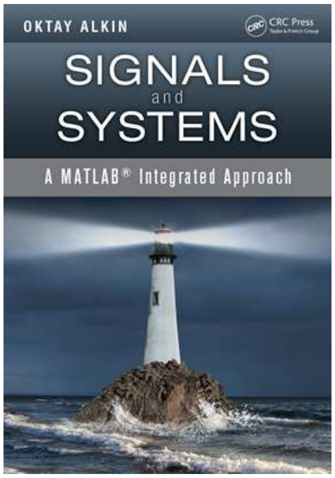

Chapter 2

Analyzing Continuous-Time Systems in the Time Domain

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CRC Press
Taylor & Francis Group

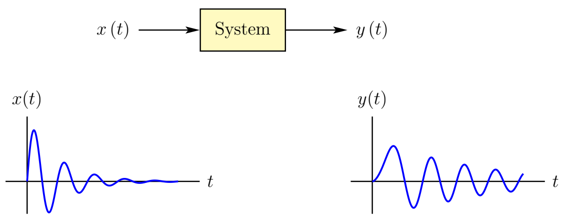
Objectives

- Develop the notion of a *continuous-time system*.
- Learn simplifying assumptions made in the analysis of systems. Discuss the concepts of *linearity* and *time-invariance*, and their significance.
- Explore the use of differential equations for representing continuous-time systems.
- Develop methods for solving differential equations to compute the output signal of a system in response to a specified input signal.
- Learn to represent a differential equation in the form of a block diagram that can be used as the basis for simulating a system.
- Discuss the significance of the *impulse response* as an alternative description form for linear and time-invariant systems.
- Learn how to compute the output signal for a linear and time-invariant system using *convolution*.
- Learn the concepts of causality and stability as they relate to physically realizable and usable systems.

Introduction

System

In general, a system is any physical entity that takes in a set of one or more physical signals and, in response, produces a new set of one or more physical signals.

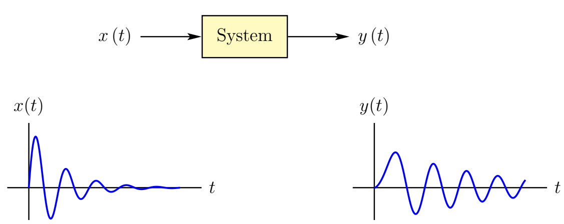


A system can be viewed as any physical entity that defines the cause-effect relationships between a set of signals known as *inputs* and another set of signals known as *outputs*.

Mathematical modeling

The mathematical model of a system is a function, formula or algorithm (or a set of functions, formulas, algorithms) to approximately recreate the same cause-effect relationship between the mathematical models of the input and the output signals.

Introduction (continued)



$$y(t) = \mathrm{Sys}\{x(t)\}$$

Some examples:

$$y(t) = K\, x(t)$$

$$y(t) = x(t-\tau)$$

$$y(t) = K\, [x(t)]^2$$

Linearity in continuous-time systems

Conditions for linearity

$$\mathrm{Sys}\{x_1(t) + x_2(t)\} = \mathrm{Sys}\{x_1(t)\} + \mathrm{Sys}\{x_2(t)\}$$

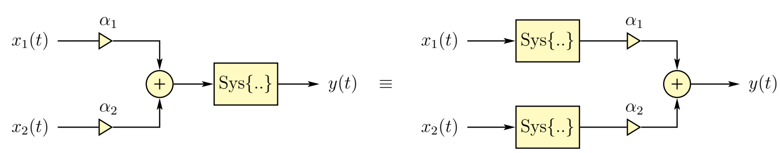
$$\mathrm{Sys}\{\alpha_1\, x_1(t)\} = \alpha_1\ \mathrm{Sys}\{x_1(t)\}$$

$x_1(t)$, $x_2(t)$: Any two input signals; α_1: Arbitrary constant gain factor.

Superposition principle (combine the two conditions into one)

$$\mathrm{Sys}\{\alpha_1\, x_1(t) + \alpha_2\, x_2(t)\} = \alpha_1\ \mathrm{Sys}\{x_1(t)\} + \alpha_2\ \mathrm{Sys}\{x_2(t)\}$$

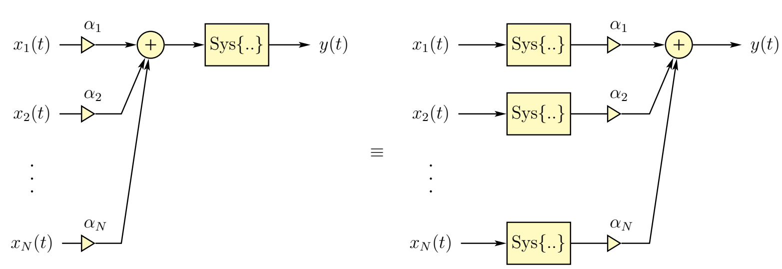
$x_1(t)$, $x_2(t)$: Any two input signals; α_1, α_2: Arbitrary constant gain factors.



Linearity in continuous-time systems (continued)

If superposition works for the weighted sum of any two input signals, it also works for an arbitrary number of input signals.

$$\mathrm{Sys}\left\{\sum_{i=1}^{N} \alpha_i\, x_i(t)\right\} = \sum_{i=1}^{N} \alpha_i\ \mathrm{Sys}\{x_i(t)\} = \sum_{i=1}^{N} \alpha_i\, y_i(t)$$



Example 2.1

Testing linearity of continuous-time systems

Four different systems are described below. For each, determine if the system is linear or not:

a. $y(t) = 5\,x(t)$

b. $y(t) = 5\,x(t) + 3$

c. $y(t) = 3\,[x(t)]^2$

d. $y(t) = \cos(x(t))$

Solution:

a.

$$\begin{aligned} y(t) =& 5\,x(t) \\ =& 5\,[\alpha_1\,x_1(t) + \alpha_2\,x_2(t)] \\ =& \alpha_1\,[5\,x_1(t)] + \alpha_2\,[5\,x_2(t)] \\ =& \alpha_1\,y_1(t) + \alpha_2\,y_2(t) \end{aligned}$$

Superposition principle holds; therefore this system is linear.

b.

$$\begin{aligned} y(t) =& 5\,x(t) + 3 \\ =& 5\,\alpha_1\,x_1(t) + 5\,\alpha_2\,x_2(t) + 3 \end{aligned}$$

Superposition principle does not hold true. The system in part (b) is not linear.

c.

$$\begin{aligned} y(t) &= 3\,[\alpha_1\,x_1(t) + \alpha_2\,x_2(t)]^2 \\ &= 3\alpha_1^2\,[x_1(t)]^2 + 6\alpha_1\alpha_2 x_1(t)\,x_2(t) \\ &\quad + 3\alpha_2^2\,[x_2(t)]^2 \end{aligned}$$

Superposition principle does not hold true. The system in part (c) is not linear.

Example 2.1 (continued)

d.

$$y(t) = \cos[\alpha_1\,x_1(t) + \alpha_2\,x_2(t)]$$

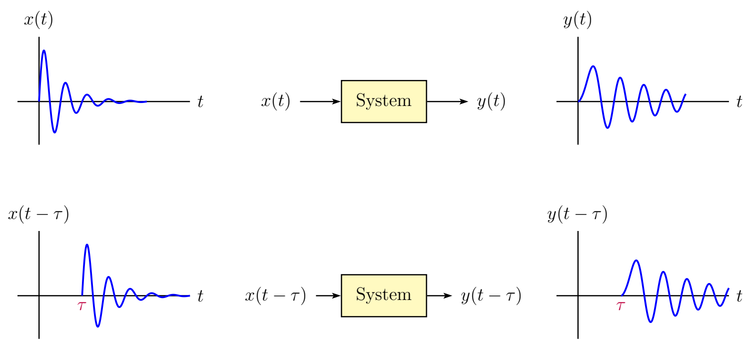
Superposition principle does not hold true. The system in part (d) is not linear.

MATLAB Exercise 2.1

Time-invariance in continuous-time systems

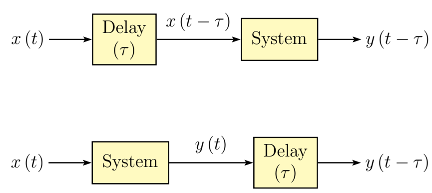
Condition for time-invariance

$$\mathrm{Sys}\{x(t)\} = y(t) \quad \text{implies that} \quad \mathrm{Sys}\{x(t-\tau)\} = y(t-\tau)$$



Time-invariance in continuous-time systems (continued)

Alternatively, time invariance can be explained by the equivalence of the two system configurations shown:



Example 2.2

Testing time-invariance of continuous-time systems

Three different systems are described below. For each, determine if the system is time-invariant or not:

a. $y(t) = 5\,x(t)$

b. $y(t) = 3\,\cos(x(t))$

c. $y(t) = 3\,\cos(t)\,x(t)$

Solution:

a. $\mathrm{Sys}\{x(t-\tau)\} = 5\,x(t-\tau) = y(t-\tau)$ Time-invariant.

b. $\mathrm{Sys}\{x(t-\tau)\} = 3\,\cos\left(x(t-\tau)\right) = y(t-\tau)$ Time-invariant.

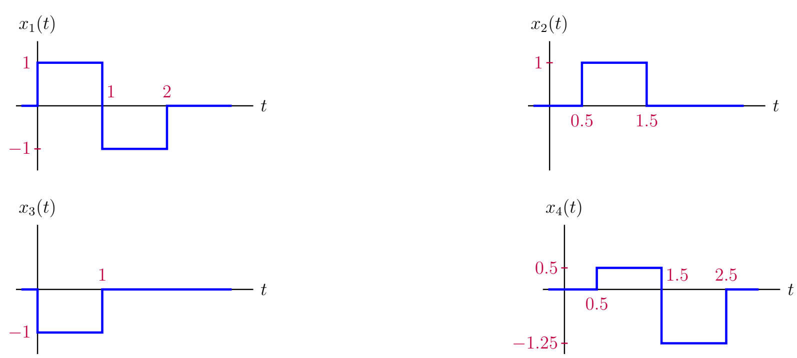
c. $\mathrm{Sys}\{x(t-\tau)\} = 3\,\cos(t)\,x(t-\tau) \neq y(t-\tau)$ Not time-invariant.

MATLAB Exercise 2.2

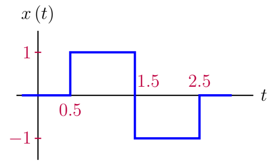
Example 2.3

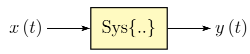
Using linearity property

A continuous-time system is known to be linear. Whether the system is time-invariant or not is not known. Assume that the responses of the system to four input signals $x_1(t)$, $x_2(t)$ $x_3(t)$ and $x_4(t)$ shown below are known. Discuss how the information provided can be used for finding the response of this system to the signal $x(t)$ shown.



Example 2.3 (continued)





Solution:

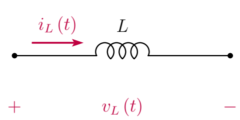
$$x(t) = 0.6\,x_2(t) + 0.8\,x_4(t) \qquad \Rightarrow \qquad y(t) = 0.6\,y_2(t) + 0.8\,y_4(t)$$

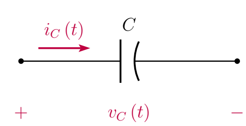
Differential equations for continuous-time systems

Example:

$$\frac{d^2y}{dt^2} + 3x(t)\,\frac{dy}{dt} + y(t) - 2x(t) = 0$$

Many physical components have mathematical models that involve integral and differential relationships between signals:





Ideal inductor:

$$v_L(t) = L\,\frac{di_L(t)}{dt}$$

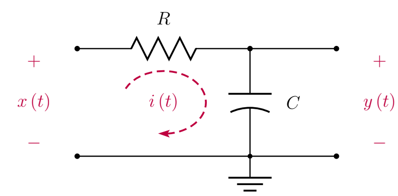
Ideal capacitor:

$$i_C(t) = C\,\frac{dv_C(t)}{dt}$$

Example 2.4

Differential equation for simple RC circuit

Find a differential equation to describe the input-output relationship for the first-order RC circuit shown.



Solution:

We know that

$$v_R(t) = R\,i(t) \quad \text{and} \quad i(t) = C\,\frac{dy(t)}{dt}$$

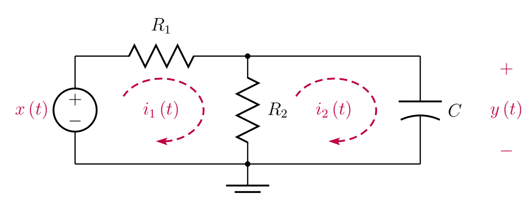
Use KVL to obtain

$$RC\,\frac{dy(t)}{dt} + y(t) = x(t) \qquad \Rightarrow \qquad \frac{dy(t)}{dt} + \frac{1}{RC}\,y(t) = \frac{1}{RC}\,x(t)$$

Example 2.5

Another RC circuit

Find a differential equation to describe the input-output relationship for the first-order RC circuit shown.



Solution:

Apply KVL:

$$-x(t) + R_1\,i_1(t) + R_2\,[i_1(t) - i_2(t)] = 0$$

$$R_2\,[i_2(t) - i_1(t)] + y(t) = 0$$

$$i_2(t) = C\,\frac{dy(t)}{dt}\,, \qquad i_1(t) = C\,\frac{dy(t)}{dt} + \frac{1}{R_2}\,y(t)$$

$$-x(t) + R_1 C\,\frac{dy(t)}{dt} - \frac{R_1 + R_2}{R_2}\,y(t) = 0$$

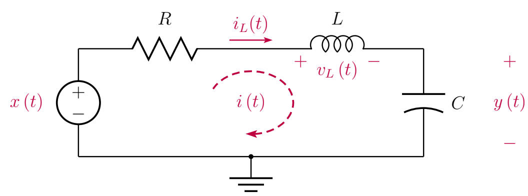
Rearrange terms

$$\frac{dy(t)}{dt} + \frac{R_1 + R_2}{R_1 R_2 C}\,y(t) = \frac{1}{R_1 C}\,x(t)$$

Example 2.6

Differential equation for RLC circuit

Find a differential equation to describe the input-output relationship for the RLC circuit shown.



