LTI Systems and Filtering

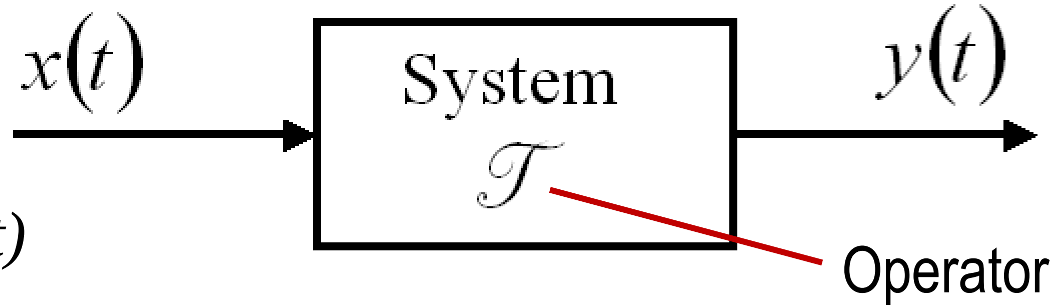
Changhai Wang

c.wang@hw.ac.uk

Linear Time-Invariant (LTI) Systems

$$y(t) = \mathrm{T}\left[x(t)\right]$$

Mapping of *x(t)* to *y(t)*



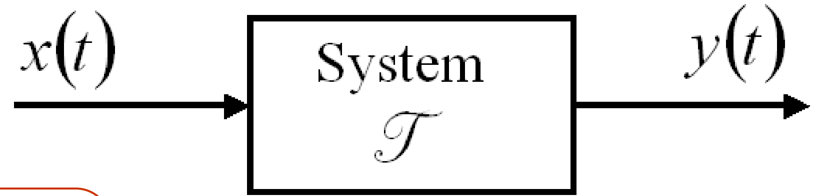
$$\mathrm{T}\left[x_1(t) + x_2(t)\right] = \mathrm{T}\left[x_1(t)\right] + \mathrm{T}\left[x_2(t)\right] = y_1(t) + y_2(t)$$

$$\mathrm{T}\left[\alpha\, x(t)\right] = \alpha\, \mathrm{T}\left[x(t)\right] = \alpha\, y(t)$$

$$\mathrm{T}\left[x(t - t_0)\right] = y(t - t_0)$$

Impulse response and response to an arbitrary input

$$y(t) = \mathrm{T}\,[x(t)]$$



$$h(t) = \mathrm{T}\,[\delta(t)]$$ *impulse response*

If $h(t) = 0$ for $t < 0$ the system is called ***causal***.

$$x(t) = \delta(t) * x(t) = \int_{-\infty}^{\infty} \delta(t-\tau)\,x(\tau)\,d\tau = \int_{-\infty}^{\infty} \delta(\tau)\,x(t-\tau)\,d\tau$$

$$\mathrm{T}\,[x(t)] = \int_{-\infty}^{\infty} \mathrm{T}\,[\delta(t-\tau)]\,x(\tau)\,d\tau = \int_{-\infty}^{\infty} h(t-\tau)\,x(\tau)\,d\tau = h(t) * x(t)$$

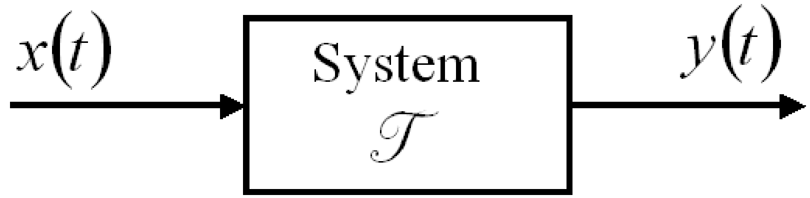
The response $y(t)$ of an LTI system to an arbitrary input $x(t)$ is given by

$$\boxed{\mathrm{y(t)} = h(t) * x(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau}$$

Commutative property of convolution

Frequency response

$$y(t) = \mathsf{T}\left[x(t)\right]$$



$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} x(t-\tau)h(t)\,d\tau$$

Let us apply the Fourier transform

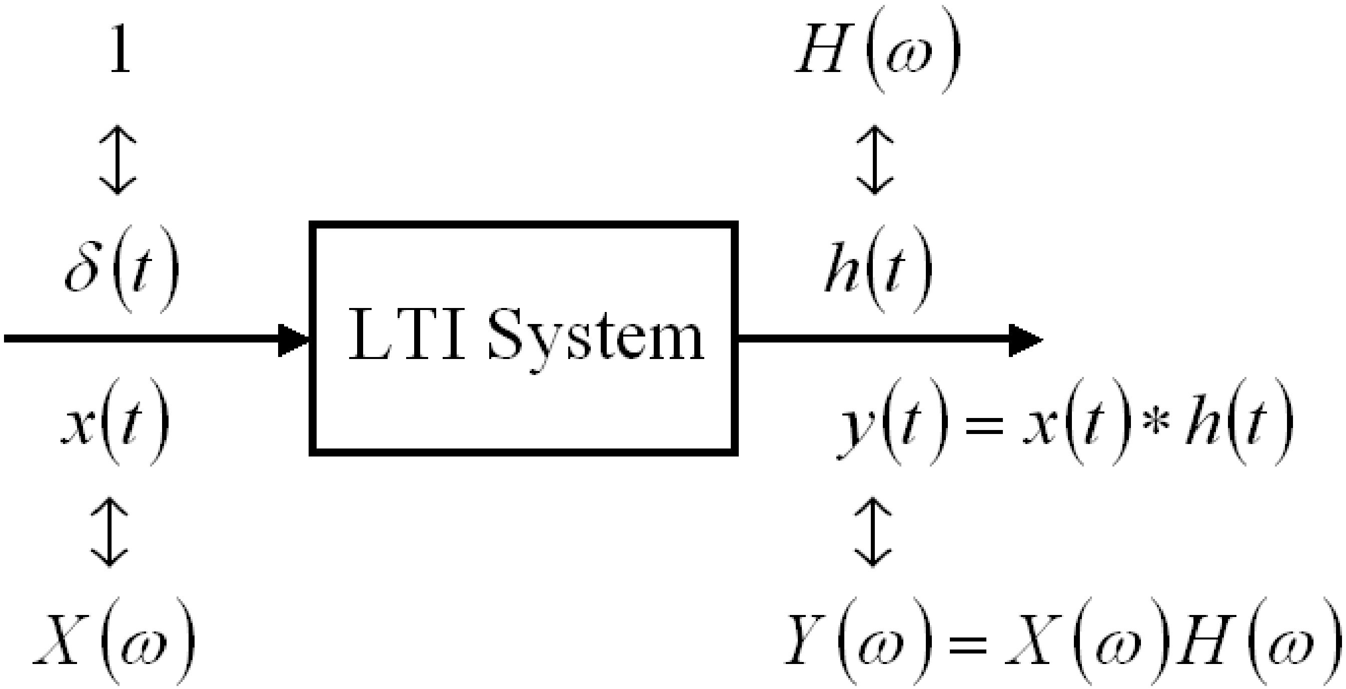
$$X(\omega) = \mathsf{F}\left[x(t)\right] \quad H(\omega) = \mathsf{F}\left[h(t)\right] \quad Y(\omega) = \mathsf{F}\left[y(t)\right]$$

