Two-sided (bilateral) Laplace transform and its properties

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Modified by

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Two-sided (bilateral) Laplace transform 1

$$\boxed{X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt}$$

s is a complex number

$$x(t) \xrightarrow{L} X(s) \qquad X(s) = L[x(t)]$$

Why do we need it?

- We would like to process signals that don't have Fourier transform (infinite energy signals);
- Sometimes we need to deal with unstable systems (control applications).

Complex-valued function $\longrightarrow$ $X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$

The domain of $X(s)$ is complex plane (s is complex)

Two-sided (bilateral) Laplace transform 2

Complex-valued function → $X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$

The domain of $X(s)$ is complex plane (s is complex)

May only converge for some s.

Two parts: (1) Region of Convergence (ROC) and (2) $X(s)$

$$x(t) = e^{at}u(t) \quad X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt = \int_{0}^{\infty} e^{(a-s)t} dt = \frac{1}{a-s} e^{(a-s)t} \Big|_{0}^{\infty} = \frac{1}{s-a}$$

$$\text{ROC:} \quad \mathrm{Re}\{s\} > \mathrm{Re}\{a\}$$

$$x(t) = -e^{at}u(-t) \quad X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt = \int_{-\infty}^{0} -e^{(a-s)t} dt$$

$$= -\frac{1}{a-s} e^{(a-s)t} \Big|_{-\infty}^{0} = \frac{1}{s-a} \qquad \text{ROC:} \quad \mathrm{Re}\{s\} < \mathrm{Re}\{a\}$$

This example demonstrates why we need to consider ROC in addition to $X(s)$

Two-sided (bilateral) Laplace transform 3

Complex-valued function → $X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$

The domain of $X(s)$ is complex plane (s is complex)

The range of values of the complex variable s for which the Laplace transform converges is called the **region of convergence** (ROC).

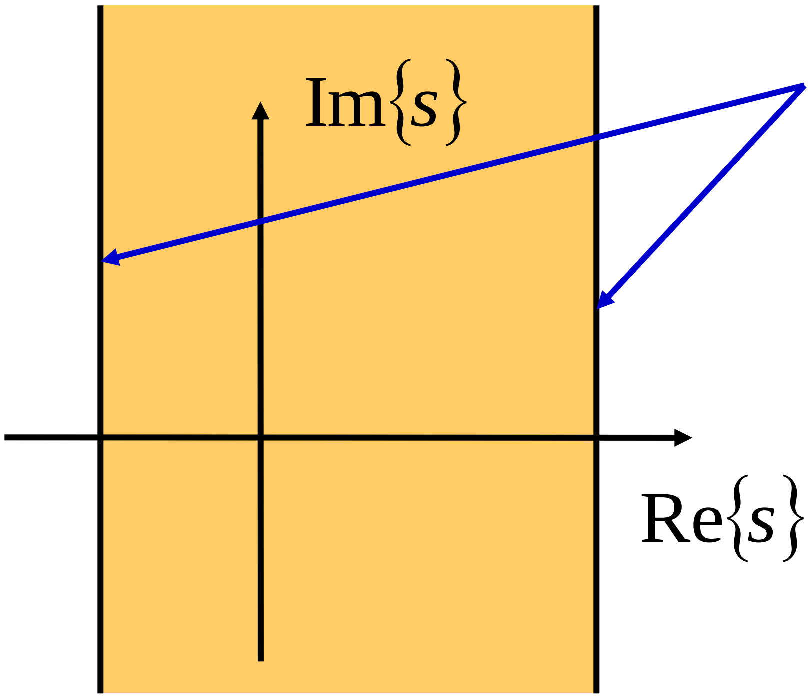
In order for the Laplace transform to be unique for each signal $x(t)$, **the ROC must be specified as part of the transform**.

$$x(t) = e^{at} u(t) \quad X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt = \int_{0}^{\infty} e^{(a-s)t} dt = \frac{1}{a-s} e^{(a-s)t} \Big|_{0}^{\infty} = \frac{1}{s-a}$$

$$\text{ROC:} \quad \mathrm{Re}\{s\} > \mathrm{Re}\{a\}$$

Region of Convergence (ROC)

ROC: always vertical strips



Left or right boundary is not always present

An example:

$$x(t) = e^{at}u(t) \qquad X(s) = \int_{-\infty}^{\infty} x(t)\,e^{-st}\,dt = \int_{0}^{\infty} e^{(a-s)t}\,dt = \frac{1}{a-s}\,e^{(a-s)t}\Big|_{0}^{\infty} = \frac{1}{s-a}$$

$$\mathrm{ROC}: \quad \mathrm{Re}\{s\} > \mathrm{Re}\{a\}$$

Examples of two-sided Laplace transforms

$$x(t)=\delta(t-\tau) \quad X(s)=\int_{-\infty}^{\infty} x(t)\,e^{-st}dt=e^{-s\tau} \quad \mathrm{ROC:}\ \mathrm{Re}\{s\}>-\infty$$

$$x(t)=u(t-\tau) \quad X(s)=\int_{\tau}^{\infty} e^{-st}dt=\frac{e^{-s\tau}}{s} \quad \mathrm{ROC:}\ \mathrm{Re}\{s\}>0$$

$$x(t)=e^{at}u(t) \quad X(s)=\int_{0}^{\infty} e^{(a-s)t}dt=\frac{1}{s-a} \quad \mathrm{ROC:}\ \mathrm{Re}\{s\}>\mathrm{Re}\{a\}$$

$$x(t)=-e^{at}u(-t)$$

$$X(s)=\int_{-\infty}^{0} e^{(a-s)t}dt=\frac{1}{s-a} \quad \mathrm{ROC:}\ \mathrm{Re}\{s\}<\mathrm{Re}\{a\}$$

More examples

$$x(t)=e^{-|t|}=e^{-t}u(t)+e^{t}u(-t)$$

$$\text{causal part}:\ e^{-t}u(t)\xrightarrow{L}\frac{1}{s+1}\quad \text{ROC}:\ \text{Re}\{s\}>-1$$

$$\text{anti-causal part}:\ e^{t}u(-t)\xrightarrow{L}-\frac{1}{s-1}\quad \text{ROC}:\ \text{Re}\{s\}<1$$

$$X(s)=\frac{1}{s+1}-\frac{1}{s-1}\quad \text{ROC}:\ \text{-}1<\text{Re}\{s\}<1$$

$$x(t)=2e^{-t}u(t)+3e^{-5t}u(t)\quad X(s)=\frac{2}{s+1}+\frac{3}{s+5}\quad \text{ROC}:\ \text{Re}\{s\}>-1$$

$$x(t)=\cos(\omega_0 t)u(t)\quad X(s)=\frac{1/2}{s-j\omega_0}+\frac{1/2}{s+j\omega_0}=\frac{s}{s^2+\omega_0^2}$$

$$\text{ROC}:\ \text{Re}\{s\}>0$$

Relationship to Fourier transform 1

$$x(t) \xrightarrow{F} X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \qquad X(\omega) = F[x(t)]$$

$$x(t) = F^{-1}[X(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

It is not always possible to calculate the Fourier transform of a signal $x(t)$ by integration. For example, if the signal is of finite power rather than finite energy the classical Fourier transform does not exist (the integral does not converge). A possible solution consists of multiplying $x(t)$ by a convergence factor exp(-σt):

$$x_\sigma(t) = x(t) e^{-\sigma t}$$

$$X_\sigma(\omega) = \int_{-\infty}^{\infty} x_\sigma(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$s = \sigma + j\omega$ Motivation behind using bilateral (two-sided) Laplace transform

