Circuit Modelling

Felix Dreßler
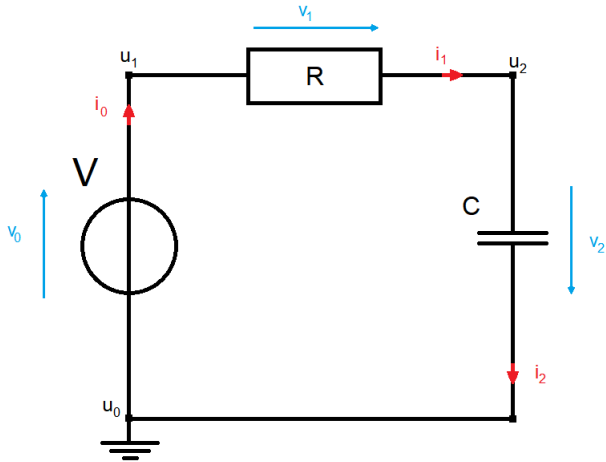
Example: Charging of a capacitor:

Figure: charging capacitor with series resistor and voltage source

Formulating a Mathematical Model

Network Topology

For a circuit with l nodes and k edges, define the incidence matrix $\tilde{A} = (\tilde{a}_{ij}) \in \mathbb{R}^{k \times l}$:

$$\tilde{a}_{ij} = \begin{cases} 1 & \text{edge } j \text{ starts at node } i, \\ -1 & \text{edge } j \text{ ends at node } i, \\ 0 & \text{else.} \end{cases}$$

By grounding node 0, i.e. $u_0 = 0$ we obtain the reduced incidence matrix $\rightarrow A$.

Formulating a Mathematical Model

Energy Conservation Laws
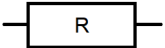
- **Kirchhoff's voltage law (KVL):**
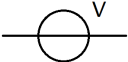 The sum of voltages along each loop of the network must equal to zero.
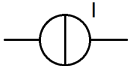
  $$\rightarrow A^\top u = v. \tag{1}$$

- **Kirchhoff's current law (KCL):**
  For any node, the sum of currents flowing into the node is equal to the sum of currents flowing out of the node.

  $$\rightarrow Ai = 0. \tag{2}$$

Name	Symbol	Component Law
Resistor	R	$v = R\,i$ or $i = G\,u$
Capacitor	C	$Q = C\,v$ and by derivation in t $I = C\,\frac{\mathrm{d}}{\mathrm{d}t}v = C\,v'$
Inductor	L	$\Phi = L\,i$ and by derivation in t $v = L\,i'$
Voltage Source	V	$v = v_{src}$
Current Source	I	$i = i_{src}$

Formulating a Mathematical Model

Modified Nodal Analysis - MNA

Rearrange reduced incidence matrix into

$$A = (A_R A_C A_L A_V A_I)$$

A_R, A_C, A_L, A_V and A_I ... columns related to components
Represent voltages:

$$v = A^\top u$$

$\rightarrow$ rearrange v into $v = (v_R, v_C, v_L, v_{src}, v_I)$ and i into $i = (i_R, i_C, i_L, i_V, i_{src})$. Rewrite component relations:

$$i_R = G\, v_R = G\, A_R^\top u,$$
$$i_C = C\, v_C' = C\, A_C^\top u'.$$

Kirchhoffs current law:

$$A_C i_C + A_R i_R + A_L i_L + A_V i_V = -A_I i_{src}.$$

Together with component relations combine to

$$A_C C A_C^\top u' + A_R G A_R^\top u + A_L i_L + A_V i_V = -A_I i_{src},$$
$$L i_L' - A_L^\top u = 0,$$
$$-A_V^\top u = -v_{src}.$$

In matrix form:

$$\begin{pmatrix} A_C C A_C^\top & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{pmatrix} * \begin{pmatrix} u' \\ i_L' \\ i_V' \end{pmatrix} + \begin{pmatrix} A_R G A_R^\top & A_L & A_V \\ -A_L^\top & 0 & 0 \\ -A_V^\top & 0 & 0 \end{pmatrix} * \begin{pmatrix} u \\ i_L \\ i_V \end{pmatrix} = \begin{pmatrix} -A_I i_{src} \\ 0 \\ -v_{src} \end{pmatrix}. \quad (3)$$

$\rightarrow$ differential and algebraic variables

Differential Algebraic Equations

Types of DAEs

In the most general form a DAE can be written as: Find $y : \mathbb{R} \to \mathbb{R}^n$ such that

$$F(t, y(t), y'(t)) = 0, \qquad \forall t \in I \tag{4}$$

with $F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ sufficiently smooth and I the time-interval.

