Laplace Transform in LTI Systems

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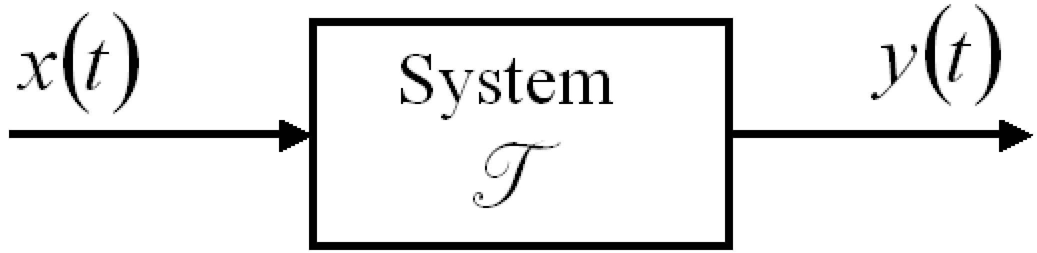
Content

- Transfer function.
- LTI systems in series and parallel. Inverse systems and linear feedback systems. Stability.
- Operational amplifiers, transfer functions for circuits with op amps.
- Bode plots.

Linear Time-Invariant (LTI) Systems

$$y(t) = \mathsf{T}\left[x(t)\right]$$



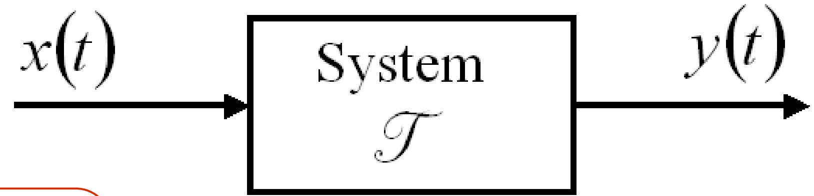
$$\mathsf{T}\left[x_1(t) + x_2(t)\right] = \mathsf{T}\left[x_1(t)\right] + \mathsf{T}\left[x_2(t)\right] = y_1(t) + y_2(t)$$

$$\mathsf{T}\left[\alpha\, x(t)\right] = \alpha\, \mathsf{T}\left[x(t)\right] = \alpha\, y(t)$$

$$\mathsf{T}\left[x(t - t_0)\right] = y(t - t_0)$$

Impulse response and response to an arbitrary input

$$y(t) = \mathrm{T}\,[x(t)]$$



$$h(t) = \mathrm{T}\,[\delta(t)]$$ *impulse response*

If $h(t) = 0$ for $t < 0$ the system is called ***causal***.

$$x(t) = \delta(t) * x(t) = \int_{-\infty}^{\infty} \delta(t-\tau)\,x(\tau)\,d\tau = \int_{-\infty}^{\infty} \delta(\tau)\,x(t-\tau)\,d\tau$$

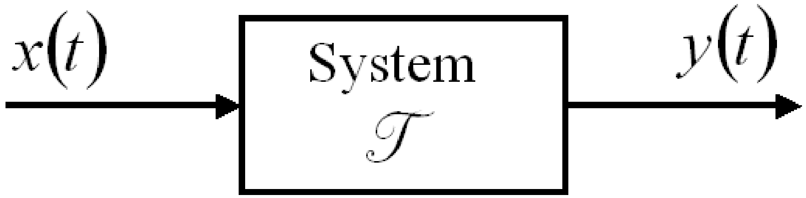
$$\mathrm{T}\,[x(t)] = \int_{-\infty}^{\infty} \mathrm{T}\,[\delta(t-\tau)]\,x(\tau)\,d\tau = \int_{-\infty}^{\infty} h(t-\tau)\,x(\tau)\,d\tau = h(t) * x(t)$$

The response $y(t)$ of an LTI system to an arbitrary input $x(t)$ is given by

$$\boxed{y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} x(t-\tau)\,h(\tau)\,d\tau}$$

Frequency response

$$y(t) = \mathsf{T}\left[x(t)\right]$$



$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} x(t-\tau)h(t)\,d\tau$$

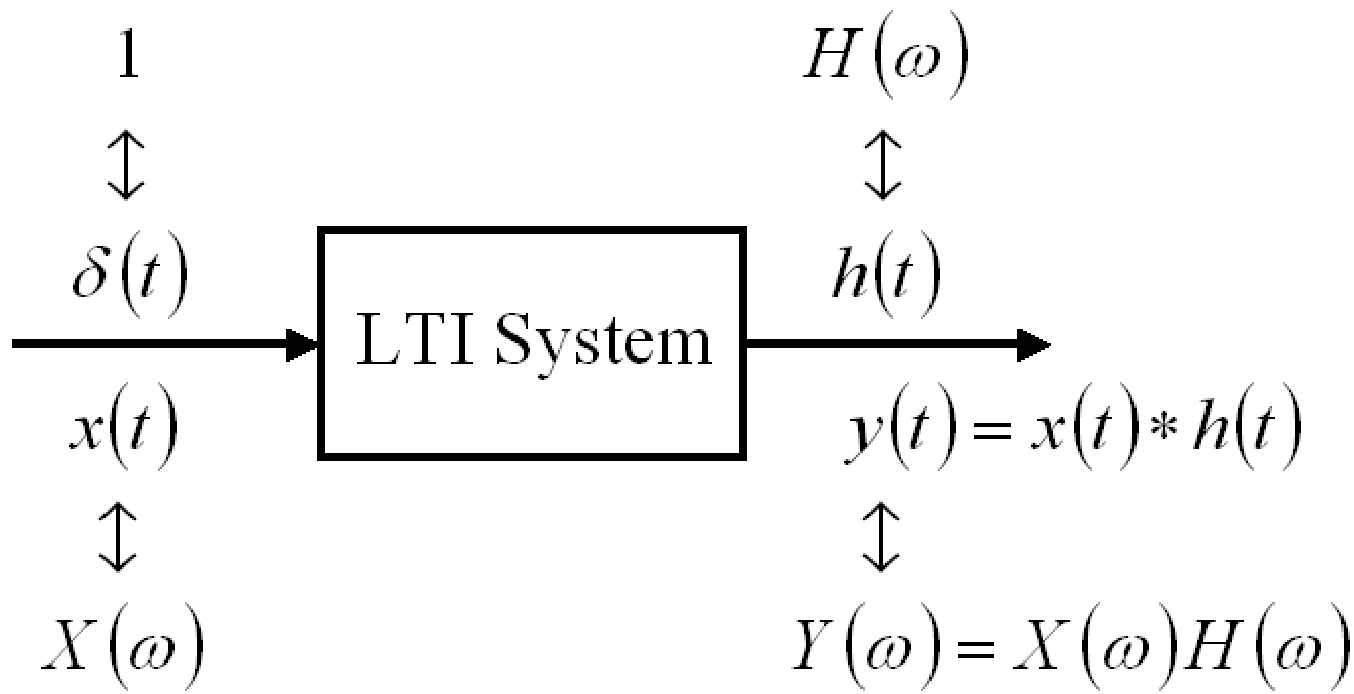
Let us apply the Fourier transform

$$X(\omega) = \mathsf{F}\left[x(t)\right] \quad H(\omega) = \mathsf{F}\left[h(t)\right] \quad Y(\omega) = \mathsf{F}\left[y(t)\right]$$

$$y(t) = h(t) * x(t) \quad \rightarrow \quad Y(\omega) = X(\omega)H(\omega)$$

$$\boxed{H(\omega) = \frac{Y(\omega)}{X(\omega)}}$$ *frequency response* (or *frequency transfer function*)

