

Circuit Modelling



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Example: Charging of a capacitor:

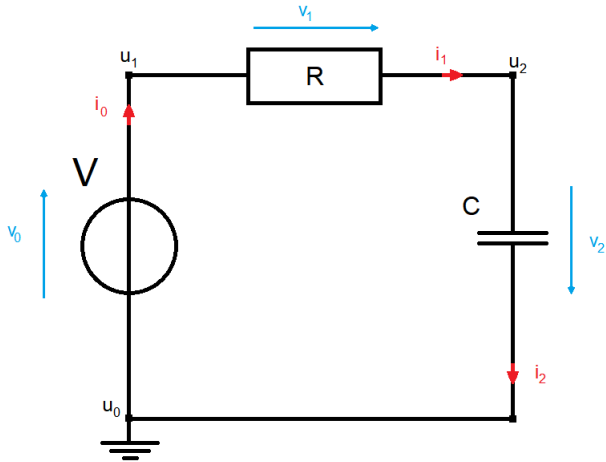


Figure: charging capacitor with series resistor and voltage source

Formulating a Mathematical Model



Network Topology

For a circuit with l nodes and k edges, define the incidence matrix $\tilde{A} = (\tilde{a}_{ij}) \in \mathbb{R}^{k \times l}$:

$$\tilde{a}_{ij} = \begin{cases} 1 & \text{edge } j \text{ starts at node } i, \\ -1 & \text{edge } j \text{ ends at node } i, \\ 0 & \text{else.} \end{cases}$$

By grounding node 0, i.e. $u_0 = 0$ we obtain the reduced incidence matrix $\rightarrow A$.

Formulating a Mathematical Model



Energy Conservation Laws

- **Kirchhoff's voltage law (KVL):**

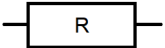


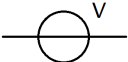
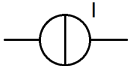
The sum of voltages along each loop of the network must equal to zero.

$$\rightarrow A^T u = v. \quad (1)$$

- **Kirchhoff's current law (KCL):**

For any node, the sum of currents flowing into the node is equal to the sum of currents flowing out of the node.

$$\rightarrow A i = 0. \quad (2)$$

| Name | Symbol | Component Law |
|----------------|---|---|
| Resistor |  | $v = R i \quad \text{or} \quad i = G u$ |
| Capacitor |  | $Q = C v \quad \text{and by derivation in t} \quad I = C \frac{d}{dt} v = C v'$ |
| Inductor |  | $\Phi = L i \quad \text{and by derivation in t} \quad v = L i'$ |
| Voltage Source |  | $v = v_{src}$ |
| Current Source |  | $i = i_{src}$ |

Formulating a Mathematical Model



Modified Nodal Analysis - MNA

Rearrange reduced incidence matrix into

$$A = (A_R A_C A_L A_V A_I)$$

A_R , A_C , A_L , A_V and A_I ... columns related to components

Represent voltages:

$$v = A^T u$$

→ rearrange v into $v = (v_R, v_C, v_L, v_{src}, v_I)$ and i into $i = (i_R, i_C, i_L, i_V, i_{src})$. Rewrite component relations:

$$\begin{aligned} i_R &= G v_R = G A_R^T u, \\ i_C &= C v'_C = C A_C^T u'. \end{aligned}$$

Kirchhoffs current law:

$$A_C i_C + A_R i_R + A_L i_L + A_V i_V = -A_I i_{src}.$$

Together with component relations combine to

$$\begin{aligned}A_C C A_C^\top u' + A_R G A_R^\top u + A_L i_L + A_V i_V &= -A_I i_{src}, \\ L i_L' - A_L^\top u &= 0, \\ -A_V^\top u &= -v_{src}.\end{aligned}$$

