MoHBF Analytical Sheet 2

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1 Exercise 1 (Whitening Filters)

We have to show that:

$$W^{-1}W^{-1} = C \tag{1}$$

and C is defined by:

$$C = U\Sigma U^T \tag{2}$$

To come to this result we first have to define the Whitening Filter. It can be found by the steps which are applied during Whitening.

First the data (x) is rotated into PCA-Space

$$y = U^T x \tag{3}$$

then it is scaled to Unit variance

$$s = \Sigma^{-\frac{1}{2}} y \tag{4}$$

and then rotated back to the original space:

$$z = Us = \underbrace{U\Sigma^{-\frac{1}{2}}U^T}_{W} x \tag{5}$$

and therefore the inverse of W is:

$$W^{-1} = U^{T^{-1}} \Sigma^{\frac{1}{2}} U^{-1} \tag{6}$$

Finally we show $W^{-1}W^{-1} = C$ by using the property $U^T = U^{-1}$ of an orthonormal matrix:

$$\begin{aligned}
W^{-1}W^{-1} &= U^{T^{-1}} \Sigma^{\frac{1}{2}} \underbrace{U^{-1}U^{T^{-1}}}_{I^{-1}=I} \Sigma^{\frac{1}{2}} U^{-1} \\
&= U^{T^{-1}} \Sigma^{\frac{1}{2}} I \, \Sigma^{\frac{1}{2}} U^{-1} \\
&= U^{T^{-1}} \Sigma U^{-1} \\
&= U\Sigma U^T \\
&= C
\end{aligned}$$

2 Exercise 2 (Independence and correlation)

2.1 independent $\Rightarrow$ uncorrelated

X, Y are independent and real variables and therefore it holds that:

$$P(X,Y) = P(X)P(Y) \tag{7}$$

f, g are functions $\mathbb{R} \to \mathbb{R}$. In order to show that $cov((f(X)), g(Y)) = 0$ it has to be shown that:

$$\begin{aligned}
\mathbb{E}(f(X)g(Y)) &= \iint f(X)g(Y) \underbrace{p(X,Y)}_{\text{independent}} \, dX dY \\
&= \iint f(X)g(Y)p(X)p(Y) \, dX dY \\
&= \iint f(X)p(X) \, dX g(Y)p(Y) \, dY \\
&= \int f(X)p(X) \, dX \int g(Y)p(Y) \, dY \\
&= \mathbb{E}(f(X))\mathbb{E}(g(Y))
\end{aligned} \tag{8}$$

2.2 depended but uncorrelated

$$\begin{aligned}
cov(X, X^2) &= \mathbb{E}(X^3) - \underbrace{\mathbb{E}(x)}_{0} \mathbb{E}(x^2) \\
&= \mathbb{E}(X^3)
\end{aligned} \tag{9}$$

The argument for showing that $\mathbb{E}(X^3) = 0$ is to say that x^3 is an odd function and that $p(X)$ is an even function so that the integral over this product sums up to 0. And therefore:

$$\int x^3 p(x) \, dx = 0 \tag{10}$$

3 Exercise 3 Gain Control

$X_1, X_2, V \sim \mathcal{N}(0, 1)$ and

$$Z_1 := X_1 V, Z_2 := X_2 V. \tag{11}$$

3.1

$$\begin{aligned}
cov(Z_1, Z_2) &= \mathbb{E}(X_1 V X_2 V) - \mathbb{E}(X_1 V)\mathbb{E}(X_2 V) \\
&= \mathbb{E}(\underbrace{X_1 X_2}_{independent} V^2) - \mathbb{E}(X_1)\mathbb{E}(V)\mathbb{E}(X_2)\mathbb{E}(V) \\
&= \underbrace{\mathbb{E}(X_1 X_2)}_{0} \mathbb{E}(V^2) - \underbrace{\mathbb{E}(X_1)\mathbb{E}(V)\mathbb{E}(X_2)\mathbb{E}(V)}_{0} \\
&= 0
\end{aligned}$$

3.2

Using that the expectancy of the square of a normally distributed random variable is 1.

$$
\begin{aligned}
cov(Z_1^2, Z_2^2) &= \mathbb{E}(X_1^2 V^2 X_1^2 V^2) - \mathbb{E}(X_1^2 V^2)\mathbb{E}(X_1^2 V^2) \\
&= \mathbb{E}(\underbrace{X_1^2 X_1^2}_{independent} V^4) - \mathbb{E}(\underbrace{X_1^2 V^2}_{independent})\mathbb{E}(\underbrace{X_1^2 V^2}_{independent}) \\
&= \underbrace{\mathbb{E}(X_1^2)\mathbb{E}(X_1^2)}_{1}\mathbb{E}(V^4) - \underbrace{\mathbb{E}(X_1^2)\mathbb{E}(X_1^2)}_{1}\mathbb{E}(V^2)\mathbb{E}(V^2) \\
&= \mathbb{E}({V^2}^2) - \mathbb{E}(V^2)^2 \\
&= var(V^2) \\
&\neq 0
\end{aligned}
$$