$$y(t) = h(t) * x(t) \quad \rightarrow \quad Y(\omega) = X(\omega)H(\omega)$$

$$\boxed{H(\omega) = \frac{Y(\omega)}{X(\omega)}}$$

frequency response (or *transfer function*)

Frequency response



$$\mathrm{T}\left[e^{j\omega t}\right] = \int_{-\infty}^{\infty} h(\tau) e^{j\omega(t-\tau)} d\tau = \left[\int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau\right] e^{j\omega t} = H(\omega) e^{j\omega t}$$

Complex exponential

$$\boxed{\mathrm{T}\left[e^{j\omega t}\right] = H(\omega) e^{j\omega t}}$$

Filter characteristics of LTI systems

$$y(t) = \mathrm{T}\left[x(t)\right] \qquad y(t) = h(t) * x(t)$$

$$h(t) \leftrightarrow H(\omega)$$

$$\mathrm{T}\left[e^{j\omega t}\right] = H(\omega)e^{j\omega t}$$

$H(\omega)$ is called the frequency response

$$H(\omega) = \left|H(\omega)\right| e^{j\theta_h(\omega)}$$

If impulse response $h(t)$ is real-valued, then

$$H(-\omega) = H^*(\omega)$$

which means

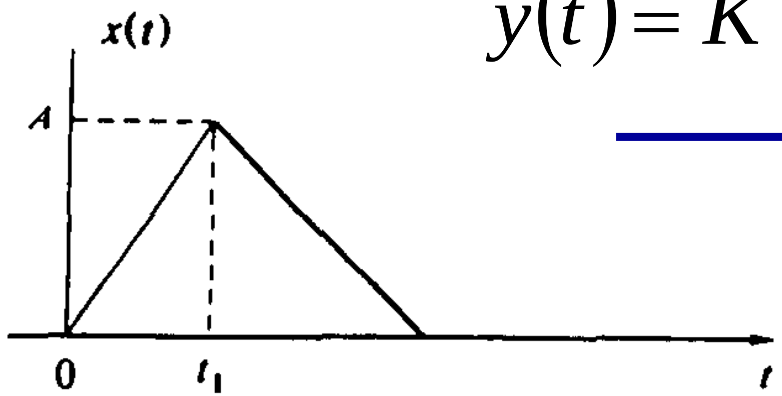
$$\left|H(-\omega)\right| = \left|H(\omega)\right| \qquad \theta_h(-\omega) = -\theta_h(\omega)$$

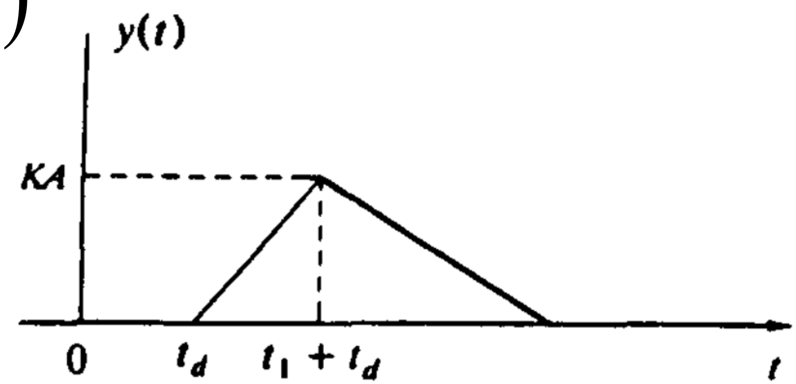
an even function an odd function

Transmission of signals through LTI systems

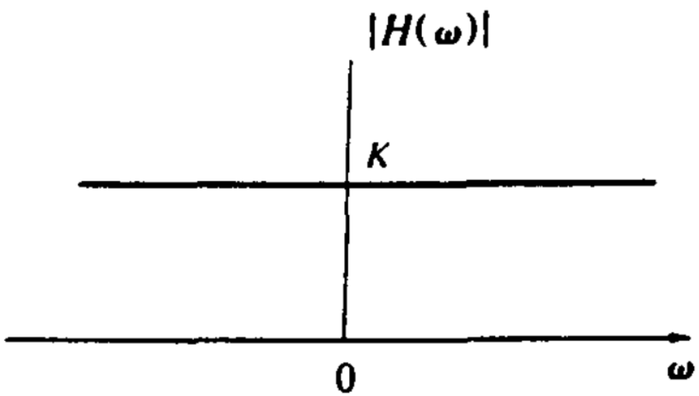
For distortionless transmission through a system, we require the exact input signal shape be reproduced at the output

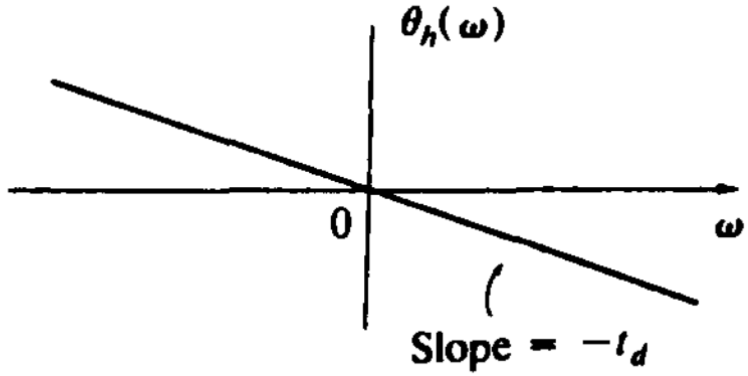
$$y(t) = K\,x(t - t_d)$$





$$Y(\omega) = K\,e^{-j\omega t_d}\,X(\omega) \qquad H(\omega) = K\,e^{-j\omega t_d}$$





