JOHANNES KEPLER
UNIVERSITY LINZ

Author / Eingereicht von
**Felix Dreßler**
k12105003

Submission / Angefertigt am
**Institute** / **Institut**

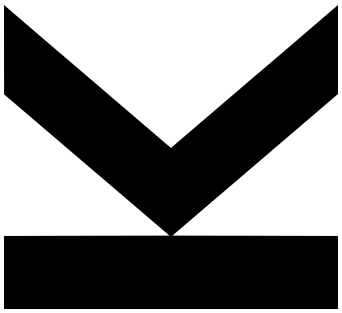
Thesis Supervisor / First Supervisor / BeurteilerIn / ErstbeurteilerIn / ErstbetreuerIn
DI Dr. **Name**

Second Supervisor / ZweitbeurteilerIn / ZweitbetreuerIn
**Name**

Assistant Thesis Supervisor / Mitbetreuung
**Name**

Month 2024 / Monat 2024

# Circuit Modelling



Subtitle / Untertitel

**JOHANNES KEPLER**
**UNIVERSITY LINZ**
Altenbergerstraße 69
4040 Linz, Austria
www.jku.at
DVR 0093696

# Kurzfassung

Kurzfassung auf Deutsch.

# Abstract

Abstract in English.

# Contents

# List of Figures

# List of Tables

# 1 Introduction

This chapter should include information about what circuit modelling wants to achieve as well as giving an overview of what this bachelor-thesis is about.

What is this thesis about?
Modelling and numerically solving systems that arrise from electrical circuits with RLC elements. Furthermore it will briefly discuss on expanding this baseline with more complicated electrical components. What is the goal of this thesis?
The goal of this thesis is to give insight into industrial standards concerning circuit modelling. It aims to elaborate on the underlying concepts of MNA as well as on the most commonly used numerical methods.

# 2 Formulating a Mathematical Model

based on DAE lecture and modelling and discretiztation of elec circuit problems.

## 2.1 Network Topology

An electrical circuit is usually considered as a graph $(N, E)$ where $N = (n_0, n_1, n_2, ...)$ are the Nodes and $E = (e_{ij})_{ij}$ are the edges, where for some $i$ and some $j$ we have that $e_{ij} = (n_j, n_k)$ is the edge from node $j$ to node $k$. We can store this information in an *incidence matrix* $\tilde{A} = (\tilde{a}_{ij})_{ij}$ which is defined by

$$\tilde{a}_{ij} = \begin{cases} 1 & \text{edge } j \text{ starts at node } i, \\ -1 & \text{edge } j \text{ ends at node } i, \\ 0 & \text{else.} \end{cases}$$

We call $u = (u_0, u_1, u_2, ...)$ the corresponding potentials to the nodes $N$. The difference of these potentials is the voltage at the associated edge. To fix the absolute values of these potentials we have to set one node to a fixed potential. We will do that by "grounding" the node $n_0$, this means we set the potential $u_0 := 0$. This grounding of a node allows us to remove the corresponding row from the incidence matrix to get the *reduced incidence matrix* $A$. The vector $v = (v_{ij})_{ij}$ represents the voltages at the edges. For some $i$ and some $j$ the voltage at edge $ij$ is $v_{ij} = u_i - u_j$.

We will later see, that the components of an electrical crircuit, which will be installed along the edges, describe a relationship between the edges current and its voltage. Thus a current vector $i = (i_1, i_2, i_3, ...)$ containing the currents along the edges is required.

## 2.2 Energy Conservation Laws

To fully fix all the variables that arrise in the model of an electrical circuit we will need some *conservation laws*:

- **Kirchhoff's voltage law (KVL):**
  The sum of voltages along each loop of the network must equal to zero. Using the incidence matrix $A$ this law can be formulated as

$$A^\top * u = v. \tag{2.1}$$

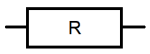
- **Kirchhoff's current law (KCL):**
  For any node, the sum of currents flowing into the node is equal to the sum of currents flowing out of the node. Using the incidence matrix $A$ again, this law can be formulated as

$$A * i = 0. \tag{2.2}$$

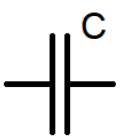
## 2.3 Electrical Components and their relations

Electrical components are described by equations relating their edge voltage $v$ to their edge current $i$.

- **Resistor**
  $v = R * i$ or $i = G * u$

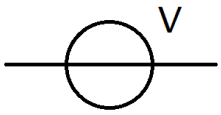



- **Capacitor**
  $Q = C * v \rightarrow I = C * \dot{v}$

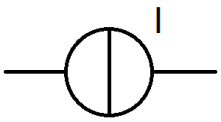


- **Coil**
  $\Phi = L * i \rightarrow v = L * \dot{i}$


- **Voltage Source**
  $v = v_{src}$




- **Current Source**
  $i = i_{src}$




- Diode - to be filled with information after the rest is complete
- Transistor

## 2.4 Modified Nodal Analysis - MNA

num gew dgl steif n steif - seite 422
modelling discr circ prob - seite 19

To analyse the network further we will sort the reduced incidence matrix $A$ such that is has the block form

$$A = (A_R A_C A_L A_V A_I)$$

where $A_R$, $A_C$, $A_L$, $A_V$ and $A_I$ all include the collumns that are related to the resistors, capacitors, coils, voltage sources and current sources.

To mathematically describe the circuit we will use *modified nodal analysis* (or short MNA). MNA uses the node voltages as well as the currents of the coils and the voltage sources as unknowns and is based on the conservation laws

$$Ai(t) = 0 \tag{2.3}$$

$$v = A^\top u(t) \tag{2.4}$$

as well as on the voltage-current relations of the electrical components. By replacing all edge-currents with their respective voltage-current relation and ell egde-voltages with their node-potentials we obtain the MNA-equations

$$\begin{aligned} A_C C A_C^\top \dot{u} + A_R G A_R^\top u + A_L i_L + A_V i_V + A_I i_{src} &= 0 \\ L\dot{i_L} - A_L^\top u &= 0 \\ -A_V^\top + v_{src} &= 0. \end{aligned}$$

In matrix form these read as

$$\begin{pmatrix} A_C C A_C^\top & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{pmatrix} * \begin{pmatrix} \dot{u} \\ \dot{i_L} \\ \dot{i_V} \end{pmatrix} + \begin{pmatrix} A_R G A_R^\top & A_L & A_V \\ -A_L^\top & 0 & 0 \\ -A_V^\top & 0 & 0 \end{pmatrix} * \begin{pmatrix} u \\ i_L \\ i_V \end{pmatrix} = \begin{pmatrix} -A_I i_{src} \\ 0 \\ -v_{src} \end{pmatrix},$$

where the diagonal matrices $C$, $G$ and $L$ contain the capacities, conductivities and inductivities.

