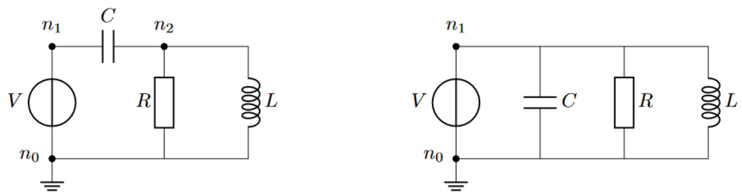
Numerical Methods for Differential-Algebraic Equations

Exercise Sheet 7, June 26, 2024, 10:15-11:45, Room S2 346

We study the modeling of electric circuits by the modified nodal analysis (MNA), the index of the resulting differential algebraic equations, and their simulation.

General setting. The circuit is described by a finite directed and connected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ with $\mathcal{N} = \{n_0, \ldots, n_N\}$ the set of nodes and $\mathcal{E} = \{e_1, \ldots, e_M\}$ the set of edges. The connectivity is represented by the reduced incidence matrix A. Each edge of the graph represent a physical devie, i.e., a resistor (R), capacitor (C), inductor (L), voltage source (V) or current source (I). Accordingly, the (reduced) incidence matrix is split as $A = [A_C, A_R, A_L, A_V, A_I]$.

Exercise 1. Setup the reduced incidence matrix for the following two circuits.



Exercise 2. The modified nodal analysis leads to a system

$$A_C C A_C^\top \dot{u} + A_R G A_R^\top u + A_L^\top i_L + A_V^\top i_V = -A_I i_{src} \tag{1}$$

$$L \dot{i}_L - A_L u = 0 \tag{2}$$

$$-A_V^\top u = -v_{src} \tag{3}$$

Set up the corresponding systems for the circuits of the previous example.

Under some (simple) topological conditions on the connectivity of the circuit, the MNA leads to a regular system of DAEs whose index can be characterized in detail.

Theorem (see GFtM'2005). Let $A = [A_C, A_R, A_L, A_V, A_I]$ denote the reduced incidence matrix of the connected graph $\mathcal{G}$ representing the circuit. Assume that C, G, L are symmetric and positive definite matrices representing the capacitances, conductances, and inductances of the corresponding devices. Then

(i) The MNA leads to a regular DAE of the form $\mathcal{M}\dot{x} + \mathcal{A}x = b(t)$ if, and only if,

$$N(A_V) = \{0\} \qquad \text{and} \qquad N([A_C, A_R, A_L, A_V]^\top) = \{0\}. \tag{I2}$$

Moreover, the index is bounded by $\nu = \nu_d = \nu_p = \mathrm{ind}(\mathcal{M}, \mathcal{A}) \le 2$.

(ii) Let $[Q_C, P_C]$ be orthogonal such that $P_C^\top Q_C = 0$, $A_C^\top Q_C = 0$, and $A_C^\top Q_C$ is regular. Then the MNA system has index $\nu \leq 1$ if, and only if

(I1) $$N(Q_C^\top A_V) = \{0\} \qquad \text{and} \qquad N([A_C, A_R, A_V]^\top) = \{0\}.$$

(iii) The index is $\nu = 0$ if, and only if,

(I0) $$A_V = \emptyset \qquad \text{and} \qquad N(A_C^\top) = \{0\}.$$

The first part of (i) and assertion (iii) were already proven in the lecture. For the second part of (i), we refer to the literature.

Proof (ii). We split $u = P_C v_C + Q_C u_c$ and correspondingly split (1) by multiplying from left with $P_C^\top$ and $Q_C^\top$, respectively. This yields

$$\begin{aligned} P_C^\top A_C C A_C^\top P_C \dot{u}_C + P_C^\top [A_R G A_R^\top (P_C u_C + Q_C v_C) + A_L i_L + A_V i_V] &= -P_C^\top A_I i_{src} \\ L \dot{I}_L - A_L^\top [P_C u_C + Q_C v_C] &= 0 \\ Q_C^\top [A_R G A_R^\top (P_C u_C + Q_C v_C) + A_L i_L + A_V i_V] &= -Q_C^\top A_I i_{src} \\ -A_V^\top i_V &= -v_{src}. \end{aligned}$$

