Circuit Modelling

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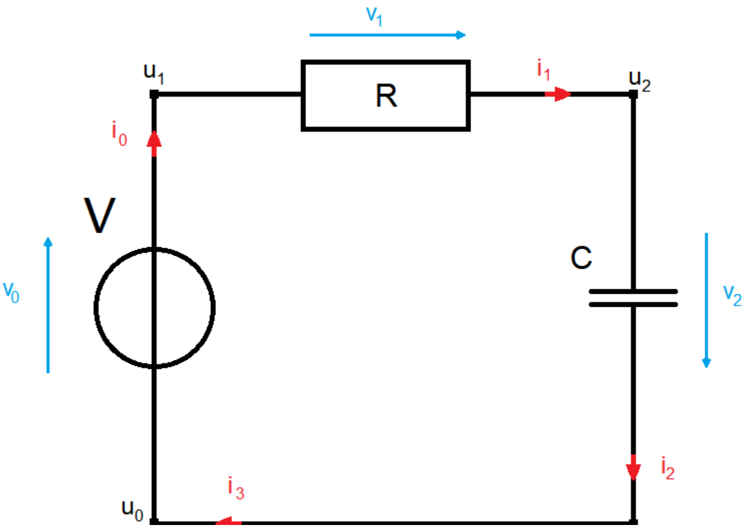
Formulating a Mathematical Model

Formulating a Mathematical Model

Network Topology

Introducing example: Charging of a capacitor:



Incidence Matrix $A = (a_{ij}) \in \mathbb{R}^{k \times l}$:

$$\tilde{a}_{ij} = \begin{cases} 1 & \text{edge } j \text{ starts at node } i, \\ -1 & \text{edge } j \text{ ends at node } i, \\ 0 & \text{else.} \end{cases}$$

With $N = (n_0, n_1, n_2, ..., n_k)$ nodes and $E = \{e_j : j = 1, ..., l\}$ edges, where $|N| = k$ is the number of nodes and $|E| = l$
$u = (u_0, u_1, u_2, ...)$ the corresponding electrical potentials to the nodes.
ground one node → reduced incidence matrix

Formulating a Mathematical Model

Energy Conservation Laws

- **Kirchhoff's voltage law (KVL):**
  The sum of voltages along each loop of the network must equal to zero. Using the incidence matrix A this law can be formulated as

$$A^{\top} u = v. \tag{1}$$

- **Kirchhoff's current law (KCL):**
  For any node, the sum of currents flowing into the node is equal to the sum of currents flowing out of the node. Using the incidence matrix A again, this law can be formulated as

$$Ai = 0. \tag{2}$$

Formulating a Mathematical Model

Electrical Components and their Relations

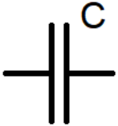
- **Resistor**

$$v = R\,i \quad \text{or} \quad i = G\,u. \tag{3}$$

Figure: resistor symbol

- **Capacitor**

$$Q = C\,v \quad \text{and by derivation in t} \quad I = C\,\frac{d}{dt}v = C\,v'. \tag{4}$$



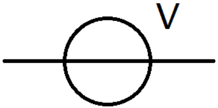
- **Inductor (Coil)**

$$\Phi = L\, i \quad \text{and by derivation in t} \quad v = L\, i'. \tag{5}$$

Figure: inductor symbol

- **Voltage Source**

$$v = v_{src} \tag{6}$$



- **Current Source**

$$i = i_{src} \tag{7}$$

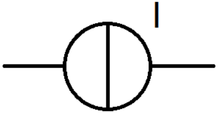


Figure: current source symbol

Formulating a Mathematical Model

Modified Nodal Analysis - MNA

To analyse the network further we will rearrange the columns of the reduced incidence matrix A such that it has the block form

$$A = (A_R A_C A_L A_V A_I)$$

where A_R, A_C, A_L, A_V and A_I include the columns that are related to the resistors, capacitors, coils, voltage sources and current sources, respectively.
The voltages can be represented using the node-potentials

$$v = A^{\top} u$$

The vector v can thus be rearranged into $v = (v_R, v_C, v_L, v_{src}, v_I)$. In a similar way we also rearrange the current vector into $i = (i_R, i_C, i_L, i_V, i_s rc)$. Using the sorted incidence matrix blocks we can rewrite the resistor current relation as