In matrix form:

$$\begin{pmatrix} A_C C A_C^\top & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{pmatrix} * \begin{pmatrix} u' \\ i_L' \\ i_V' \end{pmatrix} + \begin{pmatrix} A_R G A_R^\top & A_L & A_V \\ -A_L^\top & 0 & 0 \\ -A_V^\top & 0 & 0 \end{pmatrix} * \begin{pmatrix} u \\ i_L \\ i_V \end{pmatrix} = \begin{pmatrix} -A_I i_{src} \\ 0 \\ -v_{src} \end{pmatrix}. \quad (3)$$

→ differential and algebraic variables

Differential Algebraic Equations



Types of DAEs

In the most general form a DAE can be written as: Find $y : \mathbb{R} \rightarrow \mathbb{R}^n$ such that

$$F(t, y(t), y'(t)) = 0, \quad \forall t \in I \quad (4)$$

with $F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ sufficiently smooth and I the time-interval.

Linear systems with constant coefficients

find y such that

$$Ay'(t) + By(t) = f(t), \quad (5)$$

with $A, B \in \mathbb{R}^{n \times n}$, A singular, B and $f : \mathbb{R} \rightarrow \mathbb{R}^n$ a function in time. \rightarrow differential and algebraic variables

Equivalence transformations lead to

$$\begin{aligned}u'(t) + Ru(t) &= s(t), \\ Nv'(t) + v(t) &= q(t),\end{aligned}\tag{6}$$

where N is a nilpotent matrix and the matrix R is regular.

First equation is an ODE \rightarrow unique solution $u(t)$ for any initial values. Construct explicit solution for second equation: $q(t) \in C^{k-1}([t_0, t_1])$:

$$\begin{aligned}
 v(t) &= q(t) - Nv'(t) = q(t) - N(\underbrace{q(t) - Nv'(t)}_{=v(t)})' = q - Nq' + N^2v'' \\
 &= q - Nq' + N^2(q - Nv')'' = q - Nq' + N^2q'' - N^3v''' \\
 &\vdots \\
 &= q - Nq' + \dots + (-1)^{k-1}N^{k-1} \underbrace{\frac{d^{k-1}}{dt^{k-1}}q}_{:=q^{(k-1)}} + (-1)^{k-1} \underbrace{N^k v^{(k)}}_{=0} \\
 &= \sum_{i=0}^{k-1} (-1)^i N^i q^{(i)}(t)
 \end{aligned}$$

\rightarrow differentiation index $v = k$.

Differential Algebraic Equations



Index of a Differential Algebraic Equation

Definition

Consider the differential algebraic equation (4) to be uniquely solvable and F sufficiently smooth. For a given $m \in \mathbb{N}$ consider the equations

$$\begin{aligned} F(t, y, y') &= 0, \\ \frac{dF(t, y, y')}{dt} &= 0, \\ &\vdots \\ \frac{d^m F(t, y, y')}{dt^m} &= 0. \end{aligned}$$

The smallest natural number m for which the above system results in an explicit system of ordinary differential equations (ODEs), i.e. it has the form

$$y' = \phi(t, y),$$

Topological conditions:

- The resulting equations have index $\nu \leq 2$, if the circuit neither contains loops of only voltage sources nor cutsets of only current sources.
- They have index $\nu \leq 1$, if the circuit contains neither loops of only capacitors and/or voltage sources nor cutsets of only inductors and/or current sources.
- They have index $\nu = 0$, if every node in the circuit is connected to the reference node (ground) through a path containing only capacitors.

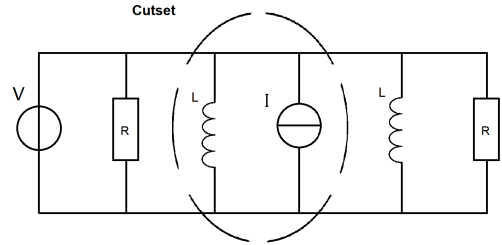
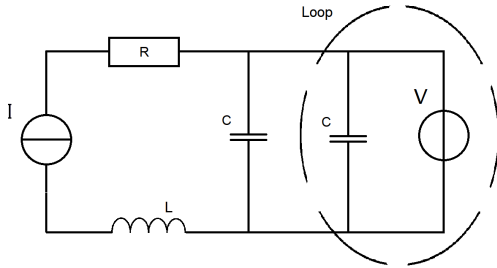


Figure: Illustration of a cutset and a loop.

