Circuit Modelling

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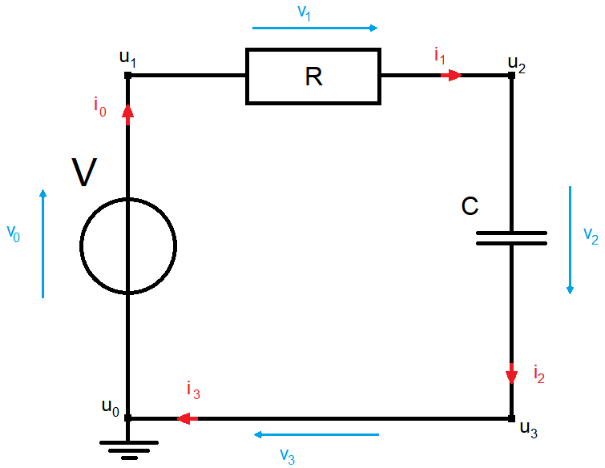
Introducing example: Charging of a capacitor:



Formulating a Mathematical Model

Network Topology

Define the incidence matrix $A = (a_{ij}) \in \mathbb{R}^{k \times l}$:

$$\tilde{a}_{ij} = \begin{cases} 1 & \text{edge } j \text{ starts at node } i, \\ -1 & \text{edge } j \text{ ends at node } i, \\ 0 & \text{else.} \end{cases}$$

By grounding node 0, i.e. $u_0 = 0$ we obtain the reduced incidence matrix.

Formulating a Mathematical Model

Energy Conservation Laws

- **Kirchhoff's voltage law (KVL):**
  The sum of voltages along each loop of the network must equal to zero.

$$\rightarrow A^\top u = v. \tag{1}$$

- **Kirchhoff's current law (KCL):**
  For any node, the sum of currents flowing into the node is equal to the sum of currents flowing out of the node.

$$\rightarrow Ai = 0. \tag{2}$$

Formulating a Mathematical Model

Electrical Components and their Relations

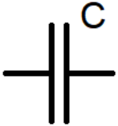
- **Resistor**

$$v = R\,i \quad \text{or} \quad i = G\,u. \tag{3}$$

Figure: resistor symbol

- **Capacitor**

$$Q = C\,v \quad \text{and by derivation in t} \quad I = C\,\frac{d}{dt}v = C\,v'. \tag{4}$$



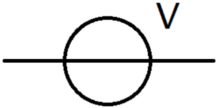
- **Inductor (Coil)**

$$\Phi = L\,i \quad \text{and by derivation in t} \quad v = L\,i'. \tag{5}$$

Figure: inductor symbol

- **Voltage Source**

$$v = v_{src} \tag{6}$$



- **Current Source**

$$i = i_{src} \tag{7}$$

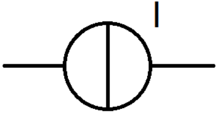


Figure: current source symbol

Formulating a Mathematical Model

Modified Nodal Analysis - MNA

Combining the component relations with the reduced incidence matrix and the Kirchhoff's laws we get:

$$\begin{aligned} A_C C A_C^\top u' + A_R G A_R^\top u + A_L i_L + A_V i_V &= -A_I i_{src}, \\ L i_L' - A_L^\top u &= 0, \\ -A_V^\top u &= -v_{src}. \end{aligned}$$

In matrix form:

$$\begin{pmatrix} A_C C A_C^\top & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{pmatrix} * \begin{pmatrix} u' \\ i_L' \\ i_V' \end{pmatrix} + \begin{pmatrix} A_R G A_R^\top & A_L & A_V \\ -A_L^\top & 0 & 0 \\ -A_V^\top & 0 & 0 \end{pmatrix} * \begin{pmatrix} u \\ i_L \\ i_V \end{pmatrix} = \begin{pmatrix} -A_I i_{src} \\ 0 \\ -v_{src} \end{pmatrix}. \tag{8}$$

Differential Algebraic Equations

Types of DAEs

In the most general form a DAE can be written as: Find $y : \mathbb{R} \to \mathbb{R}^n$ such that

$$F(t, y(t), y'(t)) = 0, \qquad \forall t \in I \tag{9}$$

with $F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ sufficiently smooth and I the time-interval.