$$i_R = G\, v_R = G\, A_R^{\top} u.$$

Analogously, we rewrite the capacitor relation as

$$i_C = C\, v_C' = C\, A_C^{\top} u'.$$

Combining this with the component law for inductors (5) and the potential-voltage relation for voltage sources (6) we finally get the modified nodal analysis equations

$$
\begin{aligned}
A_C C A_C^\top u' + A_R G A_R^\top u + A_L i_L + A_V i_V &= -A_I i_{src}, \\
L i_L' - A_L^\top u &= 0, \\
-A_V^\top u &= -v_{src}.
\end{aligned}
$$

In matrix form they read as

$$
\begin{pmatrix} A_C C A_C^\top & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{pmatrix} \ast \begin{pmatrix} u' \\ i_L' \\ i_V' \end{pmatrix} + \begin{pmatrix} A_R G A_R^\top & A_L & A_V \\ -A_L^\top & 0 & 0 \\ -A_V^\top & 0 & 0 \end{pmatrix} \ast \begin{pmatrix} u \\ i_L \\ i_V \end{pmatrix} = \begin{pmatrix} -A_I i_{src} \\ 0 \\ -v_{src} \end{pmatrix}. \tag{8}
$$

Differential Algebraic Equations

Differential Algebraic Equations

Types of DAEs

In the most general form a DAE can be written as: Find $y : \mathbb{R} \to \mathbb{R}^n$ such that

$$F(t, y(t), y'(t)) = 0, \qquad \forall t \in I \tag{9}$$

with $F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ sufficiently smooth and I the time-interval.

- **Linear systems with constant coeffiecients** are systems of the form: find y such that

$$Ay'(t) + By(t) = f(t), \tag{10}$$

with $A, B \in \mathbb{R}^{n\times n}$, A singular, B regular and $f : \mathbb{R} \to \mathbb{R}^n$ a function in time.

- **Linear time dependent systems** are systems of the form: find y such that

$$A(t)y'(t) + B(t)y(t) = f(t),$$

with $A, B : \mathbb{R} \to \mathbb{R}^{n\times n}$, $f : \mathbb{R} \to \mathbb{R}^n$ functions, such that for every $t \in \mathbb{R}$ the matrix $A(t)$ is singular and the matrix $B(t)$ regular.

- **Structured (non-linear) systems** are semi-explicit systems of the form: find (y, z) such that

$$y'(t) = f(t, y(t), z(t)), \tag{11}$$
$$0 = g(t, y(t), z(t)), \tag{12}$$

with $f : \mathbb{R} \to \mathbb{R}^n$ and $g : \mathbb{R} \to \mathbb{R}^d$ functions.

Differential Algebraic Equations

Weierstrass-Kronecker Normalform

prerequisites:

Definition

The matrix pencil $\{A, B\}$ is called *regular* if there exists some $c \in \mathbb{R}$, such that $(cA + B)$ is regular (i.e. $det(cA + B) \neq 0$), otherwise it is called singular.

Theorem

For every matrix $Q \in \mathbb{R}^{n \times n}$ there exists a regular matrix $T \in \mathbb{C}^{n \times n}$, such that

$$T^{-1}QT = J = diag(J_1, \ldots, J_r) \quad \text{with} \quad J_i = \begin{pmatrix} \lambda_i & 1 & & 0 \\ 0 & \lambda_i & \ddots & \vdots \\ & \ddots & \ddots & 1 \\ 0 & \ldots & 0 & \lambda_i \end{pmatrix} \in \mathbb{C}^{m_i \times m_i}$$

and $n = m_1 + \ldots + m_r$.

Theorem

Let $\{A, B\}$ be a regular matrix pencil. There exist $P, Q \in \mathbb{C}^{n\times n}$ such that

$$PAQ = \begin{pmatrix} I_d & 0 \\ 0 & N \end{pmatrix}, \quad PBQ = \begin{pmatrix} R & 0 \\ 0 & I_{n-d} \end{pmatrix}$$

where

$$N = diag(N_1, \dots, N_r) \quad with \quad N_i = \begin{pmatrix} 0 & 1 & & 0 & \\ & \ddots & \ddots & & \\ & & & 0 & 1 \\ 0 & & & 0 & \end{pmatrix} \in \mathbb{R}^{n_i \times n_i}$$

and R has Jordan Normalform. By I_k we denote the identity matrix of size $k \times k$.

