Frequency Response and Bode Plots

Changhai Wang

c.wang@hw.ac.uk

Frequency response

$$x(t) \longrightarrow \boxed{h(t) \leftrightarrow H(s)} \longrightarrow y(t)$$

$$y(t) = h(t) * x(t) \qquad H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

$$x(t) = e^{st} \Rightarrow y(t) = h(t) * e^{st} = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau = H(s) e^{st}$$

$$x(t) = e^{j\omega t} \Rightarrow y(t) = H(j\omega) e^{j\omega t}$$

We know that $H(j\omega)$ is called the **frequency response**.

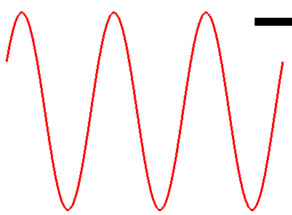
Often we write $H(\omega)$ instead of $H(j\omega)$.

$$\boxed{x(t) = \sum_k c_k e^{j\omega_k t} \Rightarrow y(t) = \sum_k c_k H(\omega_k) e^{j\omega_k t}}$$

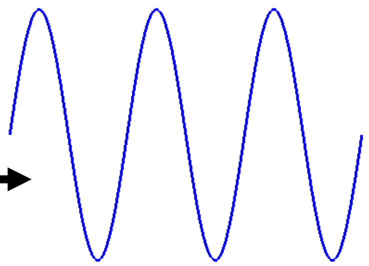
LTI, transfer function and frequency response

$$x(t) \longrightarrow \boxed{h(t) \xleftrightarrow{LT} H(s)} \longrightarrow y(t)$$

$$x(t) = e^{j\omega t} \Rightarrow y(t) = H(j\omega)e^{j\omega t} \qquad H(j\omega) = |H|e^{j\varphi} = |H|\angle\varphi$$



$$\boxed{\begin{array}{l} e^{j\omega t} \to H(j\omega)e^{j\omega t} \\ \sin(\omega t) \to A\sin(\omega t + \varphi) \\ A = |H(j\omega)| \end{array}}$$



A natural way to study properties of a LTI system consists of visualising the gain $|H(j\omega)|$ and phase shift φ as functions of angular frequency ω.

Frequency response and Bode plots

$$x(t) \longrightarrow \boxed{h(t) \leftrightarrow H(s)} \longrightarrow y(t)$$

$$x(t) = e^{j\omega t} \Rightarrow y(t) = H(j\omega)e^{j\omega t} \qquad H(j\omega) = |H|e^{j\varphi} = |H|\angle\varphi$$

Visualising the magnitude and phase of $H(\omega)$ is needed for many applications including circuit analysis, filter design, etc.

It is common to use logarithmic plots of $H(j\omega)$ instead of linear plots.

The logarithmic plots are called **Bode plots** in honour of **Hendrik W. Bode**, an American engineer, inventor and scientist of Dutch ancestry, who used them extensively in his work on amplifiers at Bell Labs.

$\lvert H\rvert$	$20\log\lvert H\rvert$ (dB)
0.1	– 20.00
0.2	–13.98
0.4	– 7.96
0.6	– 4.44
1.0	0.0
2.0	6.02
3.0	9.54
5.0	13.98
10.0	20.00
100.0	40.00

An example: Bode plots for $H(\omega)=1/(1+j\omega/\omega_0)$

$$H=\frac{1}{1+j\,\omega/\omega_0}\qquad |H|=\frac{1}{\sqrt{1+(\omega/\omega_0)^2}}\qquad \varphi=-\tan^{-1}(\omega/\omega_0)$$

$$20\log_{10}|H|=-20\log_{10}\sqrt{1+(\omega/\omega_0)^2}$$

ω_0 is called a corner frequency

$$\omega \ll \omega_0 \quad\Rightarrow\quad 1+(\omega/\omega_0)^2\approx 1 \qquad 20\log_{10}|H|\approx 0$$

for low frequencies

$$\omega \gg \omega_0 \quad\Rightarrow\quad 1+(\omega/\omega_0)^2\approx(\omega/\omega_0)^2$$

$$20\log_{10}|H|\approx 20\log_{10}\omega_0-20\log_{10}\omega$$

for large frequencies

$$\boxed{20\log_{10}|H(\omega)|\approx\begin{cases}0 & \omega<\omega_0\\ 20\log_{10}\omega_0-20\log_{10}\omega & \omega>\omega_0\end{cases}}$$

A piece-wise linear function

An example: Bode plots for $H(\omega)=1/(1+j\omega/\omega_0)$

$$H=\frac{1}{1+j\omega/\omega_0} \qquad |H|=\frac{1}{\sqrt{1+(\omega/\omega_0)^2}} \qquad \varphi=-\tan^{-1}(\omega/\omega_0)$$

$$20\log_{10}|H(\omega)|\approx\begin{cases}0 & \omega<\omega_0\\ 20\log_{10}\omega_0-20\log_{10}\omega & \omega>\omega_0\end{cases}$$

