B39SB Time and Frequency Signal Analysis

One-sided (unilateral) Laplace Transform for Solving Differential Equations and Circuit Analysis

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One-sided Laplace transform and its applications

$$\boxed{X(s) = \int_0^\infty x(t)\, e^{-st}\, dt}$$

- One-sided Laplace transform and its properties.
- Applications to initial value problems for ordinary differential equations.
- Applications to circuit analysis and, in particular, to transients.
- Transfer function.
- LTI systems in series and parallel. Inverse systems and linear feedback systems.
- Operational amplifiers, transfer functions for circuits with op amps.
- Bode plots, impulse response.
- Stability.

One-sided Laplace transform

$$\boxed{X(s)=\int_{0}^{\infty} x(t)\,e^{-st}\,dt \quad \text{unilateral (one-sided) Laplace transform}}$$

$$x(t) \xrightarrow{L} X(s) \qquad X(s)=L\left[x(t)\right]$$

One-sided LT = two-sided LT applied to a causal signal (a signal that does not start before t=0 is, i.e. $x(t)$=0 for t<0.)

$$X(s)=\int_{0}^{\infty} x(t)\,e^{-st}\,dt=\int_{-\infty}^{\infty} x(t)\,u(t)\,e^{-st}\,dt$$

One-sided Laplace transform: examples

$$\boxed{X(s)=\int_0^\infty x(t)\,e^{-st}\,dt \quad \text{unilateral (one-sided) Laplace transform}}$$

$$L[\delta(t)]=\int_0^\infty \delta(t)e^{-st}dt=1 \qquad L[u(t)]=\int_0^\infty 1e^{-st}dt=-\frac{1}{s}e^{-st}\Big|_0^\infty=\frac{1}{s}$$

$$L\left[e^{-at}\right]=\int_0^\infty e^{-at}\,e^{-st}dt=\frac{1}{s+a}$$

$$L[\cos\omega t]=\frac{1}{2}\left(\frac{1}{s-j\omega}+\frac{1}{s-j\omega}\right)=\frac{s}{s^2+\omega^2} \qquad L[\sin\omega t]=\frac{\omega}{s^2+\omega^2}$$

$$x(t)=t, \quad L[x(t)]=\int_0^\infty t\,e^{-st}dt \overset{\text{by parts}}{=} -\frac{1}{s}\int_0^\infty e^{-st}d\left(e^{-st}\right)$$

$$=-\frac{1}{s}t\,e^{-st}\Big|_0^\infty+\frac{1}{s}\int_0^\infty e^{-st}dt=\frac{1}{s^2}$$

Initial and final value theorems

$$X(s) = \int_0^\infty x(t)\, e^{-st}\, dt \quad \text{unilateral (one-sided) Laplace transform}$$

$$\boxed{x(0) = \lim_{s\to\infty} s\, X(s) \qquad x(\infty) = \lim_{s\to 0} s\, X(s)}$$

$$\int_0^\infty \frac{dx}{dt} e^{-st} dt \overset{\text{by parts}}{=} \left[x(t) e^{-st}\right]_0^\infty + s\int_0^\infty x(t) e^{-st} dt = sX(s) - x(0)$$

$$\int_0^\infty \frac{dx}{dt} e^{-st} dt \xrightarrow[s\to\infty]{} 0 \quad \Rightarrow \quad x(0) = \lim_{s\to\infty} s\, X(s)$$

$$\int_0^\infty \frac{dx}{dt} e^{-st} dt \xrightarrow[s\to 0]{} \int_0^\infty \frac{dx}{dt} dt = x(\infty) - x(0) \quad \Rightarrow \quad x(\infty) = \lim_{s\to 0} s\, X(s)$$

Laplace transform of derivatives

$$X(s) = \int_0^\infty x(t) e^{-st} dt \quad \text{unilateral (one-sided) Laplace transform}$$