Solution:

Apply KVL:

$$-x(t) + R\,i(t) + v_L(t) + y(t) = 0$$

$$i(t) = C\,\frac{dy(t)}{dt}\,, \qquad v_L(t) = L\,\frac{di(t)}{dt} = LC\,\frac{d^2y(t)}{dt^2}$$

$$-x(t) + RC\,\frac{dy(t)}{dt} + LC\,\frac{d^2y(t)}{dt^2} + y(t) = 0$$

Rearrange terms:

$$\frac{d^2y(t)}{dt^2} + \frac{R}{L}\,\frac{dy(t)}{dt} + \frac{1}{LC}\,y(t) = \frac{1}{LC}\,x(t)$$

Constant-coefficient ordinary differential equations

General constant-coefficient differential equation for a CTLTI system:

$$a_N\,\frac{d^Ny(t)}{dt^N} + a_{N-1}\,\frac{d^{N-1}y(t)}{dt^{N-1}} + \ldots + a_1\,\frac{dy(t)}{dt} + a_0\,y(t) =$$

$$b_M\,\frac{d^Mx(t)}{dt^M} + b_{M-1}\,\frac{d^{M-1}x(t)}{dt^{M-1}} + \ldots + b_1\,\frac{dx(t)}{dt} + b_0\,x(t)$$

Constant-coefficient ordinary differential equation in closed summation form

$$\sum_{k=0}^{N} a_k\,\frac{d^ky(t)}{dt^k} = \sum_{k=0}^{M} b_k\,\frac{d^kx(t)}{dt^k}$$

Initial conditions:

$$y(t_0)\,, \quad \left.\frac{dy(t)}{dt}\right|_{t=t_0}, \quad \ldots, \quad \left.\frac{d^{N-1}y(t)}{dt^{N-1}}\right|_{t=t_0}$$

Example 2.7

Checking linearity and time-invariance of a differential equation

Determine whether the first-order constant-coefficient differential equation

$$\frac{dy(t)}{dt} + a_0\, y(t) = b_0\, x(t)$$

represents a CTLTI system.

Solution:

Let input signals $x_1(t)$ and $x_2(t)$ produce the responses $y_1(t)$ and $y_2(t)$ respectively:

$$\frac{dy_1(t)}{dt} + a_0\, y_1(t) = b_0\, x_1(t) \qquad \text{and} \qquad \frac{dy_2(t)}{dt} + a_0\, y_2(t) = b_0\, x_2(t)$$

Construct a new input signal

$$x_3(t) = \alpha_1\, x_1(t) + \alpha_2\, x_2(t)$$

For linearity we need

$$y_3(t) = \alpha_1\, y_1(t) + \alpha_2\, y_2(t)$$

Example 2.7 (continued)

It can be shown that

$$\frac{dy_3(t)}{dt} + a_0\, y_3(t) = b_0\, x_3(t)$$

Is this sufficient?

What happens at $t = t_0$, the time instant at which the initial conditions are specified?

Suppose the initial value of $y(t)$ is given as $y(t_0) = y_0$. We must have

$$y_1(t_0) = y_0\ , \quad y_2(t_0) = y_0\ , \quad y_3(t_0) = y_0$$

but we also need

$$y_3(t_0) = \alpha_1\, y_1(t_0) + \alpha_2\, y_2(t_0)$$

For linearity: $y_0 = 0$.

Check for time-invariance:

$$\frac{dy(t-\tau)}{dt} + a_0\, y(t-\tau) = b_0\, x(t-\tau)$$

The system is time-invariant.

Constant-coefficient ordinary differential equations (continued)

Constant-coefficient differential equation for a CTLTI system

The differential equation

$$\sum_{k=0}^{N} a_k \, \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \, \frac{d^k x(t)}{dt^k}$$

represents a CTLTI system provided that all initial conditions are equal to zero:

$$y(t_0) = 0 \,, \qquad \left.\frac{dy(t)}{dt}\right|_{t=t_0} = 0 \,, \qquad \ldots, \qquad \left.\frac{d^{N-1} y(t)}{dt^{N-1}}\right|_{t=t_0} = 0$$

It is typical, but not required, to have $t_0 = 0$.

Solution of the first-order differential equation

Solution of the first-order differential equation

The differential equation

$$\frac{dy(t)}{dt} + \alpha \, y(t) = r(t) \;, \qquad y(t_0): \text{ specified}$$

is solved as

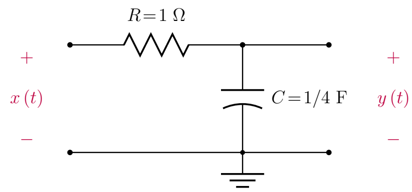
$$y(t) = e^{-\alpha(t-t_0)} \, y(t_0) + \int_{t_0}^{t} e^{-\alpha(t-\tau)} \, r(\tau) \, d\tau$$

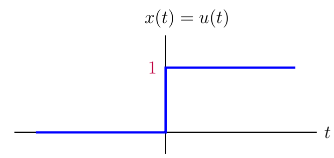
Even though this result is only applicable to a first-order differential equation, it is also useful for working with higher order systems through the use of *state-space* models.

Example 2.8

Unit-step response of the simple RC circuit

For the RC circuit shown, assume the initial value of the output at time $t = 0$ is $y(0) = 0$. Determine the response of the system to a unit-step function, i.e., $x(t) = u(t)$.





Solution:

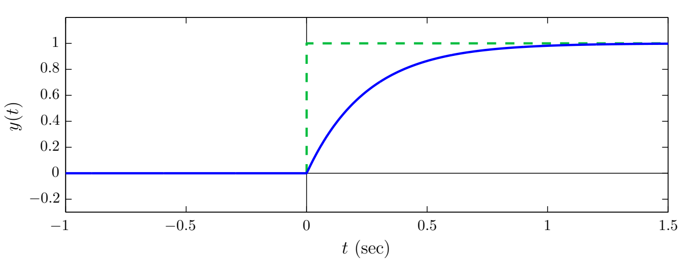
$$\frac{dy(t)}{dt} + 4\,y(t) = 4\,u(t)$$

$$y(t) = \int_0^t e^{-4(t-\tau)}\,4\,u(t)\,d\tau = 4\,e^{-4t}\int_0^t e^{4\tau}\,d\tau = 1 - e^{-4t} \qquad \text{for } t \geq 0$$

Example 2.8 (continued)

In compact form:

$$y(t) = \left(1 - e^{-4t}\right)\,u(t)$$

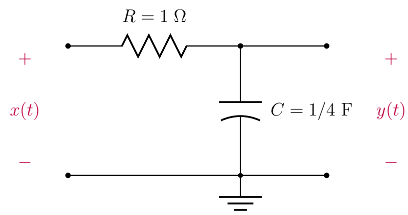


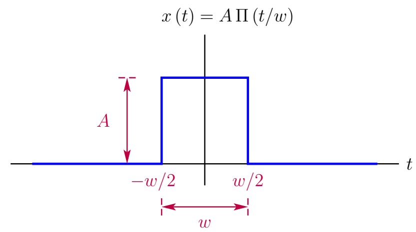
Example 2.9

Pulse response of the simple RC circuit

Determine the response of the RC circuit shown to a rectangular pulse signal

$$x(t) = A\,\Pi\,(t/w)$$





Solution:

Differential equation: $\dfrac{dy(t)}{dt} + 4\,y(t) = 4A\,\Pi\,(t/w)$

Initial value: $y(-w/2) = 0.$

Example 2.9 (continued)

Output signal:

$$y(t) = \int_{-w/2}^{t} e^{-4(t-\tau)}\,4A\,\Pi\,(\tau/w)\,d\tau$$

Case 1: $-\dfrac{w}{2} < t \le \dfrac{w}{2}$

$$y(t) = 4A\,e^{-4t}\int_{-w/2}^{t} e^{4\tau}\,d\tau = A\,\left[1 - e^{-2w}\,e^{-4t}\right]$$

Case 2: $t > \dfrac{w}{2}$

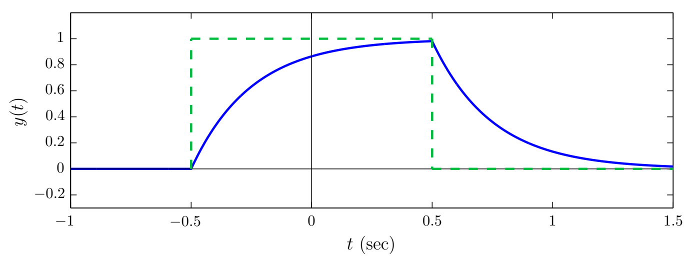
$$y(t) = 4A\,e^{-4t}\int_{-w/2}^{w/2} e^{4\tau}\,d\tau = A\,e^{-4t}\,\left[e^{2w} - e^{-2w}\right]$$

Example 2.9 (continued)

Complete response:

$$y(t) = \begin{cases} A\left[1 - e^{-2w}\, e^{-4t}\right], & -\dfrac{w}{2} < t \le \dfrac{w}{2} \\ A\, e^{-4t}\left[e^{2w} - e^{-2w}\right], & t > \dfrac{w}{2} \end{cases}$$

The signal $y(t)$ is shown for $A = 1$ and $w = 1$.



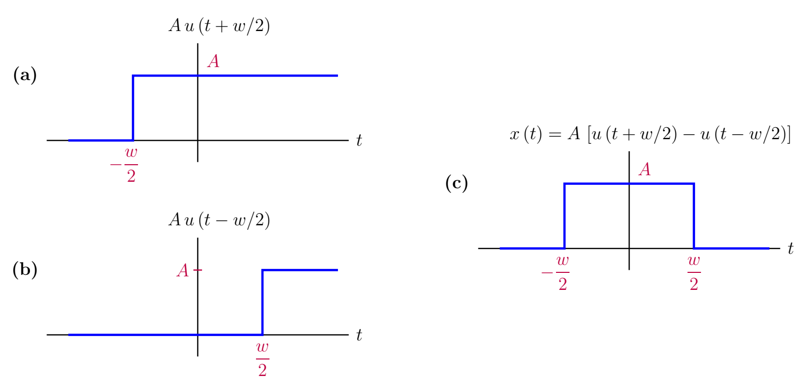
Example 2.10

Pulse response of the simple RC circuit revisited

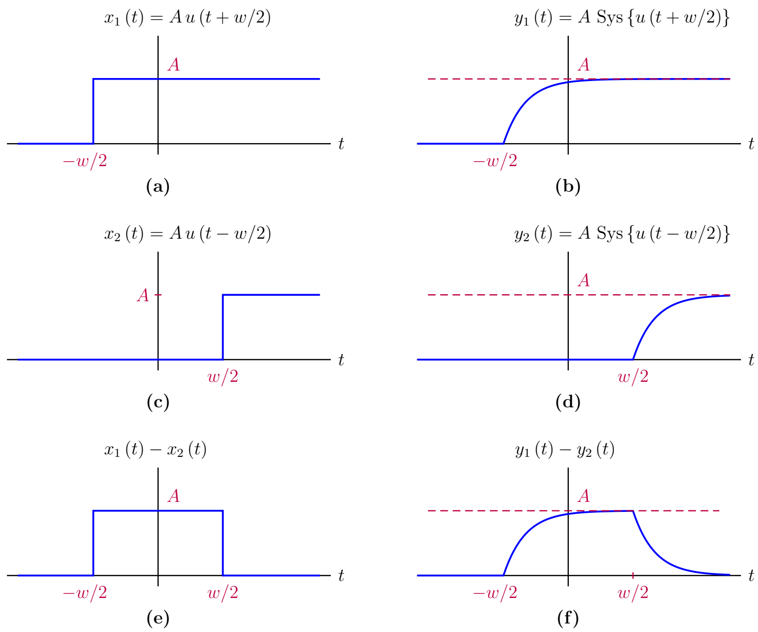
Rework the problem in Example 2.9 by making use of the unit-step response found in Example 2.8 along with linearity and time-invariance properties of the RC circuit.

Solution: Express the pulse signal as the difference of two unit-step signals:

$$x(t) = A\,\Pi(t/w) = A\,u\left(t + \frac{w}{2}\right) - A\,u\left(t - \frac{w}{2}\right)$$



Example 2.10 (continued)



Example 2.10 (continued)

Unit-step response:

$$\mathrm{Sys}\{u(t)\} = \left(1 - e^{-4t}\right)\, u(t)$$

Response to the pulse input:

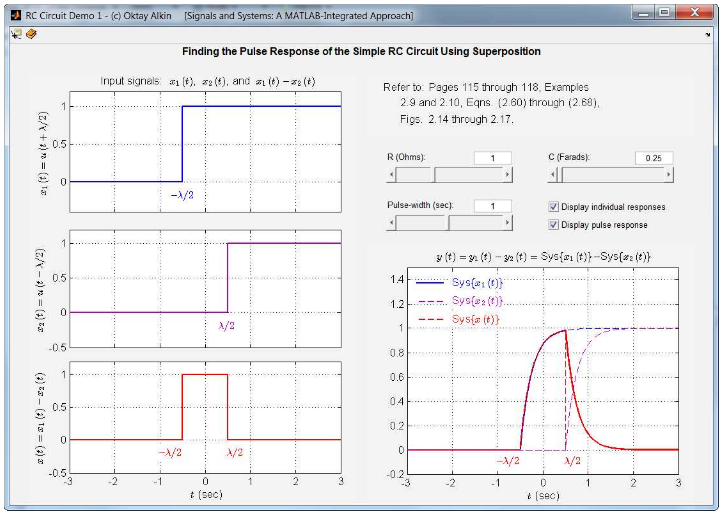
$$\mathrm{Sys}\{x(t)\} = A\ \mathrm{Sys}\left\{u\left(t+\frac{w}{2}\right)\right\} - A\ \mathrm{Sys}\left\{u\left(t-\frac{w}{2}\right)\right\}$$

$$\mathrm{Sys}\{x(t)\} = A\,\left[1 - e^{-4(t+w/2)}\right]\, u\left(t+\frac{w}{2}\right) - A\,\left[1 - e^{-4(t-w/2)}\right]\, u\left(t-\frac{w}{2}\right)$$

MATLAB Exercise 2.3

Interactive demo: rc_demo1.m

Experiment with the superposition principle by varying the circuit parameters R and C as well as the pulse width w.