Frequency response



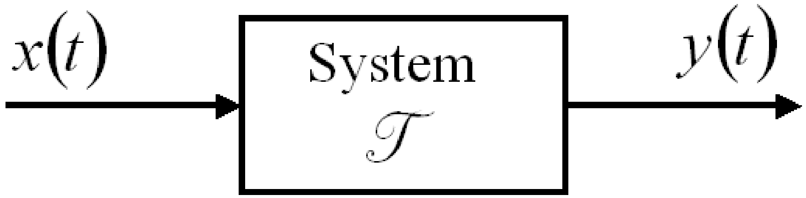
$$\mathsf{T}\left[e^{j\omega t}\right]=\int_{-\infty}^{\infty}h(\tau)e^{j\omega(t-\tau)}d\tau=\left[\int_{-\infty}^{\infty}h(\tau)e^{-j\omega\tau}d\tau\right]e^{j\omega t}=H(\omega)e^{j\omega t}$$

$$\boxed{\mathsf{T}\left[e^{j\omega t}\right]=H(\omega)e^{j\omega t}}$$

$$x(t)=\sum_{\omega}c_{\omega}e^{j\omega t}\rightarrow\sum_{\omega}c_{\omega}H(\omega)e^{j\omega t}$$

Transfer Function

$$y(t) = \mathrm{T}\,[x(t)]$$



$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} x(t-\tau)\,h(t)\,d\tau$$

Assume that we deal with causal systems and signals and apply the Laplace transform

$$X(s) = L[x(t)] \quad H(s) = L[h(t)] \quad Y(s) = L[y(t)]$$

$$y(t) = h(t) * x(t) \quad \to \quad Y(s) = X(s)H(s)$$

$$\boxed{H(s) = \frac{Y(s)}{X(s)} \quad \textit{transfer function}}$$

Transfer Function Examples 1

Consider and LTI system for which

$$\frac{dy}{dt} = \frac{dx}{dt} - 5x(t)$$

Assume that the system is causal and find the impulse response $h(t)$.

Solution.

Applying the Laplace transform yields

$$sY(s) = sX(s) - 5X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s-5}{s} = 1 - \frac{5}{s}$$

$$h(t) = \delta(t) - 5u(t)$$

Transfer Function Examples 2

Consider and LTI system for which

$$\frac{dy}{dt}+3y(t)=\frac{dx}{dt}+x(t)$$

Assume that the system is causal and find the impulse response $h(t)$.

Solution.

Applying the Laplace transform yields

$$sY(s)+3Y(s)=sX(s)+X(s)$$

$$H(s)=\frac{Y(s)}{X(s)}=\frac{s+1}{s+3}=1-\frac{2}{s+3}$$

$$h(t)=\delta(t)-2e^{-3t}u(t)$$

Transfer Function Examples 3

Consider and LTI system for which

$$\frac{d^2y}{dt^2}+\frac{dy}{dt}-6y(t)=x(t)$$

Assume that the system is causal and find the impulse response $h(t)$.

Solution.

Applying the Laplace transform yields

$$s^2Y(s)+sY(s)-6Y(s)=X(s)$$

$$H(s)=\frac{Y(s)}{X(s)}=\frac{1}{s^2+s-6}=\frac{1}{(s+3)(s-2)}=\frac{A}{s+3}+\frac{B}{s-2}$$

$$H(s)=-\frac{1}{5}\frac{1}{s+3}+\frac{1}{5}\frac{1}{s-2} \qquad h(t)=-\frac{1}{5}\left(e^{-3t}-e^{2t}\right)u(t)$$

Transfer Functions for Circuit analysis and Design

The **transfer function** $H(s)$ is the ratio of the output response $Y(s)$ to the input excitation $X(s)$, assuming all initial conditions are zero.

$$\boxed{H(s) = \frac{Y(s)}{X(s)}}$$

A circuit can have many transfer functions:

$H(s) = V_o(s)/V_i(s)$ Voltage gain

$H(s) = I_o(s)/V_i(s)$ Current gain

$H(s) = V(s)/I(s)$ Impedance

$H(s) = I(s)/V(s)$ Admittance

If the input is the unit impulse response, $x(t) = \delta(t)$, then $X(s) = 1$ and $Y(s) = H(s)$ or $y(t) = h(t)$, where $h(t) = \mathrm{L}^{-1}[H(s)]$ is called the unit impulse response.

Transfer Functions: Example 1

Find the transfer function of the LTI system and its impulse response.

$$x(t) = 2e^{-t}u(t) \longrightarrow \boxed{\begin{array}{c} H(s) = ? \\ h(t) = ? \end{array}} \longrightarrow y(t) = 10\, e^{-t}\cos(4t)\, u(t)$$

Transfer Functions: Example 1

Find the transfer function of the LTI system and its impulse response.

$$x(t) = 2e^{-t}u(t) \longrightarrow \boxed{\begin{matrix} H(s) = ? \\ h(t) = ? \end{matrix}} \longrightarrow y(t) = 10\,e^{-t}\cos(4t)\,u(t)$$

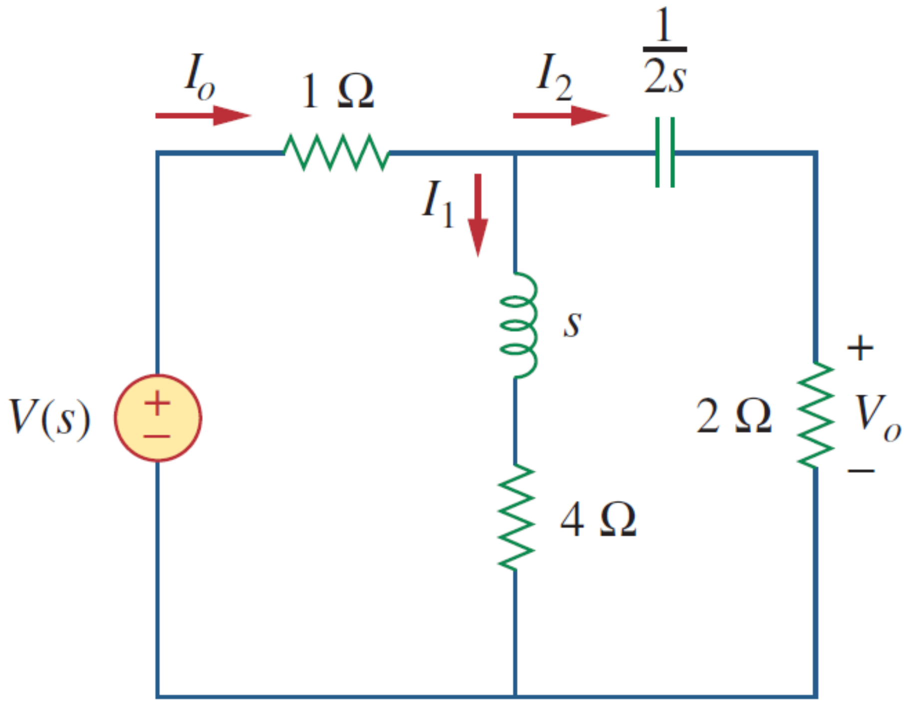
$$X(s) = \frac{2}{s+1} \qquad Y(s) = \frac{10(s+1)}{(s+1)^2+16} \qquad H(s) = \frac{5(s+1)^2}{(s+1)^2+16}$$

