Fourier Series and Fourier Transform
Frequency response
Sampling
Amplitude and Angle Modulation

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Energy and power signals

Determine whether the following signals are energy signals, power signals, or neither. Justify your answers.

$$x(t) = 3e^{-5t}u(t) \qquad y(t) = 7\cos(5t) \qquad z(t) = 3\sqrt{t}\,u(t)$$

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$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = 9\int_0^{\infty} e^{-10t} dt = \frac{9}{10} < \infty$$

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |y(t)|^2 dt, \quad \frac{1}{T} \int_{-T/2}^{T/2} |y(t)|^2 dt = \frac{49}{T} \int_{-T/2}^{T/2} \cos^2(5t)\, dt = \frac{49}{T} \int_{-T/2}^{T/2} \frac{1+\cos(10\,t)}{2} dt$$

$$= \frac{49}{2} + \frac{49}{20T} \sin(10\,t)\Big|_{-T/2}^{T/2} = \frac{49}{2} + \frac{49}{20T}\sin(10T) \xrightarrow{T\to\infty} \frac{49}{2}$$

$$\frac{1}{T}\int_{-T/2}^{T/2} |z(t)|^2 dt = \frac{9}{T}\int_0^{T/2} t\, dt = \frac{9}{2T} t^2\Big|_0^{T/2} = \frac{9}{8}T \xrightarrow{T\to\infty} \infty$$

Power and Energy

Find the power of a sinusoid $C\sin(\omega_0 t+\theta)$

Solution.

$$\frac{1}{T}\int_{-T/2}^{T/2}|x(t)|^2\,dt=\frac{1}{T}\int_{-T/2}^{T/2}C^2\sin^2(\omega_0 t+\theta)\,dt=\frac{C^2}{T}\int_{-T/2}^{T/2}\frac{1-\cos 2(\omega_0 t+\theta)}{2}\,dt$$

$$=\frac{C^2}{2}-\frac{1}{2T}\int_{-T/2}^{T/2}\cos 2(\omega_0 t+\theta)\,dt\xrightarrow{T\to\infty}\frac{C^2}{2}$$

Fourier series

Find the complex Fourier coefficients of

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$$x(t) = \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right) + \left[\frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)\right]^2$$

$$= \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t} + \frac{1}{4}\left(e^{j2\omega_0 t} + 2 + e^{-j2\omega_0 t}\right)$$

$$c_0 = \frac{1}{2}, \quad c_1 = -\frac{j}{2}, \quad c_{-1} = \frac{j}{2}, \quad c_2 = c_{-2} = \frac{1}{4}$$

Fourier transform

Find the Fourier transform of $e^{-|t|}\cos(10t)$

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$$x(t)\cos(\omega_0 t) \xleftrightarrow{\mathrm{F}} \frac{1}{2}\left[X(\omega-\omega_0)+X(\omega+\omega_0)\right]$$

$$x(t)=e^{-|t|} \qquad X(\omega)=\mathrm{F}\left[x(t)\right]=\int_{-\infty}^{\infty} e^{-|t|}e^{-j\omega t}dt$$

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$$\int_{-\infty}^{\infty} e^{-|t|}e^{-j\omega t}dt=\int_{0}^{\infty} e^{-t}e^{-j\omega t}dt+\int_{-\infty}^{0} e^{t}e^{-j\omega t}dt$$

$$=-\frac{1}{1+j\omega}e^{-(1+j\omega)t}\Big|_{0}^{\infty}+\frac{1}{1-j\omega}e^{(1-j\omega)t}\Big|_{-\infty}^{0}=\frac{1}{1+j\omega}+\frac{1}{1-j\omega}=\frac{2}{1+\omega^2}$$

$$x(t)\cos(10t) \xleftrightarrow{\mathrm{F}} \frac{1}{1+(\omega-10)^2}+\frac{1}{1+(\omega+10)^2}$$

Fourier Transform

Find the Fourier transform of $x(t)=\sin^2\omega_0 t$

Solution.

$$x(t)=\sin^2\omega_0 t=\frac{1}{2}-\frac{1}{2}\cos 2\omega_0 t=\frac{1}{2}-\frac{1}{4}e^{j2\omega_0 t}-\frac{1}{4}e^{-j2\omega_0 t}$$

$$\mathrm{F}[1]=2\pi\delta(\omega)$$

$$\mathrm{F}\left[e^{j2\omega_0 t}\right]=2\pi\delta(\omega-2\omega_0) \qquad \mathrm{F}\left[e^{-j2\omega_0 t}\right]=2\pi\delta(\omega+2\omega_0)$$

$$\mathrm{F}[x(t)]=\pi\delta(\omega)-\frac{\pi}{2}\delta(\omega-2\omega_0)-\frac{\pi}{2}\delta(\omega+2\omega_0)$$

Fourier series and Fourier Transform

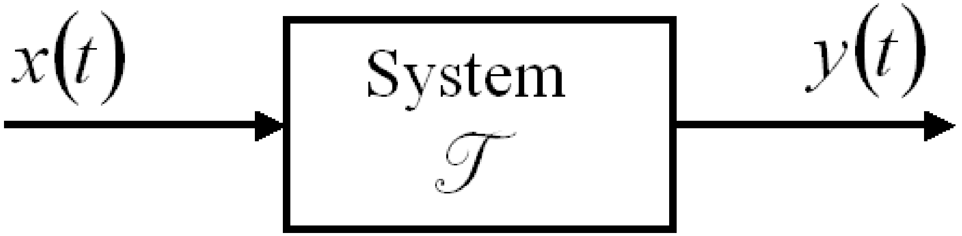
Find the Fourier transform of $x(t) = e^{-at}u(t)\cos\omega_0 t$ where $u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$