Solution of the general differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Initial conditions:

$$y(t_0)\ , \quad \left.\frac{dy(t)}{dt}\right|_{t=t_0}\ , \quad \ldots, \quad \left.\frac{d^{N-1}y(t)}{dt^{N-1}}\right|_{t=t_0}$$

General solution:

$$y(t) = y_h(t) + y_p(t)$$

- $y_h(t)$ is the *homogeneous solution* of the differential equation (natural response).
- $y_p(t)$ is the *particular solution* of the differential equation.
- $y(t) = y_h(t) + y_p(t)$ is the *forced solution* of the differential equation (forced response).

Finding the natural response of a continuous-time system

Homogeneous differential equation:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0$$

First-order homogeneous differential equation:

$$\frac{dy(t)}{dt} + \alpha\, y(t) = 0$$

Solution:

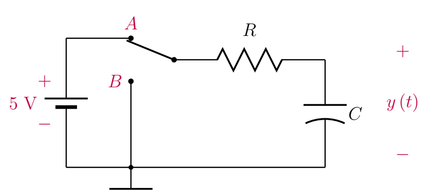
$$y(t) = c\, e^{-\alpha t}$$

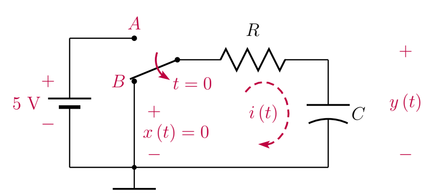
The constant c must be determined based on the desired initial value of $y(t)$ at $t = t_0$.

Example 2.11

Natural response of the simple RC circuit

Consider the RC circuit shown. Element values are $R = 1\ \Omega$ and $C = 1/4$ F. Input terminals of the circuit are connected to a battery that supplies the circuit with an input voltage of 5 V up to the time instant $t = 0$. The switch is moved from position A to position B at $t = 0$ ensuring that $x(t) = 0$ for $t \geq 0$. Find the output signal as a function of time.





Solution:

Homogeneous differential equation:

$$\frac{dy(t)}{dt} + 4\, y(t) = 0$$

Example 2.11 (continued)

Homogeneous solution is of the form:

$$y_h(t) = c\,e^{-st} = c\,e^{-4t}\,, \qquad \text{for } t \geq 0$$

Satisfy initial value:

$$y_h(0) = c\,e^{-4(0)} = c = 5$$

Natural response:

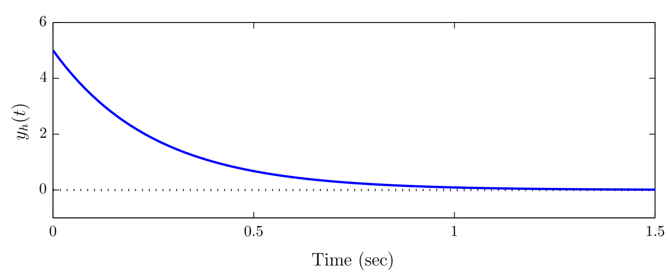
$$y_h(t) = 5\,e^{-4t}\,, \quad \text{for } \; t \geq 0$$

In compact form:

$$y_h(t) = 5\,e^{-4t}\,u(t)$$

Example 2.11 (continued)

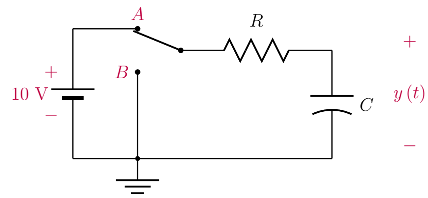
$$y_h(t) = 5\,e^{-4t}\,u(t)$$

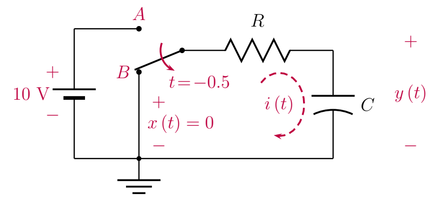


Example 2.12

Changing the start time in Example 2.11

Rework the problem in Example 2.11 with one minor change: The initial value of the output signal is specified at the time instant $t = -0.5$ seconds instead of at $t = 0$, and its value is $y(-0.5) = 10$.





Solution:

General form of the homogeneous solution:

$$y_h(t) = c\,e^{-4t}$$

To satisfy $y_h(-0.5) = 10$:

$$y_h(-0.5) = c\,e^{-4(-0.5)} = c\,e^2 = 10 \qquad \Rightarrow \qquad c = \frac{10}{e^2} = 1.3534$$

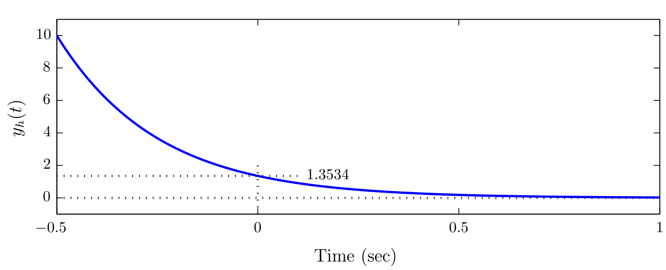
Example 2.12 (continued)

Homogeneous solution is

$$y_h(t) = 1.3534\,e^{-4t}\,, \quad \text{for} \;\; t \geq -0.5$$

In compact form:

$$y_h(t) = 1.3534\,e^{-4t}\,u(t+0.5)$$



Finding the natural response of a continuous-time system (continued)

General homogeneous differential equation:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0$$

Characteristic equation

$$\sum_{k=0}^{N} a_k s^k = 0$$

To obtain the characteristic equation, substitute:

$$\frac{d^k y(t)}{dt^k} \quad \rightarrow \quad s^k$$

Finding the natural response of a continuous-time system (continued)

Write the characteristic equation in open form:

$$a_N s^N + a_{N-1} s^{N-1} + \ldots + a_1 s + a_0 = 0$$

In factored form:

$$a_N (s - s_1)(s - s_2) \ldots (s - s_N) = 0$$

Homogeneous solution (assuming roots are distinct):

$$y_h(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} + \ldots + c_N e^{s_N t} = \sum_{k=1}^{N} c_k e^{s_k t}$$

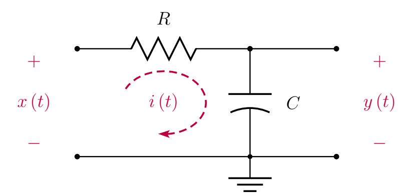
Unknown coefficients $c_1, c_2, \ldots, c_N$ are determined from the initial conditions.

Terms $e^{s_k t}$ are called the *modes of the system*.

Example 2.13

Time constant concept

Explore the natural response of the RC circuit as a function of circuit parameters and the initial voltage of the capacitor.



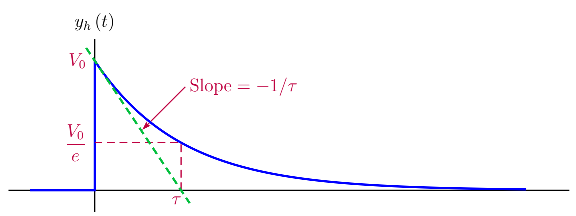
Solution:

The characteristic equation is $s + \dfrac{1}{RC} = 0$

If $y(0) = V_0$, the natural response is $y_h(t) = V_0\, e^{-t/RC}\, u(t)$

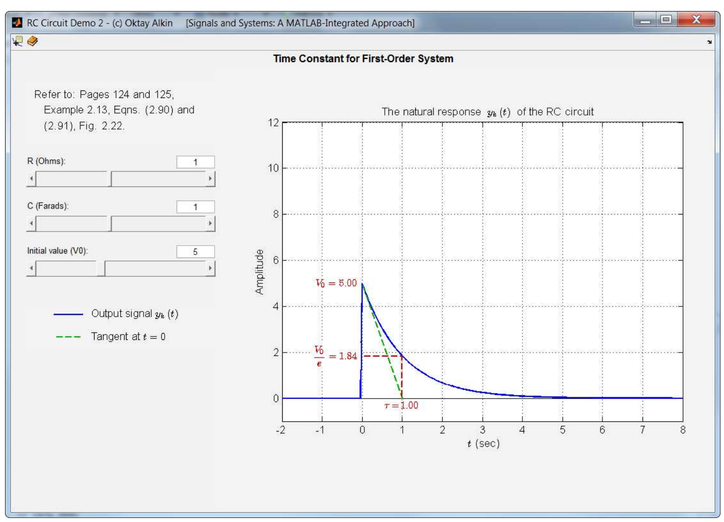
Define the *time constant* as $\tau = RC$, so that

$$y_h(t) = V_0\, e^{-t/\tau}\, u(t)$$



Interactive demo: rc_demo2.m

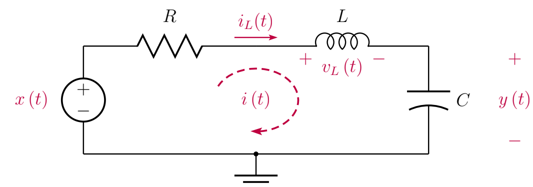
Experiment by varying the circuit parameters R and C as well as the initial voltage V_0.



Example 2.14

Natural response of second-order system

For the RLC circuit let the element values be $R = 5\ \Omega$, $L = 1$ H and $C = 1/6$ F. Initial values are $i(0) = 2$ A and $y(0) = 1.5$ V. No external input signal is applied to the circuit, therefore $x(t) = 0$. Determine the output voltage $y(t)$.



Solution:

Homogeneous differential equation:

$$\frac{d^2y(t)}{dt^2} + 5\,\frac{dy(t)}{dt} + 6\,y(t) = 0$$

Characteristic equation:

$$s^2 + 5s + 6 = 0 \qquad \Rightarrow \qquad s_1 = -2\,, \quad s_2 = -3$$

Homogeneous solution:

$$y_h(t) = c_1\,e^{-2t} + c_2\,e^{-3t} \quad \text{for } t \geq 0$$

Chapter 2
Solving Differential Equations
Finding the natural response of a continuous-time system

Example 2.14 (continued)

Evaluate $y_h(t)$ for $t = 0$:

$$y_h(0) = c_1 e^{-2(0)} + c_2 e^{-3(0)} = c_1 + c_2 = 1.5$$

Use the initial value of the inductor current:

$$i(0) = C\,\frac{dy_h(t)}{dt}\bigg|_{t=0} = 2 \qquad \Rightarrow \qquad \frac{dy_h(t)}{dt}\bigg|_{t=0} = \frac{i(0)}{C} = \frac{2}{1/6} = 12$$

Differentiate the homogeneous solution found:

$$\frac{dy_h(t)}{dt}\bigg|_{t=0} = \left[-2c_1 e^{-2t} - 3c_2 e^{-3t}\right]\bigg|_{t=0} = -2c_1 - 3c_2 = 12$$

Solve for c_1 and c_2:

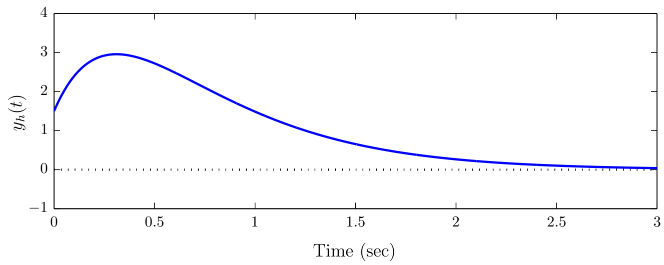
$$c_1 = 16.5\,, \quad \text{and} \quad c_2 = -15$$

Natural response:

$$y_h(t) = 16.5\,e^{-2t} - 15\,e^{-3t}\,, \quad t \geq 0$$

Example 2.14 (continued)

$$y_h(t) = 16.5\,e^{-2t} - 15\,e^{-3t}\,, \quad t \geq 0$$



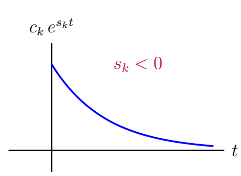
Roots of characteristic polynomial

Case 1: All roots are distinct and real-valued.

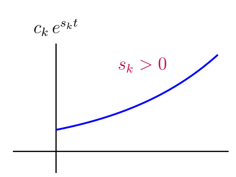
Homogeneous solution:

$$y_h(t) = \sum_{k=1}^{N} c_k\, e^{s_k t}$$

$s_k < 0 \quad \Rightarrow \quad$ Decaying exponential



$s_k > 0 \quad \Rightarrow \quad$ Growing exponential



Roots of characteristic polynomial (continued)

Case 2: Characteristic polynomial has complex-valued roots.

Since the coefficients of the characteristic polynomial are real-valued, any complex roots must appear in the form of conjugate pairs.