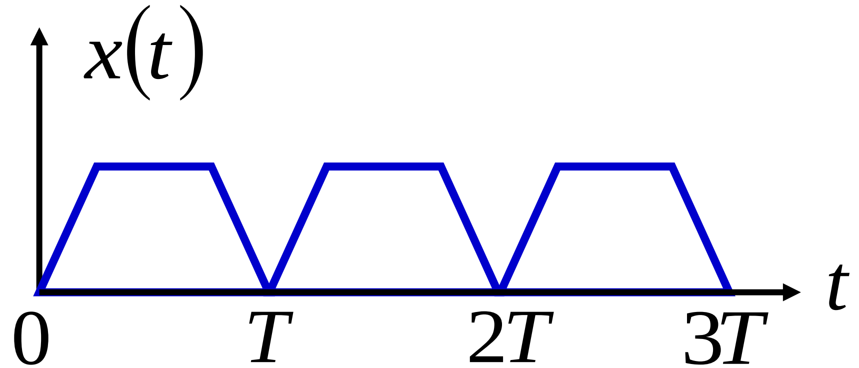
$$L[dx/dt] = s\,X(s) - x(0)$$

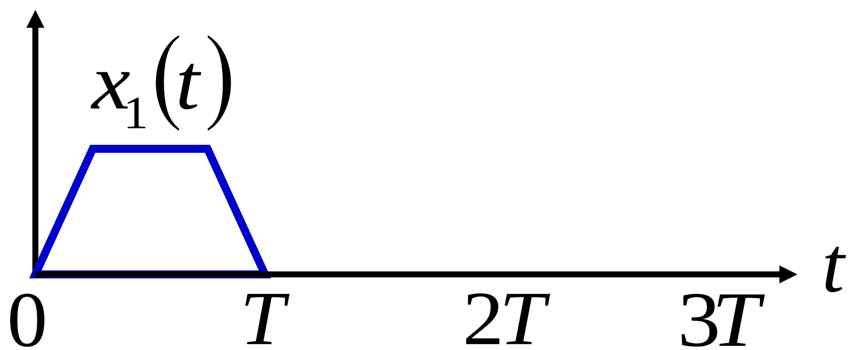
$$L[d^2x/dt^2] = s^2\,X(s) - s\,x(0) - x'(0)$$

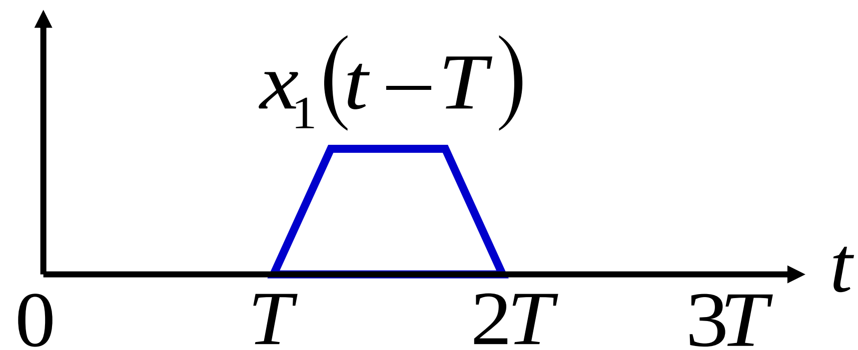
$$L[d^3x/dt^3] = s^3\,X(s) - s^2 x(0) - s\,x'(0) - x''(0)$$

$$\ldots$$

Time periodicity





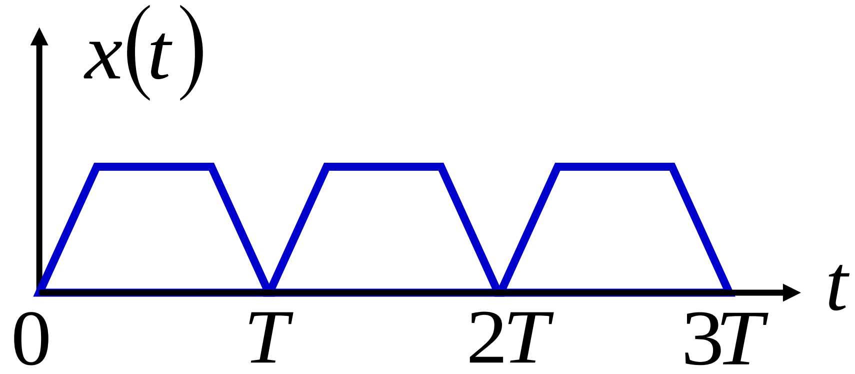


$$x_1(t) = x(t)\left[u(t) - u(t-T)\right]$$

$$x_1(t) = \begin{cases} x(t) & 0 < t < T \\ 0 & \text{otherwise} \end{cases}$$

$$x(t) = x_1(t) + x_1(t-T) + x_1(t-2T) + \ldots$$

Time periodicity



$$x_1(t) = x(t)\left[u(t) - u(t-T)\right] \qquad x_1(t) = \begin{cases} x(t) & 0 < t < T \\ 0 & \text{otherwise} \end{cases}$$

$$x(t) = x_1(t) + x_1(t-T) + x_1(t-2T) + \dots$$

$$F(s) = F_1(s) + F_1(s)e^{-Ts} + F_1(s)e^{-2Ts} + F_1(s)e^{-3Ts} + \dots$$

$$= F_1(s)\left(1 + e^{-Ts} + e^{-2Ts} + e^{-3Ts} + \dots\right) = \frac{F_1(s)}{1 - e^{-Ts}}$$

