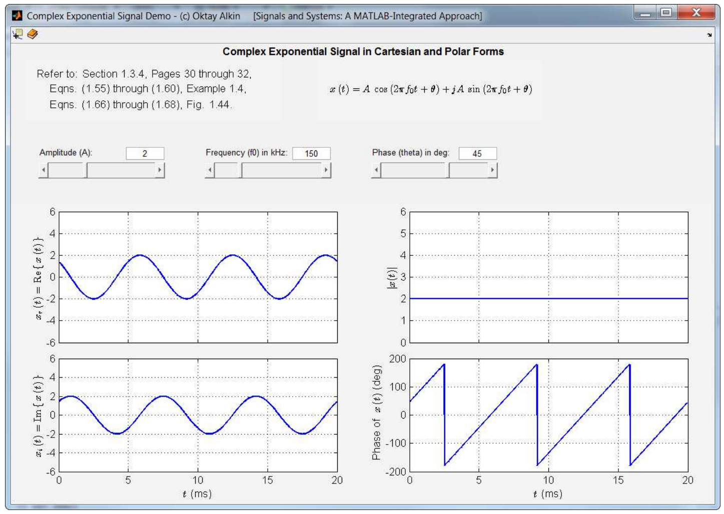
Graph the components in Cartesian and polar representations of this signal.



MATLAB Exercise 1.4

Interactive demo: cexp_demo

Experiment by varying the parameters A, f_0 and θ.



Example 1.5

Periodicity of continuous-time sinusoidal signals

Consider two continuous-time sinusoidal signals

$$x_1(t) = A_1 \sin(2\pi f_1 t + \theta_1)\ , \quad x_2(t) = A_2 \sin(2\pi f_2 t + \theta_2)$$

Determine the conditions under which the sum signal

$$x(t) = x_1(t) + x_2(t)$$

is also periodic. Also, determine the fundamental period of the signal $x(t)$ as a function of the relevant parameters of $x_1(t)$ and $x_2(t)$.

Solution:

$$x_1(t + m_1 T_1) = x_1(t)\ , \quad T_1 = 1/f_1$$

$$x_2(t + m_2 T_2) = x_2(t)\ , \quad T_2 = 1/f_2$$

$$x_1(t + T_0) + x_2(t + T_0) = x_1(t) + x_2(t)$$

$$T_0 = m_1 T_1 = m_2 T_2 \qquad \Rightarrow \qquad \frac{1}{f_0} = \frac{m_1}{f_1} = \frac{m_2}{f_2}$$

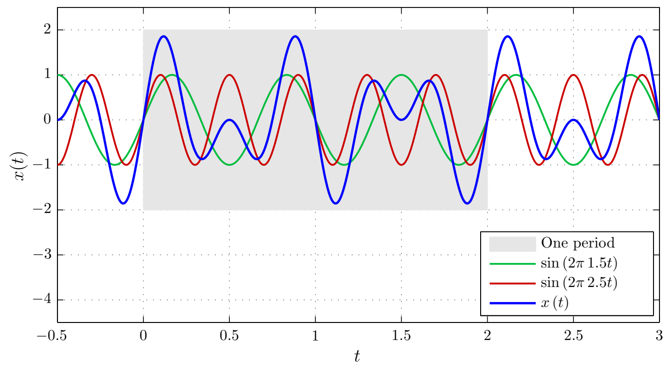
Example 1.6

More on the periodicity of sinusoidal signals

Discuss the periodicity of the signals

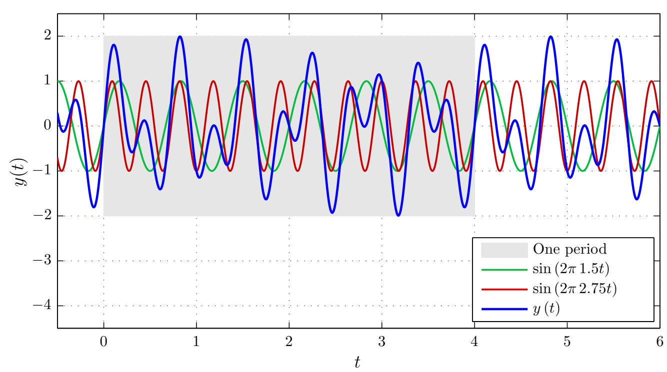
a. $x(t) = \sin(2\pi\, 1.5\, t) + \sin(2\pi\, 2.5\, t)$
b. $y(t) = \sin(2\pi\, 1.5\, t) + \sin(2\pi\, 2.75\, t)$

Solution - Part (a): $f_0 = 0.5$ Hz, $T_0 = 2$ seconds



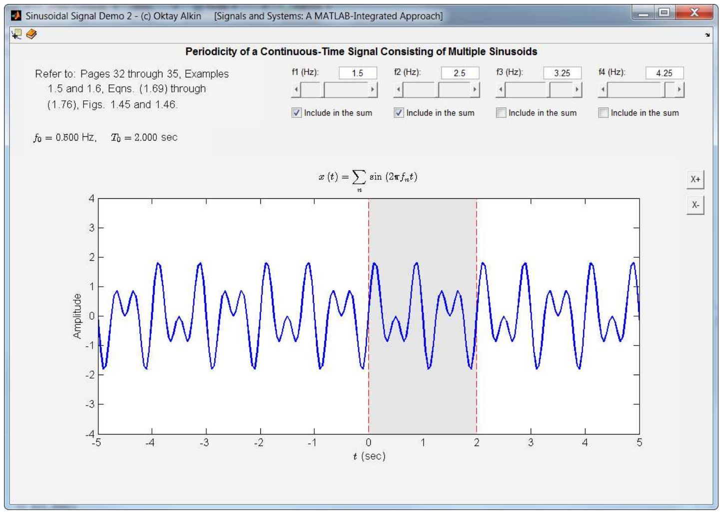
Example 1.6 (continued)

Solution - Part (b): $f_0 = 0.25$ Hz, $T_0 = 4$ seconds



Interactive demo: sin_demo2

Experiment by varying the frequencies of sinusoids.



Energy computations

Normalized energy of a signal

$$E_x = \int_{-\infty}^{\infty} x^2(t)\, dt$$

if the integral can be computed.

Normalized energy of a complex signal

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

if the integral can be computed.

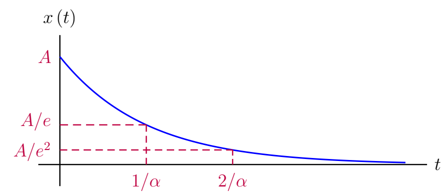
Example 1.7

Energy of a right-sided exponential signal

Compute the normalized energy of the right-sided exponential signal

$$x\,(t) = A\,e^{-\alpha t}\,u(t)$$

where $\alpha > 0$.



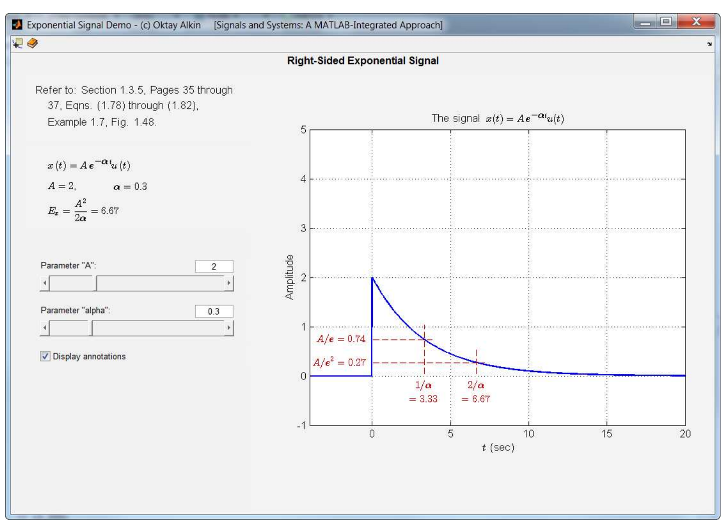
Solution:

$$E_x = \int_0^{\infty} A^2\,e^{-2\alpha t}\,dt = A^2\,\frac{e^{-2\alpha t}}{-2\alpha}\bigg|_0^{\infty} = \frac{A^2}{2\alpha}$$

The restriction $\alpha > 0$ is necessary since, without it, we could not have evaluated the integral.

