One-sided (unilateral) Laplace Transform
Laplace Circuit Analysis
Solving Differential Equations with LT

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One-sided Laplace transform and its applications

$$\boxed{X(s) = \int_0^\infty x(t)\, e^{-st}\, dt}$$

- One-sided Laplace transform and its properties.
- Applications to initial value problems for ordinary differential equations.
- Applications to circuit analysis and, in particular, to transients.
- Transfer function.
- LTI systems in series and parallel. Inverse systems and linear feedback systems.
- Operational amplifiers, transfer functions for circuits with op amps.
- Bode plots, impulse response.
- Stability.

One-sided Laplace transform

$$\boxed{X(s)=\int_0^{\infty} x(t)\,e^{-st}\,dt \quad \text{unilateral (one-sided) Laplace transform}}$$

$$x(t)\xrightarrow{L}X(s) \qquad X(s)=L[x(t)]$$

Two-sided (bilateral) Laplace transform and its properties were studied within B39SA Signals and Systems

$$X(s)=\int_{-\infty}^{\infty} x(t)\,e^{-st}\,dt \quad \text{two-sided Laplace transform}$$

One-sided LT = two-sided LT applied to a causal signal (a signal that does not start before t=0 is, i.e. $x(t)$=0 for t<0.)

$$X(s)=\int_0^{\infty} x(t)\,e^{-st}\,dt=\int_{-\infty}^{\infty} x(t)\,u(t)\,e^{-st}\,dt$$

Fourier and Laplace transforms

$$x(t) \xrightarrow{F} X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \qquad X(\omega) = F[x(t)]$$

$$x(t) = F^{-1}[X(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

It is not always possible to calculate the Fourier transform of a signal $x(t)$ by integration. For example, if the signal is of finite power rather than finite energy the classical Fourier transform does not exist (the integral does not converge). A possible solution consists of multiplying $x(t)$ by a convergence factor exp($-\sigma t$):

$$x_\sigma(t) = x(t) e^{-\sigma t}$$

$$X_\sigma(\omega) = \int_{-\infty}^{\infty} x_\sigma(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$s = \sigma + j\omega$$

Motivation behind using bilateral (two-sided) Laplace transform

Bilateral Laplace transform

$$x(t) \xrightarrow{L} X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \qquad X(s) = L[x(t)]$$

$$x_\sigma(t) = x(t) e^{-\sigma t}$$

$$X_\sigma(\omega) = \int_{-\infty}^{\infty} x_\sigma(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$s = \sigma + j\omega$$

$$x(t) = e^{\sigma t} x_\sigma(t) = e^{\sigma t} \frac{1}{2\pi} \int_{-\infty}^{\infty} X_\sigma(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_\sigma(\omega) e^{(\sigma + j\omega)t} d\omega$$

$$= \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

inverse Laplace transform

Bilateral Laplace transform: need for ROC

$$x(t) = e^{at}u(t)$$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{0}^{\infty} e^{(a-s)t}dt = \frac{1}{a-s}e^{(a-s)t}\Big|_{0}^{\infty} = \frac{1}{s-a}$$

$$\text{ROC:} \quad \operatorname{Re}(s) > \operatorname{Re}(a) \quad \text{causal signal}$$

$$x(t) = -e^{at}u(-t)$$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{0} -e^{(a-s)t}dt = -\frac{1}{a-s}e^{(a-s)t}\Big|_{-\infty}^{0} = \frac{1}{s-a}$$

$$\text{ROC:} \quad \operatorname{Re}(s) < \operatorname{Re}(a) \quad \text{anti-causal signal}$$

These ambiguities can be removed if the signal $x(t)$ is assumed to be *one-sided* or *causal*, i.e., $x(t) = 0$ if $t < 0$.

One-sided Laplace transform

$$\boxed{X(s)=\int_0^{\infty} x(t)\,e^{-st}\,dt \quad \text{unilateral (one-sided) Laplace transform}}$$

$$x(t)\xrightarrow{L}X(s) \qquad X(s)=L[x(t)]$$

One-sided Laplace transform for circuit analysis

$$X(s) = \int_0^{\infty} x(t)\, e^{-st}\, dt \quad \text{unilateral (one-sided) Laplace transform}$$

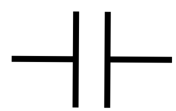
$$L[dx/dt] = s\,X(s) - x(0)$$

R

$$v = Ri$$

$$V(s) = R\,I(s)$$

C



$$i = C\frac{dv}{dt}$$

$$V(s) = \frac{1}{sC} I(s) + \frac{v(0)}{s}$$

L

$$v = L\frac{di}{dt}$$

$$V(s) = sL\left(I(s) - i(0)\right)$$

One-sided Laplace transform for circuit analysis

$$X(s) = \int_0^\infty x(t)\, e^{-st}\, dt \quad \text{unilateral (one-sided) Laplace transform}$$

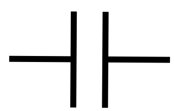
$$\int_0^\infty \frac{dx}{dt} e^{-st} dt \overset{\text{by parts}}{=} \left[x(t)e^{-st}\right]_0^\infty + s\int_0^\infty x(t)e^{-st}dt = sX(s) - x(0)$$

R

$$v = Ri$$

$$V(s) = RI(s)$$

C



$$i = C\frac{dv}{dt}$$

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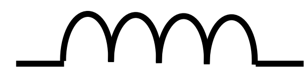
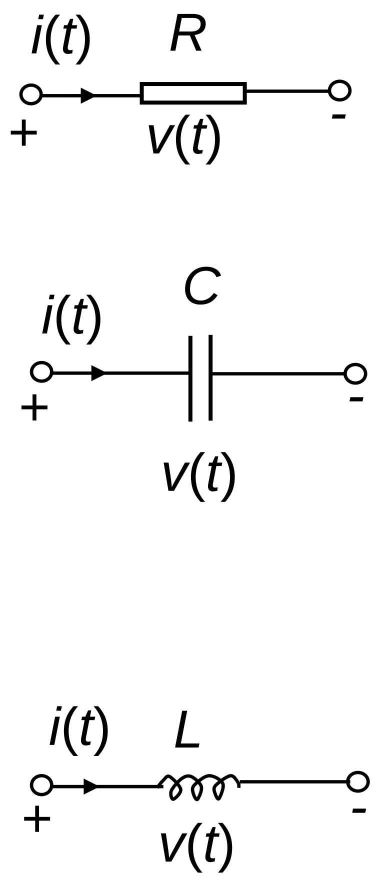


