Signals and Spectra

Sampling

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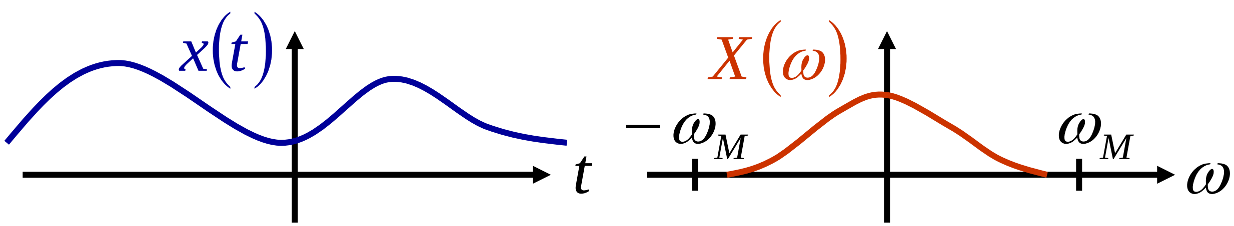
Shannon's Sampling Theorem

A ***band-limited*** signal is a signal $x(t)$ for which the Fourier transform of $x(t)$ is identically zero above a certain frequency ω_M.

$$x(t) \leftrightarrow X(\omega) = 0 \quad \text{for} \quad |\omega| > \omega_M = 2\pi\, f_M$$



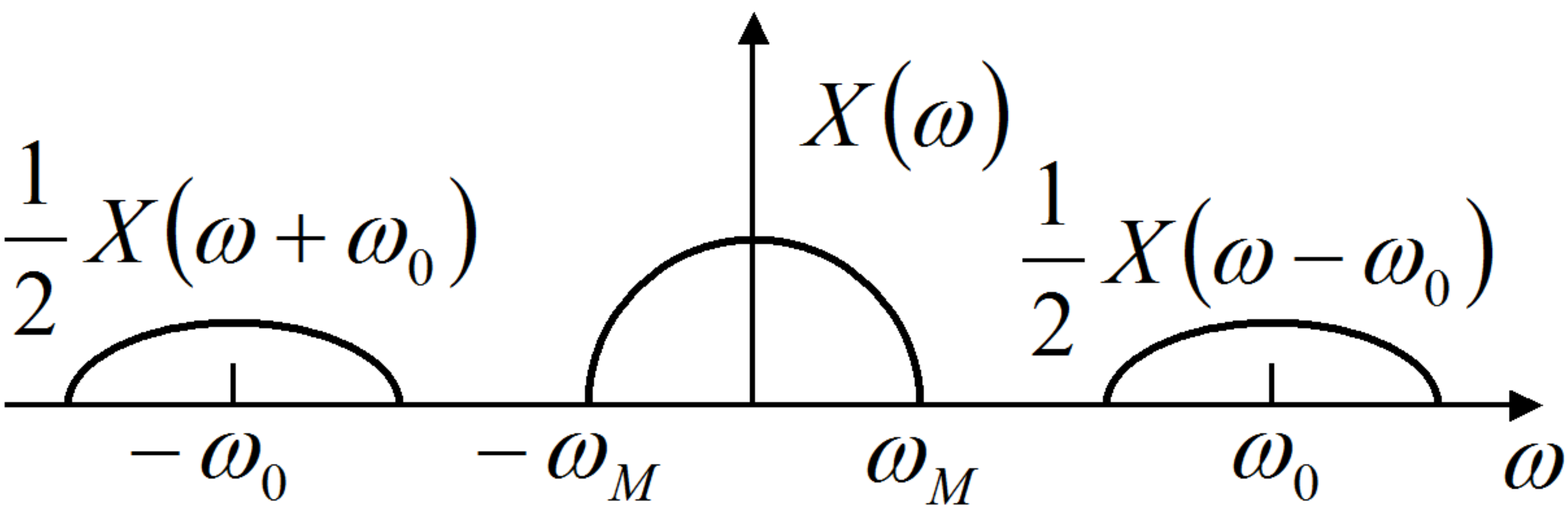
> If a signal $x(t)$ is a real-valued band-limited signal (no frequency components higher than f_M Hz), then $x(t)$ can be uniquely determined from its values $x(nT_s)$ sampled at uniform intervals T_s, where $f_s > 2f_M$

Claude Shannon was an American mathematician and electrical engineer. He is generally regarded as the father of the information age.

Modulation Theorem

Modulation theorem. If $x(t) \leftrightarrow X(\omega)$ then

$$x(t)\cos\omega_0 t \leftrightarrow \frac{1}{2}X(\omega-\omega_0)+\frac{1}{2}X(\omega+\omega_0)$$



Modulation Theorem

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Proof:

$$\cos\omega_0 t = \frac{1}{2}\left(e^{j\omega_0 t}+e^{-j\omega_0 t}\right)$$

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Proof:

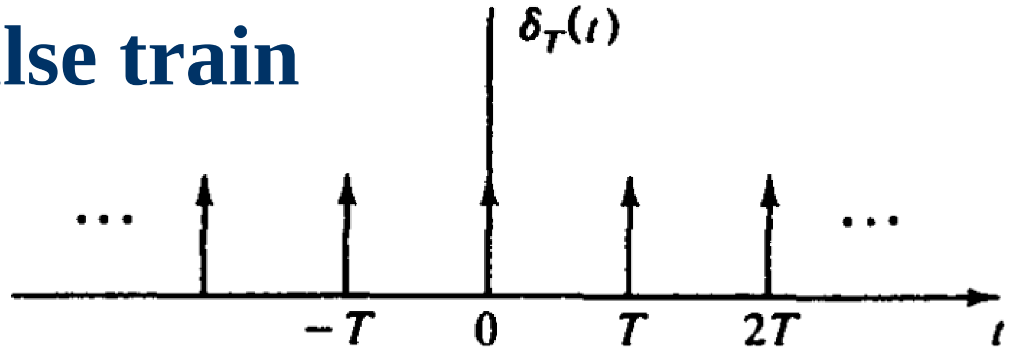
$$\cos\omega_0 t = \frac{1}{2}\left(e^{j\omega_0 t}+e^{-j\omega_0 t}\right)$$

$$\mathcal{F}\left[x(t)\cos\omega_0 t\right] = \mathcal{F}\left[\frac{1}{2}x(t)e^{j\omega_0 t}+\frac{1}{2}x(t)e^{-j\omega_0 t}\right]$$

$$=\frac{1}{2}X(\omega-\omega_0)+\frac{1}{2}X(\omega+\omega_0)$$

Fourier series of impulse train

Find the complex Fourier series of the unit impulse train



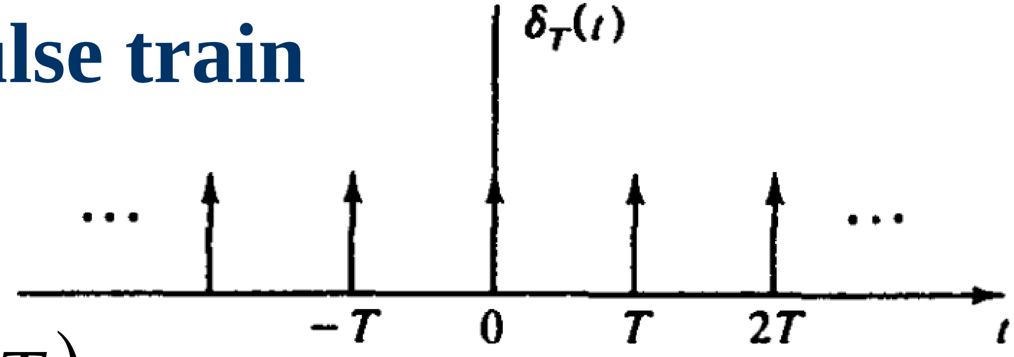
$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Solution.