## 2.5 Energy bilanz??????????

## 2.6 Charge/Flux oriented formulation of MNA

Again using KCL 2.3 and the component equations we formulate a system of equations. (**more here**) What is flux and charge - explain either here or witht the components

$$A_C\dot{q} + A_R r(A_R^\top u, t) + A_L i_L + A_V i_V + A_I i(A^\top u, \dot{q}, i_L, i_V, t) = 0 \tag{2.5}$$
$$\dot{\phi} - A_L^\top u = 0 \tag{2.6}$$
$$v(A^\top u, \dot{q}, i_L, i_V, t) - A_V^\top u = 0 \tag{2.7}$$
$$q - q_C(A_C^\top u) = 0 \tag{2.8}$$
$$\phi - \phi_L(i_L) = 0 \tag{2.9}$$

Using

node potentials $u$,

branch currents through voltage and flux controlled elements $i_V$ and $i_L$,

charges and fluxes $q$ and $\phi$,

voltage dependent resistors $r$,

voltage and current dependent charge and flux sources $q_C$ and $\phi_L$,

controlled current and voltage sources $i_{src}$ and $v_{src}$.

**Charge/flux oriented or conventional MNA?** On which formulation — charge/flux oriented or conventional — should the numerical discretization be based, if MNA is used for the automatic generation of network equations? From a structural aspect, the conventional MNA formulation yields a standard form of numerical integration problems, while the charge/flux oriented formulation does not. There are however several reasons, not to transform (4.1) into (4.2) before applying numerical discretization schemes, although they are analytically equivalent:

Structure. (4.1) is of linear-implicit nonlinear form, while (4.2) is of nonlinear-implicit nonlinear form. This may have an impact on the choice of a suitable integrator.

Numerics. Information on the charge/flux level is lost in the conventional approach, and charge conservation may only be maintained approximately in numerical integration schemes.

Implementation. Implicit numerical integration schemes for the conventional MNA equations (4.2) require second partial derivatives of $q_C$ und $\phi_L$. These derivative informations, however, are not available in circuit simulation packages, may even not exist because of the lack of smoothness in transistor models.

**Figure 2.1:**

# 3 Differential Algebraic Equations

von num gew dgl niochtsteife steife und dae

## 3.1 Abstract Problem

$$F(t, y(t), y'(t)) = 0 \qquad \forall t \in I \tag{3.1}$$

with $F : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ sufficiently smooth. We will focus on linear time invaraiant systems of the form

$$Eu'(t) = Au(t) + f(t).$$

With $E, A \in \mathbb{R}^{n \times n}$. If $E$ is regular, then this system is just an ordinary differential equation, thus we assume $E$ to be singular to obtain a "true" DAE.

In the following chapters we will discuss important analytical properties of this such sytems. We will discuss the solvability and the index of these systems.

## 3.2 Types of DAEs

- **Linear systems with constant coeffiecients**
  are systems of the form

$$Ay'(t) + By(t) = f(t) \tag{3.2}$$

with $A, B \in \mathbb{R}^{n \times n}$, $A$ singular and $f(t)$ a function.

- **linear time dependent systems**

  $$A(t)y'(t) + B(t)y(t) = f(t)$$

  with $A(t), B(t), f(t)$ functions.

- **structured (non-linear) systems**
  are semi-explicit systems of the form

  $$y'(t) = f(t, y(t), z(t)), 0 = g(t, y(t), z(t))$$

  with $f$ and $g$ functions.

For our analysis of electrical networks we will focus on linear systems with constant coefficients. (where do the other systems arrise? what are they about?)

### 3.2.1 Weierstraß-Kronecker Normalform

kapitel aus buch seite 399 num gew dgl steif, kapitel 13.2.2

To determine the solvability of a linear system with constant coefficients 3.2 we first need to introduce a Normalform for the system, the *Weierstraß-Kronecker Normalform*. This Normalform is dependant on the family $\{A, B\} := \{\mu A + B | \mu \in \mathbb{R}\}$, which is called the *matrix pencil* of the DAE.

**Definition 1.** *The matrix pencil $\{A, B\}$ is called* regular *if there exists some $c \in \mathbb{R}$, such that $(cA + B)$ is regular ($det(cA + B) \neq 0$), otherwise it is called singular.*

**Theorem 1** (Jordan Normalform). *For every matrix $Q \in \mathbb{R}^{\times}$ there exists a regular matrix $T \in \mathbb{C}^{n\times n}$, such that*

$$T^{-1}QT = J = diag(J_1, ..., J_r) \quad with \quad J_i = \begin{pmatrix} \lambda_i & 1 & & 0 \\ 0 & \lambda_i & \ddots & \vdots \\ & \ddots & \ddots & 1 \\ 0 & \dots & 0 & \lambda_i \end{pmatrix} \in \mathbb{C}^{m_i \times m_i}$$

*and $n = m_1 + ... + m_r$.*

quote that from somewhere.

translate that:

The matrix $J$ is called Jordan Normalform of $Q$, the $J_i$ are called Jordan Blocks, where $\lambda_i$ are the eigenvalues of $Q$. The matrix $J$ is uniquely determined by $Q$ except for the arrangement of the diagonal blocks. If $Q$ posesses only real eigenvalues, then $T$ can also be choosen from the reals.
A transformation from $A$ and $B$ in 3.2 enables a seperation into differential and algebraic variables.

aus buch seite 401

**Theorem 2.** *Let $\{A, B\}$ be a regular matrix pencil. There exist $P, Q \in \mathbb{C}^{n\times n}$ such that*

$$PAQ = \begin{pmatrix} I_d & 0 \\ 0 & N \end{pmatrix}, \quad PBQ = \begin{pmatrix} R & 0 \\ 0 & I_{n-d} \end{pmatrix}$$