Linear systems with constant coefficients
find y such that

$$Ay'(t) + By(t) = f(t), \tag{10}$$

with $A, B \in \mathbb{R}^{n \times n}$, A singular, B regular and $f : \mathbb{R} \to \mathbb{R}^n$ a function in time.

Linear time dependent systems are systems of the form: find y such that

$$A(t)y'(t) + B(t)y(t) = f(t),$$

with $A, B : \mathbb{R} \to \mathbb{R}^{n \times n}$, $f : \mathbb{R} \to \mathbb{R}^n$ functions, $\forall t \in \mathbb{R}$: $A(t)$ is singular and $B(t)$ regular.

Differential Algebraic Equations

Weierstrass-Kronecker normalform

prerequisites:

Definition

The matrix pencil $\{A, B\}$ is called *regular* if there exists some $c \in \mathbb{R}$, such that $(cA + B)$ is regular (i.e. $\det(cA + B) \neq 0$), otherwise it is called singular.

We get a system of the form

$$\begin{aligned} u'(t) + Ru(t) &= s(t), \\ Nv'(t) + v(t) &= q(t), \end{aligned} \tag{11}$$

where $PAQ = \begin{pmatrix} I & \\ & N \end{pmatrix}$ and $PBQ = \begin{pmatrix} R & \\ & I \end{pmatrix}$.

Definition

The nilpotency index k of the matrix N from the Weierstraß-Kronecker Normalform of a matrix pencil $\{A, B\}$ with A singular is called the *Kronecker-Index* of $\{A, B\}$, which we denote by $\mathrm{ind}\{A, B\}$. Note that for A regular we set $\mathrm{ind}\{A, B\} = 0$.

Differential Algebraic Equations

Index of a Differential Algebraic Equation

Definition

Consider the differential algebraic equation (9) to be uniquely locally solvable and F sufficiently smooth. For a given $m \in \mathbb{N}$ consider the equations

$$
\begin{aligned}
F(t, y, y') &= 0, \\
\frac{dF(t, y, y')}{dt} &= 0, \\
&\vdots \\
\frac{d^m F(t, y, y')}{dt^m} &= 0.
\end{aligned}
$$

The smallest natural number m for which the above system results in an explicit system of ordinary differential equations (ODEs), i.e. it has the form

$$y' = \phi(t, y),$$

Definition

Let $y(t)$ be the exact solution of (9) and $\tilde{y}(t)$ be the solution of the perturbed system $F(t, \tilde{y}, \tilde{y}') = \delta(t)$. The smallest number $k \in \mathbb{N}$ such that

$$\|y(t) - \tilde{y}(t)\| \leq C \left(\|y(t_0) - \tilde{y}(t_0)\| + \sum_{j=0}^{k} \max_{t_0 \leq \xi \leq T} \left\| \int_{t_0}^{\xi} \frac{d^j \delta}{d\tau^j}(\tau) d\tau \right\| \right)$$

for all $\tilde{y}(t)$, is called the **perturbation index** of this system.

Index Analysis of the Modified Nodal Analysis

Topological Conditions

Theorem (Index conditions)

Let the matrices of the capacitances, inductances and resistances be positive definite.

- *If*

$$\ker([A_R, A_C, A_V, A_L]^\top) = \{0\} \quad \textit{and} \quad \ker(A_V) = \{0\} \tag{12}$$

holds, then the MNA (8) *leads to a system with index* $\nu \leq 2$.

- *If additionally*

$$\ker([A_R, A_C, A_V]^\top) = \{0\} \quad \textit{and} \quad \ker([A_C, A_V]) = \{0\} \tag{13}$$

holds, then the system is of index $\nu \leq 1$

- *If further*

$$\ker(A_C^\top) = \{0\} \quad \textit{and} \quad \dim(v_{src}) = 0 \tag{14}$$

holds, then the system has index $\nu = 0$.

- Condition (12) can be interpreted, as the circuit neither containing loops of voltage sources nor cutsets of current sources.
- Condition (13) can be interpreted, as the circuit containing neither loops of capacitors and/or voltage sources nor cutsets of inductors and/or current sources.
- Condition (14) can be interpreted, as every node in the circuit being connected to the reference node (ground) through a path containing only the capacitors.