It is a periodic function, its Fourier series is given by

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

Fourier series of impulse train



Find the complex Fourier series of the unit impulse train

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

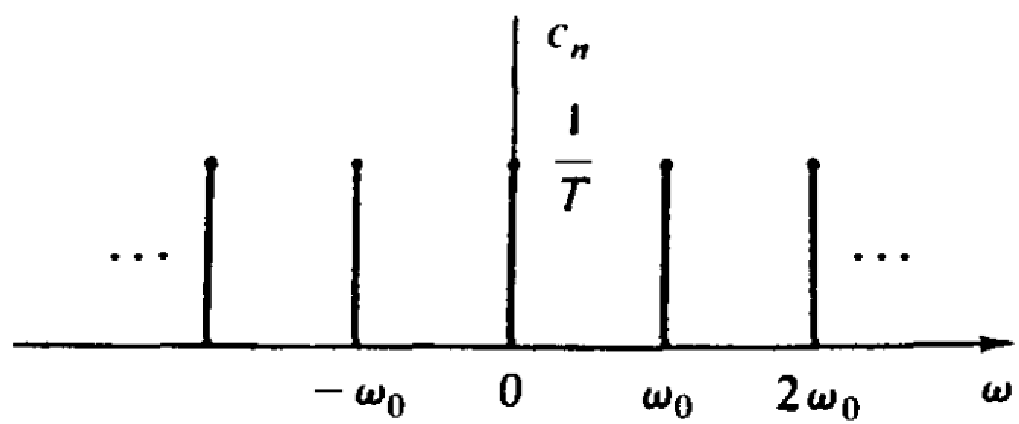
Solution.

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} \delta_T(t) e^{-jn\omega_0 t} dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jn\omega_0 t} dt = \frac{1}{T}$$

$$\boxed{\delta_T(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}}$$



Fourier transform of impulse train

Find the Fourier transform (not Fourier series) of the unit impulse train

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Solution.

$$\delta_T(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T}$$

$$1 \xrightarrow{\mathcal{F}} 2\pi\,\delta(\omega) \qquad e^{jn\omega_0 t} \xrightarrow{\mathcal{F}} 2\pi\,\delta(\omega - n\omega_0)$$

$$\mathcal{F}\left[\delta_T(t)\right] = \frac{2\pi}{T}\sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0) = \omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$$

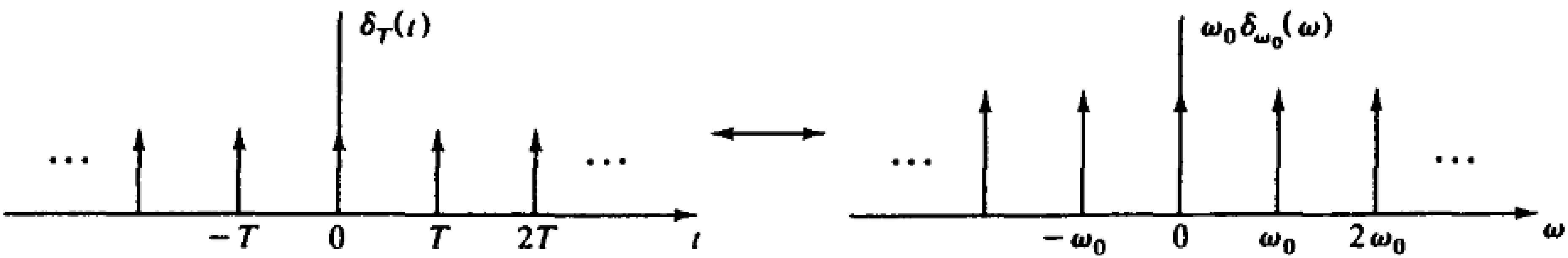
Fourier transform of impulse train

Find the Fourier transform of the unit impulse train $$\delta_T(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT)$$

Solution.

$$\delta_T(t)=\frac{1}{T}\sum_{n=-\infty}^{\infty}e^{jn\omega_0 t} \qquad \omega_0=\frac{2\pi}{T}$$

$$\mathrm{F}\left[\delta_T(t)\right]=\frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)=\omega_0\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$$



also an impulse train

Sampling Theorem

A ***band-limited*** signal is a signal $x(t)$ for which the Fourier transform of $x(t)$ is identically zero above a certain frequency ω_M.

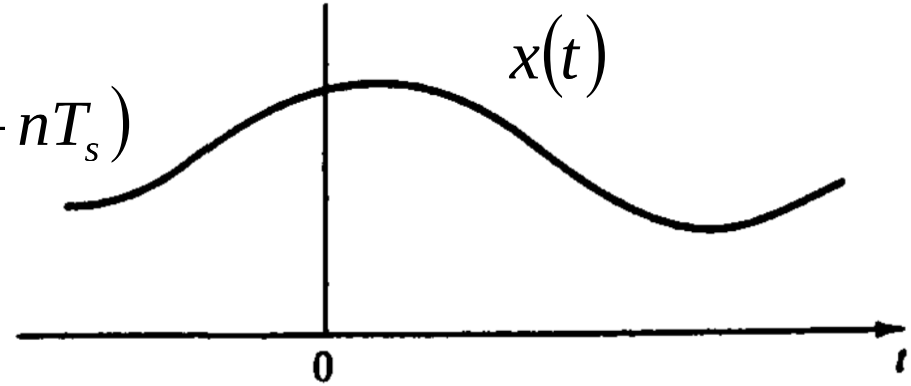
$$x(t) \leftrightarrow X(\omega) = 0 \quad \text{for} \quad |\omega| > \omega_M = 2\pi f_M$$

If a signal $m(t)$ is a real-valued band-limited signal (no frequency components higher than f_M Hz), then $m(t)$ can be uniquely determined from its values $m(nT_s)$ sampled at uniform intervals T_s, where $f_s > 2 f_M$

T_s is called the *sampling period*, f_s is called the *sampling rate*. The minimum sampling rate, $2 f_M$ samples per second, is called the ***Nyquist rate***; its reciprocal $1/(2 f_M)$ (measured in seconds) is called the ***Nyquist interval***.

Instantaneous sampling

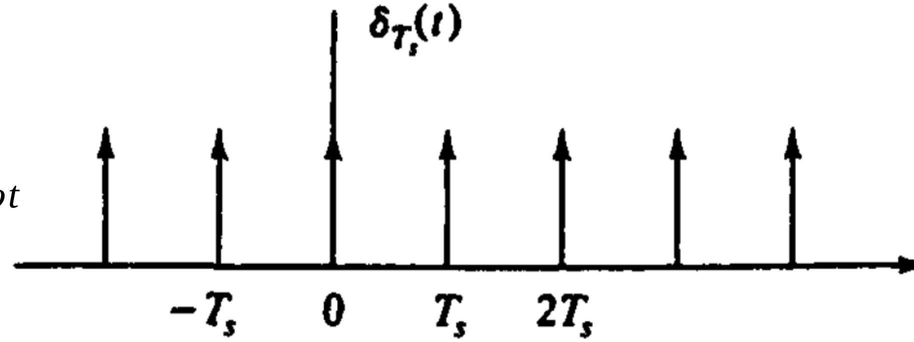
$$x_s(t) = x(t)\delta_T(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta_T(t-nT_s)$$