Linear systems with constant coefficients

find y such that

$$Ay'(t) + By(t) = f(t), \tag{5}$$

with $A, B \in \mathbb{R}^{n \times n}$, A singular, B and $f : \mathbb{R} \to \mathbb{R}^n$ a function in time. $\to$ differential and algebraic variables

Equivalence transformations lead to

$$u'(t) + Ru(t) = s(t),$$
$$Nv'(t) + v(t) = q(t),$$

(6)

where N is a nilpotent matrix and the matrix R is regular.

First equation is an ODE $\rightarrow$ unique solution $u(t)$ for any initial values. Construct explicit solution for second equation: $q(t) \in C^{k-1}([t_0, t_l])$:

$$
\begin{aligned}
v(t) &= q(t) - Nv'(t) = q(t) - N(\underbrace{q(t) - Nv'(t)}_{=v(t)})' = q - Nq' + N^2 v'' \\
&= q - Nq' + N^2(q - Nv')'' = q - Nq' + N^2 q'' - N^3 v''' \\
&\;\;\vdots \\
&= q - Nq' + ... + (-1)^{k-1} N^{k-1} \underbrace{\frac{d^{k-1}}{dt^{k-1}} q}_{:=q^{(k-1)}} + (-1)^{k-1} \underbrace{N^k v^{(k)}}_{=0} \\
&= \sum_{i=0}^{k-1} (-1)^i N^i q^{(i)}(t)
\end{aligned}
$$

$\rightarrow$ differentiation index $\nu = k$.

Differential Algebraic Equations

Index of a Differential Algebraic Equation

Definition

Consider the differential algebraic equation (4) to be uniquely solvable and F sufficiently smooth. For a given $m \in \mathbb{N}$ consider the equations

$$F(t, y, y') = 0,$$
$$\frac{dF(t, y, y')}{dt} = 0,$$
$$\vdots$$
$$\frac{d^m F(t, y, y')}{dt^m} = 0.$$

The smallest natural number m for which the above system results in an explicit system of ordinary differential equations (ODEs), i.e. it has the form

$$y' = \phi(t, y),$$

Index Analysis of the Modified Nodal Analysis

Topological Conditions

- The resulting equations have index $\nu \leq 2$, if the circuit neither contains loops of only voltage sources nor cutsets of only current sources.
- They have index $\nu \leq 1$, if the circuit contains neither loops of only capacitors and/or voltage sources nor cutsets of only inductors and/or current sources.
- They have index $\nu = 0$, if every node in the circuit is connected to the reference node (ground) through a path containing only capacitors.
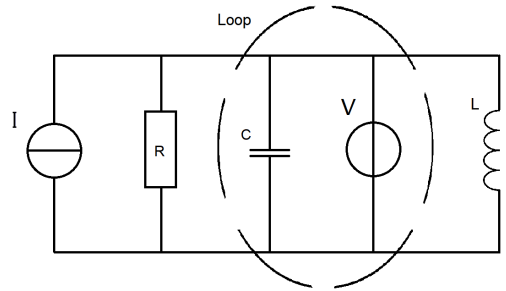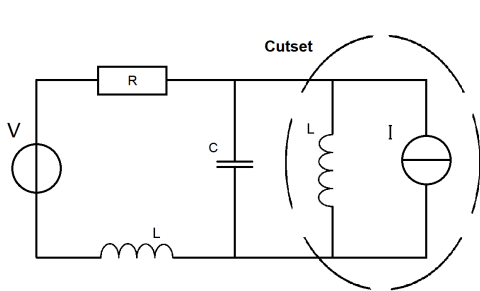
Figure: Illustration of a cutset and a loop.

Theorem (Index conditions)

Let the matrices of the capacitances, inductances and resistances be positive definite.