Filters

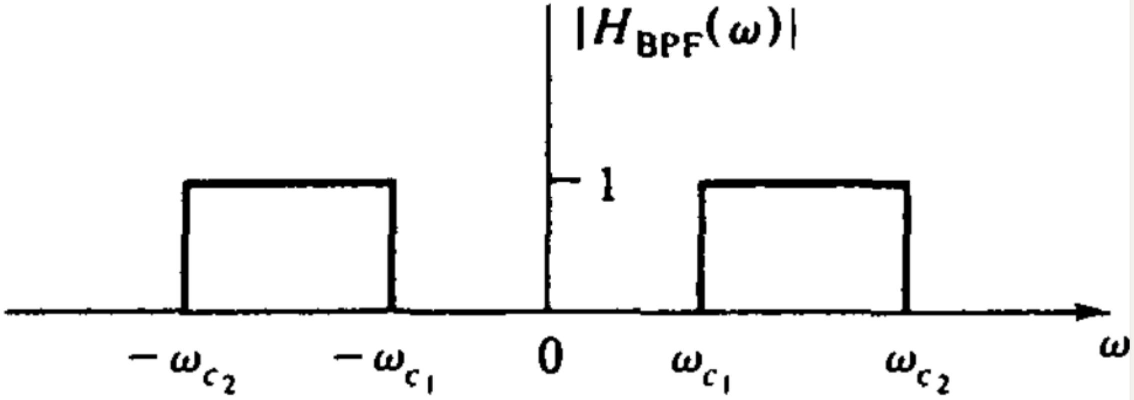
An **ideal filter** implements distortionless transmission over one or more specified frequency bands and has zero response at all other frequencies.

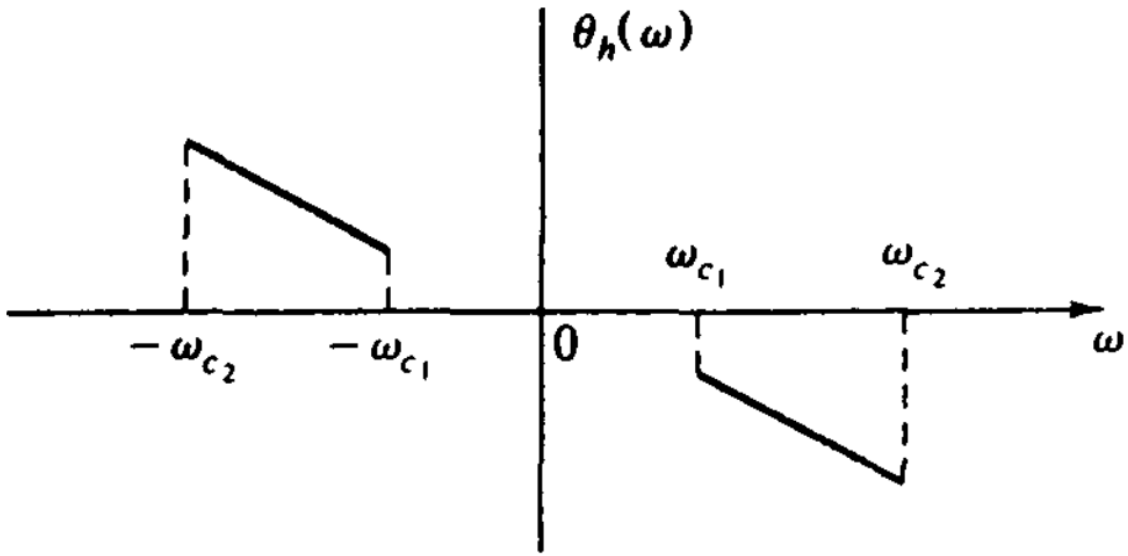
An **ideal bandpass filter** (BPF)

$$H_{\mathrm{BPF}}(\omega)=\begin{cases} e^{-j\omega t_d} & \text{for} \quad \omega_{c_1} \le |\omega| \le \omega_{c_2} \\ 0 & \text{otherwise} \end{cases}$$

ideal BPF

$$H_{\mathrm{BPF}}(\omega)=\begin{cases} e^{-j\omega t_d} & \text{for} \quad \omega_{c_1} \le |\omega| \le \omega_{c_2} \\ 0 & \text{otherwise} \end{cases}$$





Examples & Problems 1

$$y(t) = \mathrm{T}\left[x(t)\right]$$

For each of the following systems, determine whether the system is linear.

$$(1) \quad \mathrm{T}\left[x(t)\right] = x(t)\cos\omega_c t$$

$$(2) \quad \mathrm{T}\left[x(t)\right] = \left[A + x(t)\right]\cos\omega_c t$$

Examples & Problems 1

$$y(t) = \mathrm{T}\left[x(t)\right]$$

For each of the following systems, determine whether the system is linear.

$$(1) \quad \mathrm{T}\left[x(t)\right] = x(t)\cos\omega_c t$$

$$
\begin{aligned}
T[x_1(t) + x_2(t)] &= [x_1(t) + x_2(t)]\cos\omega_c t \\
&= [x_1(t)\cos\omega_c t + x_2(t)\cos\omega_c t\ ] \\
&= T[x_1(t)] + T[x_2(t)]
\end{aligned}
$$

$$T[\alpha x(t)] = [\alpha x(t)]\cos\omega_c t = \alpha\, T[x(t)]$$

This is a linear system

$$(2) \quad \mathrm{T}\left[x(t)\right] = \left[A + x(t)\right]\cos\omega_c t$$

$$T[x_1(t) + x_2(t)] = [A + x_1(t) + x_2(t)]\cos\omega_c t$$
$$\neq T[x_1(t)] + T[x_2(t)]$$

Since

$$T[x_1(t)] + T[x_2(t)] = [A + x_1(t)]\cos\omega_c t + [A + x_2(t)]\cos\omega_c t$$
$$= [2A + x_1(t) + x_2(t)]\cos\omega_c t$$

This is not a linear system

Examples & Problems 2

Consider the following system:

$$\text{T}\left[x(t)\right] = y(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t - nT)$$

Ideal sampler

Is this system linear?

Let $x(t) = x_1(t) + x_1(t)$

$$y(t) = [x_1(t) + x_2(t)]\delta_T(t) = x_1(t)\delta_T(t) + x_2(t)\delta_T(t)$$
$$= y_1(t) + y_2(t)$$

Let $x(t) = \alpha x_1(t)$, $y_1(t) = T[x_1(t)] = x_1(t)\,\delta_T(t)$

Then $y(t) = [\alpha x_1(t)]\delta_T(t) = \alpha[x_1(t)]\delta_T(t)] = \alpha y_1(t)$

So it is a linear system

Is this system time-invariant?