$$X_s(\omega) = \mathrm{F}\left[x_s(t)\right]$$

$$= \mathrm{F}\left[x(t)\right] * \mathrm{F}\left[\delta_T(t)\right]$$

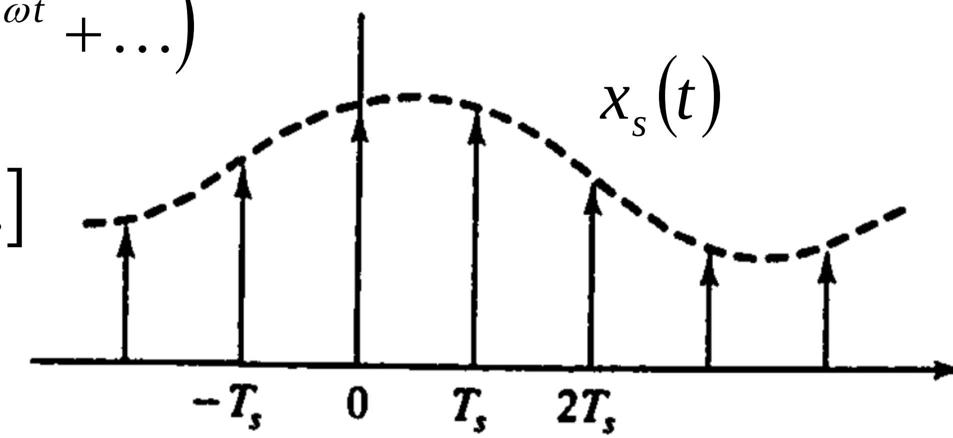
$$= X(\omega) * \omega_0 \delta_{\omega_0}(\omega)\frac{1}{2\pi}, \omega_0 = 2\pi/T$$



Alternatively: $\quad \delta_T(t) = \dfrac{1}{T}\displaystyle\sum_{n=-\infty}^{\infty} e^{jn\omega t}$

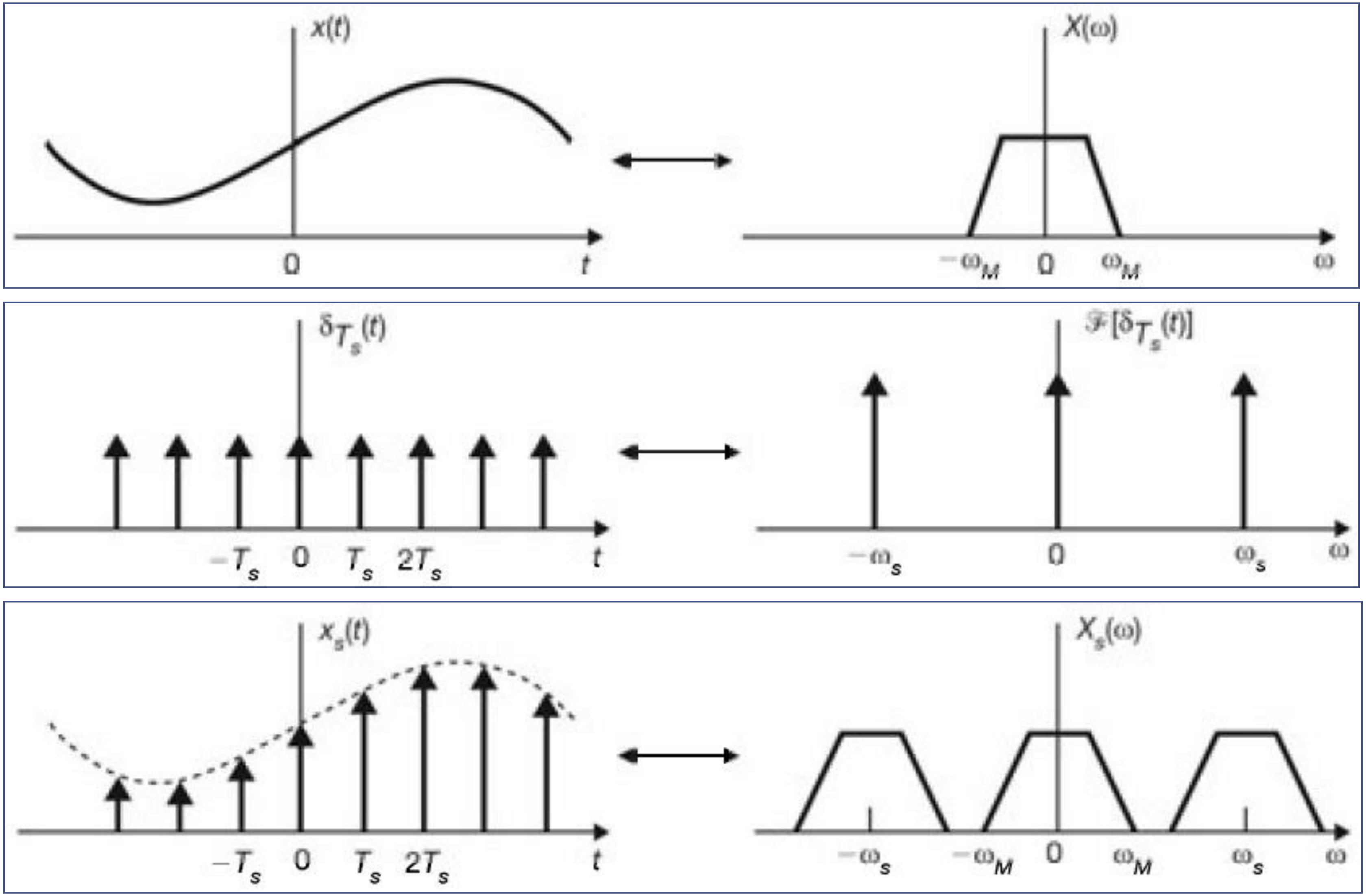
$$= \frac{1}{T}\left(1 + e^{j\omega t} + e^{-j\omega t} + e^{j2\omega t} + e^{-j2\omega t} + \ldots\right)$$

$$= \frac{1}{T}\left[1 + 2\cos\omega t + 2\cos 2\omega t + \ldots\right]$$



and then apply the Modulation Theorem

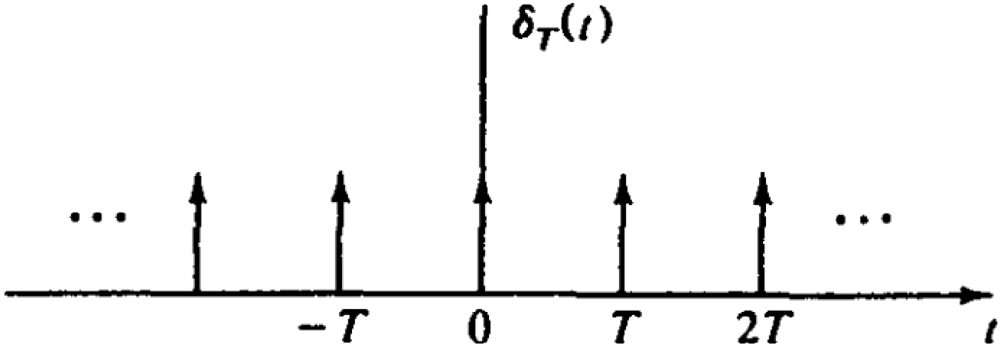
Instantaneous sampling



Can recover $X(\omega)$ from the samples $\{x(nT_s)\}$

Unit impulse train

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



$$\delta_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t} \qquad \omega = \frac{2\pi}{T} = 2\pi f$$

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} \delta_T(t) e^{-jn\omega t} dt = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jn\omega t} dt = \frac{1}{T}$$

$$\delta_T(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega t}$$

$$= \frac{1}{T} \left[1 + 2\cos\omega t + 2\cos 2\omega t + 2\cos 3\omega t + \ldots\right]$$