- *If*

$$\ker([A_R, A_C, A_V, A_L]^\top) = \{0\} \quad \text{and} \quad \ker(A_V) = \{0\} \tag{7}$$

  holds, then the MNA (3) leads to a system with index $\nu \leq 2$.

- *If additionally*

$$\ker([A_R, A_C, A_V]^\top) = \{0\} \quad \text{and} \quad \ker([A_C, A_V]) = \{0\} \tag{8}$$

  holds, then the system is of index $\nu \leq 1$

- *If further*

$$\ker(A_C^\top) = \{0\} \quad \text{and} \quad \dim(\nu_{src}) = 0 \tag{9}$$

  holds, then the system has index $\nu = 0$.

Numerical Solutions

Multistep Methods

Definition (Multistep method)

For given $\alpha_0, ..., \alpha_k$ and $\beta_0, ..., \beta_k$ the iteration rule

$$\sum_{l=0}^{k} \alpha_l y_{m+l} = h \sum_{l=0}^{k} \beta_l f(t_{m+l}, y_{m+l}), \quad m = 0, 1, ..., N - k \qquad (10)$$

is called a *linear multistep method* (linear k-step method). It is always assumed that $\alpha_k \neq 0$ and $|\alpha_0| + |\beta_k| > 0$. If $\beta_k = 0$ holds, then the method is called explicit, otherwise implicit.

Definition (generating polynomials)

$$\rho(x) := \sum_{l=0}^{k} \alpha_l x^l \qquad \text{and} \qquad \sigma(x) := \sum_{l=0}^{k} \beta_l x^l$$

Definition (Convergence order)

We say that a linear multi-step method is convergent of order $p \in \mathbb{N}$, if for a solution y of the problem and a vector $(y_j)_{j=0}^k$ created by an LMSM, we have that

$$\max_{0 \leq j \leq k} \|y(t_j) - y_j\| \leq Ch^p.$$

Where C is a constant not dependent on the step size h.

Consistency on tells us whether the method approximates the equation correctly as the step size approaches 0.

Definition

1. The set

$$S := \{z \in \mathbb{C} : \rho(\xi) - z\sigma(\xi) = 0 \implies \xi \in \mathbb{C} \text{ and } |\xi| \leq 1.$$
$$\text{If } \xi \text{ has multiplicity greater than 1, then } |\xi| < 1\} \tag{11}$$

   is called the region of stability of the method.

2. A linear multistep method is called
   - *0-stable*, if $0 \in S$.
   - $A(\alpha)$-*stable*, if it is stable in all z that lie within the set $\{z \in \mathbb{C}^- : |arg(z) - \pi| \leq \alpha\}$ for $\alpha \in (0, \frac{\pi}{2})$.

Theorem

Let $f(t, y)$ *be sufficiently smooth and the linear multi-step method be 0-stable and consistent of order* p*, then it is also convergent of order* p*.*

Numerical Solutions

Consistent Initial Values

Index $\nu = 0$**:** no additional restrictions (ODE case).
Index $\nu = 1$**:**
Rewrite system into the form

$$y'(t) = f(t, y(t), z(t)),$$
$$0 = g(t, y(t), z(t)).$$

Conditions for consistent initial values $\rightarrow g(t_0, y_0, z_0) = 0$.

Index $\nu = 2$:
Rewrite system into the form

$$y' = f(t, y(t), z(t)),$$
$$0 = g(t, y(t)).$$

Conditions for consistent initial values

$$g(t_0, y_0) = 0,$$
$$g_t(t_0, y_0) + g_y(t_0, y_0) f(t_0, y_0, z_0) = 0. \qquad \text{(hidden constraint)}$$

Numerical Solutions

Implicit Linear Multistep Formulas
BDF-k Methods

The *backward differentiation formula (BDF)* is a family of implicit linear multistep methods.

