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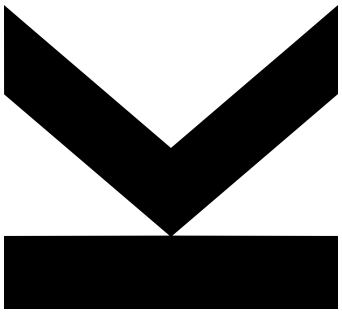
Circuit Modelling



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Kurzfassung

Kurzfassung auf Deutsch.

Abstract

Abstract in English.

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1 Introduction

This chapter should include information about what circuit modelling wants to achieve as well as giving an overview of what this bachelor-thesis is about.

What is this thesis about?
Modelling and numerically solving systems that arrise from electrical circuits with RLC elements. Furthermore it will briefly discuss on expanding this baseline with more complicated electrical components. What is the goal of this thesis?
The goal of this thesis is to give insight into industrial standards concerning circuit modelling. It aims to elaborate on the underlying concepts of MNA as well as on the most commonly used numerical methods.

2 Formulating a Mathematical Model

based on DAE lecture and [2]

To accurately represent a physical system in a mathematical model we first have to think about **how** to best formulate a mathematical model that formulates this system accurately enough. This chapter aims to elaborate on the Modified Nodal Analysis (MNA) of an electrical circuit, which is the industry standard for these kinds of models. For this we have to introduce the notions of network topology as well as some basic electrical components and laws.

2.1 Network Topology

An electrical circuit is usually considered as a graph (N, E) where $N = (n_0, n_1, n_2, ...)$ are the Nodes and $E = (e_{ij})_{ij}$ are the edges. For some i and some j we have that $e_{ij} = (n_j, n_k)$ is the edge from node j to node k. We can store this information in an *incidence matrix* $\tilde{A} = (\tilde{a}_{ij})_{ij}$ which is defined by

$$\tilde{a}_{ij} = \begin{cases} 1 & \text{edge } j \text{ starts at node } i, \\ -1 & \text{edge } j \text{ ends at node } i, \\ 0 & \text{else.} \end{cases}$$

We call $u = (u_0, u_1, u_2, ...)$ the corresponding potentials to the nodes N. The difference between the potentials at two connected nodes is called the voltage at the respective edge. To fix the absolute values of these potentials we have to set one node to a fixed potential. We will do that by "grounding" the node n_0, this means we set the potential $u_0 := 0$. This grounding of a node allows us to remove the corresponding row from the incidence matrix

to get the *reduced incidence matrix* A. The vector $v = (v_{ij})_{ij}$ represents the voltages at the edges. For some i and some j the voltage at edge ij is $v_{ij} = u_i - u_j$.

We will later see, that the components of an electrical crircuit, which will be installed along the edges, describe a relationship between the edges current and its voltage. Thus a current vector $i = (i_1, i_2, i_3, ...)$ containing the currents along the edges is required.

example

2.2 Energy Conservation Laws

To fully fix all the variables that arrise in the model of an electrical circuit we will need some *conservation laws*:

- **Kirchhoff's voltage law (KVL):**
  The sum of voltages along each loop of the network must equal to zero. Using the incidence matrix A this law can be formulated as

$$A^\top * u = v. \tag{2.1}$$

- **Kirchhoff's current law (KCL):**
  For any node, the sum of currents flowing into the node is equal to the sum of currents flowing out of the node. Using the incidence matrix A again, this law can be formulated as

$$A * i = 0. \tag{2.2}$$

2.3 Electrical Components and their relations

Electrical components are described by equations relating their edge voltage v to their edge current i. We will mainly focus on so called RLC-networks which consist of resistors, capacitors, inductances, voltage sources and current sources. Diodes and Transistors as well as other electrical components can be described in a similar way, although these would lead to a more difficult analysis of the system.

- **Resistor**
  Resistors "resist" the flow of current, which causes voltage to drop. This behaviour is described by the *resistance* R which is given in *Ohm* (Ω) and its reciprocal, the *conductance* G, which is given in *Simens* ($S = \frac{1}{\Omega}$).

  $$v = R * i \quad \text{or} \quad i = G * u.$$

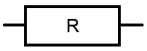


Figure 2.1: resistor symbol

- **Capacitor**
  Capacitors "store" electrical energy by accumulating electrical charge. Their characteristic equations can be described directly using the stored charge Q or indirectly using the change in charge, which is nothing other than the current I. The *capacitance* C is given in *Farads* (F).

  $$Q = C * v \quad \text{and by derivation in t} \quad I = C * \dot{v}.$$

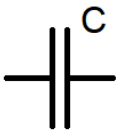


Figure 2.2: capacitor symbol

- **Inductor (Coil)**
  An electric current flowing through a conductor generates a magnetic field Φ surrounding it. This magnetic field causes a voltage drop dependant on the change in current. The *inductance* L is given in *Henry* (H).

  $$\Phi = L * i \quad \text{and by derivation in t} \quad v = L * \dot{i}.$$

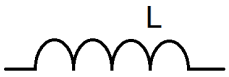


Figure 2.3: inductor symbol

- **Voltage Source**
  A voltage source supplies the system with a voltage. It can either supply varying amounts of voltage (with the special case of alternating current AC) or a fixed amount of voltage. The unit of voltage is *Volts* (V).

  $$v = v_{src}$$

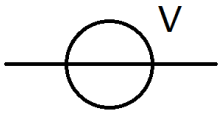


Figure 2.4: voltage source symbol

- **Current Source**
  A current source supplies the system with current. It can either supply varying amounts of current (with the special case of alternating current AC) or a fixed amount of current. The unit of current is *Ampere* (A).

  $$i = i_{src}$$

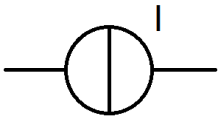


Figure 2.5: current source symbol

- Diode - to be filled with information after the rest is complete

- Transistor - unlike the other components which were all two-teminal components a transisstor is a three-terminal component

2.4 Modified Nodal Analysis - MNA

[2] and [3]

To analyse the network further we will sort the reduced incidence matrix A such that is has the block form

$$A = (A_R A_C A_L A_V A_I)$$

where A_R, A_C, A_L, A_V and A_I all include the collumns that are related to the resistors, capacitors, coils, voltage sources and current sources.