Instantaneous sampling

$$x_s(t) = x(t)\delta_T(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta_T(t - nT_s)$$

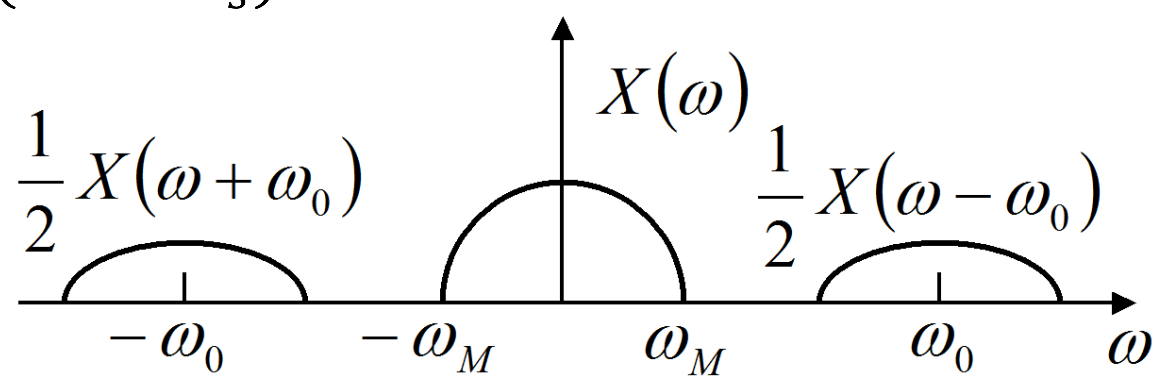
$$\delta_T(t) = \frac{1}{T}\left[1 + 2\cos\omega_s t + 2\cos 2\omega_s t + 2\cos 3\omega_s t + \ldots\right]$$

$$\omega_s = \frac{2\pi}{T} = 2\pi\ f_s$$

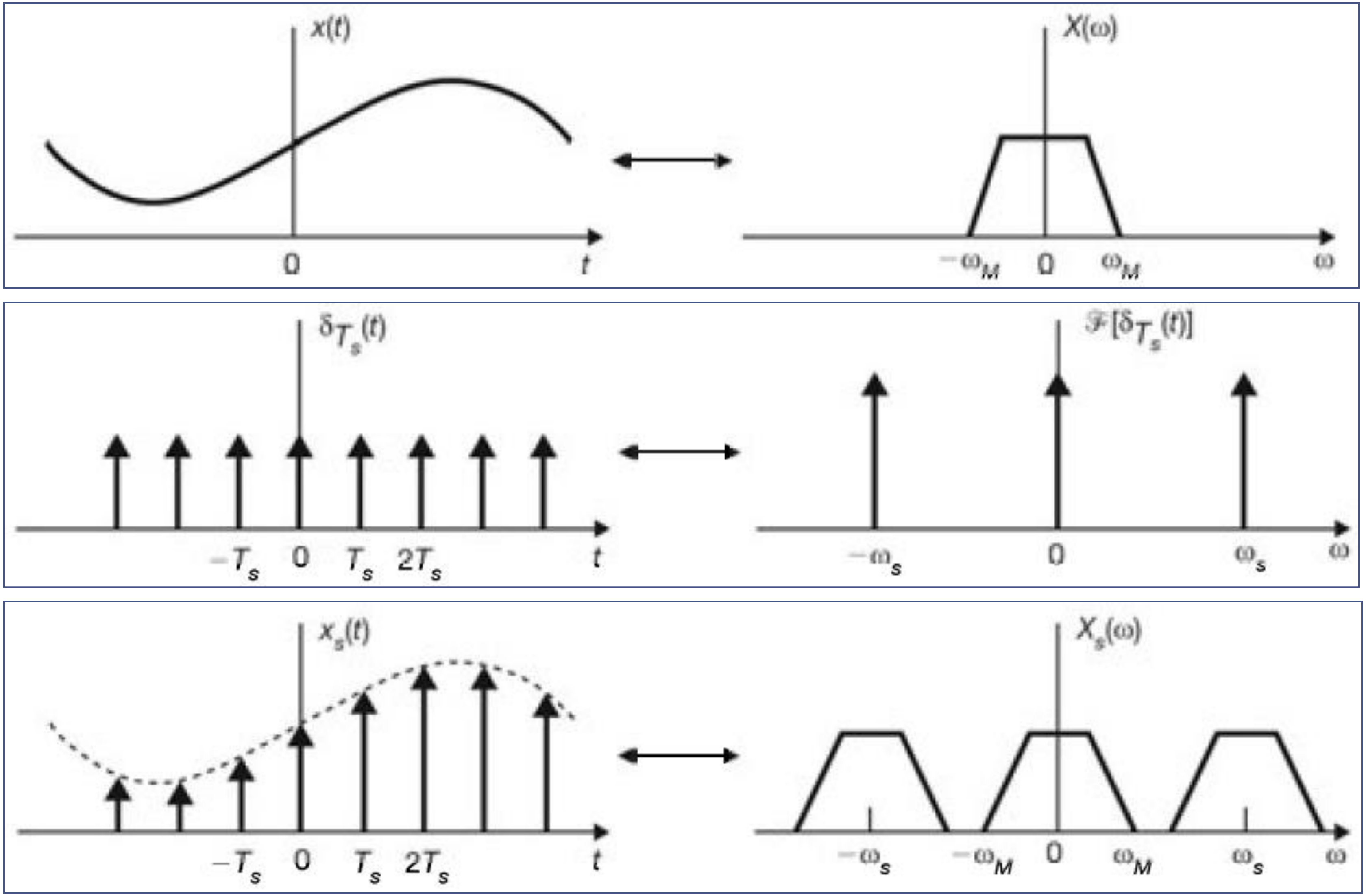
$$2x(t)\cos n\omega_s t \leftrightarrow X(\omega - n\omega_s) + X(\omega + n\omega_s)$$