Proof on blackboard

using these findings: Using the findings above we are able to transform the initial DAE (10) using the matrix P from Theorem 3. By multiplying with P from the left, we obtain

$$PAy'(t) + PBy(t) = Pf(t).$$

Setting

$$y(t) = Q\begin{pmatrix} u(t) \\ v(t) \end{pmatrix}, \quad Pf(t) = \begin{pmatrix} s(t) \\ q(t) \end{pmatrix},$$

with $u(t), s(t) : \mathbb{R} \to \mathbb{R}^d$ and $q(t), v(t) : \mathbb{R} \to \mathbb{R}^{n-d}$.

We get a system of the form

$$\begin{aligned} u'(t) + Ru(t) &= s(t), \\ Nv'(t) + v(t) &= q(t), \end{aligned} \tag{13}$$

where $PAQ = \begin{pmatrix} I & \\ & N \end{pmatrix}$ and $PBQ = \begin{pmatrix} R & \\ & I \end{pmatrix}$.

$$v(t) = q(t) - Nv'(t) = q(t) - N(\underbrace{q(t) - Nv'(t)}_{=v(t)})' = q - Nq' + N^2 v''$$

$$= q - Nq' + N^2(q - Nv')'' = q - Nq' + N^2 q'' - N^3 v'''$$

$$\vdots$$

$$= q - Nq' + \ldots + (-1)^{k-1} N^{k-1} \underbrace{\frac{d^k}{dt^k}}_{:=q^{(k-1)}} q + (-1) \underbrace{N^k v^{(k)}}_{=0}$$

$$= \sum_{i=0}^{k-1} (-1)^i N^i q^{(i)}(t) \tag{14}$$

where k is the nilpotency index of N.

Differential Algebraic Equations

Weierstrass-Kronecker Normalform

Index of a Differential Algebraic Equation

Definition

Consider the differential algebraic equation (9) to be uniquely locally solvable and F sufficiently smooth. For a given $m \in \mathbb{N}$ consider the equations

$$
\begin{aligned}
F(t, y, y') &= 0,\\
\frac{\mathrm{d}F(t, y, y')}{\mathrm{d}t} &= 0,\\
&\vdots\\
\frac{\mathrm{d}^m F(t, y, y')}{\mathrm{d}t^m} &= 0.
\end{aligned}
$$

The smallest natural number m for which the above system results in an explicit system of ordinary differential equations (ODEs), i.e. it has the form

$$y' = \phi(t, y),$$

Definition

Let $y(t)$ be the exact solution of *Abstract-DAE!!!!!!!!* and $\tilde{y}(t)$ be the solution of the perturbed system $F(t, \tilde{y}, \tilde{y}') = \delta(t)$. The smallest number $k \in \mathbb{N}$ such that

$$\|y(t) - \tilde{y}(t)\| \leq C\left(\|y(t_0) - \tilde{y}(t_0)\| + \sum_{j=0}^{k} \max_{t_0 \leq \xi \leq T} \left\| \int_{t_0}^{\xi} \frac{\mathrm{d}^j \delta}{\mathrm{d}\tau^j}(\tau) d\tau \right\|\right)$$

for all $\tilde{y}(t)$, is called the **perturbation index** of this system.

Differential Algebraic Equations

Weierstrass-Kronecker Normalform

Consistent Initial Values

index $\nu = 0$.

Case: Index $\nu = 1$.

By rewriting our system into the form

$$\begin{aligned} y'(t) &= f(t, y(t), z(t)), \\ 0 &= g(t, y(t), z(t)). \end{aligned}$$

we are able to give conditions for consistent initial values. Namely y_0 and z_0 are consistent initial values for this system, if $g(t_0, y_0, z_0) = 0$ holds.

Case: Index $\nu = 2$.
For index-2 systems we rewrite our system into

$$\begin{aligned} y' &= f(t, y(t), z(t)), \\ 0 &= g(t, y(t)). \end{aligned}$$

Consistent initial values y_0, z_0 for this case not only have to fulfill $g(t_0, y_0) = 0$ but also the *hidden constraint* $g_t(t_0, y_0) + g_y(t_0, y_0) f(t_0, y_0, z_0)$. By g_t and g_y we denote the derivative of g with respect to t or y, respectively.