Theorem (Index conditions)

Let the matrices of the capacitances, inductances and resistances be positive definite.

- If

$$\ker([A_R, A_C, A_V, A_L]^T) = \{0\} \quad \text{and} \quad \ker(A_V) = \{0\} \quad (7)$$

holds, then the MNA (3) leads to a system with index $\nu \leq 2$.

- If additionally

$$\ker([A_R, A_C, A_V]^T) = \{0\} \quad \text{and} \quad \ker([A_C, A_V]) = \{0\} \quad (8)$$

holds, then the system is of index $\nu \leq 1$

- If further

$$\ker(A_C^T) = \{0\} \quad \text{and} \quad \dim(v_{src}) = 0 \quad (9)$$

holds, then the system has index $\nu = 0$.

Numerical Solutions



Multistep Methods

Definition (Multistep method)

For given $\alpha_0, \dots, \alpha_k$ and β_0, \dots, β_k the iteration rule

$$\sum_{l=0}^k \alpha_l y_{m+l} = h \sum_{l=0}^k \beta_l f(t_{m+l}, y_{m+l}), \quad m = 0, 1, \dots, N - k \quad (10)$$

is called a *linear multistep method* (linear k-step method). It is always assumed that $\alpha_k \neq 0$ and $|\alpha_0| + |\beta_k| > 0$. If $\beta_k = 0$ holds, then the method is called explicit, otherwise implicit.

Definition (generating polynomials)

$$\rho(x) := \sum_{l=0}^k \alpha_l x^l \quad \text{and} \quad \sigma(x) := \sum_{l=0}^k \beta_l x^l$$

Definition (Convergence order)

We say that a linear multi-step method is convergent of order $p \in \mathbb{N}$, if for a solution y of the problem and a vector $(y_j)_{j=0}^k$ created by an LMSM, we have that

$$\max_{0 \leq j \leq k} \|y(t_j) - y_j\| \leq Ch^p.$$

Where C is a constant not dependent on the step size h .

Consistency on tells us whether the method approximates the equation correctly as the step size approaches 0.

Definition

1. The set

$$S := \{z \in \mathbb{C} : \rho(\xi) - z\sigma(\xi) = 0 \implies \xi \in \mathbb{C} \text{ and } |\xi| \leq 1. \quad (11)$$

If ξ has multiplicity greater than 1, then $|\xi| < 1\}$

is called the region of stability of the method.

2. A linear multistep method is called

- *0-stable*, if $0 \in S$.
- *$A(\alpha)$ -stable*, if it is stable in all z that lie within the set $\{z \in \mathbb{C}^- : |\arg(z) - \pi| \leq \alpha\}$ for $\alpha \in (0, \frac{\pi}{2})$.

Theorem

Let $f(t, y)$ be sufficiently smooth and the linear multi-step method be 0-stable and consistent of order p , then it is also convergent of order p .

Numerical Solutions



Consistent Initial Values

Index $\nu = 0$: no additional restrictions (ODE case).

Index $\nu = 1$:

Rewrite system into the form

$$\begin{aligned}y'(t) &= f(t, y(t), z(t)), \\ 0 &= g(t, y(t), z(t)).\end{aligned}$$

Conditions for consistent initial values $\rightarrow g(t_0, y_0, z_0) = 0$.