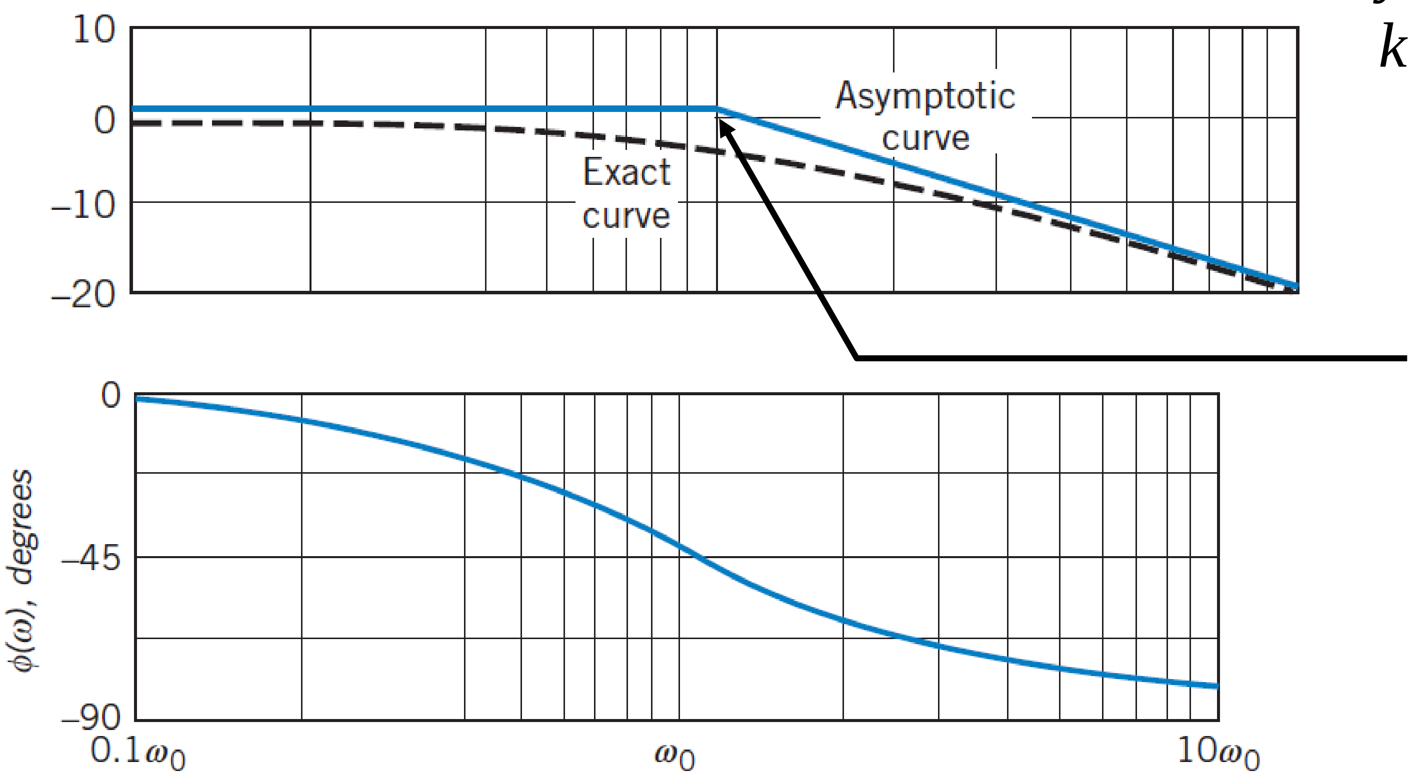
Two straight lines

$y=c+k\,x$

$x=\log_{10}\omega$

$y=20\log_{10}|H(\omega)|$

$k=0$ and $k=-1$



corner frequency ω_0

An example: Bode plots for H(ω)=1/(1+jω/10)

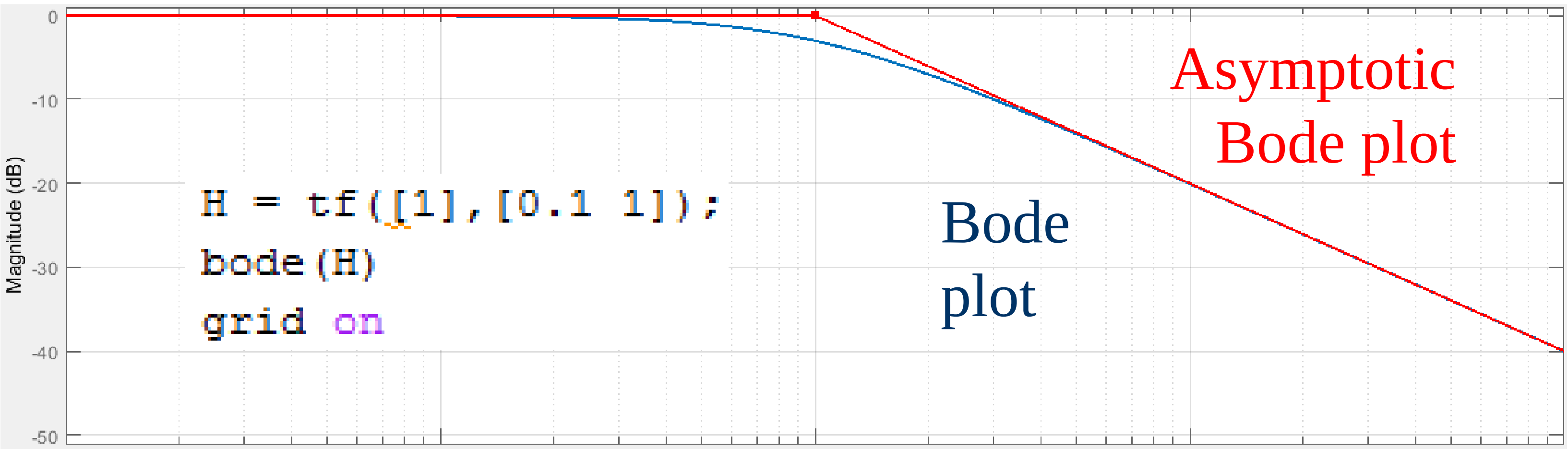
$$H(\omega)=\frac{1}{1+j\,\omega/10} \quad |H(\omega)|=\frac{1}{\sqrt{1+(\omega/10)^2}} \quad \varphi=-\tan^{-1}(\omega/10)$$