Index Analysis of the Modified Nodal Analysis

Index Analysis of the Modified Nodal Analysis

General Index Analysis

content...

Index Analysis of the Modified Nodal Analysis

Topological Conditions

*Theorem (Index conditions [**shashkov_tuprints27452**])*

Let the matrices of the capacitances, inductances and resistances be positive definite.

- *If*

  $$ker([A_R, A_C, A_V, A_L]^{\top}) = \{0\} \quad and \quad ker(A_V) = \{0\} \tag{15}$$

  holds, then the MNA (8) *leads to a system with index $\nu \leq 2$.*

- *If additionally*

  $$ker([A_R, A_C, A_V]^{\top}) = \{0\} \quad and \quad ker([A_C, A_V]) = \{0\} \tag{16}$$

  holds, then the system is of index $\nu \leq 1$

- *If further*

  $$ker(A_C^{\top}) = \{0\} \quad and \quad dim(v_{src}) = 0 \tag{17}$$

  holds, then the system has index $\nu = 0$.

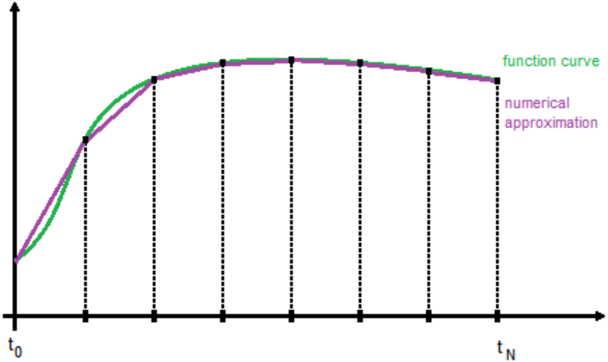
- Condition (15) can be interpreted, as the circuit neither containing loops of voltage sources nor cutsets of current sources.
- Condition (16) can be interpreted, as the circuit containing neither loops of capacitors and/or voltage sources nor cutsets of inductors and/or current sources.
- Condition (17) can be interpreted, as every node in the circuit being connected to the reference node (ground) through a path containing only the capacitors.

Numerical Solutions

general initial value problem Find y, such that

$$y'(t) = f(t, y), \quad t \in [t_0, t_l], \tag{18}$$

$$y(t_0) = y_0. \tag{19}$$



Numerical Solutions

Single-Step Methods

Definition

A numerical method to approximate a differential equation 18 on a time-grid $t_0, \ldots, t_l$ with the intermediate values $y_0, \ldots, y_l$ is called a single-step method if it is of the form

$$y_{j+1} = y_j + h_j \phi(t_j, y_j, y_{j+1}, h_j). \tag{20}$$

We call ϕ the *procedural function*. If ϕ does not depend on y_{j+1}, then the method is called *explicit*, otherwise it is called *implicit*.

Numerical Solutions

Single-Step Methods

Consistency, Stability and Convergence

Definition

Let y_{m+1} be the result of one step of a single step method (20) with the exact start-vector $y_m = y(t_m)$ then

$$\delta_{m+1} = \delta(t_m + h) = y(t_{m+1}) - \tilde{y}_{m+1}, \quad m = 0, \ldots, N-1 \tag{21}$$

is called the *local discretization error* of the single step method at the point t_{m+1}.

Definition

A single-step method is called *consistent* if for all initial value problems (18)

$$\lim_{h \to 0} \frac{\|\delta(t+h)\|}{h} = 0 \quad \text{for} \quad t_0 \leq t \leq t_l \tag{22}$$

holds.

It is called *consistent of order p*, if for a sufficiently smooth function f

$$\|\delta(t+h)\| \leq Ch^{p+1} \quad \text{for all} \quad h \in (0, H] \quad \text{and} \quad t_0 \leq t \leq t_l - h \tag{23}$$

holds with C independent of h.