Part of the homogeneous solution that is due to a conjugate pair of roots:

$$\begin{aligned} y_{h1}(t) &= c_{1a}\, e^{s_{1a}t} + c_{1b}\, e^{s_{1b}t} \\ &= c_{1a}\, e^{(\sigma_1 + j\omega_1)t} + c_{1b}\, e^{(\sigma_1 - j\omega_1)t} \end{aligned}$$

Coefficients c_{1a} and c_{1b} must form a complex conjugate pair as well.

$$c_{1a} = |c_1|\; e^{j\theta_1} \quad \text{and} \quad c_{1b} = |c_1|\; e^{-j\theta_1}$$

$$y_{h1}(t) = 2\,|c_1|\; e^{\sigma_1 t} \cos(\omega_1 t + \theta_1)$$

Using the appropriate trigonometric identity:

$$y_{h1}(t) = d_1\, e^{\sigma_1 t}\, \cos(\omega_1 t) + d_2\, e^{\sigma_1 t}\, \sin(\omega_1 t)$$

Roots of characteristic polynomial (continued)

- A pair of complex conjugate roots for the characteristic polynomial leads to a solution component in the form of a cosine signal multiplied by an exponential signal.
- The oscillation frequency of the cosine signal is determined by ω_1, the imaginary part of the complex roots.
- The real part of the complex roots, σ_1, impacts the amplitude of the solution. If $\sigma_1 < 0$, then the amplitude of the cosine signal decays exponentially over time. In contrast, if $\sigma_1 > 0$, the amplitude of the cosine signal grows exponentially over time.

Roots of characteristic polynomial (continued)

Case 3: Characteristic polynomial has some multiple roots.

$$a_N\,(s-s_1)\,(s-s_2)\,\ldots\,(s-s_N)=0$$

What if $s_2=s_1$?

$$y_h\,(t)=c_{11}\,e^{s_1 t}+c_{12}\,t\,e^{s_1 t}+\text{other terms}$$

A root of multiplicity r requires r terms in the homogeneous solution:

$$y_h\,(t)=c_{11}\,e^{s_1 t}+c_{12}\,t\,e^{s_1 t}+\ldots+c_{1r}\,t^{r-1}\,e^{s_1 t}+\text{other terms}$$

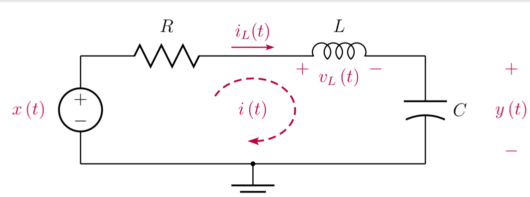
Example 2.15

Natural response of second-order system revisited

For the RLC circuit shown, the initial inductor current is $i\,(0)=0.5$ A, and the initial capacitor voltage is $y\,(0)=2$ V. No external input signal is applied to the circuit, therefore $x\,(t)=0$. Determine the output voltage $y\,(t)$ if

a. the element values are $R=2\ \Omega$, $L=1$ H and $C=1/26$ F,

b. the element values are $R=6\ \Omega$, $L=1$ H and $C=1/9$ F.



Solution:

Using specified initial value of the inductor current:

$$i\,(0)=C\left.\frac{dy_h\,(t)}{dt}\right|_{t=0}=0.5\qquad\Rightarrow\qquad\left.\frac{dy_h\,(t)}{dt}\right|_{t=0}=\frac{0.5}{C}$$

Example 2.15 (continued)

a.
Homogeneous differential equation:

$$\frac{d^2y(t)}{dt^2} + 2\,\frac{dy(t)}{dt} + 26\,y(t) = 0$$

Characteristic equation:

$$s^2 + 2s + 26 = 0 \qquad \Rightarrow \qquad s_1 = -1 + j5\,, \quad s_2 = -1 - j5$$

Natural response:

$$y_h(t) = d_1\,e^{-t}\,\cos(5t) + d_2\,e^{-t}\,\sin(5t)$$

Impose initial conditions:

$$y_h(0) = d_1 = 2$$

$$\left.\frac{dy_h(t)}{dt}\right|_{t=0} = -d_1 + 5d_2 = 13 \qquad \Rightarrow \qquad d_2 = 3$$

Natural response:

$$y_h(t) = 2\,e^{-t}\,\cos(5t) + 3\,e^{-t}\,\sin(5t) \quad \text{for } t \geq 0$$

Example 2.15 (continued)

b.
Homogeneous differential equation:

$$\frac{d^2y(t)}{dt^2} + 6\,\frac{dy(t)}{dt} + 9\,y(t) = 0$$

Characteristic equation:

$$s^2 + 6s + 9 = 0 \quad \Rightarrow \quad (s+3)^2 = 0$$

Homogeneous solution:

$$y_h(t) = c_{11}\,e^{-3t} + c_{12}\,te^{-3t} \quad \text{for } t \geq 0$$

Impose initial conditions:

$$c_{11} = 2$$

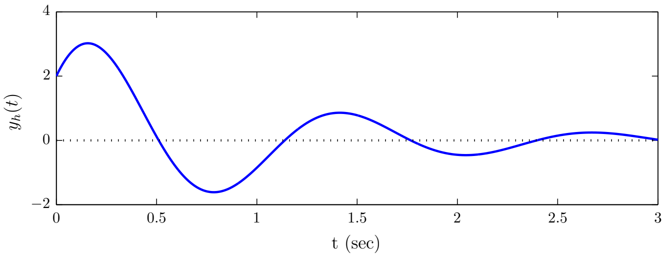
$$\left.\frac{dy_h(t)}{dt}\right|_{t=0} = -3\,c_{11} + c_{12} = 4.5 \qquad \Rightarrow \qquad c_{12} = 10.5$$

Natural response:

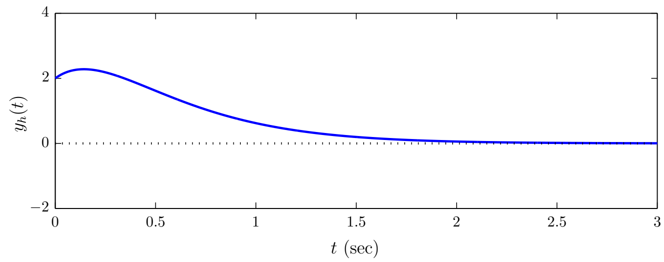
$$y_h(t) = 2\,e^{-3t} + 10.5\,te^{-3t} \quad \text{for } t \geq 0$$

Example 2.15 (continued)

Homogeneous solution for part (a):

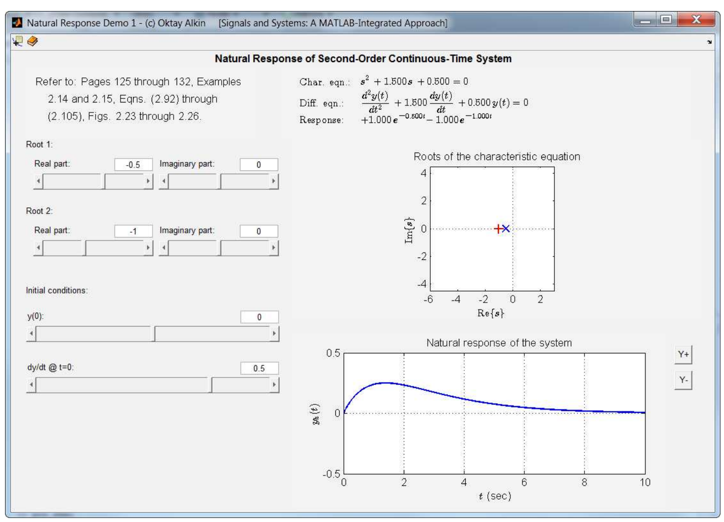


Homogeneous solution for part (b):



Interactive demo: nr_demo1.m

Experiment by varying the locations of the two roots s_1 and s_2 on the complex plane.



Finding the forced response of a continuous-time system

Choosing a particular solution for various input signals

Input signal	Particular solution
K (constant)	k_1
$K\,e^{at}$	$k_1\,e^{at}$
$K\,\cos(at)$	$k_1\,\cos(at) + k_2\,\sin(at)$
$K\,\sin(at)$	$k_1\,\cos(at) + k_2\,\sin(at)$
$K\,t^n$	$k_n t^n + k_{n-1} t^{n-1} + \ldots + k_1 t + k_0$

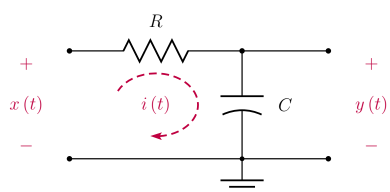
Example 2.16

Forced response of the first-order system for sinusoidal input

Determine the output signal of the RC circuit shown in response to a sinusoidal input signal in the form

$$x(t) = A\,\cos(\omega t)$$

with amplitude $A = 20$ and radian frequency $\omega = 8$ rad/s. The initial value of the output signal is $y(0) = 5$.



Solution:

Differential equation:

$$\frac{dy(t)}{dt} + 4\,y(t) = 4\,x(t)$$

Homogeneous solution is in the form

$$y_h(t) = c\,e^{-4t} \quad \text{for } t \geq 0$$

Do not determine c yet!

Example 2.16 (continued)

Particular solution is in the form

$$y_p(t) = k_1 \cos(\omega t) + k_2 \sin(\omega t)$$

Particular solution $y_p(t)$ must satisfy the differential equation:

$$\frac{dy_p(t)}{dt} = -\omega k_1 \sin(\omega t) + \omega k_2 \cos(\omega t)$$

$$-\omega k_1 \sin(\omega t) + \omega k_2 \cos(\omega t) + 4\left[k_1 \cos(\omega t) + 4k_2 \sin(\omega t)\right] = A \cos(\omega t)$$

In compact form:

$$(4k_1 + \omega k_2 - A) \cos(\omega t) + (4k_2 - \omega k_1) \sin(\omega t) = 0$$

Solve for k_1 and k_2:

$$k_1 = \frac{4A}{16+\omega^2}, \quad k_2 = \frac{A\omega}{16+\omega^2}$$

Forced solution:

$$y(t) = y_h(t) + y_f(t) = ce^{-4t} + \frac{4A}{16+\omega^2} \cos(\omega t) + \frac{A\omega}{16+\omega^2} \sin(\omega t)$$

Example 2.16 (continued)

Using numerical values $A = 20$ and $\omega = 8$ rad/s:

$$y(t) = c\, e^{-4t} + \cos(8t) + 2 \sin(8t)$$

Impose the initial condition $y(0) = 5$:

$$y(0) = 5 = c + \cos(0) + 2 \sin(0) \qquad \Rightarrow \qquad c = 4$$

Complete solution:

$$y(t) = 4\, e^{-4t} + \cos(8t) + 2 \sin(8t) \quad \text{for } t \ge 0$$

$$y(t) = y_t(t) + y_{ss}(t)$$

Transient component:

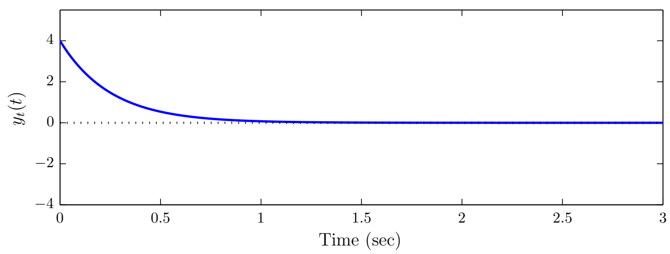
$$y_t(t) = 4\, e^{-4t}, \qquad \lim_{t\to\infty} \{y_t(t)\} = 0$$

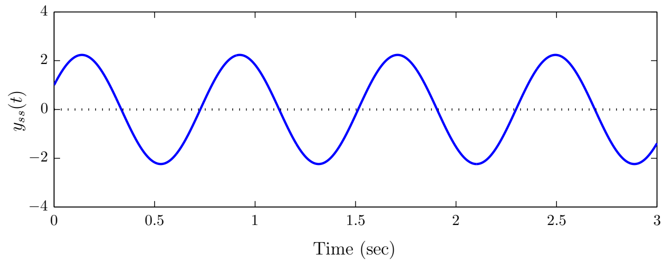
Steady-state component:

$$y_{ss}(t) = \cos(8t) + 2 \sin(8t)$$

Example 2.16 (continued)

$$y_t\,(t) = 4\,e^{-4t}\,, \quad y_{ss}\,(t) = \cos\,(8t) + 2\,\sin\,(8t)$$

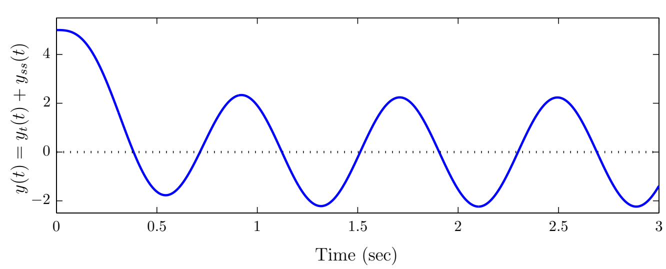




Example 2.16 (continued)

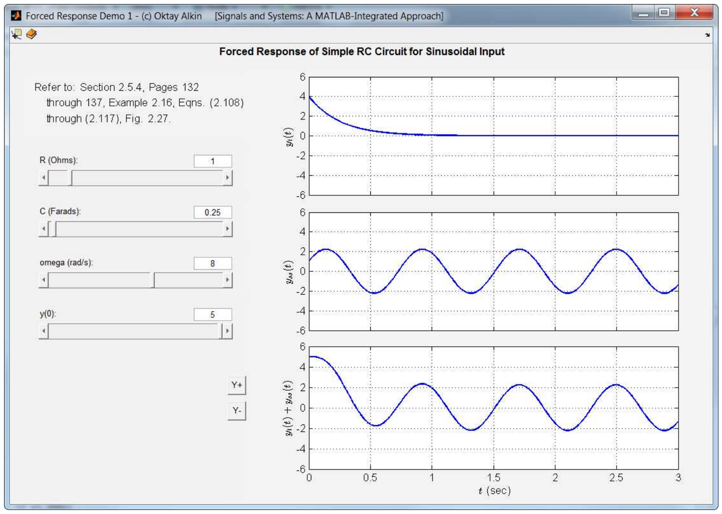
Complete solution:

$$y\,(t) = 4\,e^{-4t} + \cos\,(8t) + 2\,\sin\,(8t) \quad \text{for } t \ge 0$$



Interactive demo: fr_demo1.m

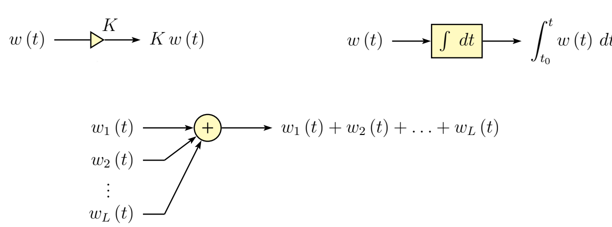
Experiment by varying the circuit parameters R and C, the radian frequency ω and the initial value $y(0)$. Observe the effects on transient response $y_t(t)$, the steady-state response $y_{ss}(t)$ and the total forced response $y(t) = y_t(t) + y_{ss}(t)$.



