Signals and Spectra

Fourier transforms and continuous spectra

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Fourier transforms

Forward transform, analysis equation

$$\boxed{X(\omega) = \mathcal{F}\left[x(t)\right] = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt}$$

Inverse transform, synthesis equation

$$\boxed{x(t) = \mathcal{F}^{-1}\left[X(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega}$$

Fourier transform pair

$$x(t) \leftrightarrow X(\omega)$$

Fourier transforms

Sometimes it will be more convenient to express the Fourier transform of a signal in terms of the frequency *f* in Hz rather than the radian frequency ω in rad/s.

$$\boxed{X(f) = \mathcal{F}\left[x(t)\right] = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt}$$

$$\boxed{x(t) = \mathcal{F}^{-1}\left[X(f)\right] = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df}$$

$$d\omega = 2\pi\, df$$

Frequency spectra

$$X(\omega) = \mathcal{F}[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

In general, the Fourier transform $X(\omega)$ is a complex function of angular velocity ω

$$X(\omega) = |X(\omega)|e^{j\theta(\omega)}$$

$|X(\omega)|$ is called the ***continuous amplitude spectrum*** of $x(t)$

$\theta(\omega)$ is called the ***continuous phase spectrum*** of $x(t)$

If $x(t)$ is a real function of time, then

$$X(-\omega) = X^*(\omega) = |X(\omega)|e^{-j\theta(\omega)}$$

$$|X(-\omega)| = |X(\omega)|$$

an even function of ω

$$\theta(-\omega) = -\theta(\omega)$$

an odd function of ω

Energy content of a signal

The normalized energy content E of a signal $x(t)$

$$\boldsymbol{E} = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

If $\boldsymbol{E}$ is finite, then $x(t)$ is called an ***energy signal***.

For example, $x(t) = Ae^{-a|t|}$ is an energy signal.

If $\boldsymbol{E}$ is infinite, we define the normalized average power $\boldsymbol{P}$

$$\boldsymbol{P} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt$$

If $\boldsymbol{P}$ is finite, then $x(t)$ is referred as a ***power signal***.

For example, $x(t) = A\sin(\omega t)$ is a power signal.

A periodic signal is a power signal if its energy per period is finite.

Parseval's theorem

If $x(t)$ is an energy signal, then

$$\boxed{\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega}$$

Basic properties of Fourier transform

Linearity: $\boxed{a_1x_1(t)+a_2x_2(t)\leftrightarrow a_1X_1(\omega)+a_2X_2(\omega)}$

Time Shifting: $\boxed{x(t-t_0)\leftrightarrow X(\omega)e^{-j\omega t_0}}$

Frequency Shifting: $x(t)e^{j\omega_0 t}\leftrightarrow X(\omega-\omega_0)$

Scaling: $x(at)\leftrightarrow \dfrac{1}{|a|}X\left(\dfrac{\omega}{a}\right)$

Time-Reversal: $x(-t)\leftrightarrow X(-\omega)$

Duality: $X(t)\leftrightarrow 2\pi\, x(-\omega)$

Derivations of some properties of Fourier transform

Frequency Shifting: $x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0)$

$$\mathsf{F}\left[x(t)\,e^{j\omega_0 t}\right] = \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}dt = X(\omega - \omega_0)$$

Time-Reversal: $x(-t) \leftrightarrow X(-\omega)$

$$\mathsf{F}\left[x(-t)\right] = \int_{-\infty}^{\infty} x(-t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(\lambda)e^{j\omega\lambda}d\lambda = \int_{-\infty}^{\infty} x(\lambda)e^{-j(-\omega)\lambda}d\lambda$$

$$= X(-\omega)$$

If $x(t)$ is **real**, then $X(-\omega) = X^*(\omega)$

$$X^*(\omega) = \left(\int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}dt\right)^* = \int_{-\infty}^{\infty} x^*(t)\,e^{j\omega t}dt = \int_{-\infty}^{\infty} x(t)\,e^{j\omega t}dt = X(-\omega)$$

Derivations of some properties of Fourier transform

Duality: $X(t) \leftrightarrow 2\pi\, x(-\omega)$

$$\mathrm{F}^{-1}[X(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} dt = x(t)$$

$$\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} dt = 2\pi\, x(t)$$

$$t \rightarrow -t \quad \Rightarrow \quad \int_{-\infty}^{\infty} X(\omega) e^{-j\omega t} dt = 2\pi\, x(-t)$$

$$t \leftrightarrow \omega \quad \Rightarrow \quad \int_{-\infty}^{\infty} X(t) e^{-j\omega t} dt = 2\pi\, x(-\omega)$$