Properties of the one-sided Laplace transform

Property	$x(t)$	$X(s)$
Linearity	$c_1x_1(t)+c_2x_2(t)$	$c_1X_1(s)+c_2X_2(s)$
Scaling	$x(at)$	$X(s/a)/a$
Time shift	$x(t-a)u(t-a)$	$e^{-as}X(s)$
Frequencyshift	$e^{-at}x(t)$	$X(s+a)$
d/dt	dx/dt	$sX(s)-x(0)$
$\int d\tau$	$\int_0^t x(\tau)d\tau$	$X(s)/s$
Frequency differentiation	$t\,x(t)$	$-dX/ds$
Frequency integration	$x(t)/t$	$\int_s^\infty X(\sigma)d\sigma$
$x(t)$ is periodic	$x(t)=x(t+nT)$	$X_1(s)/\left(1-e^{-sT}\right)$
Initial value	$x(0)$	$\lim_{s\to\infty} sX(s)$
Final value	$x(\infty)$	$\lim_{s\to 0} sX(s)$
Convolution	$x_1(t)*x_2(t)$	$X_1(s)X_2(s)$

One-sided Laplace transform pairs

Defined for $t \geq 0$

It is assumed that $x(t) = 0$ for $t < 0$

$x(t)$	$X(s)$
$\delta(t)$	1
$u(t)$	$1/s$
e^{-at}	$1/(s+a)$
t	$1/s^2$
t^n	$n!/s^{n+1}$
$t^n e^{-at}$	$n!/(s+a)^{n+1}$
$\sin(\omega t)$	$\omega/(s^2+\omega^2)$
$\cos(\omega t)$	$s/(s^2+\omega^2)$
$e^{-at}\sin(\omega t)$	$\omega/\left[(s+a)^2+\omega^2\right]$
$e^{-at}\cos(\omega t)$	$(s+a)/\left[(s+a)^2+\omega^2\right]$

Laplace transform examples

$$x(t) = e^{-at}$$

$$X(s) = \int_0^\infty x(t)\, e^{-st} dt = \int_0^\infty e^{-(s+a)t} dt = \frac{1}{s+a}$$

$$x(t) = \cos \omega t$$

$$X(s) = \int_0^\infty \frac{e^{j\omega t} + e^{-j\omega t}}{2} e^{-st} dt = \frac{1}{2}\left(\frac{1}{s + j\omega} + \frac{1}{s - j\omega} \right) = \frac{s}{s^2 + \omega^2}$$

$$x(t) = \sin \omega t$$

$$X(s) = \int_0^\infty \frac{e^{j\omega t} - e^{-j\omega t}}{2j} e^{-st} dt = \frac{1}{2j}\left(\frac{1}{s - j\omega} + \frac{1}{s + j\omega} \right) = \frac{\omega}{s^2 + \omega^2}$$