Numerical Solutions

Multistep Methods

Definition

For given $\alpha_0, ..., \alpha_k$ and $\beta_0, ..., \beta_k$ the iteration rule

$$\sum_{l=0}^{k} \alpha_l y_{m+l} = h \sum_{l=0}^{k} \beta_l f(t_{m+l}, y_{m+l}), \quad m = 0, 1, ..., N-k \tag{15}$$

is called a *linear multistep method* (linear k-step method). It is always assumed that $\alpha_k \neq 0$ and $|\alpha_0| + |\beta_k| > 0$. If $\beta_k = 0$ holds, then the method is called explicit, otherwise implicit.

Definition

We say that a linear multi-step method is convergent if for a solution y of the problem a solution vector created by an LMSM y_j for $j \in 0, ..., k$ we have that

$$\lim_{h \to \infty} \max_{0 \leq j \leq k} \|y(t_j) - y_j\| = 0.$$

Numerical Solutions

Multistep Methods

further stability properties

Dahlquist test problem as a model problem, find y such that

$$y' = \lambda y, \quad t > 0 \tag{16}$$

$$y(0) = y_0 \tag{17}$$

with $\lambda \in \mathbb{C}$ and y_0 fixed.
Thus the resulting linear multistep method is of the form

$$\sum_{l=0}^{k} \alpha_l y_{n+l} = h \sum_{l=0}^{k} \beta_l \lambda y_{n+l}$$

$$\iff \sum_{l=0}^{k} [\alpha_l - h\beta_l \lambda] y_{n+l}$$

Definition

1. The set

$$S := \{z \in \mathbb{C} : \rho(\xi) - z\sigma(\xi) = 0 \implies \xi \in \mathbb{C} \text{ and } |\xi| \leq 1. \\ \text{If } \xi \text{ has multiplicity greater than 1, then } |\xi| < 1\} \tag{18}$$

   is called the region of stability of the method.

2. A linear multistep method is called
   - *0-stable*, if $0 \in S$.
   - stable in the point $z \in \mathbb{C}$, if $z \in S$.
   - $A(\alpha)$*-stable*, if it is stable in all z that lie within the set $\{z \in \mathbb{C}^- : |arg(z) - \pi| \leq \alpha\}$ for $\alpha \in (0, \frac{\pi}{2})$.

Theorem

Let $f(t, y)$ be sufficiently smooth and the linear multi-step method be zero-stable and consistent of order p, then it is also convergent of order p.

Numerical Solutions

Consistent Initial Values

index $\nu = 0$.
Case: Index $\nu = 1$.
By rewriting our system into the form

$$\begin{aligned} y'(t) &= f(t, y(t), z(t)), \\ 0 &= g(t, y(t), z(t)). \end{aligned}$$

we are able to give conditions for consistent initial values. Namely y_0 and z_0 are consistent initial values for this system, if $g(t_0, y_0, z_0) = 0$ holds.

Case: Index $\nu = 2$.
For index-2 systems we rewrite our system into

$$\begin{aligned} y' &= f(t, y(t), z(t)), \\ 0 &= g(t, y(t)). \end{aligned}$$

Consistent initial values y_0, z_0 for this case not only have to fulfill $g(t_0, y_0) = 0$ but also the *hidden constraint* $g_t(t_0, y_0) + g_y(t_0, y_0) f(t_0, y_0, z_0)$. By g_t and g_y we denote the derivative of g with respect to t or y, respectively.

Numerical Solutions

Implicit Linear Multistep Formulas

BDF-k Methods

The *backward differentiation formula (BDF)* is a family of implicit linear multistep methods. They have the general form

$$\sum_{k=0}^{s} \alpha_k y_{n+k} = h\beta f(t_{n+s}, y_{n+s}) \tag{19}$$

The BDF or BDF-k formulas for $k = 1, ..., 3$ have the following form

$$\begin{aligned} k = 1 : hf_{m+1} &= y_{m+1} - y_m \\ k = 2 : hf_{m+2} &= \frac{1}{2}(3y_{m+2} - 4y_{m+1} + y_m) \\ k = 3 : hf_{m+3} &= \frac{1}{6}(11y_{m+3} - 18y_{m+2} + 9y_{m+1} - 2y_m) \end{aligned}$$