To mathematically describe the circuit we will use *modified nodal analysis* (or short MNA). MNA uses the node voltages as well as the currents of the coils and the voltage sources as unknowns and is based on the conservation laws 2.2 and 2.1 as well as on the voltage-current relations of the electrical components. By replacing all edge-currents with their respective voltage-current relation and ell egde-voltages with their node-potentials we obtain the MNA-equations

$$\begin{aligned} A_C C A_C^\top \dot{u} + A_R G A_R^\top u + A_L i_L + A_V i_V + A_I i_{src} &= 0, \\ L\dot{i_L} - A_L^\top u &= 0, \\ -A_V^\top + v_{src} &= 0. \end{aligned}$$

In matrix form these read as

$$\begin{pmatrix} A_C C A_C^\top & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{pmatrix} * \begin{pmatrix} \dot{u} \\ \dot{i_L} \\ \dot{i_V} \end{pmatrix} + \begin{pmatrix} A_R G A_R^\top & A_L & A_V \\ -A_L^\top & 0 & 0 \\ -A_V^\top & 0 & 0 \end{pmatrix} * \begin{pmatrix} u \\ i_L \\ i_V \end{pmatrix} = \begin{pmatrix} -A_I i_{src} \\ 0 \\ -v_{src} \end{pmatrix}, \quad (2.3)$$

where the diagonal matrices C, G and L contain the capacities, conductivities and inductivities.

The resulting systems are *stiff* systems. This means that for their numerical solution special care has to be put into which methods are suitable for a solving these sytsems in a stable manner.

example

2.5 Charge/Flux oriented formulation of MNA

Again using KCL 2.2 and the component equations we formulate a system of equations. This time we use the charge and flux based formulations. This means that instead of directly using current and voltage we use the flux of the magnetic field (used in the description of inductances) and the charge (used in the description of capacitors). We obtain

$$A_C\dot{q} + A_R r(A_R^\top u, t) + A_L i_L + A_V i_V + A_I i(A^\top u, \dot{q}, i_L, i_V, t) = 0, \tag{2.4}$$

$$\dot{\phi} - A_L^\top u = 0, \tag{2.5}$$

$$v(A^\top u, \dot{q}, i_L, i_V, t) - A_V^\top u = 0, \tag{2.6}$$

$$q - q_C(A_C^\top u) = 0, \tag{2.7}$$

$$\phi - \phi_L(i_L) = 0. \tag{2.8}$$

Using

node potentials u,

branch currents through voltage and flux controlled elements i_V and i_L,

charges and fluxes q and ϕ,

voltage dependent resistors r,

voltage and current dependent charge and flux sources q_C and ϕ_L,

controlled current and voltage sources i_{src} and v_{src}.

We call this formulation the *charge-flux oriented modified nodal analysis*.

3 Differential Algebraic Equations

von [3] und DAE lecture

The resulting MNA of electrical circuits leads to a special form of differential equations, namely *differential algebraic equations* (DAE). To understand the general solvability of those systems and to find good numerical methods we first have to take a look at the theory of such DAEs.

3.1 Abstract Problem

In the most general form a DAE can be written as

$$F(t, y(t), y'(t)) = 0, \qquad \forall t \in I \tag{3.1}$$

with $F : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ sufficiently smooth. As this is usually too general one has to consider more specific types of DAEs.

3.2 Types of DAEs

Commonly systems of either of the three types below are considered when talking about DAEs.

- **Linear systems with constant coeffiecients**
  are systems of the form

$$Ay'(t) + By(t) = f(t), \tag{3.2}$$

with $A, B \in \mathbb{R}^{n\times n}$, A singular, B regular and $f(t)$ a function. If A would not be singular this equation could be transformed into an ordinary differental equation. Thus it is reasonable to assume A singular.

- **Linear time dependent systems** are systems of the form

$$A(t)y'(t) + B(t)y(t) = f(t),$$

with $A(t), B(t), f(t)$ functions.

- **Structured (non-linear) systems**
are semi-explicit systems of the form

$$y'(t) = f(t, y(t), z(t)), \tag{3.3}$$
$$0 = g(t, y(t), z(t)), \tag{3.4}$$

with f and g functions.

For our analysis of electrical networks we will focus on linear systems with constant coefficients. The other systems could for example occur when we also consider time-dependant component relations (e.g. Temperature, ...).

3.2.1 Weierstraß-Kronecker Normalform

To determine the solvability of a linear system with constant coefficients 3.2 we first need to introduce a Normalform for the system, the *Weierstraß-Kronecker Normalform*. This Normalform is dependant on the family $\{A, B\} := \{\mu A + B | \mu \in \mathbb{R}\}$, which is called the *matrix pencil* of the DAE.

Definition 1. *The matrix pencil $\{A, B\}$ is called* regular *if there exists some $c \in \mathbb{R}$, such that $(cA + B)$ is regular ($det(cA + B) \neq 0$), otherwise it is called singular.*

Theorem 1 (Jordan Normalform). *For every matrix $Q \in \mathbb{R}^{\times}$ there exists a regular matrix $T \in \mathbb{C}^{n\times n}$, such that*

$$T^{-1}QT = J = diag(J_1, ..., J_r) \quad with \quad J_i = \begin{pmatrix} \lambda_i & 1 & & 0 \\ 0 & \lambda_i & \ddots & \vdots \\ & \ddots & \ddots & 1 \\ 0 & \dots & 0 & \lambda_i \end{pmatrix} \in \mathbb{C}^{m_i \times m_i}$$

and $n = m_1 + ... + m_r$.

The matrix J is called Jordan Normalform of Q, the J_i are called Jordan Blocks, where λ_i are the eigenvalues of Q. The matrix J is uniquely determined by Q except for the arrangement of the diagonal blocks. If Q posesses only real eigenvalues, then T can also be choosen from the reals.
A transformation from A and B in 3.2 enables a seperation into differential and algebraic variables.

Theorem 2. *Let $\{A, B\}$ be a regular matrix pencil. There exist $P, Q \in \mathbb{C}^{n\times n}$ such that*

$$PAQ = \begin{pmatrix} I_d & 0 \\ 0 & N \end{pmatrix}, \quad PBQ = \begin{pmatrix} R & 0 \\ 0 & I_{n-d} \end{pmatrix}$$

where

$$N = diag(N_1, ..., N_r) \quad with \quad N_i = \begin{pmatrix} 0 & 1 & & 0 & \\ & \ddots & \ddots & & \\ & & & 0 & 1 \\ 0 & & & 0 & \end{pmatrix} \in \mathbb{R}^{n_i \times n_i}$$

and R has Jordan Normalform.