$$H(s) = \frac{5\left(s^2+2s+1\right)}{s^2+2s+17} = 5 - 20\,\frac{4}{(s+1)^2+16}$$

$$H(s) = L\left[h(t)\right] \quad \Rightarrow \quad h(t) = 5\,\delta(t) - 20\,e^{-t}\sin(4t)u(t)$$

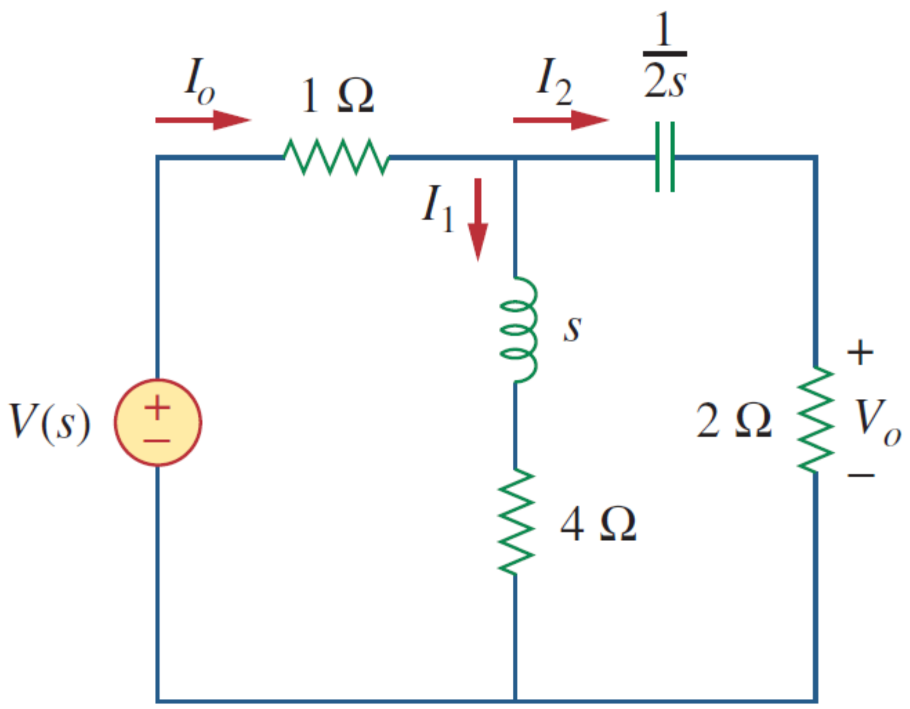
Transfer Functions: Example 2

Determine the transfer function $H(s) = V_o(s) / I_o(s)$



Transfer Functions: Example 2

Determine the transfer function $H(s) = V_o(s) \,/\, I_o(s)$

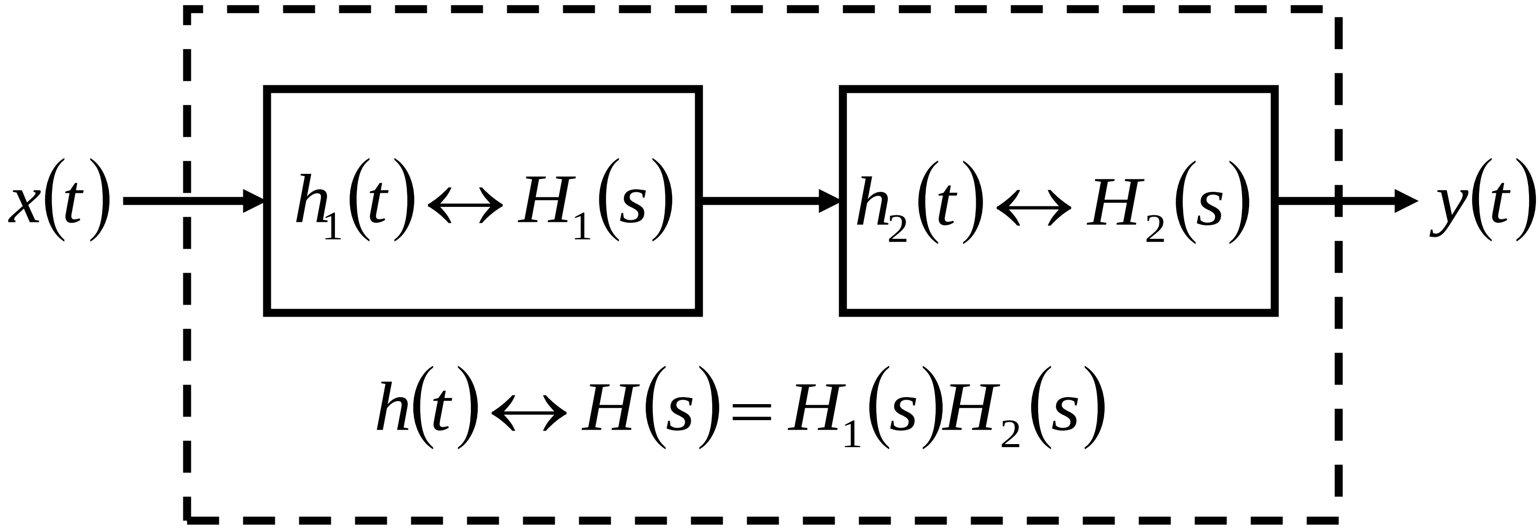


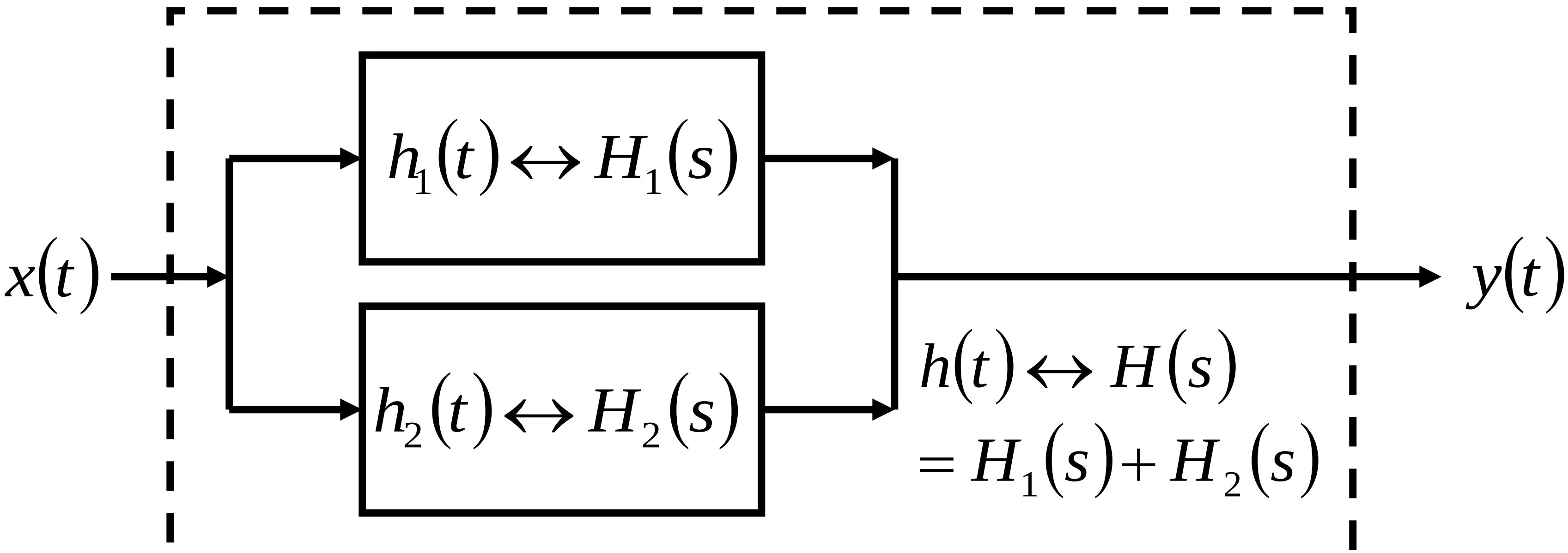
$$I_2 = \frac{(s+4)I_0}{(s+4)+(2+1/2s)}$$

$$V_0 = 2I_2 = \frac{2(s+4)I_0}{(s+4)+(2+1/2s)}$$