Relationship to Fourier transform 2

$$x(t) \xrightarrow{L} X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \qquad X(s) = L[x(t)]$$

$$x_\sigma(t) = x(t) e^{-\sigma t}$$

$$X_\sigma(\omega) = \int_{-\infty}^{\infty} x_\sigma(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$s = \sigma + j\omega$$

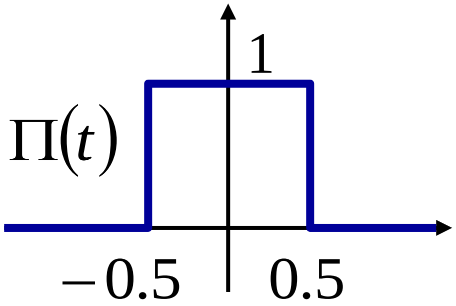
$$x(t) = e^{\sigma t} x_\sigma(t) = e^{\sigma t} \frac{1}{2\pi} \int_{-\infty}^{\infty} X_\sigma(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_\sigma(\omega) e^{(\sigma + j\omega)t} d\omega$$

$$= \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

the inverse Laplace transform
(it is not used in practice)

An example: Laplace transform of a box function

$x = \Pi(t)$



$$X(s) = \int_{-1/2}^{1/2} e^{-st}\,dt = \frac{e^{-s/2} - e^{s/2}}{s}$$

ROC: All s. ROC = the whole complex plane.

The same holds for any finite-duration signal.

Poles and Zeros of rational $X(s)$

In many applications $X(s)$ is a rational function in s:

$$X(s) = \frac{N(s)}{D(s)}$$

polynomials

$$X(s) = \frac{a_0 s^m + a_1 s^{m-1} + \ldots + a_m}{b_0 s^n + b_1 s^{n-1} + \ldots + b_n} = \frac{a_0}{b_0} \frac{(s - z_1)\ldots(s - z_m)}{(s - p_1)\ldots(s - p_n)}$$

Zeros $z_1,\ldots,z_m$ and poles $p_1,\ldots,p_n$.

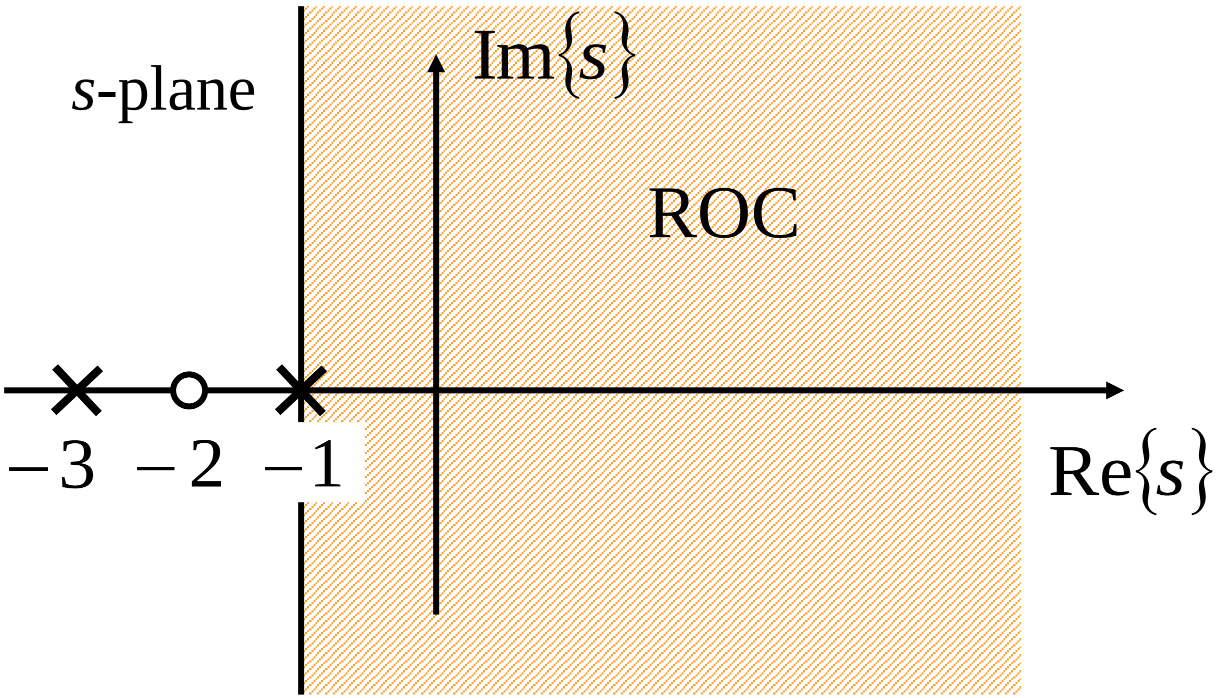
The poles of $X(s)$ lie outside the ROC, as $X(s)$ is infinite for those values of s.

The zeros may lie inside or outside the ROC.

Except for the scale factor a_0/b_0, $X(s)$ can be completely specified by its zeros and poles.

Poles and Zeros: An example

$$X(s)=\frac{3s+6}{2s^2+8s+6}=\frac{3}{2}\frac{s+2}{(s+1)(s+3)}\qquad \mathrm{Re}(s)>-1$$



Traditionally, an '**x**' is used to indicate each pole location and an '**o**' is used to indicate each zero.

Properties of ROC

P1. The ROC does not contain any poles.

P2. If $x(t)$ is a finite-duration signal, that is $x(t)$=0 except a finite interval $t_1 \le t \le t_2$, then the ROC is the entire s-plane except possibly $s = 0$ or $s = \infty$.

P3. If $x(t)$ is a right-sided signal, that is, $x(t) = 0$ for $t < t_1$, then the ROC is of the form $\mathrm{Re}(s) > \sigma_{\max}$, where $\sigma_{\max}$ equals the maximum real part of any of the poles of $X(s)$. In other words, the ROC is a half-plane to the right of the vertical line $\mathrm{Re}(s) = \sigma_{\max}$ in the s-plane.