Definition

A single-step method is called *convergent*, if for all initial value problems (18) for the *global discretization error*

$$e_m = y(t_m) - y_m$$

holds that

$$\max_m \|e_m\| \to 0 \quad \text{for} \quad h_{max} \to 0.$$

The single-step method is called to have the *convergence order p*, if

$$\max_m \|e_m\| \leq Ch_{max}^p \quad \text{for} \quad h_{max} \in (0, H] \quad \text{with} \quad t_0 \leq t_m \leq t_l$$

with the constant C not dependent on the step size h.

Definition

A single-step method is called *(discretely) stable* if for grid-functions y_h and $\tilde{y}_h$ with

$$y_{i+1} = y_i + h\phi(t_i, y_i), \tag{24}$$
$$\tilde{y}_{i+1} = \tilde{y}_i + h[\phi(t_i, \tilde{y}_i) + \theta_i], \tag{25}$$

and perturbations $\theta_i = \theta_h(t_i)$ of the right side as well as a bounded perturbation in the initial-values $y_0 - \tilde{y}_0$ the error is bounded by

$$\|y_h - \tilde{y}_h\|_{\infty,h} \leq C(\|y_0 - \tilde{y}_0\|_{l^2} + \|\theta_h\|_{\infty,h})$$

with a constant C which is not dependent on h. The norm $\|.\|_{\infty,h}$ denotes the maximum norm over the time-grid, i.e. for a function $b : T = t_0, ..., t_N \to \mathbb{R}^d$ we have $\|b\|_{\infty,h} = \max_{t \in T} \|b(t)\|$, $\|b\|$ is the euclidean norm.

Numerical Solutions

Single-Step Methods

further stability properties

Dahlquist equation, i.e. find y such that

$$y' = \lambda y, \quad t > 0 \tag{26}$$

$$y(0) = y_0 \tag{27}$$

with $\lambda \in \mathbb{C}$ and u_0 fixed.

Definition

1. If a single-step method can be written in the form

$$y_{i+1} = R(z)\, y_i, \quad z := h\lambda \tag{28}$$

   then we call $R : \mathbb{C} \to \mathbb{C}$ the *stability function* of the single-step method.

2. The set

$$S := \{z \in \mathbb{C} : |R(z)| \leq 1\} \tag{29}$$

   is called the *region of stability* of the method.

3. A single-step method is called
   - *0-stable*, if $0 \in S$.
   - *A-stable*, if $\mathbb{C}^- \subset S$.
   - *L-stable*, if $R(z) \to 0$ for $Re(z) \to -\infty$.

Numerical Solutions

Multistep Methods

Definition

For given $\alpha_0, \dots, \alpha_k$ and $\beta_0, \dots, \beta_k$ the iteration rule

$$\sum_{l=0}^{k} \alpha_l y_{m+l} = h \sum_{l=0}^{k} \beta_l f(t_{m+l}, y_{m+l}), \quad m = 0, 1, \dots, N-k \tag{30}$$

is called a *linear multistep method* (linear k-step method). It is always assumed that $\alpha_k \neq 0$ and $|\alpha_0| + |\beta_k| > 0$. If $\beta_k = 0$ holds, then the method is called explicit, otherwise implicit.

Numerical Solutions

Multistep Methods

Consistency, Convergence and Stability

Definition

Let y_{m+k} be the result of one step of the multi-step method (30) with the start-values given as the evaluations of the exact solution $y_{m+l} = y(t_{m+l})$ at $0 \leq l < k$. This means

$$\alpha_k \tilde{u}_{m+k} = \sum_{l=0}^{k-1} \left(h\beta_l f(t_{m+l}, y(t_{m+l})) - \alpha_l y(t_{m+l}) \right) + h\beta_k f(t_{m+k}, y_{m+k}).$$

Then

$$\delta_{m+k} = \delta(t_{m+k}) = y(t_{m+k}) - y_{m+k}, \quad m = 0, 1, \ldots, N-k$$

is called the *local discretization error* (local error) of the linear multi-step method, see Def. 30 at the point t_{m+k}.

Definition

A linear multi-step method is called *consistent*, if for all functions $y(t) \in C^2([t_0, t_l])$

$$\lim_{h \to 0} \frac{1}{h} L[y(t), h] = 0$$

holds. It has the *consistency order p*, if for all functions $y(t) \in C^{p+1}[t_0, t_l]$

$$L[y(t), h] = O(h^{p+1}) \quad \text{for} \quad h \to 0$$

holds.