Block diagram representation of continuous-time systems

Block diagrams for continuous-time systems are constructed using three types of components:

- Constant-gain amplifiers
- Signal adders
- Integrators



Block diagram representation of continuous-time systems (continued)

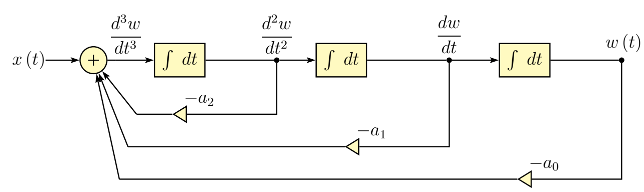
A third-order differential equation:

$$\frac{d^3y}{dt^3} + a_2\,\frac{d^2y}{dt^2} + a_1\,\frac{dy}{dt} + a_0\,y = b_2\,\frac{d^2x}{dt^2} + b_1\,\frac{dx}{dt} + b_0\,x$$

Use an intermediate variable $w(t)$ in place of $y(t)$ in the left side of the differential equation, and set the result equal to $x(t)$:

$$\frac{d^3w}{dt^3} + a_2\,\frac{d^2w}{dt^2} + a_1\,\frac{dw}{dt} + a_0\,w = x$$

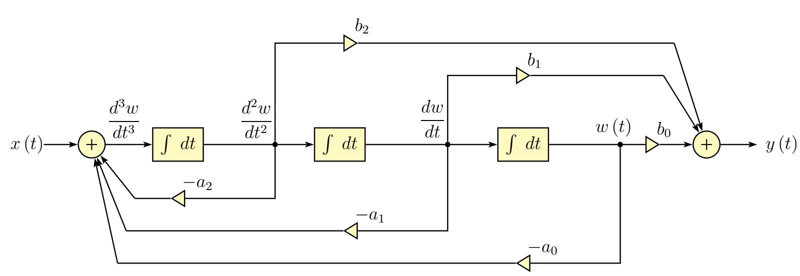
$$\frac{d^3w}{dt^3} = x - a_2\,\frac{d^2w}{dt^2} - a_1\,\frac{dw}{dt} - a_0\,w$$



Block diagram representation of continuous-time systems (continued)

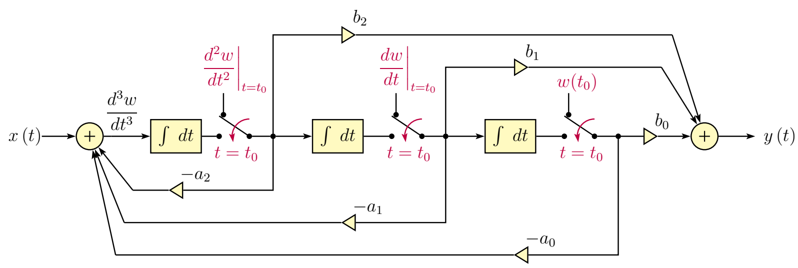
Express the signal $y(t)$ in terms of the intermediate variable $w(t)$:

$$y = b_2\,\frac{d^2w}{dt^2} + b_1\,\frac{dw}{dt} + b_0\,w$$



Block diagram representation of continuous-time systems (continued)

Imposing initial conditions:



Example 2.17

Block diagram for continuous-time system

Construct a block diagram to solve the differential equation

$$\frac{d^3y}{dt^3} + 5\,\frac{d^2y}{dt^2} + 17\,\frac{dy}{dt} + 13\,y = x + 2\,\frac{dx}{dt}$$

with the input signal $x(t) = \cos(20\pi t)$ and subject to initial conditions

$$y(0) = 1\,, \qquad \left.\frac{dy}{dt}\right|_{t=0} = 2\,, \qquad \left.\frac{d^2y}{dt^2}\right|_{t=0} = -4\,,$$

Solution:

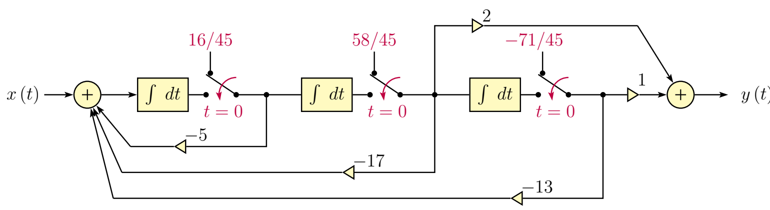
Using the intermediate variable $w(t)$:

$$\frac{d^3w}{dt^3} + 5\,\frac{d^2w}{dt^2} + 17\,\frac{dw}{dt} + 13\,w = x \qquad \text{and} \qquad y = w + 2\,\frac{dw}{dt}$$

Example 2.17 (continued)

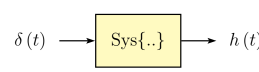
Initial conditions specified in terms of the values of y, dy/dt and d^2y/dt^2 at $t=0$ need to be expressed in terms of the integrator outputs w, dw/dt and d^2w/dt^2 at $t=0$.

$$w\left(0\right)=\frac{-71}{45}\;,\qquad \left.\frac{dw}{dt}\right|_{t=0}=\frac{58}{45}\;,\qquad \left.\frac{d^2w}{dt^2}\right|_{t=0}=\frac{16}{45}$$



Impulse response

$$h\left(t\right)=\text{Sys}\left\{\delta\left(t\right)\right\}$$



For a CTLTI system: The impulse response also constitutes a complete description of the system.

Finding the impulse response of a CTLTI system from the differential equation

1. Use a unit-step function for the input signal, and compute the forced response of the system, i.e., the *unit-step response*.
2. Differentiate the unit-step response of the system to obtain the impulse response, i.e.,

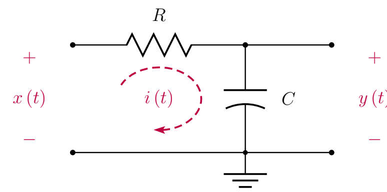
$$h\left(t\right)=\frac{dy\left(t\right)}{dt}$$

$$\text{Sys}\left\{\delta\left(t\right)\right\}=\text{Sys}\left\{\frac{du\left(t\right)}{dt}\right\}=\frac{d}{dt}\left[\text{Sys}\left\{u\left(t\right)\right\}\right]$$

Example 2.18

Impulse response of the simple RC circuit

Determine the impulse response of the first-order RC circuit shown. Assume the system is initially relaxed, that is, there is no initial energy stored in the system. (Recall that this is a necessary condition for the system to be CTLTI.)



Solution: Differential equation is

$$\frac{dy(t)}{dt} + 4\,y(t) = 4\,x(t)$$

Using the first-order solution method:

$$h(t) = \int_0^t e^{-4(t-\tau)}\,4\,\delta(\tau)\,d\tau$$

Using the sifting property of the unit-impulse function:

$$h(t) = 4\,e^{-4t}\,u(t)$$

Example 2.18 (continued)

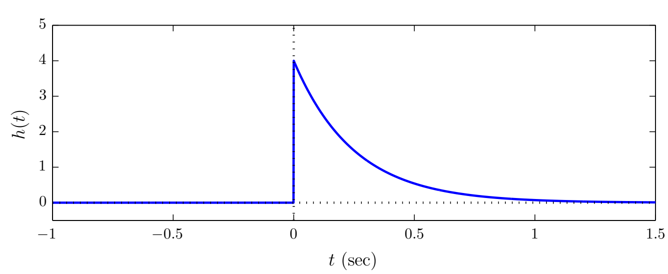
Using the the more general method that relies on the unit step response:

$$y(t) = \text{Sys}\{u(t)\} = \left(1 - e^{-4t}\right)\,u(t)$$

Differentiating $y(t)$:

$$h(t) = \frac{dy(t)}{dt} = \frac{d}{dt}\left[\left(1 - e^{-4t}\right)\,u(t)\right] = 4\,e^{-4t}\,u(t)$$

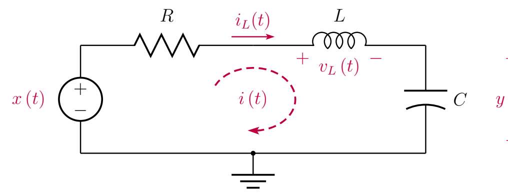
s



Example 2.19

Impulse response of a second-order system

Determine the impulse response of the RLC circuit shown. Use element values $R = 2\ \Omega$, $L = 1$ H and $C = 1/26$ F.



Solution:

Differential equation:

$$\frac{d^2y(t)}{dt^2} + 2\,\frac{dy(t)}{dt} + 26\,y(t) = 0$$

The homogeneous solution is (see Example 2.15)

$$y_h(t) = d_1\,e^{-t}\cos(5t) + d_2\,e^{-t}\sin(5t)$$

To find the unit-step response, start with the particular solution

$$y_p(t) = k_1$$

Example 2.19 (continued)

Particular solution must satisfy the differential equation, therefore $k_1 = 1$, and the complete solution is

$$\begin{aligned} y(t) =& y_h(t) + y_p(t) \\ =& d_1\,e^{-t}\cos(5t) + d_2\,e^{-t}\sin(5t) + 1 \end{aligned}$$

The system is CTLTI, and is therefore initially relaxed.

$$y(0) = d_1 + 1 = 0 \qquad \Rightarrow \qquad d_1 = -1$$

$$\left.\frac{dy_h(t)}{dt}\right|_{t=0} = 0 \qquad \Rightarrow \qquad -d_1 + 5d_2 = 0 \qquad \Rightarrow \qquad d_2 = -0.2$$

s Unit-step response is

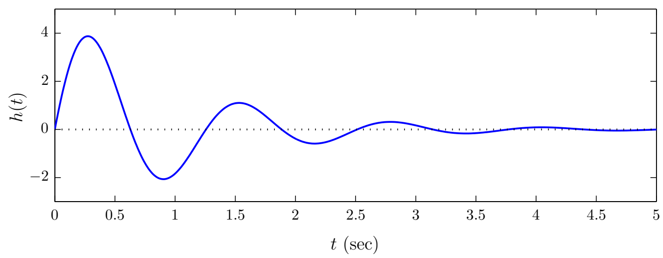
$$y(t) = y_h(t) + y_p(t) = -e^{-t}\cos(5t) - (0.2)\,e^{-t}\sin(5t) + 1 \quad \text{for } t \geq 0$$

Impulse response is

$$h(t) = \frac{dy(t)}{dt} = 5.2\,e^{-t}\sin(5t) \quad \text{for } t \geq 0$$

Example 2.19 (continued)

$$h(t) = \frac{dy(t)}{dt} = 5.2\, e^{-t} \sin(5t) \quad \text{for } t \ge 0$$



Convolution operation for CTLTI systems

The output signal $y(t)$ of a CTLTI system is equal to the convolution of its impulse response $h(t)$ with the input signal $x(t)$.

Continuous-time convolution

$$\begin{aligned} y(t) =& x(t) * h(t) = \int_{-\infty}^{\infty} x(\lambda)\; h(t-\lambda)\; d\lambda \\ =& h(t) * x(t) = \int_{-\infty}^{\infty} h(\lambda)\; x(t-\lambda)\; d\lambda \end{aligned}$$

Convolution operation for CTLTI systems (continued)

Steps involved in computing the convolution of two signals

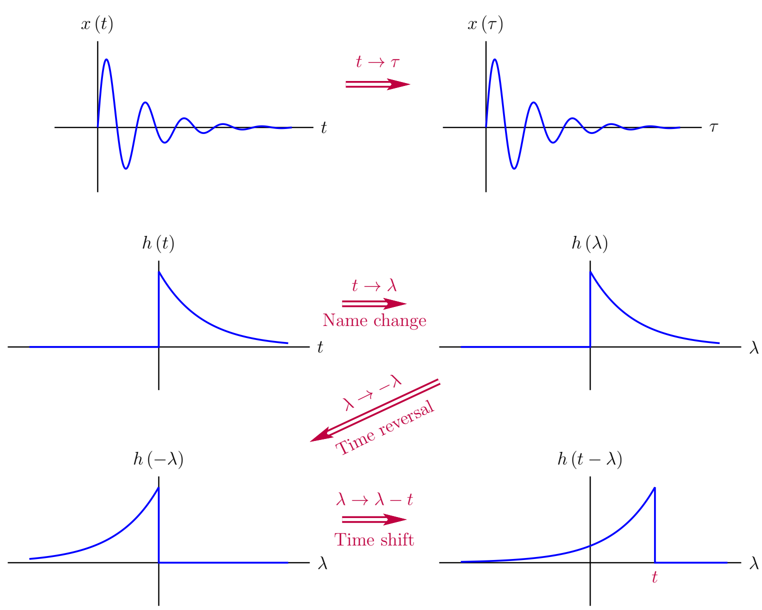
To compute the convolution of $x(t)$ and $h(t)$ at a specific time-instant t:

1. Sketch the signal $x(\lambda)$ as a function of the independent variable λ. This corresponds to a simple name change on the independent variable, and the graph of the signal $x(\lambda)$ appears identical to the graph of the signal $x(t)$.
2. For one specific value of t, sketch the signal $h(t-\lambda)$ as a function of the independent variable λ. This task can be broken down into two steps as follows:
   - 2a. Sketch $h(-\lambda)$ as a function of λ. This step amounts to time-reversal of $h(\lambda)$.
   - 2b. In $h(\lambda)$ substitute $\lambda \to \lambda - t$. This step yields

     $$h(-\lambda)\Big|_{\lambda \to \lambda - t} = h(t-\lambda)$$

     and amounts to time-shifting $h(-\lambda)$ by t.
3. Multiply the two signals in 1 and 2 to obtain $f(\lambda) = x(\lambda)\, h(t-\lambda)$.
4. Compute the area under the product $f(\lambda) = x(\lambda)\, h(t-\lambda)$ by integrating it over the independent variable λ. The result is the value of the output signal at the specific time instant t.
5. Repeat steps 1 through 4 for all values of t that are of interest.