Check it for

$$x_1(t) = \cos\frac{2\pi t}{T} \quad \text{and} \quad x_2(t) = x_1\left(t - \frac{T}{4}\right) = \sin\frac{2\pi t}{T}$$

Examples & Problems 2

$$\mathrm{T}\left[x(t)\right] = y(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT) = \sum_{n=-\infty}^{\infty}x(nT)\,\delta(t-nT)$$

$$x_1(t) = \cos\frac{2\pi t}{T} \quad \text{and} \quad x_2(t) = x_1\left(t - \frac{T}{4}\right) = \sin\frac{2\pi t}{T}$$

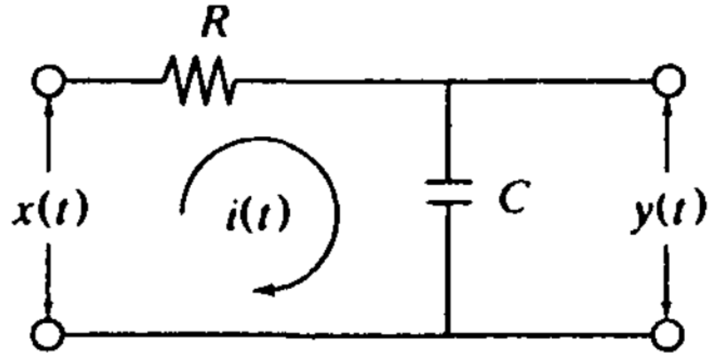
$$y_1(t) = \sum_{n=-\infty}^{\infty}\cos\frac{2\pi nT}{T}\,\delta(t-nT) = \sum_{n=-\infty}^{\infty}\delta(t-nT)$$

$$y_2(t) = \sum_{n=-\infty}^{\infty}\sin\frac{2\pi nT}{T}\,\delta(t-nT) = 0 \neq y_1\left(t - \tfrac{T}{4}\right)$$

The system is NOT time-invariant

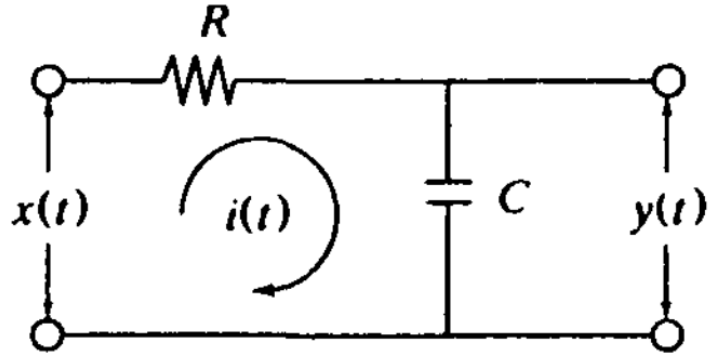
Examples & Problems 3

Find frequency response $H(\omega)$ and impulse response $h(t)$



Examples & Problems 3

Find frequency response $H(\omega)$ and impulse response $h(t)$



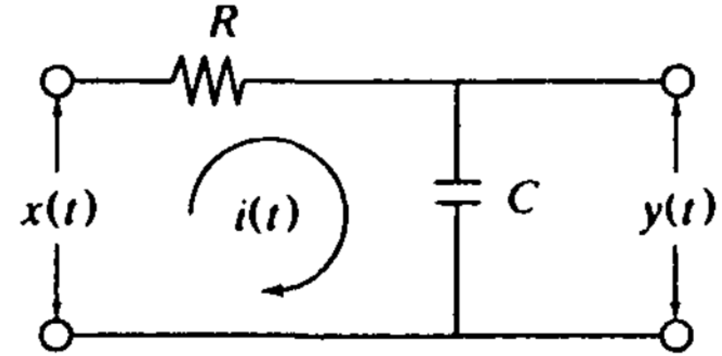
Solution. $\mathrm{T}\left[e^{j\omega t}\right] = H(\omega)e^{j\omega t}$

$$V_m \cos(\omega t + \varphi) = \mathrm{Re}\left[V_m e^{j\varphi} e^{j\omega t}\right] \qquad \mathbf{V} = V_m e^{j\varphi} \text{ phasor}$$

$$H(\omega) = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{1 + j\omega RC}$$

Now inverse Fourier transform of $H(\omega)$ yields $h(t)$.

Examples & Problems 3



Find frequency response $H(\omega)$ and impulse response $h(t)$

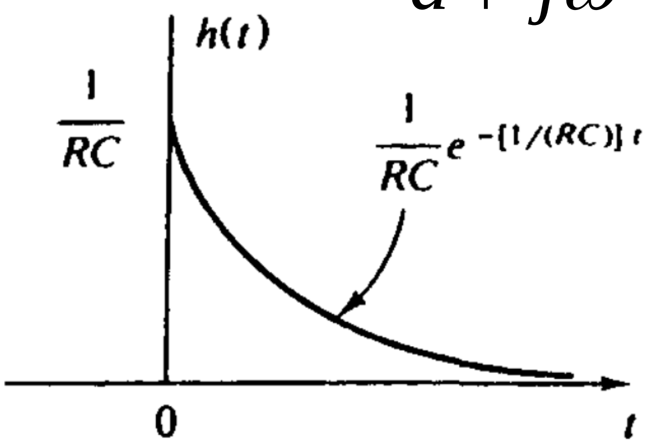
$$H(\omega) = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{1 + j\omega RC}$$

Let $\quad h(t) = ae^{-at}u(t) \quad$ Then

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt = a\int_{0}^{\infty} e^{-at}e^{-j\omega t}dt = \frac{a}{a + j\omega}$$