Solution.

$$e^{-at}u(t) \xrightarrow{F} \int_0^\infty e^{-at}e^{-j\omega t}dt = \int_0^\infty e^{-(a+j\omega)t}dt = \frac{1}{a+j\omega}$$

$$x(t) \xrightarrow{F} X(\omega) = \frac{1}{2}\left[\frac{1}{a+j(\omega-\omega_0)} + \frac{1}{a+j(\omega+\omega_0)}\right]$$

Frequency response $H(\omega)$



$$\boxed{h(t) = \mathrm{T}\left[\delta(t)\right] \quad \textit{impulse response}}$$

$$x(t) = \delta(t) * x(t) = \int_{-\infty}^{\infty} \delta(\tau)\, x(t-\tau)\, d\tau$$

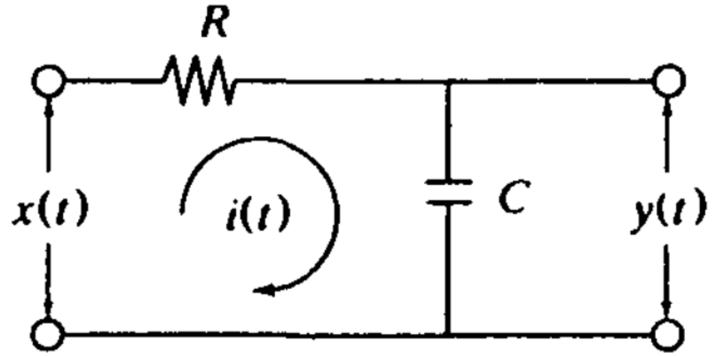
$$\mathrm{T}\left[x(t)\right] = \int_{-\infty}^{\infty} \mathrm{T}\left[\delta(\tau)\right] x(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = h(t) * x(t)$$

$$\mathrm{T}\left[e^{j\omega t}\right] = \int_{-\infty}^{\infty} h(\tau)\, e^{j\omega(t-\tau)}\, d\tau = \left[\int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau\right] e^{j\omega t} = H(\omega)\, e^{j\omega t}$$

$$\boxed{\mathrm{T}\left[e^{j\omega t}\right] = H(\omega)\, e^{j\omega t}}$$

Frequency response 1

Find frequency response $H(\omega)$



Frequency response 1

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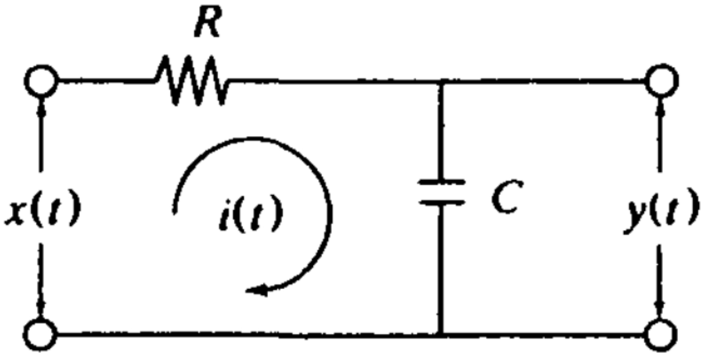
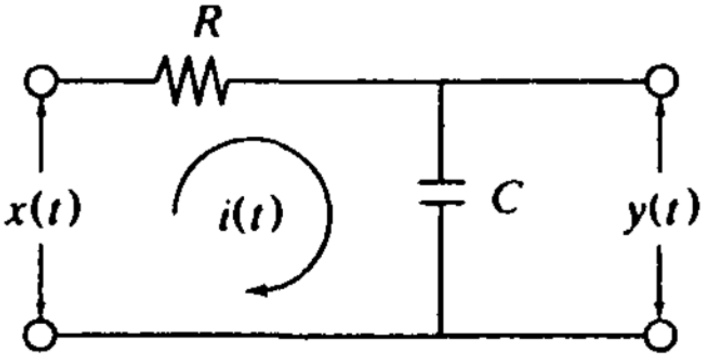


$$\mathrm{T}\left[e^{j\omega t}\right] = H(\omega)e^{j\omega t}$$

Frequency response 1

Find frequency response $H(\omega)$



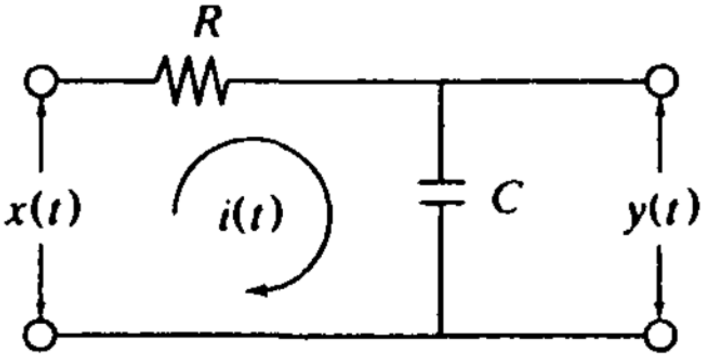
Solution. $\mathrm{T}\left[e^{j\omega t}\right] = H(\omega)e^{j\omega t}$

$$V_m \cos(\omega t + \varphi) = \mathrm{Re}\left[V_m e^{j\varphi} e^{j\omega t}\right] \quad V = V_m e^{j\varphi} \text{ phasor}$$

$$H(\omega) = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{1 + j\omega RC}$$

Now inverse Fourier transform of $H(\omega)$ yields $h(t)$, the corresponding impulse response.