Proof. Becuase $\{A, B\}$ is a regular matrix pencil, there exists $c \in \mathbb{R}$ such that $(cA + B)$ is regular. Set

$$\hat{A} := (cA + B)^{-1}A, \quad \hat{B} := (cA + B)^{-1}B.$$

Considering

$$(cA+B)^{-1}(cA+B) = I \implies (cA+B)^{-1}B + c(cA+B)^{-1}A = I,$$

we get that

$$\hat{B} = I - c\hat{A}.$$

Let $J_{\hat{A}}$ be the Jordan Normalform of $\hat{A}$, this means that there exists a regular matrix T_1 such that

$$T_1^{-1}AT_1 = J_{\hat{A}} = \begin{pmatrix} W & 0 \\ 0 & \tilde{N} \end{pmatrix}.$$

The matrix W contains the Jordanblocks with Eigenvalues which are nonzero, the matrix $\tilde{N}$ contains the Jordan blocks with Eigenvalues equal to zero, thus $\tilde{N}$ is *nilpotent*. The Jordan Normalform $J_{\hat{B}}$ of $\hat{B}$ is given by

$$T_1^{-1}\hat{B}T_1 = J_{\hat{B}} = \begin{pmatrix} I - cW & 0 \\ 0 & I - c\tilde{N} \end{pmatrix}.$$

The following two transformations will allow us to get the desired structure. First we will transform $J_{\hat{A}}$ with

$$T_2 := \begin{pmatrix} W & 0 \\ 0 & I - c\tilde{N} \end{pmatrix}$$

in

$$T_2^{-1}J_{\hat{A}} = \begin{pmatrix} I & 0 \\ 0 & (I - c\tilde{N})^{-1}\tilde{N} \end{pmatrix}$$

and $J_{\hat{B}}$ in

$$T_2^{-1}J_{\hat{B}} = \begin{pmatrix} W^{-1} - cI & 0 \\ 0 & I \end{pmatrix}.$$

Let now R be the Jordan Normalform of $(W^{-1} - cI)$ and N be the Normalform of $(I - c\tilde{N})^{-1}\tilde{N}$, this means

$$T_W^{-1}(W^{-1} - cI)T_W = R \quad \text{and} \quad T_{\tilde{N}}^{-1}(I - c\tilde{N})^{-1}\tilde{N}T_{\tilde{N}} = N$$

Considering this definition together with the Neumann-series of $(I - c\tilde{N})^{-1}$ we obtain

$$\tilde{N}(I - c\tilde{N})^{-1} = \tilde{N}(c\sum_{i=0}^{\infty}\tilde{N}^i) = \tilde{N}(c\sum_{i=0}^{k-1}\tilde{N}^i) = c\sum_{i=0}^{k-1}\tilde{N}^{i+1} = (c\sum_{i=0}^{k-1}\tilde{N}^i)\tilde{N}$$

where we used that $\tilde{N}$ is nilpotent with nilpotency index k. This shows that $\tilde{N}$ and $(I - c\tilde{N})^{-1}$ commute.

From this we can conclude that

$$N^k = [T_{\tilde{N}}^{-1}(I - c\tilde{N})^{-1}\tilde{N}T_{\tilde{N}}]^k = T_{\tilde{N}}^{-1}[(I - c\tilde{N})^{-1}\tilde{N}]^k T_{\tilde{N}} = T_{\tilde{N}}^{-1}(I - c\tilde{N})^{-k}\underbrace{\tilde{N}^k}_{=0}T_{\tilde{N}} = 0$$

We used the commutativity in the third step here. The nilpotent matrix N thus also has the nilpotency index k. A transformation with

$$T_3 := \begin{pmatrix} T_W & 0 \\ 0 & T_{\tilde{N}} \end{pmatrix}$$

transforms $T_2^{-1}J_{\hat{A}}$ into the Jordan Normalform

$$J_{\tilde{A}} := T_3^{-1}T_2^{-1}J_{\hat{A}}T_3 = T_3^{-1}T_2^{-1}T_1^{-1}\hat{A}T_1T_3 = \begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix}$$

and $T_2^{-1}J_{\hat{B}}$ into

$$J_{\tilde{B}} := T_3^{-1}T_2^{-1}J_{\hat{B}}T_3 = T_3^{-1}T_2^{-1}T_1^{-1}\hat{B}T_1T_3 = \begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix}.$$

Now set

$$P := T_3^{-1}T_2^{-1}T_1^{-1}(cA + B)^{-1} \quad \text{and} \quad Q = T_1T_3$$

to get the statement. □

Definition 2. *The nilpotency index k from the Weierstraß-Kronecker Normalform of a matrix pencil $\{A, B\}$ with A singular is called the* Kronecker-Index *of $\{A, B\}$. We write $ind\{A, B\}$. For A regular we set $ind\{A, B\} = 0$.*

Lemma 1. *The Kronecker-Index $ind\{A, B\}$ is independant of the choice of the matrices P and Q.*

Using the findings above we are able to Transform the initial DAE 3.2 using the matrix P from 2. By multiplying P from the left we obtain

$$PAy'(t) + PBy(t) = Pf(t).$$

Setting

$$y = Q\begin{pmatrix} u \\ v \end{pmatrix}, \quad Pf(t) = \begin{pmatrix} s(t) \\ q(t) \end{pmatrix} \quad \text{with} \quad u, s \in \mathbb{R}^d,$$

we get a system of the form

$$\begin{aligned} u'(t) + Ru(t) &= s(t) \\ Nv'(t) + v(t) &= q(t) \end{aligned} \tag{3.5}$$

The first equation is an ordinary differential equation of first order and posesses a unique solution $u(t)$ in $[t_0, t_l]$ for any starting values $u_0 \in \mathbb{R}^d$. Additionally setting $q(t) \in C^{k-1}([t_0, t_l])$ then differentiating the second equation in 3.5 gives

$$\begin{aligned} v(t) &= q(t) - Nv'(t) = q(t) - N(\underbrace{q(t) - Nv'(t)}_{=v(t)})' = q - Nq' + N^2v'' \\ &= q - Nq' + N^2(q - Nv')'' = q - Nq' + N^2q'' - N^3v''' \\ &\vdots \\ &= q - Nq' + ... + (-1)^{k-1}N^{k-1}q^{(k-1)} + (-1)\underbrace{N^kv^{(k)}}_{=0} \end{aligned}$$

$$= \sum_{i=0}^{k-1}(-1)^iN^iq^{(i)}(t) \tag{3.6}$$

where k is the nilpotency index of N. This expression gives an explicit solution for $v(t)$ in $[t_0, t_l]$ with $v(t) \in \mathbb{R}^{d-1}$. It shows the depencdency of the solution and its derivatives. The higher the Kronecker index k gets, the more differentiations of $q(t)$ have to be performed.

The Kronecker index k shows, that k differentiations are required to receive an ordinary differential equation.