Index $\nu = 2$:

Rewrite system into the form

$$\begin{aligned}y' &= f(t, y(t), z(t)), \\ 0 &= g(t, y(t)).\end{aligned}$$

Conditions for consistent initial values

$$\begin{aligned}g(t_0, y_0) &= 0, \\ g_t(t_0, y_0) + g_y(t_0, y_0)f(t_0, y_0, z_0) &= 0.\end{aligned}\quad \text{(hidden constraint)}$$

Numerical Solutions



Implicit Linear Multistep Formulas
BDF-k Methods

The *backward differentiation formula (BDF)* is a family of implicit linear multistep methods.

$$\sum_{k=0}^s \alpha_k y_{n+k} = h\beta f(t_{n+s}, y_{n+s})$$

The BDF or BDF-k formulas for $k = 1, \dots, 3$ have the following form

$$k = 1 : hf_{m+1} = y_{m+1} - y_m \quad (\text{implicit euler})$$

$$k = 2 : hf_{m+2} = \frac{1}{2}(3y_{m+2} - 4y_{m+1} + y_m)$$

$$k = 3 : hf_{m+3} = \frac{1}{6}(11y_{m+3} - 18y_{m+2} + 9y_{m+1} - 2y_m)$$

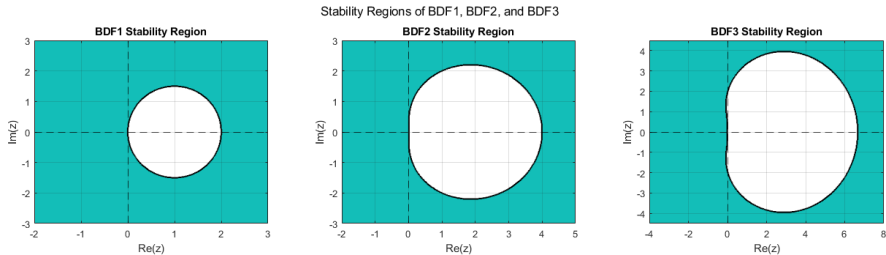


Figure: stability regions of BDF-schemes

Theorem

The BDF- k methods have consistency order $p = k$.

Corollary (Convergence rate)

The BDF- k methods with $k \leq 6$ are convergent with order k

Numerical Solutions



Implicit Linear Multistep Formulas
Trapezoidal rule

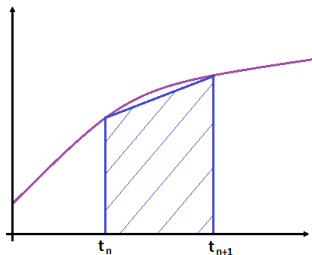


Figure: illustration of the trapezoidal rule

This procedure is repeated for small subsections of the interval $[a, b]$. Thus we obtain the iteration formula

$$y_{n+1} = y_n + \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_{n+1})].$$

The trapezoidal rule has convergence order $p = 2$.

Numerical Solutions



Numerical Examples
Example1

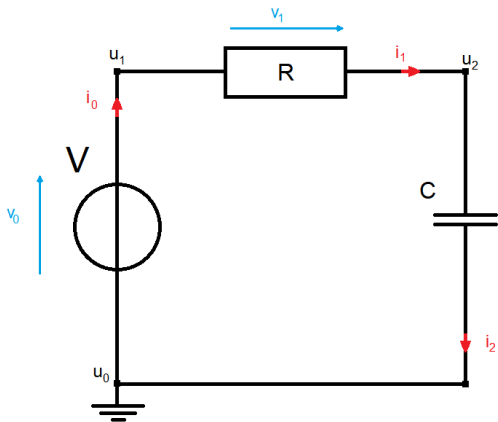


Figure: charging capacitor with series resistor and voltage source

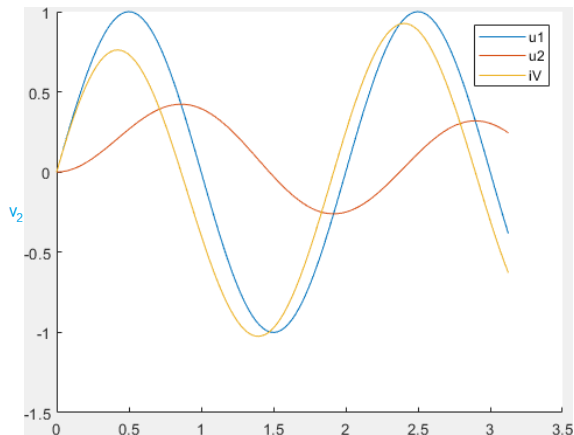


Figure: Exact solution for example 1.

