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Circuit Modelling

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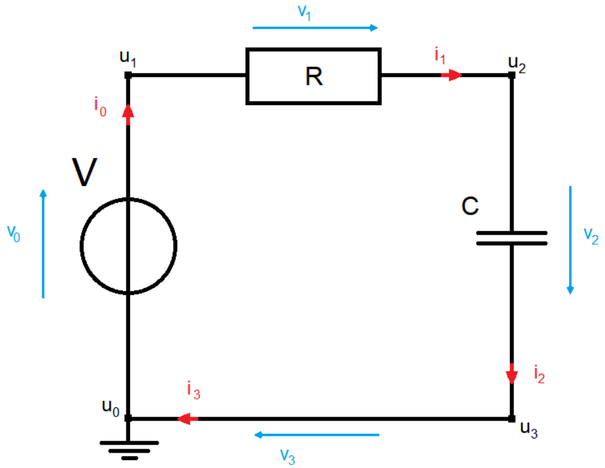
Example: Charging of a capacitor:



Formulating a Mathematical Model

Network Topology

For a circuit with l nodes and k edges, define the incidence matrix $\tilde{A} = (\tilde{a}_{ij}) \in \mathbb{R}^{k \times l}$:

$$\tilde{a}_{ij} = \begin{cases} 1 & \text{edge } j \text{ starts at node } i, \\ -1 & \text{edge } j \text{ ends at node } i, \\ 0 & \text{else.} \end{cases}$$

By grounding node 0, i.e. $u_0 = 0$ we obtain the reduced incidence matrix $\rightarrow A$.

Formulating a Mathematical Model

Energy Conservation Laws

- **Kirchhoff's voltage law (KVL):**
  The sum of voltages along each loop of the network must equal to zero.

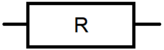
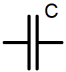
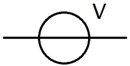
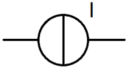
$$\rightarrow A^\top u = v. \tag{1}$$

- **Kirchhoff's current law (KCL):**
  For any node, the sum of currents flowing into the node is equal to the sum of currents flowing out of the node.

$$\rightarrow Ai = 0. \tag{2}$$

Formulating a Mathematical Model

Electrical Components and their Relations

Name	Symbol	Component Law
Resistor		$v = R\,i$ or $i = G\,u$
Capacitor		$Q = C\,v$ and by derivation in t $I = C\,\frac{d}{dt}v = C\,v'$
Inductor		$\Phi = L\,i$ and by derivation in t $v = L\,i'$
Voltage Source		$v = v_{src}$
Current Source		$i = i_{src}$

Formulating a Mathematical Model

Modified Nodal Analysis - MNA

Rearrange the columns of the reduced incidence matrix A into

$$A = (A_R A_C A_L A_V A_I)$$

A_R, A_C, A_L, A_V and A_I ... columns related to components
Represent voltages:

$$v = A^\top u$$

$\to$ rearrange v into $v = (v_R, v_C, v_L, v_{src}, v_I)$ and i into $i = (i_R, i_C, i_L, i_V, i_s rc)$. Rewrite component relations:

$$i_R = G\, v_R = G\, A_R^\top u,$$
$$i_C = C\, v_C' = C\, A_C^\top u'.$$

Kirchhoffs current law:

$$A_C i_C + A_R i_R + A_L i_L + A_V i_V = -A_I i_{src}.$$

Combining the component relations with the reduced incidence matrix and the Kirchhoff's laws we get:

$$
\begin{aligned}
A_C C A_C^\top u' + A_R G A_R^\top u + A_L i_L + A_V i_V &= -A_I i_{src}, \\
L i_L' - A_L^\top u &= 0, \\
-A_V^\top u &= -v_{src}.
\end{aligned}
$$

In matrix form:

$$
\begin{pmatrix} A_C C A_C^\top & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{pmatrix} * \begin{pmatrix} u' \\ i_L' \\ i_V' \end{pmatrix} + \begin{pmatrix} A_R G A_R^\top & A_L & A_V \\ -A_L^\top & 0 & 0 \\ -A_V^\top & 0 & 0 \end{pmatrix} * \begin{pmatrix} u \\ i_L \\ i_V \end{pmatrix} = \begin{pmatrix} -A_I i_{src} \\ 0 \\ -v_{src} \end{pmatrix}. \tag{3}
$$

$\rightarrow$ differential and algebraic variables

Differential Algebraic Equations

Types of DAEs

In the most general form a DAE can be written as: Find $y : \mathbb{R} \to \mathbb{R}^n$ such that

$$F(t, y(t), y'(t)) = 0, \qquad \forall t \in I \tag{4}$$

with $F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ sufficiently smooth and I the time-interval.