$$H(s) = \frac{V_0(s)}{I_0(s)} = \frac{4s(s+4)}{2s^2+12s+1}$$

LTI Systems in Series and Parallel





LTI Systems in Series and Parallel

Two systems are arranged in series with $h_1(t) = e^{-2t}u(t)$ and $h_2(t) = e^{-4t}u(t)$. Find the impulse response of the entire system.

Solution.

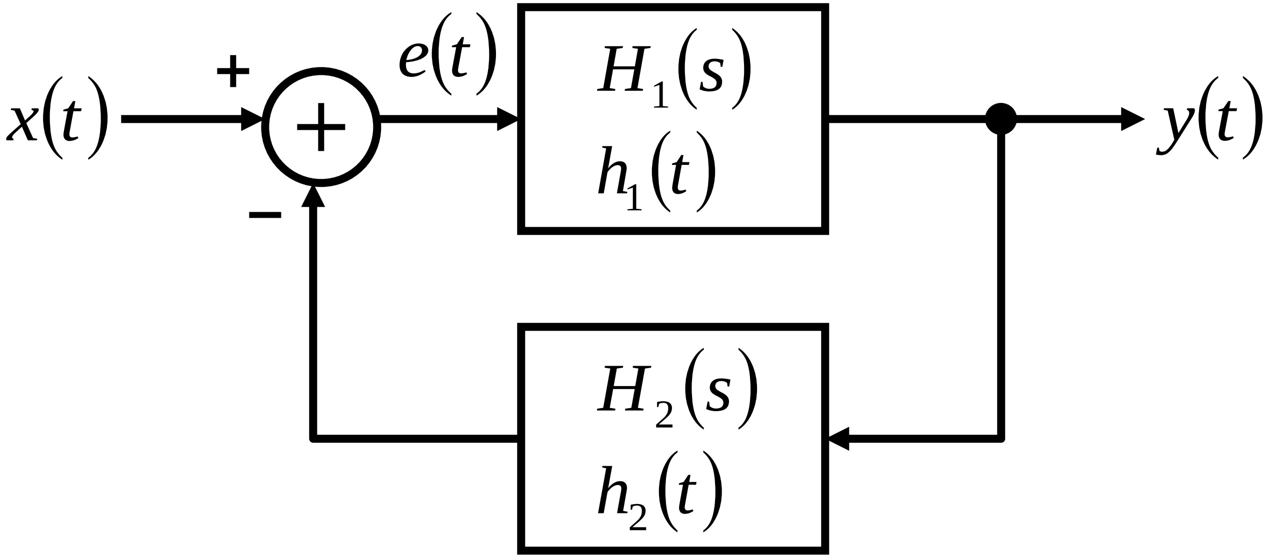
$$H_1(s) = \frac{1}{s+2} \qquad H_2(s) = \frac{1}{s+4}$$

$$H(s) = H_1(s)H_2(s) = \left(\frac{1}{s+2}\right)\left(\frac{1}{s+4}\right) = \frac{A}{s+2} + \frac{B}{s+4}$$

$$A = 1/2 = -B$$

$$H(s) = \frac{1}{2}\frac{1}{s+2} - \frac{1}{2}\frac{1}{s+4} \qquad h(t) = \frac{1}{2}\left(e^{-2t} - e^{-4t}\right)u(t)$$