| h | k = 1 | | k = 2 | | k = 3 | | trapezoidal | |
|-------|------------------------|------------------------|------------------------|------------------------|------------------------|------------------------|------------------------|------------------------|
| | err(u ₂) | err(i _V) | err(u ₂) | err(i _V) | err(u ₂) | err(i _V) | err(u ₂) | err(i _V) |
| 0.1 | 4.620×10^{-2} | 4.620×10^{-2} | 9.567×10^{-3} | 9.567×10^{-3} | 2.852×10^{-3} | 2.852×10^{-3} | 3.344×10^{-3} | 3.344×10^{-3} |
| 0.05 | 2.339×10^{-2} | 2.339×10^{-2} | 2.454×10^{-3} | 2.454×10^{-3} | 3.645×10^{-4} | 3.645×10^{-4} | 8.367×10^{-4} | 8.367×10^{-4} |
| 0.025 | 1.178×10^{-2} | 1.178×10^{-2} | 6.264×10^{-4} | 6.264×10^{-4} | 4.928×10^{-5} | 4.928×10^{-5} | 2.092×10^{-4} | 2.092×10^{-4} |

Table: Resulting errors for the BDF-k methods and the trapezoidal rule.

Definition

Let $y(t)$ be the exact solution of (4) and $\tilde{y}(t)$ be the solution of the perturbed system $F(t, \tilde{y}, \tilde{y}') = \delta(t)$. The smallest number $k \in \mathbb{N}$ such that

$$\|y(t) - \tilde{y}(t)\| \leq C \left(\|y(t_0) - \tilde{y}(t_0)\| + \sum_{j=0}^k \max_{t_0 \leq \xi \leq T} \left\| \int_{t_0}^{\xi} \frac{d^j \delta}{d\tau^j}(\tau) d\tau \right\| \right)$$

for all $\tilde{y}(t)$, is called the **perturbation index** of this system.

Dahlquist test problem as a model problem, find y such that

$$y' = \lambda y, \quad t > 0 \quad (12)$$

$$y(0) = y_0 \quad (13)$$

with $\lambda \in \mathbb{C}$ and y_0 fixed.

Thus the resulting linear multistep method is of the form

$$\begin{aligned} \sum_{l=0}^k \alpha_l y_{n+l} &= h \sum_{l=0}^k \beta_l \lambda y_{n+l} \\ \Leftrightarrow \sum_{l=0}^k [\alpha_l - h\beta_l \lambda] y_{n+l} &= 0 \end{aligned}$$

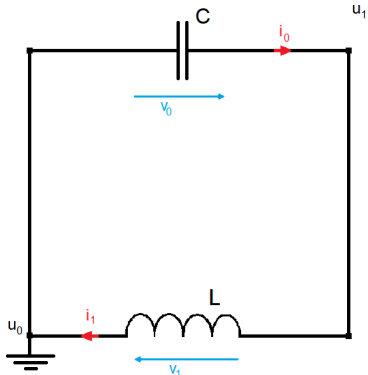


Figure: LC-circuit

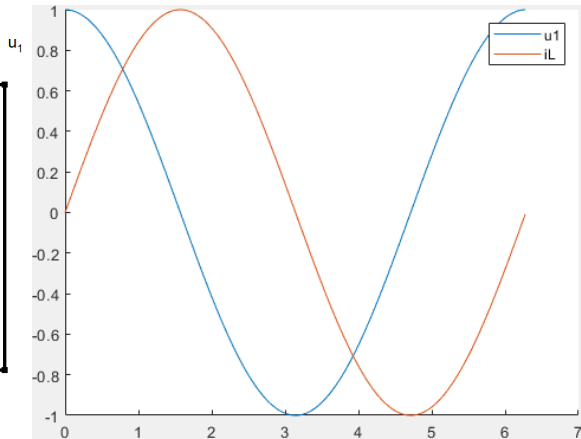


Figure: Exact solution for example 2.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} u_1' \\ i_L' \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} * \begin{pmatrix} u_1 \\ i_L \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