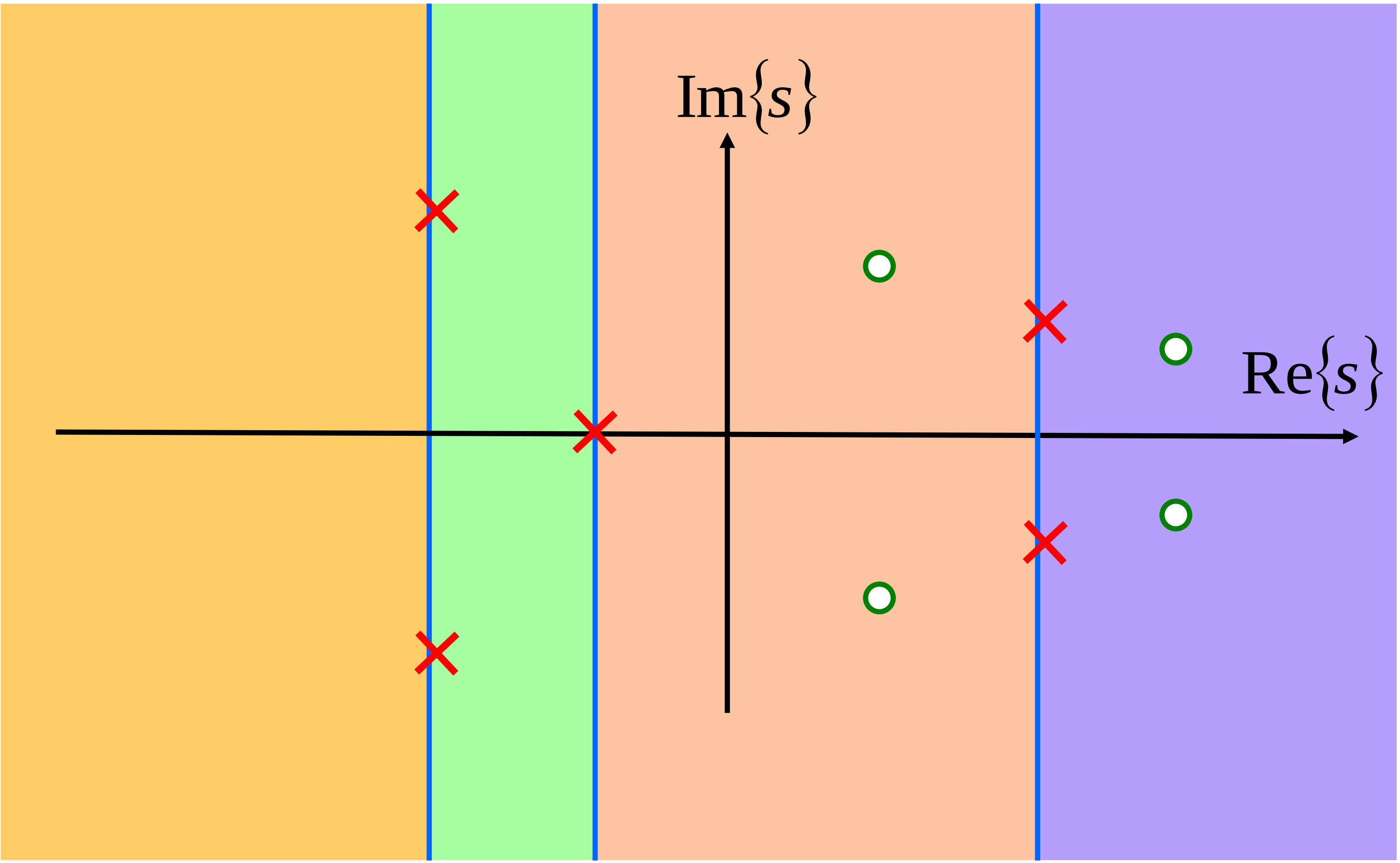
P4. If $x(t)$ is a left-sided signal, that is , $x(t) = 0$ for $t_2 < t$, then the ROC is of the form $\mathrm{Re}(s) < \sigma_{\min}$, where $\sigma_{\min}$ equals the minimum real part of any of the poles of $X(s)$.

P5. If $x(t)$ is a two-sided signal, that is, is $x(t)$ a finite-duration signal, then the ROC is of the form $\sigma_1 < \mathrm{Re}(s) < \sigma_2$, where σ_1 and σ_2 are the real parts of two poles of $X(s)$. The ROC is a vertical strip in the s-plane between the vertical lines $\mathrm{Re}(s) = \sigma_1$ and $\mathrm{Re}(s) = \sigma_2$.

Pole-zero diagrams and ROCs

$$X(s)=\frac{N(s)}{D(s)}=k\,\frac{(s-z_1)\ldots(s-z_m)}{(s-p_1)\ldots(s-p_n)}$$

Four different ROCs, four different signals have the same Laplace transform formula.



Pole-zero diagrams: a geometric interpretation

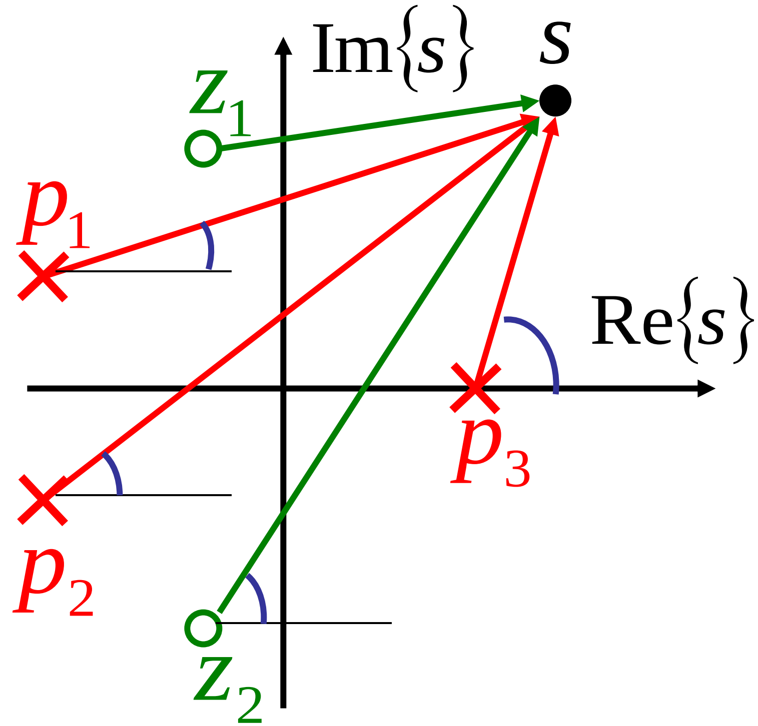
$$X(s) = \frac{N(s)}{D(s)} = k\,\frac{(s-z_1)\ldots(s-z_m)}{(s-p_1)\ldots(s-p_n)}$$

Magnitude:

$$|X(s)| = |k|\,\frac{|s-z_1|\ldots|s-z_m|}{|s-p_1|\ldots|s-p_n|}$$

Phase: $\angle X(s) = \angle k +$

$$+\sum_{\text{zeros}} \angle(s-z_i) - \sum_{\text{poles}} \angle(s-p_i)$$



Laplace Transform Pairs for Common Signals 1

$x(t)$	$X(s)$	ROC
$\delta(t)$	1	All s
$u(t)$	$\frac{1}{s}$	$\mathrm{Re(s)} > 0$
$-u(-t)$	$\frac{1}{s}$	$\mathrm{Re(s)} < 0$
$tu(t)$	$\frac{1}{s^2}$	$\mathrm{Re}(s) > 0$
$t^k u(t)$	$\frac{\mathrm{k}!}{s^{k+1}}$	$\mathrm{Re}(s) > 0$
$e^{-at} u(t)$	$\frac{1}{s+a}$	$\mathrm{Re}(s) > -\mathrm{Re}(a)$
$-e^{-at} u(-t)$	$\frac{1}{s+a}$	$\mathrm{Re}(s) < -\mathrm{Re}(a)$

