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Circuit Modelling

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Example: Charging of a capacitor:

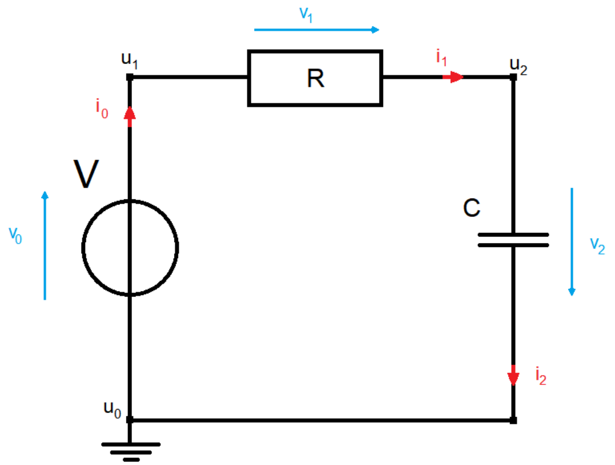


Figure: charging capacitor with series resistor and voltage source

Formulating a Mathematical Model

Network Topology

For a circuit with l nodes and k edges, define the incidence matrix $\tilde{A} = (\tilde{a}_{ij}) \in \mathbb{R}^{k \times l}$:

$$\tilde{a}_{ij} = \begin{cases} 1 & \text{edge } j \text{ starts at node } i, \\ -1 & \text{edge } j \text{ ends at node } i, \\ 0 & \text{else.} \end{cases}$$

By grounding node 0, i.e. $u_0 = 0$ we obtain the reduced incidence matrix $\rightarrow A$.

Formulating a Mathematical Model

Energy Conservation Laws

- **Kirchhoff's voltage law (KVL):**
  The sum of voltages along each loop of the network must equal to zero.

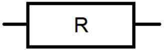
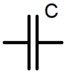
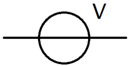
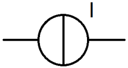
$$\to A^\top u = v. \tag{1}$$

- **Kirchhoff's current law (KCL):**
  For any node, the sum of currents flowing into the node is equal to the sum of currents flowing out of the node.

$$\to Ai = 0. \tag{2}$$

Formulating a Mathematical Model

Electrical Components and their Relations

Name	Symbol	Component Law
Resistor		$v = R\,i$ or $i = G\,u$
Capacitor		$Q = C\,v$ and by derivation in t $I = C\,\frac{d}{dt}v = C\,v'$
Inductor		$\Phi = L\,i$ and by derivation in t $v = L\,i'$
Voltage Source		$v = v_{src}$
Current Source		$i = i_{src}$

Formulating a Mathematical Model

Modified Nodal Analysis - MNA

Rearrange reduced incidence matrix into

$$A = (A_R A_C A_L A_V A_I)$$

A_R, A_C, A_L, A_V and A_I ... columns related to components
Represent voltages:

$$v = A^\top u$$

$\rightarrow$ rearrange v into $v = (v_R, v_C, v_L, v_{src}, v_I)$ and i into $i = (i_R, i_C, i_L, i_V, i_{src})$. Rewrite component relations:

$$i_R = G\, v_R = G\, A_R^\top u,$$
$$i_C = C\, v_C' = C\, A_C^\top u'.$$

Kirchhoffs current law:

$$A_C i_C + A_R i_R + A_L i_L + A_V i_V = -A_I i_{src}.$$

Kirchhoffs current law with component relations combine to

$$\begin{aligned} A_C C A_C^\top u' + A_R G A_R^\top u + A_L i_L + A_V i_V &= -A_I i_{src}, \\ L i_L' - A_L^\top u &= 0, \\ -A_V^\top u &= -v_{src}. \end{aligned}$$

In matrix form:

$$\begin{pmatrix} A_C C A_C^\top & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{pmatrix} * \begin{pmatrix} u' \\ i_L' \\ i_V' \end{pmatrix} + \begin{pmatrix} A_R G A_R^\top & A_L & A_V \\ -A_L^\top & 0 & 0 \\ -A_V^\top & 0 & 0 \end{pmatrix} * \begin{pmatrix} u \\ i_L \\ i_V \end{pmatrix} = \begin{pmatrix} -A_I i_{src} \\ 0 \\ -v_{src} \end{pmatrix}. \tag{3}$$

$\rightarrow$ differential and algebraic variables

Differential Algebraic Equations

Types of DAEs

In the most general form a DAE can be written as: Find $y : \mathbb{R} \to \mathbb{R}^n$ such that

$$F(t, y(t), y'(t)) = 0, \qquad \forall t \in I \tag{4}$$

with $F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ sufficiently smooth and I the time-interval.

Linear systems with constant coefficients

find y such that

$$Ay'(t) + By(t) = f(t), \tag{5}$$

with $A, B \in \mathbb{R}^{n \times n}$, A singular, B and $f : \mathbb{R} \to \mathbb{R}^n$ a function in time. $\to$ differential and algebraic variables

Differential Algebraic Equations

Weierstrass-Kronecker normalform

Equivalence transformations lead to

$$\begin{aligned} u'(t) + Ru(t) &= s(t), \\ Nv'(t) + v(t) &= q(t), \end{aligned} \tag{6}$$

where N is a nilpotent matrix and the matrix R is regular.