3.3 Index of a Differential Algebraic Equation

The Index of a DAE gives us insight about its numerical properties and in general about the solvability. ("The higher the index, the harder to solve")

We will consider two types of index concepts, the differentiation index and the perturbation index.

Definition 3 (differentiation index)**.** *Consider the differential algebraic equation 3.1 to be uniquely locally solvable and F suffieciently smooth differentiable. For a given $m \in \mathbb{N}$ consider*

$$\begin{aligned} F(t, y, y') &= 0, \\ \frac{\mathrm{d}F(t, y, y')}{\mathrm{d}t} &= 0, \\ &\vdots \\ \frac{\mathrm{d}^m F(t, y, y')}{\mathrm{d}t^m} &= 0. \end{aligned}$$

The smallest natural number m for which the above System results in an explicit system of the form

$$y' = \phi(t, y)$$

from which y can be determined is called ***differentiation index****.*

In the previous chapter we have already discussed, that for a DAE with constant coefficients 3.2 and a regular matrix pencil $\{A, B\}$ we need $k = ind\{A, B\}$ differentiations to receive an ordinary differential equation. This means that the Kronecker index k is equal to the differentiation index in the case of a DAE with constant coefficients.

Since numerical differentiation is an unstable procedure this index gives a measure for the numerical problems to be expected when solving such systems.

Definition 4 (perturbation index)**.** *Let $y(t)$ be the exact solution to 3.1. This problem has the* ***perturbation index*** *$k \in \mathbb{N}$ along $y(t), t_0 \leq t \leq T$ if for all $\tilde{y}(t)$ with $F(t, \tilde{y}, \tilde{y}') = \delta(t)$ the inequality*

$$||y(t) - \tilde{y}(t)|| \leq C \left(||y(t_0) - \tilde{y}(t_0)|| + \sum_{j=0}^{k} \max_{t_0 \leq \xi \leq T} \left\| \int_{t_0}^{\xi} \frac{\mathrm{d}^j \delta}{\mathrm{d}\tau^j}(\tau) d\tau \right\| \right)$$

for the smallest number k.

In the case of linear DAEs with constant coefficients where the matrix pencil A, B is regular we can transform the DAE

$$A(y'(t) - \tilde{y}'(t)) + B(y'(t) - \tilde{y}(t)) = \delta(t)$$

into

$$\begin{aligned} u'(t) - \tilde{u}'(t) + \hat{R}(u(t) - \tilde{u}(t)) &= \delta_1(t), \\ N(v'(t) - \tilde{v}'(t)) &= \delta_2(t). \end{aligned}$$

According to 3.6 the solution of the algebraic variable has the form

$$v(t) - \tilde{v}(t) = \sum_{i=0}^{k-1} (-1)^i N^i \delta_2^{(i)}(t)$$

Thus the perturbation index is the same as $ind A, B$ [3]. We see that the derivatives of the pertubation also enter the solution.

4 Index Analysis of the MNA

[2]

In the previous chapter we have seen two different kinds of Index concepts for differential algebraic equations. We have also seen that these two, even though they describe rather different structural aspects of the equation, are the same for our use-cases. This of course leads to the question : "What are the indices we can usually expect for MNA?".

This chapter aims to answer that question for the linear RLC case. Which means that the RLC components are described by linear functions with positive capacitances, inductances and resistances. Thus the matrices

$$C := \frac{\partial q_C(w)}{\partial w}, \quad L := \frac{\partial \phi_L(w)}{\partial w}, \quad G := \frac{\partial r(w)}{\partial w}$$

are positive definite and symmetric. The generalization of these results to the nonlinear case still relies on positive definiteness.

Recall that we consider the equations resulting from the analysis above. These equations are of the form 3.2:

$$Ay'(t) + By(t) = f(t).$$

Specifically the obtained equations from the Modified Nodal Analysis 2.3 are

$$\begin{pmatrix} A_C C A_C^\top & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{pmatrix} * \begin{pmatrix} \dot{u} \\ \dot{i_L} \\ \dot{i_V} \end{pmatrix} + \begin{pmatrix} A_R G A_R^\top & A_L & A_V \\ -A_L^\top & 0 & 0 \\ -A_V^\top & 0 & 0 \end{pmatrix} * \begin{pmatrix} u \\ i_L \\ i_V \end{pmatrix} = \begin{pmatrix} -A_I i_{src} \\ 0 \\ -v_{src} \end{pmatrix}.$$

4.1 General Index analysis

Assuming the system only contains linear elements or is linearized at an operating point in order to investigate the system behaviour then the corresponding network equation represents a DAE with constant coefficients 3.2. We will denote $x = (u, i_L, i_V)^\top$. The structure of the system is reliant on the matrix B, thus we consider

- **ODE-case**:
  The matrix B is regular in 3.2. This is the case iff the circuit contains no voltage sources and there are no nodes which have no path to ground via capacitors. Then the system represents a linear-implicit system of ODEs and can be transformed into the explicit ODE sytstem

  $$\dot{x} = B^{-1}(-Ax + f(t)).$$

  Thus we obtain an index of 0.

- **DAE-case**: The matrix B is singular in 3.2. This is the interesting case which we will analyse further.

For the matrix B being singular we have already obtained a representation of the form

$$u'(t) + Ru(t) = s(t),$$
$$Nv'(t) + v(t) = q(t)$$

in the previous chapter. We now consider the nilpotency index of the matrix N, we already know that this index correlates with the differentiation as well as the perturbation index. We consider two cases:

1. **The nilpotency index is 1**
   Because N is nilpotent with nipotency index $\nu = 1$ it holds that $N^1 = 0$, thus the system transforms to

   $$u'(t) + Ru(t) = s(t),$$
   $$v(t) = q(t).$$

This means that the algebraic variables are given explicitly. Thus the system is in ODE form.

2. **The nilpotency index is ≥ 2**
   This case is the situation we have seen in 3.2.1, it led to 3.6

$$v(t) = \sum_{i=0}^{k-1} (-1)^i N^i q^{(i)}(t).$$

   By differentiating one more time we receive

$$v'(t) = \sum_{i=0}^{k-1} (-1)^i N^i q^{(i+1)}(t) \tag{4.1}$$

   with the assumption that q is ν times differentiable.

We can see that an algebraic constraint has to be fulfilled by the solution. In the case of index 1 this equation is given explicitly, for index ≥ 2 it is given implicitly.

The system is also sensitive to perturbations. Small noise in the input can have arbitrarily large derivatives which can influence v as we it depends on them.