$$\sum_{k=0}^{s} \alpha_k y_{n+k} = h\beta f(t_{n+s}, y_{n+s})$$

The BDF or BDF-k formulas for $k = 1, ..., 3$ have the following form

$$k = 1 : hf_{m+1} = y_{m+1} - y_m \qquad \text{(implicit euler)}$$

$$k = 2 : hf_{m+2} = \frac{1}{2}(3y_{m+2} - 4y_{m+1} + y_m)$$

$$k = 3 : hf_{m+3} = \frac{1}{6}(11y_{m+3} - 18y_{m+2} + 9y_{m+1} - 2y_m)$$
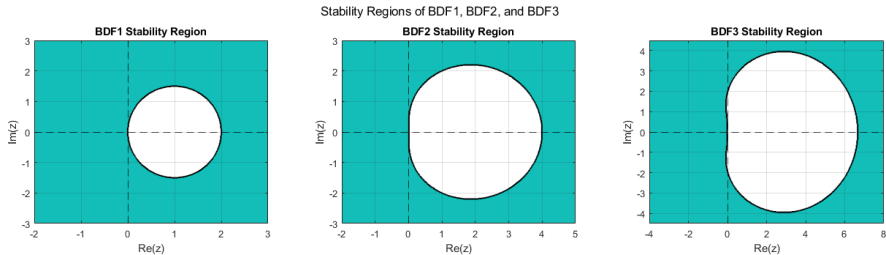
Figure: stability regions of BDF-schemes

Theorem

The BDF-k methods have consistency order $p = k$.

Corollary (Convergence rate)

The BDF-k methods with $k \leq 6$ are convergent with order k

Numerical Solutions

Implicit Linear Multistep Formulas

Trapezoidal rule

JOHANNES KEPLER
UNIVERSITY LINZ
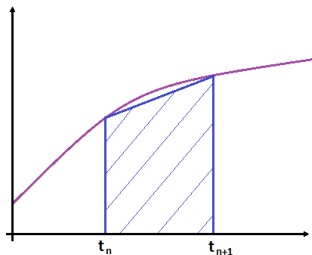
Figure: illustration of the trapezoidal rule

This procedure is repeated for small subsections of the interval $[a, b]$. Thus we obtain the iteration formula

$$y_{n+1} = y_n + \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_{n+1})].$$

The trapezoidal rule has convergence order $p = 2$.

Numerical Solutions

Numerical Examples
Example1

JOHANNES KEPLER
UNIVERSITY LINZ

Figure: charging capacitor with series resistor and voltage source
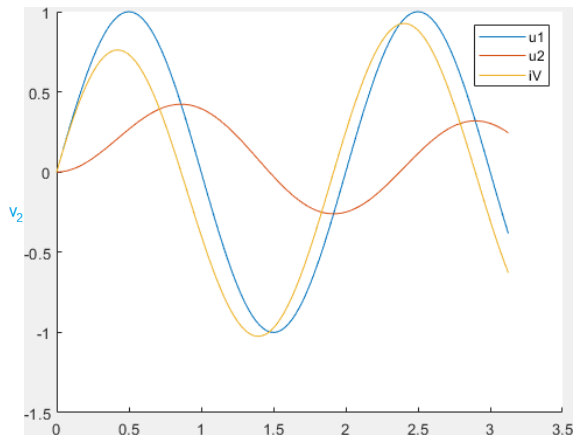
Figure: Exact solution for example 1.

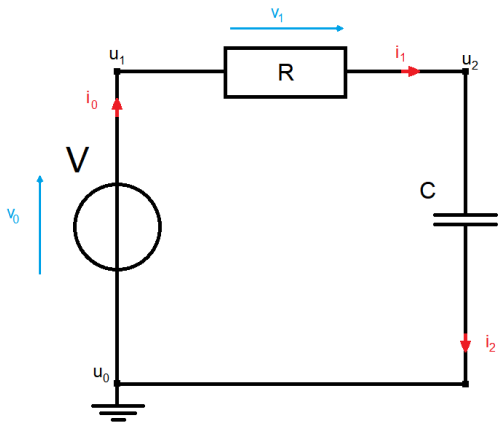| h	k = 1		k = 2		k = 3		trapezoidal	
	$err(u_2)$	$err(i_V)$	$err(u_2)$	$err(i_V)$	$err(u_2)$	$err(i_V)$	$err(u_2)$	$err(i_V)$
0.1	4.620×10^{-2}	4.620×10^{-2}	9.567×10^{-3}	9.567×10^{-3}	2.852×10^{-3}	2.852×10^{-3}	3.344×10^{-3}	3.344×10^{-3}
0.05	2.339×10^{-2}	2.339×10^{-2}	2.454×10^{-3}	2.454×10^{-3}	3.645×10^{-4}	3.645×10^{-4}	8.367×10^{-4}	8.367×10^{-4}
0.025	1.178×10^{-2}	1.178×10^{-2}	6.264×10^{-4}	6.264×10^{-4}	4.928×10^{-5}	4.928×10^{-5}	2.092×10^{-4}	2.092×10^{-4}

Table: Resulting errors for the BDF-k methods and the trapezoidal rule.