Laplace transform examples

$$x(t) = 1 \quad X(s) = 1/s$$

$$x(t) = t \quad X(s) = 1/s^2$$

$$L[dx/dt] = L[1] = s\,X(s) - x(0) = sX(s)$$

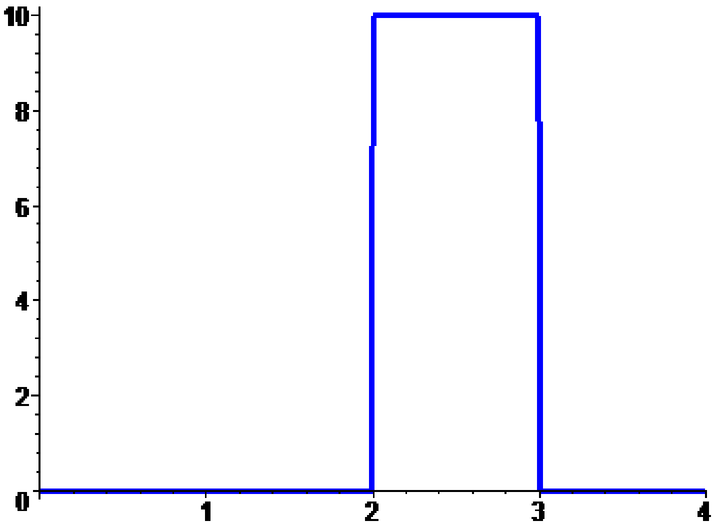
$$x(t) = t^2 \quad X(s) = 2/s^3$$

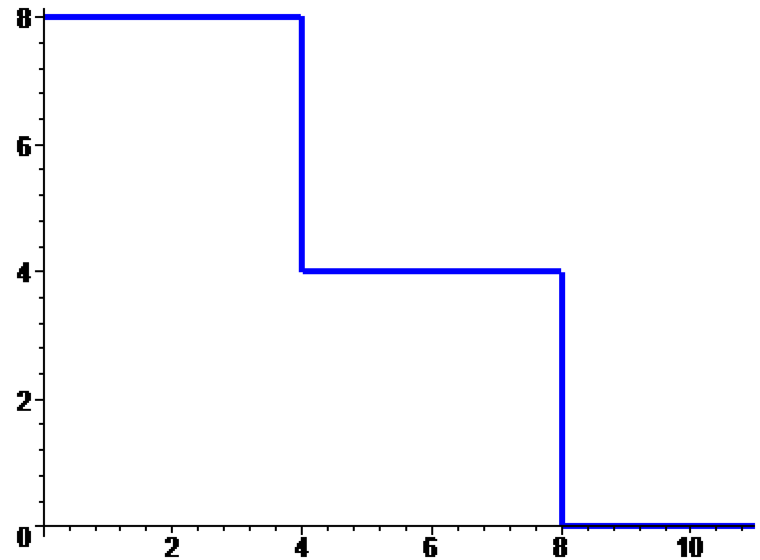
$$L[dx/dt] = L[2t] = 2L[t] = s\,X(s) - x(0) = sX(s)$$

$$x(t) = 5\cos 3t + 2\sin 5t - 6t^3$$

$$X(s) = \frac{5s}{s^2+9} + \frac{20}{s^2+25} - \frac{36}{s^4}$$

Laplace transform examples





$$x(t) = 10\left(u(t-2) - u(t-3)\right)$$

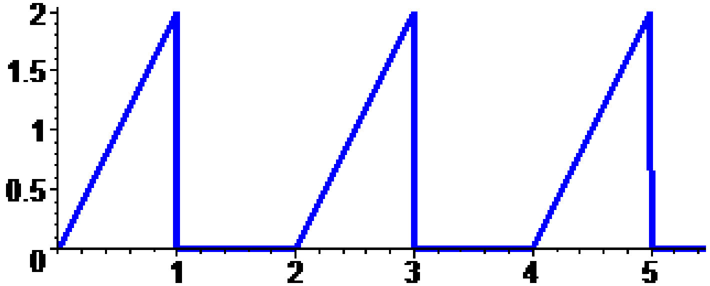
$$X(s) = 10\left(\frac{e^{-2s}}{s} - \frac{e^{-3s}}{s}\right)$$