*where*

$$N = diag(N_1, ..., N_r) \quad with \quad N_i = \begin{pmatrix} 0 & 1 & & 0 & \\ & \ddots & \ddots & & \\ & & & 0 & 1 \\ 0 & & & 0 & \end{pmatrix} \in \mathbb{R}^{n_i \times n_i}$$

*and R has Jordan Normalform.*

*Proof.* Becuase $\{A, B\}$ is a regular matrix pencil, there exists $c \in \mathbb{R}$ such that $(cA + B)$ is regular. Set

$$\hat{A} := (cA + B)^{-1}A, \quad \hat{B} := (cA + B)^{-1}B.$$

Considering

$$(cA + B)^{-1}(cA + B) = I \implies (cA + B)^{-1}B + c(cA + B)^{-1}A = I,$$

we get that

$$\hat{B} = I - c\hat{A}.$$

Let $J_{\hat{A}}$ be the Jordan Normalform of $\hat{A}$, this means that there exists a regular matrix $T_1$ such that

$$T_1^{-1}AT_1 = J_{\hat{A}} = \begin{pmatrix} W & 0 \\ 0 & \tilde{N} \end{pmatrix}.$$

The matrix $W$ contains the Jordanblocks with Eigenvalues which are nonzero, the matrix $\tilde{N}$ contains the Jordan blocks with Eigenvalues equal to zero, thus $\tilde{N}$ is *nilpotent*. The Jordan Normalform $J_{\hat{B}}$ of $\hat{B}$ is given by

$$T_1^{-1}\hat{B}T_1 = J_{\hat{B}} = \begin{pmatrix} I - cW & 0 \\ 0 & I - c\tilde{N} \end{pmatrix}.$$

The following two transformations will allow us to get the desired structure. First we will transform $J_{\hat{A}}$ with

$$T_2 := \begin{pmatrix} W & 0 \\ 0 & I - c\tilde{N} \end{pmatrix}$$

in

$$T_2^{-1}J_{\hat{A}} = \begin{pmatrix} I & 0 \\ 0 & (I - c\tilde{N})^{-1}\tilde{N} \end{pmatrix}$$

and $J_{\hat{B}}$ in

$$T_2^{-1} J_{\hat{B}} = \begin{pmatrix} W^{-1} - cI & 0 \\ 0 & I \end{pmatrix}.$$

Let now $R$ be the Jordan Normalform of $(W^{-1} - cI)$ and $N$ be the Normalform of $(I - c\tilde{N})^{-1}\tilde{N}$, this means

$$T_W^{-1}(W^{-1} - cI)T_W = R \quad \text{and} \quad T_{\tilde{N}}^{-1}(I - c\tilde{N})^{-1}\tilde{N}T_{\tilde{N}} = N$$

Considering

Die Matrizen $\widetilde{N}$ und $(I - c\widetilde{N})^{-1}$ kommutieren. Daraus folgt

$$N^k = T_{\widetilde{N}}^{-1}(I - c\widetilde{N})^{-k}\widetilde{N}^k\widetilde{N}T_{\widetilde{N}} = T_{\widetilde{N}}^{-1}\widetilde{N}^k T_{\widetilde{N}}.$$

**Figure 3.1:** fig 1, i dont get why

The nilpotent matrix $N$ thus has the nilpotency index $k$. A transformation with

$$T_3 := \begin{pmatrix} T_W & 0 \\ 0 & T_{\tilde{N}} \end{pmatrix}$$

transforms $T_2^{-1} J_{\hat{A}}$ into the Jordan Normalform

$$J_{\tilde{A}} := T_3^{-1}T_2^{-1}J_{\hat{A}}T_3 = T_3^{-1}T_2^{-1}T_1^{-1}\hat{A}T_1T_3 = \begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix}$$

and $T_2^{-1} J_{\hat{B}}$ into

$$J_{\tilde{B}} := T_3^{-1}T_2^{-1}J_{\hat{B}}T_3 = T_3^{-1}T_2^{-1}T_1^{-1}\hat{B}T_1T_3 = \begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix}.$$

Now set

$$P := T_3^{-1}T_2^{-1}T_1^{-1}(cA + B)^{-1} \quad \text{and} \quad Q = T_1T_3$$

to get the statement. □

**Definition 2.** *The nilpotency index k from the Weierstraß-Kronecker Normalform of a matrix pencil $\{A, B\}$ with A singular is called the* Kronecker-Index *of $\{A, B\}$. We write $ind\{A, B\}$. For A regular we set $ind\{A, B\} = 0$.*

next lemma needs quotation, it is lemma 13.2.1

**Lemma 1.** *The Kronecker-Index $ind\{A, B\}$ is independant of the choice of the matrices P and Q.*

Lemma 13.2.2 maybe

Using the findings above we are able to Transform the initial DAE 3.2 using the matrix $P$ from 2. By multiplying $P$ from the left we obtain

$$PAy'(t) + PBy(t) = Pf(t).$$

Setting

$$y = Q\begin{pmatrix} u \\ v \end{pmatrix}, \quad Pf(t) = \begin{pmatrix} s(t) \\ q(t) \end{pmatrix} \quad \text{with} \quad u, s \in \mathbb{R}^d,$$

we get a system of the form

$$\begin{aligned} u'(t) + Ru(t) &= s(t) \\ Nv'(t) + v(t) &= q(t) \end{aligned} \tag{3.3}$$

The first equation is an ordinary differential equation of first order and posesses a unique solution $u(t)$ in $[t_0, t_l]$ for any starting values $u_0 \in \mathbb{R}^d$. Additionally setting $q(t) \in C^{k-1}([t_0, t_l])$ then differentiating the second equation in 3.3 gives (**whyyyyyyy??????**)

**Beweis.** Angenommen, das Matrixbüschel $\{A, B\}$ hat zwei verschiedene Weierstraß-Kronecker-Normalformen, d. h., es existieren reguläre quadratische Matrizen $P, Q$ und $\widehat{P}, \widehat{Q}$, so dass gilt

$$PAQ = \begin{pmatrix} I_{d_1} & 0 \\ 0 & N \end{pmatrix}, \quad PBQ = \begin{pmatrix} R & 0 \\ 0 & I_{n-d_1} \end{pmatrix}, \tag{13.2.7}$$

und

$$\widehat{P}A\widehat{Q} = \begin{pmatrix} I_{d_2} & 0 \\ 0 & \widehat{N} \end{pmatrix}, \quad \widehat{P}B\widehat{Q} = \begin{pmatrix} \widehat{R} & 0 \\ 0 & I_{n-d_2} \end{pmatrix}. \tag{13.2.8}$$