Numerical Solutions

Numerical Examples
Gauss

Gauss-1 method or implicit midpoint rule:

$$y_{n+1} = y_n + hf(t_n + \frac{h}{2}, \frac{y_n + y_{n+1}}{2})$$

different convergence rates for algebraic and differential variables:

index $\nu = 1$		index $\nu = 2$	
differential	algebraic	differential	algebraic
2	2	2	0

Table: Convergence order for Gauss method.

h	example 1			example 2		example 3	
	$err(u_1)$ (alg)	$err(u_2)$ (diff)	$err(i_0)$ (alg)	$err(u_1)$ (diff)	$err(i_L)$ (diff)	$err(u_1)$ (diff)	$err(i_V)$ (alg)
0.1	$1.247{\times}10^{-2}$	$2.141{\times}10^{-3}$	$1.299{\times}10^{-2}$	$6.589{\times}10^{-3}$	$7.885{\times}10^{-3}$	$1.247{\times}10^{-2}$	$1.318{\times}10^{-1}$
0.05	$3.092{\times}10^{-3}$	$5.343{\times}10^{-4}$	$3.252{\times}10^{-3}$	$1.649{\times}10^{-3}$	$1.974{\times}10^{-3}$	$3.092{\times}10^{-3}$	$3.270{\times}10^{-2}$
0.025	$7.716{\times}10^{-4}$	$1.335{\times}10^{-4}$	$8.115{\times}10^{-4}$	$4.123{\times}10^{-4}$	$4.936{\times}10^{-4}$	$7.716{\times}10^{-4}$	$8.153{\times}10^{-3}$

Table: Resulting errors for the Gauss method with one stage.

Definition

Let $y(t)$ be the exact solution of (4) and $\tilde{y}(t)$ be the solution of the perturbed system $F(t, \tilde{y}, \tilde{y}') = \delta(t)$. The smallest number $k \in \mathbb{N}$ such that

$$\|y(t) - \tilde{y}(t)\| \le C \left(\|y(t_0) - \tilde{y}(t_0)\| + \sum_{j=0}^{k} \max_{t_0 \le \xi \le T} \left\| \int_{t_0}^{\xi} \frac{d^j \delta}{d\tau^j}(\tau) d\tau \right\| \right)$$

for all $\tilde{y}(t)$, is called the **perturbation index** of this system.

Dahlquist test problem as a model problem, find y such that

$$y' = \lambda y, \quad t > 0 \tag{12}$$

$$y(0) = y_0 \tag{13}$$

with $\lambda \in \mathbb{C}$ and y_0 fixed.

Thus the resulting linear multistep method is of the form

$$\sum_{l=0}^{k} \alpha_l y_{n+l} = h \sum_{l=0}^{k} \beta_l \lambda y_{n+l}$$

$$\Longleftrightarrow \sum_{l=0}^{k} [\alpha_l - h\beta_l \lambda] y_{n+l} = 0$$
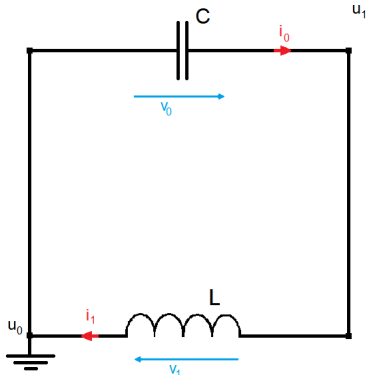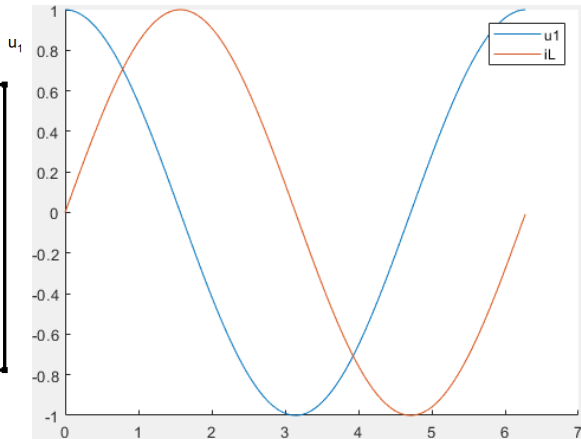
Figure: LC-circuit