$$Xs(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

Modulation Theorem

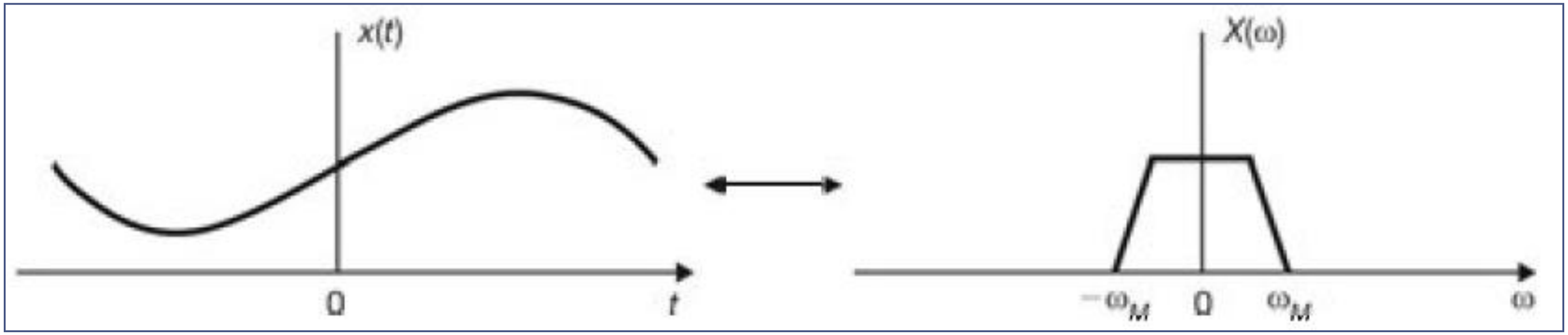


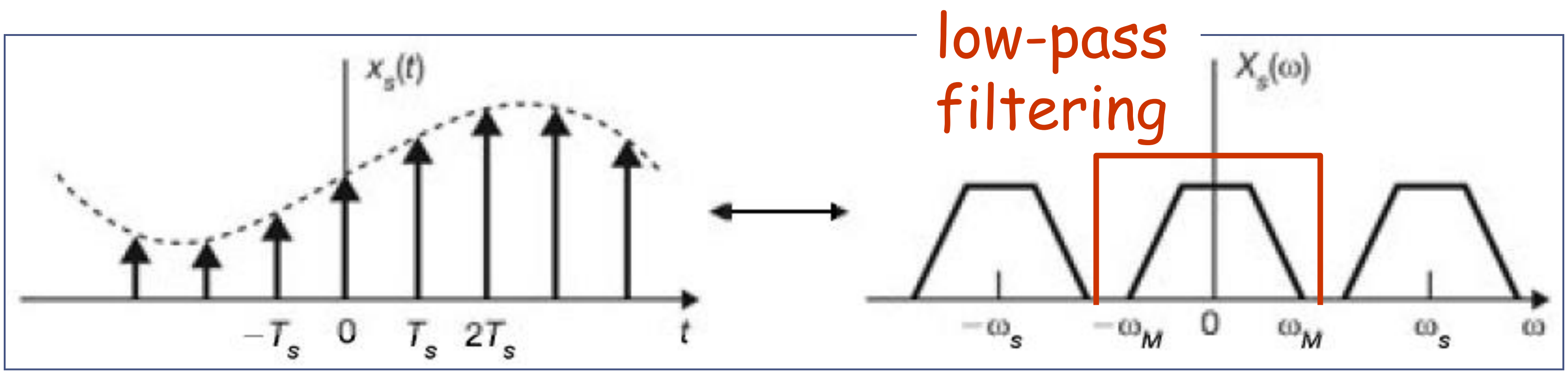
Instantaneous sampling



Can recover $X(\omega)$ from the samples $\{x(nT_s)\}$

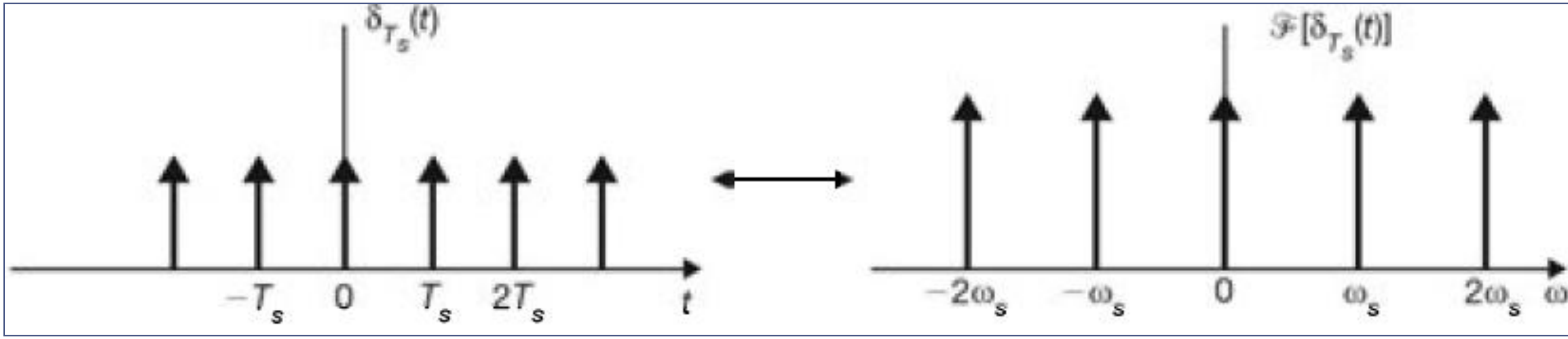
Instantaneous sampling

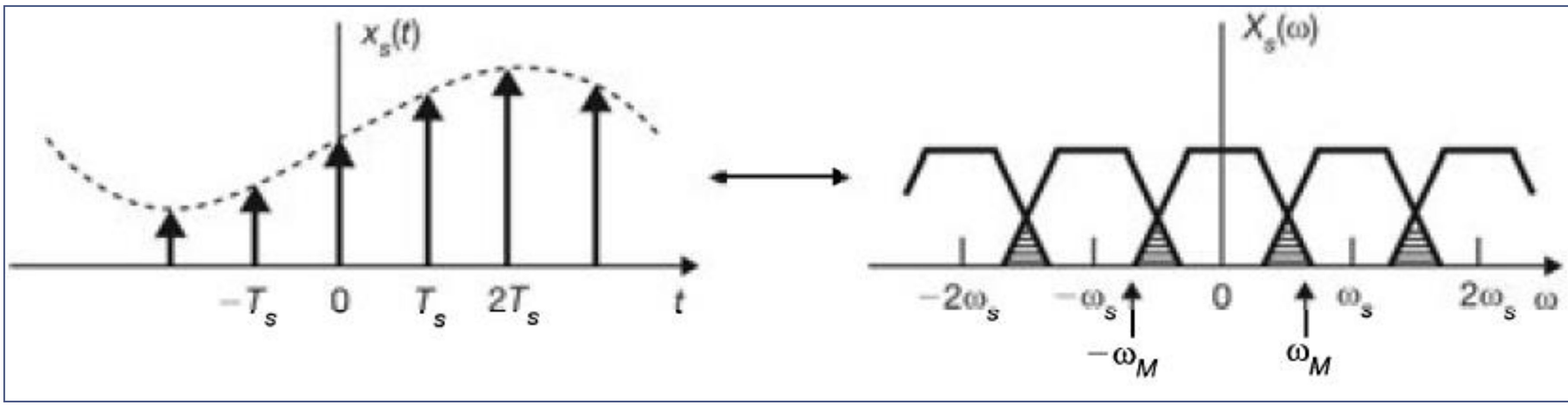




Can recover $X(\omega)$ from the samples $\{x(nT_s)\}$

Instantaneous sampling





Cannot recover $X(\omega)$ from the samples $\{x(nT_s)\}$

Sampling: Example 1

A signal

$$x(t) = \cos 200\pi t + 2\cos 320\pi t$$

is ideally sampled at f_s=300 Hz. If the sampled signal is passed through an ideal low-pass filter with a cutoff frequency of 250 Hz, what frequency components will appear in the input?

$$\omega_s = \frac{2\pi}{T_s} = 2\pi\, f_s \qquad f_s = \frac{\omega_s}{2\pi}$$

In this example $f_{\text{Nyquist}} = 320\ \text{Hz}$

Sampling: Example 1

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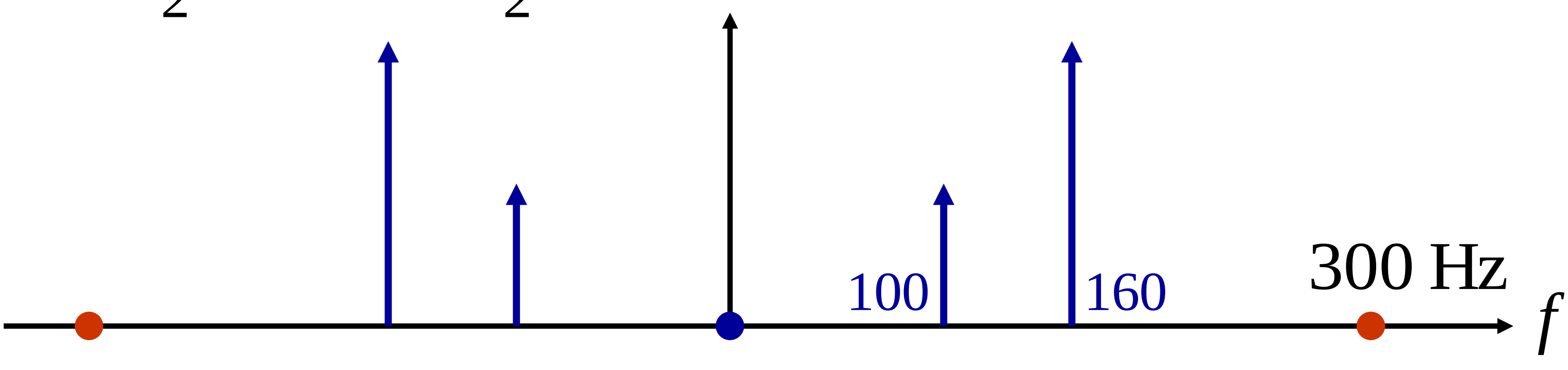
$$x(t) = \frac{1}{2}\left(e^{-j200\pi t} + e^{j200\pi t}\right) + \left(e^{-j320\pi t} + e^{j320\pi t}\right)$$