Wir betrachten das Polynom

$$\begin{aligned} p(\lambda) &= \det(\lambda A + B) = \det(P^{-1}) \det(\lambda I_{d_1} + R) \det(Q^{-1}) \\ &= \det(\widehat{P}^{-1}) \det(\lambda I_{d_2} + \widehat{R}) \det(\widehat{Q}^{-1}). \end{aligned}$$

Es gilt

$$\mathrm{Grad}(p(\lambda)) = d_1 = d_2,$$

d. h., die ersten Blöcke in (13.2.7) und (13.2.8) haben die gleiche Dimension. Mit den regulären Matrizen $\widetilde{P} := \widehat{P}P^{-1}$ und $\widetilde{Q} := \widehat{Q}^{-1}Q$ ergibt sich aus (13.2.7) und (13.2.8)

$$\widetilde{P}\begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix} = \widehat{P}AQ = \begin{pmatrix} I & 0 \\ 0 & \widehat{N} \end{pmatrix}\widetilde{Q}, \quad \widetilde{P}\begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix} = \widehat{P}BQ = \begin{pmatrix} \widehat{R} & 0 \\ 0 & I \end{pmatrix}\widetilde{Q},$$

so dass gilt

$$\begin{pmatrix} \widetilde{P}_{11} & \widetilde{P}_{12} \\ \widetilde{P}_{21} & \widetilde{P}_{22} \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & \widehat{N} \end{pmatrix}\begin{pmatrix} \widetilde{Q}_{11} & \widetilde{Q}_{12} \\ \widetilde{Q}_{21} & \widetilde{Q}_{22} \end{pmatrix}$$

und

$$\begin{pmatrix} \widetilde{P}_{11} & \widetilde{P}_{12} \\ \widetilde{P}_{21} & \widetilde{P}_{22} \end{pmatrix}\begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} \widehat{R} & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} \widetilde{Q}_{11} & \widetilde{Q}_{12} \\ \widetilde{Q}_{21} & \widetilde{Q}_{22} \end{pmatrix}.$$

**Figure 3.2:**

Daraus folgen die Beziehungen

$$\tilde{P}_{11} = \tilde{Q}_{11}, \quad \tilde{P}_{12}N = \tilde{Q}_{12}, \quad \tilde{P}_{21} = \hat{N}\tilde{Q}_{21}, \quad \tilde{P}_{22}N = \hat{N}\tilde{Q}_{22}$$

und

$$\tilde{P}_{11}R = \hat{R}\tilde{Q}_{11}, \quad \tilde{P}_{12} = \hat{R}\tilde{Q}_{12}, \quad \tilde{P}_{21}R = \tilde{Q}_{21}, \quad \tilde{P}_{22} = \tilde{Q}_{22}.$$

Damit ergibt sich

$$\tilde{P}_{21} = \hat{N}\tilde{P}_{21}R = \hat{N}^2\tilde{P}_{21}R^2 = \cdots = \hat{N}^{\hat{k}}\tilde{P}_{21}R^{\hat{k}} = 0 \text{ und } \tilde{Q}_{21} = 0.$$

Hierbei ist $\hat{k}$ der Nilpotenzindex von $\hat{N}$. Auf analoge Weise erhält man

$$\tilde{P}_{12} = \hat{R}\tilde{P}_{12}N = \hat{R}^2\tilde{P}_{12}N^2 \cdots = \hat{R}^k\tilde{P}_{12}N^k = 0,$$

wobei $k$ der Nilpotenzindex von $N$ ist. Aufgrund der Regularität der Transformationsmatrizen $\tilde{P}$ und $\tilde{Q}$ müssen die Blöcke

$$\tilde{P}_{11} = \tilde{Q}_{11} \quad \text{und} \quad \tilde{P}_{22} = \tilde{Q}_{22}$$

regulär sein. Ferner gilt jetzt

$$\tilde{Q}_{11}R = \hat{R}\tilde{Q}_{11} \quad \text{und} \quad \tilde{Q}_{22}N = \hat{N}\tilde{Q}_{22},$$

d. h., $R$ und $\hat{R}$ sowie $N$ und $\hat{N}$ sind ähnliche Matrizen. Damit folgt, dass die Jordansche Normalform von $N$ und die von $\hat{N}$ aus den gleichen nilpotenten Jordan-Blöcken bestehen und somit den Nilpotenzindex $\hat{k} = k$ besitzen. ■

**Figure 3.3:**

$$\begin{aligned} v(t) &= q(t) - Nv'(t) = q(t) - N(q(t) - Nv'(t))' = q - Nq' + N^2v'' \\ &= q - Nq' + N^2(q - Nv')'' = q - Nq' + N^2q'' - N^3v''' \\ &\vdots \\ &= q - Nq' + ... + (-1)^{k-1}N^{k-1}q^{(k-1)} + (-1)\underbrace{N^kv^{(k)}}_{=0} \end{aligned}$$

$$= \sum_{i=0}^{k-1}(-1)^iN^iq^{(i)}(t) \tag{3.4}$$

where $k$ is the nilpotency index of $N$. This expression gives an explicit solution for $v(t)$ in $[t_0, t_l]$ with $v(t) \in \mathbb{R}^{d-1}$. It shows the depencdency of the solution and its derivatives. The higher the Kronecker index $k$ gets, the more differentiations of $q(t)$ have to be performed.

The Kronecker index $k$ shows, that $k$ differentiations are required to receive an ordinary differential equation.

## 3.3 Index of a Differential Algebraic Equation

The Index of a DAE gives us insight about it's numerical properties and in general about the solvability. In general, the higher the index, the harder it is, to solve the system.

We will consider two types of index concepts, the differentiation index and the perturbation index.