Definition

We say that a linear multi-step method is convergent if for a solution y of the problem a solution vector created by an LMSM y_j for $j \in 0, \ldots, k$ we have that

$$\lim_{h \to \infty} \max_{0 \leq j \leq k} ||y(t_j) - y_j|| = 0.$$

Definition

A linear multi-step method is called (discretely) stable, if for solutions y_h and $\tilde{y}_h$ of

$$\sum_{l=0}^{k} \alpha_l y_{m+l} = h \sum_{l=0}^{k} \beta_l f(t_{m+l}, y_{m+l}), \tag{31}$$

$$\sum_{l=0}^{k} \alpha_l \tilde{y}_{m+l} = h \sum_{l=0}^{k} \beta_l f(t_{m+l}, \tilde{y}_{m+l}) + h\theta_n \tag{32}$$

and bounded initial values $y_j - \tilde{y}_j$ for $j \in 0, \ldots, k$ we have that

$$\max_{t_0 \le t_n \le T} ||y_n - \tilde{y}_n|| \le C \sum_{j=0}^{k-1} ||y_j - \tilde{y}_j|| + \max_{t_0 \le t_n \le T} ||\theta_n||.$$

Numerical Solutions

Multistep Methods

further stability properties

Dahlquist test problem as a model problem, find y such that

$$y' = \lambda y, \quad t > 0 \tag{33}$$

$$y(0) = y_0 \tag{34}$$

with $\lambda \in \mathbb{C}$ and y_0 fixed.
Thus the resulting linear multistep method is of the form

$$\sum_{l=0}^{k} \alpha_l y_{n+l} = h \sum_{l=0}^{k} \beta_l \lambda y_{n+l}$$

$$\iff \sum_{l=0}^{k} [\alpha_l - h\beta_l \lambda] y_{n+l}$$

Definition

1. The set
$$S := \{z \in \mathbb{C} : \rho(\xi) - z\sigma(\xi) = 0 \implies \xi \in \mathbb{C} \text{ and } |\xi| \leq 1. \\ \text{If } \xi \text{ has multiplicity greater than 1, then } |\xi| < 1\} \tag{35}$$
is called the region of stability of the method.
2. A linear multistep method is called
   - *0-stable*, if $0 \in S$.
   - stable in the point $z \in \mathbb{C}$, if $z \in S$.
   - *$A(\alpha)$-stable*, if it is stable in all z that lie within the set $\{z \in \mathbb{C}^- : |arg(z) - \pi| \leq \alpha\}$ for $\alpha \in (0, \frac{\pi}{2})$.

Theorem

Let $f(t, y)$ be sufficiently smooth and the linear multi-step method be zero-stable and consistent of order p, then it is also convergent of order p.

Numerical Solutions

Implicit Linear Multistep Formulas

Numerical Solutions

Implicit Linear Multistep Formulas

BDF-k Methods

The *backward differentiation formula (BDF)* is a family of implicit linear multistep methods. They have the general form

$$\sum_{k=0}^{s} \alpha_k y_{n+k} = h\beta f(t_{n+s}, y_{n+s}) \tag{36}$$

The BDF or BDF-k formulas for $k = 1, \dots, 3$ have the following form

$$k = 1 : hf_{m+1} = y_{m+1} - y_m$$
$$k = 2 : hf_{m+2} = \frac{1}{2}(3y_{m+2} - 4y_{m+1} + y_m)$$
$$k = 3 : hf_{m+3} = \frac{1}{6}(11y_{m+3} - 18y_{m+2} + 9y_{m+1} - 2y_m)$$

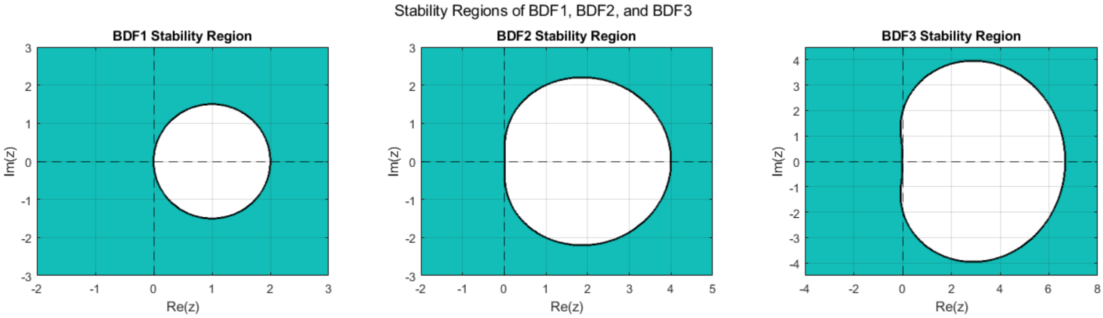


Figure: stability regions of BDF-schemes

***Theorem* ([NumerikGewöhnlicherDifferentialgleichungen])**

The BDF-k methods have consistency order p = k.