Properties of Fourier transform

Differentiation: $$\boxed{x'(t) = \frac{d}{dt}x(t) \leftrightarrow j\omega X(\omega)}$$

Integration: $$\int_{-\infty}^{t} x(\tau)\,d\tau \leftrightarrow \pi X(0)\delta(\omega) + \frac{1}{j\omega}X(\omega)$$

Convolution: $$x(t) = x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)\,d\tau$$

$$\boxed{x_1(t) * x_2(t) \leftrightarrow X_1(\omega)X_2(\omega)}$$

Multiplication: $$\boxed{x_1(t)x_2(t) \leftrightarrow \frac{1}{2\pi}X_1(\omega) * X_2(\omega)}$$

Fourier transform and differentiation

Differentiation: $x'(t) = \frac{d}{dt}x(t) \leftrightarrow j\omega X(\omega)$

$$x(t) = \mathcal{F}^{-1}[X(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\frac{d}{dt}\left(\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \boxed{j\omega X(\omega)}e^{j\omega t}d\omega$$

Now we can solve differential equations by applying the Fourier transform and solving the corresponding algebraic equations.

Fourier transform and integration

Integration: $\int_{-\infty}^{t} x(\tau)\,d\tau \leftrightarrow \pi X(0)\delta(\omega) + \frac{1}{j\omega} X(\omega)$

$$\int_{-\infty}^{t} x(\tau)\,d\tau = \int_{-\infty}^{\infty} u(t-\tau)x(\tau)\,d\tau = x(t) * u(t) \quad \text{where} \quad u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\boxed{u(t) \xleftrightarrow{\mathrm{F}} \pi\delta(\omega) + \frac{1}{j\omega}}$$

$$x(t) * u(t) \leftrightarrow X(\omega)\left(\pi\delta(\omega) + \frac{1}{j\omega}\right) = \pi X(0)\delta(\omega) + \frac{1}{j\omega} X(\omega)$$

$$\mathrm{F}\left[u(t)\right] = \pi\delta(\omega) + \frac{1}{j\omega}$$ How to prove it ?

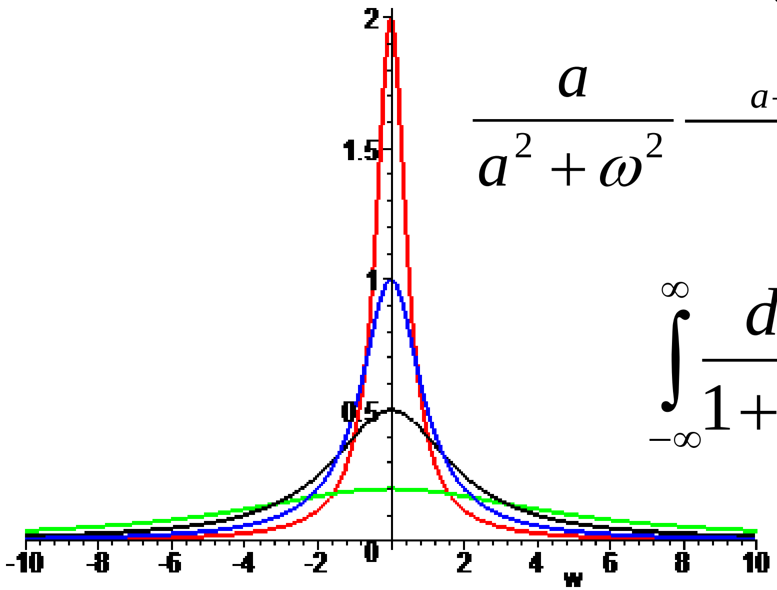
Fourier transform of step function

Unit step function $u(t)=\begin{cases}1 & t\ge 0\\ 0 & t<0\end{cases}$

$$\mathrm{F}\left[u(t)\right]=\pi\,\delta(\omega)+\frac{1}{j\omega} \quad ?$$

$$g_a(t)=\begin{cases}e^{-at} & t\ge 0\\ 0 & t<0\end{cases} \quad a>0 \qquad g_a(t)\xrightarrow{a\to 0}u(t)$$

