Circuit Modelling

Felix Dreßler

JOHANNES KEPLER UNIVERSITY LINZ
Altenberger Straße 69
4040 Linz, Austria
jku.at

Formulating a Mathematical Model

Formulating a Mathematical Model

Network Topology

Incidence Matrix $A = (a_{ij}) \in \mathbb{R}^{k \times l}$:

$$\tilde{a}_{ij} = \begin{cases} 1 & \text{edge } j \text{ starts at node } i, \\ -1 & \text{edge } j \text{ ends at node } i, \\ 0 & \text{else.} \end{cases}$$

With $N = (n_0, n_1, n_2, ..., n_k)$ nodes and $E = \{e_j : j = 1, ..., l\}$ edges, where $|N| = k$ is the number of nodes and $|E| = l$

$u = (u_0, u_1, u_2, ...)$ the corresponding electrical potentials to the nodes.

→ reduced incidence matrix

Formulating a Mathematical Model

Energy Conservation Laws

- **Kirchhoff's voltage law (KVL):**
  The sum of voltages along each loop of the network must equal to zero. Using the incidence matrix A this law can be formulated as

$$A^{\top} u = v. \tag{1}$$

- **Kirchhoff's current law (KCL):**
  For any node, the sum of currents flowing into the node is equal to the sum of currents flowing out of the node. Using the incidence matrix A again, this law can be formulated as

$$Ai = 0. \tag{2}$$

Formulating a Mathematical Model

Electrical Components and their Relations

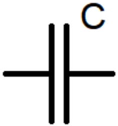
- **Resistor**

$$v = R\,i \quad \text{or} \quad i = G\,u. \tag{3}$$

Figure: resistor symbol

- **Capacitor**

$$Q = C\,v \quad \text{and by derivation in t} \quad I = C\,\frac{d}{dt}v = C\,v'. \tag{4}$$



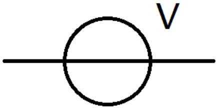
- **Inductor (Coil)**

$$\Phi = L\, i \quad \text{and by derivation in t} \quad v = L\, i'. \tag{5}$$

Figure: inductor symbol

- **Voltage Source**

$$v = v_{src} \tag{6}$$



- **Current Source**

$$i = i_{src} \tag{7}$$

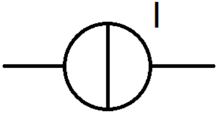


Figure: current source symbol

Formulating a Mathematical Model

Modified Nodal Analysis - MNA

$$\begin{pmatrix} A_C C A_C^{\top} & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{pmatrix} * \begin{pmatrix} u' \\ i_L' \\ i_V' \end{pmatrix} + \begin{pmatrix} A_R G A_R^{\top} & A_L & A_V \\ -A_L^{\top} & 0 & 0 \\ -A_V^{\top} & 0 & 0 \end{pmatrix} * \begin{pmatrix} u \\ i_L \\ i_V \end{pmatrix} = \begin{pmatrix} -A_I i_{src} \\ 0 \\ -v_{src} \end{pmatrix}, \tag{8}$$

Differential Algebraic Equations

Differential Algebraic Equations

Types of DAEs

In the most general form a DAE can be written as: Find $y : \mathbb{R} \to \mathbb{R}^n$ such that

$$F(t, y(t), y'(t)) = 0, \qquad \forall t \in I \tag{9}$$

with $F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ sufficiently smooth and I the time-interval.

- **Linear systems with constant coeffiecients**
  are systems of the form: find y such that
$$Ay'(t) + By(t) = f(t), \tag{10}$$
  with $A, B \in \mathbb{R}^{n\times n}$, A singular, B regular and $f : \mathbb{R} \to \mathbb{R}^n$ a function in time.
- **Linear time dependent systems** are systems of the form: find y such that
$$A(t)y'(t) + B(t)y(t) = f(t),$$
  with $A, B : \mathbb{R} \to \mathbb{R}^{n\times n}$, $f : \mathbb{R} \to \mathbb{R}^n$ functions, such that for every $t \in \mathbb{R}$ the matrix $A(t)$ is singular and the matrix $B(t)$ regular.
- **Structured (non-linear) systems**
  are semi-explicit systems of the form: find (y, z) such that
$$y'(t) = f(t, y(t), z(t)), \tag{11}$$
$$0 = g(t, y(t), z(t)), \tag{12}$$
  with $f : \mathbb{R} \to \mathbb{R}^n$ and $g : \mathbb{R} \to \mathbb{R}^d$ functions.