$$|H(\omega)|_{dB}=20\log_{10}|H(\omega)|=20\log_{10}\frac{1}{\sqrt{1+(\omega/10)^2}}=-20\log_{10}\sqrt{1+(\omega/10)^2}$$

$$\text{low frequences}: \quad \omega\to 0 \quad \Rightarrow \quad |H(\omega)|_{dB}\to 0$$

$$\text{high frequences}: \quad \omega>>10 \quad \Rightarrow \quad |H(\omega)|_{dB}\approx -20\log_{10}\frac{\omega}{10}=20-20\log_{10}\omega$$

$$\text{corner frequency}: \ \omega=\omega_c=10 \quad \Rightarrow \quad |H(\omega_c)|_{dB}=-20\log_{10}\sqrt{2}=-3\,\text{dB}$$

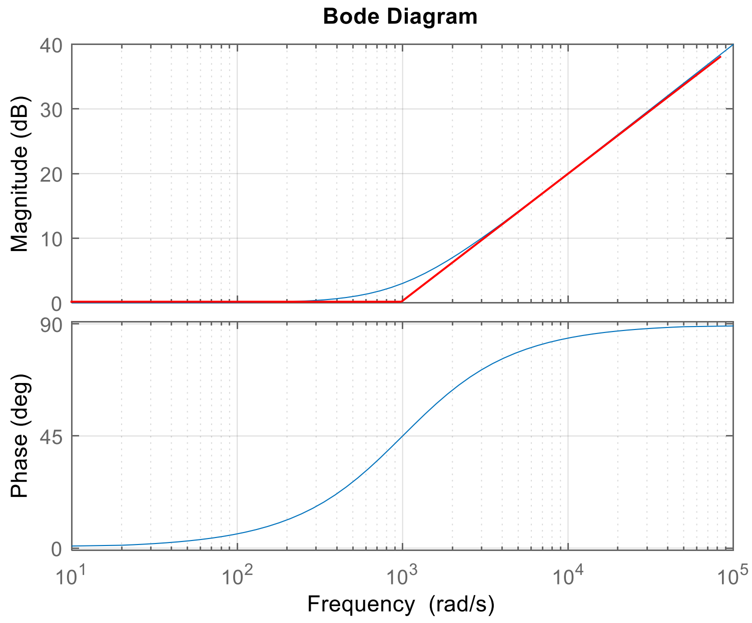


An example: Bode plots for $1+j\omega/\omega_0$

$$H = 1 + j\,\omega/\omega_0 \qquad |H| = \sqrt{1+(\omega/\omega_0)^2} \qquad \varphi = \tan^{-1}(\omega/\omega_0)$$

$$20\log_{10}|H| = 20\log_{10}\sqrt{1+(\omega/\omega_0)^2}$$

$$\omega \ll \omega_0 \quad\Rightarrow\quad 1+(\omega/\omega_0)^2 \approx 1 \qquad 20\log_{10}|H| \approx 0$$



$$\omega \gg \omega_0 \quad\Rightarrow\quad 1+(\omega/\omega_0)^2 \approx (\omega/\omega_0)^2$$

$$20\log_{10}|H| \approx 20\log_{10}\omega - 20\log_{10}\omega_0$$

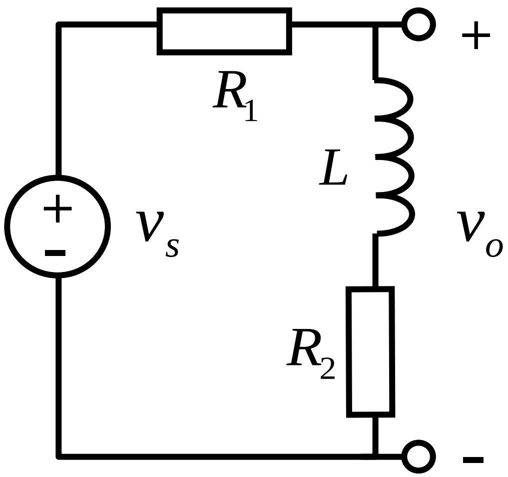
straight line: $y=c+k\,x$

$x=\log_{10}\omega \quad y=20\log_{10}|H(\omega)|$

$k = 1$, $c = -20\log_{10}\omega_0$

$$20\log_{10}|H(\omega)| \approx \begin{cases} 0 & \omega < \omega_0 \\ 20\log_{10}\omega - 20\log_{10}\omega_0 & \omega > \omega_0 \end{cases}$$

An example: Bode plots for k(1+jω/ω$_1$)/(1+jω/ω$_2$)