Impulse response



Find frequency response $H(\omega)$ and impulse response $h(t)$

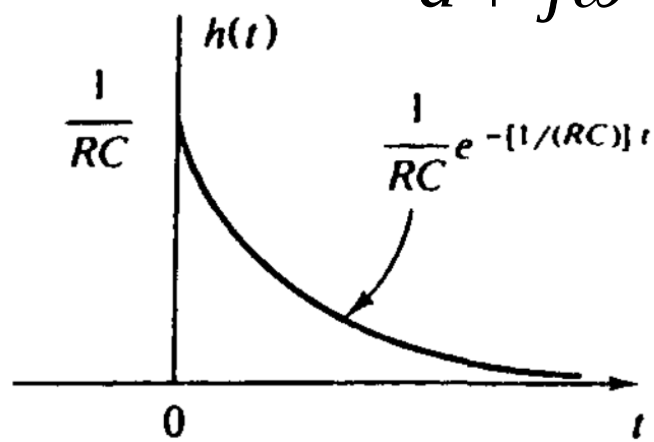
$$H(\omega)=\frac{1/j\omega C}{R+1/j\omega C}=\frac{1}{1+j\omega RC}$$

Let $h(t) = ae^{-at}u(t)$ Then

$$H(\omega)=\int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt = a\int_{0}^{\infty} e^{-at}e^{-j\omega t}dt = \frac{a}{a+j\omega}$$

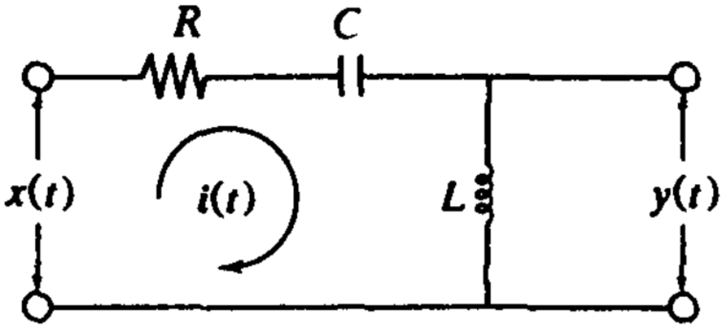
a = 1/RC

$$h(t)=\frac{1}{RC}e^{-t/(RC)}u(t)$$



Frequency response 2

Find the frequency response $H(\omega)$ of the following network



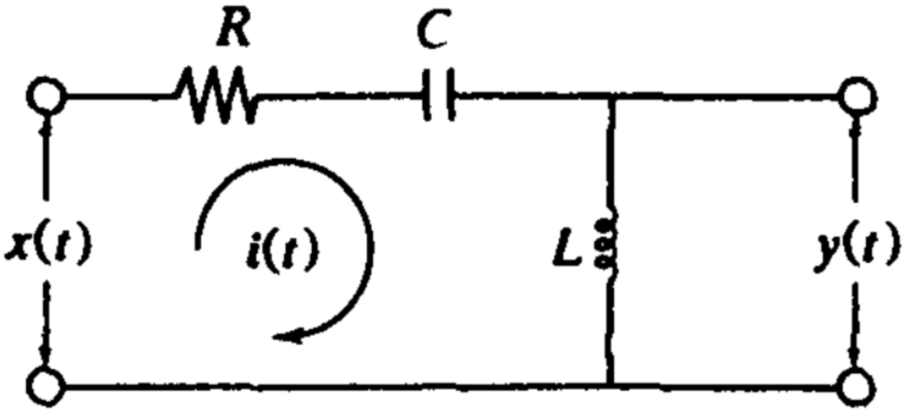
Solution.

Let $\boldsymbol{V}$ denote the phasor corresponding to $x(t)$. Then the phasor corresponding to the voltage across L is given by

$$\frac{j\omega L}{R+1/(j\omega C)+j\omega L}\boldsymbol{V}$$

Thus $\quad H(\omega)=\dfrac{j\omega L}{R+1/(j\omega C)+j\omega L}$

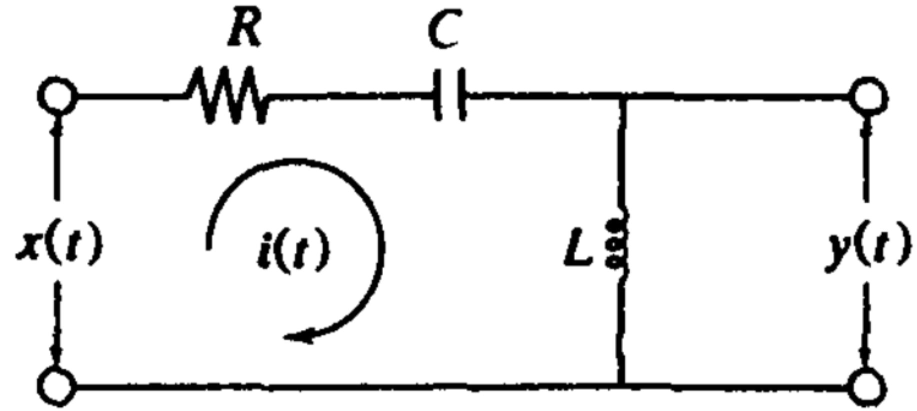
Frequency response 2



$$|H(\omega)| = -\frac{LC\omega^2}{1 - LC\omega^2 + j\omega RC}$$

Determine the response $y(t)$ of the circuit shown above when $x(t) = 5e^{-3t}u(t)$ V.