Differential Algebraic Equations

Weierstrass-Kronecker Normalform

prerequisites

Theorem

Let $\{A, B\}$ be a regular matrix pencil. There exist $P, Q \in \mathbb{C}^{n\times n}$ such that

$$PAQ = \begin{pmatrix} I_d & 0 \\ 0 & N \end{pmatrix}, \quad PBQ = \begin{pmatrix} R & 0 \\ 0 & I_{n-d} \end{pmatrix}$$

where

$$N = diag(N_1, \ldots, N_r) \quad \text{with} \quad N_i = \begin{pmatrix} 0 & 1 & & 0 & \\ & \ddots & \ddots & & \\ & & & 0 & 1 \\ 0 & & & 0 & \end{pmatrix} \in \mathbb{R}^{n_i \times n_i}$$

and R has Jordan Normalform. By I_k we denote the identity matrix of size $k \times k$.

Proof on blackboard

Differential Algebraic Equations

Index of a Differential Algebraic Equation

Definition

Consider the differential algebraic equation (9) to be uniquely locally solvable and F sufficiently smooth. For a given $m \in \mathbb{N}$ consider the equations

$$\begin{aligned} F(t, y, y') &= 0, \\ \frac{\mathrm{d}F(t, y, y')}{\mathrm{d}t} &= 0, \\ &\vdots \\ \frac{\mathrm{d}^m F(t, y, y')}{\mathrm{d}t^m} &= 0. \end{aligned}$$

The smallest natural number m for which the above system results in an explicit system of ordinary differential equations (ODEs), i.e. it has the form

$$y' = \phi(t, y),$$

Definition

Let $y(t)$ be the exact solution of *Abstract-DAE!!!!!!!!* and $\tilde{y}(t)$ be the solution of the perturbed system $F(t, \tilde{y}, \tilde{y}') = \delta(t)$. The smallest number $k \in \mathbb{N}$ such that

$$\|y(t) - \tilde{y}(t)\| \leq C\left(\|y(t_0) - \tilde{y}(t_0)\| + \sum_{j=0}^{k} \max_{t_0 \leq \xi \leq T} \left\| \int_{t_0}^{\xi} \frac{\mathrm{d}^j \delta}{\mathrm{d}\tau^j}(\tau) d\tau \right\|\right)$$

for all $\tilde{y}(t)$, is called the **perturbation index** of this system.

Differential Algebraic Equations

Consistent Initial Values

index $\nu = 0$.

Case: Index $\nu = 1$.

By rewriting our system into the form

$$
\begin{aligned}
y'(t) &= f(t, y(t), z(t)), \\
0 &= g(t, y(t), z(t)).
\end{aligned}
$$

we are able to give conditions for consistent initial values. Namely y_0 and z_0 are consistent initial values for this system, if $g(t_0, y_0, z_0) = 0$ holds.

Case: Index $\nu = 2$.
For index-2 systems we rewrite our system into

$$\begin{aligned} y' &= f(t, y(t), z(t)), \\ 0 &= g(t, y(t)). \end{aligned}$$

Consistent initial values y_0, z_0 for this case not only have to fulfill $g(t_0, y_0) = 0$ but also the *hidden constraint* $g_t(t_0, y_0) + g_y(t_0, y_0)f(t_0, y_0, z_0)$. By g_t and g_y we denote the derivative of g with respect to t or y, respectively.

Index Analysis of the Modified Nodal Analysis

Index Analysis of the Modified Nodal Analysis

General Index Analysis

content...

Index Analysis of the Modified Nodal Analysis

Topological Conditions

*Theorem (Index conditions [**shashkov_tuprints27452**])*

Let the matrices of the capacitances, inductances and resistances be positive definite.

- *If*

$$ker([A_R, A_C, A_V, A_L]^\top) = \{0\} \quad and \quad ker(A_V) = \{0\} \tag{13}$$

  holds, then the MNA (8) *leads to a system with index $\nu \leq 2$.*

- *If additionally*

$$ker([A_R, A_C, A_V]^\top) = \{0\} \quad and \quad ker([A_C, A_V]) = \{0\} \tag{14}$$

  holds, then the system is of index $\nu \leq 1$

- *If further*

$$ker(A_C^\top) = \{0\} \quad and \quad dim(v_{src}) = 0 \tag{15}$$

  holds, then the system has index $\nu = 0$.

- Condition (13) can be interpreted, as the circuit neither containing loops of voltage sources nor cutsets of current sources.
- Condition (14) can be interpreted, as the circuit containing neither loops of capacitors and/or voltage sources nor cutsets of inductors and/or current sources.
- Condition (15) can be interpreted, as every node in the circuit being connected to the reference node (ground) through a path containing only the capacitors.

Numerical Solutions

Numerical Solutions

Single-Step Methods

content...

Numerical Solutions

Multistep Methods

content...

Numerical Solutions

Implicit Linear Multistep Formulas

content...

Numerical Solutions

Numerical Examples

content...