We note that, by assumption, the matrix $A_V^\top P_C$ is injective, and hence $P_C^\top A_C C A_C^\top P_C$ is regular. Hence the above system can be brought into the form

$$\begin{aligned} \dot{y} &= f(y, z) \\ 0 &= g(y, z) \end{aligned}$$

with differential and algebraic variable $y = (u_C, i_L)$ and $z = (v_C, i_V)$. We show that $g(y, z)$ satisfies the index-1 condition: $g_z(y, z)$ regular. By differentiating the second and third condition w.r.t. $z = (v_C, i_V)$, we get

$$g_z(y, z) = \begin{pmatrix} Q_C^\top A_R G A_R^\top Q_C & A_V \\ -A_V^\top & 0 \end{pmatrix}$$

We recall that a block-matrix of the form $\begin{pmatrix} a & b^\top \\ b & 0 \end{pmatrix}$ is regular, if and only if (i) $b^\top$ is injective (resp. b is surjective) and a is regular on the nullspace $N(b)$. The first condition means $N(b^\top) = N(Q_C^\top A_V) = \{0\}$, which ammounts to the first condition in (I1). Recall that the columns of Q_C are a basis for $N(A_C^\top)$. So the second condition says that $N(A_V^\top) \cap N(A_C^\top) \cap N(A_R^\top) = \{0\}$. This is equivalent to the second condition of (I1). $\square$

Exercise 3. Use the above theorem to determine the index of the two system considered in the previous exercises.

The index-1 conditions (I1) can be interpreted as follows (see GFtM'2005):

- $N(Q_C^\top A_V) = \{0\}$ means that the circuit has no loops that only contain C and V elements;
- $N([A_C, A_R, A_V]^\top) = \{0\}$ means that the circuit has no cutsets consisting only of L and I elements.

Exercise 4. Apply the above assertions to the examples of the previous exercises.

Simulation. We now study the numerical simulation of circuit equations by means of simple RK- and multistep methods. In particular, we consider

(a) the implicit Euler (=BDF-1) method;
(b) the implicit trapezoidal rule;
(c) the BDF-2 method.

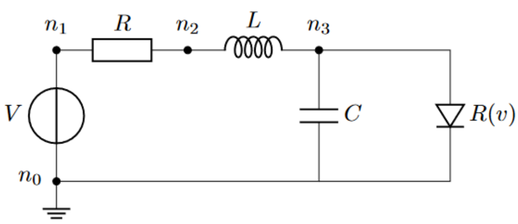
For all simulations, we use initial data $(u_0, i_L(0), i_V(0)) = 0$ and $v_{src}(t) = \sin(\pi t)$. The parameters in the remaining device models are set to $R = C = L = 1$.

Exercise 5. Discuss the consistency of the initial values for the two examples that were investigated in the previous exercises.

Exercise 6. Conduct simulations with the methods stated above up to final time $T = 10$ with time step size $h = 1, 1/2, 1/4, \ldots$ Discuss the results of your tests.

Exercise 7. Repeat your simulations with appropriate Matlab ode-solvers (e.g. `ode15s`, `ode23s`, `ode23s`, `ode23t`, `ode23b`, `ode15i`). Search in the documentation to gather information about these solvers.

Exercise 8. The following circuit represent a model for a tunnel diode oscillator



As model parameters, we choose $v_{src} = 0.25 \cdot \sin(100\pi t)$[V], $R = 1[\Omega]$, $L = 2\dot{1}0^{-3}$[H], $C = 10^{-7}$[F]. The diode is modeled as nonlinear resistor with

$$i = 1.8\, v - 8.766\, v^2 + 10.8\, v^3 =: G(v)v.$$

Setup the circuit equations using the modified nodal analysis, determine the index of the system, and conduct simulations over multiple time periods. Plot the node potentials $u_i(t)$ as functions of time and discuss the effect of the diode.