Laplace Transform Pairs for Common Signals 2

$x(t)$	$X(s)$	ROC
$te^{-at}u(t)$	$\dfrac{1}{(s+a)^2}$	$\mathrm{Re}(s) > -\mathrm{Re}(a)$
$-te^{-at}u(-t)$	$\dfrac{1}{(s+a)^2}$	$\mathrm{Re}(s) < -\mathrm{Re}(a)$
$\cos \omega_0 t u(t)$	$\dfrac{s}{s^2+\omega_0^2}$	$\mathrm{Re}(s) > 0$
$\sin \omega_0 t u(t)$	$\dfrac{\omega_0}{s^2+\omega_0^2}$	$\mathrm{Re}(s) > 0$
$e^{-at}\cos \omega_0 t u(t)$	$\dfrac{s+a}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}(s) > -\mathrm{Re}(a)$
$e^{-at}\sin \omega_0 t u(t)$	$\dfrac{\omega_0}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}(s) > -\mathrm{Re}(a)$

Properties of the two-sided Laplace transform 1

Linearity. $x_1(t) \leftrightarrow X_1(s)$, ROC = R_1, $x_2(t) \leftrightarrow X_2(s)$, ROC = R_2. Then $a_1x_1(t)+ a_2x_2(t) \leftrightarrow a_1X_1(s)+ a_2X_2(s)$, ROC = $R' \supset R_1 \cap R_2$

Time Shifting. $x(t) \leftrightarrow X(s)$, ROC = R. Then $x(t\text{-}t_0) \leftrightarrow \exp(-st_0)X(s)$, ROC = $R' = R$.

Shifting in the *s*-Domain. $x(t) \leftrightarrow X(s)$, ROC = R.

Then $\exp(st_0)\, x(t) \leftrightarrow X(s\text{-}s_0)$, ROC = $R' = R + Re(s_0)$.

Time Scaling. $x(t) \leftrightarrow X(s)$, ROC = $R' = R$.

Then $x(at) \leftrightarrow (1/|a|)\, X(s/a)$, ROC = $R' = aR$

Time Reversal. $x(t) \leftrightarrow X(s)$, ROC = R.

Then $x(-t) \leftrightarrow X(-s)$, ROC = $R' = -R$.

Properties of the two-sided Laplace transform 2

Differentiation in the Time Domain. $x(t) \leftrightarrow X(s)$, ROC = R.

Then $dx(t)/dt \leftrightarrow sX(s)$, ROC = $R' \supset R$.

Differentiation in the *s*-Domain. $x(t) \leftrightarrow X(s)$, ROC = R.

Then $-tx(t)/dt \leftrightarrow dX(s)/ds$, ROC = $R' = R$.

Integration in the Time Domain. $x(t) \leftrightarrow X(s)$, ROC = R.

Then

$$\int_{-\infty}^{t} x(\tau)\,d\tau \leftrightarrow \frac{1}{s} X(s) \qquad \text{ROC} = R' = R \cap \{\mathrm{Re}(s)>0\}$$

Convolution. $x_1(t) \leftrightarrow X_1(s)$, ROC = R_1, $x_2(t) \leftrightarrow X_2(s)$, ROC = R_2.

Then $x_1(t)*x_2(t) \leftrightarrow X_1(s)X_2(s)$, ROC = $R' \supset R_1 \cap R_2$

Properties of the two-sided Laplace transform 3

PROPERTY	SIGNAL	TRANSFORM	ROC
	$x(t)$	$X(s)$	R
	$x_1(t)$	$X_1(s)$	R_1
	$x_2(t)$	$X_2(s)$	R_2
Linearity	$a_1x_1(t) + a_2x_2(t)$	$a_1 X_1(s) + a_2 X_2(s)$	$R' \supset R_1 \cap R_2$
Time shifting	$x(t-t_0)$	$e^{-st_0}\ X(s)$	$R' = R$
Shifting in s	$e^{s_0 t}\ x(t)$	$X(s-s_0)$	$R' = R + \mathrm{Re}(s_0)$
Time scaling	$x(at)$	$\frac{1}{\lvert a \rvert} X(a)$	$R' = aR$
Time reversal	$x(-t)$	$X(-s)$	$R' = -R$
Differentiation in t	$\frac{dx(t)}{dt}$	$sX(s)$	$R' \supset R$
Differentiation in s	$-tx(t)$	$\frac{dX(s)}{ds}$	$R' = R$
Integration	$\int_{-\infty}^{t} x(\tau)\, d\tau$	$\frac{1}{s} X(s)$	$R' \supset R \cap \{\mathrm{Re}(s) > 0\}$
Convolution	$x_1(t) * x_2(t)$	$X_1(s)\, X_2(s)$	$R' \supset R_1 \cap R_2$

Laplace transform properties: Time-shifting

$$x(t) \xrightarrow{L} X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$\boxed{x(t-t_0) \xleftrightarrow{L} e^{-st_0} X(s)}$$

$$X(s) = \int_{-\infty}^{\infty} x(t-t_0) e^{-st} dt = e^{-st_0} \int_{-\infty}^{\infty} x(t-t_0) e^{-s(t-t_0)} dt = e^{-st_0} X(s)$$

Laplace transform properties: Time-Reversal

$$x(t) \xrightarrow{L} X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$\boxed{x(-t) \overset{L}{\leftrightarrow} X(-s)}$$

$$x(-t) = \frac{1}{2\pi j} \int_{-\infty}^{\infty} X(s) e^{s(-t)} ds = \frac{1}{2\pi j} \int_{-\infty}^{\infty} X(s) e^{(-s)t} ds$$

$$= \frac{1}{2\pi j} \int_{-\infty}^{\infty} X(-s) e^{(-s)t} d(-s) = \frac{1}{2\pi j} \int_{-\infty}^{\infty} X(s') e^{(s')t} d(s')$$

$$x(-t) \overset{L}{\leftrightarrow} X(s') = X(-s)$$

Laplace transform properties: Differentiation

$$\boxed{\frac{dx(t)}{dt} \xleftrightarrow{L} s\,X(s)}$$

$$x(t) \xrightarrow{L} X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \qquad x_\sigma(t) = x(t)\,e^{-\sigma t}$$

$$X_\sigma(\omega) = \int_{-\infty}^{\infty} x_\sigma(t)\,e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t)\,e^{-(\sigma+j\omega)t}dt$$

$$= \int_{-\infty}^{\infty} x(t)\,e^{-st}dt, \quad s = \sigma + j\omega$$

$$x(t) = e^{\sigma t}x_\sigma(t) = e^{\sigma t}\frac{1}{2\pi}\int_{-\infty}^{\infty} X_\sigma(\omega)e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_\sigma(\omega)e^{(\sigma+j\omega)t}d\omega = \frac{1}{2\pi\,j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds$$

$$x(t) = \frac{1}{2\pi\,j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds$$

$$\frac{d}{dt}e^{st} = s\,e^{st}$$

$$\frac{dx}{dt} = \frac{1}{2\pi\,j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\frac{d}{dt}\left(e^{st}\right)ds = \frac{1}{2\pi\,j}\int_{\sigma-j\infty}^{\sigma+j\infty} s\,X(s)e^{st}ds$$