$$v = L\frac{di}{dt}$$

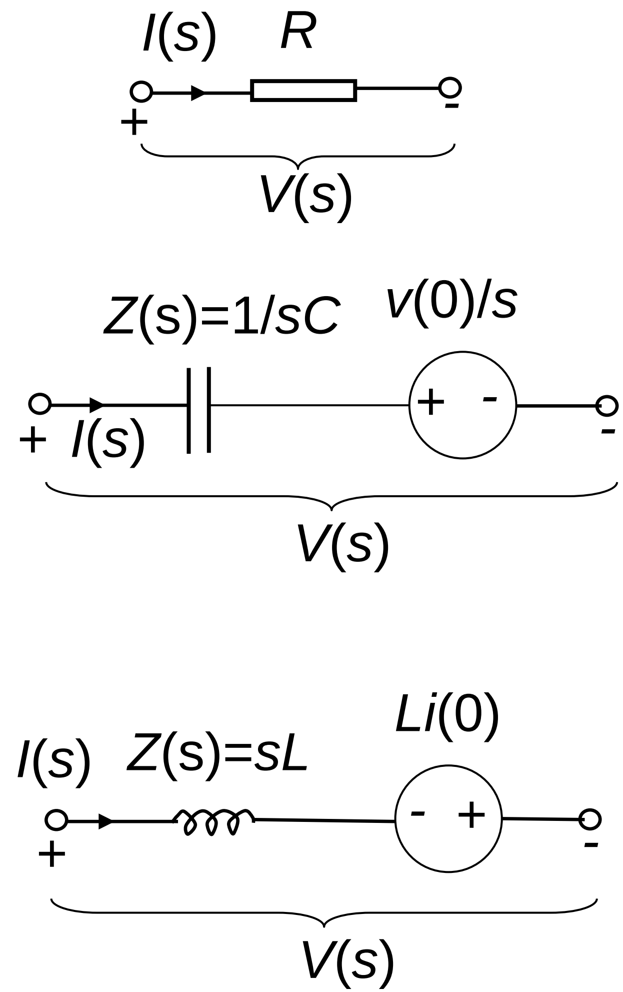
$$V(s) = L\left(sI(s) - i(0)\right)$$

Circuits in time domain and s-domain

Time domain circuit

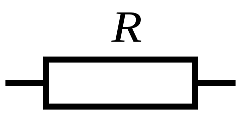


s-domain circuit



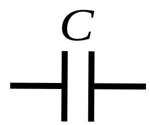
Laplace transform for circuit analysis

$$X(s)=\int_0^\infty x(t)\,e^{-st}\,dt \qquad L[dx/dt]=s\,X(s)-x(0)$$



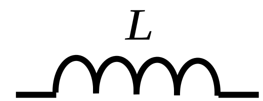
$$v=Ri$$

$$V(s)=RI(s)$$



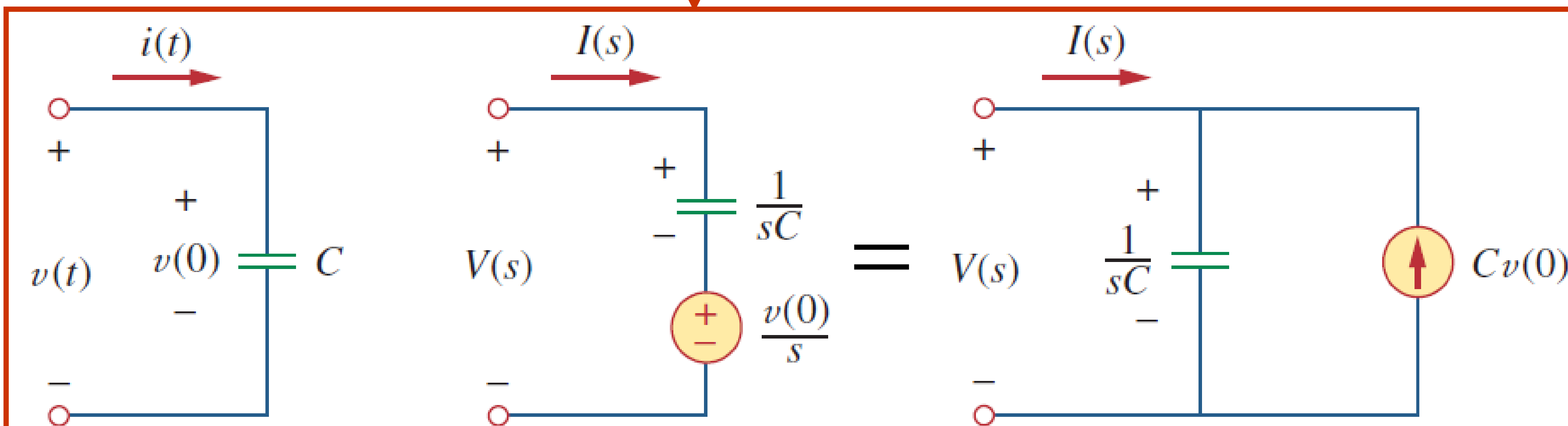
$$i=C\frac{dv}{dt}$$

$$V(s)=\frac{1}{sC}I(s)+\frac{v(0)}{s}$$



$$v=L\frac{di}{dt}$$

$$V(s)=L\left(sI(s)-i(0)\right)$$



Laplace transform for circuit analysis

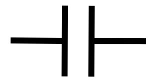
$$X(s)=\int_0^{\infty} x(t)\,e^{-st}\,dt \qquad L\left[dx/dt\right]=s\,X(s)-x(0)$$