$$X(\omega) = \frac{2\pi}{2}\delta(\omega + 200\pi) + \frac{2\pi}{2}\delta(\omega - 200\pi) + \delta(\omega + 320\pi) + \delta(\omega - 320\pi)$$

$$X(f) = \frac{1}{2}\delta(f + 100) + \frac{1}{2}\delta(f - 100) + \delta(f + 160) + \delta(f - 160)$$



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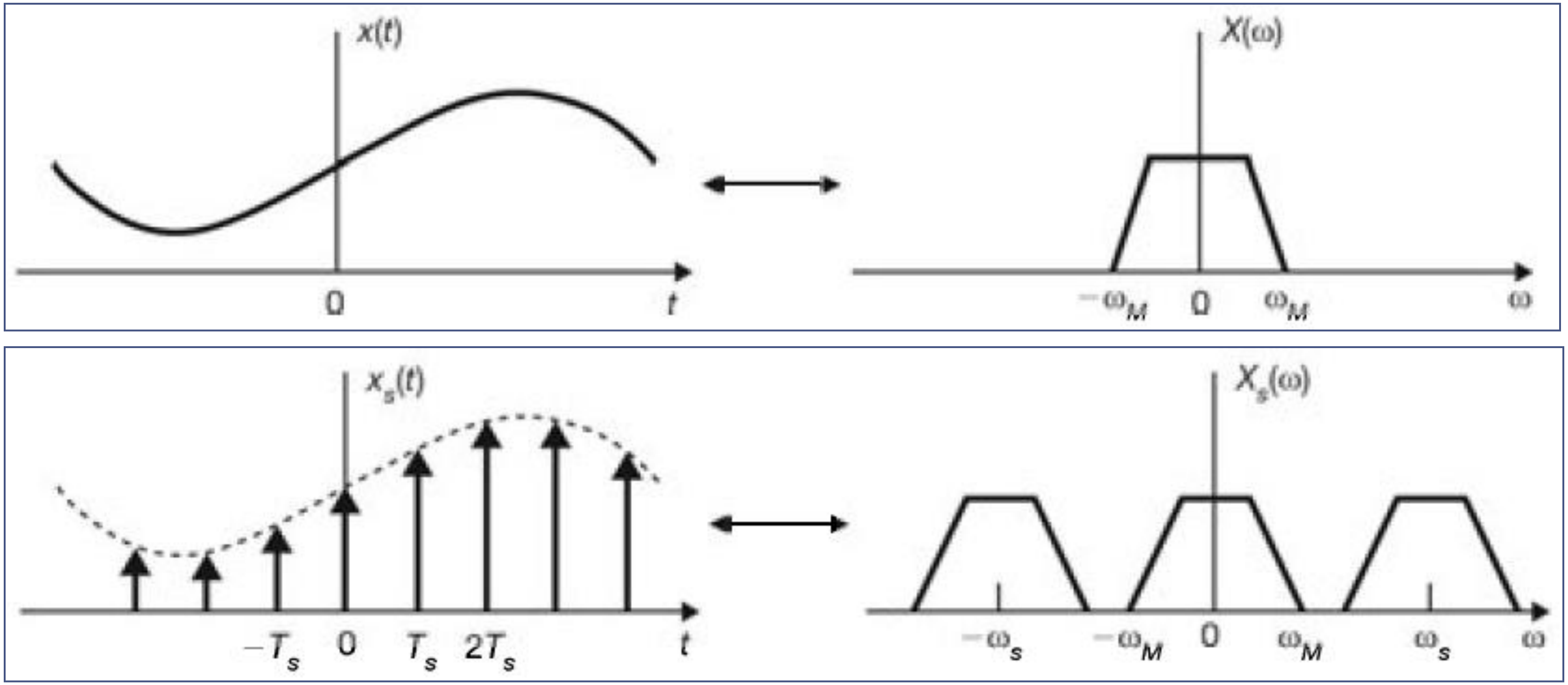
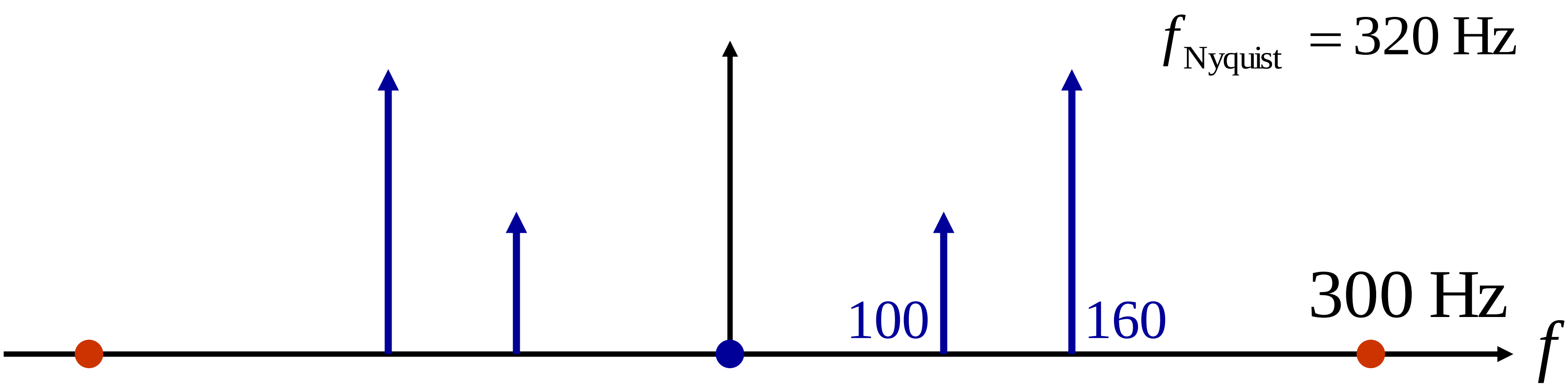