$$\mathrm{F}\left[g_a(t)\right]=\frac{1}{a+j\omega}=\frac{a-j\omega}{a^2+\omega^2}=\frac{a}{a^2+\omega^2}-j\frac{\omega}{a^2+\omega^2}$$



$$\frac{a}{a^2+\omega^2}\xrightarrow{a\to 0}\pi\,\delta(\omega) \qquad -j\frac{\omega}{a^2+\omega^2}\xrightarrow{a\to 0}\frac{1}{j\omega}$$

$$\int_{-\infty}^{\infty}\frac{d\omega}{1+\omega^2}=\arctan\omega\Big|_{-\infty}^{\infty}=\frac{\pi}{2}+\frac{\pi}{2}=\pi$$

Fourier transforms of step and signum functions

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \qquad \mathcal{F}[u(t)] = \pi\delta(\omega) + \frac{1}{j\omega} \quad ?$$

$$u(t) = \frac{1}{2} + \frac{1}{2}\operatorname{sgn}(t) \qquad \operatorname{sgn}(t) = \begin{cases} 1 & t \geq 0 \\ -1 & t < 0 \end{cases}$$

$$\mathcal{F}[u(t)] = \frac{1}{2}\mathcal{F}[1] + \frac{1}{2}\mathcal{F}[\operatorname{sgn}(t)]$$

Differentiation property of FT

$$\frac{d}{dt}sgn(t) = 2\delta(t) \leftrightarrow 2 = j\omega F[sgn(t)]$$

$$F[sgn(t)] = \frac{2}{j\omega}$$

Duality property of FT applied to δ(t)

$$F^{-1}[\delta(\omega)] = \frac{1}{2\pi} \Longrightarrow F[1] = 2\pi\delta(\omega) \Longrightarrow \frac{1}{2}F[1] = \pi\delta(\omega)$$

Therefore $\boxed{F[u(t)] = \pi\delta(\omega) + \frac{1}{j\omega}}$

Fourier transform and convolution

Convolution: $x_1(t) * x_2(t) \leftrightarrow X_1(\omega) X_2(\omega)$

$$\mathrm{F}\left[x_1(t) * x_2(t)\right] = \int_{-\infty}^{\infty} \left[x_1(t) * x_2(t)\right] e^{-j\omega t} d t$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) d\tau \right] e^{-j\omega t} d t$$

$$= \int_{-\infty}^{\infty} x_1(\tau) \left[\int_{-\infty}^{\infty} x_2(t-\tau) e^{-j\omega t} d t \right] d\tau = \int_{-\infty}^{\infty} x_1(\tau) X_2(\omega) e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau) e^{-j\omega\tau} d\tau \, X_2(\omega) = X_1(\omega) X_2(\omega)$$

Applications of Convolution Theorem

$$x_1(t) * x_2(t) \leftrightarrow X_1(\omega) X_2(\omega)$$

$$\underbrace{g(x,y)}_{\text{blurred image}} = \underbrace{f(x,y)}_{\text{original image}} * h(x,y)$$

$$G(u,v) = \mathcal{F}\left[g(x,y)\right]$$
$$F(u,v) = \mathcal{F}\left[f(x,y)\right]$$
$$H(u,v) = \mathcal{F}\left[h(x,y)\right]$$

$$G(u,v) = F(u,v)H(u,v)$$
$$F(u,v) = G(u,v)/H(u,v)$$

$$\mathcal{F}^{-1}\left[\frac{G(u,v)}{H(u,v)}\right]$$

g = Blurred and Noisy

deconwnr(g,h,K)

Parseval's theorem

If $x(t)$ is an energy signal, then

$$\boxed{\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega}$$

We prove it assuming that $x(t)$ is real

Proof. Let us prove a stronger statement. Namely, let us show that if

$$x_1(t) \leftrightarrow X_1(\omega) \quad \text{and} \quad x_2(t) \leftrightarrow X_2(\omega)$$

then $$\int_{-\infty}^{\infty} x_1(t)\, x_2(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\omega) X_2(-\omega)\, d\omega$$

Parseval's theorem

$$\boxed{\begin{gathered} x_1(t) \leftrightarrow X_1(\omega) \quad \text{and} \quad x_2(t) \leftrightarrow X_2(\omega) \\ \int_{-\infty}^{\infty} x_1(t)\, x_2(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\omega) X_2(-\omega)\, d\omega \end{gathered}}$$