**Definition 3** (differentiation index)**.** *Consider the differential algebraic equation 3.1 to be uniquely locally solvable and F suffieciently smooth differentiable. For a given $m \in \mathbb{N}$ consider*

$$\begin{aligned} F(t, y, y') &= 0, \\ \frac{\mathrm{d}F(t, y, y')}{\mathrm{d}t} &= 0, \\ &\vdots \\ \frac{\mathrm{d}^m F(t, y, y')}{\mathrm{d}t^m} &= 0. \end{aligned}$$

*The smallest natural number m for which the above System results in an explicit system of the form*

$$y' = \phi(t, y)$$

*from which y can be determined is called* ***differentiation index****.*

In the previous chapter we have already discussed, that for a DAE with constant coefficients 3.2 and a regular matrix pencil $\{A, B\}$ we need $k = ind\{A, B\}$ differentiations to receive an ordinary differential equation. This means that the Kronecker index $k$ is equal to the differentiation index in the case of a DAE with constant coefficients.

**Definition 4** (perturbation index). *Let $y(t)$ be the exact solution to 3.1. This problem has the **perturbation index** $k \in \mathbb{N}$ along $y(t), t_0 \leq t \leq T$ if for all $\tilde{y}(t)$ with $F(t, \tilde{y}, \tilde{y}') = \delta(t)$ the inequality*

$$||y(t) - \tilde{y}(t)|| \leq C \left( ||y(t_0) - \tilde{y}(t_0)|| + \sum_{j=0}^{k} \max_{t_0 \leq \xi \leq T} \left\| \int_{t_0}^{\xi} \frac{\mathrm{d}^j \delta}{\mathrm{d}\tau^j}(\tau) d\tau \right\| \right)$$

*for the smallest number k.*

for lin const nilpot ind = ind + proof

**Bemerkung 13.3.5.** Im Fall linearer differential-algebraischer Systeme (13.2.1) mit regulärem Matrixbüschel $\{A, B\}$ kann die DAE

$$A(y'(t) - \widetilde{y}'(t)) + B(y(t) - \widetilde{y}(t)) = \delta(t)$$

nach Folgerung 13.2.1 transformiert werden in

$$\begin{aligned} u'(t) - \widetilde{u}'(t) + \widehat{R}(u(t) - \widetilde{u}(t)) &= \delta_1(t) \\ N(v'(t) - \widetilde{v}'(t)) &= \delta_2(t). \end{aligned} \tag{13.3.29}$$

Aus der Lösungsdarstellung von (13.3.29), vgl. (13.2.11), folgt, dass der Störungsindex $pi$ gleich dem Kronecker-Index $k$ ist. Mit Bemerkung 13.3.1 gilt damit $k = di = pi$. □

Bezüglich des Störungsindex von (13.3.8) gilt der

**Figure 3.4:**

# 4 Index Analysis of the MNA

der ane satz do + proof hopefully

seite 22 kap 7 netw top and dae ind for rlc - modelling and discretization circ prob

In the linear case, the capacitances, inductances and resistances are symmetrical positive definite (We consider them as matrices in this case). In the nonlinear case ... . Then it is called an RLC network.

In our further analysis we will only consider such RLC networks.

## 4.1 General Index analysis

Assuming the system only contains linear elements or is linearized at an operating point in order to investigate the system behaviour then the corresponding network equation represent a DAE with constant coefficients 3.2. Ww will denote $x = (ui_L i_V)^\top$.

- **ODE-case**:
  The matrix $B$ is regular in 3.2. This is the case if

ODE-case: $B$ regular. This holds, iff the circuit contains no voltage sources and there are no nodes which have no path to ground via capacitors. In this case, system (6.1) represents a

**Figure 4.1:**

  , then the system represents alinear-implicit system of ODEs and can be transformed into the explicit ODE sytstem

$$\dot{x} = B^{-1}(-Ax + f(t))$$

- **DAE-case**: The matrix $B$ is singular in 3.2. In the following we will assume $D$ to be regular. Multiplying with $D^{-1}$ from the left side produces
  nevermind thats the same as we did in the 3.2 chapter. - reference to that an explain in these terms. - page 20

## 4.2 Topological Conditions

For this we consider RLC networks with independant voltage and current sources (**what does that mean?**). To obtain the perturbation index of the MNA we perturb the right-hand side of
reference to Tischendorf

From analyzing the MNA some conditions to the circuit topology can be obtained. We will be concidering the impact of loops containing only capacitances and voltage sources as well as cutsets containing only inductances and current sources.

**Theorem 3** (Index-1 condition)**.** *[***Tischendorf2004Topological***] Let the matrices of the capacitances, inductances and resistances respectively be positive definite. If the network neither cointains* inductance-current-source cutsets *nor (controlled?)* capacitance-voltage-source loops*, then the MNA leads to an index-1 DAE.*

**Theorem 4** (Index-2 condition)**.** *[***Tischendorf2004Topological***] If the Network contains* inductance-current-source cutsets *or* capacitance-voltage-source loops *except for capacitance-only loops, then the MNA leads to an index-2 DAE.*

# 5 Numerical Solutions

This chapter focuses on the numerical solution of the mentioned systems. (based on the DAE-lecture) It will include Code that illustrates some of the shown procedures.

We will first focus on the general methods used for solving a more general problem

$$y'(t) = f(t, y), \quad t \in [t_0, t_l], \tag{5.1}$$

$$y(t_0) = y_0. \tag{5.2}$$

We will presume that the function $f(t, y)$ is continuous and Lipschitz, thus for every $y_0$ it is uniquely solvable in $[t_0, t_l]$.

Numerische Verfahren arbeiten mit einer *Diskretisierung*, d. h., man betrachtet eine Zerlegung des Integrationsintervalls

$$t_0 < t_1 < t_2 \cdots < t_N \leq t_e$$

und Näherungen $u_m \approx y(t_m)$, $m = 0, 1, \ldots, N$. Die $t_m$ heißen *Gitterpunkte*, $I_h = \{t_0, t_1, \ldots, t_N\}$ heißt *Punktgitter*, die $h_m := t_{m+1} - t_m$ heißen *Schrittweiten* und $h_{max} = \max h_m$ charakterisiert die *Feinheit* des Gitters $I_h$. Sind die Schrittweiten $h_m$ konstant ($h_m = h$ für alle $m = 0, \ldots, N-1$), so heißt das Gitter *äquidistant*, andernfalls spricht man von einem *nichtäquidistanten* Gitter. Die Näherungen lassen sich auch als Funktionswerte einer sogenannten *Gitterfunktion* $u_h$ interpretieren.