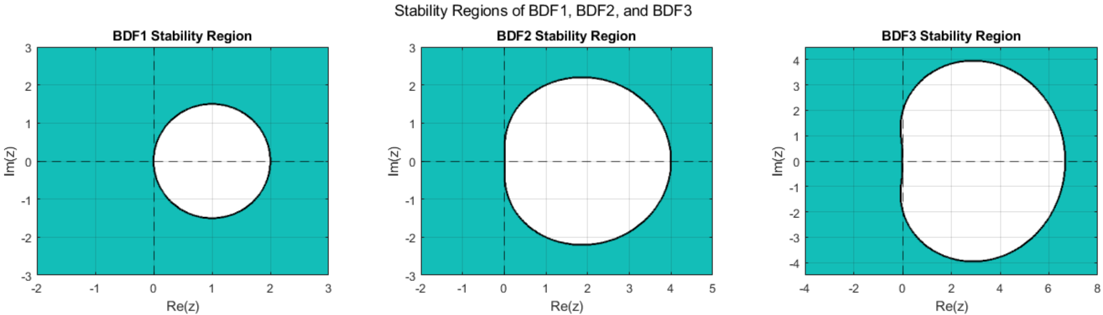


Figure: stability regions of BDF-schemes

Theorem

The BDF-k methods have consistency order $p = k$.

Numerical Solutions

Implicit Linear Multistep Formulas

Trapezoidal rule

This procedure is repeated for small subsections of the interval $[a, b]$. Thus we obtain the iteration formula

$$u_h(t+h) = u_h(t) + \frac{h}{2}[f(t, u_h(t)) + f(t+h, u_h(t+h))].$$

Numerical Solutions

Numerical Examples

Example1

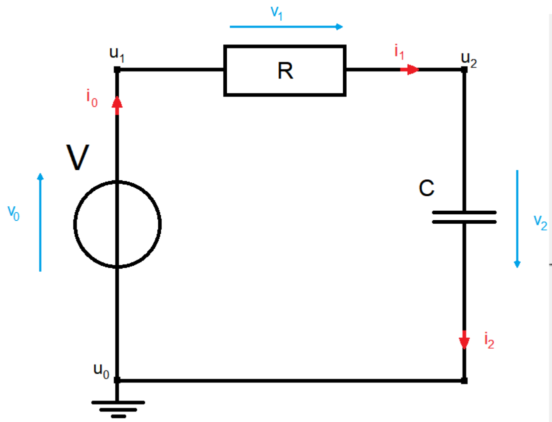


Figure: charging capacitor with series resistor and voltage source

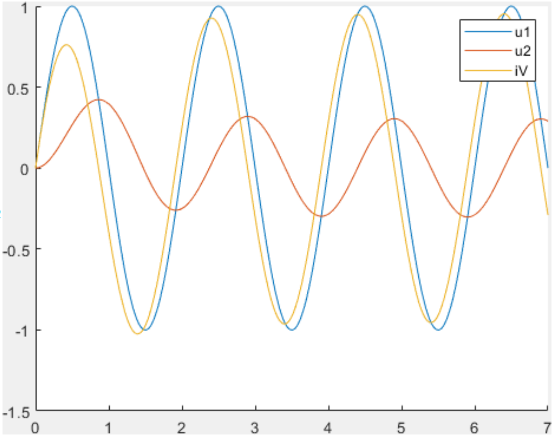


Figure: Exact solution for example 1.

h	k = 1		k = 2		k = 3		trapezoidal	
	u2	iV	u2	iV	u2	iV	u2	iV
0.1	4.620×10^{-2}	4.620×10^{-2}	9.567×10^{-3}	9.567×10^{-3}	3.057×10^{-3}	3.057×10^{-3}	3.344×10^{-3}	3.344×10^{-3}
0.05	2.339×10^{-2}	2.339×10^{-2}	2.454×10^{-3}	2.454×10^{-3}	6.083×10^{-4}	6.083×10^{-4}	8.367×10^{-4}	8.367×10^{-4}
0.025	1.178×10^{-2}	1.178×10^{-2}	6.264×10^{-4}	6.264×10^{-4}	1.672×10^{-4}	1.672×10^{-4}	2.092×10^{-4}	2.092×10^{-4}

Table: Resulting errors for the BDF-k methods and ther trapezoidal rule.