| h | k = 1 | | k = 2 | | k = 3 | | trapezoidal | |
|-------|------------------------|------------------------|------------------------|------------------------|------------------------|------------------------|------------------------|------------------------|
| | err(u ₁) | err(i _l) | err(u ₁) | err(i _l) | err(u ₁) | err(i _l) | err(u ₁) | err(i _l) |
| 0.1 | 2.659×10^{-1} | 2.106×10^{-1} | 1.686×10^{-2} | 1.917×10^{-2} | 1.286×10^{-3} | 1.042×10^{-3} | 4.007×10^{-3} | 5.140×10^{-3} |
| 0.05 | 1.446×10^{-1} | 1.125×10^{-1} | 4.266×10^{-3} | 5.067×10^{-3} | 1.501×10^{-4} | 1.131×10^{-4} | 1.003×10^{-3} | 1.301×10^{-3} |
| 0.025 | 7.543×10^{-2} | 5.817×10^{-2} | 1.070×10^{-3} | 1.294×10^{-3} | 1.460×10^{-5} | 9.674×10^{-6} | 2.507×10^{-4} | 3.268×10^{-4} |

Table: Resulting errors for the BDF-k methods and the trapezoidal rule.

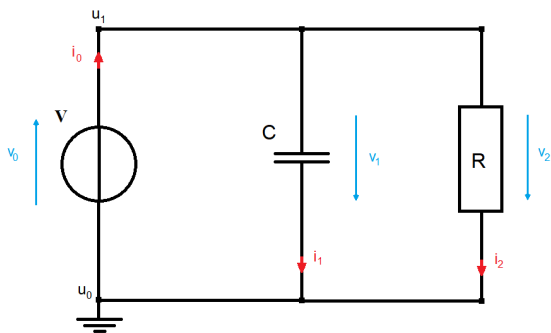


Figure: Current source with capacitor and resistor.

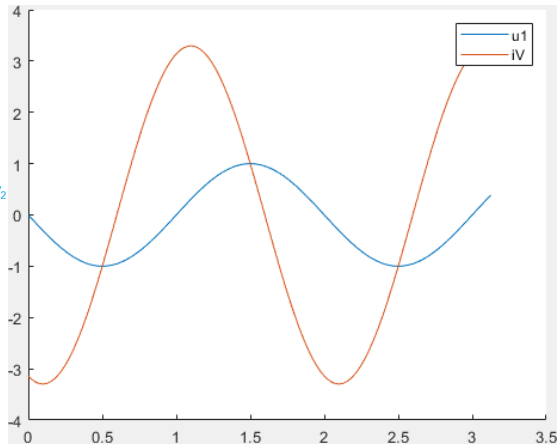


Figure: Exact solution for example 3.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} * \begin{pmatrix} u_1' \\ i_V' \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} u_1 \\ i_V \end{pmatrix} = \begin{pmatrix} 0 \\ -\sin(\pi t) \end{pmatrix}.$$

| h | k = 1 err(i_V) | k = 2 err(i_V) | k = 3 err(i_V) | trapezoidal err(i_V) |
|-------|------------------------|------------------------|------------------------|-----------------------------|
| 0.1 | 4.894×10^{-1} | 1.023×10^{-1} | 2.403×10^{-2} | 5.219×10^{-2} |
| 0.05 | 2.462×10^{-1} | 2.577×10^{-2} | 3.034×10^{-3} | 1.295×10^{-2} |
| 0.025 | 1.233×10^{-1} | 6.456×10^{-3} | 4.029×10^{-4} | 3.232×10^{-3} |

Table: Resulting errors for the BDF-k methods and the trapezoidal rule.