Interactive demo: exp_demo

Experiment by varying the parameters A and α.



Time averaging operator

Time average of a signal periodic with period T_0

$$\langle x(t) \rangle = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)\, dt$$

Time average of an aperiodic signal

$$\langle x(t) \rangle = \lim_{T \to \infty} \left[\frac{1}{T} \int_{-T/2}^{T/2} x(t)\, dt \right]$$

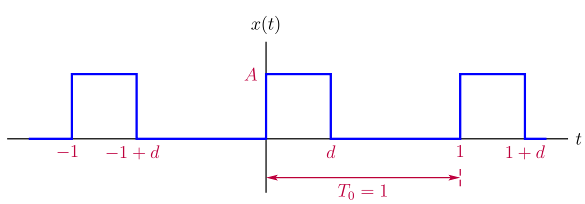
Chapter 1
Continuous-Time Signals
Energy and power definitions

Example 1.8

Time average of a pulse-train

Compute the time average of a periodic pulse-train with an amplitude of A and a period of $T_0 = 1$ s, defined by the equations

$$x(t) = \begin{cases} A\,, & 0 < t < d \\ 0\,, & d < t < 1 \end{cases} \quad \text{and} \quad x(t + k\,T_0) = x(t)\,, \quad \text{all } t, \text{ all integer } k$$

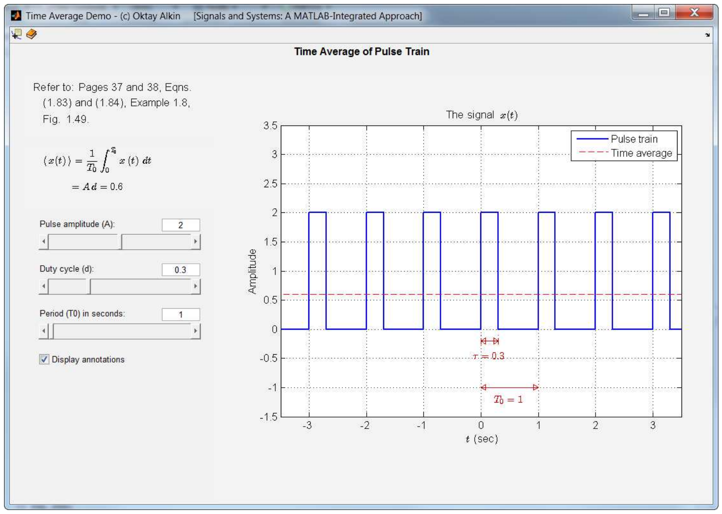


Solution:

$$\langle x(t) \rangle = \int_0^1 x(t)\, dt = \int_0^d (A)\, dt + \int_d^1 (0)\, dt = A\, d$$

Interactive demo: tavg_demo

Experiment by varying the amplitude A and duty cycle d.



Power computations

Normalized instantaneous power (real signal)

$$p_{\text{norm}}(t) = x^2(t)$$

Normalized instantaneous power (complex signal)

$$p_{\text{norm}}(t) = |x(t)|^2$$

Normalized average power (real signal)

$$P_x = \left\langle x^2(t) \right\rangle$$

Normalized average power (complex signal)

$$P_x = \left\langle |x(t)|^2 \right\rangle$$

Example 1.9

Power of a sinusoidal signal

Determine the normalized average power of the signal

$$x(t) = A\,\sin(2\pi f_0 t + \theta)$$

Solution:

$$\begin{aligned} P_x &= f_0 \int_{-1/2f_0}^{1/2f_0} A^2 \sin^2\left(2\pi f_0 t + \theta\right)\,dt \\ &= f_0 \int_{-1/2f_0}^{1/2f_0} \frac{A^2}{2}\,dt - f_0 \int_{-1/2f_0}^{1/2f_0} \frac{A^2}{2} \cos\left(4\pi f_0 t + 2\theta\right)\,dt \\ &= \frac{A^2}{2} \end{aligned}$$

Example 1.10

Right-sided exponential signal revisited

Compute normalized energy or normalized average power of the signal

$$x\,(t) = A\,e^{-\alpha t}\,u\,(t)$$

as appropriate for $\alpha = 0$.

Solution: For $\alpha > 0$, the normalized average power of the signal is $P_x = 0$. For $\alpha = 0$ we get $x\,(t) = A\,u\,(t)$. Therefore

$$\begin{aligned} P_x &= \lim_{T\to\infty} \left[\frac{1}{T} \int_{-T/2}^{T/2} x^2\,(t)\,dt\right] \\ &= \lim_{T\to\infty} \left[\frac{1}{T} \int_{-T/2}^{0} (0)\,dt + \frac{1}{T} \int_{0}^{T/2} A^2\,dt\right] \\ &= \lim_{T\to\infty} \left[\frac{A^2}{2}\right] = \frac{A^2}{2} \end{aligned}$$

and $E_x \to \infty$.

Energy signals vs. power signals

- Energy signals are those that have finite energy, and zero power.
  $E_x < \infty$, and $P_x = 0$.
- Power signals are those that have finite power and infinite energy.
  $E_x \to \infty$, and $P_x < \infty$.

RMS value of a signal

Mathematical definition

The *root-mean-square (RMS)* value of a signal $x(t)$ is defined as

$$X_{RMS} = \left[\left\langle x^2(t)\right\rangle\right]^{1/2}$$

Example 1.11 - RMS value of a sinusoidal signal

Determine the RMS value of the signal $x(t) = A \sin(2\pi f_0 t + \theta)$

<u>Solution:</u>
Recall that

$$P_x = \frac{A^2}{2}$$

It follows that

$$X_{RMS} = \sqrt{P_x} = \frac{A}{\sqrt{2}}$$

Example 1.12

RMS value of a multitone signal

Determine the RMS value and the normalized average power of the signal

$$x(t) = a_1 \cos(2\pi f_1 t + \theta_1) + a_2 \cos(2\pi f_2 t + \theta_2)$$

where the two frequencies are distinct, i.e., $f_1 \neq f_2$.

Solution:

$$\left\langle x^2(t) \right\rangle = \frac{a_1^2}{2} + \frac{a_2^2}{2}$$

$$X_{RMS} = \sqrt{\frac{a_1^2}{2} + \frac{a_2^2}{2}}$$

$$P_x = \frac{a_1^2}{2} + \frac{a_2^2}{2}$$

Symmetry properties

Even symmetry

A real-valued signal is said to have *even symmetry* if it has the property

$$x(-t) = x(t)$$

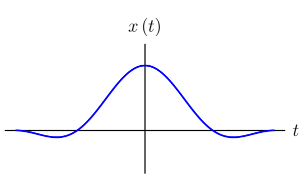
for all values of t.

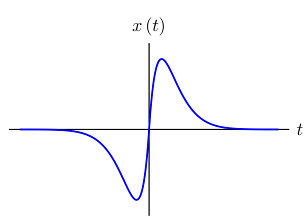
Odd symmetry

A real-valued signal is said to have *odd symmetry* if it has the property

$$x(-t) = -x(t)$$

for all values of t.