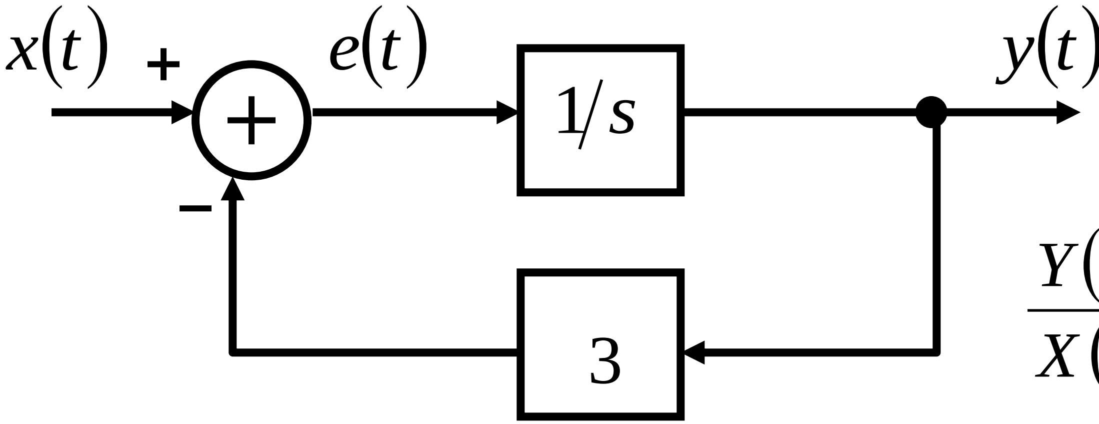
Block Diagram Representations



$$Y(s) = H_1(s)\left[X(s) - H_2(s)Y(s)\right] = H_1(s)X(s) - H_1(s)H_2(s)Y(s)$$

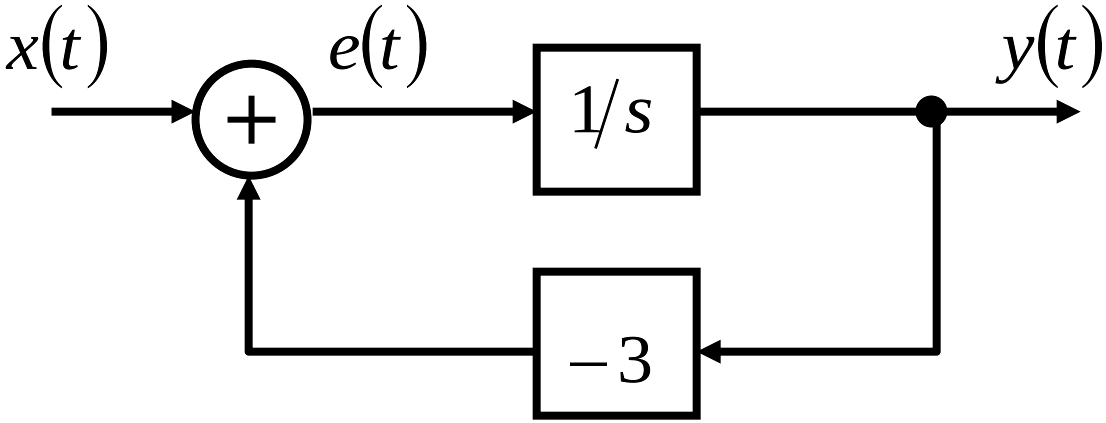
$$\frac{Y(s)}{X(s)} = H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

Block Diagram Representations: Example 1

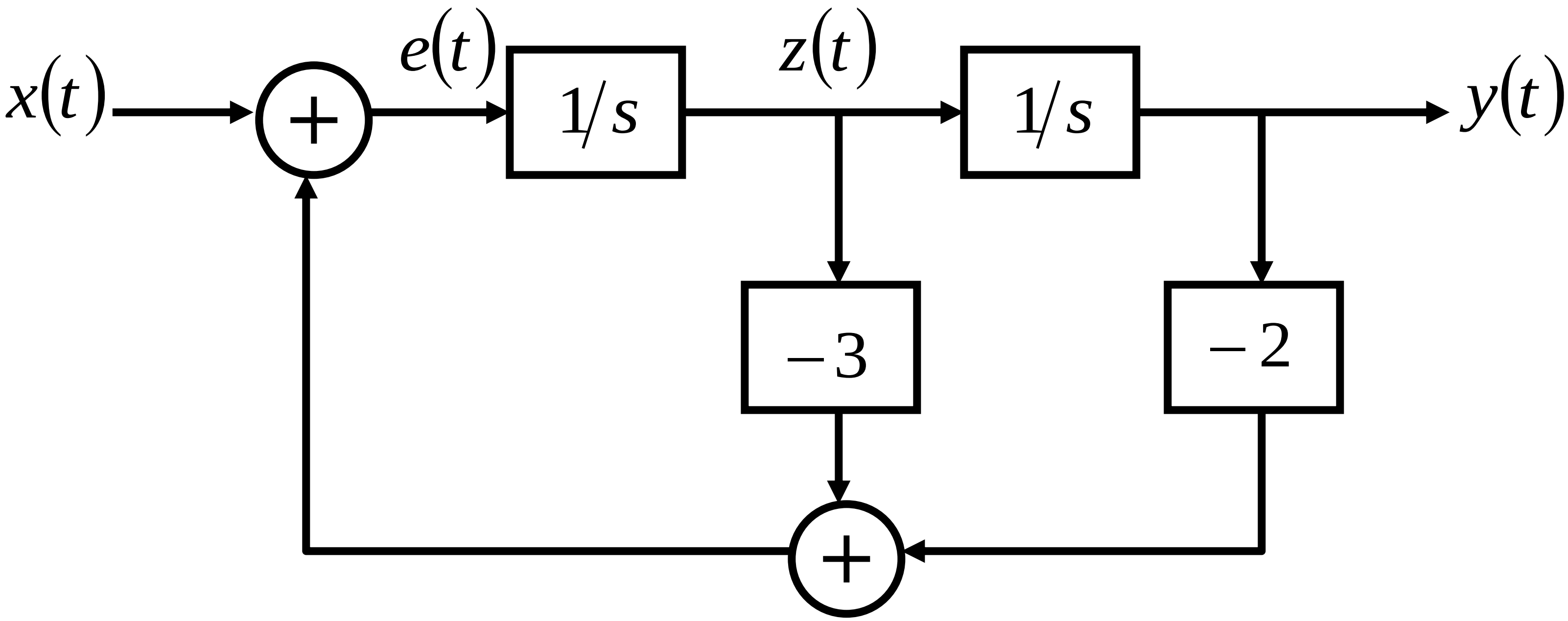


$$\frac{Y(s)}{X(s)} = H(s) = \frac{1/s}{1+3/s} = \frac{1}{s+3}$$

$$\frac{dy}{dt} + 3y(t) = x(t)$$



Block Diagram Representations: Example 2

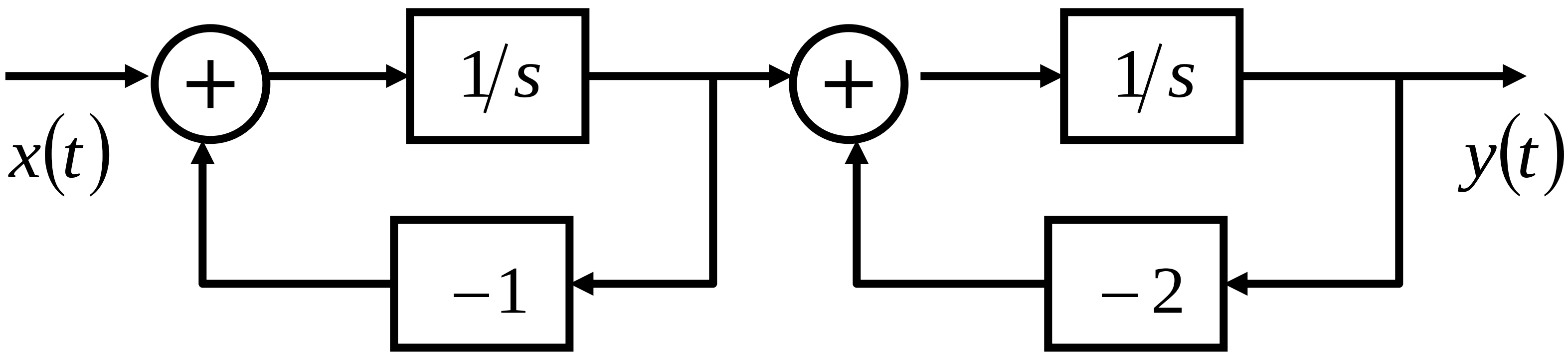


$$X\frac{1}{s^2}-Y\frac{2}{s^2}-Z\frac{3}{s^2}=Y \qquad Y=Z\frac{1}{s}$$

$$X\frac{1}{s^2}=Y+Y\frac{2}{s^2}+Y\frac{3}{s} \qquad Y=X\frac{1/s^2}{1+3/s+2/s^2}$$

$$H(s)=\frac{1}{s^2+3s+2} \qquad \frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y(t)=x(t)$$