Frequency response 2



$$H(\omega) = -\frac{LC\omega^2}{1 - LC\omega^2 + j\omega RC}$$

Determine the response $y(t)$ of the circuit shown above when $x(t) = 5e^{-3t}u(t)$ V.

$$x(t) \overset{F}{\longleftrightarrow} X(\omega) \qquad y(t) \overset{F}{\longleftrightarrow} Y(\omega)$$

$$Y(\omega) = H(\omega)X(\omega)$$

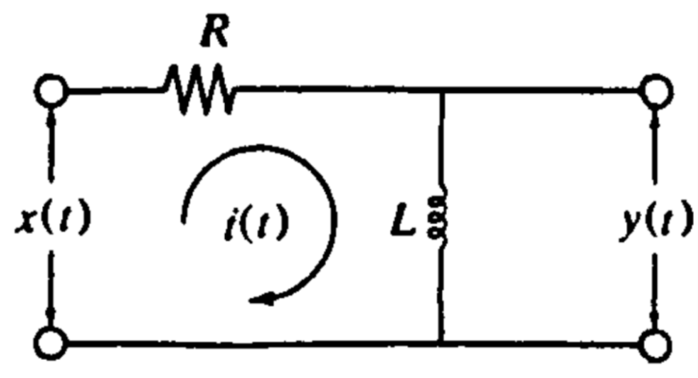
$$x(t) = 5e^{-3t}u(t) \overset{F}{\longleftrightarrow} \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = 5\int_{0}^{\infty} e^{-3t}e^{-j\omega t}dt$$

$$= -\frac{5}{3+j\omega}e^{-(3+j\omega)t}\Big|_0^\infty = \frac{5}{3+j\omega}$$

Frequency response and filtering

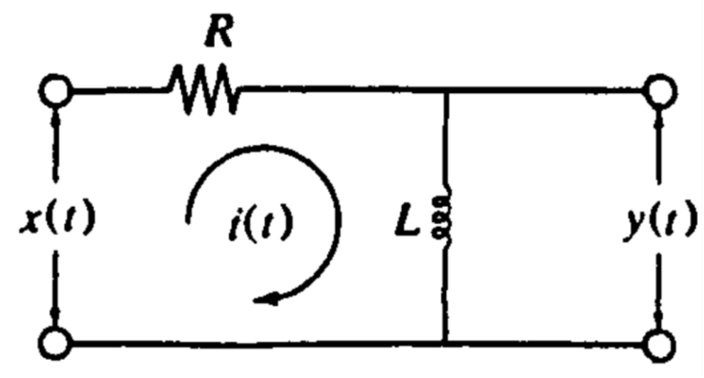
Find frequency response $H(\omega)$

Show that this *RL* network acts as a high-pass filter.



Frequency response and filtering

Find frequency response $H(\omega)$

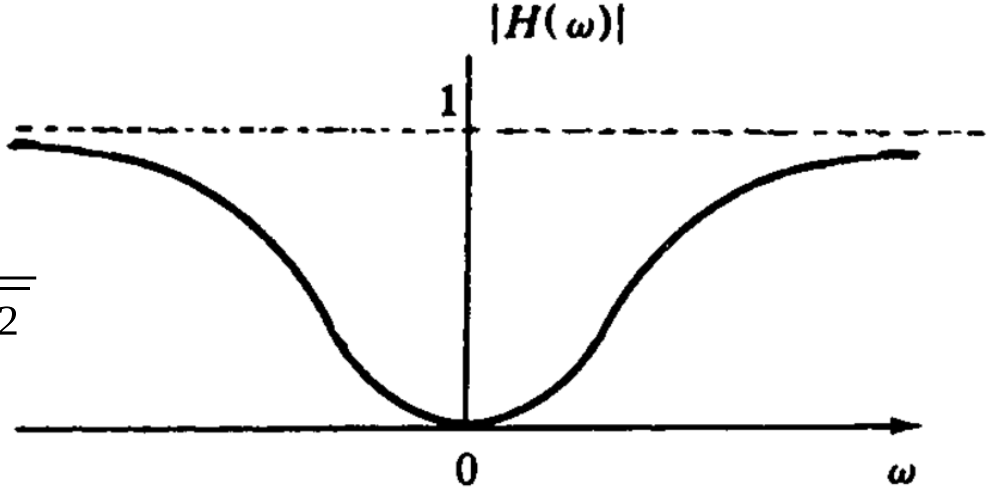


Show that this RL network acts as a high-pass filter.

$$H(\omega) = \frac{j\omega L}{R + j\omega L} = \frac{j(\omega/\omega_0)}{1 + j(\omega/\omega_0)}, \quad \omega_0 = R/L$$

$$|H(\omega)| = \left| \frac{j(\omega/\omega_0)}{1 + j(\omega/\omega_0)} \right|$$