Numerical Solutions

Implicit Linear Multistep Formulas

Trapezoidal rule

This procedure is repeated for small subsections of the interval $[a, b]$. Thus we obtain the iteration formula

$$u_h(t+h) = u_h(t) + \frac{h}{2}[f(t, u_h(t)) + f(t+h, u_h(t+h))].$$

Numerical Solutions

Numerical Examples

Numerical Solutions

Numerical Examples

Example1

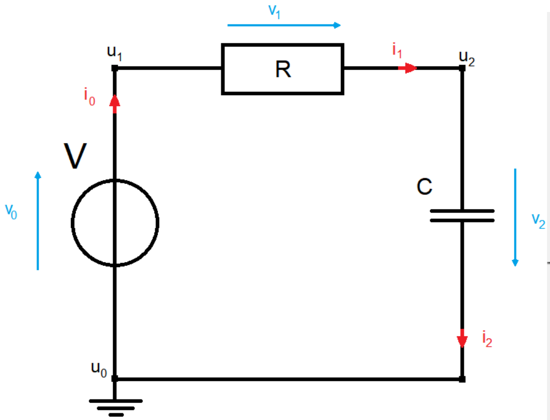


Figure: charging capacitor with series resistor and voltage source

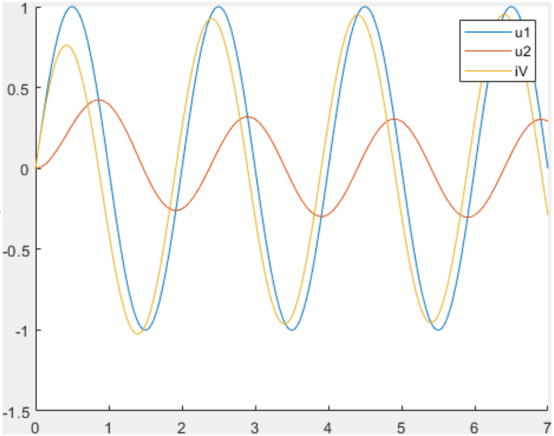


Figure: Exact solution for example 1.

h	k = 1		k = 2		k = 3		trapezoidal	
	u2	iV	u2	iV	u2	iV	u2	iV
0.1	4.620×10^{-2}	4.620×10^{-2}	9.567×10^{-3}	9.567×10^{-3}	3.057×10^{-3}	3.057×10^{-3}	3.344×10^{-3}	3.344×10^{-3}
0.05	2.339×10^{-2}	2.339×10^{-2}	2.454×10^{-3}	2.454×10^{-3}	6.083×10^{-4}	6.083×10^{-4}	8.367×10^{-4}	8.367×10^{-4}
0.025	1.178×10^{-2}	1.178×10^{-2}	6.264×10^{-4}	6.264×10^{-4}	1.672×10^{-4}	1.672×10^{-4}	2.092×10^{-4}	2.092×10^{-4}

Table: Resulting errors for the BDF-k methods and ther trapezoidal rule.