**Definition 2.1.1.** Unter einem *Diskretisierungsverfahren* zur Approximation der Lösung $y(t)$ des Anfangswertproblems (2.0.1) versteht man eine *Verfahrensvorschrift*, die jedem Punktgitter $I_h$ eine Gitterfunktion $u_h : I_h \to \mathbb{R}^n$ zuordnet. □

**Figure 5.1:**

from book circuit modelling 35-47

## 5.1 Single-Step-Methods

page 46 modelling circuit

page 20 num gew dgl

**Definition 5.** *A numerical method to approximate a differential equation 5.1 on a time-grid $t_0, ..., t_l$ with the intermediate values $y_0, ..., y_l$ is called a single-step method if it is from the form*

$$y_{j+1} = y_j + h_j \phi(t_j, y_j, y_{j+1}, h_j). \tag{5.3}$$

*With the* procedural function $\phi$*. If $\phi$ is not dependent on $y_{j+1}$ then the method is called* explicit*, otherwise it is called* implicit*.*

### 5.1.1 Consistency and Convergence

**Definition 6.** *Let $\tilde{y}_{m+1}$ be the result of one step of 5.3 with the exact start-vector $y_m = y(t_m)$ then*

$$le_{m+1} = le(t_m + h) = y(t_{m+1}) - \tilde{y}_{m+1}, \quad m = 0, ..., N-1 \tag{5.4}$$

*is called the* local discretization error *of the single step method at the point $t_{m+1}$.*

**Definition 7.** *A single-step method is called* consistent *if for all initial value problems 5.1*

$$\lim_{h \to 0} \frac{||le(t+h)||}{h} = 0 \quad for \quad t_0 \leq t \leq t_l \tag{5.5}$$

*holds.*
*It is called* consistent of order p*, if for a sufficiently smooth function $f$*

$$||le(t+h)|| \leq Ch^{p+1} \quad for\ all \quad h \in (0, H] \quad and \quad t_0 \leq t \leq t_l - h \tag{5.6}$$

*holds with C not dependent on h.*

**Definition 8.** *A single-step method is called* convergent, *if for all initial value problems 5.1 for the* global discretization error

$$e_m = y(t_m) - y_h(t_m)$$

*holds that*

$$\max_m ||e_m|| \to 0 \quad for \quad h_{max} \to 0.$$

*The single-step method is called to have the* convergence order $p$, *if*

$$\max_m ||e_m|| \leq Ch_max^p \quad for \quad h_{max} \in (0, H] \quad with \quad t_0 \leq t_m \leq t_l$$

*with the constant C not dependent on the step size h.*

### 5.1.2 Runge-Kutta Methods

this chapter gives a brief overview for Runge-Kutta Methods, then presents some of those methods.

**Definition 9.** *Let $s \in \mathbb{N}$. A single-step method of the form*

$$y_{m+1} = y_m + h \sum_{i=1}^{s} b_i f(t_m + c_i h, y_{m+1}^{(i)}) \tag{5.7}$$

$$y_{m+1}^{(i)} = y_m + \sum_{j=1}^{s} a_{ij} f(t_m + c_j h, y_{m+1}^{(j)}) \tag{5.8}$$

*is called a* Runge-Kutta Method *with s steps.*

We usually collect the coefficients into the vectors and matrices $c = (c_1, ..., c_s)$, $A = (a_{ij})_{ij}$ and $b = (b_1, ..., b_s)$.

heißt *s-stufiges Runge-Kutta-Verfahren* (*s-stufiges RK-Verfahren*). Dabei ist $c = (c_1, \ldots, c_s)^\top$ der *Knotenvektor*, $A = (a_{ij})$ die *Verfahrensmatrix*, $b = (b_1, \ldots, b_s)^\top$ der *Gewichtsvektor* und $s$ die *Stufenzahl*. □

Ist die Verfahrensmatrix $A$ eine strikt untere Dreiecksmatrix, d. h. $a_{ij} = 0$ für alle $j \geq i$, so lassen sich die Hilfsgrößen $u_{m+1}^{(i)}$, $i = 1, \ldots, s$, der Reihe nach explizit aus den Gleichungen (2.4.2) berechnen. Es liegt ein *explizites Runge-Kutta-Verfahren* vor. Ist mindestens ein $a_{ij} \neq 0$, $j \geq i$, so liegt ein *implizites Runge-Kutta-Verfahren* vor. Zur Berechnung von $u_{m+1}^{(i)}$ hat man dann i. Allg. ein nichtlineares Gleichungssystem zu lösen, was zu einem höheren Rechenaufwand führt. Derartige Verfahren können aber sehr gute Stabilitätseigenschaften besitzen, wir betrachten sie in Teil II.
Eine äquivalente Formulierung eines $s$-stufigen RK-Verfahrens ist gegeben durch

$$\begin{aligned} u_{m+1} &= u_m + h \sum_{i=1}^{s} b_i k_i(t_m, u_m, h) \\ k_i(t_m, u_m, h) &= f(t_m + c_i h, u_m + h \sum_{j=1}^{s} a_{ij} k_j(t_m, u_m, h)), \quad i = 1, \ldots, s. \end{aligned} \tag{2.4.3}$$

Die Darstellung (2.4.2) basiert auf den *Stufenwerten* $u_{m+1}^{(i)}$ und die Darstellung (2.4.3) auf den *Steigungswerten* $k_i(t_m, u_m, h)$.

**Lemma 2.4.1.** *Ein RK-Verfahren ist konsistent genau dann, wenn*

$$\sum_{i=1}^{s} b_i = 1$$

*gilt.*

**Figure 5.2:**

Die Koeffizienten eines $s$-stufigen RK-Verfahrens werden üblicherweise durch das von Butcher eingeführte Parameterschema („*Butcher-Schema*“) charakterisiert.

$$\begin{array}{c|ccc} c_1 & a_{11} & \cdots & a_{1s} \\ \vdots & \vdots & & \vdots \\ c_s & a_{s1} & \cdots & a_{ss} \\ \hline & b_1 & \ldots & b_s \end{array} \quad \text{bzw. in Matrixschreibweise} \quad \begin{array}{c|c} c & A \\ \hline & b^\top \end{array}$$

Tabelle 2.4.1: Butcher-Schema eines $s$-stufigen RK-Verfahrens

**Figure 5.3:**

## 5.2 Multistep-Methods

based on chapter 4 of book num gew dgl steif nichtsteif
Linear multistep methods use approxiamtions $u_{m+l}$ along the gridpoints $t_{m+l}$, $l = 0, 1, ..., k-1$ to calculate the new approximation $u_{m+k}$ at $t_{m+k}$. WE will first discuss topics related to the order of the methods depending on its parameters, stability and convergence.