Decomposition into even and odd components

$$x(t) = x_e(t) + x_o(t)$$

Even component:

$$x_e(t) = \frac{x(t) + x(-t)}{2} \qquad \Rightarrow \qquad x_e(-t) = x_e(t)$$

Odd component:

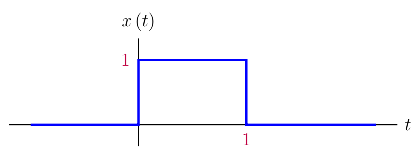
$$x_o(t) = \frac{x(t) - x(-t)}{2} \qquad \Rightarrow \qquad x_o(-t) = -x_o(t)$$

Example 1.13

Even and odd components of a rectangular pulse

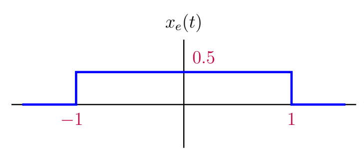
Determine even and odd components of the rectangular pulse signal

$$x(t) = \Pi\left(t - \tfrac{1}{2}\right) = \begin{cases} 1\,, & 0 < t < 1 \\ 0\,, & \text{otherwise} \end{cases}$$

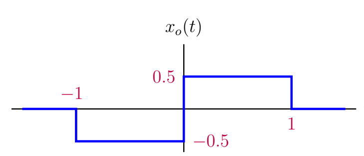


Solution:

$$x_e(t) = \frac{\Pi\left(t - \frac{1}{2}\right) + \Pi\left(-t - \frac{1}{2}\right)}{2} = \frac{1}{2}\,\Pi(t/2)$$



$$x_o(t) = \frac{\Pi\left(t - \frac{1}{2}\right) - \Pi\left(-t - \frac{1}{2}\right)}{2}$$

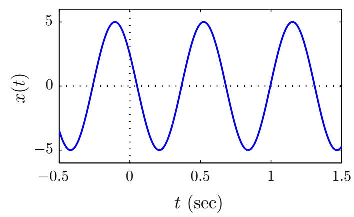


Example 1.14

Even and odd components of a sinusoidal signal

Determine even and odd components of the signal

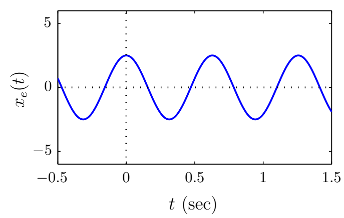
$$x(t) = 5\cos(10t + \pi/3)$$



Solution:

Even component

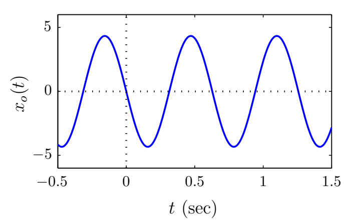
$$x_e(t) = \frac{5}{2}\cos(10t+\pi/3) + \frac{5}{2}\cos(-10t+\pi/3)$$
$$= 2.5\cos(10t)$$



Example 1.14 (continued)

Odd component

$$x_o(t) = \frac{5}{2}\cos(10t+\pi/3) - \frac{5}{2}\cos(-10t+\pi/3)$$
$$= -4.3301\sin(10t)$$



Symmetry properties for complex signals

Conjugate symmetry

A complex-valued signal is said to be *conjugate symmetric* if it satisfies

$$x(-t) = x^*(t)$$

for all t.

Conjugate antisymmetry

A complex-valued signal is said to be *conjugate antisymmetric* if it satisfies

$$x(-t) = -x^*(t)$$

for all t.

$$x(t) = x_E(t) + x_O(t)$$

Conjugate symmetric component:

$$x_E(t) = \frac{x(t) + x^*(-t)}{2}$$

Conjugate antisymmetric component:

$$x_O(t) = \frac{x(t) - x^*(-t)}{2}$$

Example 1.15

Symmetry of a complex exponential signal

Consider the complex exponential signal

$$x(t) = A\, e^{j\omega t}\,, \qquad A: \text{ real}$$

What symmetry property does this signal have, if any?

Solution: Time reverse the signal:

$$x(-t) = A\, e^{-j\omega t}$$

Conjugate the signal:

$$x^*(t) = \left(A\, e^{j\omega t}\right)^* = A\, e^{-j\omega t}$$

Since $x(-t) = x^*(t)$, the signal $x(t)$ is conjugate symmetric.

Graphical representation of sinusoidal signals using phasors

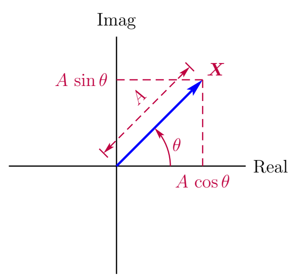
$$x(t) = A\cos(2\pi f_0 t + \theta)$$

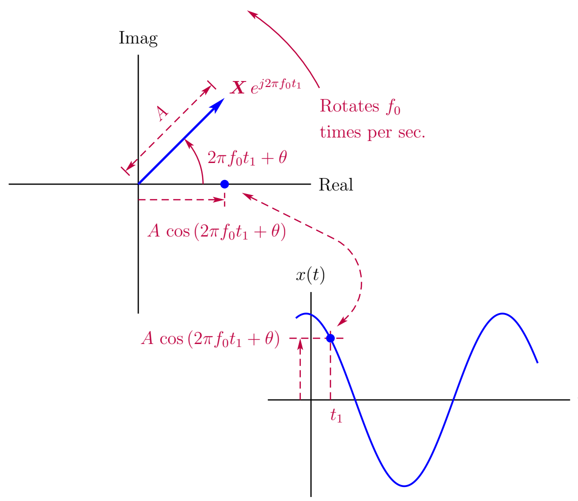
Let the phasor X be defined as

$$X \triangleq A\,e^{j\theta}$$

so that

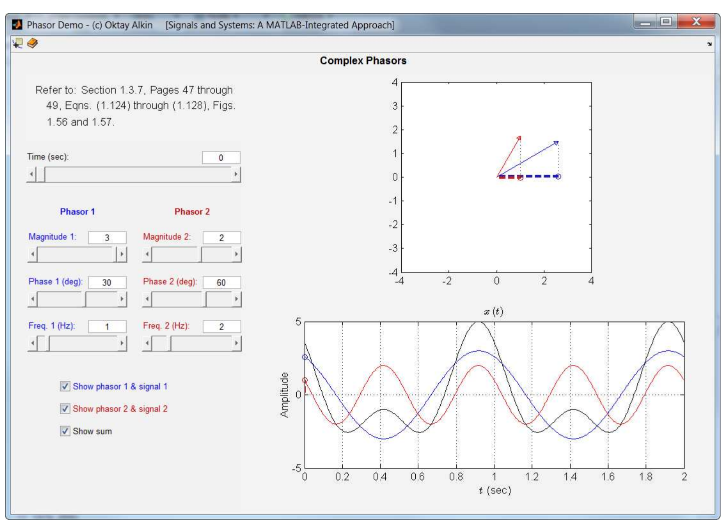
$$x(t) = \text{Re}\left\{ A\,e^{j(2\pi f_0 t + \theta)} \right\}$$
$$= \text{Re}\left\{ X\,e^{j2\pi f_0 t} \right\}$$





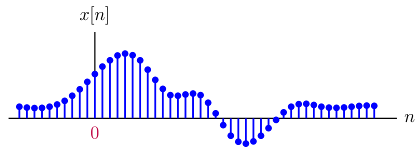
Interactive demo: phs_demo

Experiment by varying the parameters of the two phasors.

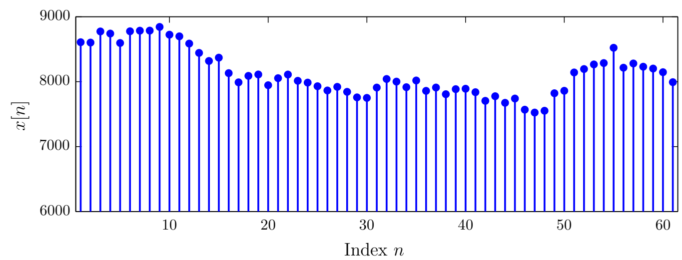


Discrete-time signals

A discrete-time signal.



Dow Jones Industrial Average for the first three months of 2003.