Indeed $\quad \mathcal{F}\left[x_1(t)\, x_2(t)\right] = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} X_1(\lambda) X_2(\omega - \lambda)\, d\lambda$

$$\int_{-\infty}^{\infty} \left[x_1(t)\, x_2(t)\right] e^{-j\omega t}\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\lambda) X_2(\omega - \lambda)\, d\lambda$$

Setting $\omega=0$ yields

$$\boxed{\int_{-\infty}^{\infty} \left[x_1(t)\, x_2(t)\right] dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\lambda) X_2(-\lambda)\, d\lambda}$$

Parseval's theorem

$$x_1(t) \leftrightarrow X_1(\omega) \quad \text{and} \quad x_2(t) \leftrightarrow X_2(\omega)$$

$$\int_{-\infty}^{\infty} x_1(t)\, x_2(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\omega) X_2(-\omega)\, d\omega$$

If $x(t)$ is **real**, then $X(-\omega) = X^*(\omega)$

$$x_1(t) = x(t) \quad \text{and} \quad x_2(t) = x^*(t) = x(t) \leftrightarrow X(\omega)$$

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} x_1(t)\, x_2(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) X(-\omega)\, d\omega$$

$$\boxed{\int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega}$$

Parseval's theorem, general case

$$x_1(t) \leftrightarrow X_1(\omega) \quad \text{and} \quad x_2(t) \leftrightarrow X_2(\omega)$$

$$\int_{-\infty}^{\infty} x_1(t)\, x_2(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\omega) X_2(-\omega)\, d\omega$$

In the general case, $x^*(t) \xleftrightarrow{\;\;F\;\;} X^*(-\omega)$

$$\boxed{\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\omega) X_2^*(\omega)\, d\omega}$$

Parseval's theorem, general case

$$x_1(t) \leftrightarrow X_1(\omega) \quad \text{and} \quad x_2(t) \leftrightarrow X_2(\omega)$$

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In the general case, $x^*(t) \xleftrightarrow{\mathcal{F}} X^*(-\omega)$

$$\boxed{\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\omega) X_2^*(\omega)\, d\omega}$$

$$X^*(-\omega) = \left(\int_{-\infty}^{\infty} x(t) e^{j\omega t} dt \right)^* = \int_{-\infty}^{\infty} x^*(t) e^{-j\omega t} dt = \mathcal{F}\left[x^*(t)\right]$$

Impulse function (Dirac delta function)

$$\boxed{\delta(t): \quad \int_{-\infty}^{\infty} \delta(t)\varphi(t)\,dt = \varphi(0)}$$

$$\int_{-\infty}^{\infty} \varphi(t)\delta(t-t_0)\,dt = \varphi(t_0)$$

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

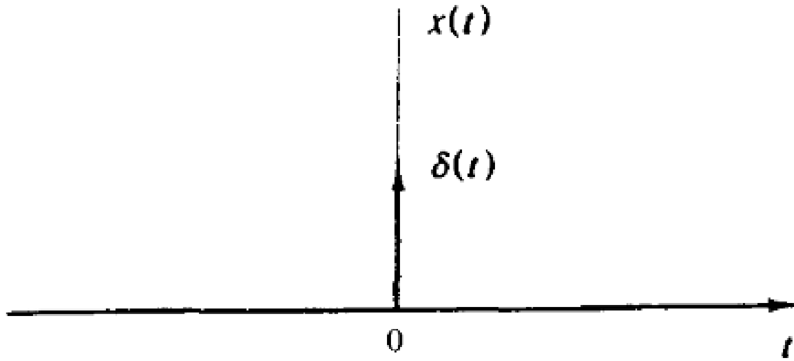
$$\delta(-t) = \delta(t)$$

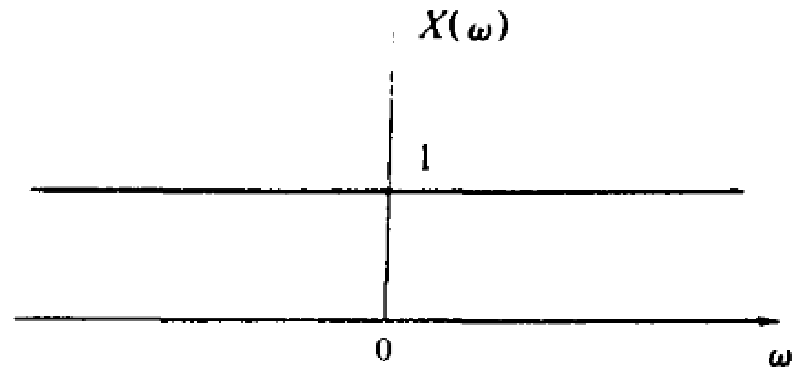
$$x(t) * \delta(t-t_0) = x(t-t_0)$$

Impulse function (Dirac delta function)

$$\mathcal{F}\left[\delta(t)\right] = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