Figure: Exact solution for example 2.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} u_1' \\ i_L' \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} * \begin{pmatrix} u_1 \\ i_L \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

h	k = 1		k = 2		k = 3		trapezoidal	
	$err(u_1)$	$err(i_l)$	$err(u_1)$	$err(i_l)$	$err(u_1)$	$err(i_l)$	$err(u_1)$	$err(i_l)$
0.1	$2.659{\times}10^{-1}$	$2.106{\times}10^{-1}$	$1.686{\times}10^{-2}$	$1.917{\times}10^{-2}$	$1.286{\times}10^{-3}$	$1.042{\times}10^{-3}$	$4.007{\times}10^{-3}$	$5.140{\times}10^{-3}$
0.05	$1.446{\times}10^{-1}$	$1.125{\times}10^{-1}$	$4.266{\times}10^{-3}$	$5.067{\times}10^{-3}$	$1.501{\times}10^{-4}$	$1.131{\times}10^{-4}$	$1.003{\times}10^{-3}$	$1.301{\times}10^{-3}$
0.025	$7.543{\times}10^{-2}$	$5.817{\times}10^{-2}$	$1.070{\times}10^{-3}$	$1.294{\times}10^{-3}$	$1.460{\times}10^{-5}$	$9.674{\times}10^{-6}$	$2.507{\times}10^{-4}$	$3.268{\times}10^{-4}$

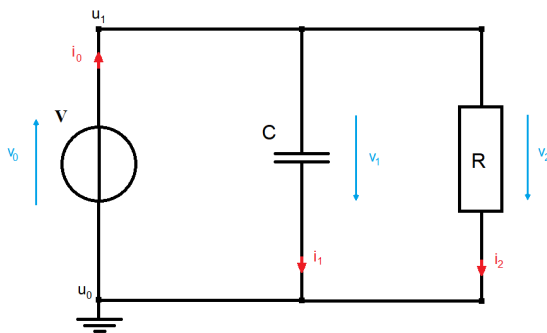Table: Resulting errors for the BDF-k methods and the trapezoidal rule.

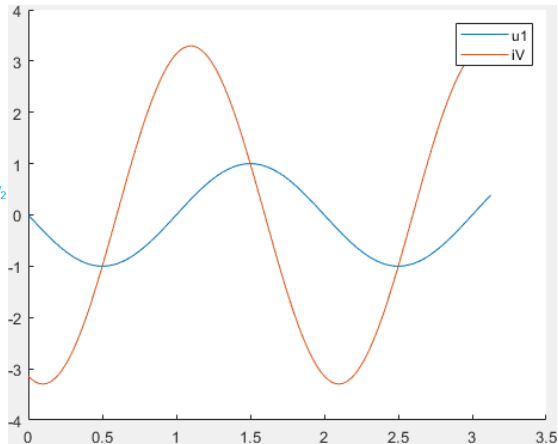Figure: Current source with capacitor and resistor.

Figure: Exact solution for example 3.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} * \begin{pmatrix} u_1' \\ i_V' \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} u_1 \\ i_V \end{pmatrix} = \begin{pmatrix} 0 \\ -\sin(\pi t) \end{pmatrix}.$$

h	k = 1	k = 2	k = 3	trapezoidal
	$err(i_V)$	$err(i_V)$	$err(i_V)$	$err(i_V)$
0.1	4.894×10^{-1}	1.023×10^{-1}	2.403×10^{-2}	5.219×10^{-2}
0.05	2.462×10^{-1}	2.577×10^{-2}	3.034×10^{-3}	1.295×10^{-2}
0.025	1.233×10^{-1}	6.456×10^{-3}	4.029×10^{-4}	3.232×10^{-3}

Table: Resulting errors for the BDF-k methods and the trapezoidal rule.