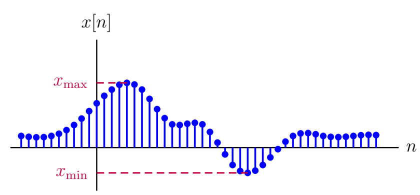


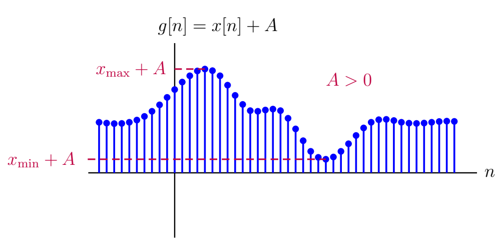
MATLAB Exercise 1.6

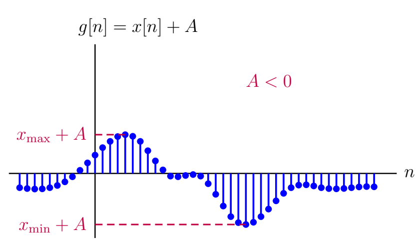
Signal operations

Addition of a constant offset

$$g[n] = x[n] + A$$



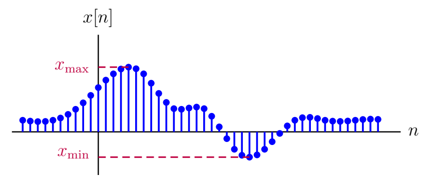


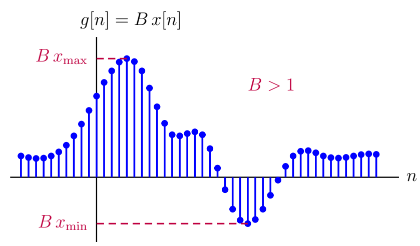


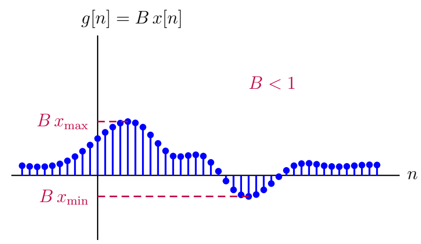
Signal operations (continued)

Multiplication by a constant gain factor

$$g[n] = B\,x[n]$$



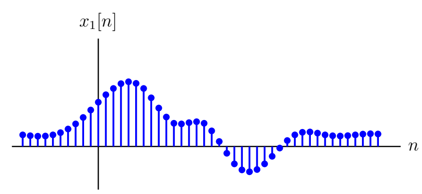


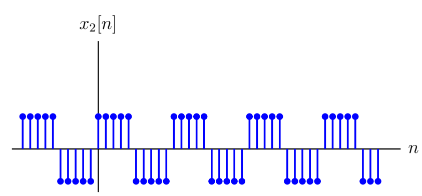


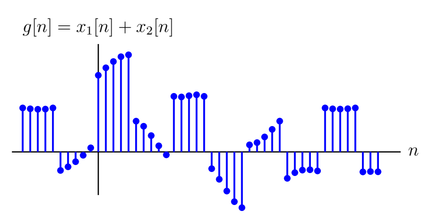
Signal operations (continued)

Adding two signals

$$g[n] = x_1[n] + x_2[n]$$



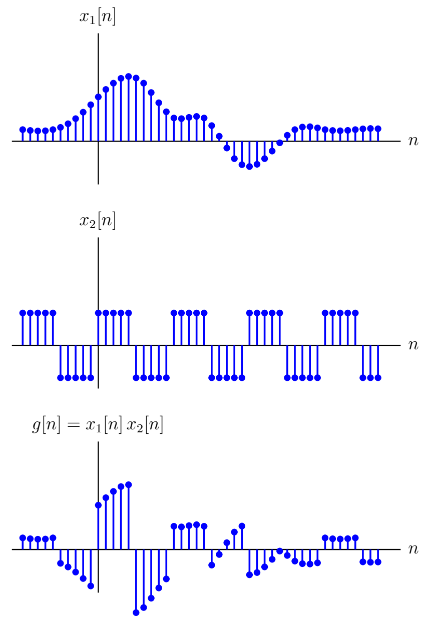




Signal operations (continued)

Multiplying two signals

$$g[n] = x_1[n]\, x_2[n]$$

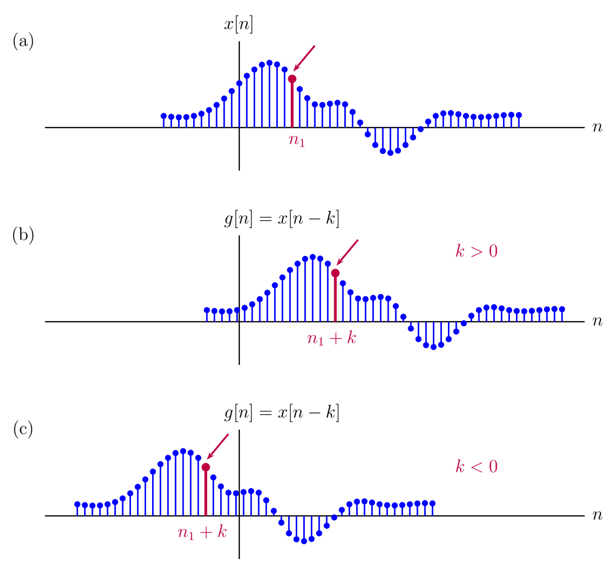


Signal operations (continued)

Time shifting

$$g[n] = x[n-k]$$

k: Integer

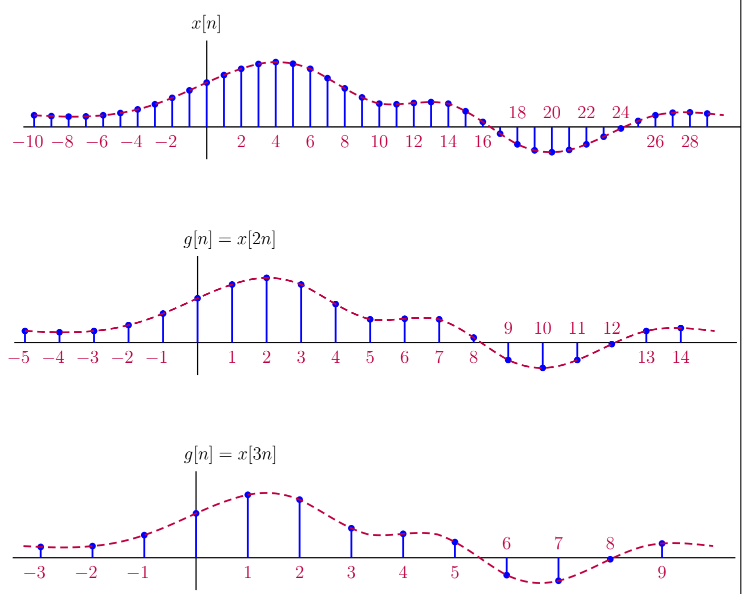


Signal operations (continued)

Time scaling

$$g[n] = x[kn]$$

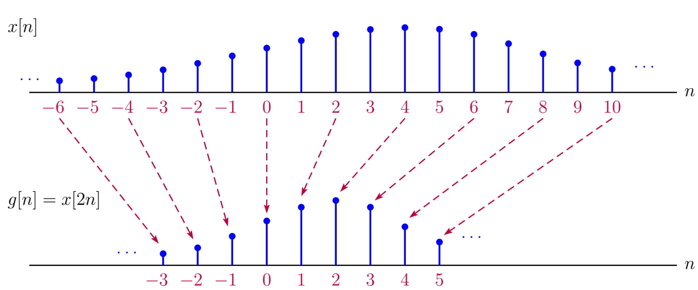
k: integer



Signal operations (continued)

Time scaling example (downsampling)

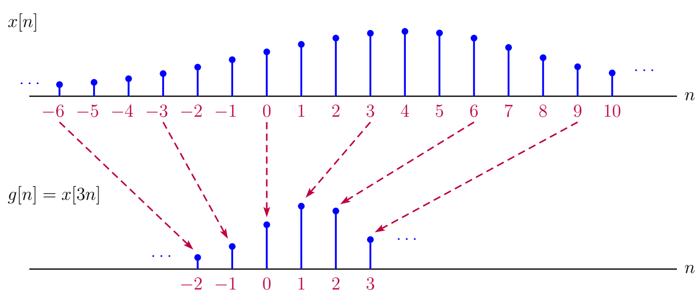
$$g[n] = x[2n]$$



Signal operations (continued)

Time scaling example (downsampling)

$$g[n] = x[3n]$$

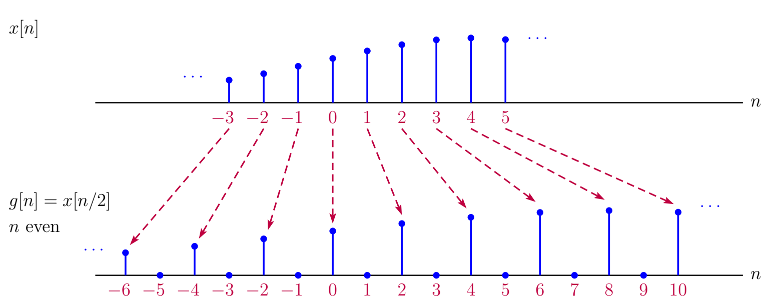


Signal operations (continued)

An alternative form of time scaling (upsampling)

$g[n] = x[n/2]$ How do we handle odd values of n?