**Definition 10.** *For given $\alpha_0, ..., \alpha_k$ and $\beta_0, ..., \beta_k$ the iteration rule*

$$\sum_{l=0}^{k} \alpha_l u_{m+l} = h \sum_{l=0}^{k} \beta_l f(t_{m+l}, u_{m+l}), \quad m = 0, 1, ..., N-k \tag{5.9}$$

*is called a* linear multistep method *(linear k-step method). It is always assumed that $\alpha_k \neq 0$ and $|\alpha_0| + |\beta_k| > 0$. If $\beta_k = 0$ holds, then the method is called explicit, otherwise implicit.*

Durch die Forderung $|\alpha_0| + |\beta_0| > 0$ ist die Schrittzahl $k$ eindeutig festgelegt. Im Falle $\beta_k = 0$ lässt sich die Näherungsfolge $\{u_{m+k}\}$, $m = 0, 1, \ldots, N - k$, direkt berechnen. Die Verfahrensvorschrift (4.2.1) liefert demzufolge für jedes äquidistante Gitter $I_h$ eine eindeutig bestimmte Gitterfunktion $u_h(t)$. Ist $\beta_k \neq 0$, so hat man zur Bestimmung von $u_{m+k}$ ein i. Allg. nichtlineares Gleichungssystem der Form

$$u_{m+k} = h\frac{\beta_k}{\alpha_k} f(t_{m+k}, u_{m+k}) + v, \tag{4.2.2}$$

zu lösen, wobei der von $u_{m+k}$ unabhängige Vektor $v$ durch

$$v = \frac{1}{\alpha_k} \sum_{l=0}^{k-1} \Big( h\beta_l f(t_{m+l}, u_{m+l}) - \alpha_l u_{m+l} \Big)$$



gegeben ist. Zur Lösung von (4.2.2) verwendet man für nichtsteife Systeme Funktionaliteration, d. h.

$$u_{m+k}^{(\varkappa+1)} = h\frac{\beta_k}{\alpha_k} f(t_{m+k}, u_{m+k}^{(\varkappa)}) + v, \quad \varkappa = 0, 1, \ldots$$

Unter der Schrittweiteneinschränkung

$$h \left| \frac{\beta_k}{\alpha_k} \right| L < 1, \tag{4.2.3}$$

wobei $L$ eine Lipschitz-Konstante für $f(t, y)$ darstellt, konvergiert die Folge $\{u_{m+k}^{(\varkappa)}\}$ bei beliebig vorgegebenem Startvektor $u_{m+k}^{(0)}$ gegen die eindeutige Lösung von (4.2.2). Für ein nichtsteifes Anfangswertproblem ist die Bedingung (4.2.3) an $h$ keine wesentliche Einschränkung.

Ein lineares Mehrschrittverfahren setzt sich zusammen aus zwei Bestandteilen:

1. Der *Startphase* zur Berechnung der Näherungswerte $u_1, \ldots, u_{k-1}$ in den Gitterpunkten $t_l = t_0 + lh$, $l = 1, \ldots, k-1$, die mit einem Einschrittverfahren, z. B. mit einem expliziten Runge-Kutta-Verfahren, oder mit Mehrschrittformeln niedriger Schrittzahlen und sehr kleinen Schrittweiten bestimmt werden können.
2. Der *Laufphase*, d. h. einer Mehrschrittformel (4.2.1) zur sukzessiven Berechnung der Approximationen $u_{m+k}$ in den Gitterpunkten $t_{m+k}$.

**Figure 5.4:**

A linear multi-step method consists of two parts:

1. In the *starting-phase* approximations $u_1, ..., u_{k-1}$ for the first $k-1$ gridpoints $t_l = t_0 + th, l = 1, ..., k-1$ are calculated using a single-step method. For example using an explicit Runge-Kutta Methodor a multi-step method with fewer steps.

2. In the *run-phase* the multi-step formula is used to determine new approximations $u_{m+k}$ for the gridpoint $t_{m+k}$

For theoretical analysis of the multi-step methods we consider the generating polynomials

$$\rho(x) := \sum_{l=0}^{k} \alpha_l x^l \tag{5.10}$$

$$\sigma(x) := \sum_{l=0}^{k} \beta_l x^l \tag{5.11}$$

Eine zentrale Rolle bei der theoretischen Untersuchung linearer Mehrschrittverfahren spielen die beiden *erzeugenden Polynome*

$$\rho(\xi) := \alpha_k \xi^k + \alpha_{k-1}\xi^{k-1} + \cdots + \alpha_0$$
$$\sigma(\xi) := \beta_k \xi^k + \beta_{k-1}\xi^{k-1} + \cdots + \beta_0.$$

Sie wurden erstmals von Dahlquist [79] zur Stabilitätsuntersuchung linearer Mehrschrittverfahren verwendet. Mit den erzeugenden Polynomen lassen sich die Konsistenzbedingungen (4.2.7) in der Form

$$\rho(1) = 0 \text{ und } \rho'(1) = \sigma(1). \tag{4.2.11}$$

**Figure 5.5:**

### 5.2.1 Consistency and order

local discretization error - def 4.2.2

**Definition 11.** *Let $\tilde{u}_{m+k}$ be the result of one step of the multi-step method 5.9 with the start-vectors $u_m, u_{m+1}, ..., u_{m+k-1}$ lying on the exact solution $y(t)$ of the problem* ***reference****. This means*

$$\alpha_k \tilde{u}_{m+k} = \sum_{l=0}^{k-1} (h\beta_l f(t_{m+l}, y(t_{m+l})) - \alpha_l y(t_{m+l})) + h\beta_k f(t_{m+k}, \tilde{u}_{m+k}).$$

*Then*

$$le_{m+k} = le(t_{m+k}) = y(t_{m+k}) - \tilde{u}_{m+k}, \quad m = 0, 1, ..., N-k$$

*is called the* local discretization error *(local error) of the linear multi-step method 5.9 at the point* $t_{m+k}$.