Linear systems with constant coefficients

find y such that

$$Ay'(t) + By(t) = f(t), \tag{5}$$

with $A, B \in \mathbb{R}^{n \times n}$, A singular, B regular and $f : \mathbb{R} \to \mathbb{R}^n$ a function in time. $\to$ differential and algebraic variables

Differential Algebraic Equations

Weierstrass-Kronecker normalform

Definition

The matrix pencil $\{A, B\}$ is called *regular* if there exists some $c \in \mathbb{R}$, such that $(cA + B)$ is regular (i.e. $\det(cA + B) \neq 0$), otherwise it is called singular.

Equivalence transformations lead to

$$\begin{aligned} u'(t) + Ru(t) &= s(t), \\ Nv'(t) + v(t) &= q(t), \end{aligned} \tag{6}$$

where N is a nilpotent matrix and the matrix R is regular.

First equation of (6) is a first order ordinary differential equation and posesses a unique solution $u(t)$ in $[t_0, t_l]$ for any initial values $u_0 \in \mathbb{R}^d$. Now let $q(t) \in C^{k-1}([t_0, t_l])$ and differentiate the second equation in (6):

$$
\begin{aligned}
v(t) &= q(t) - Nv'(t) = q(t) - N(\underbrace{q(t) - Nv'(t)}_{=v(t)})' = q - Nq' + N^2 v'' \\
&= q - Nq' + N^2(q - Nv')'' = q - Nq' + N^2 q'' - N^3 v''' \\
&\vdots \\
&= q - Nq' + ... + (-1)^{k-1} N^{k-1} \underbrace{\frac{d^k}{dt^k} q}_{:=q^{(k-1)}} + (-1) \underbrace{N^k v^{(k)}}_{=0} \\
&= \sum_{i=0}^{k-1} (-1)^i N^i q^{(i)}(t)
\end{aligned}
$$

$\rightarrow$ differentiation index ν. (There exist other index concepts, for our case these are all

Differential Algebraic Equations

Index of a Differential Algebraic Equation

Definition

Consider the differential algebraic equation (4) to be uniquely solvable and F sufficiently smooth. For a given $m \in \mathbb{N}$ consider the equations

$$
\begin{aligned}
F(t, y, y') &= 0, \\
\frac{dF(t, y, y')}{dt} &= 0, \\
&\vdots \\
\frac{d^m F(t, y, y')}{dt^m} &= 0.
\end{aligned}
$$

The smallest natural number m for which the above system results in an explicit system of ordinary differential equations (ODEs), i.e. it has the form

$$
y' = \phi(t, y),
$$

Index Analysis of the Modified Nodal Analysis

Topological Conditions

- The resulting equations have index $\nu \leq 2$, if the circuit neither contains loops of voltage sources nor cutsets of current sources.
- They have index $\nu \leq 1$, if the circuit contains neither loops of capacitors and/or voltage sources nor cutsets of inductors and/or current sources.
- They have index $\nu = 0$, if every node in the circuit is connected to the reference node (ground) through a path containing only capacitors.

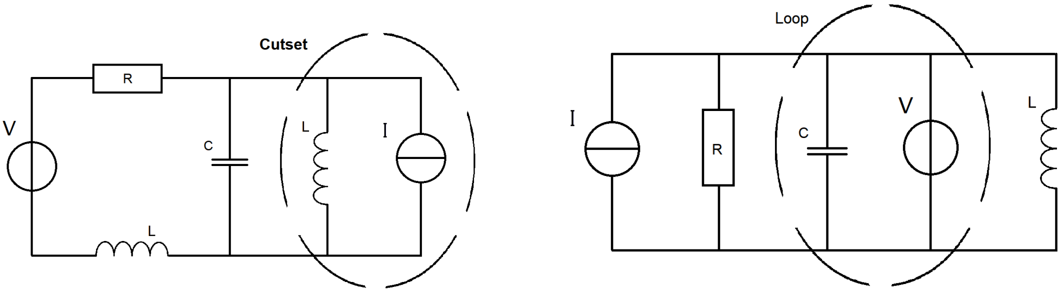


Figure: Illustration of a cutset and a loop.