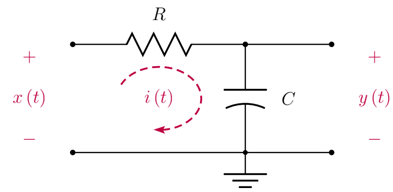
Convolution operation for CTLTI systems (continued)



Example 2.20

Unit-step response of RC circuit revisited

Compute the unit-step response of the simple RC circuit using the convolution operation.



Solution:

Impulse response of the RC circuit is

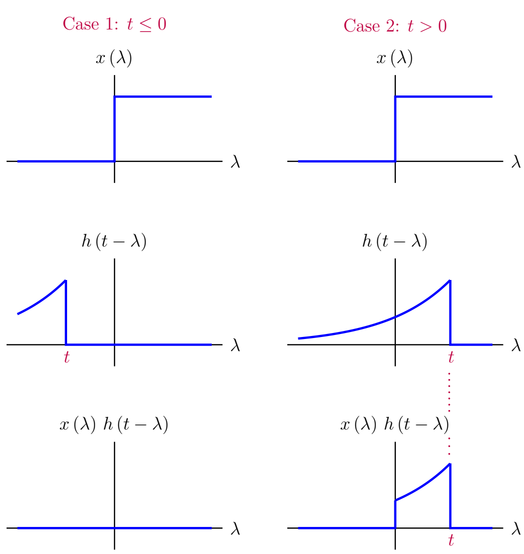
$$h(t) = \frac{1}{RC}\, e^{-t/RC}\, u(t)$$

Output of the system in response to input $x(t)$:

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)\, h(t-\lambda)\, d\lambda$$

Functions needed: $x(\lambda)$ and $h(t-\lambda)$.

Example 2.20 (continued)



Example 2.20 (continued)

<u>Case 1:</u> $t \leq 0$

Functions $x(\lambda)$ and $h(t-\lambda)$ do not overlap anywhere. Therefore

$$y(t) = 0, \quad \text{for } t \leq 0$$

<u>Case 2:</u> $t > 0$

Functions $x(\lambda)$ and $h(t-\lambda)$ overlap for values of λ in the interval $(0, t)$. In this interval $x(\lambda) = 1$ and $h(t-\lambda) = \dfrac{1}{RC}\, e^{-(t-\lambda)/RC}$. Therefore

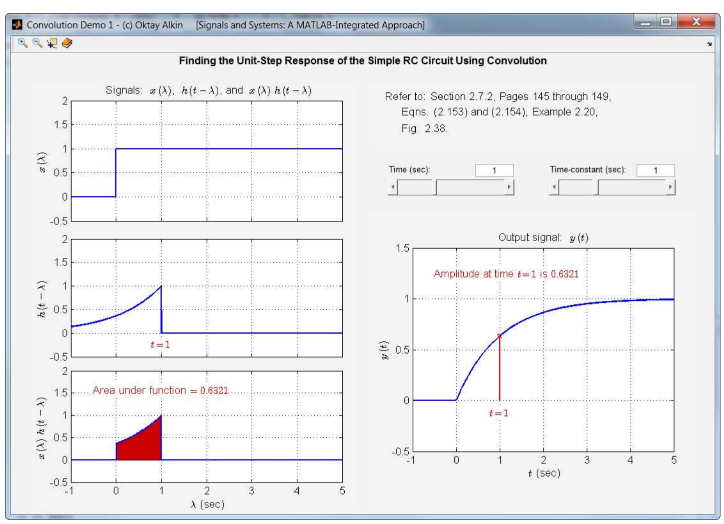
$$y(t) = \int_0^t \frac{1}{RC}\, e^{-(t-\lambda)/RC}\, d\lambda = 1 - e^{-t/RC}, \quad \text{for} \quad t > 0$$

Combine the two cases through the use of a unit-step function:

$$y(t) = \left(1 - e^{-t/RC}\right) u(t)$$

Interactive demo: conv_demo1.m

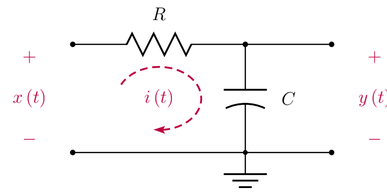
Vary t and observe the waveforms and their overlaps.



Example 2.21

Pulse response of RC circuit revisited

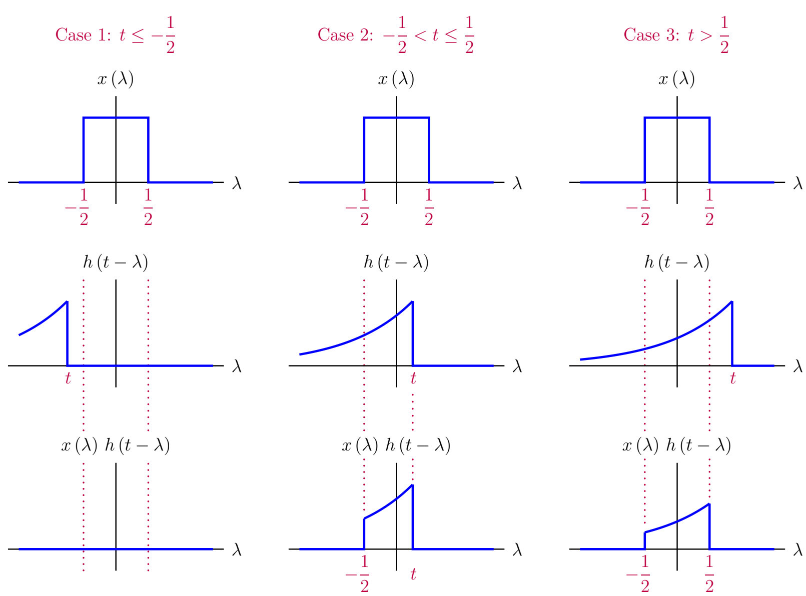
Using convolution, determine the response of the RC circuit to a unit-pulse input signal $x(t) = \Pi(t)$.



Solution:

It is useful to sketch the functions involved in the convolution integral, namely $x(\lambda)$ and $h(t-\lambda)$. Three distinctly different possibilities for the time variable t will be considered.

Example 2.21 (continued)



Example 2.21 (continued)

<u>Case 1:</u> $t \le -\frac{1}{2}$

Functions $x(\lambda)$ and $h(t-\lambda)$ do not overlap. Therefore

$$y(t) = 0\,, \quad \text{for} \quad t \le -\frac{1}{2}$$

<u>Case 2:</u> $-\frac{1}{2} < t \le \frac{1}{2}$

Functions $x(\lambda)$ and $h(t-\lambda)$ overlap in the range $-\frac{1}{2} < \lambda \le t$. Therefore

$$y(t) = \int_{-1/2}^{t} \frac{1}{RC}\, e^{-(t-\lambda)/RC}\, d\lambda = \left(1 - e^{-(t+1/2)/RC}\right)\,, \quad \text{for} \quad -\frac{1}{2} < t \le \frac{1}{2}$$

<u>Case 3:</u> $t > \frac{1}{2}$

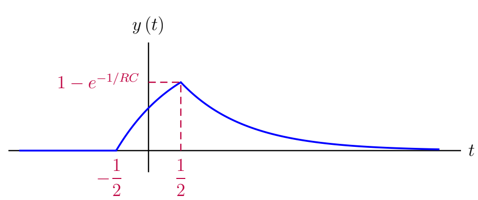
Functions $x(\lambda)$ and $h(t-\lambda)$ overlap in the range $-\frac{1}{2} < \lambda \le \frac{1}{2}$. Therefore

$$y(t) = \int_{-1/2}^{1/2} \frac{1}{RC}\, e^{-(t-\lambda)/RC}\, d\lambda = e^{-t/RC}\left(e^{1/2RC} - e^{-1/2RC}\right)\,, \quad \text{for} \quad t > \frac{1}{2}$$

Chapter 2
Impulse Response and Convolution
Convolution operation for CTLTI systems

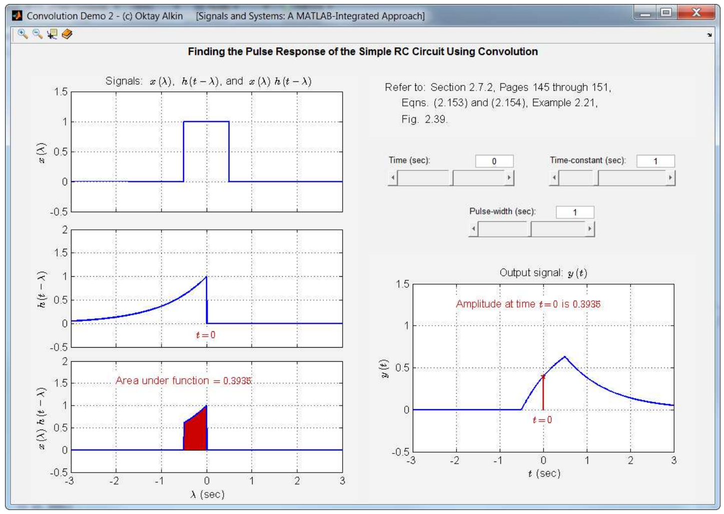
Example 2.21 (continued)

$$y(t) = \begin{cases} 0\,, & t \le -\frac{1}{2} \\ \left(1 - e^{-(t+1/2)/RC}\right)\,, & -\frac{1}{2} < t \le \frac{1}{2} \\ e^{-t/RC}\left(e^{1/2RC} - e^{-1/2RC}\right)\,, & t > \frac{1}{2} \end{cases}$$



Interactive demo: conv_demo2.m

Vary t and observe the waveforms and their overlaps.



Example 2.22

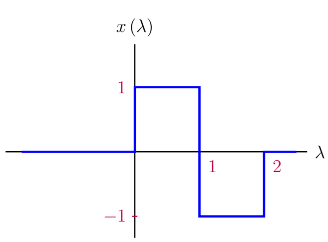
A more involved convolution problem

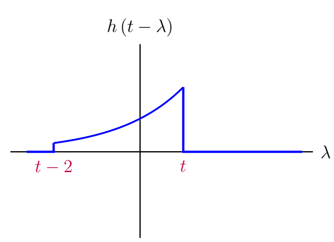
Impulse response of a CTLTI system is $h(t) = e^{-t}\,[u(t) - u(t-2)]$. The input signal is

$$x(t) = \Pi(t-0.5) - \Pi(t-1.5) = \begin{cases} 1\,, & 0 \le t < 1 \\ -1\,, & 1 \le t < 2 \\ 0\,, & \text{otherwise} \end{cases}$$

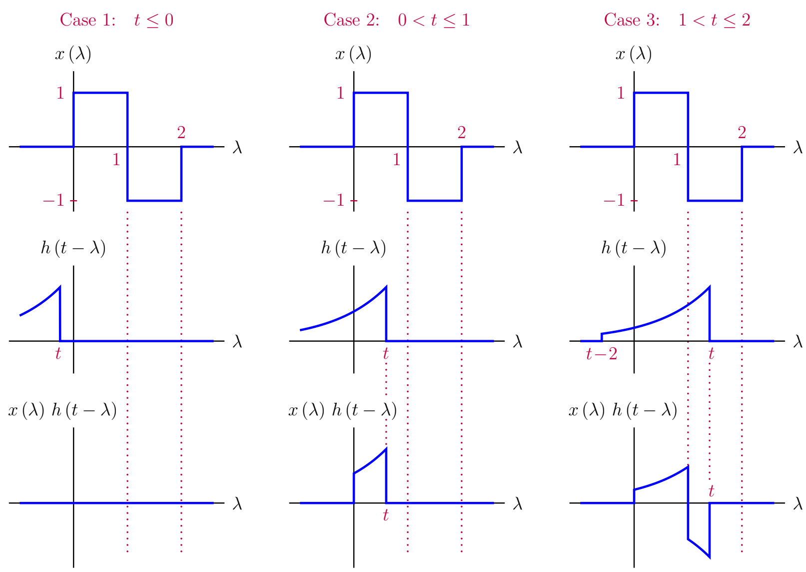
Determine the output signal $y(t)$ using convolution.