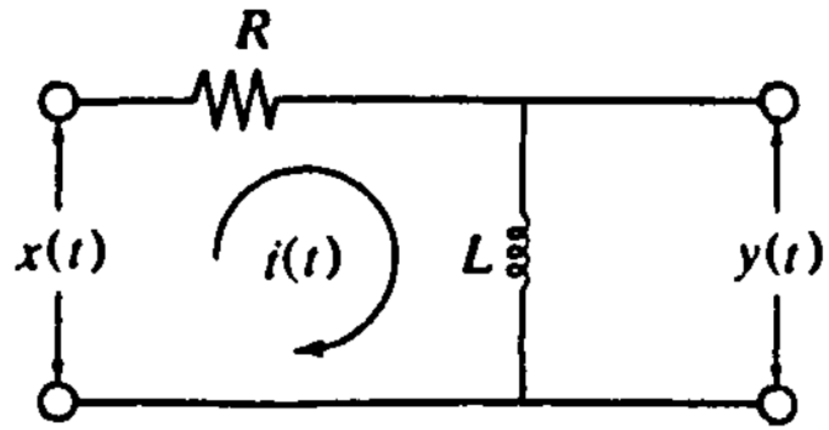
a = 1/RC

$$h(t) = \frac{1}{RC}e^{-t/(RC)}u(t)$$



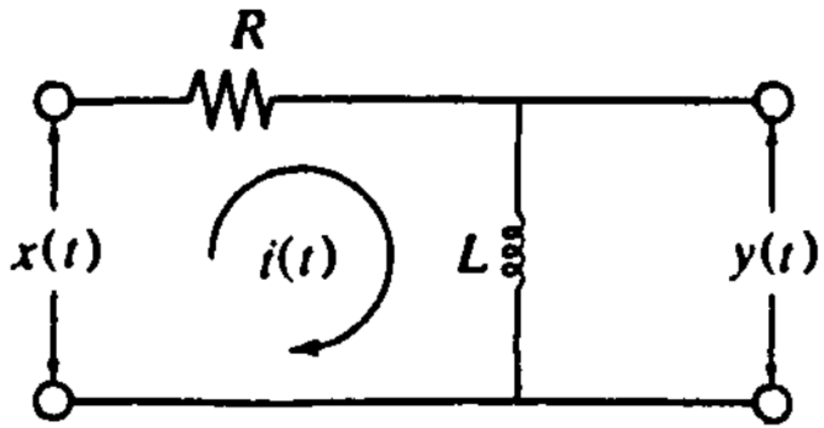
Examples & Problems 4

Show that this *RL* network is a high-pass filter.



Examples & Problems 4

Show that this *RL* network is a high-pass filter.

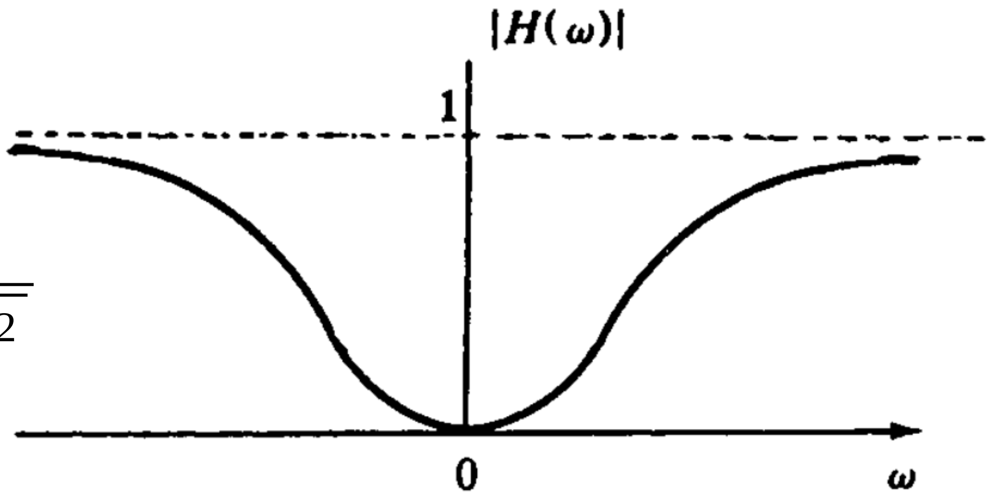


Solution.

$$H(\omega) = \frac{j\omega L}{R + j\omega L} = \frac{j(\omega/\omega_0)}{1 + j(\omega/\omega_0)}, \quad \omega_0 = R/L$$

$$|H(\omega)| = \left| \frac{j(\omega/\omega_0)}{1 + j(\omega/\omega_0)} \right|$$

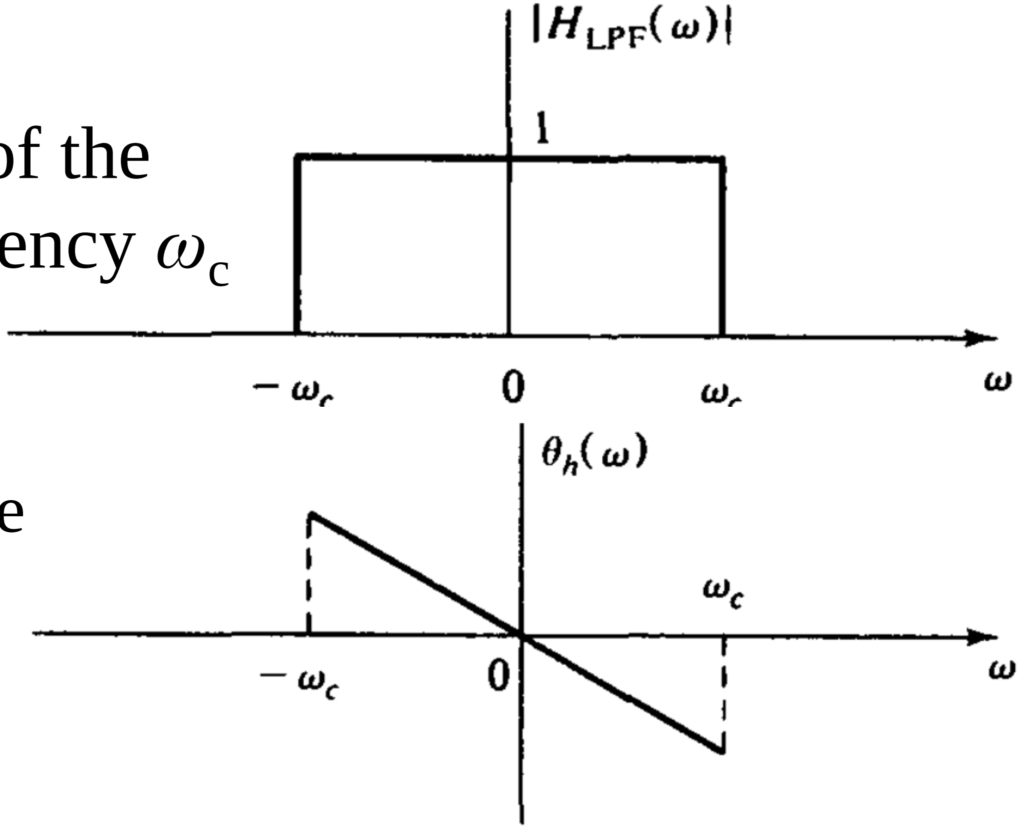
$$= \frac{|j(\omega/\omega_0)|}{|1 + j(\omega/\omega_0)|} = \frac{(\omega/\omega_0)}{\sqrt{1 + (\omega/\omega_0)^2}}$$