Laplace transform properties: Differentiation in s

$$x(t) \xrightarrow{L} X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$\boxed{-t\, x(t) \xleftrightarrow{L} \frac{dX(s)}{ds}}$$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \qquad \frac{d}{ds} X(s) = \int_{-\infty}^{\infty} x(t) \frac{d}{ds}\left(e^{-st}\right) dt$$

$$= \int_{-\infty}^{\infty} -t\, x(t) e^{-st} dt$$

Laplace transform properties: Convolution

$$x(t) \xrightarrow{L} X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$\boxed{x_1(t) * x_2(t) \xrightarrow{L} X_1(s) X_2(s)}$$

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) d\tau$$

$$L[x_1(t) * x_2(t)] \xrightarrow{L} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) d\tau \right) e^{-st} dt$$

$$= \int_{-\infty}^{\infty} x_1(\tau) \left(\int_{-\infty}^{\infty} x_2(t-\tau) e^{-st} \, dt \right) d\tau = \int_{-\infty}^{\infty} x_1(\tau) e^{-s\tau} X_2(s) d\tau$$

$$X_2(s) \int_{-\infty}^{\infty} x_1(\tau) e^{-s\tau} d\tau = X_1(s) X_2(s)$$

Laplace transform examples 1

$$x(t) \xrightarrow{L} X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

Derive the Laplace transforms of the following signals:

$$x_1(t) = \delta(t) \qquad x_2(t) = u(t) \qquad x_3(t) = \delta'(t)$$

$$X_1(s) = \int_{-\infty}^{\infty} \delta(t) e^{-st} dt = 1 \quad \text{ROC: all } s$$

$$X_2(s) = \int_{-\infty}^{\infty} u(t) e^{-st} dt = \int_{0}^{\infty} e^{-st} dt = \frac{1}{s} \quad \text{ROC: } \mathrm{Re}(s) > 0$$

$$X_3(s) = \int_{-\infty}^{\infty} \delta'(t) e^{-st} dt = -\int_{-\infty}^{\infty} \delta(t) \frac{d}{dt}\left(e^{-st}\right) dt = s \quad \text{ROC: all } s$$

integration by parts

Laplace transform examples 2

$$x(t) \xrightarrow{L} X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

Derive the Laplace transforms of the following signals:

$$x_4(t) = t\,u(t) \qquad x_5(t) = e^{-at}u(t) \qquad x_6(t) = t\,e^{-at}u(t)$$

$$X_4(s) = \int_{-\infty}^{\infty} t\,u(t) e^{-st} dt = \int_{0}^{\infty} t\,e^{-st} dt = -\frac{1}{s}\int_{0}^{\infty} t\,d\left(e^{-st}\right) = \frac{1}{s}\int_{0}^{\infty} e^{-st}\,dt = \frac{1}{s^2}$$

$\mathrm{ROC:Re}(s) > 0$

integration by parts

$$X_5(s) = \int_{-\infty}^{\infty} e^{-at}\,e^{-st} dt = \int_{-\infty}^{\infty} e^{-(s+a)t} dt = \frac{1}{s+a} \qquad \mathrm{ROC:}\ \mathrm{Re}(s) > -a$$

$$X_6(s) = \int_{-\infty}^{\infty} t\,e^{-at} e^{-st} dt = \frac{1}{(s+a)^2} \qquad \mathrm{ROC:}\ \mathrm{Re}(s) > -a$$

differentiation in s property (can be also done by integration by parts)

Laplace transform examples 3

$$x(t) \xrightarrow{L} X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

Derive the Laplace transforms of the following signals:

$$x_7(t) = \cos(\omega_0 t) u(t) \qquad x_8(t) = e^{-at} \cos(\omega_0 t) u(t)$$

$$X_7(s) = \int_{-\infty}^{\infty} \cos(\omega_0 t) u(t) e^{-st} dt = \int_{0}^{\infty} \cos(\omega_0 t) e^{-st} dt$$

$$= \int_{0}^{\infty} \frac{1}{2}\left(e^{-j\omega_0 t} + e^{j\omega_0 t}\right) e^{-st} dt = \frac{1}{2}\left(\frac{1}{s + j\omega_0} + \frac{1}{s - j\omega_0}\right) = \frac{s}{s^2 + \omega_0^2}$$

$$\text{ROC: } \mathrm{Re}(s) > 0$$

$$X_8(s) = \int_{-\infty}^{\infty} e^{-at} \cos(\omega_0 t) u(t) e^{-st} dt = \frac{s + a}{(s + a)^2 + \omega_0^2}$$

$$\text{ROC: } \mathrm{Re}(s) > -a$$

Rational Laplace transform 1

An important special case: $X(s)=\frac{N(s)}{D(s)}=k\frac{(s-z_1)\ldots(s-z_m)}{(s-p_1)\ldots(s-p_n)}$

If m<n, then X(s) is called a proper rational function. Otherwise it is called an improper rational function. Below we deal with proper rational functions.

Simple Pole Case. Assume that all zeros of D(s) are distinct. Partial fraction decomposition: then X(s) can be written as

$$X(s)=\frac{c_1}{s-p_1}+\ldots+\frac{c_n}{s-p_n}, \quad c_k=(s-p_k)X(s)\big|_{s=p_k}$$

Multiple Pole Case. Assume that D(s) has multiple roots: it contains factors of the form $(s\text{-}p_i)^r$. Then the partial fraction decomposition of X(s) consists of terms

$$\frac{\lambda_1}{s-p_i}+\frac{\lambda_2}{(s-p_i)^2}+\ldots+\frac{\lambda_r}{(s-p_i)^r}, \quad \lambda_{r-k}=\frac{1}{k!}\frac{d^k}{ds^k}\left[(s-p_i)^r X(s)\right]\Big|_{s=p_i}$$

Rational Laplace transform 2

$$X(s) = \frac{N(s)}{D(s)} = k\,\frac{(s-z_1)\ldots(s-z_m)}{(s-p_1)\ldots(s-p_n)}$$

Improper rational function: $m \geq n$

$$X(s) = \frac{N(s)}{D(s)} = Q(s) + \frac{R(s)}{D(s)}$$

$\frac{R(s)}{D(s)}$ is a proper rational function

The inverse Laplace transform of Q(s) can be computed by using the transformation pairs

$$\frac{d^k\delta(t)}{dt^k} \leftrightarrow s^k \quad k = 1,2,3,\ldots$$

Rational Laplace transform examples 1

$x(t)$	$X(s)$	ROC
$\delta(t)$	1	All s
$u(t)$	$\frac{1}{s}$	$\mathrm{Re}(s) > 0$
$-u(-t)$	$\frac{1}{s}$	$\mathrm{Re}(s) < 0$
$tu(t)$	$\frac{1}{s^2}$	$\mathrm{Re}(s) > 0$
$t^k u(t)$	$\frac{k!}{s^{k+1}}$	$\mathrm{Re}(s) > 0$
$e^{-at} u(t)$	$\frac{1}{s+a}$	$\mathrm{Re}(s) > -\mathrm{Re}(a)$
$-e^{-at} u(-t)$	$\frac{1}{s+a}$	$\mathrm{Re}(s) < -\mathrm{Re}(a)$