R

$$v=R\,i$$

$$V(s)=R\,I(s)$$

C



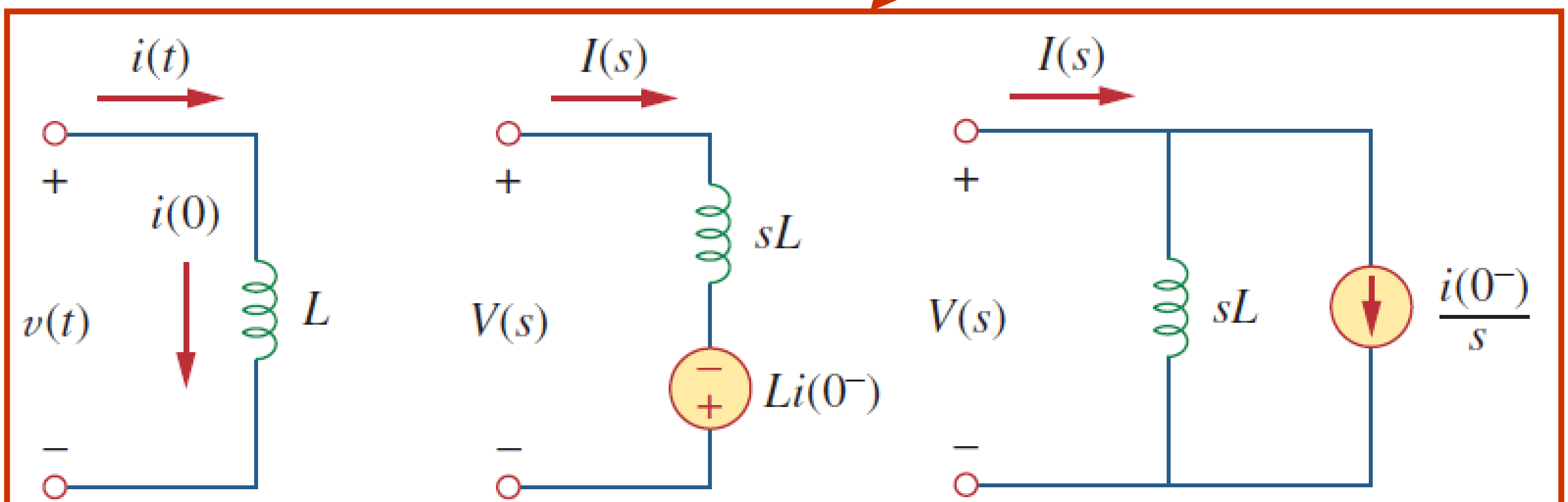
$$i=C\frac{dv}{dt}$$

$$V(s)=\frac{1}{sC}I(s)+\frac{v(0)}{s}$$

L

$$v=L\frac{di}{dt}$$

$$V(s)=L\left(sI(s)-i(0)\right)$$



One-sided Laplace transform pairs

Defined for t $\geq$ 0

It is assumed that $x(t) = 0$ for $t < 0$

$x(t)$	$X(s)$
$\delta(t)$	1
$u(t)$	$1/s$
e^{-at}	$1/(s+a)$
t	$1/s^2$
t^n	$n!/s^{n+1}$
$t^n e^{-at}$	$n!/(s+a)^{n+1}$
$\sin(\omega t)$	$\omega/(s^2+\omega^2)$
$\cos(\omega t)$	$s/(s^2+\omega^2)$
$e^{-at}\sin(\omega t)$	$\omega/\left[(s+a)^2+\omega^2\right]$
$e^{-at}\cos(\omega t)$	$(s+a)/\left[(s+a)^2+\omega^2\right]$

Laplace and inverse Laplace transforms with matlab

```
>> syms t
>> x1 = 1/sqrt(t);
>> X1 = laplace(x1)

X1 =

pi^(1/2)/s^(1/2)
```

```
>> syms s a
>> X = 1/(s+a);
>> x = ilaplace(X)

x =

exp(-a*t)
```

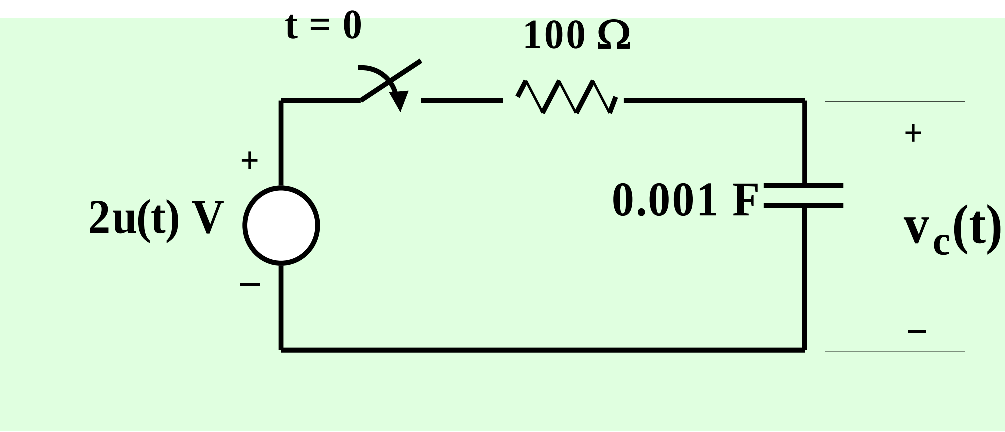
```
>> syms f(t) s
>> Df = diff(f(t),t);
>> laplace(Df,t,s)

ans =

s*laplace(f(t), t, s) - f(0)
```

Laplace circuit analysis: Example 1

Assume $v_c(0) = -4$ V. Use Laplace transform to find $v_c(t)$.



The time domain circuit

Laplace circuit analysis: Example 1

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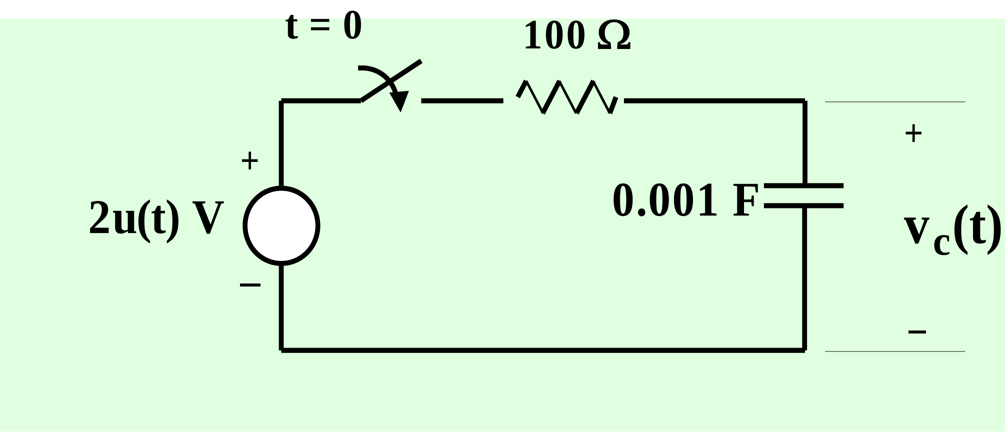
$$L\left[\frac{dx}{dt}\right] = s\,X(s) - x(0)$$