Examples & Problems 5

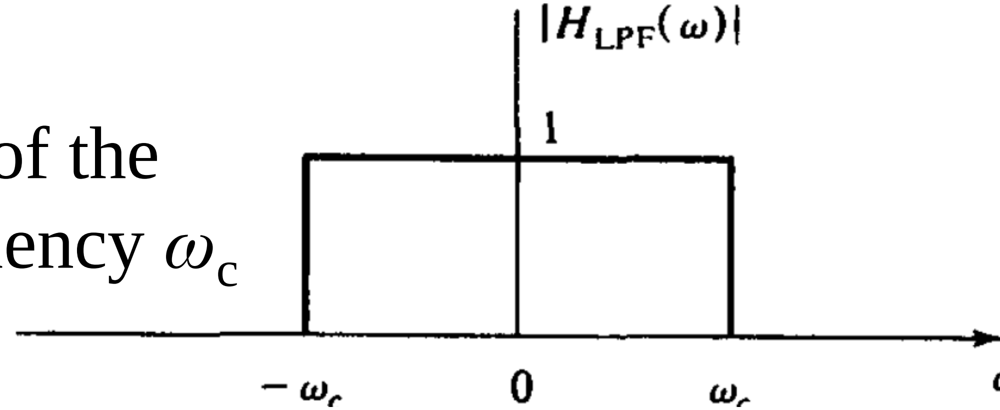
Find the impulse response of the ideal LPF with cutoff frequency ω_c

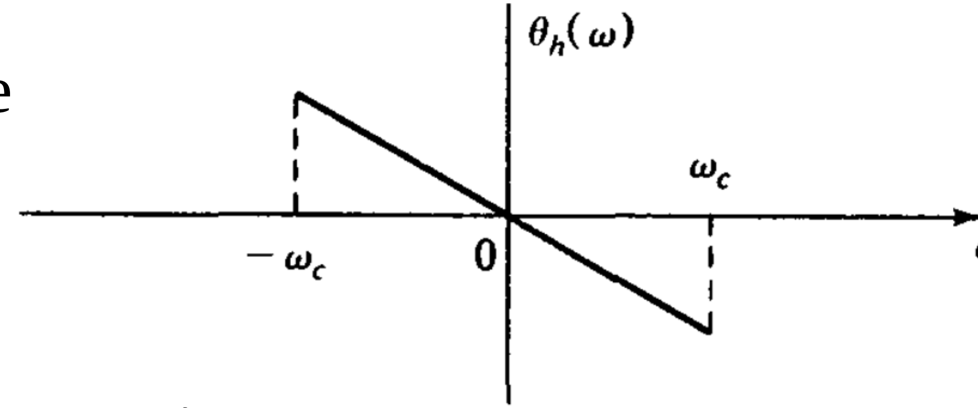
$$H_{\text{LPF}}(\omega)=\begin{cases} e^{-j\omega t_d} & |\omega|\le\omega_c \\ 0 & \text{otherwise}\end{cases}$$



Examples & Problems 5

Find the impulse response of the ideal LPF with cutoff frequency ω_c





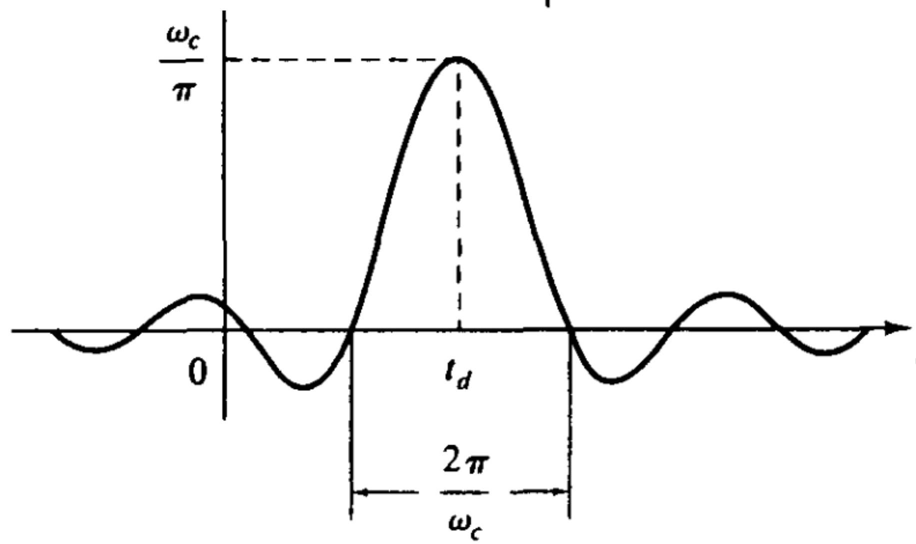
$$H_{\text{LPF}}(\omega) = \begin{cases} e^{-j\omega t_d} & |\omega| \le \omega_c \\ 0 & \text{otherwise} \end{cases}$$

Solution.

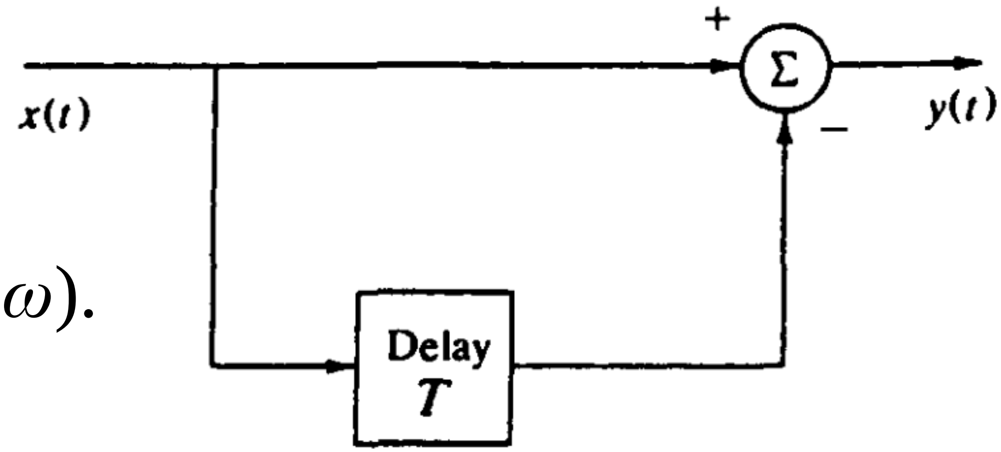
$$H(\omega) = \mathcal{F}\left[h(t)\right]$$