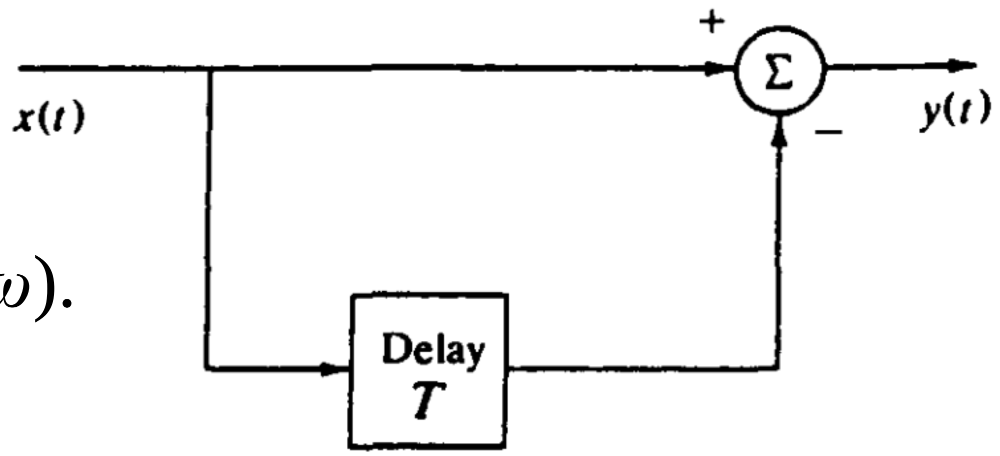
$$= \frac{|j(\omega/\omega_0)|}{|1 + j(\omega/\omega_0)|} = \frac{(\omega/\omega_0)}{\sqrt{1 + (\omega/\omega_0)^2}}$$



Frequency response and impulse response

$$y(t) = x(t) - x(t - T)$$

Find impulse response $h(t)$
and frequency response $H(\omega)$.



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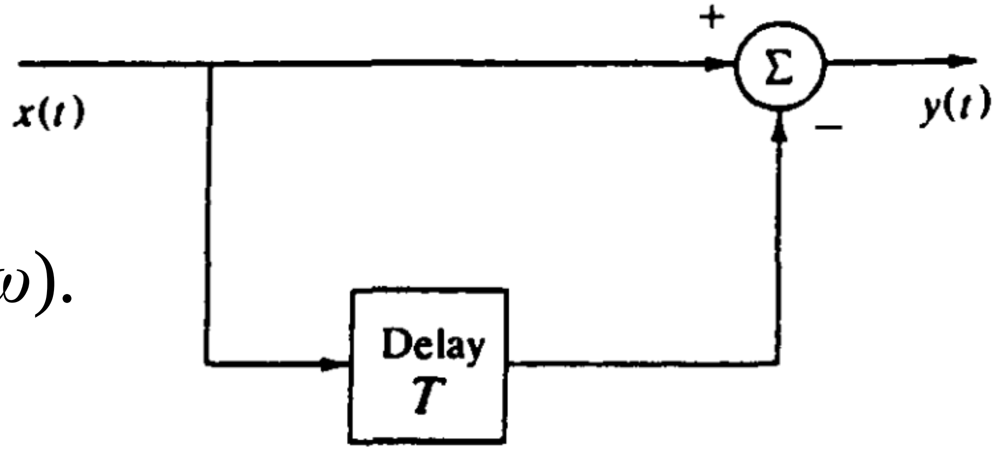


Find impulse response $h(t)$
and frequency response $H(\omega)$.

$$y(t) = x(t) - x(t-T)$$

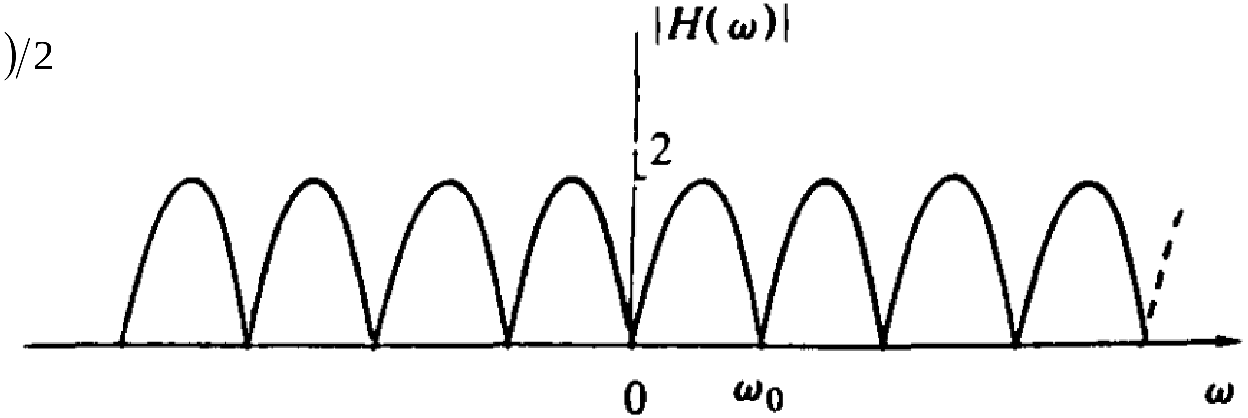
$$h(t) = \delta(t) - \delta(t-T)$$

$$x(t) = e^{j\omega t} \quad y(t) = e^{j\omega t} - e^{j\omega(t-T)} = H(\omega)e^{j\omega t} \quad H(\omega) = 1 - e^{-j\omega T}$$

$$H(\omega) = \mathsf{F}[h(t)] = 1 - e^{-j\omega T} = e^{-j\omega T/2}\left(e^{j\omega T/2} - e^{-j\omega T/2}\right)$$

$$= 2\sin\left(\frac{\omega T}{2}\right)e^{-j(\omega T+\pi)/2}$$

Comb filter



Amplitude modulation 1

A signal is band-limited to ω_M. It is frequency-translated by multiplying it by the carrier signal $\cos \omega_c t$. Find ω_c so that the bandwidth of the transmitted signal is 2 percent of the carrier frequency ω_c.