This means no severe numerical problems arise from index-1 systems. Implicit numerical integration schemes for stiff systems are feasable. On the other hand severe nunmerical problems may occur in the case that the system has index greater or equal to 2. The hidden constraints can only be resolved by an unstable differentiation process.

As the Kronecker index $k = ind\{A, B\}$ defines the behaviour of the system theoretically as well as numerically it is called the *algebraic index* of the linear implicit system.

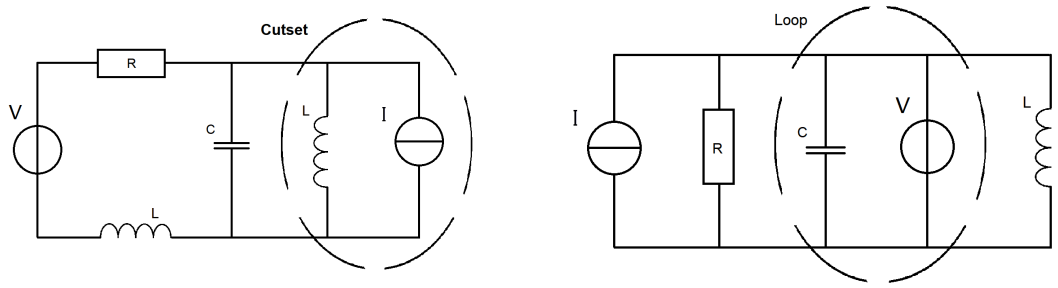
4.2 Topological Conditions

[4], [2] [1]

From analyzing the MNA some conditions to the circuit topology can be obtained. We will be considering the impact of special arrangements of components on the index of the system. In [4] they present very interesting results about the index of MNA equations.

Theorem 3 (Index-1 condition). *[4] Let the matrices of the capacitances, inductances and resistances respectively be positive definite. If the network neither cointains* inductance-current-source cutsets *nor (controlled?)* capacitance-voltage-source loops, *then the MNA leads to an index-1 DAE.*

Where inductance-current-source cutsets and capacitance-voltage-source loops are subcircuits of the form (Seite 481 (searchbar) in basic circuit theory)



(a) inductance-current-source cutset

(b) capacitance-voltage-source loop

Figure 4.1: cutset and loop

This means that if our circuit does not contain any loops or cutsets of the above form the resulting MNA has index 1.

Theorem 4 (Index-2 condition). *[4] If the Network contains* inductance-current-source cutsets *or* capacitance-voltage-source loops *except for capacitance-only loops, then the MNA leads to an index-2 DAE.*

These results can also be interpreted in a more formal way. As they are just descriptions of the topology of the circuit we can reformulate the theorems in terms of the incidence matrix A.

By perturbing the right hand side of of 2.3 with a slight perturbation $\delta = (\delta_C, \delta_L, \delta_V)^\top$ we get a corresponding solution $x^\delta = (u^\delta, i_l^\delta, i_V^\delta)^\top$. One can show that the difference of this perturbed solution to the solution x of the unperturbed system is bounded by

$$||x^\delta - x(t)|| \leq C * \left(||x^\delta(0) - x(0)|| + \max_{0\leq\tau\leq t} ||\delta(\tau)|| + \max_{0\leq\tau\leq t} ||Q_{CRV}^\top \dot{\delta_C}|| + \max_{0\leq\tau\leq t} ||\tilde{Q}_{V-C}^\top \dot{\delta_V}|| \right)$$

using orthogonal projectorsQ_C, Q_{CRV} and $\tilde{Q}_{V-C}$ onto $ker(A_C^\top)$, $ker(A_C A_R A_V)^\top$ and $ker(Q_C^\top A_V)$ respectively.

Theorem 3 can be interpreted as the two conditions

$$ker Q_C^\top A_v = 0, \tag{4.2}$$

$$ker(A_C A_R A_V)^\top = 0. \tag{4.3}$$

This means that for "reasonable" RLC circuits (satisfying the prerequisites from the above theorems) the index will not exceed 2. We will only consider such circuits in this document.

If one of those conditions is violated we are in the case of Theorem 4 and thus have index 2.

We will apply those results to some examples:

examples

5 Numerical Solutions

As we are usually interested in finding real (or complex) valued solutions of our systems we also have to look into solving them numerically. This chapter focuses on the numerical solution of the mentioned systems.

We will first focus on methods used to solve a more general problem

$$y'(t) = f(t, y), \quad t \in [t_0, t_l], \tag{5.1}$$
$$y(t_0) = y_0. \tag{5.2}$$

For this we presume that the function $f(t, y)$ is continuous and Lipschitz, thus *Picard-Lindelöf* gives us, that for every y_0 it is uniquely solvable in $[t_0, t_l]$.

Numerical Methods work by discretization, this means we divide the time-intervall into

$$t_0 < t_1 < ... < t_N \leq t_l$$

and consider approximations $y_m \approx y(t_m)$ for $m = 1, ..., N$. We call this a *time-grid*. The step-size between two grid-points $t_i - t_j$ might not be choosen equidistant. This can be usefull in certain situations. For simplicity we will non the less assume the stepsize to be equidistant.

5.1 Single-Step-Methods

The first class of numerical methods we will have a look at are single-step methods. These methods use the previous approximated value y_j and (for implicit methods) also the current approximated value to determine the current value through a *procedural function*.

Definition 5. *A numerical method to approximate a differential equation 5.1 on a time-grid $t_0, ..., t_l$ with the intermediate values $y_0, ..., y_l$ is called a single-step method if it is from the form*

$$y_{j+1} = y_j + h_j \phi(t_j, y_j, y_{j+1}, h_j). \tag{5.3}$$

With the procedural function ϕ. *If ϕ is not dependent on y_{j+1} then the method is called* explicit, *otherwise it is called* implicit.

5.1.1 Consistency, Stability and Convergence

To compare different single-step methods we have to define some notions to compare their quality. This leads to the definition of the error of the mehtod, its consistency and its convergence.

Definition 6. *Let $\tilde{y}_{m+1}$ be the result of one step of 5.3 with the exact start-vector $y_m = y(t_m)$ then*

$$le_{m+1} = le(t_m + h) = y(t_{m+1}) - \tilde{y}_{m+1}, \quad m = 0, ..., N-1 \tag{5.4}$$

is called the local discretization error *of the single step method at the point t_{m+1}.*

The error encodes how far off the numerical method is from the true value of the solution. In real applications this solution is usually not known. This shows how important error bounds for numerical methods can be.