Find the inverse Laplace transforms of $X(s) = \dfrac{1}{s+1}$ $\quad \mathrm{ROC:}\ \mathrm{Re}(s) > -1$

From the above table: $x(t) = e^{-t} u(t)$

Rational Laplace transform examples 2

$x(t)$	$X(s)$	ROC
$\delta(t)$	1	All s
$u(t)$	$\frac{1}{s}$	$\mathrm{Re}(s) > 0$
$-u(-t)$	$\frac{1}{s}$	$\mathrm{Re}(s) < 0$
$tu(t)$	$\frac{1}{s^2}$	$\mathrm{Re}(s) > 0$
$t^k u(t)$	$\frac{k!}{s^{k+1}}$	$\mathrm{Re}(s) > 0$
$e^{-at} u(t)$	$\frac{1}{s+a}$	$\mathrm{Re}(s) > -\mathrm{Re}(a)$
$-e^{-at} u(-t)$	$\frac{1}{s+a}$	$\mathrm{Re}(s) < -\mathrm{Re}(a)$

Find the inverse Laplace transforms of $X(s) = \dfrac{1}{s+1}$ ROC: $\mathrm{Re}(s) < -1$

From the above table: $x(t) = -e^{-t}u(-t)$

Rational Laplace transform examples 3

$te^{-at}u(t)$	$\dfrac{1}{(s+a)^2}$	$\mathrm{Re}(s) > -\mathrm{Re}(a)$
$-te^{-at}u(-t)$	$\dfrac{1}{(s+a)^2}$	$\mathrm{Re}(s) < -\mathrm{Re}(a)$
$\cos\omega_0 t u(t)$	$\dfrac{s}{s^2+\omega_0^2}$	$\mathrm{Re}(s) > 0$
$\sin\omega_0 t u(t)$	$\dfrac{\omega_0}{s^2+\omega_0^2}$	$\mathrm{Re}(s) > 0$
$e^{-at}\cos\omega_0 t u(t)$	$\dfrac{s+a}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}(s) > -\mathrm{Re}(a)$
$e^{-at}\sin\omega_0 t u(t)$	$\dfrac{\omega_0}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}(s) > -\mathrm{Re}(a)$

Find the inverse Laplace transforms of $X(s) = \dfrac{s}{s^2+4}$ ROC: $\mathrm{Re}(s) > 0$

From the above table: $x(t) = \cos(2t)u(t)$

Rational Laplace transform examples 4a

Find the inverse Laplace transform of

$$X(s)=\frac{2s+4}{s^2+4s+3} \quad \mathrm{ROC:}\ \mathrm{Re}(s)>-1$$

$$X(s)=\frac{2s+4}{s^2+4s+3}=2\frac{s+2}{(s+1)(s+3)}=\frac{c_1}{s+1}+\frac{c_2}{s+3}$$

$$c_1=(s+1)X(s)\big|_{s=-1}=1 \quad c_2=(s+3)X(s)\big|_{s=-3}=1$$

$$X(s)=\frac{1}{s+1}+\frac{1}{s+3}$$

$$\boxed{\mathrm{ROC:}\ \mathrm{Re}(s)>-1 \quad \Rightarrow \quad x(t)=\left(e^{-t}+e^{-3t}\right)u(t)}$$

Rational Laplace transform examples 4b

Find the inverse Laplace transform of

$$X(s)=\frac{2s+4}{s^2+4s+3}\quad \mathrm{ROC:}\ \mathrm{Re}(s)<-3$$

$$X(s)=\frac{2s+4}{s^2+4s+3}=2\frac{s+2}{(s+1)(s+3)}=\frac{c_1}{s+1}+\frac{c_2}{s+3}$$

$$c_1=(s+1)X(s)\big|_{s=-1}=1\quad c_2=(s+3)X(s)\big|_{s=-3}=1$$

$$X(s)=\frac{1}{s+1}+\frac{1}{s+3}$$

$$\boxed{\mathrm{ROC:}\ \mathrm{Re}(s)<-3\quad\Rightarrow\quad x(t)=-\left(e^{-t}+e^{-3t}\right)u(-t)}$$

Rational Laplace transform examples 4c

Find the inverse Laplace transform of

$$X(s) = \frac{2s+4}{s^2+4s+3} \quad \mathrm{ROC}:\ -3 < \mathrm{Re}(s) < -1$$

$$X(s) = \frac{2s+4}{s^2+4s+3} = 2\frac{s+2}{(s+1)(s+3)} = \frac{c_1}{s+1} + \frac{c_2}{s+3}$$

$$c_1 = (s+1)X(s)\Big|_{s=-1} = 1 \quad c_2 = (s+3)X(s)\Big|_{s=-3} = 1$$

$$X(s) = \frac{1}{s+1} + \frac{1}{s+3}$$

$$\boxed{\mathrm{ROC}:\ -3 < \mathrm{Re}(s) < -1 \quad \Rightarrow \quad x(t) = -e^{-t}u(-t) + e^{-3t}u(t)}$$

Rational Laplace transform examples 5

Find the inverse Laplace transform of

$$X(s)=\frac{5s+13}{s\left(s^2+4s+13\right)} \quad \mathrm{ROC:}\ \mathrm{Re}(s)>0$$

$$X(s)=\frac{5s+13}{s(s+2-j3)(s+2+j3)}=\frac{c_1}{s}+\frac{c_2}{s-(-2+j3)}+\frac{c_3}{s-(-2-j3)}$$

$$c_1=s\,X(s)\big|_{s=0}=1 \quad c_2=(s+2-j3)X(s)\big|_{s=-2+j3}=-\frac{1}{2}(1+j)$$

$$c_3=(s+2+j3)X(s)\big|_{s=-2-j3}=-\frac{1}{2}(1-j)$$

$$X(s)=\frac{1}{s}-\frac{1}{2}\frac{1+j}{s-(-2+j3)}-\frac{1}{2}\frac{1-j}{s-(-2-j3)}$$

$$\boxed{\mathrm{ROC:}\ \mathrm{Re}(s)>0 \quad x(t)=u(t)-\frac{1+j}{2}e^{(-2+j3)t}u(t)-\frac{1-j}{2}e^{(-2-j3)t}u(t)}$$