$$i = C\frac{dv_c}{dt}$$

$$I(s) = C\left[sV_c(s) - v_c(0)\right]$$

$$V_c(s) = \frac{1}{s}\left[\frac{I(s)}{C} + v_c(0)\right]$$

Laplace circuit analysis: Example 1

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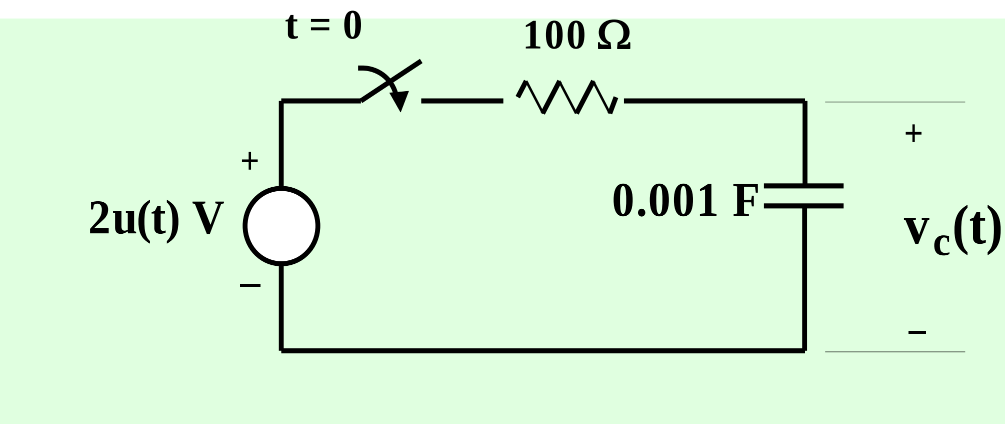

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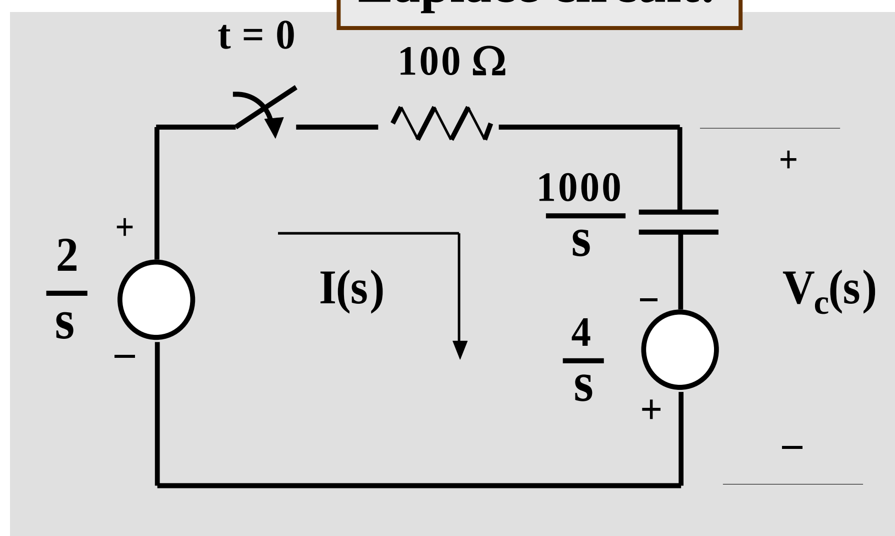
Laplace circuit:

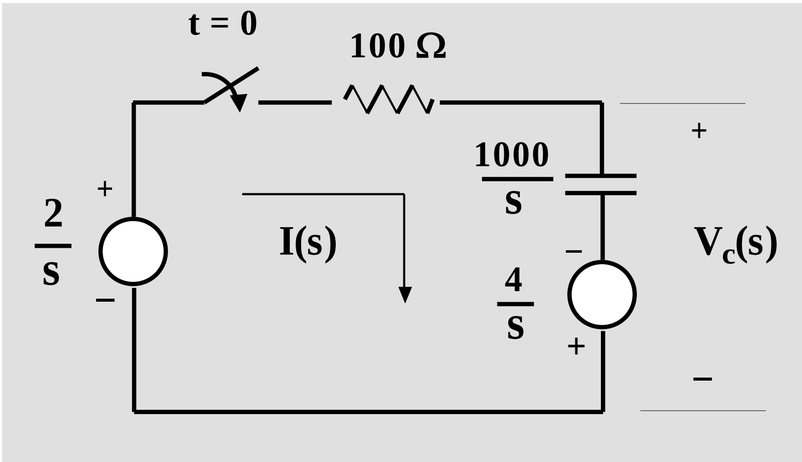


Kirchhoff voltage law in s-domain

$$\frac{2}{s} + \frac{4}{s} = I(s)\left[100 + \frac{1000}{s}\right]$$

$$100\,I(s) = \frac{6}{s+10}$$

Laplace circuit analysis: Example 1



1

$$\frac{2}{s} - 100I(s) - V_c(s) = 0$$

$$\frac{2}{s} - \frac{6}{s+10} = V_c(s)$$

2

$$V_c(s) = \frac{-4s+20}{s(s+10)} = \frac{A}{s} + \frac{B}{s+10}$$

$$V_c(s) = \frac{2}{s} - \frac{6}{s+10}$$

$$v(t) = \left[2 - 6e^{-10t}\right]u(t)$$

Check the boundary conditions

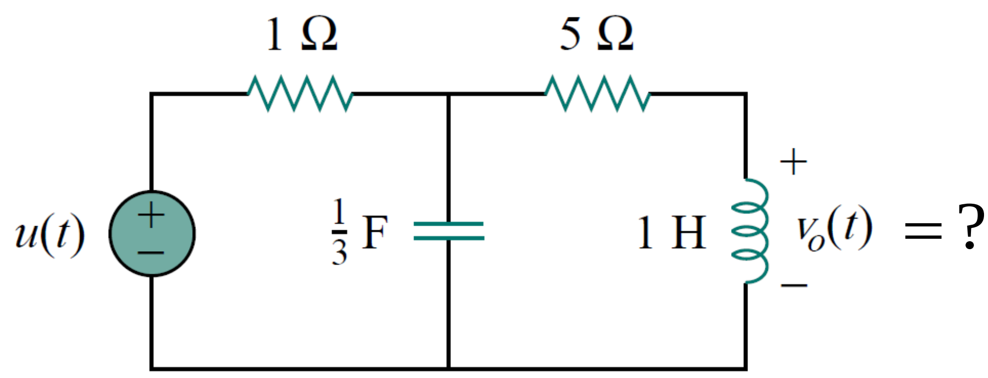
3

$$v_C(0) = -4\,\mathrm{V}$$

$$v_C(\infty) = 2\,\mathrm{V}$$

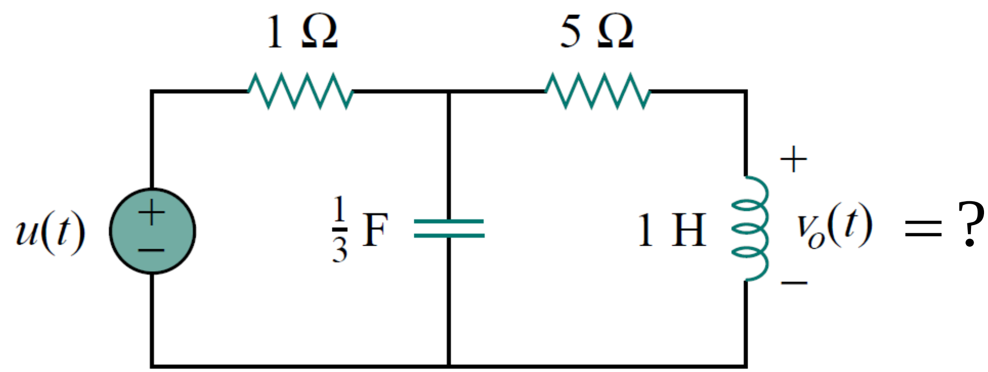
Laplace circuit analysis: Example 2

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$



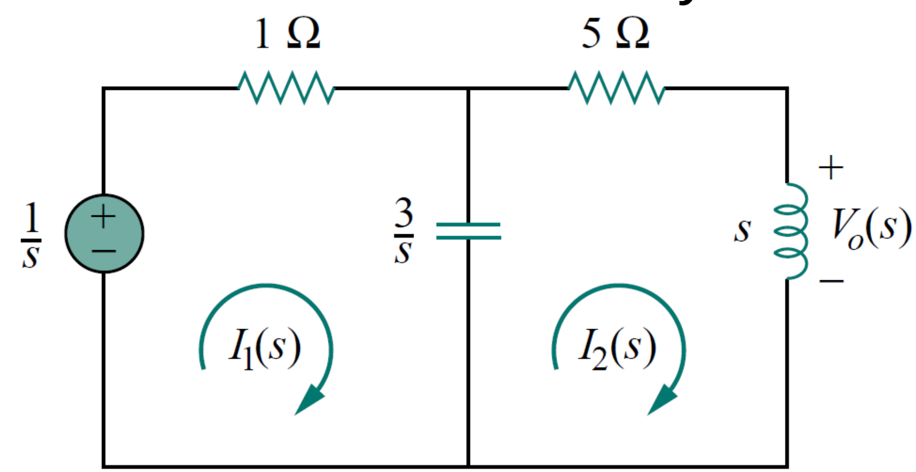
Find $v_o(t)$. Assuming no current in the circuit at $t = 0$.

Laplace circuit analysis: Example 2



$$u(t)=\begin{cases}1 & t>0\\ 0 & t<0\end{cases}$$

Let us use mesh analysis



$$\frac{1}{s}=\left(1+\frac{3}{s}\right)I_1-\frac{3}{s}I_2$$

$$0=-\frac{3}{s}I_1+\left(s+5+\frac{3}{s}\right)I_2$$

$$I_2(s)=\frac{3}{s^3+8s^2+18s}$$

$$V_0(s)=sI_2(s)$$

$$v_0(t)=\frac{3}{\sqrt{2}}e^{-4t}\sin\sqrt{2}\,t \text{ V}$$

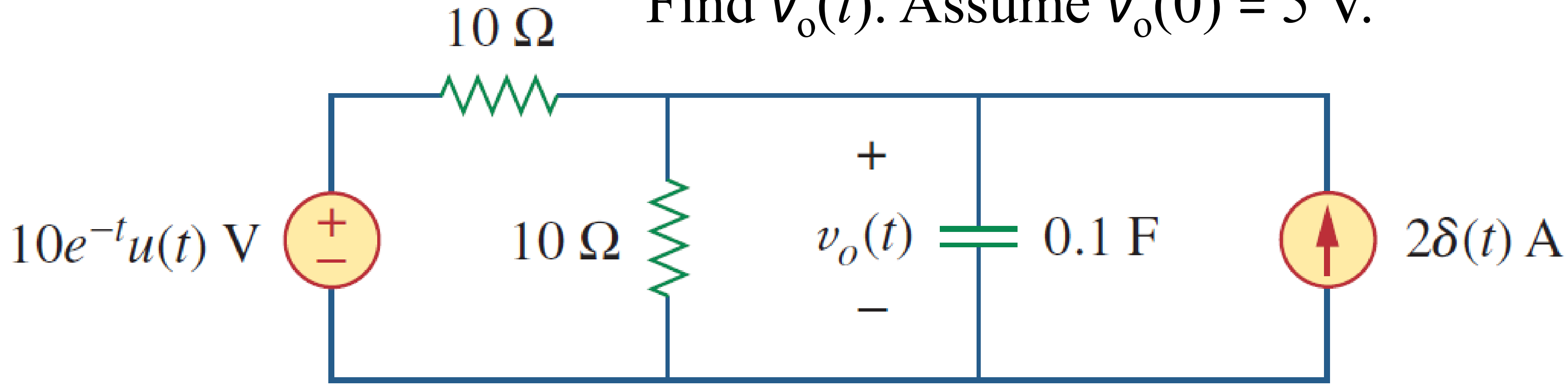
```
>> syms s
>> X = 3*s/(s^3+8*s^2+18*s);
>> x = ilaplace(X)

x =

(3*2^(1/2)*exp(-4*t)*sin(2^(1/2)*t))/2
```

Laplace circuit analysis: Example 3

Find $v_o(t)$. Assume $v_o(0) = 5$ V.