First equation is an ODE $\rightarrow$ unique solution $u(t)$ for any initial values. Construct explicit solution for second equation: $q(t) \in C^{k-1}([t_0, t_l])$:

$$
\begin{aligned}
v(t) &= q(t) - Nv'(t) = q(t) - N(\underbrace{q(t) - Nv'(t)}_{=v(t)})' = q - Nq' + N^2 v'' \\
&= q - Nq' + N^2(q - Nv')'' = q - Nq' + N^2 q'' - N^3 v''' \\
&\vdots \\
&= q - Nq' + ... + (-1)^{k-1} N^{k-1} \underbrace{\frac{d^{k-1}}{dt^{k-1}} q}_{:= q^{(k-1)}} + (-1)^{k-1} \underbrace{N^k v^{(k)}}_{=0} \\
&= \sum_{i=0}^{k-1} (-1)^i N^i q^{(i)}(t)
\end{aligned}
$$

$\rightarrow$ differentiation index $\nu = k$.

Differential Algebraic Equations

Index of a Differential Algebraic Equation

Definition

Consider the differential algebraic equation (4) to be uniquely solvable and F sufficiently smooth. For a given $m \in \mathbb{N}$ consider the equations

$$\begin{aligned} F(t, y, y') &= 0, \\ \frac{dF(t, y, y')}{dt} &= 0, \\ &\vdots \\ \frac{d^m F(t, y, y')}{dt^m} &= 0. \end{aligned}$$

The smallest natural number m for which the above system results in an explicit system of ordinary differential equations (ODEs), i.e. it has the form

$$y' = \phi(t, y),$$

Index Analysis of the Modified Nodal Analysis

Topological Conditions

- The resulting equations have index $\nu \leq 2$, if the circuit neither contains loops of only voltage sources nor cutsets of only current sources.
- They have index $\nu \leq 1$, if the circuit contains neither loops of only capacitors and/or voltage sources nor cutsets of only inductors and/or current sources.
- They have index $\nu = 0$, if every node in the circuit is connected to the reference node (ground) through a path containing only capacitors.

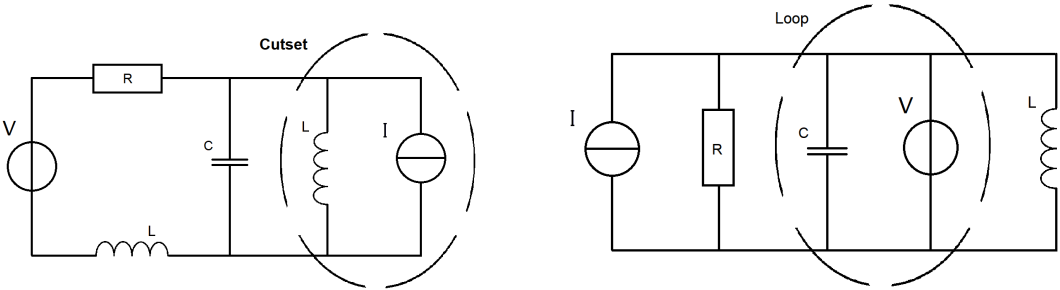


Figure: Illustration of a cutset and a loop.

Theorem (Index conditions)

Let the matrices of the capacitances, inductances and resistances be positive definite.

- *If*

$$\ker([A_R, A_C, A_V, A_L]^\top) = \{0\} \quad \textit{and} \quad \ker(A_V) = \{0\} \tag{7}$$

  holds, then the MNA (3) *leads to a system with index* $\nu \leq 2$*.*

- *If additionally*

$$\ker([A_R, A_C, A_V]^\top) = \{0\} \quad \textit{and} \quad \ker([A_C, A_V]) = \{0\} \tag{8}$$

  holds, then the system is of index $\nu \leq 1$

- *If further*

$$\ker(A_C^\top) = \{0\} \quad \textit{and} \quad \dim(v_{src}) = 0 \tag{9}$$

  holds, then the system has index $\nu = 0$*.*

Numerical Solutions

Multistep Methods

Definition (Multistep method)

For given $\alpha_0, ..., \alpha_k$ and $\beta_0, ..., \beta_k$ the iteration rule

$$\sum_{l=0}^{k} \alpha_l y_{m+l} = h \sum_{l=0}^{k} \beta_l f(t_{m+l}, y_{m+l}), \quad m = 0, 1, ..., N-k \tag{10}$$

is called a *linear multistep method* (linear k-step method). It is always assumed that $\alpha_k \neq 0$ and $|\alpha_0| + |\beta_k| > 0$. If $\beta_k = 0$ holds, then the method is called explicit, otherwise implicit.