$$h(t) = \mathcal{F}^{-1}\left[H(\omega)\right]$$

$$h_{\text{LPF}}(t) = \frac{\sin \omega_c (t - t_d)}{\pi (t - t_d)}$$



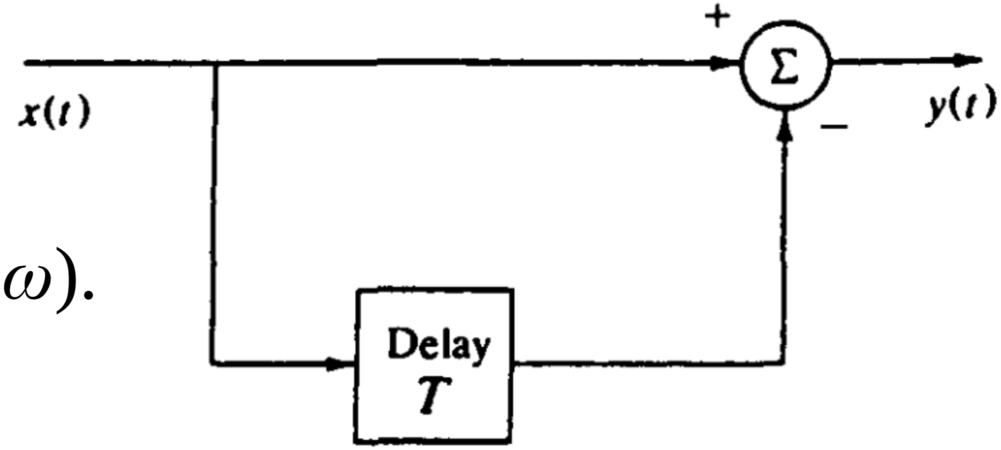
Examples & Problems 6



Find impulse response $h(t)$
and frequency response $H(\omega)$.

$$y(t) = x(t) - x(t-T)$$

Examples & Problems 6



Find impulse response $h(t)$ and frequency response $H(\omega)$.

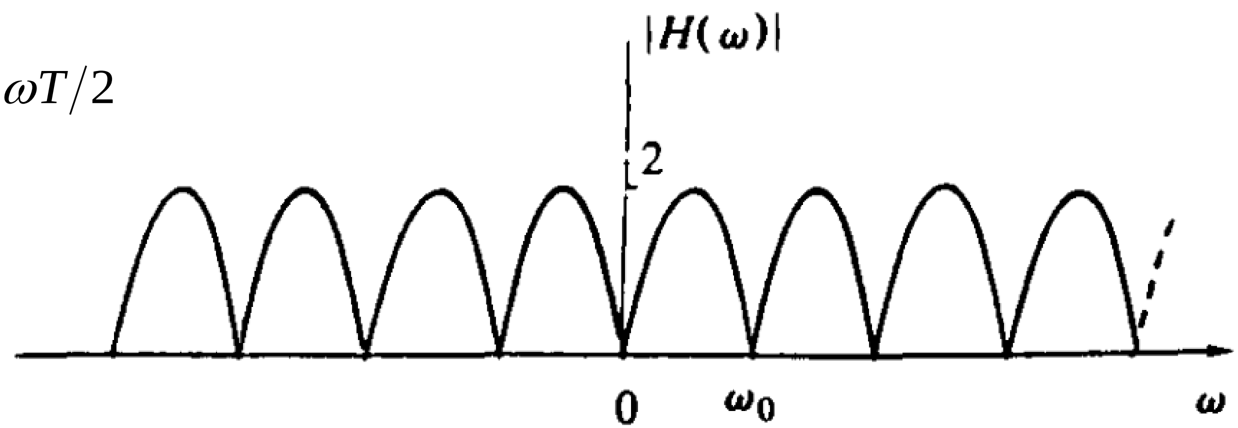
$$y(t) = x(t) - x(t-T)$$

<u>Solution.</u>

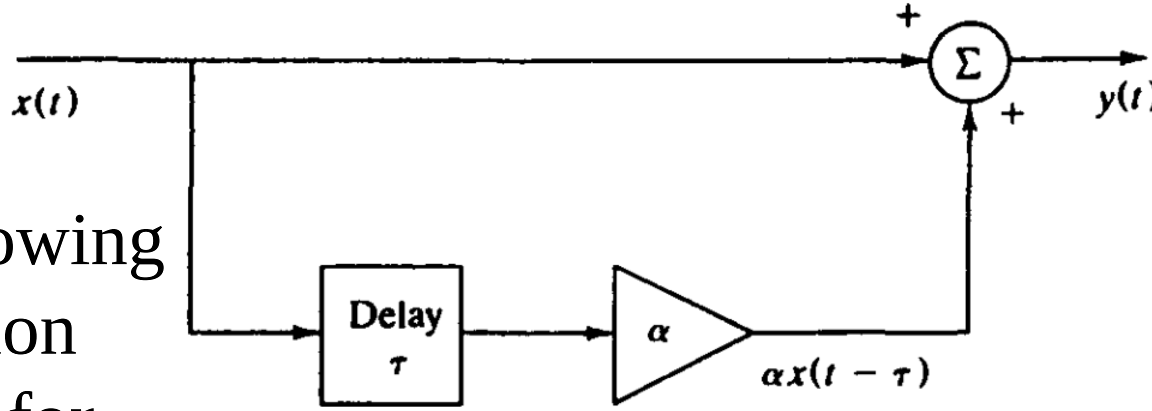
$$h(t) = \delta(t) - \delta(t-T)$$

$$H(\omega) = \mathcal{F}\left[h(t)\right] = 1 - e^{-j\omega T} = e^{-j\omega T/2}\left(e^{j\omega T/2} - e^{-j\omega T/2}\right)$$

$$= 2\sin\left(\frac{\omega T}{2}\right)e^{-j\omega T/2}$$



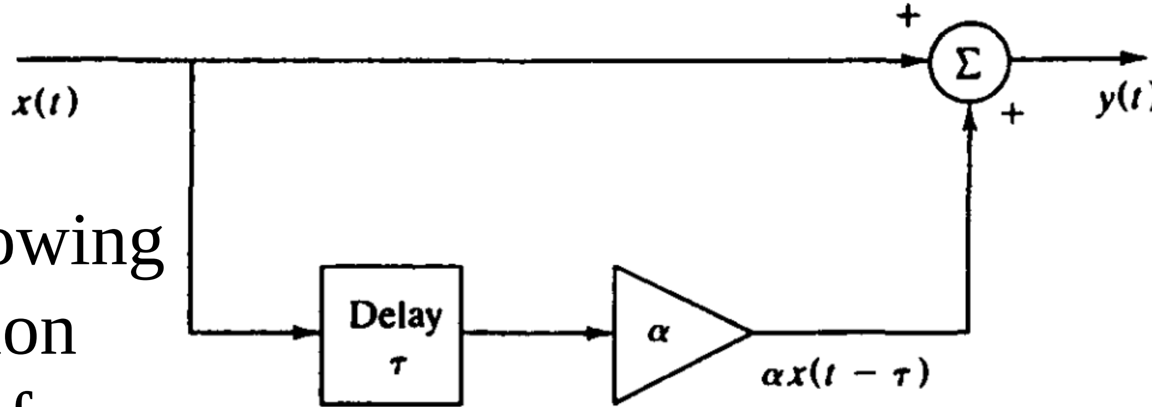
Examples & Problems 7



Find $H(\omega)$ for the following multipath communication cannel and plot $|H(\omega)|$ for α=1 and α= 0.5.

