Analytical exercises for the module "Models of Higher Brain Function"

Ingo Fründ

Natural Image Statistics II
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Exercise 1 Whitening filter

In the lecture, we derived a whitening filter from a PCA decomposition of the covariance matrix. This whitening matrix is sometimes also called inverse matrix square root of the covariance matrix.

To this end, we decompose the covariance matrix $\mathbf{C}$ of our images as

$$\mathbf{U}\mathbf{\Sigma}\mathbf{U}^T = \mathbf{C},$$

where $\mathbf{U}$ contains the principal component vectors as columns and $\mathbf{\Sigma}$ is a diagonal matrix with the variances of the data in principal component space on the diagonal. Note that the principal component vectors are orthonormal, that is $\mathbf{U}\mathbf{U}^T = \mathbf{U}^T\mathbf{U} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix.

1. Show that for the whitening matrix $\mathbf{W}$ from the lecture we have

$$\mathbf{W}^{-1}\mathbf{W}^{-1} = \mathbf{C}.$$

Note, that you first have to find the form of the whitening matrix $\mathbf{W}$.

Exercise 2 Independence and correlation

The covariance of two random variables X and Y is defined as

$$\mathrm{cov}(X, Y) := \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y),$$

where $\mathbb{E}$ denotes expectation. If $\mathrm{cov}(X, Y) \neq 0$, we say that X and Y are *correlated*.

If two random variables are independent, we can write

$$\mathbb{P}(X, Y) = \mathbb{P}(X)\mathbb{P}(Y).$$

In this exercise, we see how these two concepts are related. The differences between independence and correlation are also a reason for the differences between the results from principal component analysis and independent component analysis on natural images.

1. Suppose X and Y are independent, real random variables, and $f, g : \mathbb{R} \to \mathbb{R}$ are (not necessarily linear) functions. Show that

$$\mathrm{cov}(f(X), g(Y)) = \mathbb{E}(f(X)g(Y)) - \mathbb{E}(f(X)\mathbb{E}(g(X)) = 0.$$

   That does in particular mean that $\mathrm{cov}(X, Y) = 0$.

2. The two random variables X and X^2 are obviously dependent. Show that if X follows a normal distribution with mean 0 and standard deviation 1, then X and X^2 are uncorrelated.

Gain control 3

In the lecture, we learned about a nonlinear dependency: variance correlation. In this exercise, we will explicitely study a probabilistic model that generates data that show variance correlations.

Consider three independent random variables X_1, X_2, and V, with $X_1, X_2, V \sim \mathcal{N}(0, 1)$. Use these to define two variables

$$Z_1 := X_1 V, \quad Z_2 := X_2 V.$$

Show the following properties of this model:

1. $\mathrm{cov}(Z_1, Z_2) = 0$,
2. $\mathrm{cov}(Z_1^2, Z_2^2) \neq 0$,
3. $\mathrm{cov}(Z_1^2, Z_2^2|V) = \mathbb{E}(Z_1^2 Z_2^2|V) - \mathbb{E}(Z_1^2|V)\mathbb{E}(Z_2^2|V) = 0$,
4. $\mathrm{var}(Z_1|V) = V^2$,
5. $\mathrm{cov}(\mathrm{var}(Z_1|V), Z_2^2) = \mathrm{cov}(V^2, Z_2^2) > 0$.

Property 1 indicates that in this model Z_1 and Z_2 are still uncorrelated. Property 2 demonstrates that Z_1 and Z_2 are not independent (they violate the independence property proved in the first question of Exercise 2. Furthermore, property 2 implies that the variance of Z_1 depends on the value of Z_2 like in the lecture. Property 3 demonstrates that knowledge about V abolishes the higher order correlation shown in property 2. Properties 4 and 5 show that the variance of Z_1 can be approximated by a linear function of Z_2^2 as in the lecture. Note the conditioned expectations are still random variables!