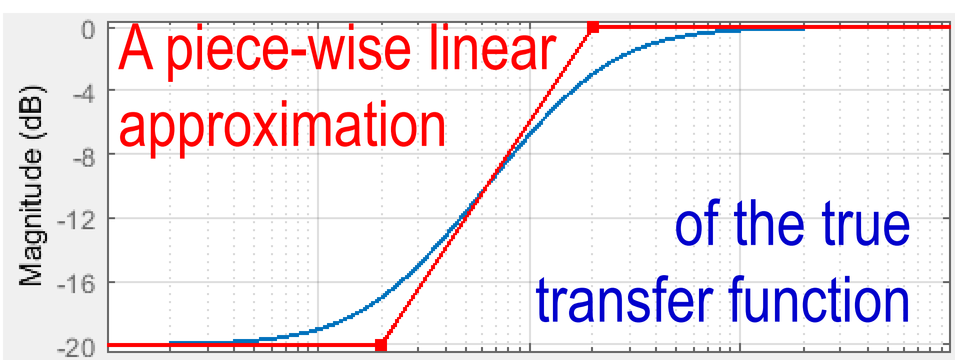


$$H(j\omega)=\frac{V_o}{V_s}=\frac{R_2+j\omega L}{R_1+R_2+j\omega L}=k\,\frac{1+j\,\omega/\omega_1}{1+j\,\omega/\omega_2}$$

$$H(j\omega)\approx\begin{cases} k & \omega<\omega_1 \\ k\,\omega/\omega_1 & \omega_1<\omega<\omega_2 \\ k\,\omega_2/\omega_1 & \omega_2<\omega \end{cases}$$

Assume that $\omega_1 < \omega_2$

$$20\log_{10}|H|\approx\begin{cases} 20\log_{10}k & \omega<\omega_1 \\ (20\log_{10}k-20\log_{10}\omega_1)+20\log_{10}\omega & \omega_1<\omega<\omega_2 \\ (20\log_{10}k-20\log_{10}\omega_1)+20\log_{10}\omega_2 & \omega_2<\omega \end{cases}$$

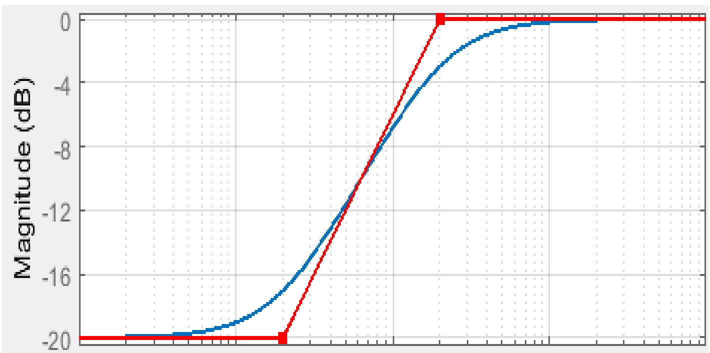


$$\log_{10}|H|=\log_{10}k$$
$$+\log_{10}|1+j\,\omega/\omega_1|$$
$$-\log_{10}|1+j\,\omega/\omega_2|$$

An example: Bode plots for $k(1+j\omega/\omega_1)/(1+j\omega/\omega_2)$

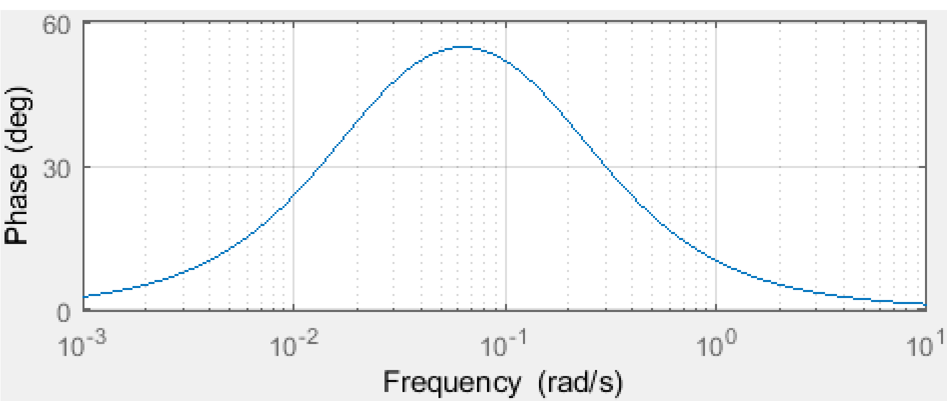
$$H = k\frac{1+j\omega/\omega_1}{1+j\omega/\omega_2}$$

$$20\log_{10}|H| = 20\log_{10}k + 20\log_{10}|1+j\omega/\omega_1| - 20\log_{10}|1+j\omega/\omega_2|$$



$20\log_{10}|H(\omega)|$ is approximated by a piece-wise linear function of $\log_{10}\omega$.

phase

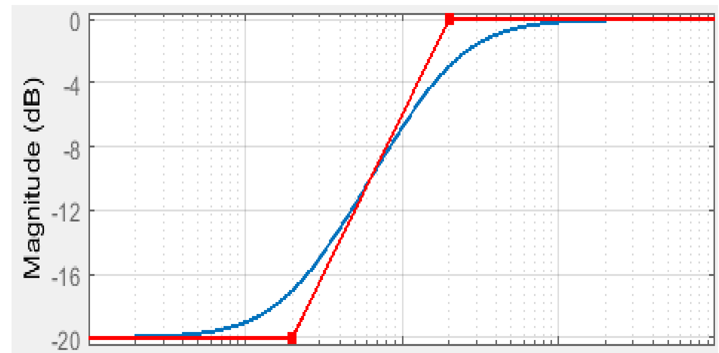


An example: Bode plots for k(1+jω/ω$_1$)/(1+jω/ω$_2$)

$$H = k\left(1 + j\,\omega/\omega_1\right)/\left(1 + j\,\omega/\omega_2\right) \quad \omega_1 < \omega_2$$

$$20\log_{10}|H| = 20\log_{10} k + 20\log_{10}|1 + j\,\omega/\omega_1| - 20\log_{10}|1 + j\,\omega/\omega_2|$$

$$\begin{aligned} 20\log_{10}|H| \approx\ & 20\log_{10} k \\ & + 20\log_{10}\omega - 20\log_{10}\omega_1 \\ & - 20\log_{10}\omega + 20\log_{10}\omega_2 \end{aligned}$$



$20\log_{10}|H(\omega)|$ is approximated by a piece-wise linear function of $\log_{10}\omega$.