Solution: Functions involved in the convolution integral are:

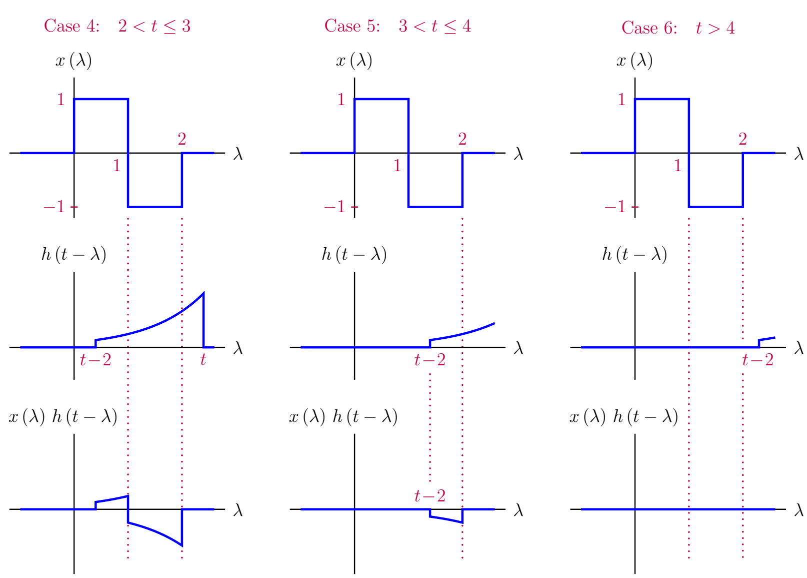




Example 2.22 (continued)



Example 2.22 (continued)



Example 2.22 (continued)

Case 1: $t \le 0$

Functions $x(\lambda)$ and $h(t-\lambda)$ do not overlap. Therefore

$$y(t) = 0\,, \quad \text{for} \quad t \le 0$$

Case 2: $0 < t \le 1$

Functions $x(\lambda)$ and $h(t-\lambda)$ overlap for $0 < \lambda \le t$. Therefore

$$y(t) = \int_0^t (1)\; e^{-(t-\lambda)}\, d\lambda = 1 - e^{-t}\,, \quad \text{for} \quad 0 < t \le 1$$

Case 3: $1 < t \le 2$

Functions $x(\lambda)$ and $h(t-\lambda)$ overlap for $0 < \lambda \le t$. Therefore

$$y(t) = \int_0^1 (1)\; e^{-(t-\lambda)}\, d\lambda + \int_1^t (-1)\; e^{-(t-\lambda)}\, d\lambda$$
$$= -1 + 4.4366\, e^{-t}\,, \quad \text{for} \quad 1 < t \le 2$$

Example 2.22 (continued)

Case 4: $2 < t \le 3$

Functions $x(\lambda)$ and $h(t-\lambda)$ overlap for $t-2 < \lambda \le 2$. Therefore

$$y(t) = \int_{t-2}^1 (1)\; e^{-(t-\lambda)}\, d\lambda + \int_1^2 (-1)\; e^{-(t-\lambda)}\, d\lambda$$
$$= -0.1353 - 1.9525\, e^{-t}\,, \quad \text{for} \quad 2 < t \le 3$$

Case 5: $3 < t \le 4$

Functions $x(\lambda)$ and $h(t-\lambda)$ overlap for $t-2 < \lambda \le 2$. Therefore

$$y(t) = \int_{t-2}^2 (-1)\; e^{-(t-\lambda)}\, d\lambda = 0.1353 - 7.3891\, e^{-t}\,, \quad \text{for} \quad 3 < t \le 4$$

Case 6: $t > 4$

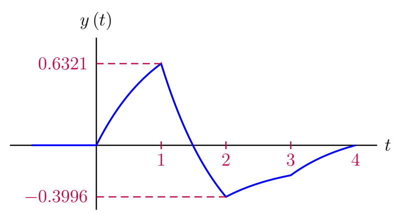
Functions $x(\lambda)$ and $h(t-\lambda)$ do not overlap. Therefore

$$y(t) = 0\,, \quad \text{for} \quad t > 4$$

Example 2.22 (continued)

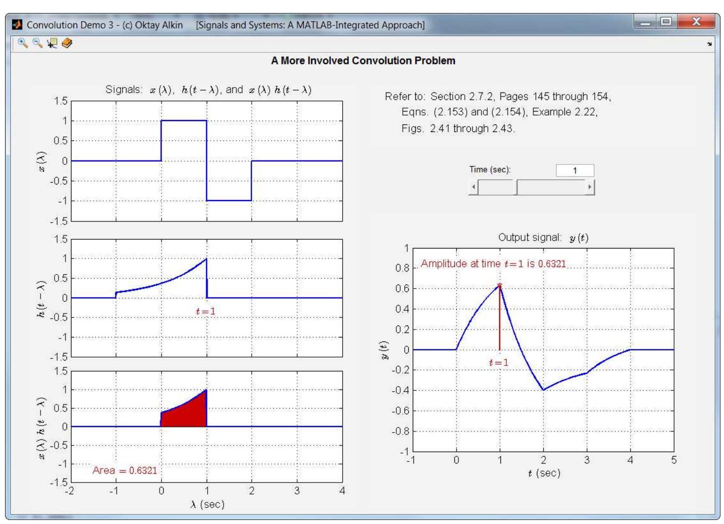
In compact form:

$$y(t) = \begin{cases} 0\,, & t<0 \text{ or } t>4 \\ 1-e^{-t}\,, & 0<t\le 1 \\ -1+4.4366\,e^{-t}\,, & 1<t\le 2 \\ -0.1353-1.9525\,e^{-t}\,, & 2<t\le 3 \\ 0.1353-7.3891\,e^{-t}\,, & 3<t\le 4 \end{cases}$$



Interactive demo: conv_demo3.m

Vary t and observe the waveforms and their overlaps.



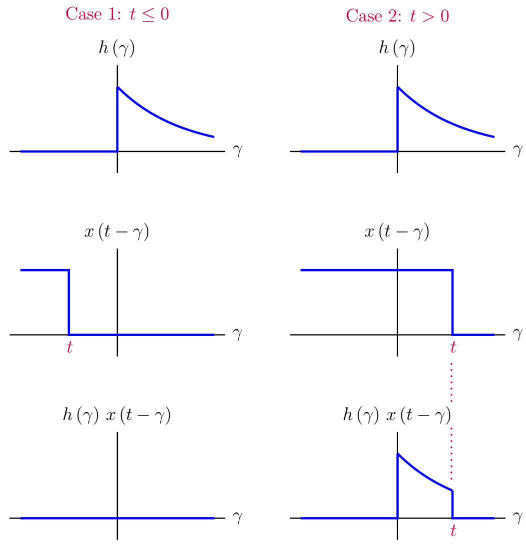
Example 2.23

Using alternative form of convolution

Find the unit-step response of the RC circuit with impulse response

$$h(t) = \frac{1}{RC}\, e^{-t/RC}\, u(t)$$

using the alternative form of the convolution integral.



Solution:

$$y(t) = \int_0^t h(\gamma)\, x(t-\gamma)\, d\gamma$$

Example 2.23 (continued)

For $t \le 0$ the two functions do not overlap. Therefore

$$y(t) = 0\,, \quad \text{for } t \le 0$$

For $t > 0$, the two functions $h(\gamma)$ and $x(t-\gamma)$ overlap in the interval $(0, t)$. Therefore

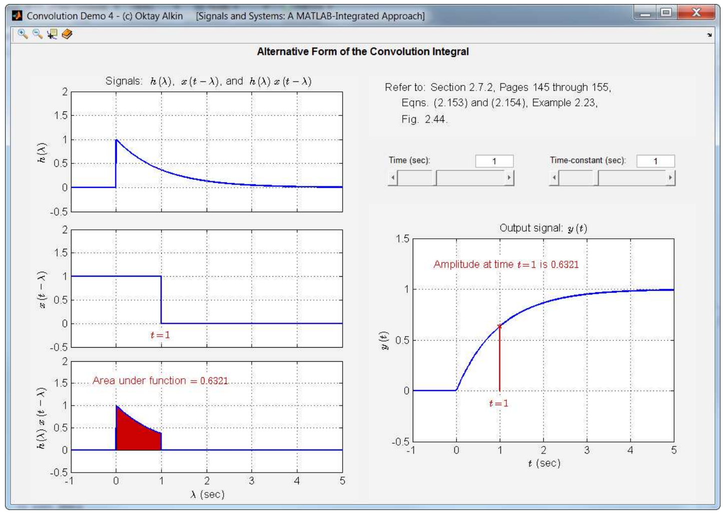
$$y(t) = \int_0^t \frac{1}{RC}\, e^{-\gamma/RC}\, d\gamma = 1 - e^{-t/RC}\,, \quad \text{for} \quad t > 0$$

In compact form:

$$y(t) = \left(1 - e^{-t/RC}\right)\, u(t)$$

Interactive demo: conv_demo4.m

Vary t and observe the waveforms and their overlaps.



Causality in continuous-time systems

Causal system

A system is said to be causal if the current value of the output signal depends only on current and past values of the input signal, but not on its future values.

CTLTI system:

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\lambda)\, x(t-\lambda)\, d\lambda$$

For $\lambda < 0$, the term $x(t-\lambda)$ refers to future values of the input signal.

Causality in CTLTI systems

For a CTLTI system to be causal, the impulse response of the system must be equal to zero for all negative values of its argument.

$$h(t) = 0 \quad \text{for all} \quad t < 0$$

Stability in continuous-time systems

Stable system

A system is said to be *stable* in the *bounded-input bounded-output (BIBO)* sense if any bounded input signal is guaranteed to produce a bounded output signal.

$$\left|x(t)\right| < B_x < \infty \quad \text{implies that} \quad \left|y(t)\right| < B_y < \infty$$

CTLTI system:

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\lambda)\, x(t-\lambda)\, d\lambda$$

Stability in CTLTI systems

For a CTLTI system to be stable, its impulse response must be *absolute integrable*.

$$\int_{-\infty}^{\infty} \left|h(\lambda)\right|\, d\lambda < \infty$$

Example 2.24

Stability of a first-order continuous-time system

Evaluate the stability of the first-order CTLTI system described by the differential equation

$$\frac{dy(t)}{dt} + a\, y(t) = x(t)$$

where a is a real-valued constant.

Solution:

Impulse response:

$$h(t) = e^{-at}\, u(t)$$

Check for stability:

$$\int_{-\infty}^{\infty} \left|h(\lambda)\right|\, d\lambda = \int_{0}^{\infty} e^{-a\lambda}\, d\lambda = \frac{1}{a} \quad \text{provided that} \quad a > 0$$

The system is stable if $a > 0$.

Approximate numerical solution of a differential equation

First-order linear differential equation:

$$\frac{dy(t)}{dt} + \frac{1}{RC}\,y(t) = \frac{1}{RC}\,x(t)$$

Rearrange terms:

$$\frac{dy(t)}{dt} = -\frac{1}{RC}\,y(t) + \frac{1}{RC}\,x(t)$$

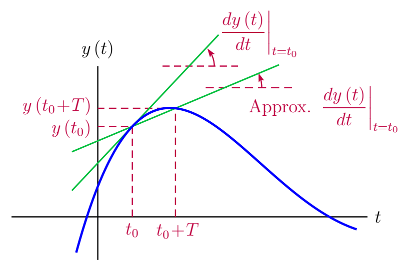
General form:

$$\frac{dy(t)}{dt} = g\left[t, y(t)\right] \qquad \text{where} \qquad g\left[t, y(t)\right] = -\frac{1}{RC}\,y(t) + \frac{1}{RC}\,x(t)$$

Approximate the derivative

$$\left.\frac{dy(t)}{dt}\right|_{t=t_0} \approx \frac{y(t_0+T) - y(t_0)}{T}\,,$$

T: Small step size



Approximate numerical solution of a differential equation (continued)

$$\frac{y(t_0+T) - y(t_0)}{T} \approx g\left[t_0, y(t_0)\right] \qquad \Rightarrow \qquad y(t_0+T) \approx y(t_0) + T\,g\left[t_0, y(t_0)\right]$$

For the RC circuit, using $t_0 = 0$:

$$\begin{aligned} y(T) &\approx y(0) + T\,g\left[0, y(0)\right] \\ &= y(0) + T\left[-\frac{1}{RC}\,y(0) + \frac{1}{RC}\,x(0)\right] \end{aligned}$$

and

$$\begin{aligned} y(2T) &\approx y(T) + T\,g\left[T, y(T)\right] \\ &= y(T) + T\left[-\frac{1}{RC}\,y(T) + \frac{1}{RC}\,x(T)\right] \end{aligned}$$

This is known as the *Euler method*. More sophisticated methods exist with better accuracy.

MATLAB Exercise 2.4 MATLAB Exercise 2.5

MATLAB Exercise 2.1

Testing linearity of continuous-time systems

Simulate the four systems considered in Example 2.1, and test them using signals generated in MATLAB.

Solution:

If a system is linear

$$x(t) = \alpha_1\, x_1(t) + \alpha_2\, x_2(t) \qquad \Rightarrow \qquad y(t) = \alpha_1\, y_1(t) + \alpha_2\, y_2(t)$$

$x_1(t)$, $x_2(t)$: Arbitrary signals
α_1, α_2: Arbitrary constants

MATLAB Exercise 2.1 (continued)

Create test signals:

```
>>  t = [0:0.01:5];
>>  x1 = cos(2*pi*5*t);
>>  x2 = exp(-0.5*t);
```

Construct and graph $x(t)$ with $\alpha_1 = 2$ and $\alpha_2 = 1.25$:

```
>>  alpha1 = 2;
>>  alpha2 = 1.25;
>>  x = alpha1*x1+alpha2*x2;
>>  plot(t,x);
```

Simulate the first system:

```
>>  sys_a = @(x) 5*x;
>>  y1 = sys_a(x1);
>>  y2 = sys_a(x2);
>>  y_exp = alpha1*y1+alpha2*y2;  % Expected output if system is linear
>>  y_act = sys_a(x);             % Actual output
```

MATLAB Exercise 2.1 (continued)

Complete script:

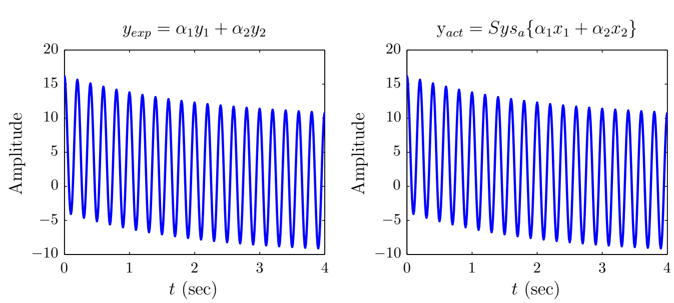
```
% Script: matex_2_1.m
%
t = [0:0.01:4];                 % Create a time vector.
x1 = cos(2*pi*5*t);             % Test signal 1.
x2 = exp(-0.5*t);               % Test signal 2.
alpha1 = 2;                     % Set parameters alpha1
alpha2 = 1.25;                  %   and alpha2.
x = alpha1*x1+alpha2*x2;        % Combined signal.
% Define anonymous functions for the systems in Example 2.1.
sys_a = @(x) 5*x;
sys_b = @(x) 5*x+3;
sys_c = @(x) 3*x.*x;
sys_d = @(x) cos(x);
% Test the system in part (a) of Example 2.1.
y1 = sys_a(x1);
y2 = sys_a(x2);
y_exp = alpha1*y1+alpha2*y2;  % Expected response for a linear system.
y_act = sys_a(x);               % Actual response.
clf;                            % Clear figure.
```


MATLAB Exercise 2.1 (continued)

Script "matex_2_1.m" continued:

```
subplot(1,2,1);
plot(t,y_exp);                  % Graph expected response.
title('y_{exp} = \alpha_1 y_1 + \alpha_2 y_2')
xlabel('t (sec)');  ylabel('Amplitude');
subplot(1,2,2);
plot(t,y_act);                  % Graph actual response.
title('y_{act} = Sys_a\{\alpha_1 x_1 + \alpha_2 x_2\}')
xlabel('t (sec)');  ylabel('Amplitude');
```

Example 2.1



MATLAB Exercise 2.2

Testing time-invariance of continuous-time systems

Simulate the three systems considered in Example 2.2, and test them using signals generated in MATLAB.