Theorem (Index conditions)

Let the matrices of the capacitances, inductances and resistances be positive definite.

- *If*

$$\ker([A_R, A_C, A_V, A_L]^\top) = \{0\} \quad \textit{and} \quad \ker(A_V) = \{0\} \tag{7}$$

holds, then the MNA (3) *leads to a system with index* $\nu \leq 2$*.*

- *If additionally*

$$\ker([A_R, A_C, A_V]^\top) = \{0\} \quad \textit{and} \quad \ker([A_C, A_V]) = \{0\} \tag{8}$$

holds, then the system is of index $\nu \leq 1$

- *If further*

$$\ker(A_C^\top) = \{0\} \quad \textit{and} \quad \dim(v_{src}) = 0 \tag{9}$$

holds, then the system has index $\nu = 0$*.*

Numerical Solutions

Multistep Methods

Definition (Multistep method)

For given $\alpha_0, ..., \alpha_k$ and $\beta_0, ..., \beta_k$ the iteration rule

$$\sum_{l=0}^{k} \alpha_l y_{m+l} = h \sum_{l=0}^{k} \beta_l f(t_{m+l}, y_{m+l}), \quad m = 0, 1, ..., N-k \tag{10}$$

is called a *linear multistep method* (linear k-step method). It is always assumed that $\alpha_k \neq 0$ and $|\alpha_0| + |\beta_k| > 0$. If $\beta_k = 0$ holds, then the method is called explicit, otherwise implicit.

Definition (generating polynomials)

$$\rho(x) := \sum_{l=0}^{k} \alpha_l x^l \qquad \text{and} \qquad \sigma(x) := \sum_{l=0}^{k} \beta_l x^l$$

Definition (Convergence order)

We say that a linear multi-step method is convergent of order $p \in \mathbb{N}$, if for a solution y of the problem and a vector $(y_j)_{j=0}^{k}$ created by an LMSM, we have that

$$\max_{0 \leq j \leq k} \|y(t_j) - y_j\| \leq Ch^p.$$

Where C is a constant not dependent on the step size h.

Definition (Consistency order)

A linear multi-step method is called *consistent with order p*, if for all functions $y(t) \in C^{p+1}[t_0, t_l]$

$$L[y(t), h] = \mathcal{O}(h^{p+1}) \quad \text{for} \quad h \to 0$$

holds.

Numerical Solutions

Multistep Methods

Stability properties

Definition

1. The set
$$S := \{z \in \mathbb{C} : \rho(\xi) - z\sigma(\xi) = 0 \implies \xi \in \mathbb{C} \text{ and } |\xi| \leq 1. \quad \text{If } \xi \text{ has multiplicity greater than 1, then } |\xi| < 1\} \tag{11}$$
is called the region of stability of the method.
2. A linear multistep method is called
    - *0-stable*, if $0 \in S$.
    - stable in the point $z \in \mathbb{C}$, if $z \in S$.
    - $A(\alpha)$*-stable*, if it is stable in all z that lie within the set $\{z \in \mathbb{C}^- : |arg(z) - \pi| \leq \alpha\}$ for $\alpha \in (0, \frac{\pi}{2})$.

Theorem

Let $f(t, y)$ be sufficiently smooth and the linear multi-step method be zero-stable and consistent of order p, then it is also convergent of order p.

Numerical Solutions

Consistent Initial Values

Index $\nu = 0$: no additional restrictions (ODE case).
Index $\nu = 1$:
By rewriting our system into the form

$$\begin{aligned} y'(t) &= f(t, y(t), z(t)), \\ 0 &= g(t, y(t), z(t)). \end{aligned}$$

we are able to give conditions for consistent initial values. Namely y_0 and z_0 are consistent initial values for this system, if $g(t_0, y_0, z_0) = 0$ holds.