$\log_{10}\omega_1$ $\log_{10}\omega_2$ $\log_{10}\omega$

$\boxed{20\log_{10} k}$

$\boxed{\begin{array}{c} 20\log_{10} k \\ -20\log_{10}\omega_1 + 20\log_{10}\omega \end{array}}$

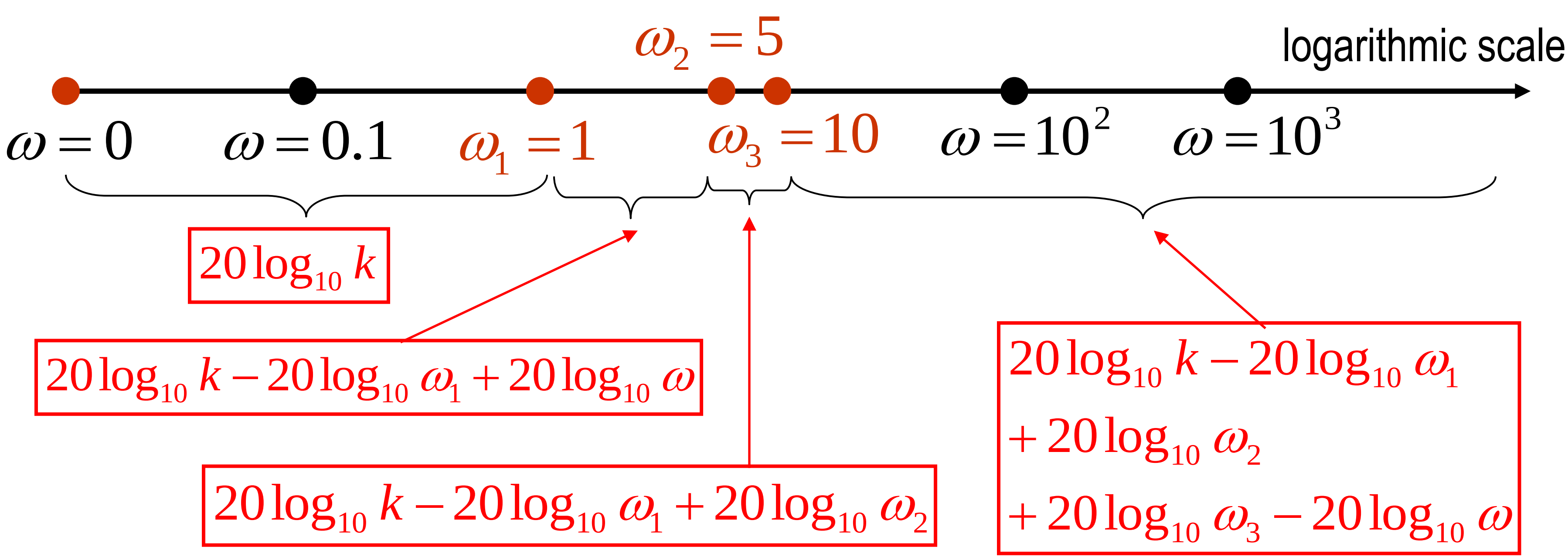
$\boxed{\begin{array}{c} 20\log_{10} k \\ +\cancel{20\log_{10}\omega} - 20\log_{10}\omega_1 \\ -\cancel{20\log_{10}\omega} + 20\log_{10}\omega_2 \end{array}}$

An example: Bode plot for 300(5+jω)/(-ω²+j11ω+10)

$$H(j\omega) = 300\frac{5+j\omega}{(1+j\omega)(10+j\omega)} = 150\frac{1+j\omega/5}{(1+j\omega)(1+j\omega/10)}$$

$$\left|H(\omega)\right|_{dB} = 20\log_{10}\left|H(\omega)\right| = 20\log_{10}\left|150\right| + 20\log_{10}\left|1+j\omega/5\right|$$
$$-20\log_{10}\left|1+j\omega\right| - 20\log_{10}\left|1+j\omega/10\right|$$