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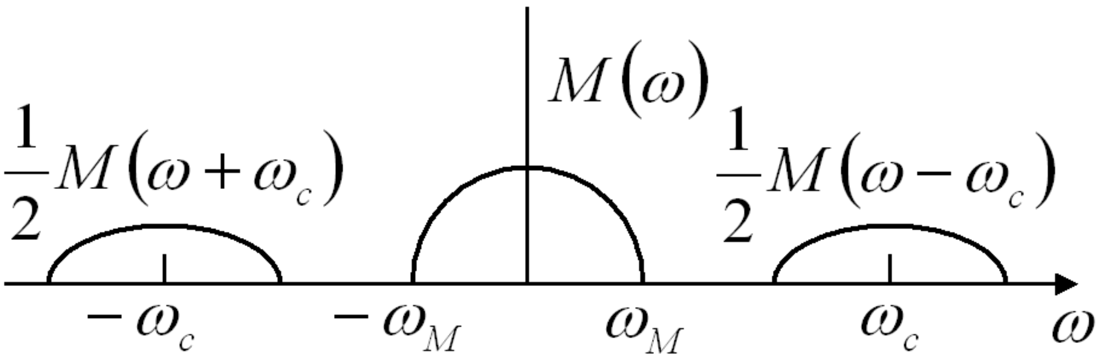

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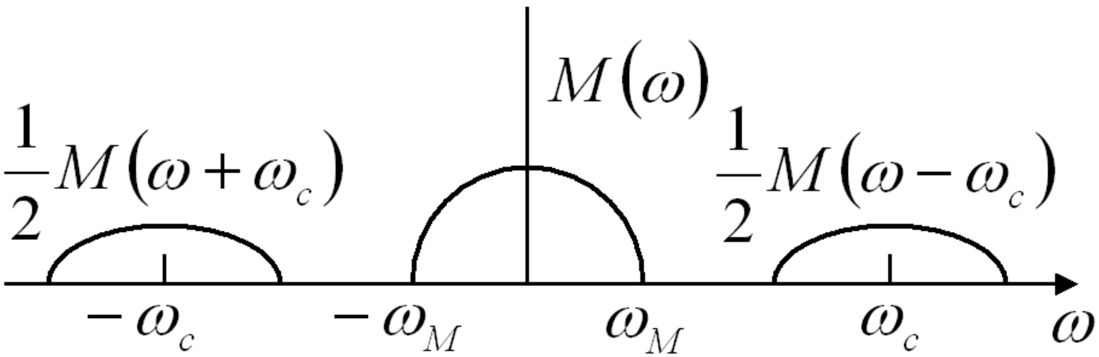

$$m(t)\cos\omega_c t \leftrightarrow \frac{1}{2}M(\omega-\omega_c)+\frac{1}{2}M(\omega+\omega_c)$$

$$2\omega_M = \omega_c/50$$

$$\omega_c = 100\,\omega_M \qquad f_c = 100\,f_M$$

Amplitude modulation 2

A single-tone audio signal $m(t) = A_m \cos\omega_m t$ is modulated by a high-frequency carrier wave $c(t) = A_c \cos\omega_c t$, where ω_c is much bigger than ω_m. Obtain and sketch the spectrum of the resulting DSB modulated signal.

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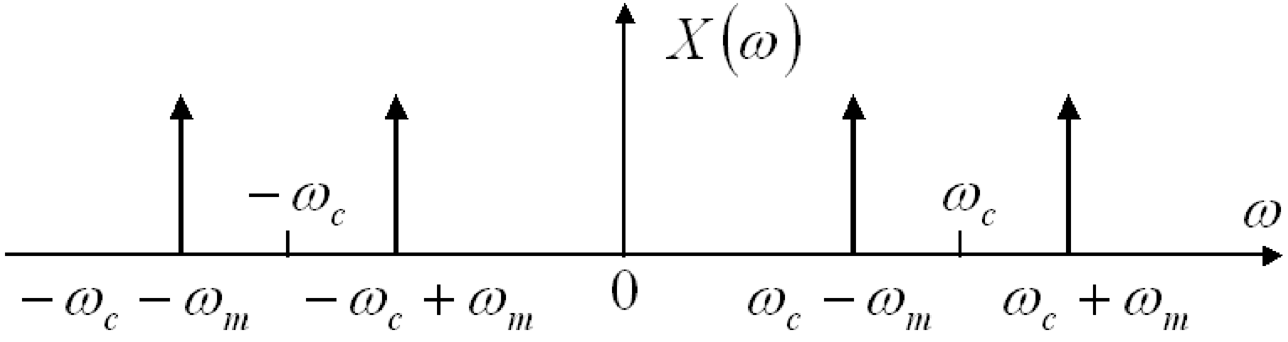
$$x(t) = m(t)c(t) = A_m A_c \cos\omega_m t \cos\omega_c t$$

$$= \frac{A_m A_c}{2}\left[\cos(\omega_c - \omega_m)t + \cos(\omega_c + \omega_m)t\right]$$

$$x(t) \leftrightarrow X(\omega) = \frac{A_m A_c}{2}\pi[\delta(\omega - \omega_c + \omega_m) + \delta(\omega + \omega_c - \omega_m)$$

$$+ \delta(\omega - \omega_c - \omega_m) + \delta(\omega + \omega_c + \omega_m)]$$



AM and envelope detection

An AM signal is of the form

$$x_{\mathrm{AM}}(t) = \left[1 + \alpha\cos\omega_m t + \alpha\cos 2\omega_m t\right]\cos\omega_c t \qquad \alpha > 0$$