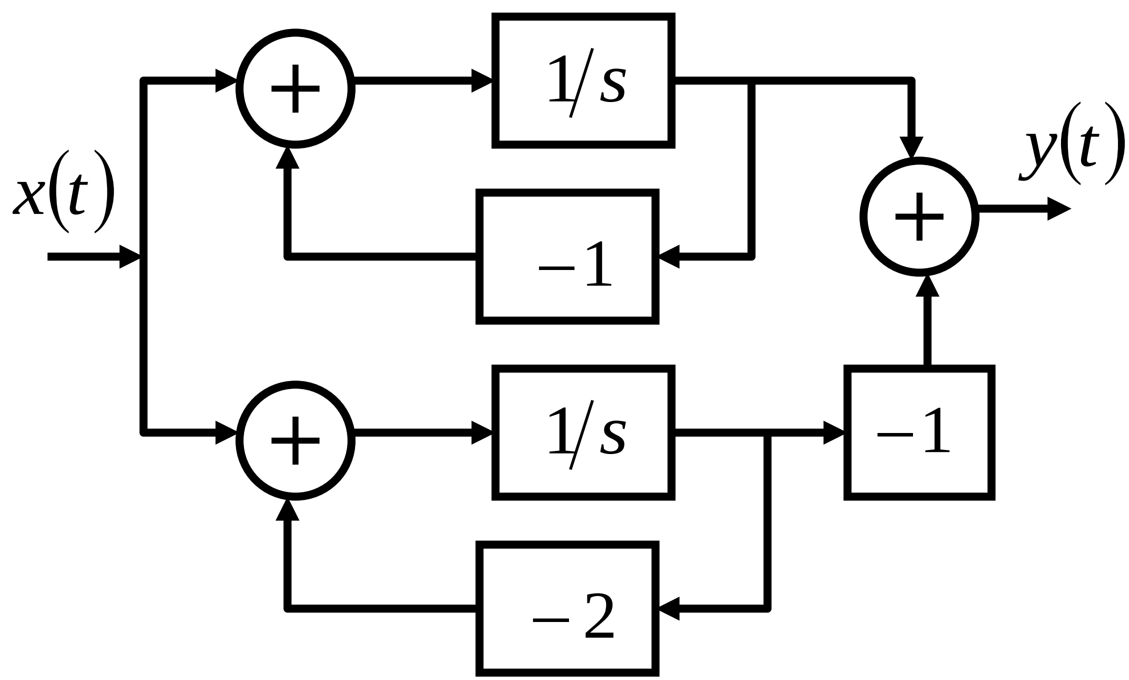
Block Diagram Representations: Example 3



$$H(s) = \frac{1}{s^2 + 3s + 2}$$

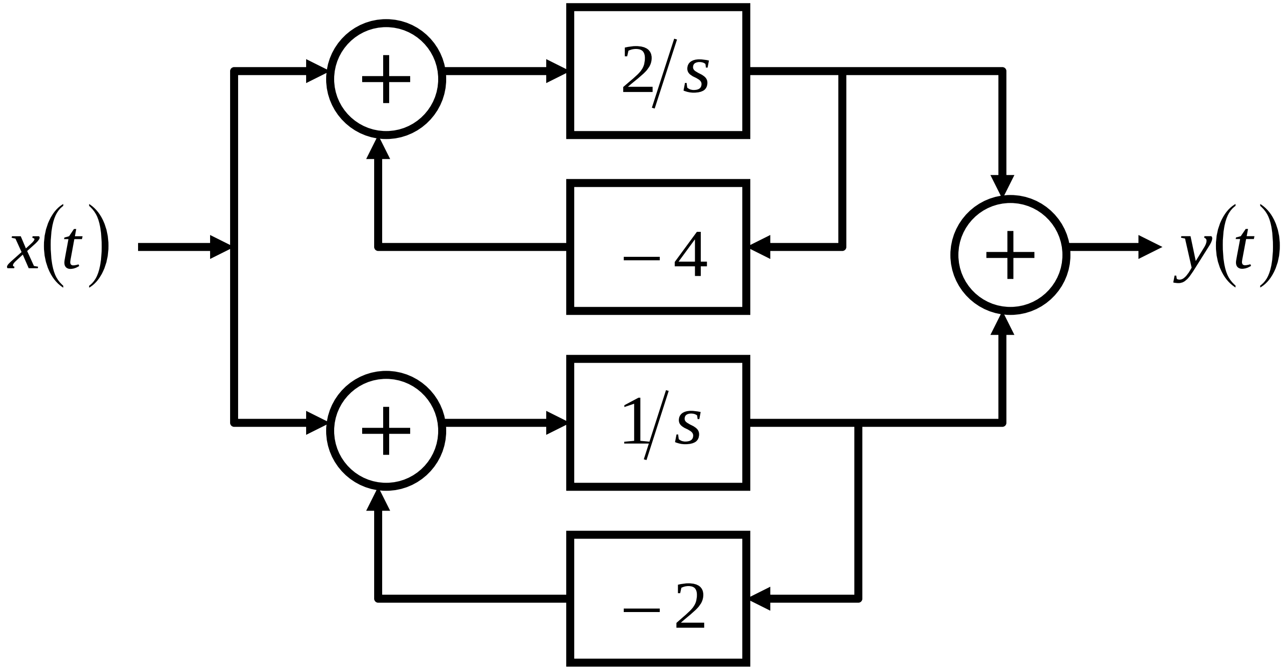
$$= \frac{1}{(s+1)(s+2)}$$

$$= \frac{1}{(s+1)} - \frac{1}{(s+2)}$$



Block Diagram Representations: Example 4

Determine a differential equation relating the input $x(t)$ to the output $y(t)$ of the system shown below.