Laplace circuit analysis: Example 3

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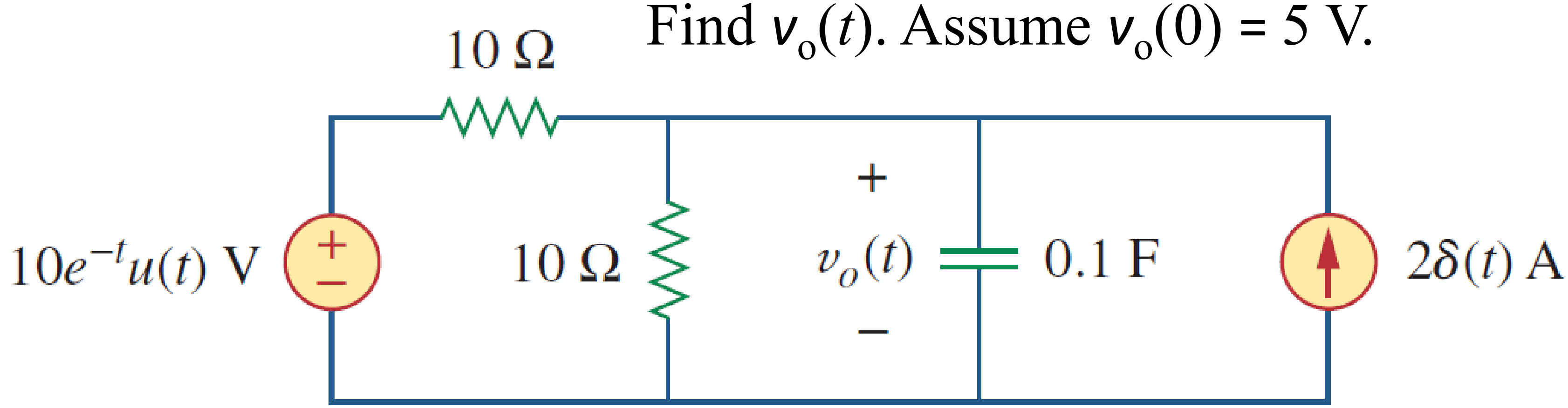
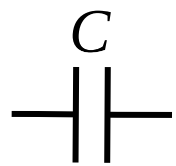
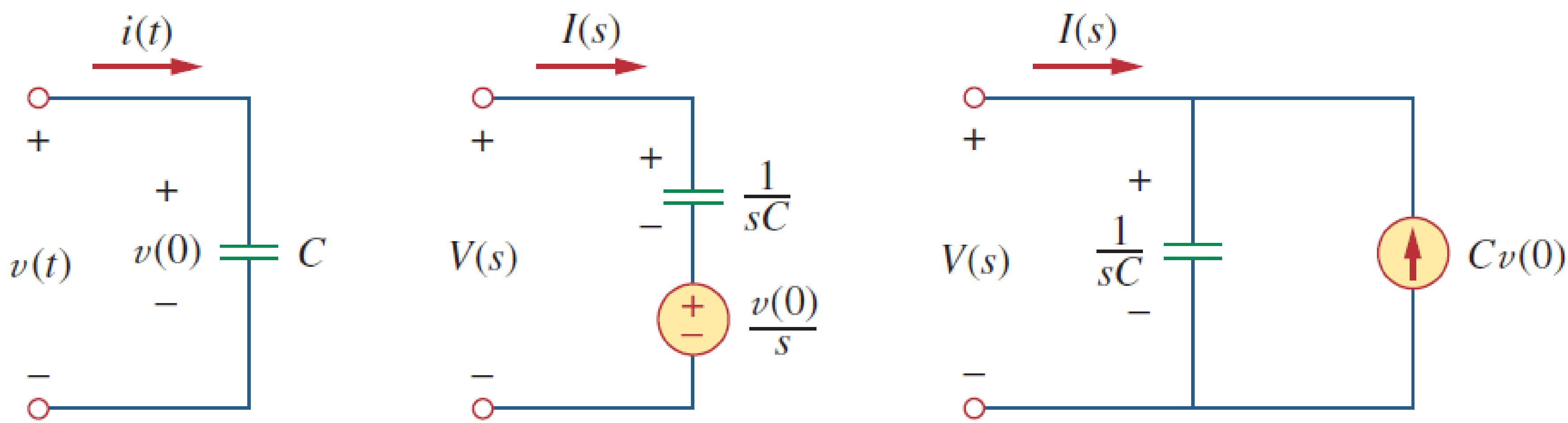


$$L\left[\frac{dx}{dt}\right] = s\,X(s) - x(0)$$



$$i = C\frac{dv}{dt}$$

$$V(s) = \frac{1}{sC}I(s) + \frac{v(0)}{s}$$



Laplace circuit analysis: Example 3

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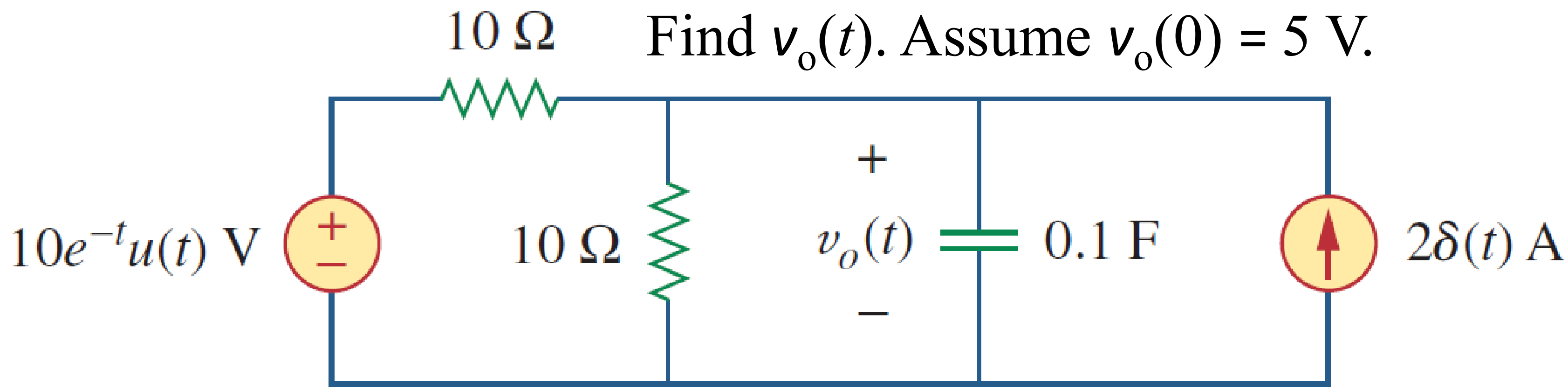
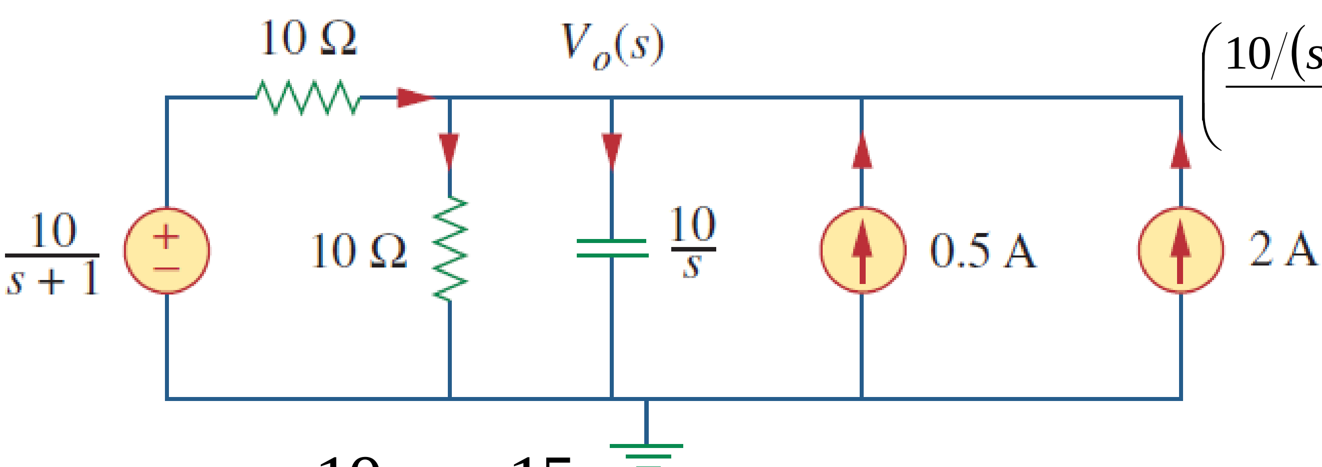