Show that, to avoid distortions (if an envelope detector is used), $\alpha < 8/9$

AM and envelope detection

$$x_{\mathrm{AM}}(t) = [1 + \alpha\cos\omega_m t + \alpha\cos 2\omega_m t]\cos\omega_c t \qquad \alpha > 0$$

$$A(t) = 1 + \alpha\cos\omega_m t + \alpha\cos 2\omega_m t > 0 \quad \text{or} \quad \min A(t) > 0$$

$$\frac{dA(t)}{dt} = -\alpha\omega_m \sin\omega_m t - 2\alpha\omega_m \sin 2\omega_m t = -\alpha\omega_m \sin\omega_m t[1 + 4\cos\omega_m t] = 0$$

$$\sin\omega_m t = 0$$

$$\omega_m t = 0, \pi, 2\pi, \ldots \qquad \cos\omega_m t = -1/4$$

$$A(t) = 1 + 2\alpha \quad \text{for} \quad \omega_m t = 0, 2\pi, 4\pi, \ldots$$

$$A(t) = 1 \quad \text{for} \quad \omega_m t = \pi, 3\pi, 5\pi, \ldots$$

OR

$$\cos\omega_m t = -1/4$$

$$\cos 2\omega_m t = -7/8$$

$$A(t) = 1 - 9\alpha/8$$

$$\min A(t) = 1 - 9\alpha/8 > 0$$

$$\alpha < 8/9$$

Sampling: Example 1

A signal

$$x(t) = \cos 200\pi t + 2\cos 320\pi t$$

is ideally sampled at f_s=300 kHz. If the sampled signal is passed through an ideal low-pass filter with a cutoff frequency of 250 kHz, what frequency components will appear in the input?

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Answer: 100- 140- 160- and 200 Hz components

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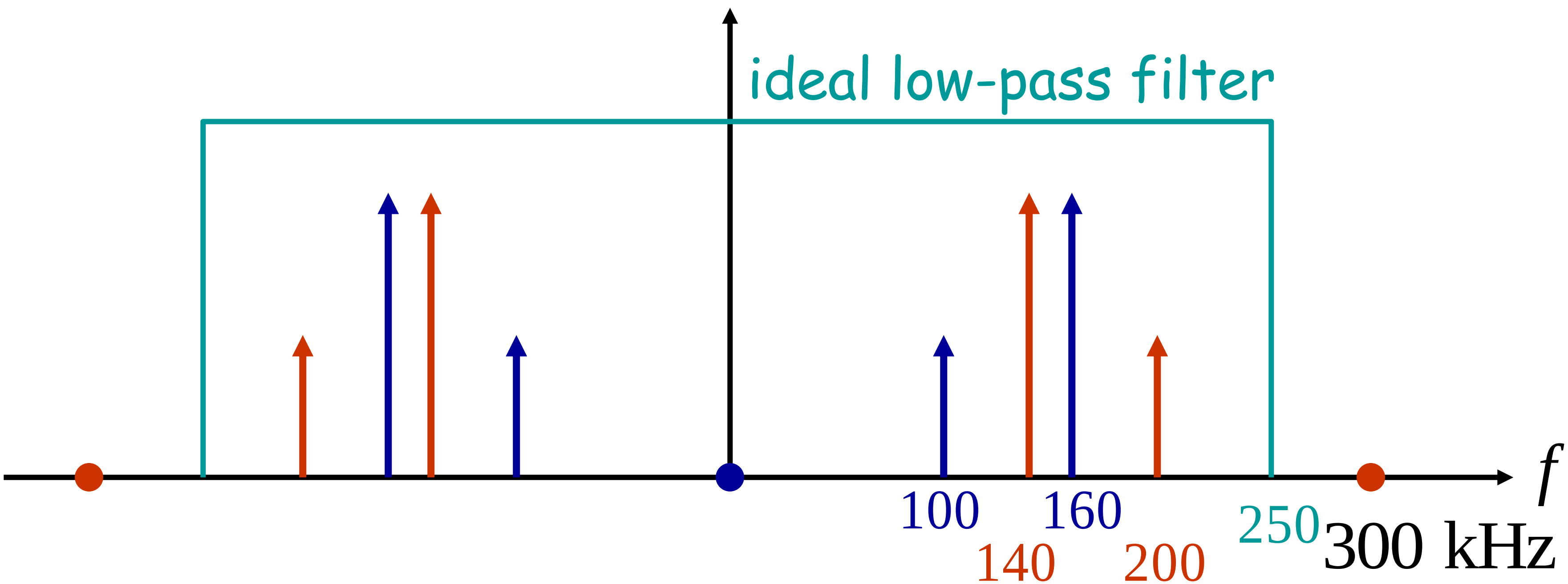

$140 = 300 - 160 \qquad 200 = 300 - 100$

Sampling: Example 2

The signals

$$x_1(t) = 10\cos 100\pi t \quad \text{and} \quad x_2(t) = 10\cos 50\pi t$$

are both sampled with f_s=75 Hz. Show that the two sequences of samples are identical.