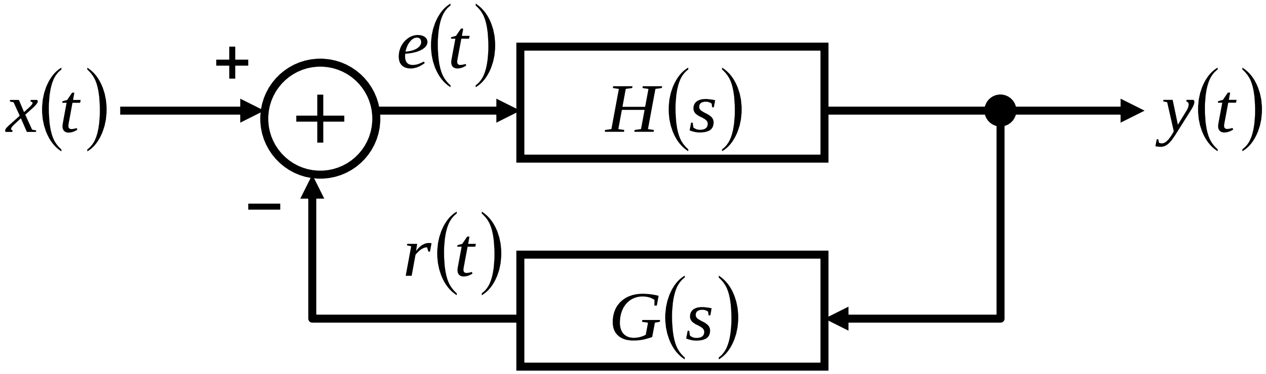


Solution.

$$H(s) = \frac{2/s}{1+8/s} + \frac{1/s}{1+2/s} = \frac{2}{s+8} + \frac{1}{s+2} = \frac{3s+12}{s^2+10s+16}$$

$$\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 16y(t) = 3\frac{dx}{dt} + 12x(t)$$

Linear Feedback Systems



$$Y(s) = H(s)X(s) - H(s)G(s)Y(s)$$

$$\frac{Y(s)}{X(s)} = Q(s) = \frac{H(s)}{1 + G(s)H(s)}$$

Inverse System Design:

Assume that K is sufficiently large.

So the feedback system approximates the inverse of the system with system function $P(s)$

$$H(s) = K \quad G(s) = P(s)$$

$$Q(s) = \frac{K}{1 + K\,P(s)} \approx \frac{1}{P(s)}$$

Inverse Systems: Removing echoes from acoustic signals

Oppenheim-Willsky, Problem 2.64. If an auditorium has a perceptible echo, then an initial acoustic impulse will be followed by attenuated versions of the sound at regularly spaced intervals. A good model for this phenomenon is an LTI system with an impulse response consisting of a train of impulses. h_0 is the gain of the initial impulse, h_2 and h_3… are that of the echos resulting from the initial impulse:

$$h(t) = \sum_{k=0}^{\infty} h_k \delta(t - kT)$$

Assume that h_0=1, h_1=1/2, and the others are zeros.

If y(t) denotes the actual signal heard without processing to remove the echoes,

- $y(t) = x(t) * h(t)$

Inverse Systems: Removing echoes from acoustic signals

If an LTI system with impulse response of g(t), is used to remove the echoes from y(t), the original sound signal x(t) is given by

$$x(t) = y(t) * g(t)$$

Then we have $$\mathrm{g}(t) * h(t) = \delta(t)$$

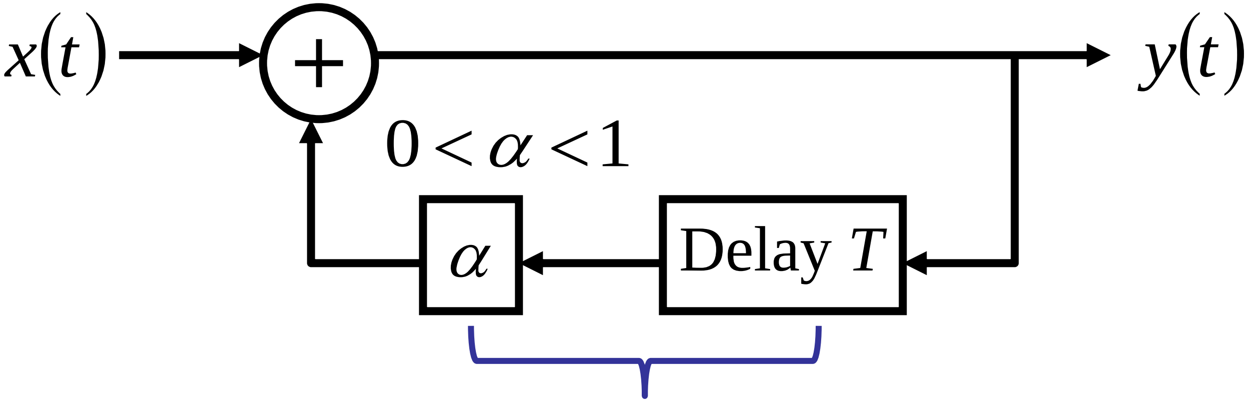
$$G(s)H(s) = 1$$

$$H(s) = 1 + e^{-Ts}/2$$

$$G(s) = \frac{1}{H(s)} = \frac{1}{1 + e^{-Ts}/2} = 1 - \frac{e^{-Ts}}{2} + \frac{e^{-2Ts}}{2^2} - \frac{e^{-3Ts}}{2^3} + \cdots$$

$$g(t) = \delta(t) + \sum_{k=1}^{\infty} \left(-\frac{1}{2}\right)^k \delta\,(t - kT)$$

A model for generation (representation) of echoes):