Definition 7. *A single-step method is called* consistent *if for all initial value problems 5.1*

$$\lim_{h \to 0} \frac{||le(t+h)||}{h} = 0 \quad for \quad t_0 \leq t \leq t_l \tag{5.5}$$

holds.
It is called consistent of order p*, if for a sufficiently smooth function f*

$$||le(t+h)|| \leq Ch^{p+1} \quad \textit{for all} \quad h \in (0, H] \quad \textit{and} \quad t_0 \leq t \leq t_l - h \tag{5.6}$$

holds with C not dependent on h.

Consistency aims to give insight in how similar the problem that the numerical methods solves it to the real problem that we want the solution from.

Definition 8. *A single-step method is called* convergent*, if for all initial value problems 5.1 for the* global discretization error

$$e_m = y(t_m) - y_h(t_m)$$

holds that

$$\max_m ||e_m|| \to 0 \quad \textit{for} \quad h_{max} \to 0.$$

The single-step method is called to have the convergence order *p, if*

$$\max_m ||e_m|| \leq Ch_max^p \quad \textit{for} \quad h_{max} \in (0, H] \quad \textit{with} \quad t_0 \leq t_m \leq t_l$$

with the constant C not dependent on the step size h.

As the name suggestes convergence tries to quantify how far off a numerical solution is from the real solution of a system. A very interesting result follows if we also require the Single-Step Method to be stable.

Definition 9. *A Single-Step Method is called* (discretely) stable *if for grid-functions y_h and $\tilde{y}_h$ with*

$$y_{i+1} = y_i + h\phi(t_i, y_i), \tag{5.7}$$
$$\tilde{y}_{i+1} = \tilde{y}_i + h[\phi(t_i, \tilde{y}_i) + \theta_i], \tag{5.8}$$

and perturbations $\theta_i = \theta_h(t_i)$ of the right side as well as a bounded perturbation in the staring-values $y_0 - \tilde{y}_0$ the Error is bounded by

$$||y_h - \tilde{y}_h||_{\infty,h} \leq C(||y_0 - \tilde{y}_0||_{l^2} + ||\theta_h||_{\infty,h}$$

with a constant C which is not dependent on h.

For Single-Step Methods which are consistent and stable we obtain the following convergence theorem.

Theorem 5 (Lax-Richtmyer)**.** *A consistent (with order p) and discretely stable Single-Step Method is convergent (with order p). (assuming smoothness of the solution y)*

This theorem is due to Lax and Richtmyer. The converse of this statement is also true.

5.1.2 further stability properties

from numpdgl skript
In this section we consider the following simple problem

$$u' = \lambda u, \quad t > 0 \tag{5.9}$$
$$u(0) = u_0 \tag{5.10}$$

with $\lambda \in \mathbb{C}$ and u_0 fixed.

Definition 10. *1. If a single-step method can be written in the form*

$$u_{i+1} = R(z)u_i, \quad z := h * y \tag{5.11}$$

then we call $R : \mathbb{C} \to \mathbb{C}$ the stability function *of the single-step method.*

2. The set

$$S := z \in \mathbb{C} : |R(z)| \leq 1 \tag{5.12}$$

is called the region of stability of the method.

3. *A single-step method is called*
   - 0-stable, *if* $0 \in S$.
   - A-stable, *if* $\mathbb{C}^- \subset S$.
   - L-stable, *if* $R(z) \to 0$ *for* $Re(z) \to -\infty$.

5.1.3 Runge-Kutta Methods

A very prevalent family of numerical single-step methods are the *Runge-Kutta* methods.

Definition 11. *Let* $s \in \mathbb{N}$. *A single-step method of the form*

$$y_{m+1} = y_m + h \sum_{i=1}^{s} b_i f(t_m + c_i h, y_{m+1}^{(i)}) \tag{5.13}$$

$$y_{m+1}^{(i)} = y_m + \sum_{j=1}^{s} a_{ij} f(t_m + c_j h, y_{m+1}^{(j)}) \tag{5.14}$$

is called a Runge-Kutta Method *with s steps.*

We usually collect the coefficients into the vectors and matrices $c = (c_1, ..., c_s)$, $A = (a_{ij})_{ij}$ and $b = (b_1, ..., b_s)$.

If A is a strictly lower triangle matrix, this means for all $j \geq i$ holds $a_{ij} = 0$ then the Runge-Kutta method is explicit, otherwise it is implicit. In general implicit Runge-Kutta methods might need more computational effort because to calculate $y_m^{(i)}$ a nonlinear system of equations has to be solved. But in contrast those methods can also lead to very good stability characteristics.

Lemma 2. *[3] A Runge-Kutta mehtod is consistent, if and only if*

$$\sum_{i=1}^{s} b_i = 1$$

The coefficients of a Runge-Kutta method are usually represented in the *Butchertableau*, which was introduced by John C. Butcher and has the following form.

$$\begin{array}{c|ccc} c_1 & a_{11} & \dots & a_{1s} \\ \vdots & \vdots & & \vdots \\ c_s & a_{s1} & \dots & a_{ss} \\ \hline & b_1 & \dots & b_s \end{array} \quad \text{of in matrix form} \quad \begin{array}{c|c} c & A \\ \hline & b \end{array}$$

Their stability can directly be derived from the butcher tableau. The stability function of an arbitrary m-stage Runge-Kutta method has the form

$$R(z) = 1 + zb^\top (I_m - zA)^{-1} e$$

where $e = (1, ..., 1)$.

Remark: The trapezoidal rule is a widely used Runge-Kutta method which we can also consider as a multistep-method. It will be discussed later on.

5.2 Multistep-Methods

based on chapter 4 of book num gew dgl steif nichtsteif
Linear multistep methods use approxiamtions u_{m+l} along the gridpoints t_{m+l}, $l = 0, 1, ..., k-1$ to calculate the new approximation u_{m+k} at t_{m+k}. We will first discuss topics related to the order of the methods depending on its parameters, stability and convergence.

Definition 12. *For given $\alpha_0, ..., \alpha_k$ and $\beta_0, ..., \beta_k$ the iteration rule*

$$\sum_{l=0}^{k} \alpha_l u_{m+l} = h \sum_{l=0}^{k} \beta_l f(t_{m+l}, u_{m+l}), \quad m = 0, 1, ..., N-k \tag{5.15}$$

is called a linear multistep method *(linear k-step method). It is always assumed that* $\alpha_k \neq 0$ *and* $|\alpha_0| + |\beta_k| > 0$*. If* $\beta_k = 0$ *holds, then the method is called explicit, otherwise implicit.*

A linear multi-step method consists of two parts:

1. In the *starting-phase* approximations $u_1, ..., u_{k-1}$ for the first $k-1$ gridpoints $t_l = t_0 + th, l = 1, ..., k-1$ are calculated using a single-step method. For example using an explicit Runge-Kutta Method or a multi-step method with fewer steps.
2. In the *run-phase* the multi-step formula is used to determine new approximations u_{m+k} for the gridpoint t_{m+k}

For theoretical analysis of the multi-step methods we consider the generating polynomials

$$\rho(x) := \sum_{l=0}^{k} \alpha_l x^l \tag{5.16}$$

$$\sigma(x) := \sum_{l=0}^{k} \beta_l x^l \tag{5.17}$$

5.2.1 Consistency, Stability and Convergence

[3]

Again we need to define the properties consistency, stability and convergence for this method.