Let us use Nodal Analysis: $$\frac{10/(s+1)-V_0}{10}+2+0.5=\frac{V_0}{10}+\frac{V_0}{10/s}$$

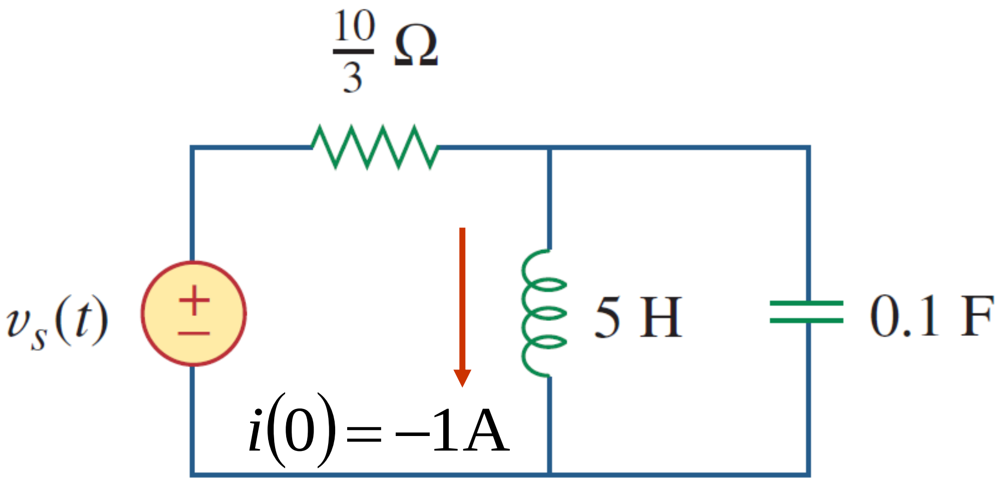


$$\left(\frac{10/(s+1)-V_0}{10}+2=\frac{V_0}{10}+\frac{V_0-5/s}{10/s}\right)$$

$$V_0(s)=\frac{10}{s+1}+\frac{15}{s+2}$$

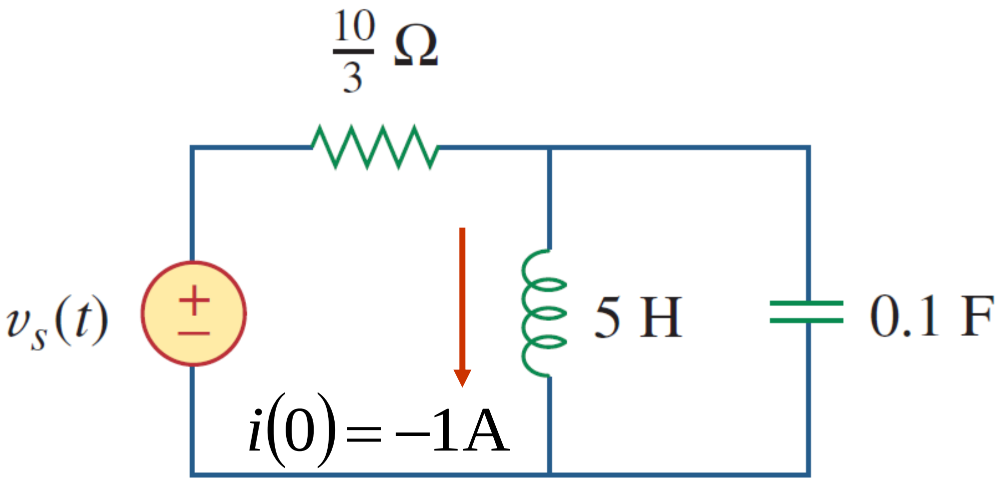
$$v_0(t)=\left(10\,e^{-t}+15\,e^{-2t}\right)u(t)\mathrm{V}$$

Laplace circuit analysis: Example 4



Find the value of the voltage across the capacitor assuming that $v_s(t)=10u(t)$ V and assume that at $t = 0$, current -1A flows through the inductor and +5 V is across the capacitor.