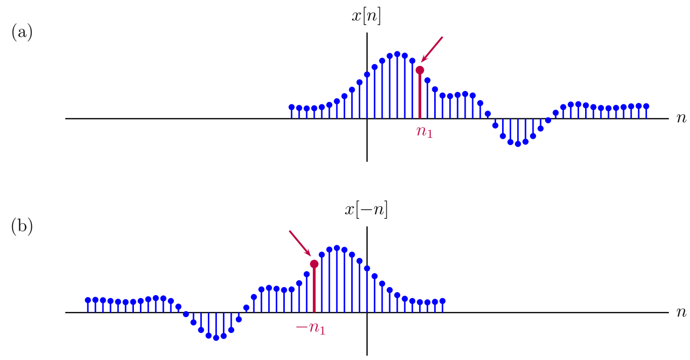
$$g[n] = \begin{cases} x[n/2]\ , & \text{if } n/2 \text{ is integer} \\ 0\ , & \text{otherwise} \end{cases}$$



Signal operations (continued)

Time reversal

$$g[n] = x[-n]$$



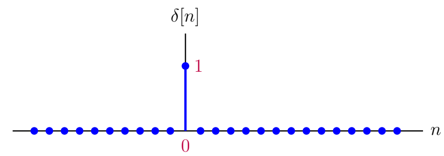
Basic building blocks

- Unit-impulse function
- Unit-step function
- Unit-ramp function
- Sinusoidal signals

Unit-impulse function

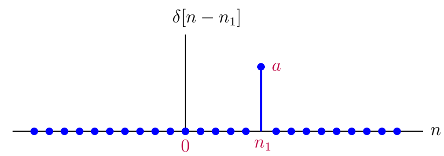
Mathematical definition

$$\delta[n] = \begin{cases} 1\,, & n = 0 \\ 0\,, & n \neq 0 \end{cases}$$



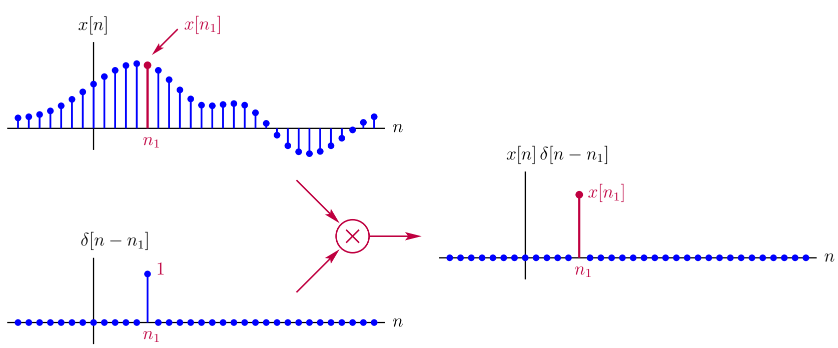
Scaling and time shifting:

$$a\,\delta[n - n_1] = \begin{cases} a\,, & n = n_1 \\ 0\,, & n \neq n_1 \end{cases}$$



Sampling property of the unit-impulse function

$$x[n]\,\delta[n - n_1] = x[n_1]\,\delta[n - n_1]$$



$$x[n]\,\delta[n - n_1] = \begin{cases} x[n_1]\,, & n = n_1 \\ 0\,, & n \neq n_1 \end{cases}$$

Sifting property of the unit-impulse function

$$\sum_{n=-\infty}^{\infty} x[n]\,\delta[n-n_1] = x[n_1]$$

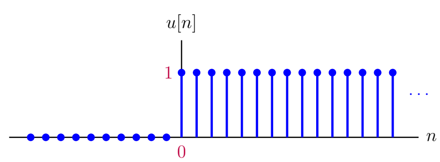
Using the sampling property:

$$\begin{aligned}\sum_{n=-\infty}^{\infty} x[n]\,\delta[n-n_1] = & \sum_{n=-\infty}^{\infty} x[n_1]\,\delta[n-n_1] \\ =& x[n_1] \sum_{n=-\infty}^{\infty} \delta[n-n_1] \\ =& x[n_1]\end{aligned}$$

Unit-step function

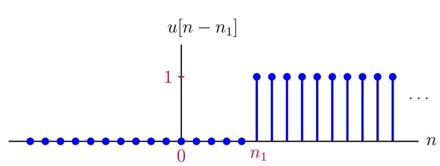
Mathematical definition

$$u[n] = \begin{cases} 1\,, & n \geq 0 \\ 0\,, & n < 0 \end{cases}$$



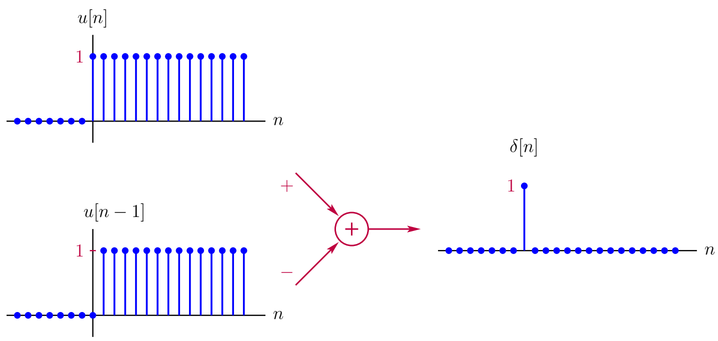
Time shifting the unit-step function:

$$u[n-n_1] = \begin{cases} 1\,, & n \geq n_1 \\ 0\,, & n < n_1 \end{cases}$$



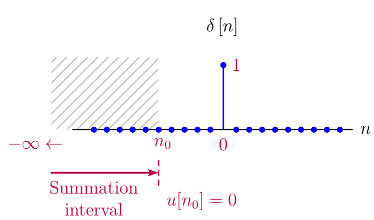
Relationship between unit-step and unit-impulse functions