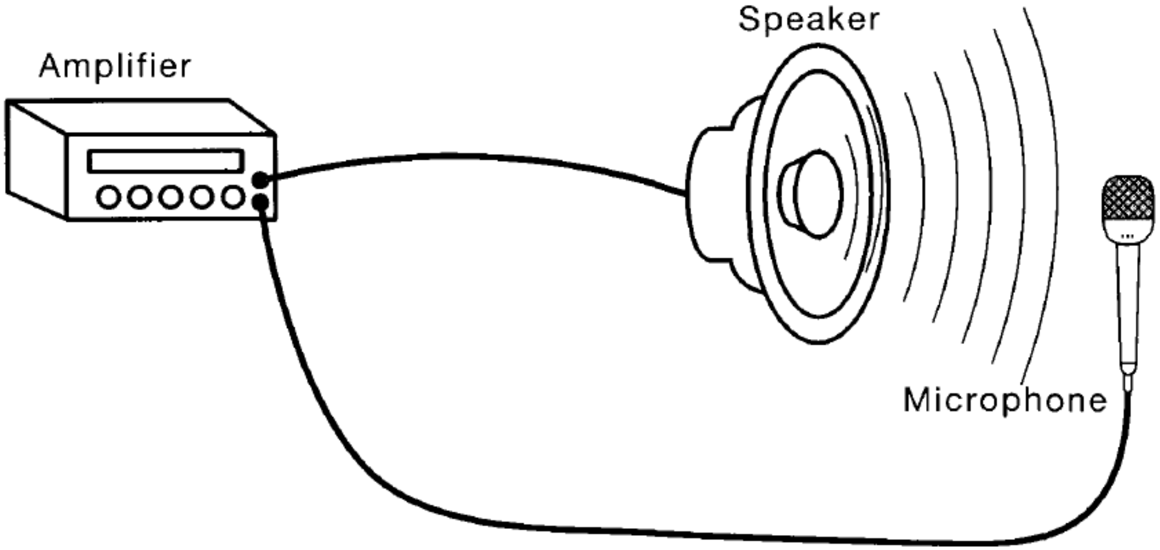
- $G(s)$ – echo generation

$$Y(s) = X(s) + G(s)Y(s) \qquad \frac{Y(s)}{X(s)} = Q(s) = \frac{1}{1-G(s)}$$

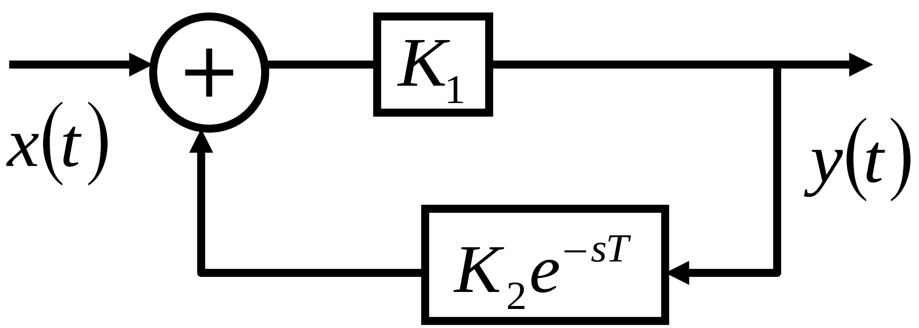
$$G(s) = \alpha e^{-Ts} \qquad Q(s) = \frac{1}{1-\alpha e^{-Ts}} = 1 + \alpha e^{-Ts} + \alpha^2 e^{-2Ts} + \dots$$

$$q(t) = \sum_{k=0}^{\infty} \alpha^k \delta(t - kT) \quad \text{the same form as} \quad h(t) = \sum_{k=0}^{\infty} h_k \delta(t - kT)$$

Linear Feedback Systems: amplifier, speaker, microphone



The system can be modelled as



$$Q(s) = \frac{K_1}{1 - K_1 K_2 e^{-sT}}$$

Stability for LTI Systems

A system is said to be stable in the bounded-input bounded-output (BIBO) sense if any bounded input signal is guaranteed to produce a bounded output signal.

$$x(t) \longrightarrow \boxed{\;h(t) \quad H(s)\;} \longrightarrow y(t) \qquad |x(t)| < B_x \quad \Rightarrow \quad |y(t)| < B_y$$

$$|y(t)| = \left| \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \right| \le \int_{-\infty}^{\infty} |h(\tau)||x(t-\tau)|\,d\tau \le B_x \int_{-\infty}^{\infty} |h(\tau)|\,d\tau$$

Thus, if $\displaystyle\int_{-\infty}^{\infty} |h(\tau)|\,d\tau < \infty$ the system is stable.

Stability for LTI Systems

Let
$$H(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s-p_1)(s-p_2)\ldots(s-p_n)}$$

$H(s)$ must meet two requirements for the system to be stable

1. The degree of $N(s)$ must be less than the degree of $D(s)$
2. All the poles $p_1, \ldots, p_n$ must have negative real parts

Then we have (for the sake of simplicity, assume that we have only simple roots)

$$h(t) = (k_1 e^{p_1 t} + k_2 e^{p_2 t} + \cdots + k_n e^{p_n t})u(t)$$

and the system is stable if all the poles $p_1, \ldots, p_n$ have negative real parts.

Stability: Examples

$$x(t) = e^{-3t}u(t) \quad \rightarrow \quad y(t) = \left[e^{-t} - e^{-2t}\right]u(t)$$

$$X(s) = \frac{1}{s+3} \qquad Y(s) = \frac{1}{(s+1)(s+2)} \qquad H(s) = \frac{s+3}{(s+1)(s+2)}$$

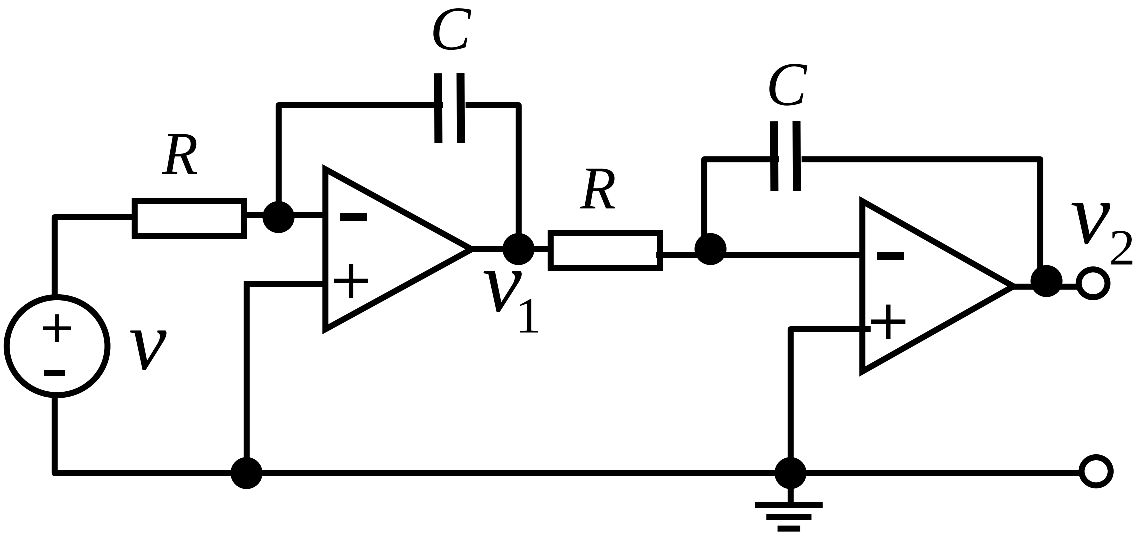
The system is stable since the poles are -1 and -2.

An active filter has the transfer function $H(s) = \dfrac{k}{s^2 + s(4-k) + 1}$

$$p_{1,2} = \frac{-(4-k) \pm \sqrt{(4-k)^2 - 4}}{2}$$

The filter is stable when –(4 – *k) <0, so k* < 4.

Stability: Examples



$$\frac{v-0}{R}=\frac{0-v_1}{1/(sC)} \quad \Rightarrow \quad v_1=-\frac{v}{s\,RC}$$

$$\frac{v_1-0}{R}=\frac{0-v_2}{1/(sC)} \quad \Rightarrow \quad v_2=-\frac{v_1}{s\,RC}$$

$$v_2=\frac{v}{s^2(RC)^2}$$

This circuit is said to be unstable since the poles are 0 and thus the real parts are not negative.