Examples & Problems 7

Find $H(\omega)$ for the following multipath communication cannel and plot $|H(\omega)|$ for α=1 and α= 0.5.



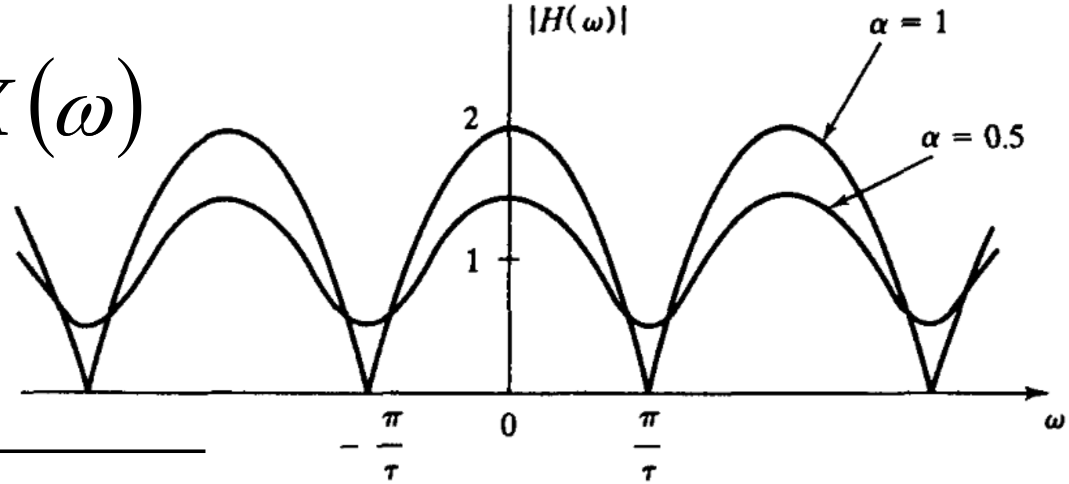
$$H(\omega)=Y(\omega)/X(\omega)$$

$$H(\omega)=1+\alpha e^{-j\omega\tau}$$

Solution.

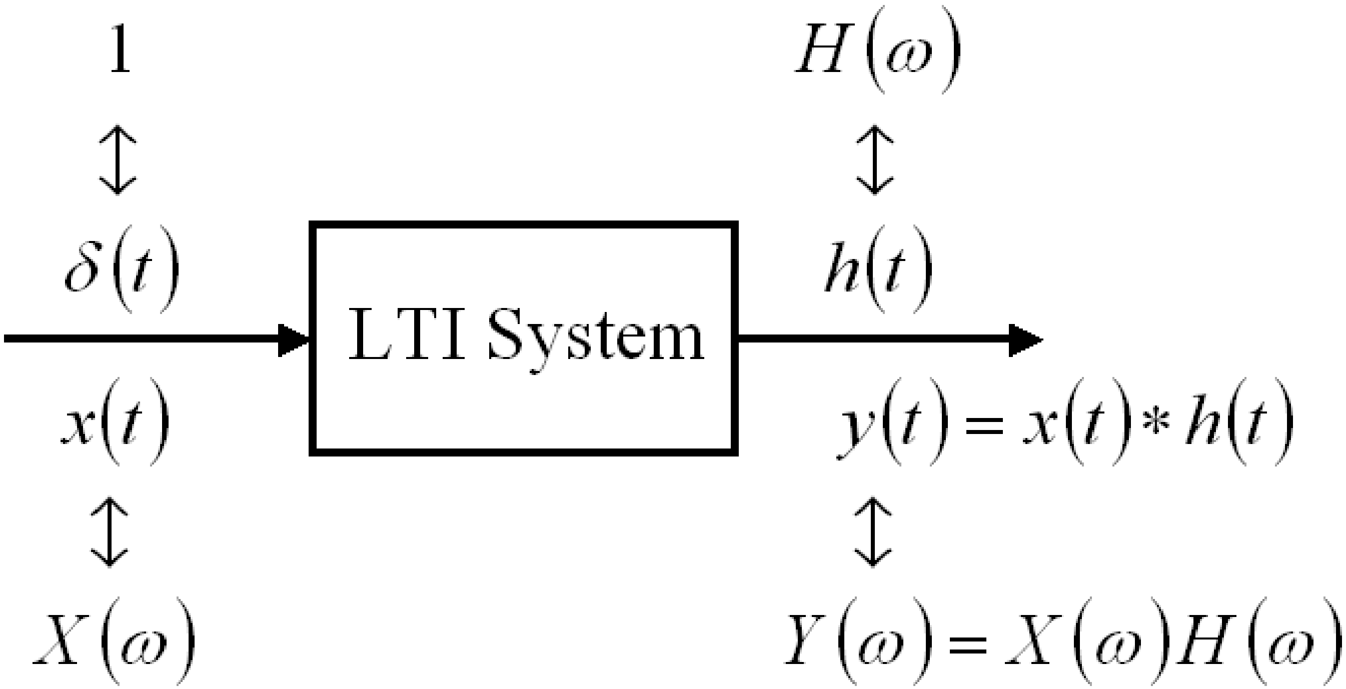
$$y(t)=x(t)+\alpha\, x(t-\tau)$$

$$Y(\omega)=X(\omega)+\alpha\, e^{-j\omega\tau}X(\omega)$$



$$|H(\omega)|=\sqrt{1+\alpha^2+2\alpha\cos\omega\tau}$$

Frequency response



$$\mathrm{T}\left[e^{j\omega t}\right] = \int_{-\infty}^{\infty} h(\tau) e^{j\omega(t-\tau)} d\tau = \left[\int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau\right] e^{j\omega t} = H(\omega) e^{j\omega t}$$

$$\boxed{\mathrm{T}\left[e^{j\omega t}\right] = H(\omega) e^{j\omega t}}$$