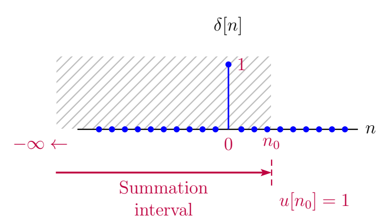
$$\delta[n] = u[n] - u[n-1]$$



Relationship between unit-step and unit-impulse functions

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$





An alternative approach: $\quad u[n] = \sum_{k=0}^{\infty} \delta[n-k]$

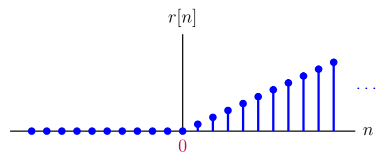
Unit-ramp function

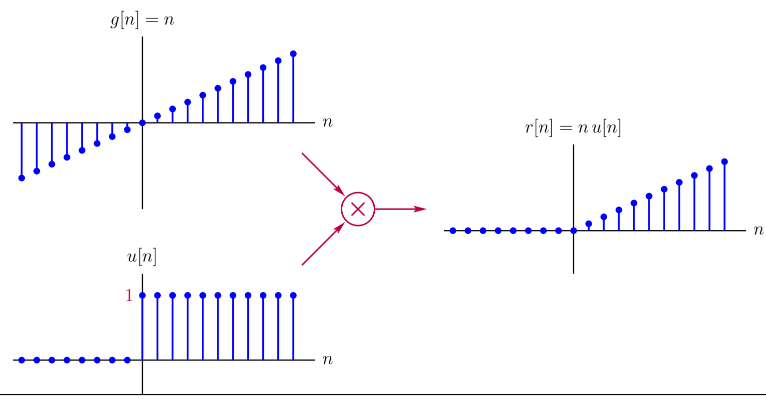
Mathematical definition

$$r[n] = \begin{cases} n\,, & n \geq 0 \\ 0\,, & n < 0 \end{cases}$$

or, equivalently

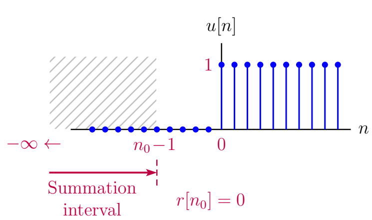
$$r[n] = n\,u[n]$$

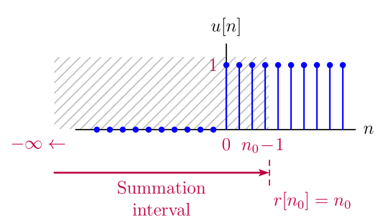




Constructing a unit-ramp from a unit-step

$$r[n] = \sum_{k=-\infty}^{n-1} u[k]$$





Sinusoidal signals

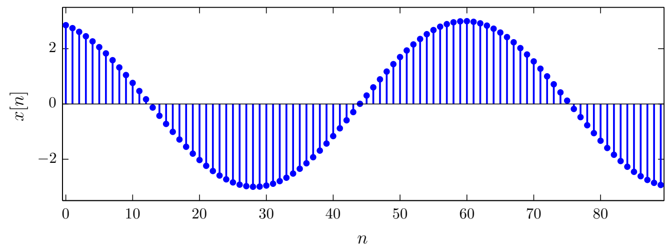
Sinusoidal signal

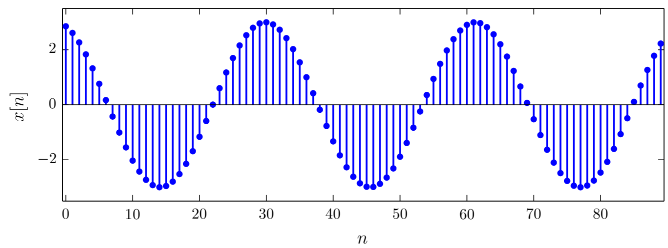
$$x[n] = A\cos(\Omega_0 n + \theta)$$

A : Amplitude

Ω_0 : Angular frequency (radians)

θ : Phase (radians)





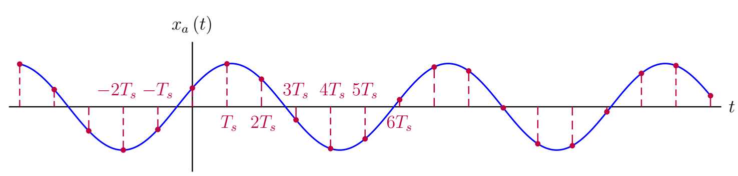
Characteristics of discrete-time sinusoids

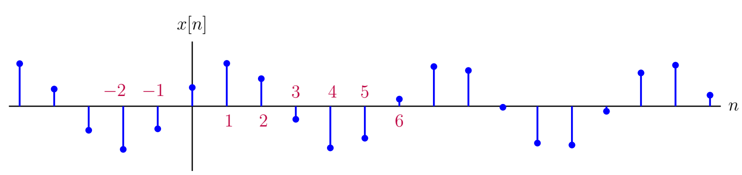
- For continuous-time sinusoidal signal $x_a(t) = A\cos(\omega_0 t)$: ω_0 is in rad/s.
- For discrete-time sinusoidal signal $x[n] = A\cos(\Omega_0 n)$: Ω_0 is in radians.

$$x_a(t) = A\cos(\omega_0 t + \theta)$$

$$\begin{aligned} x[n] &= x_a(nT_s) \\ &= A\cos(\omega_0 T_s n + \theta) \\ &= A\cos(2\pi f_0 T_s n + \theta) \end{aligned}$$

$$\begin{aligned} \Omega_0 &= \omega_0 T_s \\ F_0 &= f_0 T_s \\ \Omega_0 &= 2\pi F_0 \end{aligned}$$





Impulse decomposition for discrete-time signals

Impulse decomposition

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

Let

$$x_k[n] = x[k]\,\delta[n-k] = \begin{cases} x[k]\,, & n = k \\ 0\,, & n \neq k \end{cases}$$

For the signal $x[n]$

$$x[n] = \{\, 3.7,\ \underset{\uparrow}{1.3},\ -1.5,\ 3.4,\ 5.9 \,\}$$

the components $x_k[n]$ are

$$\begin{aligned} &\vdots \\ x_{-1}[n] &= \{\, 3.7,\ \underset{\uparrow}{0},\ 0,\ 0,\ 0 \,\} \\ x_0[n] &= \{\, 0,\ \underset{\uparrow}{1.3},\ 0,\ 0,\ 0 \,\} \\ x_1[n] &= \{\, 0,\ \underset{\uparrow}{0},\ -1.5,\ 0,\ 0 \,\} \\ x_2[n] &= \{\, 0,\ \underset{\uparrow}{0},\ 0,\ 3.4,\ 0 \,\} \\ x_3[n] &= \{\, 0,\ \underset{\uparrow}{0},\ 0,\ 0,\ 5.9 \,\} \end{aligned}$$

Real vs. complex signals

Complex signal in Cartesian form

$$x[n] = x_r[n] + j\,x_i[n]$$

Complex signal in polar form

$$x[n] = |x[n]|\; e^{j\measuredangle x[n]}$$

$$|x[n]| = \left(x_r^2[n] + x_i^2[n]\right)^{1/2}$$

$$\measuredangle x[n] = \tan^{-1}\left(\frac{x_i[n]}{x_r[n]}\right)$$

$$x_r[n] = |x[n]|\,\cos\left(\measuredangle x[n]\right)$$

$$x_i[n] = |x[n]|\,\sin\left(\measuredangle x[n]\right)$$

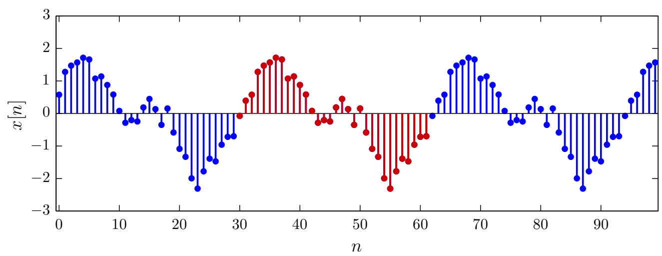
Periodic signals

Definition

A signal is said to be *periodic* if it satisfies

$$x[n+N] = x[n]$$

for all integer n and for a specific value of $N \neq 0$.



MATLAB Exercise 1.7

If a signal is periodic with period N, then it is also periodic with periods of $2N, 2N, \ldots, kN, \ldots$ where k is any integer.

Chapter 1
Discrete-Time Signals
Signal classifications

Example 1.16

Periodicity of a discrete-time sinusoidal signal

Check the periodicity of the following discrete-time signals:

a. $x[n] = \cos(0.2n)$

b. $x[n] = \cos(0.2\pi n + \pi/5)$

c. $x[n] = \cos(0.3\pi n - \pi/10)$

Solution:

a. $F_0 = \dfrac{0.2}{2\pi}$, $N = \dfrac{k}{F_0} = 10\pi k$ cannot be integer. $\Rightarrow$ Not periodic.

b. $F_0 = 0.1$, $N = \dfrac{k}{F_0} = 10k$ $\Rightarrow$ Periodic with $N = 10$ samples.

c. $F_0 = 0.15$, $N = \dfrac{k}{F_0} = \dfrac{k}{0.15}$ $\Rightarrow$ Periodic with $N = 20$ samples.

Example 1.17

Periodicity of a multitone discrete-time sinusoidal signal

Comment on the periodicity of the two-tone discrete-time signal

$$x[n] = 2\,\cos(0.4\pi n) + 1.5\,\sin(0.48\pi n)$$

Solution:

$$x[n] = x_1[n] + x_2[n]$$

$$x_1[n] = 2\,\cos(\Omega_1 n)\ , \quad \Omega_1 = 0.4\pi \text{ rad}$$

$$x_2[n] = 1.5\,\sin(\Omega_2 n)\ , \quad \Omega_2 = 0.48\pi \text{ rad}$$

$$\text{For } x_1[n]: \quad N_1 = \frac{k_1}{F_1} = \frac{k_1}{0.2}\ , \quad k_1 = 1\ , \quad N_1 = 5$$

$$\text{For } x_2[n]: \quad N_2 = \frac{k_2}{F_2} = \frac{k_2}{0.24}\ , \quad k_2 = 6\ , \quad N_2 = 25$$

The signal $x[n]$ is periodic with $N = 25$ samples.