Numerical Solutions

Numerical Examples

Example 2

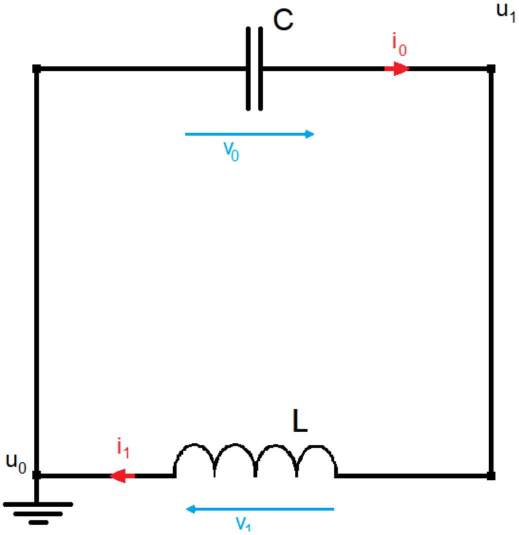


Figure: LC circuit

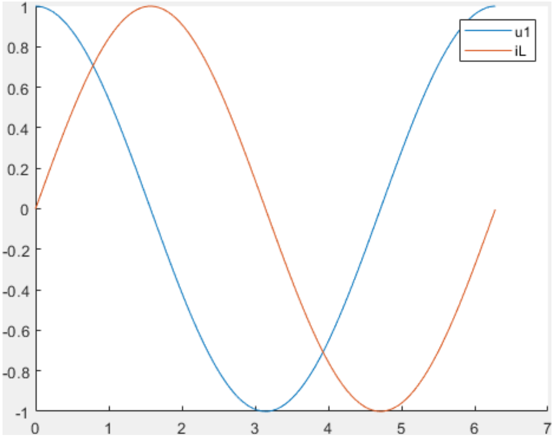


Figure: Exact solution for example 2.

h	k = 1		k = 2		k = 3		trapezoidal	
	u1	iL	u1	iL	u1	iL	u1	iL
0.1	7.145×10^{-1}	6.905×10^{-1}	7.763×10^{-2}	8.060×10^{-2}	5.395×10^{-3}	5.180×10^{-3}	1.963×10^{-2}	2.087×10^{-2}
0.05	4.659×10^{-1}	4.448×10^{-1}	1.964×10^{-2}	2.066×10^{-2}	5.938×10^{-4}	5.579×10^{-4}	4.912×10^{-3}	5.224×10^{-3}
0.025	2.695×10^{-1}	2.551×10^{-1}	4.924×10^{-3}	5.216×10^{-3}	5.773×10^{-5}	4.740×10^{-5}	1.228×10^{-3}	1.308×10^{-3}

Table: Resulting errors for the BDF-k methods and ther trapezoidal rule.

Numerical Solutions

Numerical Examples

Example 3

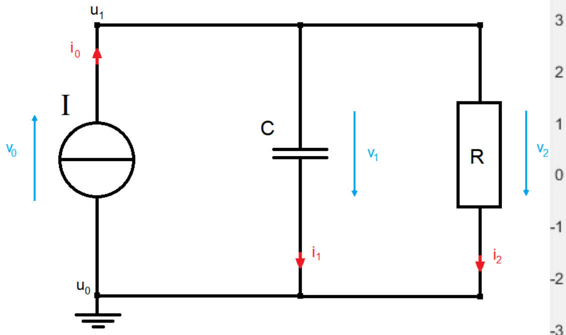


Figure: Current source with capacitor and resistor.

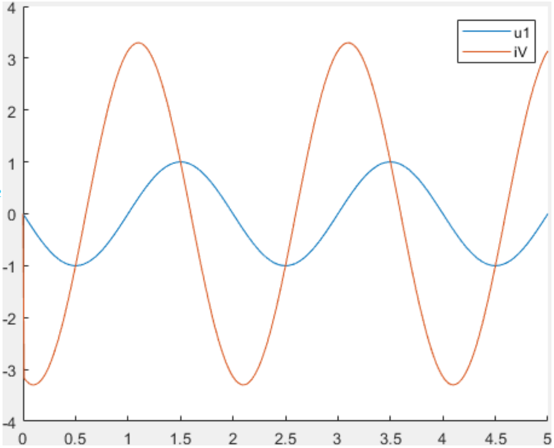


Figure: Exact solution for example 3.

h	k = 1 iV	k = 2 iV	k = 3 iV	trapezoidal iV
0.1	4.894×10^{-1}	1.023×10^{-1}	2.530×10^{-2}	5.219×10^{-2}
0.05	2.462×10^{-1}	2.577×10^{-2}	6.426×10^{-3}	1.295×10^{-2}
0.025	1.233×10^{-1}	6.456×10^{-3}	1.613×10^{-3}	3.232×10^{-3}

Table: Resulting errors for the BDF-k methods and the trapezoidal rule.