$$x(t) = 8u(t) - 4u(t-4) - 4u(t-8)$$

$$X(s) = \frac{4}{s}\left(2 - e^{-4s} - e^{-8s}\right)$$

Laplace transform examples

Calculate the Laplace transform of the periodic function



$$x_1(t) = 2t[u(t) - u(t-1)] = 2tu(t) - 2(t-1)u(t-1) - 2u(t-1)$$

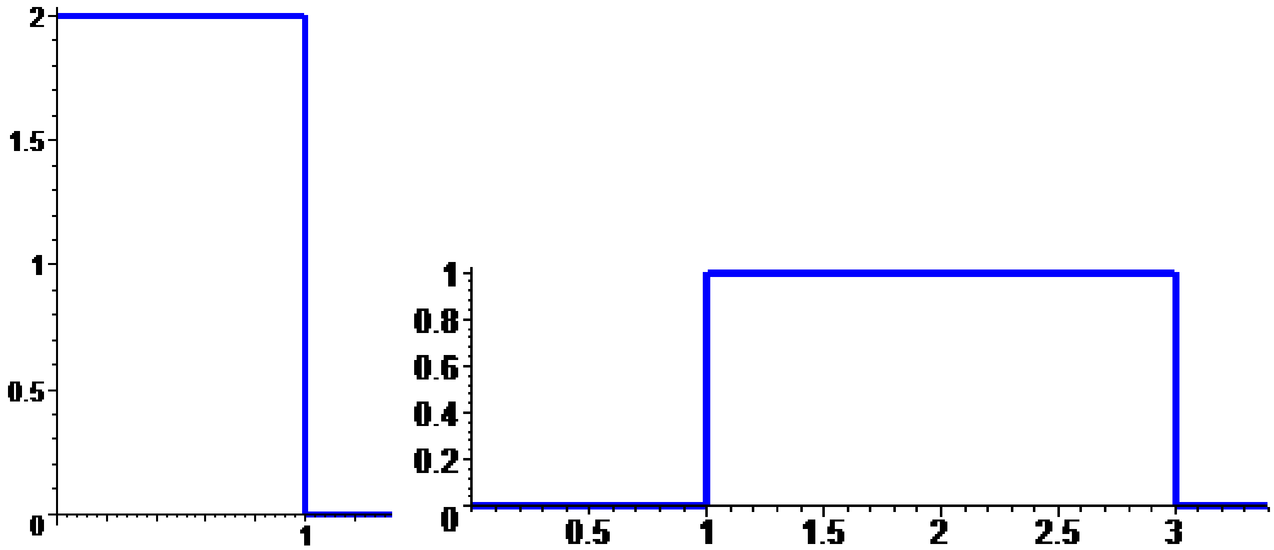
$$F_1(s) = \frac{2}{s^2} - \frac{2}{s^2}e^{-s} - \frac{2}{s}e^{-s} = \frac{2}{s^2}(1\text{-}\ e^{-s} - se^{-s})$$

$$F_1(s) = \frac{2}{s^2} - \frac{2}{s^2}e^{-s} - \frac{2}{s}e^{-s} = \frac{2}{s^2}(1\text{-}\ e^{-s} - se^{-s})$$

$$F(s) = \frac{F_1(s)}{1-e^{-Ts}} = \frac{2}{s^2(1-e^{2s})}(1\text{-}\ e^{-s} - se^{-s})$$

Laplace transform examples

Use the Laplace transform to find the convolution of two signals



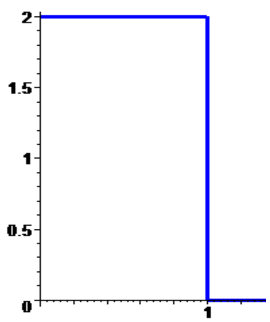
$$x_1(t) = 2[u(t) - u(t-1)] \quad X_1(s) = \frac{2}{s} - \frac{2e^{-s}}{s} = \frac{2}{s}\left(1 - e^{-s}\right)$$

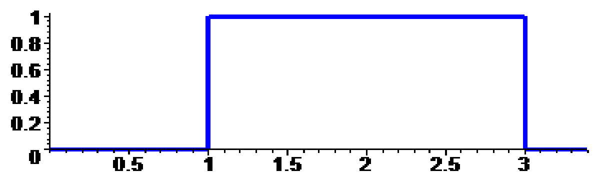
$$x_2(t) = u(t-1) - u(t-3) \quad X_2(s) = \frac{e^{-s}}{s} - \frac{e^{-3s}}{s} = \frac{e^{-s}}{s}\left(1 - e^{-2s}\right)$$

$$X_1(s)X_2(s) = \frac{2e^{-s}}{s^2}\left(1 - e^{-s}\right)\left(1 - e^{-2s}\right)$$