Energy computations

Normalized energy of a signal

$$E_x = \sum_{-\infty}^{\infty} x^2[n]$$

if the result of the summation can be computed.

Normalized energy of a complex signal

$$E_x = \sum_{-\infty}^{\infty} |x[n]|^2$$

if the result of the summation can be computed.

Time averaging operator

Time average of a signal periodic with period N

$$\langle x[n] \rangle = \frac{1}{N} \sum_{n=0}^{N-1} x[n]$$

Time average of an aperiodic signal

$$\langle x[n] \rangle = \lim_{M \to \infty} \left[\frac{1}{2M+1} \sum_{n=-M}^{M} x[n] \right]$$

Power computations

Normalized avg. power (real signal)

$$P_x = \left\langle x^2[n] \right\rangle$$

Periodic signal:

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} x^2[n]$$

Non-periodic signal:

$$P_x = \lim_{M \to \infty} \left[\frac{1}{2M+1} \sum_{n=-M}^{M} x^2[n] \right]$$

Normalized avg. power (complex signal)

$$P_x = \left\langle |x[n]|^2 \right\rangle$$

Periodic signal:

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} \left| x[n] \right|^2$$

Non-periodic signal:

$$P_x = \lim_{M \to \infty} \left[\frac{1}{2M+1} \sum_{n=-M}^{M} \left| x[n] \right|^2 \right]$$

Energy signals vs. power signals

- Energy signals are those that have finite energy, and zero power.
  $E_x < \infty$, and $P_x = 0$.
- Power signals are those that have finite power and infinite energy.
  $E_x \to \infty$, and $P_x < \infty$.

Symmetry properties

Even symmetry

A real-valued signal is said to have *even symmetry* if it has the property

$$x[-n] = x[n]$$

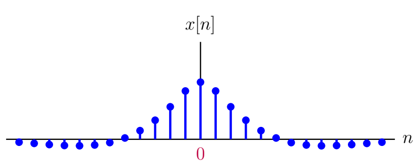
for all integer values of n.

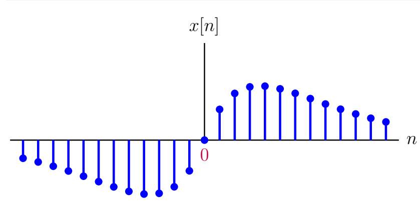
Odd symmetry

A real-valued signal is said to have *odd symmetry* if it has the property

$$x[-n] = -x[n]$$

for all integer values of n.





Decomposition into even and odd components

$$x[n] = x_e[n] + x_o[n]$$

Even component:

$$x_e[n] = \frac{x[n] + x[-n]}{2} \qquad \Rightarrow \qquad x_e[-n] = x_e[n]$$

Odd component:

$$x_o[n] = \frac{x[n] - x[-n]}{2} \qquad \Rightarrow \qquad x_o[-n] = -x_o[n]$$

Symmetry properties for complex signals

Conjugate symmetry

A complex-valued signal is said to be *conjugate symmetric* if it satisfies

$$x[-n] = x^*[n]$$

for all integer n.

Conjugate antisymmetry

A complex-valued signal is said to be *conjugate antisymmetric* if it satisfies

$$x[-n] = -x^*[n]$$

for all integer n.

$$x[n] = x_E[n] + x_O[n]$$

Conjugate symmetric component:

$$x_E[n] = \frac{x[n] + x^*[-n]}{2}$$

Conjugate antisymmetric component:

$$x_O[n] = \frac{x[n] - x^*[-n]}{2}$$

MATLAB Exercise 1.1

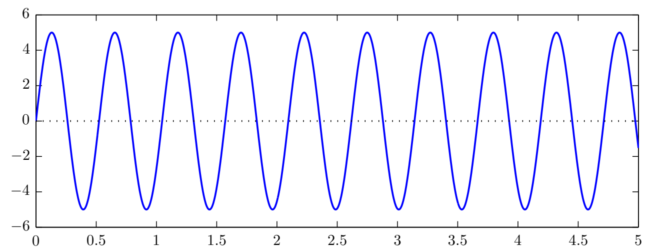
Computing and graphing continuous-time signals - Part (a)

Compute the signal

$$x_1(t) = 5\,\sin(12t)$$

at 500 points in the time interval $0 \le t \le 5$, and graph the result.

```
t = linspace(0,5,500);
x1 = 5*sin(12*t);
plot(t,x1);
```



MATLAB Exercise 1.1 (continued)

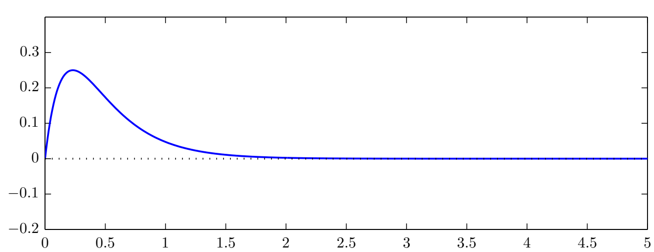
Computing and graphing continuous-time signals - Part (b)

Compute and graph the signal

$$x_2(t) = \begin{cases} e^{-3t} - e^{-6t}\,, & t \ge 0 \\ 0\,, & t < 0 \end{cases}$$

in the time interval $0 \le t \le 5$ seconds, using a time increment of $\Delta t = 0.01$ seconds.

```
t = [0:0.01:5];
x2 = exp(-3*t)-exp(-6*t);
plot(t,x2);
```



MATLAB Exercise 1.1 (continued)

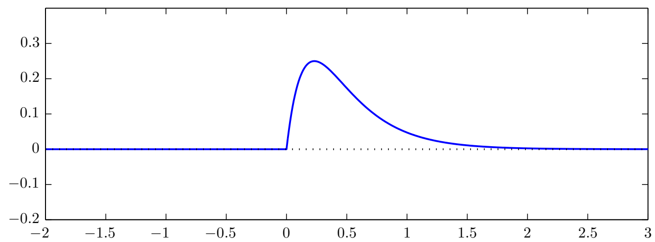
Computing and graphing continuous-time signals - Part (c)

Compute and graph the signal

$$x_2(t) = \begin{cases} e^{-3t} - e^{-6t} , & t \geq 0 \\ 0 , & t < 0 \end{cases}$$

in the time interval $-2 \leq t \leq 3$ seconds, using a time increment of $\Delta t = 0.01$ seconds.

```
t = [-2:0.01:3];
x2 = (exp(-3*t)-exp(-6*t)).*(t>=0);
plot(t,x2);
```



MATLAB Exercise 1.1 (continued)

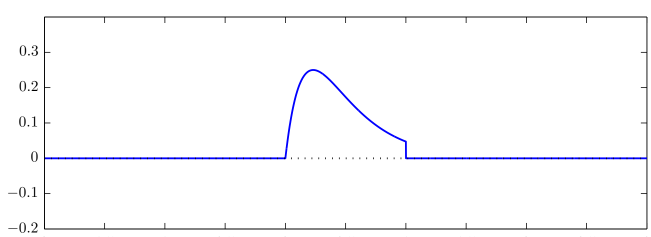
Computing and graphing continuous-time signals - Part (d)

Compute and graph the signal

$$x_3(t) = \begin{cases} e^{-3t} - e^{-6t} , & 0 \leq t \leq 1 \\ 0 , & \text{otherwise} \end{cases}$$

in the time interval $-2 \leq t \leq 3$ seconds, using a time increment of $\Delta t = 0.01$ seconds.

```
t = [-2:0.01:3];
x3 = (exp(-3*t)-exp(-6*t)).*((t>=0)&(t<=1));
plot(t,x3);
```

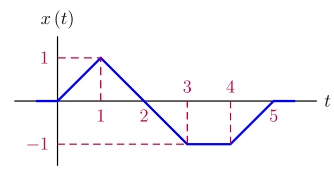


Mathematical modeling

MATLAB Exercise 1.2

Describing signals using piecewise linear segments

Consider the signal $x(t)$ shown. Describe the signal by specifying the endpoints of linear segments and interpolating between them.