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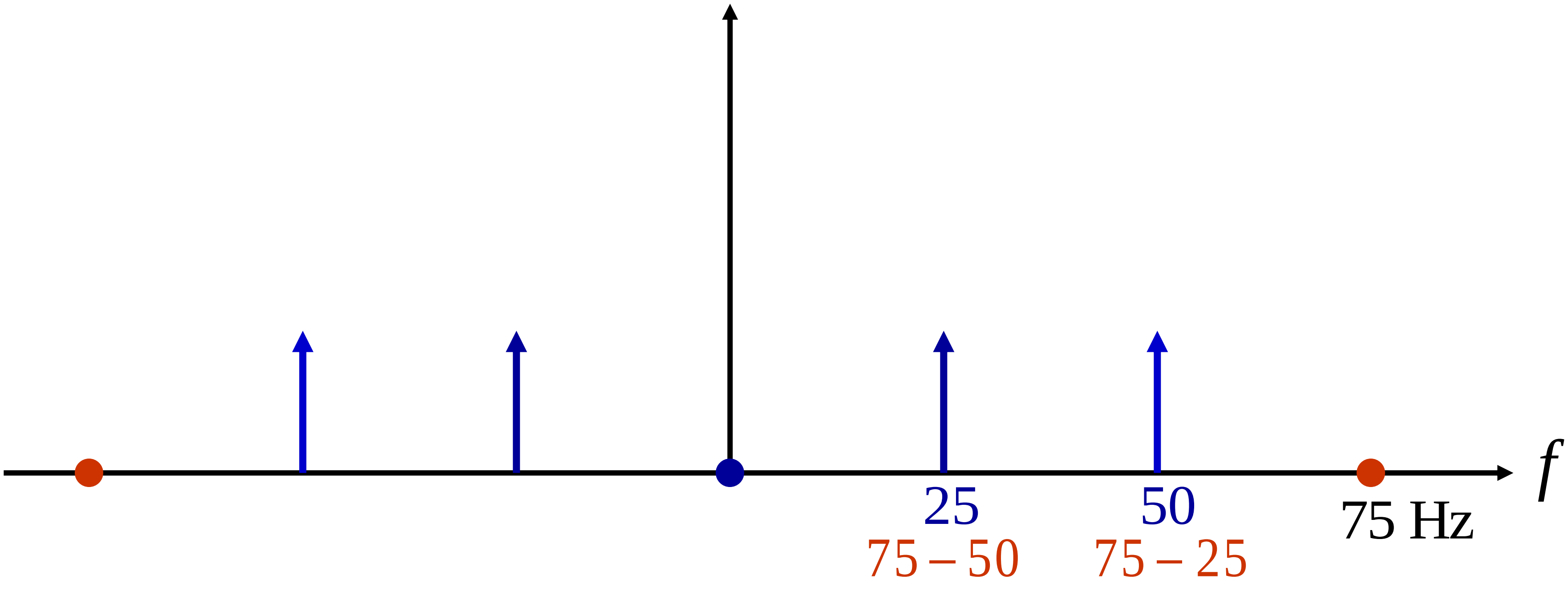

Angle modulation 1

Find the instantaneous frequency of

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$$\omega_i(t) = \frac{d}{dt}\left[300\pi t^2 + 500\pi t + 200\right] = 600\pi t + 500\pi \text{ radians/seconds}$$

Angle modulation 2

An interesting signal having a linearly varying frequency is the chirp signal. It is expressed in the general form as $x(t) = \mathrm{Re}\left[Ae^{j\theta(t)}\right]$ and has a phase given by $\theta(t) = \omega_a t^2 + \omega_b t + \varphi$. Such signals are called **chirps** because they sound like the chirp of a bird.

1. For $A = 2,\ \omega_a = 1000\pi,\ \omega_b = 1500\pi$ and $\varphi = 100$, express $x(t)$ in the sinusoidal form.
2. Plot the instantaneous frequency versus time for the period $0 \le t \le 2$.

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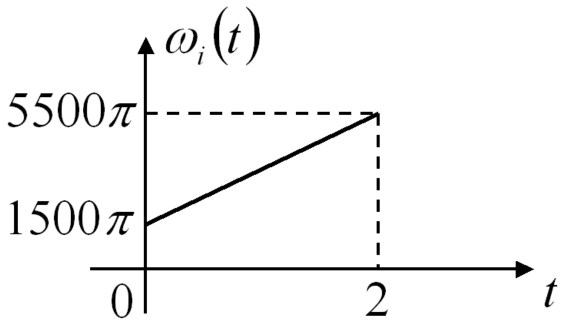
2. Plot the instantaneous frequency versus time for the period $0 \le t \le 2$.

$$x(t) = \mathrm{Re}\left[2e^{j\theta(t)}\right] = 2\cos\theta(t) = 2\cos\left[1000\pi t^2 + 1500\pi t + 100\right]$$

$$\omega_i(t) = \frac{d}{dt}\left[1000\pi t^2 + 1500\pi t + 100\right]$$

$$= 2000\pi t + 1500\pi \quad \text{radians/seconds}$$



Angle modulation 3

An angle-modulated signal is given by $x(t) = 5\cos(12000t), \quad 0 \le t \le 1$

Let the carrier frequency be ω_c=10000 rad/s.

If $x(t)$ is a FM signal with k_f=500, determine the modulating signal $m(t)$

If $x(t)$ is a PM signal with k_p=500, determine the modulating signal $m(t)$

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$$x_{\mathrm{FM}}(t) = A_c \cos\left[\omega_c t + k_f \int_{-\infty}^{t} m(\lambda)\,d\lambda\right]$$

$$12000t = 10000t + 500\int_0^t m(\lambda)\,d\lambda$$

$$m(t) = 4$$

$$x_{\mathrm{PM}}(t) = A_c \cos[\omega_c t + k_p m(t)]$$

$$12000t = 10000t + 500m(t)$$

$$m(t) = 4t$$