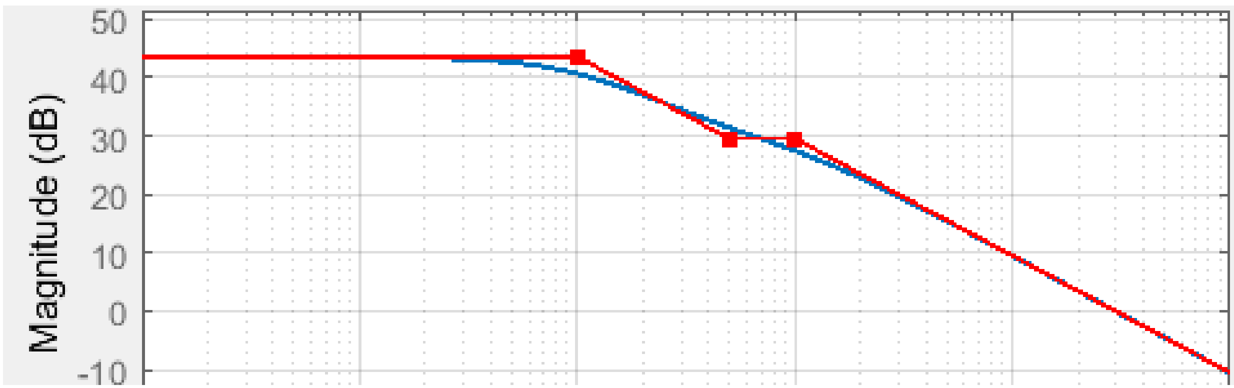
The first term is just a constant, $k=20\log_{10}150$. For the other three terms, there are three corner frequencies: $\omega_1=1$, $\omega_2=5$ and $\omega_3=10$



An example: Bode plot for 300(5+jω)/(-ω²+j11ω+10)

$$\left|H(\omega)\right|_{dB} = 20\log_{10}\left|H(\omega)\right| = 20\log_{10}\left|150\right| + 20\log_{10}\left|1 + j\,\omega/5\right|$$

$$-20\log_{10}\left|1 + j\omega\right| - 20\log_{10}\left|1 + j\,\omega/10\right|$$



logarithmic scale

$\omega = 0$ $\quad \omega = 0.1 \quad \omega_1 = 1 \quad \omega_2 = 5 \quad \omega_3 = 10 \quad \omega = 10^2 \quad \omega = 10^3$

$\boxed{20\log_{10} k}$

$\boxed{20\log_{10} k - 20\log_{10}\omega_1 + 20\log_{10}\omega}$

$\boxed{20\log_{10} k - 20\log_{10}\omega_1 + 20\log_{10}\omega_2}$

$$\boxed{\begin{aligned}&20\log_{10} k - 20\log_{10}\omega_1\\&+20\log_{10}\omega_2\\&+20\log_{10}\omega_3 - 20\log_{10}\omega\end{aligned}}$$

Bode plots: A general case

Frequency response function $H(j\omega) = \frac{K(\ )(\ )(\ )\ldots}{(\ )(\ )(\ )\ldots}$

$$|H(j\omega)| = \frac{K|(\ )||(\ )||(\ )|\ldots}{|(\ )||(\ )||(\ )|\ldots}$$

Three types of terms:

$$\angle|H(j\omega)| = \angle|(\ )| + \angle|(\ )| + \ldots - \angle|(\ )| - \angle|(\ )| - \ldots$$

1. First order terms $(1+j\omega/\omega_0)$
2. Second order terms $(j\omega/\omega_0)^2+2\zeta\omega/\omega_0+1$
3. Terms $j\omega$

Numerator terms (+)
Denominator terms (-)

Magnitude in dB = $20\log_{10}|H(j\omega)| = 20\log_{10}K$
$+ 20\log_{10}|(\ldots)| + \ldots\ldots$ numerator terms
$- 20\log_{10}|(\ldots)| - \ldots\ldots$ denominator terms

K is referred to as a gain term, it does not depend on frequency.

Bode plots: quadratic terms

$$H(j\omega)=\frac{1}{(j\omega/\omega_0)^2+2\zeta\,\omega/\omega_0+1}=\frac{\omega_0^2}{(j\omega)^2+2\zeta\omega\omega_0+\omega_0^2}$$

$$\text{Magnitude}=20\log_{10}\omega_0^2-20\log_{10}\left|(j\omega)^2+2\zeta\omega\omega_0+\omega_0^2\right|$$

$$\text{As }\ \omega\to 0,\quad \text{magnitude}\to 0\,\text{dB}$$

$$\text{As }\ \omega\to\infty,\quad \text{magnitude}\approx 20\log_{10}\omega_0^2-20\log_{10}\omega^2$$