Rational Laplace transform examples 5 continued

Find the inverse Laplace transform of

$$X(s) = \frac{5s+13}{s\left(s^2+4s+13\right)} \quad \text{ROC: } \operatorname{Re}(s) > 0$$

$$\boxed{\text{ROC: } \operatorname{Re}(s) > 0 \quad x(t) = u(t) - \frac{1+j}{2} e^{(-2+j3)t} u(t) - \frac{1-j}{2} e^{(-2-j3)t} u(t)}$$

$$e^{(-2\pm j3)t} = e^{-2t}\left(\cos 3t \pm j\sin 3t\right)$$

$$\boxed{x(t) = \left(1 - e^{-2t}\left[\cos 3t - \sin 3t\right]\right) u(t)}$$

Rational Laplace transform examples 6

Find the inverse Laplace transform of

$$X(s)=\frac{2s+1}{s+2} \quad \mathrm{ROC:}\ \mathrm{Re}(s)>-2$$

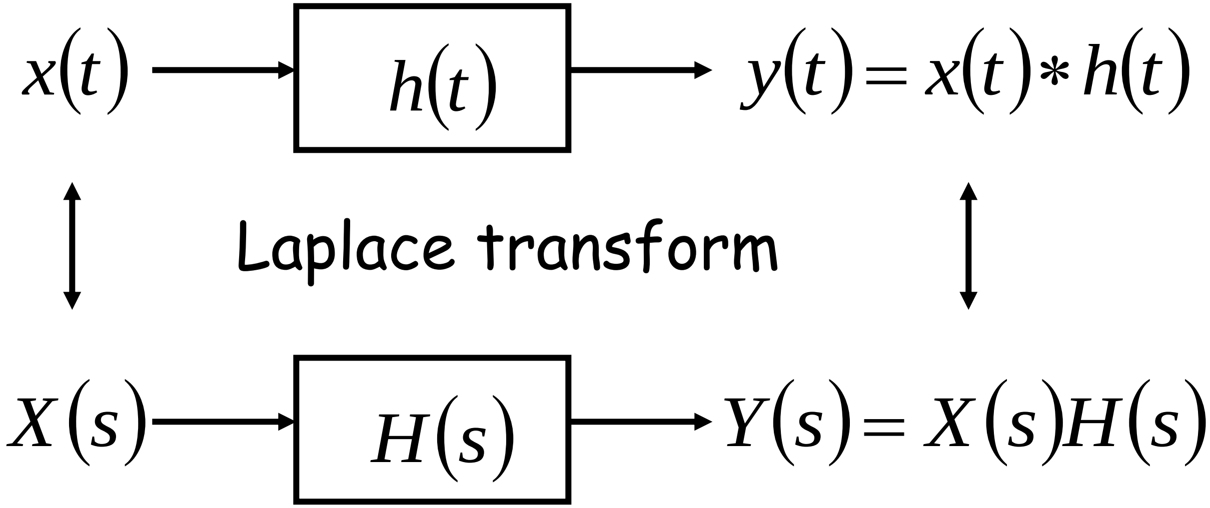
$$X(s)=\frac{2s+1}{s+2}=2-\frac{3}{s+2} \qquad x(t)=2\delta(t)-3e^{-2t}u(t)$$

Find the inverse Laplace transform of

$$X(s)=\frac{s^3+2s^2+6}{s^2+3s}=s-1+\frac{2}{s}+\frac{1}{s+3} \quad \mathrm{ROC:}\ \mathrm{Re}(s)>0$$

$$x(t)=\delta'(t)-\delta(t)+\left(2+e^{-3t}\right)u(t)$$

The System Function of a continuous-time LTI system



A **causal** system does not respond to an input event until that event actually occurs.

$$h(t) = 0 \quad t < 0$$

System Function examples 1

$$\begin{array}{ccccc} x(t) & \longrightarrow & \boxed{h(t)} & \longrightarrow & y(t)=x(t)*h(t) \\ \updownarrow & & \text{Laplace transform} & & \updownarrow \\ X(s) & \longrightarrow & \boxed{H(s)} & \longrightarrow & Y(s)=X(s)H(s) \end{array}$$

$$u(t) \rightarrow \boxed{h(t)} \rightarrow 2e^{-3t}u(t) \qquad h(t)=?$$

$$X(s)=\frac{1}{s} \quad \mathrm{Re}(s)>0 \qquad Y(s)=\frac{2}{s+3} \quad \mathrm{Re}(s)>-3$$

$$H(s)=\frac{Y(s)}{X(s)}=\frac{2s}{s+3}=2-\frac{6}{s+3} \quad \mathrm{Re}(s)>-3$$

$$h(t)=2\delta(t)-6e^{-3t}u(t)$$

Alternatively: $\delta(t) \rightarrow \boxed{h(t)} \rightarrow h(t) \qquad x'(t) \rightarrow \boxed{h(t)} \rightarrow y'(t)$

$$\delta(t)=\frac{d}{dt}u(t) \qquad \frac{d}{dt}\left(2e^{-3t}u(t)\right)=2\delta(t)-6e^{-3t}u(t)=h(t)$$

System Function examples 2

$$
\begin{tikzcd}
x(t) \arrow[r] \arrow[d, leftrightarrow] & \boxed{h(t)} \arrow[r] \arrow[d, phantom, "\text{Laplace transform}"] & y(t) = x(t) * h(t) \arrow[d, leftrightarrow] \\
X(s) \arrow[r] & \boxed{H(s)} \arrow[r] & Y(s) = X(s)H(s)
\end{tikzcd}
$$

$$u(t) \rightarrow \boxed{h(t)} \rightarrow 2e^{-3t}u(t) \qquad h(t) = 2\delta(t) - 6e^{-3t}u(t)$$

$$x(t) = e^{-t}u(t) \qquad y(t) = ?$$

$$X(s) = \frac{1}{s+1} \quad \mathrm{Re}(s) > -1 \qquad H(s) = \frac{2s}{s+3}$$

$$Y(s) = X(s)H(s) = \frac{1}{s+1}\frac{2s}{s+3} = -\frac{1}{s+1} + \frac{3}{s+3}$$

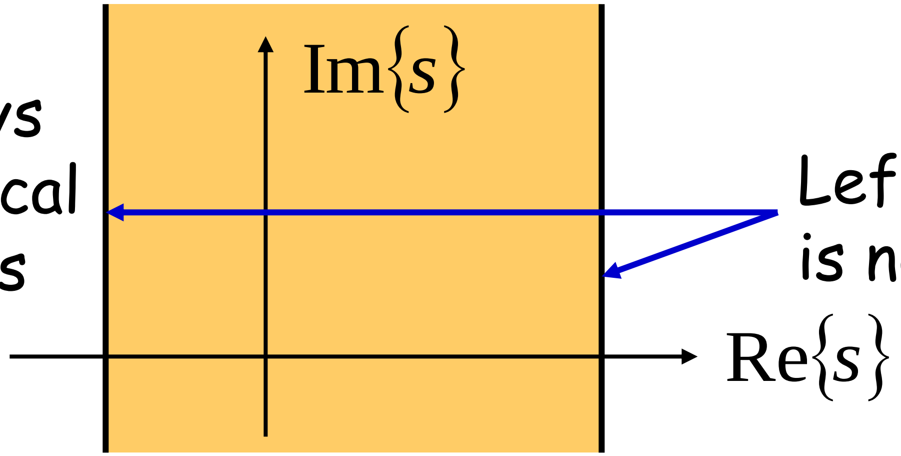
$$y(t) = \left(-e^{-t} + 3e^{-3t}\right)u(t)$$

Alternatively:

$$y(t) = h(t) * x(t) = \left(2\delta(t) - 6e^{-3t}u(t)\right) * \left(e^{-t}u(t)\right) = \ldots$$

Region of Convergence (ROC) and BIBO stability

ROC: always vertical strips



Left or right boundary is not always present

Stability (bounded input yields bounded output, **BIBO**):

$$y(t) = h(t) * x(t) = \int h(\tau)\, x(t-\tau)\, d\tau$$

$$|x(t)| \le M \quad \Rightarrow \quad |y(t)| \le M \int |h(\tau)|\, d\tau$$

$$\boxed{\int_{-\infty}^{\infty} |h(t)|\, dt < \infty}$$

Stability (bounded input yields bounded output, BIBO): Let h(t) be the impulse response. The system is stable iff the ROC of H(s) contains the imaginary axis in interior.