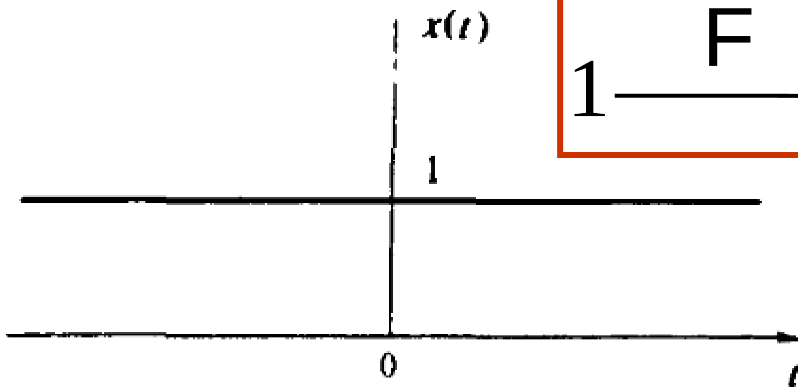
$$\boxed{\delta(t) \xrightarrow{\mathcal{F}} 1}$$

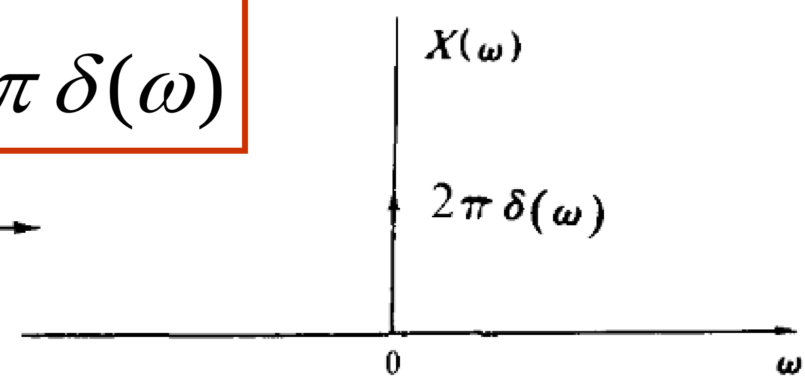




$$x(t) = \mathcal{F}^{-1}\left[X(\omega)\right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} dt$$

$$\boxed{1 \xrightarrow{\mathcal{F}} 2\pi\,\delta(\omega)}$$





Example 1

Find the Fourier transform of the signal

$$x(t) = e^{-a|t|} = \begin{cases} e^{-at} & t > 0 \\ e^{at} & t < 0 \end{cases}$$