<u>Solution:</u>

If the system under consideration is time-invariant we need

$$\text{Sys}\{x(t)\} = y(t) \qquad \Rightarrow \qquad \text{Sys}\{x(t-\tau)\} = y(t-\tau)$$

for any arbitrary time shift τ.

Create and graph the test signal $x(t) = e^{-0.5t}\,u(t)$ and its time shifted version:

```
>>  t = [0:0.01:10];
>>  x = @(t) exp(-0.5*t).*(t>=0);
>>  plot(t,x(t),t,x(t-2));
```


MATLAB Exercise 2.2 (continued)

Simulate the system:

```
>>  sys_c = @(x) 3*cos(t).*x;
>>  y1 = sys_c(x(t));
>>  y2 = sys_c(x(t-2));
>>  plot(t,y1,'b-',t,y2,'r:');
```

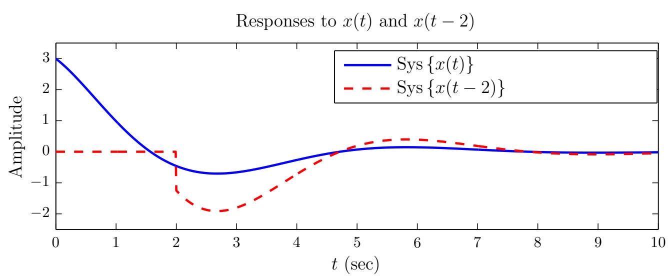
Complete script:

```
% Script matex_2_2.m
%
t = [0:0.01:10];                % Create a time vector.
x = @(t) exp(-0.5*t).*(t>=0);  % Anonymous function for test signal.
% Define anonymous functions for the systems in Example 2-2.
sys_a = @(x) 5*x;
sys_b = @(x) 3*cos(x);
sys_c = @(x) 3*cos(t).*x;
% Test the system in part (c) of Example 2.2.
y1 = sys_c(x(t));
y2 = sys_c(x(t-2));
```

MATLAB Exercise 2.2 (continued)

Script “matex_2_2.m” continued:

```
clf;                              % Clear figure.
plot(t,y1,'b-',t,y2,'r:');        % Graph the two responses.
title('Responses to x(t) and x(t-2)')
xlabel('t (sec)');
ylabel('Amplitude');
legend('Sys\{x(t)\}','Sys\{x(t-2)\}');
```



Example 2.2

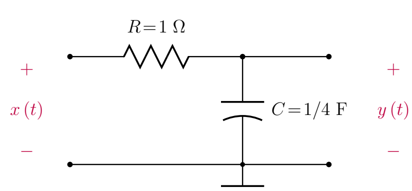
MATLAB Exercise 2.3

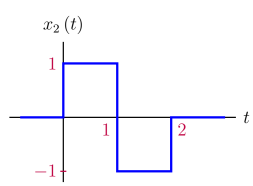
Using linearity to determine the response of the RC circuit

The response of the simple RC circuit to a unit-step signal was found in Example 2.8 to be

$$y_u(t) = \text{Sys}\{u(t)\} = \left(1 - e^{-4t}\right) u(t)$$

Using superposition, compute and graph the response of the circuit to the signal $x_2(t)$ shown.





Solution:

Define an anonymous function to compute $y_u(t)$:

```
yu = @(t) (1-exp(-4*t)).*(t>=0);
```

MATLAB Exercise 2.3 (continued)

Express the signal $x_2(t)$ through unit-step functions:

$$x_2(t) = u(t) - 2\,u(t-1) + u(t-2)$$

Complete script:

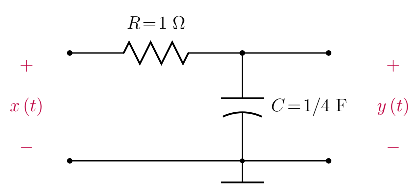
```
% Script: matex_2_3b.m
%
% Anonymous function for unit-step response.
yu = @(t) (1-exp(-4*t)).*(t>=0);
t = [-5:0.01:5];              % Vector of time instants.
y2 = yu(t)-2*yu(t-1)+yu(t-2); % Compute response to x2(t)].
plot(t,y2);
```

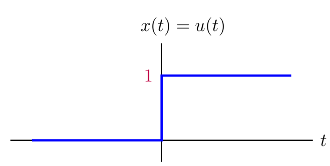
Example 2.10

MATLAB Exercise 2.4

Numerical solution of the RC circuit using Euler method

Use the Euler method to find an approximate numerical solution for the RC circuit problem of Example 2.8, and compare it to the exact solution that was found.





Solution:

For the specified input signal, the differential equation of the circuit is

$$\frac{dy(t)}{dt} + 4\,y(t) = 4\,u(t)$$

With $y(0) = 0$, the exact solution for the output signal is

$$y(t) = \left(1 - e^{-4t}\right)\,u(t)$$

MATLAB Exercise 2.4 (continued)

To use the Euler method, write the differential equation in the form

$$\frac{dy(t)}{dt} = g(t, y(t)) \ , \qquad g(t, y(t)) = -4\ y(t) + 4\, u(t)$$

The Euler method approximation $\hat{y}(t)$ is

$$\begin{aligned}\hat{y}\left((k+1)T_s\right) =& \hat{y}(kT_s) + T_s\, g(kT_s, \hat{y}(kT_s)) \\ =& \hat{y}(kT_s) + T_s\ \left(-4\,\hat{y}(kT_s) + 4\,u(kT_s)\right)\end{aligned}$$

Percent error:

$$\varepsilon(kT_s) = \frac{\hat{y}(kT_s) - y(kT_s)}{y(kT_s)} \times 100$$

MATLAB Exercise 2.4 (continued)

Complete script:

```
% Script: matex_2_4.m
%
Ts = 0.1;          % Time increment
t = [0:Ts:1];      % Vector of time instants
% Compute the exact solution.
y = 1-exp(-4*t);   % Eqn.(2.186)
% Compute the approximate solution using Euler method.
yhat = zeros(size(t));
yhat(1) = 0;      % Initial value.
for k = 1:length(yhat)-1,
  g = -4*yhat(k)+4;          % Eqn.(2.188)
  yhat(k+1) = yhat(k)+Ts*g;  % Eqn.(2.189)
end;
% Graph exact and approximate solutions.
clf;
subplot(211);
plot(t,y,'-',t,yhat,'ro'); grid;
title('Exact and approximate solutions for RC circuit');
```

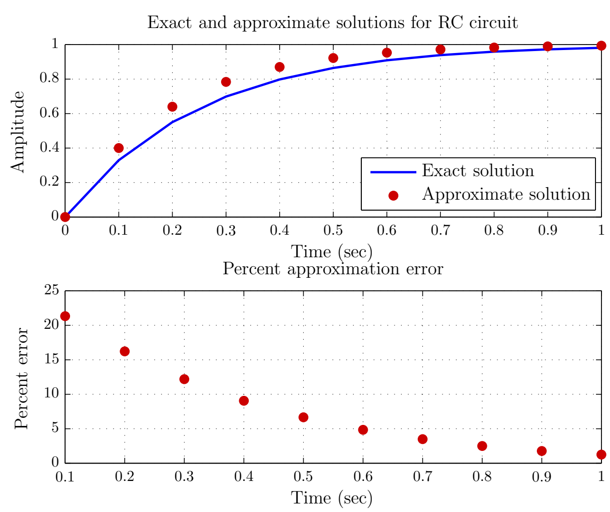
MATLAB Exercise 2.4 (continued)

Script "`matex_2_4.m`" continued:

```
xlabel('Time (sec)');
ylabel('Amplitude');
legend('Exact solution','Approximate solution','Location','SouthEast');
% Compute and graph the percent approximation error.
err_pct = (yhat-y)./y*100;
subplot(212);
plot(t(2:length(t)),err_pct(2:length(t)),'ro'); grid
title('Percent approximation error');
xlabel('Time (sec)');
ylabel('Error (%)');
```

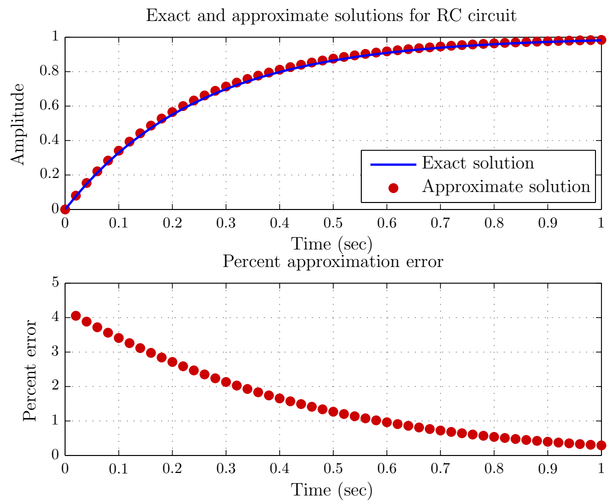

MATLAB Exercise 2.4 (continued)

Actual and approximate solutions for the RC circuit and the percent error for $\Delta t = 0.1$ seconds.



MATLAB Exercise 2.4 (continued)

Actual and approximate solutions for the RC circuit and the percent error for $\Delta t = 0.02$ s.



Approx. Num. Solution

MATLAB Exercise 2.5

Improved numerical solution of the RC circuit

Solve the approximation problem of MATLAB Exercise 2.4 using function ode45(..)

Solution:

Start by developing a function rc1(..) to compute the right side $g\left[t, y\left(t\right)\right]$ of the differential equation.

```
function ydot = rc1(t,y)
  ydot = -4*y+4;
end
```

MATLAB Exercise 2.5 (continued)

Complete script:

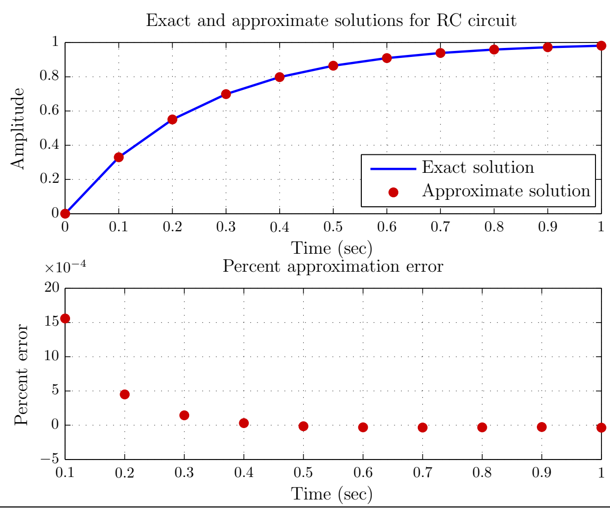
```
% Script: matex_2_5a.m
%
t = [0:0.1:1];     % Vector of time instants
% Compute the exact solution.
y = 1-exp(-4*t);   % Eqn.(2.187)
% Compute the approximate solution using ode45().
[t,yhat] = ode45(@rc1,t,0);
% Graph exact and approximate solutions.
clf;
subplot(211);
plot(t,y,'-',t,yhat,'ro'); grid;
title('Exact and approximate solutions for RC circuit');
xlabel('Time (sec)');
ylabel('Amplitude');
legend('Exact solution','Approximate solution','Location','SouthEast');
% Compute and graph the percent approximation error.
err_pct = (yhat-y')./y'*100;
subplot(212);
plot(t(2:max(size(t))),err_pct(2:max(size(t))),'ro'); grid
```

Chapter 2
MATLAB Exercises

MATLAB Exercise 2.5 (continued)

Script "matex_2_5a.m" continued:

```
title('Percent approximation error');
xlabel('Time (sec)');
ylabel('Percent error');
```



MATLAB Exercise 2.5 (continued)

Modified script that uses an anonymous function instead of “rc1.m”.

```
% Script: matex_2_5b.m
%
t = [0:0.1:1];      % Vector of time instants
% Compute the exact solution.
y = 1-exp(-4*t);    % Eqn.(2.187)
% Compute the approximate solution using ode45().
rc2 = @(t,y) -4*y+4;
[t,yhat] = ode45(rc2,t,0);
% Graph exact and approximate solutions.
clf;
subplot(211);
plot(t,y,'-',t,yhat,'ro'); grid;
title('Exact and approximate solutions for RC circuit');
xlabel('Time (sec)');
ylabel('Amplitude');
legend('Exact solution','Approximate solution','Location','SouthEast');
% Compute and graph the percent approximation error.
err_pct = (yhat-y').y'*100;
```

Chapter 2
MATLAB Exercises

MATLAB Exercise 2.5 (continued)

Script “matex_2_5b.m” continued:

```
subplot(212);
plot(t(2:max(size(t))),err_pct(2:max(size(t))),'ro'); grid
title('Percent approximation error');
xlabel('Time (sec)');
ylabel('Percent error');
```

Approx. Num. Solution