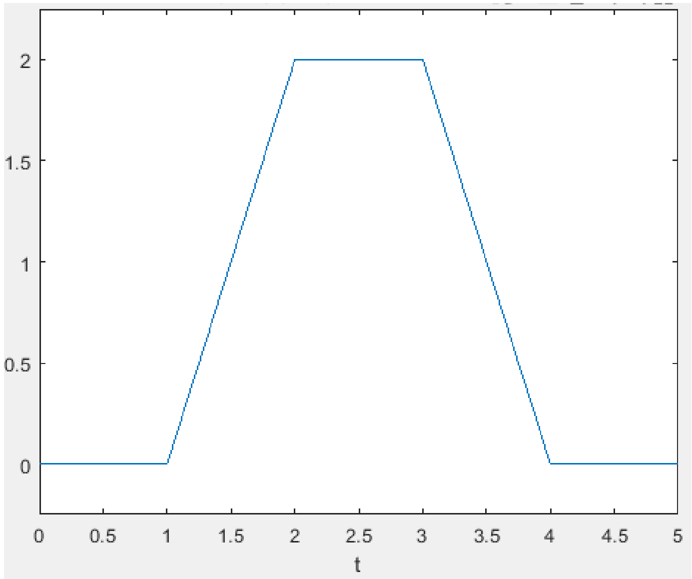
Laplace transform examples

Use the Laplace transform to find the convolution of two signals






```
syms X1 X2 x1 x2 t s
x1 = 2*heaviside(t)-2*heaviside(t-1)
x2 = heaviside(t-1)-heaviside(t-3)
X1 = laplace(x1,t,s)
X2 = laplace(x2,t,s)
x = ilaplace(X1*X2)
ezplot(x, [0,5])
```



LT and differential equations 1

Solve the differential equation

$$\frac{dx}{dt} + x = 9e^{2t}, \quad x(0) = 3$$

Solution:

$$L\left[\frac{dx}{dt}\right] + L[x] = 9L\left[e^{2t}\right] \qquad x(0) = 3$$

$$L\left[\frac{dx}{dt}\right] = sX(0) - x(0)$$

$$sX(s) - 3 + X(s) = \frac{9}{s-2}$$

$$X(s) = \frac{3}{(s+1)(s-2)} = -\frac{1}{S+1} + \frac{1}{S-2}$$

$$x(t) = -e^{-t}u(t) + e^{2t}u(t)$$

LT and differential equations 2

Solve the differential equation

$$\frac{d^2x}{dt^2}+5\frac{dx}{dt}+6x=2e^{-t}, \quad x(0)=1, \quad \frac{dx}{dt}(0)=0$$

LT and differential equations 2

Solve the differential equation

$$\frac{d^2x}{dt^2}+5\frac{dx}{dt}+6x=2e^{-t},\quad x(0)=1,\quad \frac{dx}{dt}(0)=0$$

Solution:

$$L\left[\frac{d^2x}{dt^2}\right]+L\left[5\frac{dx}{dt}\right]+L[6x]=L\left[2e^{-t}\right]\qquad x(0)=1,\quad \frac{dx}{dt}(0)=0$$

$$L[dx/dt]=s\,X(s)-x(0)$$

$$L\left[d^2x/dt^2\right]=sL[dx/dt]-x'(0)=s^2\,X(s)-s\,x(0)-x'(0)$$

LT and differential equations 2

Solve the differential equation

$$\frac{d^2x}{dt^2}+5\frac{dx}{dt}+6x=2e^{-t},\quad x(0)=1,\quad \frac{dx}{dt}(0)=0$$

Solution:

$$L\left[\frac{d^2x}{dt^2}\right]+L\left[5\frac{dx}{dt}\right]+L[6x]=L\left[2e^{-t}\right]\qquad x(0)=1,\quad \frac{dx}{dt}(0)=0$$

$$L[dx/dt]=s\,X(s)-x(0)$$

$$L\left[d^2x/dt^2\right]=sL[dx/dt]-x'(0)=s^2\,X(s)-s\,x(0)-x'(0)$$

$$\left[s^2X(s)-s\,x(0)-x'(0)\right]+5[X(s)-x(0)]+6X(s)=\frac{2}{s+1}$$

$$\left(s^2+5s+6\right)X(s)=\frac{2}{s+1}+s+5\qquad X(s)=\frac{1}{s+1}+\frac{1}{s+2}-\frac{1}{s+3}$$

$$x(t)=e^{-t}+e^{-2t}-e^{-3t},\quad t\geq 0$$

LT and differential equations 3

Solve the differential equation (linear, with constant coefficients)

$$\frac{d^3x}{dt^3}+5\frac{d^2x}{dt^2}+17\frac{dx}{dt}+13x=1, \quad x(0)=1=\frac{dx}{dt}(0), \quad \frac{d^2x}{dt^2}(0)=0$$

LT and differential equations 3

Solve the differential equation (linear, with constant coefficients)

$$\frac{d^3x}{dt^3} + 5\frac{d^2x}{dt^2} + 17\frac{dx}{dt} + 13x = 1, \quad x(0) = 1 = \frac{dx}{dt}(0), \quad \frac{d^2x}{dt^2}(0) = 0$$