$$\omega_s = 2\pi / T_s = 2\pi\ f_s$$



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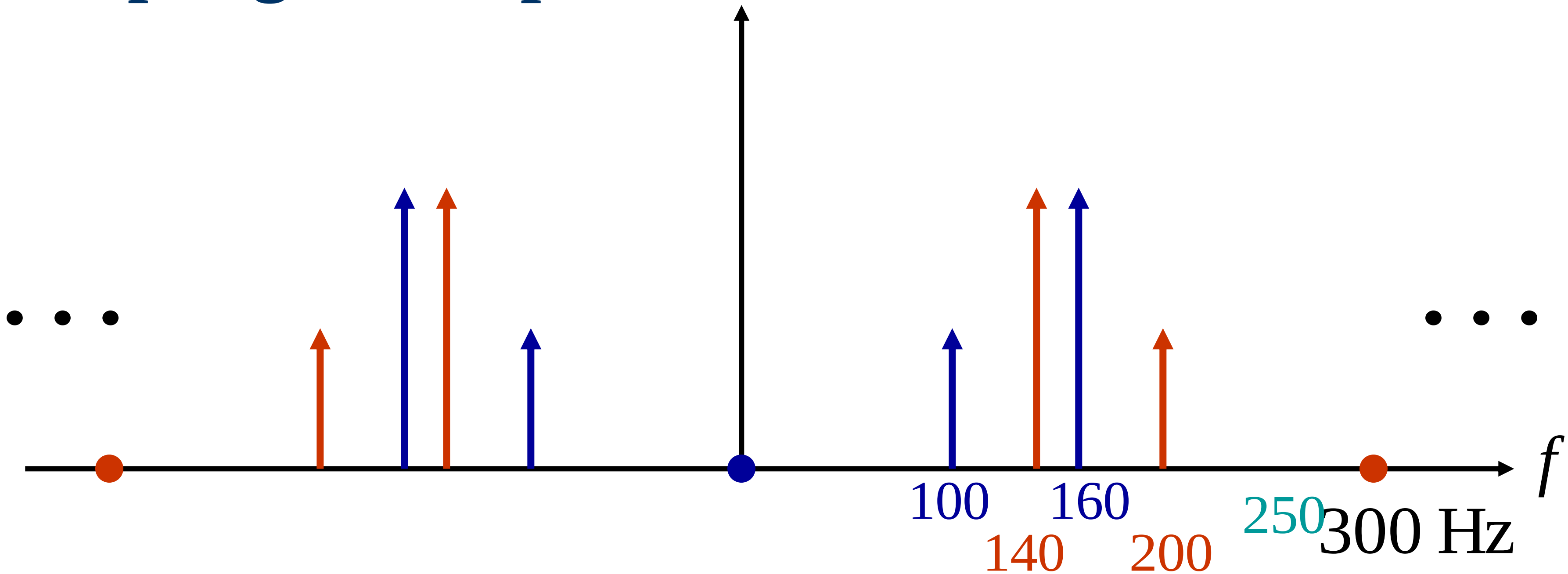
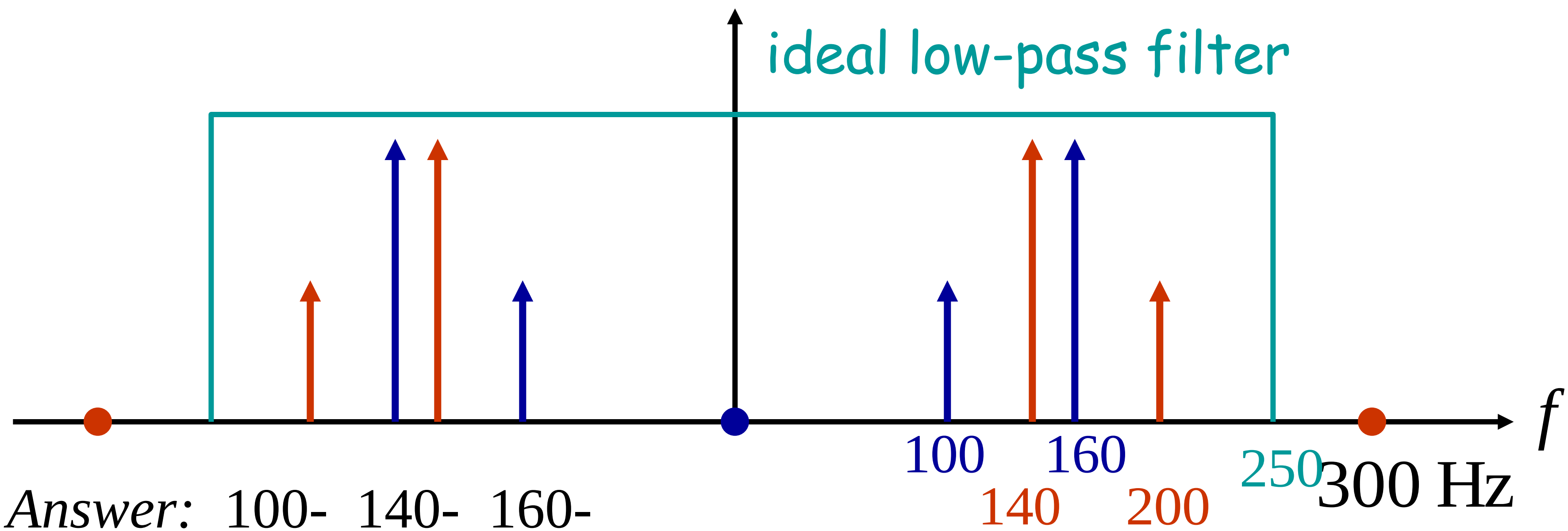


There are an infinite number of frequency components in the sampled signal, given by $\pm|mf_s \pm f_0|$, m = 0, 1, 2, 3….., f_0 is the frequency of the sinusoid signal that is sampled.

Sampling: Example 1

A signal $$x(t) = \cos 200\pi t + 2\cos 320\pi t$$

is ideally sampled at f_s=300 Hz. If the sampled signal is passed through an ideal low-pass filter with a cutoff frequency of 250 Hz, what frequency components will appear in the input?



$140 = 300 - 160$ $\quad$ $200 = 300 - 100$

Answer: 100- 140- 160- and 200-Hz components

Sampling: Example 2

The signals

$$x_1(t) = 10\cos 100\pi t \quad \text{and} \quad x_2(t) = 10\cos 50\pi t$$

are both sampled with f_s=75 Hz. Show that the two sequences of samples are identical.