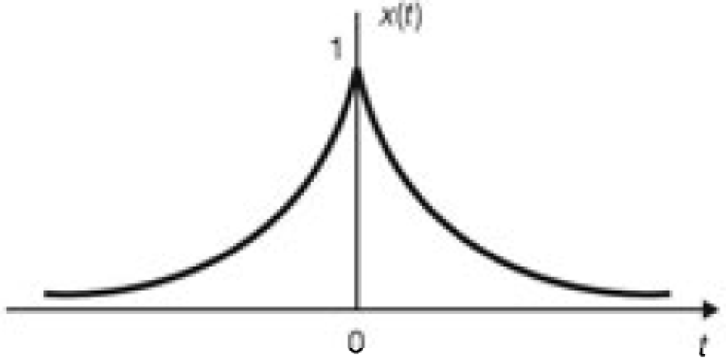
Example 1

Find the Fourier transform of the signal

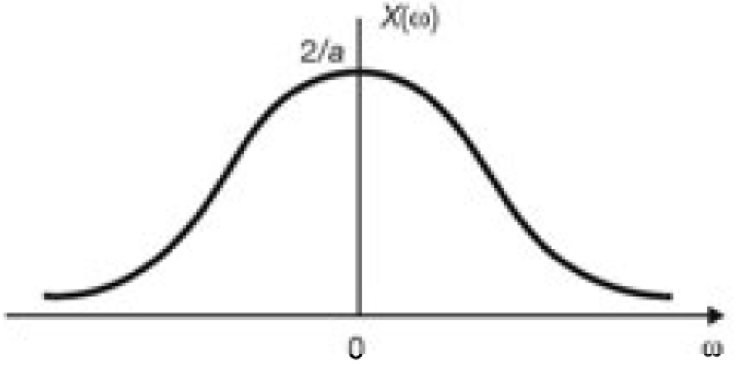
$$x(t) = e^{-a|t|} = \begin{cases} e^{-at} & t > 0 \\ e^{at} & t < 0 \end{cases}$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-\infty}^{0} e^{at} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt$$

$$= \frac{1}{a - j\omega} + \frac{1}{a + j\omega} = \frac{2a}{a^2 + \omega^2}$$

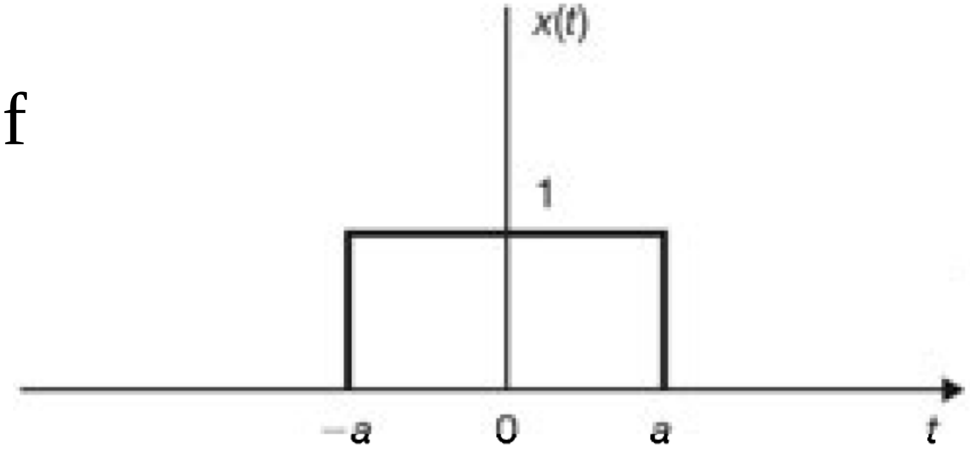


$\longleftrightarrow$



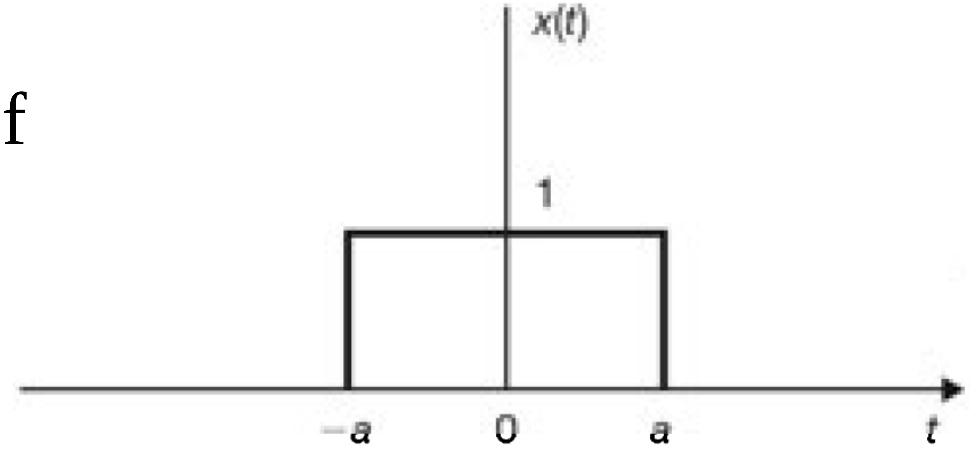
Example 2

Find the Fourier transform of the rectangular pulse signal



Example 2

Find the Fourier transform of the rectangular pulse signal



$$x(t) = p_a(t) = \begin{cases} 1 & |t| < a \\ 0 & |t| > a \end{cases}$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt = \int_{-a}^{a} e^{-j\omega t} dt = \frac{1}{j\omega}\left(e^{j\omega a} - e^{-j\omega a}\right) = 2a\,\frac{\sin(\omega a)}{\omega a}$$