Index $\nu = 2$:
For index-2 systems we rewrite our system into

$$\begin{aligned} y' &= f(t, y(t), z(t)), \\ 0 &= g(t, y(t)). \end{aligned}$$

Consistent initial values y_0, z_0 for this case not only have to fulfill $g(t_0, y_0) = 0$ but also the *hidden constraint* $g_t(t_0, y_0) + g_y(t_0, y_0) f(t_0, y_0, z_0)$. By g_t and g_y we denote the derivative of g with respect to t or y, respectively.

Numerical Solutions

Implicit Linear Multistep Formulas

BDF-k Methods

The *backward differentiation formula (BDF)* is a family of implicit linear multistep methods. They have the general form

$$\sum_{k=0}^{s} \alpha_k y_{n+k} = h\beta f(t_{n+s}, y_{n+s}) \tag{12}$$

The BDF or BDF-k formulas for $k = 1, ..., 3$ have the following form

$$k = 1 : hf_{m+1} = y_{m+1} - y_m \qquad \text{(implicit euler)}$$

$$k = 2 : hf_{m+2} = \frac{1}{2}(3y_{m+2} - 4y_{m+1} + y_m)$$

$$k = 3 : hf_{m+3} = \frac{1}{6}(11y_{m+3} - 18y_{m+2} + 9y_{m+1} - 2y_m)$$

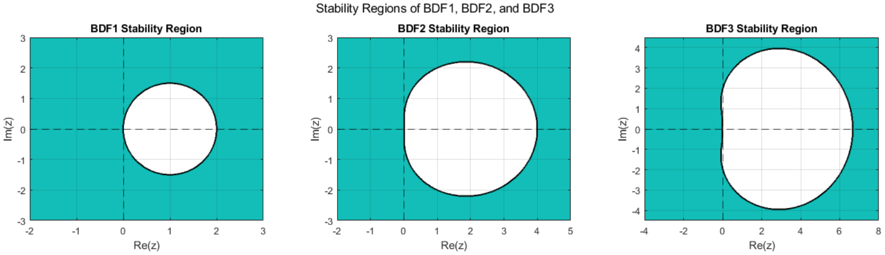


Figure: stability regions of BDF-schemes

Theorem

The BDF-k methods have consistency order $p = k$.

Corollary (Convergence rate)

The BDF-k methods with $k \leq 6$ *are convergent with order* k

Numerical Solutions

Implicit Linear Multistep Formulas

Trapezoidal rule

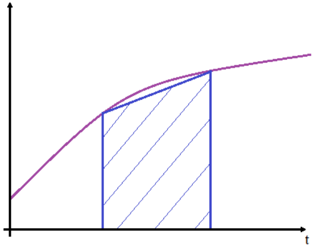


Figure: illustration of the trapezoidal rule

This procedure is repeated for small subsections of the interval $[a, b]$. Thus we obtain the iteration formula

$$u_h(t+h) = u_h(t) + \frac{h}{2}[f(t, u_h(t)) + f(t+h, u_h(t+h))].$$

The trapezoidal rule has convergence order $p = 2$.

Numerical Solutions

Numerical Examples

Example1

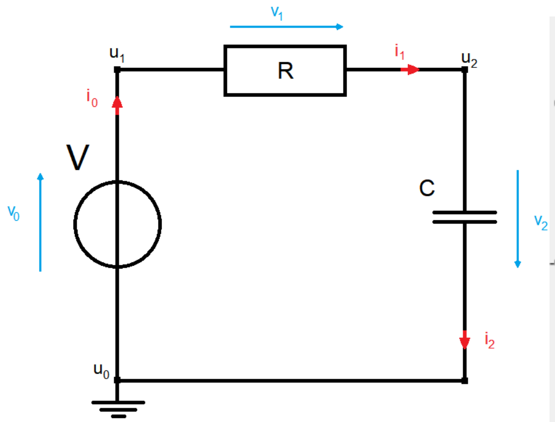


Figure: charging capacitor with series resistor and voltage source

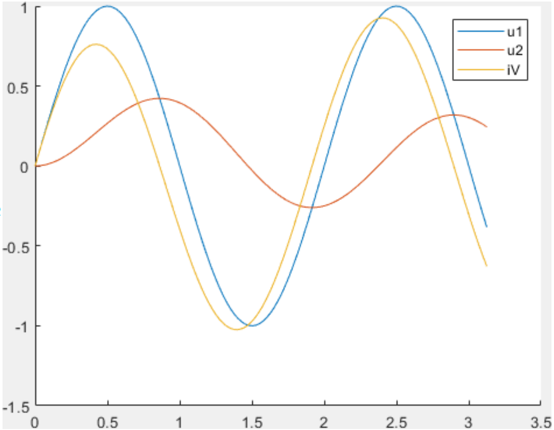


Figure: Exact solution for example 1.