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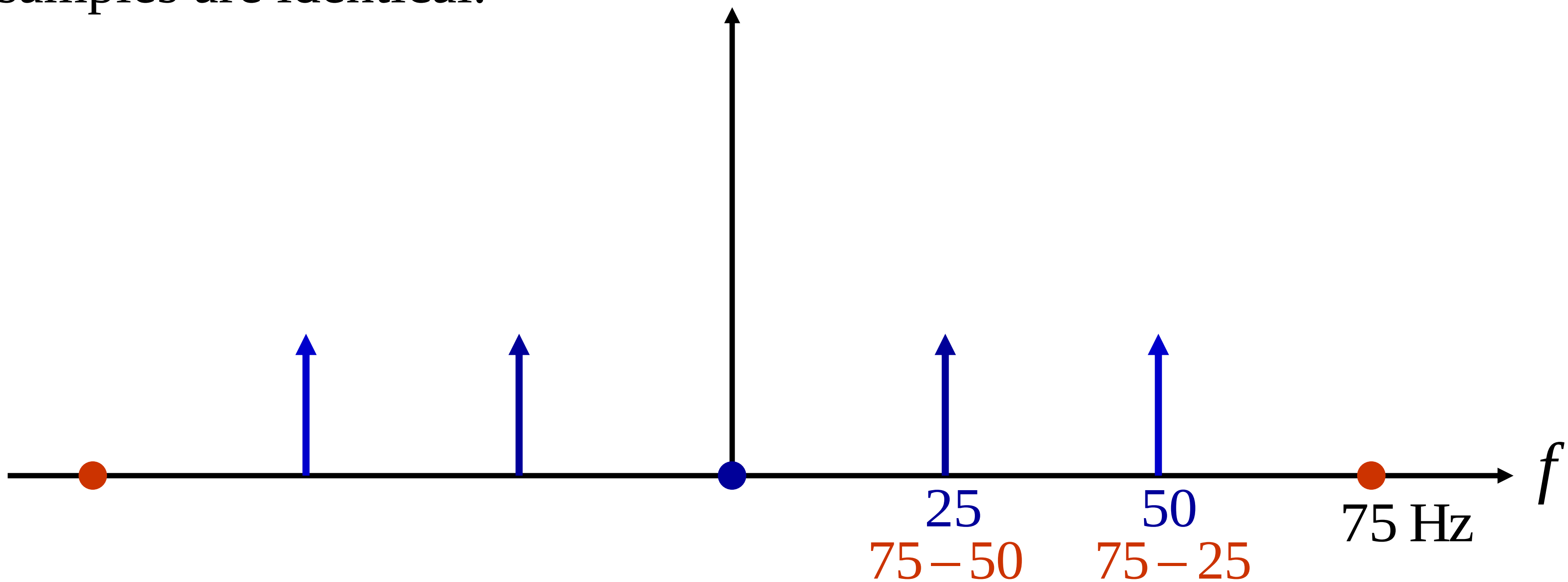
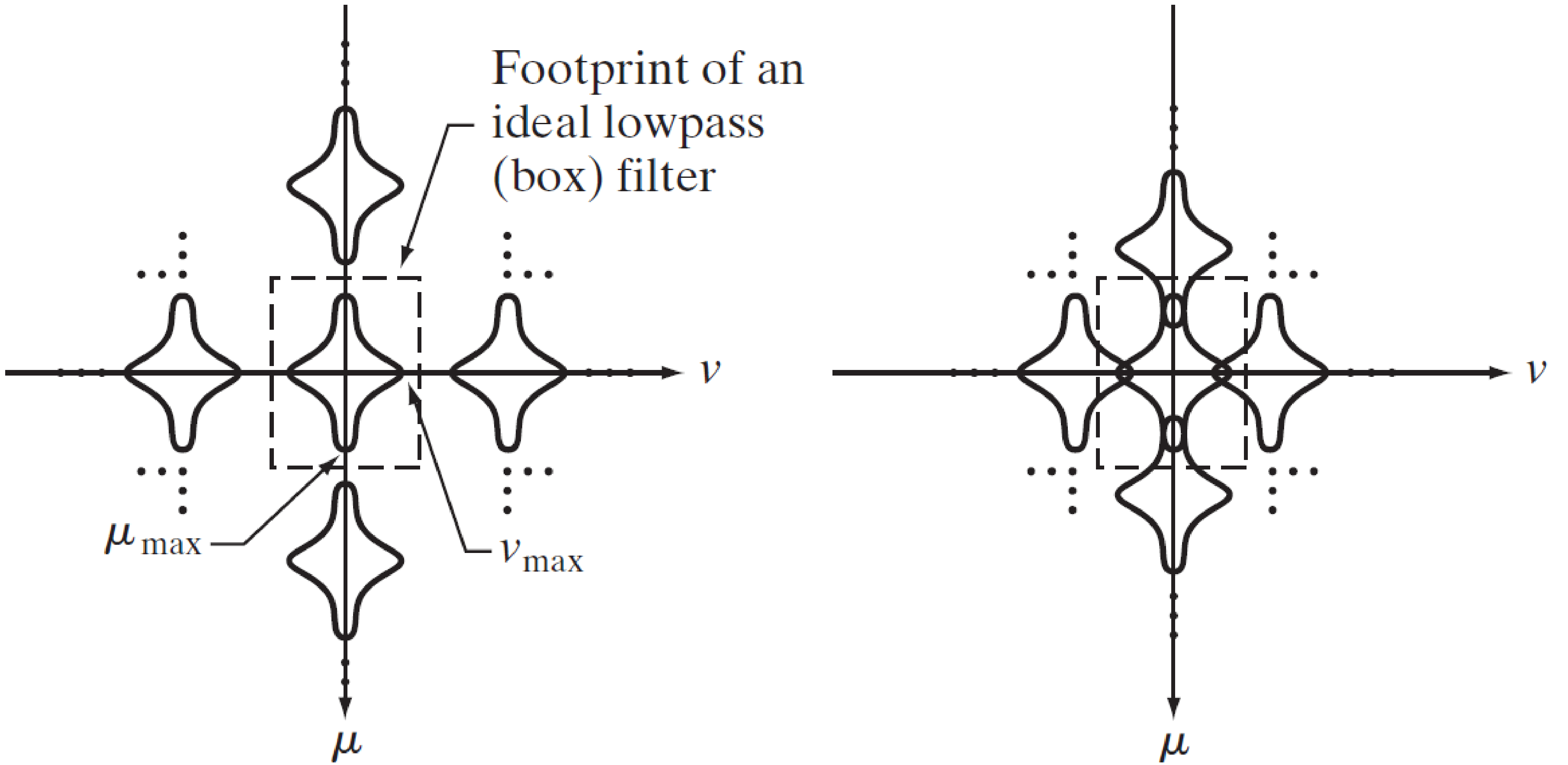
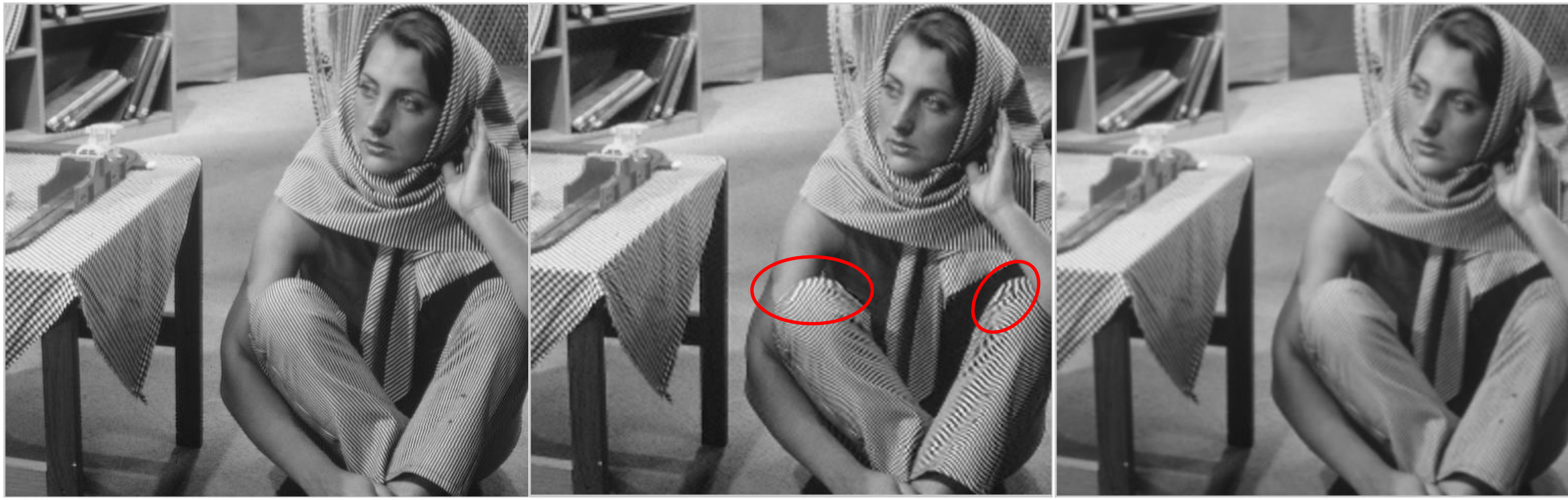


$$f_{\text{Nyquist}} = 100 \text{ Hz}$$

Aliasing in Images.



Aliasing in Images



Down sampled by factor 2.
Without and with lowpass prefiltering

Down sampled by factor 3.
Without and with lowpass prefiltering