Solution sketch:

$$L[dx/dt] = s\,X(s) - x(0)$$

$$L\left[d^2x/dt^2\right] = sL[dx/dt] - x'(0) = s^2\,X(s) - s\,x(0) - x'(0)$$

$$L\left[d^3x/dt^3\right] = sL\left[d^2x/dt^2\right] - x''(0) = \ldots$$

LT and differential equations 3

Solve the differential equation (linear, with constant coefficients)

$$\frac{d^3x}{dt^3} + 5\frac{d^2x}{dt^2} + 17\frac{dx}{dt} + 13x = 1, \quad x(0) = 1 = \frac{dx}{dt}(0), \quad \frac{d^2x}{dt^2}(0) = 0$$

Solution sketch:

$$L[dx/dt] = s\,X(s) - x(0)$$

$$L\left[d^2x/dt^2\right] = sL[dx/dt] - x'(0) = s^2\,X(s) - s\,x(0) - x'(0)$$

$$L\left[d^3x/dt^3\right] = sL\left[d^2x/dt^2\right] - x''(0) = \ldots$$

$$X(s) = \frac{s^3 + 6s^2 + 22s + 1}{s(s+1)\left(s^2 + 4s + 13\right)} = \frac{1}{13}\frac{1}{s} + \frac{8/5}{s+1} - \frac{1}{65}\frac{7 + 44s}{s^2 + 4s + 13}$$

$$x(t) = \frac{1}{13} + \frac{8}{5}e^{-t} - \frac{1}{65}e^{-2t}(44\cos 3t - 27\sin 3t), \quad t \geq 0$$

LT and differential equations 3

Solve the differential equation (linear, with constant coefficients)

$$\frac{d^3x}{dt^3}+5\frac{d^2x}{dt^2}+17\frac{dx}{dt}+13x=1, \quad x(0)=1=\frac{dx}{dt}(0), \quad \frac{d^2x}{dt^2}(0)=0$$

Matlab Solution:

```
clear all; close all; clc;

syms x(t)
Dx = diff(x,t);
D2x = diff(x,t,2);
D3x = diff(x,t,3);
ode = D3x+5*D2x+17*Dx+13*x == 1;
conds = [x(0)==1 Dx(0)==1 D2x(0)==0];
x = dsolve(ode,conds)
pretty(x)
```

```
(8*exp(-t))/5 - (44*cos(3*t)*exp(-2*t))/65 + (27*sin(3*t)*exp(-2*t))/65 + 1/13

8 exp(-t)   cos(3 t) exp(-2 t) 44   sin(3 t) exp(-2 t) 27    1
--------- - --------------------- + --------------------- + --
    5                65                      65              13
```

LT and differential equations 4

Solve the **system of differential equations** (linear, with constant coefficients)

$$\frac{dx}{dt}+\frac{dy}{dt}+5x+3y=e^{-t},\quad 2\frac{dx}{dt}+\frac{dy}{dt}+x+y=3,\quad x(0)=2,\quad y(0)=1$$

$$L[dx/dt]=s\,X(s)-x(0)\quad L[dy/dt]=s\,Y(s)-y(0)$$

$$\begin{cases} s\,X(s)-x(0)+s\,Y(s)-y(0)+5X(s)-3Y(s)=1/(s+1) \\ 2[\,sX(s)-x(0)]+s\,Y(s)-y(0)+X(s)+Y(s)=3/s \end{cases}$$

LT and differential equations 5

$$\frac{d^2 v(t)}{dt^2} + 6\frac{dv(t)}{dt} + 8v(t) = 2u(t), \quad v(0) = 1, \quad \frac{dv}{dt}(0) = -2$$

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

LT and differential equations 6

$$\frac{d^2v(t)}{dt^2}+6\frac{dv(t)}{dt}+8v(t)=2u(t),\quad v(0)=1,\quad \frac{dv}{dt}(0)=-2$$

$$u(t)=\begin{cases}1 & t>0\\ 0 & t<0\end{cases}$$

$$\left[s^2V(s)-sv(0)-v'(0)\right]+6\left[sV(s)-v(0)\right]+8V(s)=\frac{2}{s}$$

$$s^2V(s)-s+2+6sV(s)-6+8V(s)=\frac{2}{s}$$

$\ldots$