h	k = 1		k = 2		k = 3		trapezoidal	
	$\mathrm{err}(u_2)$	$\mathrm{err}(i_V)$	$\mathrm{err}(u_2)$	$\mathrm{err}(i_V)$	$\mathrm{err}(u_2)$	$\mathrm{err}(i_V)$	$\mathrm{err}(u_2)$	$\mathrm{err}(i_V)$
0.1	4.620×10^{-2}	4.620×10^{-2}	9.567×10^{-3}	9.567×10^{-3}	2.852×10^{-3}	2.852×10^{-3}	3.344×10^{-3}	3.344×10^{-3}
0.05	2.339×10^{-2}	2.339×10^{-2}	2.454×10^{-3}	2.454×10^{-3}	3.645×10^{-4}	3.645×10^{-4}	8.367×10^{-4}	8.367×10^{-4}
0.025	1.178×10^{-2}	1.178×10^{-2}	6.264×10^{-4}	6.264×10^{-4}	4.928×10^{-5}	4.928×10^{-5}	2.092×10^{-4}	2.092×10^{-4}

Table: Resulting errors for the BDF-k methods and the trapezoidal rule.

Numerical Solutions

Numerical Examples

Gauss

$$y_{n+1} = y_n + hf(t_n + \frac{h}{2}, \frac{y_n + y_{n+1}}{2})$$

number of stages s	index $\nu = 1$ differential	index $\nu = 1$ algebraic	index $\nu = 2$ differential	index $\nu = 2$ algebraic
even	$2s$	$s+1$	$s+1$	$s-1$
odd		s	s	$s-2$

Table: Convergence order for Gauss method.

h	example 1			example 2		example 3	
	$\mathrm{err}(u_1)$ (alg)	$\mathrm{err}(u_2)$ (diff)	$\mathrm{err}(i_0)$ (alg)	$\mathrm{err}(u_1)$ (diff)	$\mathrm{err}(i_L)$ (diff)	$\mathrm{err}(u_1)$ (diff)	$\mathrm{err}(i_V)$ (alg)
0.1	6.257×10^{-1}	1.041×10^{-1}	6.154×10^{-1}	6.589×10^{-3}	7.885×10^{-3}	6.257×10^{-1}	1.251×10^{3}
0.05	3.138×10^{-1}	5.181×10^{-2}	3.081×10^{-1}	1.649×10^{-3}	1.974×10^{-3}	3.138×10^{-1}	2.511×10^{3}
0.025	1.570×10^{-1}	2.578×10^{-2}	1.547×10^{-1}	4.123×10^{-4}	4.936×10^{-4}	1.570×10^{-1}	5.025×10^{3}

Table: Resulting errors for the Gauss method with one stage.

Definition

Let $y(t)$ be the exact solution of (4) and $\tilde{y}(t)$ be the solution of the perturbed system $F(t, \tilde{y}, \tilde{y}') = \delta(t)$. The smallest number $k \in \mathbb{N}$ such that

$$\|y(t) - \tilde{y}(t)\| \leq C \left(\|y(t_0) - \tilde{y}(t_0)\| + \sum_{j=0}^{k} \max_{t_0 \leq \xi \leq T} \left\| \int_{t_0}^{\xi} \frac{d^j \delta}{d\tau^j}(\tau) d\tau \right\| \right)$$

for all $\tilde{y}(t)$, is called the **perturbation index** of this system.

Dahlquist test problem as a model problem, find y such that

$$y' = \lambda y, \quad t > 0 \tag{13}$$

$$y(0) = y_0 \tag{14}$$

with $\lambda \in \mathbb{C}$ and y_0 fixed.
Thus the resulting linear multistep method is of the form

$$\sum_{l=0}^{k} \alpha_l y_{n+l} = h \sum_{l=0}^{k} \beta_l \lambda y_{n+l}$$

$$\iff \sum_{l=0}^{k} [\alpha_l - h\beta_l \lambda] y_{n+l} = 0$$