BIBO stability

$$\boxed{\int_{-\infty}^{\infty} \left|h(t)\right| dt < \infty}$$

$$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt \qquad s = j\omega$$

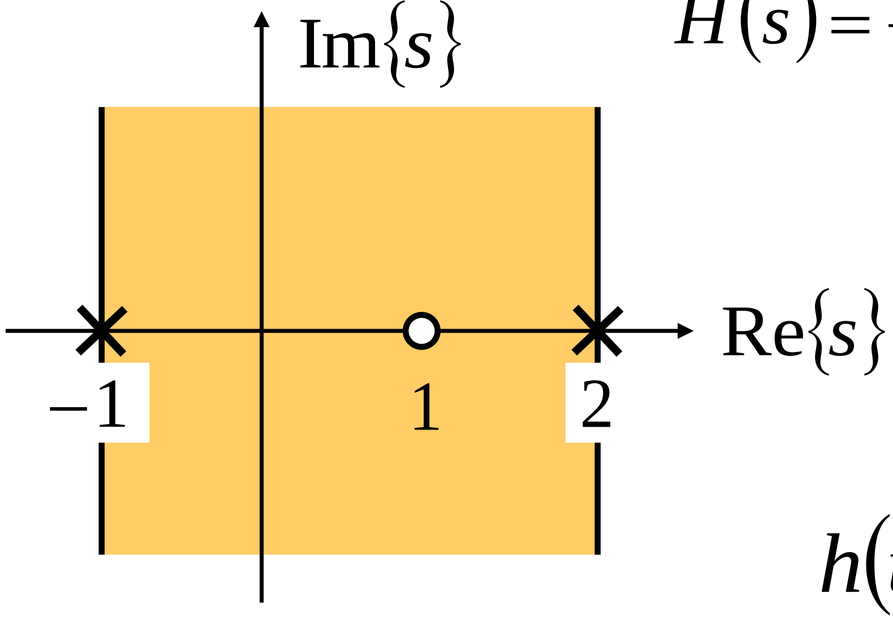
$$\left|H(j\omega)\right| = \left|\int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt\right| \le \int_{-\infty}^{\infty} \left|h(t) e^{-j\omega t}\right| dt = \int_{-\infty}^{\infty} \left|h(t)\right| dt < \infty$$

If the system is stable, then H(s) converges for s = jω. That is, for a stable continuous-time LTI system, the ROC of H(s) must contain the imaginary axis s = jω.

BIBO stability: An example

Consider a continuous-time LTI system whose system function is $H(s)=\dfrac{s-1}{(s+1)(s-2)}$

Let the system be BIBO stable. Find its ROC and the impulse response function h(t).



$$H(s)=\frac{s-1}{(s+1)(s-2)}=\frac{2}{3}\frac{1}{s+1}+\frac{1}{3}\frac{1}{s-2}$$

$$h(t)=\frac{2}{3}e^{-t}u(t)-\frac{1}{3}e^{2t}u(-t)$$

LTI Systems described by Linear Constant-Coefficient Differential Equations

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

$$\sum_{k=0}^{N} a_k s^k Y(s) = \sum_{k=0}^{M} b_k s^k X(s)$$

$$Y(s)\sum_{k=0}^{N} a_k s^k = X(s)\sum_{k=0}^{M} b_k s^k \qquad H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

Hence, H(s) is always rational. H(s) is called system equation (also called transfer function). Note that the ROC of H(s) is not specified by the differential equation but must be inferred with additional requirements on the system such as the causality or the stability.

LTI Systems described by Linear Constant-Coefficient Differential Equations: An example

Consider a continuous-time LTI system for which the input $x(t)$ and output $y(t)$ are related by

$$y''(t)+y'(t)-2y(t)=x(t)$$

(a) Find the system function $H(s)$.

(b) Determine the impulse response $h(t)$ for each of the following three cases: (i) the system is casual, (ii) the system is stable, (iii) the system is neither casual nor stable.

(a) We have $$s^2Y(s)+sY(s)-2Y(s)=X(s)$$

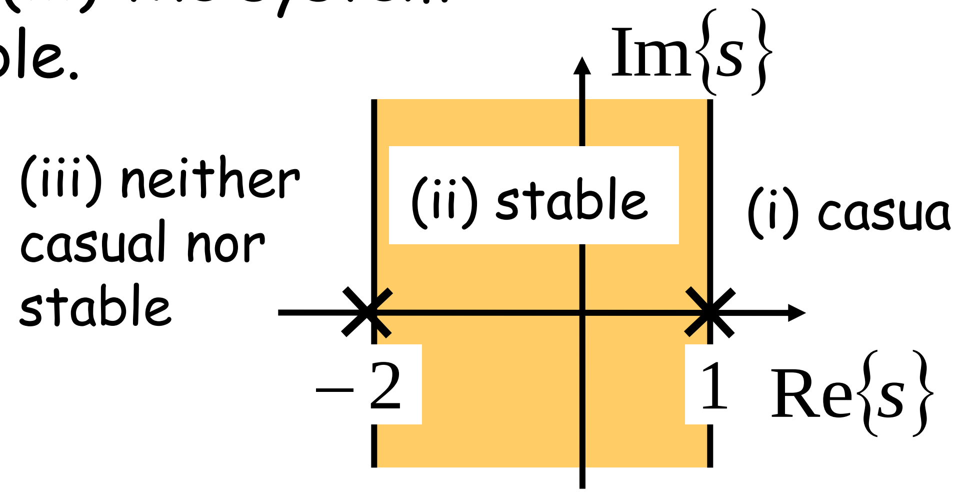
$$H(s)=\frac{Y(s)}{X(s)}=\frac{1}{s^2+s-2}=\frac{1}{(s+2)(s-1)}$$

LTI Systems described by Linear Constant-Coefficient Differential Equations: An example

Consider a continuous-time LTI system for which the input $x(t)$ and output $y(t)$ are related by $y''(t) + y'(t) - 2y(t) = x(t)$

(b) Determine the impulse response $h(t)$ for each of the following three cases: (i) the system is casual, (ii) the system is stable, (iii) the system is neither casual nor stable.

$$H(s) = -\frac{1}{3}\frac{1}{s+2} + \frac{1}{3}\frac{1}{s-1}$$



$$(i)\ \ h(t) = -\frac{1}{3}\left(e^{-2t} - e^{t}\right)u(t) \qquad (ii)\ \ h(t) = -\frac{1}{3}e^{-2t}u(t) - \frac{1}{3}e^{t}u(-t)$$

$$(iii)\ \ h(t) = \frac{1}{3}e^{-2t}u(-t) - \frac{1}{3}e^{t}u(-t)$$

Causality and stability

$$X(s)=\frac{N(s)}{D(s)}=k\,\frac{(s-z_1)\ldots(s-z_m)}{(s-p_1)\ldots(s-p_n)}$$

A rational Laplace transform is both causal and stable if all poles are in the left half-plane (the poles have negative real parts)