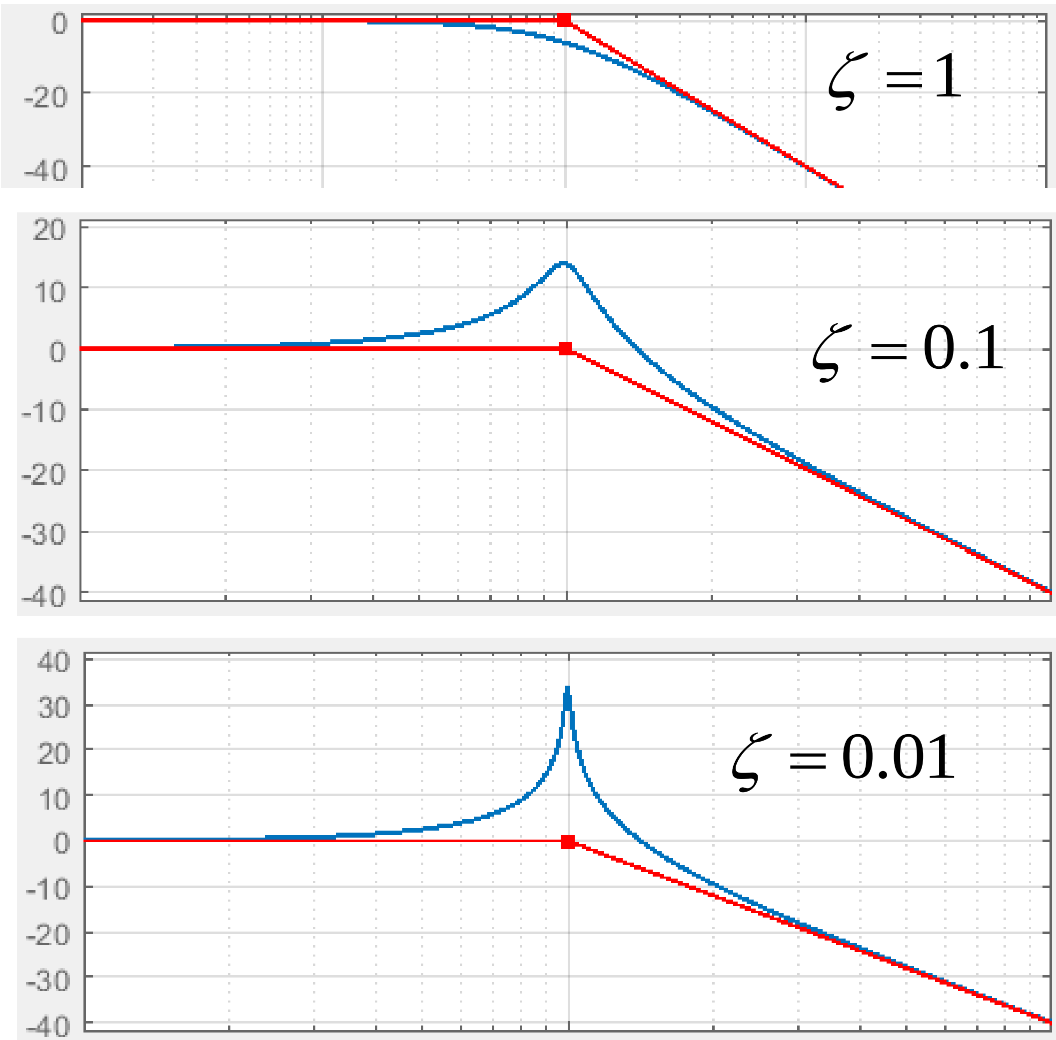
$$=\boxed{20\log_{10}\omega_0^2-40\log_{10}\omega}$$

The equations of a straight line of slope = -40 dB/decade. This line intersects the 0 dB axis when $\omega = \omega_0$

Unlike the first order terms the quadratic terms however the error at the corner points can be considerable depending on the value of ζ (damping factor or ratio).

Bode plots: quadratic terms

$$H(j\omega) = \frac{1}{(j\omega/\omega_0)^2 + 2\zeta\,\omega/\omega_0 + 1}$$



Unlike the first order terms, for the quadratic terms, the error at the corner points can be considerable depending on the value of ζ.

Bode plots: An example

Consider frequency response $$H(j\omega)=\frac{100(1+j\omega)}{(10+j\omega)(100+j\omega)}$$

To obtain the Bode plot for $H(j\omega)$, we rewrite it as follows

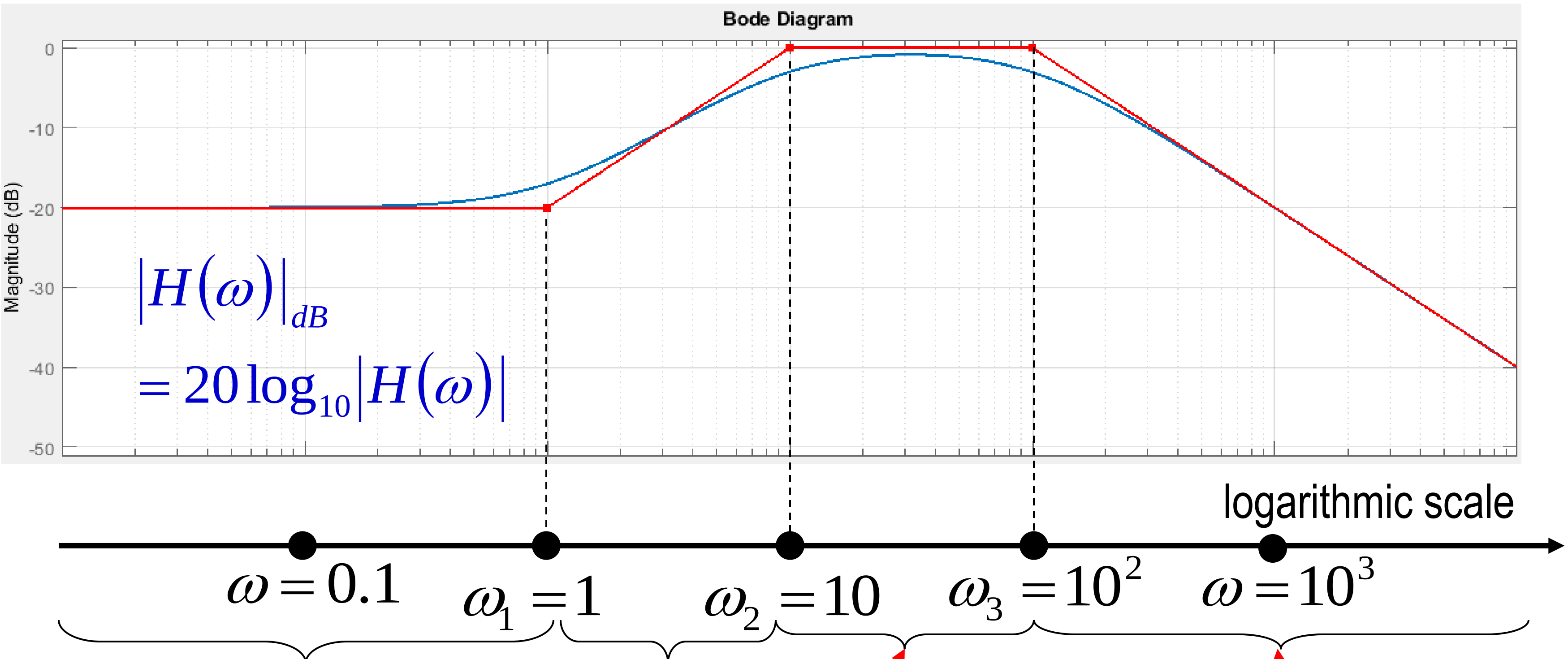
$$H(j\omega)=\left(\frac{1}{10}\right)\left(\frac{1}{1+j\omega/10}\right)\left(\frac{1}{1+j\omega/100}\right)(1+j\omega)$$

