Numerical Methods for Differential-Algebraic Equations

Exercise Sheet 2, May 2, 2024, 10:15-11:45, Room S2 346

We consider initial value problems of the form

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0. \tag{1}$$

The function f is assumed to be sufficiently smooth in both arguments.

Exercise 1. Putting $z(t) = (y(t), t)$, we obtain the autonomous system

$$z'(s) := \begin{pmatrix} y'(s) \\ t'(s) \end{pmatrix} = \begin{pmatrix} f(t(s), y(s)) \\ 1 \end{pmatrix} =: g(z(s)), \qquad z(0) = \begin{pmatrix} y_0 \\ t_0 \end{pmatrix}. \tag{2}$$

Show that the two formulations (1) and (2) are equivalent, i.e., solutions of the one correspond to solutions of the other, and vice versa.

Exercise 2. We consider RK-methods with $y^0 = y_0$, $t^n = nh$, and

$$y^{n+1} = y^n + h \sum\nolimits_{j=1}^{m} \beta_j f(t^n + \gamma_j h, g_j^n) \tag{3}$$

$$g_j^n = y^n + h \sum\nolimits_{l=1}^{m} \alpha_{jl} f(t^n + \gamma_k h, g_k^n), \qquad j = 1, \dots, m. \tag{4}$$

Assume that $\sum_j \beta_j = 1$ and that the scheme is *autonomy-invariant*, i.e.,

$$\sum\nolimits_{l=1}^{m} \alpha_{jl} = \gamma_j, \qquad j = 1, \dots, m. \tag{5}$$

Show that the numerical solutions for (1) and (2) obtained such schemes coincide.

Exercise 3 (g-form vs. k-form)**.**
Let α_{ij}, β_j, and γ_j be the coefficients of a RK-scheme. Consider the method

$$y^{n+1} = y^n + h \sum\nolimits_j \beta_j k_j^n \tag{6}$$

with intermediate slopes k_j^n defined by

$$k_j^n = f(t^n + \gamma_j h, y^n + h \sum\nolimits_l \alpha_{jl} k_l^n). \tag{7}$$

Show that this k-form (6)–(7) is equivalent to the g-form (3)–(4).

Exercise 4. Determine the Butcher-Tableaus for the following methods:

(a) Method of Heun: $y^{n+1} = y^n + \frac{h}{2}[f(t^n, y^n) + f(t^n + h, y^n + hf(t^n, y^n))]$.

(b) Implicit trapezoidal rule: $y^{n+1} = y^n + \frac{h}{2}[f(t^n, y^n) + f(t^n + h, y^{n+1})]$

Are the schemes explicit or explicit? What is the number of stages?

Exercise 5. Consider the following Butcher-Tableaus:

$$(a)\quad \begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array} \qquad\qquad (b)\quad \begin{array}{c|cc} \frac{1}{3} & \frac{5}{12} & -\frac{1}{12} \\ 1 & \frac{3}{4} & \frac{1}{4} \\ \hline & \frac{3}{4} & \frac{1}{4} \end{array}$$

Formulate the corresponding RK-schemes in g- and k-form. Are the schemes explicit or explicit? What is the number of stages?

Exercise 6. Using Taylor-series, one can show the following facts (see Strehmel, Weiner, Podhaisky): A Runge-Kutta method is consistent with order $p \geq 1$, if

(O1) $\sum_j \beta_j = 1$.

The order is $p \geq 2$, if additionally, the conditions

(AI) $\sum_k \alpha_{jk} = \gamma_j, \quad j = 1, \ldots, m;$
(O2) $\sum_{j,k} \beta_j \alpha_{jk} = \frac{1}{2};$

hold. For order $p \geq 3$, the additional conditions

(O3a) $\sum_{j,k,l} \beta_j \alpha_{jk} \alpha_{jl} = \frac{1}{3};$

(O3b) $\sum_{j,k,l} \beta_j \alpha_{jk} \alpha_{kl} = \frac{1}{6};$

have to be satisfied, and for order $p \geq 4$, one requires

(O4a) $\sum_{j,k,l,m} \beta_j \alpha_{jk} \alpha_{jl} \alpha_{jm} = \frac{1}{4};$

(O4b) $\sum_{j,k,l,m} \beta_j \alpha_{jk} \alpha_{jl} \alpha_{lm} = \frac{1}{8};$

(O4c) $\sum_{j,k,l,m} \beta_j \alpha_{jk} \alpha_{kl} \alpha_{km} = \frac{1}{12};$

(O4d) $\sum_{j,k,l,m} \beta_j \alpha_{jk} \alpha_{kl} \alpha_{lm} = \frac{1}{24}.$

Use these conditions to determine the order of the methods in Exercise 4 and 5.

Exercise 7. Implement MATLAB codes for the numerical solution of

(a) $y'(t) = -y(t) + \sin(t)$, $y(0) = 0$;

(b) $y'(t) = \sin(y(t))$, $y(0) = 1$;

with the methods of Exercise 4 and 5. Test the correctness of the codes and determine the convergence rates by comparing to the analytical solutions on $[0, 5]$.

Bonus. Consider the extension of your code to nonlinear systems of differential equations and implicit RK-methods of arbitrary order. Formulate appropriate test problems and verify the correctness of your code.