Definition 13. *Let* $\tilde{y}_{m+k}$ *be the result of one step of the multi-step method 5.15 with the start-vectors* $y_m = y(t_m)$*. This means*

$$\alpha_k \tilde{u}_{m+k} = \sum_{l=0}^{k-1} (h\beta_l f(t_{m+l}, y(t_{m+l})) - \alpha_l y(t_{m+l})) + h\beta_k f(t_{m+k}, \tilde{u}_{m+k}).$$

Then

$$le_{m+k} = le(t_{m+k}) = y(t_{m+k}) - \tilde{u}_{m+k}, \quad m = 0, 1, ..., N-k$$

is called the local discretization error *(local error) of the linear multi-step method 5.15 at the point* t_{m+k}.

We will assign the linear difference operator

$$L[y(t), h] = \sum_{l=0}^{k} \left(\alpha_l y(t + lh) - h\beta_l y'(t + lh) \right) \tag{5.18}$$

to the local discretization error. Using this we gain the following definition.

Definition 14. *A linear multi-step method is called* preconsistent *if for all functions* $y(t) \in C^1[t_0, t_l]$

$$\lim_{h \to 0} L[y(t), h] = 0$$

holds. It is called consistent, *if for all functions* $y(t) \in C^2[t_0, t_l]$

$$\lim_{h \to 0} \frac{1}{h} L[y(t), h] = 0$$

holds. It has the consistency order p, *if for all functions* $y(t) \in C^{p+1}[t_0, t_l]$

$$L[y(t), h] = \mathcal{O}(h^{p+1}) \quad for \quad h \to 0$$

holds.

From the generating polynomials we can derive simple consistency conditions

$$\rho(1) = 0 \quad \text{and} \quad \rho'(1) = \sigma(1).$$

Of course we also need convergence of such methods.

Definition 15. *We say that a linear multi-step method is convergent if for a solution y of the problem a solution vector created by an LMSV y_j for $j \in 0, ..., k$ we have that*

$$\lim_{h \to \infty} \max_{0 \leq j \leq k} ||y(t_j) - y_j|| = 0.$$

The discrete stability is also very similar to the single-step methods.

Definition 16. *A linear multi-step method is called (discretely) stable, if for solutions u_h and $\tilde{u}_h$ of*

$$\sum_{l=0}^{k} \alpha_l u_{m+l} = h \sum_{l=0}^{k} \beta_l f(t_{m+l}, u_{m+l}), \tag{5.19}$$

$$\sum_{l=0}^{k} \alpha_l \tilde{u}_{m+l} = h \sum_{l=0}^{k} \beta_l f(t_{m+l}, \tilde{u}_{m+l}) + h\theta_n \tag{5.20}$$

and bounded initial values $y_j - \tilde{y}_j$ for $j \in 0, ..., k$ we have that

$$\max_{t_0 \leq t_n \leq T} ||y_n - \tilde{y}_n|| \leq C \sum_{j=0}^{k-1} ||y_j - \tilde{y}_j|| + \max_{t_0 \leq t_n \leq T} ||\theta_n||.$$

Definition 17. *A linear multi-step method is called* zero-stable *if all solutions of the difference equation*

$$\sum_{l=0}^{k} \alpha_l u_{m+l} = 0$$

are bounded.

Lax-Richtmyer

Theorem 6. *A linear multi-step method is zero-stable, if and only if the polynomial $\rho(x)$ fullfills the "root-condition", this means:*

1. *All roots $\bar{x}$ of $\rho(x)$ are within the unit-circle $|\bar{x}| \leq$ in the complex plane.*

2. *All roots $\bar{x}$ with $|x| = 1$ are singular.*

Theorem 7 (DAE lecture)**.** *A linear multistep method is stable if and only if it is zero-stable.*

5.2.2 further stability properties

In this section we consider the following simple problem

$$u' = \lambda u, \quad t > 0 \tag{5.21}$$

$$u(0) = u_0 \tag{5.22}$$

with $\lambda \in \mathbb{C}$ and u_0 fixed.

Asessment for stiff equations lt wikipedia (LMSM)

Thus the resulting linear multistep method is of the form

$$\sum_{l=0}^{k} \alpha_l u_{n+l} = h \sum_{l=0}^{k} \beta_l \lambda u_{n+l}$$

$$\iff \sum_{l=0}^{k} [\alpha_l - h\beta_l \lambda] u_{n+l}$$

Using this we define the following important stability notions.

Definition 18. 1. *The set*

$$S := z \in \mathbb{C} : \rho(\xi) - z\sigma(\xi) = 0 \implies \xi \textit{fullfills root criteria} \tag{5.23}$$

is called the region of stability of the method.

2. *A linear multistep method is called*

   - 0-stable, *if* $0 \in S$.

- *stable in the point* $z \in \mathbb{C}$, *if Wurzelbedingung erfüllt bei* z, *aber doch eigentlich* z *in* S *oder?*
- $A(\alpha)$-stable, *if it is stable in all* z *that lie within the set* $\{z \in \mathbb{C}^- : |arg(z) - \pi| \leq \alpha\}$ *for* $\alpha \in (0, \frac{\pi}{2})$.

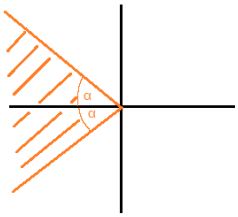


Figure 5.1

5.3 Implicit linear multi-step formulas

These kinds of multi-step methods are conventionally used to numerically solve the systems obtained using modified nodal analysis. Our assumption that we only consider network equations arising from networks consisting of RLC components as well as controlled sources which keep the index between 1 and 2 still holds.