The endpoints of the signal under consideration are

$$(t_p, x_p) = \{(-1,0), (0,0), (1,1), (3,-1), (4,-1), (5,0), (6,0)\}$$

```
tp = [-1,0,1,3,4,5,6];
xp = [0,0,1,-1,-1,0,0];
t = [-1:0.01:6];
x = interp1(tp,xp,t,'linear');
plot(t,x,'b-',tp,xp,'ro'); grid;
```

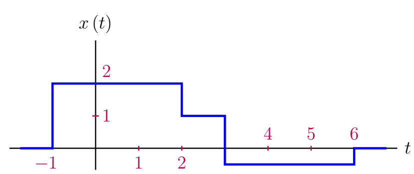
Example 1.2

MATLAB Exercise 1.3

Signal operations for continuous-time signals

Using MATLAB compute and graph the signal

$$x(t) = \begin{cases} 2\,, & -1 \le t < 2 \\ 1\,, & 2 \le t < 3 \\ -0.5\,, & 3 \le t \le 6 \end{cases}$$




```
t = [-10:0.01:10];
x = 2*((t>=-1)&(t<2))+1*((t>=2)&(t<3))-0.5*((t>=3)&(t<=6));
plot(t,x);
```

MATLAB Exercise 1.3 (continued)

Signal operations for continuous-time signals

Use an anonymous function for $x(t)$. Compute and graph the signals $g(t) = x(2t-5)$ and $h(t) = x(-4t+2)$.

```
sx = @(t) 2*((t>=-1)&(t<2))+1*((t>=2)&(t<3))-0.5*((t>=3)&(t<=6));
plot(t,sx(2*t-5));
plot(t,sx(-4*t+2));
```

▸ Example 1.3

MATLAB Exercise 1.4

a. Periodic square wave

One period of a square-wave signal is defined as

$$x(t) = \begin{cases} 1\,, & 0 < t < T/2 \\ -1\,, & T/2 < t < T \end{cases}$$

Compute and graph a square-wave signal with period $T = 1$.

```
t = [-1:0.01:10];
x = square(2*pi*t);
plot(t,x);
grid;
```

b. Periodic square wave

A square-wave signal with a duty cycle of $0 < d \le 1$ is defined as

$$x(t) = \begin{cases} 1\,, & 0 < t < Td \\ -1\,, & Td < t < T \end{cases}$$

Compute and graph a square-wave signal with period $T = 1$ s and duty cycle $d = 0.2$.

```
t = [-1:0.01:10];
x = square(2*pi*t,20);
plot(t,x);
grid;
```

MATLAB Exercise 1.4 (continued)

c. Periodic sawtooth waveform

One period of a sawtooth waveform is

$$x(t) = t/T \quad \text{for} \quad 0 < t < T$$

Compute and graph a sawtooth signal with period $T = 1.5$ seconds.

```
t = [-1:0.01:10];
x = sawtooth(2*pi*t/1.5);
plot(t,x);
grid;
```


MATLAB Exercise 1.4 (continued)

d. Arbitrary periodic waveform

A signal $x(t)$ that is periodic with period $T = 2.5$ s is defined through the following:

$$x(t) = \begin{cases} t\,, & 0 \le t < 1 \\ e^{-5(t-1)}\,, & 1 \le t < 2.5 \end{cases}, \qquad \text{and} \qquad x(t + 2.5k) = x(t)$$

Compute and graph this signal in the time interval $-2 < t < 12$ seconds.

```
t = [-2:0.01:12];
x1 = @(t) t.*((t>=0)&(t<1))+exp(-5*(t-1)).*((t>=1)&(t<2.5));
x = x1(mod(t,2.5));
plot(t,x);
grid;
```

Example 1.4

MATLAB Exercise 1.5

Functions for basic building blocks:

Unit-step $u(t)$

```
function x = ss_step(t)
  x = 1*(t>=0);
```

Unit-ramp $r(t)$

```
function x = ss_ramp(t)
  x = t.*(t>=0);
```

Unit-pulse $\Pi(t)$

```
function x = ss_pulse(t)
  x = ss_step(t+0.5)-ss_step(t-0.5);
```

Unit-triangle $\Lambda(t)$

```
function x = ss_tri(t)
  x = ss_ramp(t+1)-2*ss_ramp(t)+ss_ramp(t-1);
```


MATLAB Exercise 1.6

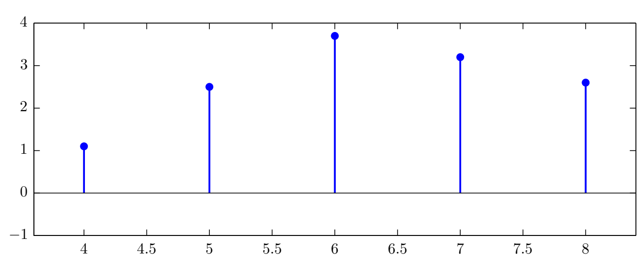
Part (a)

Compute and graph the signal

$$x_1[n] = \{\, 1.1,\ \underset{\substack{\uparrow \\ n=5}}{2.5},\ 3.7,\ 3.2,\ 2.6 \,\}$$

for the index range $4 \le n \le 8$.

```
n = [4:8];
x1 = [1.1,2.5,3.7,3.2,2.6];
stem(n,x1);
```



MATLAB Exercise 1.6 (continued)

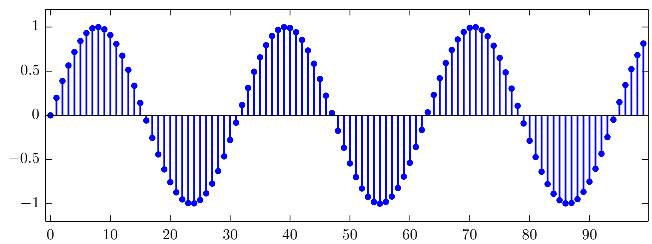
Part (b)

Compute and graph the signal

$$x_2[n] = \sin(0.2n)$$

for the index range $n = 0, 1, \ldots, 99$.

```
n = [0:99];
x2 = sin(0.2*n);
stem(n,x2);
```



MATLAB Exercise 1.6 (continued)

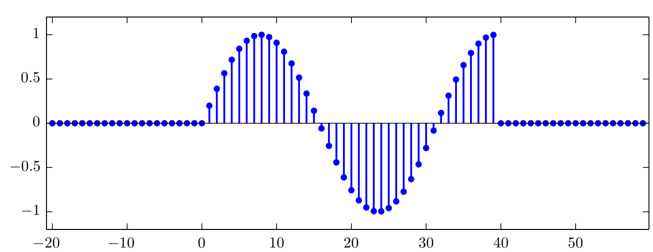
Part (c)

Compute and graph the signal

$$x_3[n] = \begin{cases} \sin(0.2n) \ , & n = 0, \ldots, 39 \\ 0 \ , & \text{otherwise} \end{cases}$$

for the interval $n = -20, \ldots, 59$.

```
n = [-20:59];
x3 = sin(0.2*n).*((n>=0)&(n<=39));
stem(n,x3);
```



▸ Discrete-time signals

MATLAB Exercise 1.7

Periodic extension of a discrete-time signal

Given $x[n]$ for $n = 0, \ldots, N-1$, let $\tilde{x}[n] = \sum_{m=-\infty}^{\infty} x[n+mN]$

```
function xtilde = ss_per(x,idx)
  N = length(x);    % Period of the signal.
  n = mod(idx,N);   % Modulo indexing.
  nn = n+1;         % MATLAB indices start with 1.
  xtilde = x(nn);
end
```

Test the function ss_per(..) with the signal $x[n] = n$; $n = 0, \ldots, 4$:

```
x = [0,1,2,3,4]
n = [-15:15]
xtilde = ss_per(x,n)
stem(n,xtilde)
```

```
x = [0,1,2,3,4]
n = [-15:15]
xtilde = ss_per(x,-n)
stem(n,xtilde)
```

Periodic signals