The Bode plot for $20\log_{10}|H(j\omega)|$ is the sum of the Bode plots corresponding to each of the factors.

The constant factor 1/10 accounts for an offset of –20 dB at each frequency. The $(1+j\omega)$ factor has the corner (break) frequency at $\omega=1$ and produces the 20 dB/decade rise that starts at $\omega=1$ and is cancelled by the 20 dB/decade decay that starts at the corner frequency at $\omega=10$ and is due to the $1/(1+j\omega/10)$ factor. Finally, the $1/(1+j\omega/100)$ factor contributes another corner frequency at $\omega=100$ and a subsequent decay at the rate of 20 dB/decade.

Bode plots: An example

$$H(j\omega)=\left(\frac{1}{10}\right)\left(\frac{1}{1+j\omega/10}\right)\left(\frac{1}{1+j\omega/100}\right)(1+j\omega)$$



$$\boxed{20\log_{10}k}$$

$$\boxed{20\log_{10}k-20\log_{10}\omega_1+20\log_{10}\omega}$$

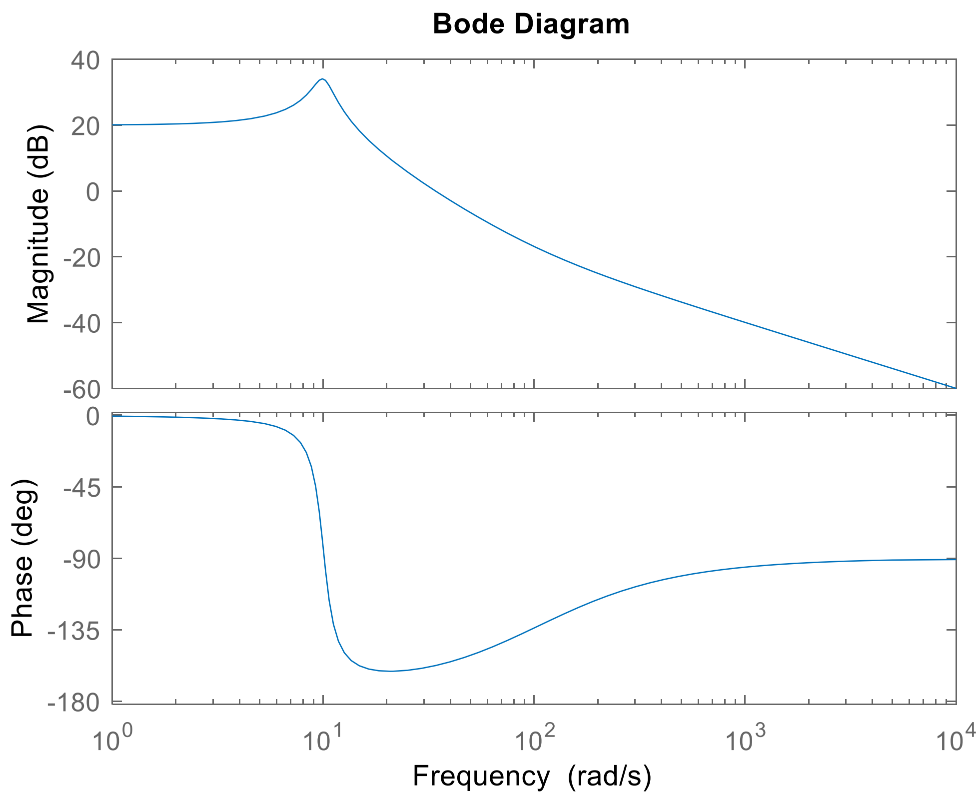
$$\boxed{20\log_{10}k-20\log_{10}\omega_1+20\log_{10}\omega_2}$$

$$\boxed{\begin{aligned}&20\log_{10}k-20\log_{10}\omega_1\\&+20\log_{10}\omega_2\\&+20\log_{10}\omega_3-20\log_{10}\omega\end{aligned}}$$

Bode Plot Using Matlab

$$H(s) = \frac{10s + 1000}{s^2 + 2s + 100},$$ num=[10 100]; den=[1 2 100]

Bode(num, den) or Bode(tf([10 100], [1 2 100]))



Impulse Response Using Matlab

$$H(s) = \frac{10s + 1000}{s^2 + 2s + 100},$$ num=[10 100]; den=[1 2 100]

impulse(num, den) or impulse(tf([10 100], [1 2 100]))