Laplace circuit analysis: Example 4



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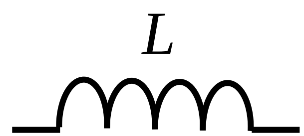
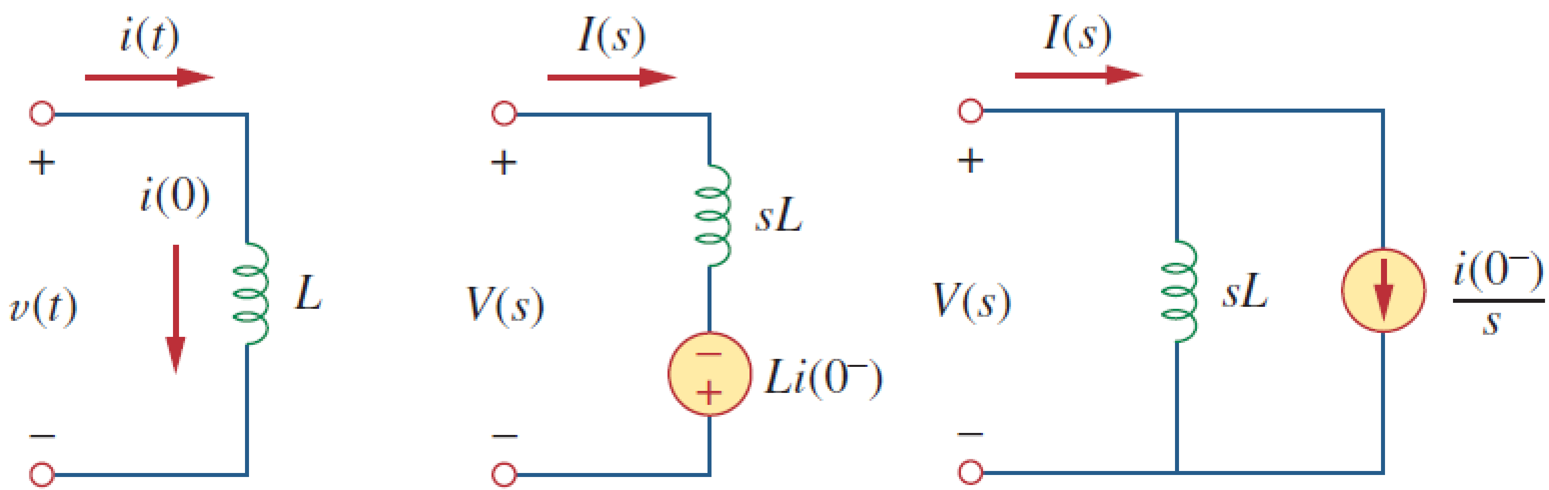
$$L\left[\frac{dx}{dt}\right]=s\,X(s)-x(0)$$

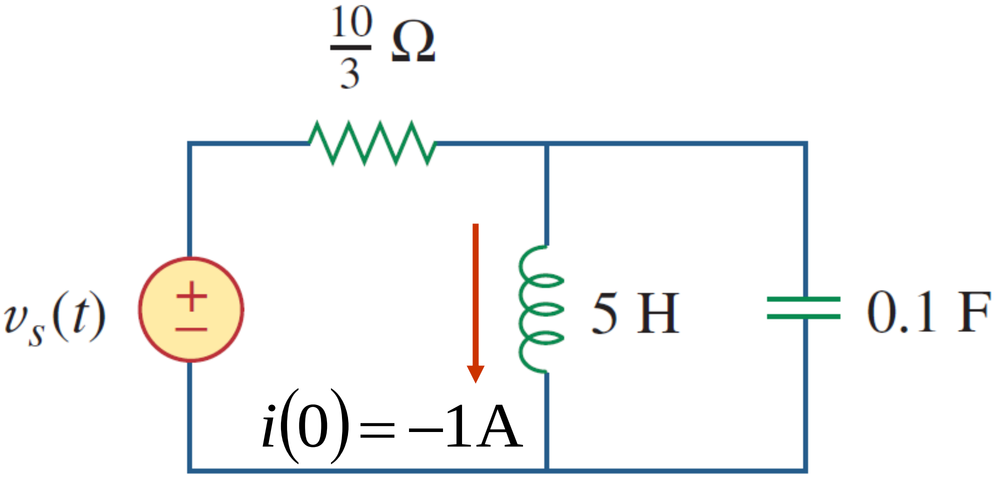


$$v=L\frac{di}{dt}$$

$$V(s)=L\big(sI(s)-i(0)\big)$$

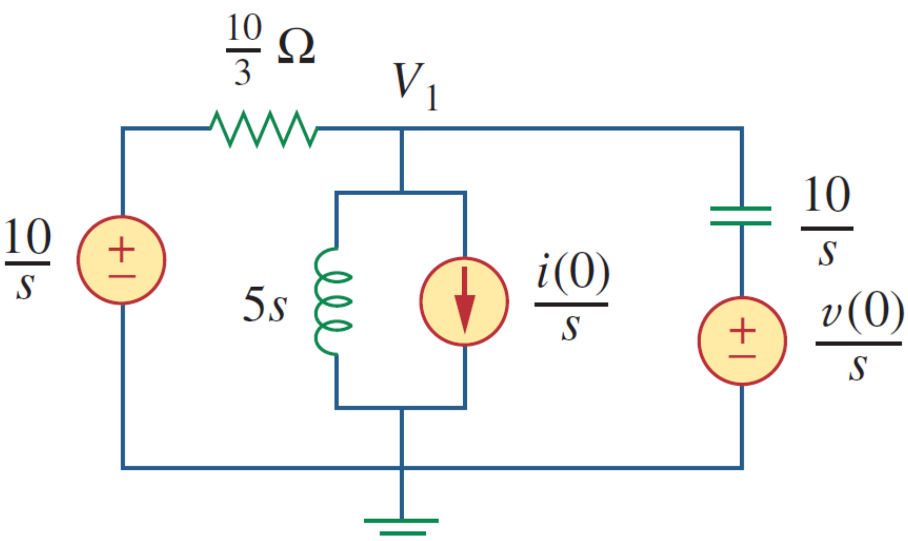


Laplace circuit analysis: Example 4



Find the value of the voltage across the capacitor assuming that $v_s(t)=10u(t)$ V and assume that at $t = 0$, current -1A flows through the inductor and +5 V is across the capacitor.

Convert the circuit to s-domain and use Nodal Analysis:



$$\frac{V_1 - 10/s}{10/3} + \frac{V_1 - 0}{5s} + \frac{i(0)}{s} + \frac{V_1 - v_C(0)/s}{10/s} = 0$$

$$\frac{1}{10}\left(s + 3 + \frac{2}{s}\right)V_1 = \frac{3}{s} + \frac{1}{s} + \frac{5}{10}$$

$$\left(s^2 + 3s + 2\right)V_1 = 40 + 5s$$

$$V_1 = \frac{40 + 5s}{(s+1)(s+2)} = \frac{35}{s+1} - \frac{30}{s+2}$$

$$v_1(t) = \left(35\,e^{-t} - 30\,e^{-2t}\right)u(t)\,\text{V}$$