We consider the equations in charge/flux oriented formulation

$$0 = \underbrace{\begin{pmatrix} A_c & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix}}_{=:A} \underbrace{\begin{pmatrix} \dot{q} \\ \dot{\phi} \end{pmatrix}}_{=:\dot{y}} + \underbrace{\begin{pmatrix} A_R r(A_R^\top u, t) + A_L i_L + A_V i_V + A_I i(u, i_L, i_V, t) \\ -A_L^\top u \\ v(u, i_L, i_V, t) - A_V^\top u \end{pmatrix}}_{=:f(x,t)},$$

$$\underbrace{\begin{pmatrix} q \\ \phi \end{pmatrix}}_{=:y} = \underbrace{\begin{pmatrix} q_C(A_C^\top u) \\ \phi_L(i_L) \end{pmatrix}}_{=:g(x,t)},$$

or simply

$$0 = F(\dot{y}(t), x(t), t) := A\dot{y}(t) + f(x(t), t),$$
$$0 = y(t) - g(x(t)).$$

with $x := (u, i_L, i_V)^\top$ containing the unknowns.

The conventional approach to solve those systems numerically can be split into three main steps:

1. Computation of consistent initial values

   The first step is to compute consistant inital values (x_0, y_0) at the initial time point t_0. In the index-1 case this can be done by solving a so called steady state problem. Steady state means that we consider a not time-dependent circuit (DC operating point). This means we have to solve

   $$F(0, x_0, t_0) = 0$$

   for x_0 and then set $y_0 = g(x_0)$.

2. Numerical integration based on multi-step schemes

   Using the consistent initial values we obtain solutions of the network equations at discrete timepoints $t_1, t_2, ...$ by integration with linear multi-step methods.

3. Transformation of the DAE into a nonlinear system of equations.

   The numerical solution is now reduced into solving a system of equations of the form

   $$F(\alpha_k g(x_k) + r_k, x_k, t_k) = 0$$

   which can be solved iteratively using Newton's method.

In the following two subchapters we will give a short overview over the two most commonly used methods for solving those systems.

5.3.1 BDF-schemes

chapter 9.2 numerik book wikipedia

The most commonly used numerical methods for solving the systems that arise in electrical circuits are the BDF-scheme and the trapezoidal rule.

We will not give a deeper look into their construction but will only state their properties. BDF schemes are appealing because they save function evaluation as much as possible, since this is very costly in circuit simulation.

The *backward differentiation fooormula (BDF)* is a family of implicit linear multistep methods. They have the general form

$$\sum_{k=0}^{s} \alpha_k y_{n+k} = h\beta f(t_{n+s}, y_{n+s}) \tag{5.24}$$

Since we are interested in the unknown y_{n+s} which is used to evaluate f, this method is implicit. The coefficients α_k and β are chosen, so that the method achieves order s which is the maximum possible.

The BDF or BDF-k formulas for $k = 1, ...6$ have the following form

$$\begin{aligned}
k = 1 &: hf_{m+1} = u_{m+1} - u_m \\
k = 2 &: hf_{m+2} = \frac{1}{2}(3u_{m+2} - 4iu_{m+1} + u_m) \\
k = 3 &: hf_{m+3} = \frac{1}{6}(11u_{m+3} - 18u_{m+2} + 9u_{m+1} - 2u_m) \\
k = 4 &: hf_{m+4} = \frac{1}{12}(25u_{m+4} - 48u_{m+3} + 36u_{m+2} - 16u_{m+1} + 3u_m) \\
k = 5 &: hf_{m+5} = \frac{1}{260}(137u_{m+5} - 300u_{m+4} + 300u_{m+3} - 200u_{m+2} + 75u_{m+1} - 12u_m) \\
k = 6 &: hf_{m+6} = \frac{1}{60}(147u_{m+6} - 360u_{m+5} + 450u_{m+4} - 400u_{m+3} + 225u_{m+2} - 72u_{m+1} + 10u_m)
\end{aligned}$$

Methods with $s > 6$ are not zero-stable. Indeed in reality methods with order greater than 3 are rarely used because of their low smoothness properties.

BDF schemes have consistency order $p = k$. They are methods for solving stiff equations, thus their stability is indicated by their region of absolute stability. Unfortunately not all BDF-schemes are A stable, but their stability region still contains a large part of the complex left half-plane (see figure 5.2 They are the most efficient linear multistep methods of this kind.

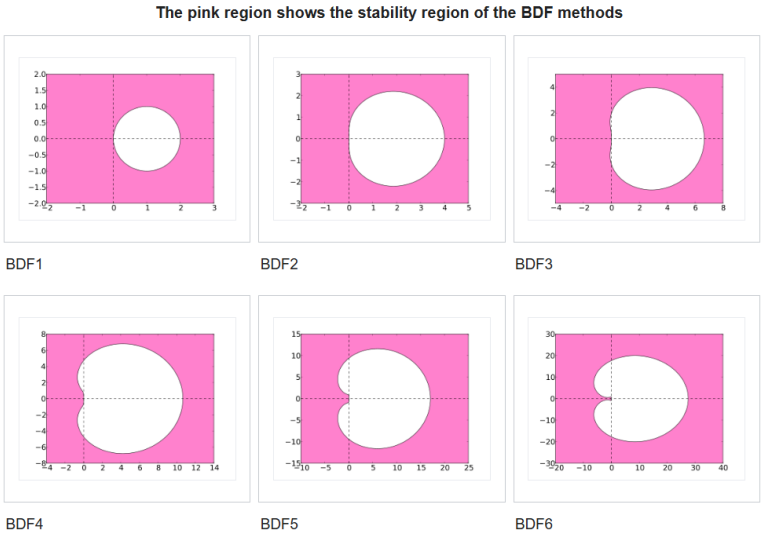


Figure 5.2

The first timestep is always performed by BDF1 (implicit Euler scheme) as a starting procedure.

5.3.2 trapezoidal rule

The trapezoidal rule is a somewhat natural alternative to BDF2 since it is A-stable and as a linear multistep method of order 2 the one with the smallest leading error coefficient. ([2])

It works by approximating the region under the graph of the function $f(x)$ as a trapezoid, hence the name. It follows that

$$\int_a^b f(x)dx \approx (b-a)\frac{1}{2}(f(a)+f(b))$$

This procedure is repeated for small subsections of the Intervall $[a, b]$. Thus we obtain the iteration formula

$$u_h(t+h) = u_h(t) + \frac{h}{2}[f(t, u_h(t)) + f(t+h, u_h(t+h))].$$

This iteration rule can also be formulated using the butcher tableau

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

Because $u_h(t+h)$ appears in f again we see that this is an implicit method. The butcher tableau confirms this as well.

5.4 Numerical Examples

In this section we will give two explicit examples of circuits being solved using either of the above mentioned methods and compare the convergence speed between the methods as well as comparing the suitability of the methods for systems of different index.

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