We will assign the linear difference opreator

$$L[y(t), h] = \sum_{l=0}^{k} \left( \alpha_l y(t + lh) - h\beta_l y'(t + lh) \right) \tag{5.12}$$

to the local discretization error. Using this we gain the following definition.

**Definition 12.** *A linear multi-step method is called* preconsistent *if for all functions* $y(t) \in C^1[t_0, t_l]$

$$\lim_{h \to 0} L[y(t), h] = 0$$

*holds. It is called* consistent, *if for all functions* $y(t) \in C^2[t_0, t_l]$

$$\lim_{h \to 0} \frac{1}{h} L[y(t), h] = 0$$

*holds. It has the* consistency order p, *if for all functions* $y(t) \in C^{p+1}[t_0, t_l]$

$$L[y(t), h] = \mathcal{O}(h^{p+1}) \quad \textit{for} \quad h \to 0$$

*holds.*

schreiben. Die Bedingung für die Präkonsistenz lautet

$$\rho(1) = \sum_{l=0}^{k} \alpha_l = 0.$$

Führt man den Shiftoperator (Verschiebungsoperator) $E_h$ durch

$$E_h u_m = u_{m+1}, \quad E_h f(t_m, u_m) = f(t_{m+1}, u_{m+1}) \tag{4.2.12}$$

ein, so lässt sich das lineare Mehrschrittverfahren (4.2.1) in der kompakten Form

$$\rho(E_h)u_m = h\sigma(E_h)f(t_m, u_m)$$

schreiben. Die erzeugenden Polynome $\rho$ und $\sigma$ definieren also Differenzenoperatoren

$$\rho(E_h)u_m = \sum_{l=0}^{k} \alpha_l u_{m+l}, \quad \sigma(E_h)f(t_m, u_m) = \sum_{l=0}^{k} \beta_l f(t_{m+l}, u_{m+l}).$$

**Lemma 4.2.2.** *Ein lineares Mehrschrittverfahren besitzt die Konsistenzordnung $p$ genau dann, wenn die Bedingung*

$$\rho(e^h) - h\sigma(e^h) = \mathcal{O}(h^{p+1}), \quad h \to 0 \tag{4.2.13}$$

*bzw.*

$$\frac{\rho(\xi)}{\ln \xi} - \sigma(\xi) = \mathcal{O}((\xi - 1)^p), \quad \xi \to 1 \tag{4.2.14}$$

*erfüllt ist.*

**Figure 5.6:**

### 5.2.2 Convergence and stability

**Definition 13.** *A linear multi-step method is called* zero-stable *if all solutions of the difference equation*

$$\sum_{l=0}^{k} \alpha_l u_{m+l} = 0$$

*are bounded.*

**Theorem 5.** *A linear multi-step method is zero-stable, if and only if the polynomial $\rho(x)$ fullfills the root-condition, this means:*

1. *All roots $\bar{x}$ of $\rho(x)$ are within the unit-circle $|\bar{x}| \leq$ in the complex plane.*
2. *All roots $\bar{x}$ with $|x| = 1$ are singular.*

**from circuit book below, above from modelling book**

## 5.3 Implicit linear multi-step formulas

These kinds of multi-step methods are conventionally used to numerically solve the systems obtained using modified nodal analysis.

The conventional approach can be split into three main steps:

1. Computation of consistent initial values
2. numerical integration based on multi-step schemes
3. transformation of the DAE into a nonlinear system and its numerical solutioon by Newton's procedure (??????????????????will not be discussed further because not very specific)

**Consistant initial values**

Consistent initial values. The first step in the transient analysis is to compute consistent initial values $(x_0, y_0)$ for the initial time point $t_0$. In the index-1 case, this can be done by performing a steady state (DC operating point) analysis, i.e. to solve

$$\mathcal{F}(0, x_0, t_0) = 0 \tag{10.2}$$

for $x_0$ and then set $y_0 := g(x_0)$. If there are no controlled sources, the Jacobian $\partial\mathcal{F}/\partial x$ of (10.2) with respect to $x_0$ reads

$$\frac{\partial \mathcal{F}}{\partial x} = \begin{pmatrix} \widetilde{G}(A_R^\top u_0, t_0) & A_L & A_V \\ -A_L^\top & 0 & 0 \\ -A_V^\top & 0 & 0 \end{pmatrix}$$

with the definition $\widetilde{G}(A_R^\top u, t) := A_R G(A_R^\top u, t) A_R^\top$ already introduced in Section 4. Since $\ker(\partial\mathcal{F}/\partial x) = \ker(A_R, A_L, A_V)^\top \times \ker(A_L, A_V)$ holds, the matrix is only regular, if there are neither loops of independent voltage sources and/or inductors, nor cutsets of independent current sources and/or capacitors. If these topological conditions are violated, no steady state solution can be computed, and so most circuit analysis programs check and refuse these circuit configurations. Additional assumptions are implied in the case of controlled sources. But note that in the nonlinear case the Jacobian matrix also may become numerically singular, e. g. due to vanishing partial derivatives or in the case of bifurcation.

An approach always feasible in the index-1 case is to extract the algebraic constraints using the projector $Q_C$ onto $\ker A_C^\top$:

$$\begin{aligned} Q_C^\top(A_R r(A_R^\top u, t) + A_L \jmath_L + A_V \jmath_V + A_I i(u, \jmath_L, \jmath_V, t)) &= 0 \\ v(u, \jmath_L, \jmath_V, t) - A_V^\top u &= 0. \end{aligned}$$

If the index-1 topological conditions hold, this nonlinear system uniquely defines for $t = t_0$ the algebraic components $Q_C u_0$ and $\jmath_{V,0}$ for given (arbitrary) differential components $(I - Q_C)u_0$ and $\jmath_{L,0}$. The derivatives $\dot{y}_0$ have then to be chosen such that $A\dot{y}_0 + f(x_0, t_0) = 0$ holds.

**Figure 5.7:**

**Nunmerical integration.**

### 5.3.1 BDF schemes and trapezoidal rule