Definition (generating polynomials)

$$\rho(x) := \sum_{l=0}^{k} \alpha_l x^l \qquad \text{and} \qquad \sigma(x) := \sum_{l=0}^{k} \beta_l x^l$$

Definition (Convergence order)

We say that a linear multi-step method is convergent of order $p \in \mathbb{N}$, if for a solution y of the problem and a vector $(y_j)_{j=0}^k$ created by an LMSM, we have that

$$\max_{0 \leq j \leq k} \|y(t_j) - y_j\| \leq Ch^p.$$

Where C is a constant not dependent on the step size h.

Consistency on tells us whether the method approximates the equation correctly as the step size approaches 0.

Numerical Solutions

Multistep Methods

Stability properties

Definition

1. The set
$$S := \{z \in \mathbb{C} : \rho(\xi) - z\sigma(\xi) = 0 \implies \xi \in \mathbb{C} \text{ and } |\xi| \le 1. \\ \text{If } \xi \text{ has multiplicity greater than 1, then } |\xi| < 1\} \tag{11}$$
is called the region of stability of the method.
2. A linear multistep method is called
   - *0-stable*, if $0 \in S$.
   - stable in the point $z \in \mathbb{C}$, if $z \in S$.
   - $A(\alpha)$-*stable*, if it is stable in all z that lie within the set $\{z \in \mathbb{C}^- : |\arg(z) - \pi| \le \alpha\}$ for $\alpha \in (0, \frac{\pi}{2})$.

Theorem

Let $f(t, y)$ *be sufficiently smooth and the linear multi-step method be 0-stable and consistent of order* p*, then it is also convergent of order* p*.*

Numerical Solutions

Consistent Initial Values

Index $\nu = 0$**:** no additional restrictions (ODE case).
Index $\nu = 1$**:**
Rewrite system into the form

$$\begin{aligned} y'(t) &= f(t, y(t), z(t)), \\ 0 &= g(t, y(t), z(t)). \end{aligned}$$

Conditions for consistent initial values $\rightarrow g(t_0, y_0, z_0) = 0$.

Index $\nu = 2$:
Rewrite system into the form

$$
\begin{aligned}
y' &= f(t, y(t), z(t)), \\
0 &= g(t, y(t)).
\end{aligned}
$$

Conditions for consistent initial values

$$
\begin{aligned}
g(t_0, y_0) &= 0, \\
g_t(t_0, y_0) + g_y(t_0, y_0) f(t_0, y_0, z_0) &= 0. \qquad \text{(hidden constraint)}
\end{aligned}
$$

Numerical Solutions

Implicit Linear Multistep Formulas

BDF-k Methods

The *backward differentiation formula (BDF)* is a family of implicit linear multistep methods.

$$\sum_{k=0}^{s} \alpha_k y_{n+k} = h\beta f(t_{n+s}, y_{n+s})$$

The BDF or BDF-k formulas for $k = 1, ..., 3$ have the following form

$$k = 1 : hf_{m+1} = y_{m+1} - y_m \qquad \text{(implicit euler)}$$

$$k = 2 : hf_{m+2} = \frac{1}{2}(3y_{m+2} - 4y_{m+1} + y_m)$$

$$k = 3 : hf_{m+3} = \frac{1}{6}(11y_{m+3} - 18y_{m+2} + 9y_{m+1} - 2y_m)$$

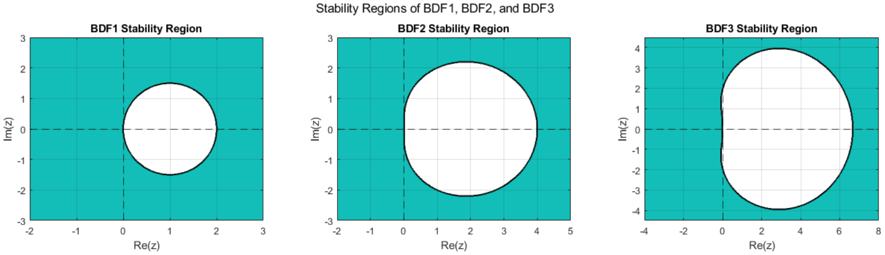


Figure: stability regions of BDF-schemes

Theorem

The BDF-k methods have consistency order $p = k$.

Corollary (Convergence rate)

The BDF-k methods with $k \leq 6$ *are convergent with order* k

Numerical Solutions

Implicit Linear Multistep Formulas

Trapezoidal rule

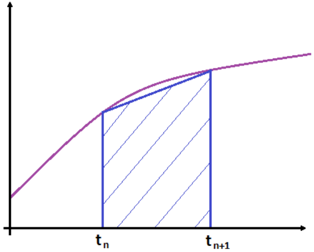


Figure: illustration of the trapezoidal rule

This procedure is repeated for small subsections of the interval $[a, b]$. Thus we obtain the iteration formula

$$y_{n+1} = y_n + \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_{n+1})].$$

The trapezoidal rule has convergence order $p = 2$.

Numerical Solutions

Numerical Examples

Example1

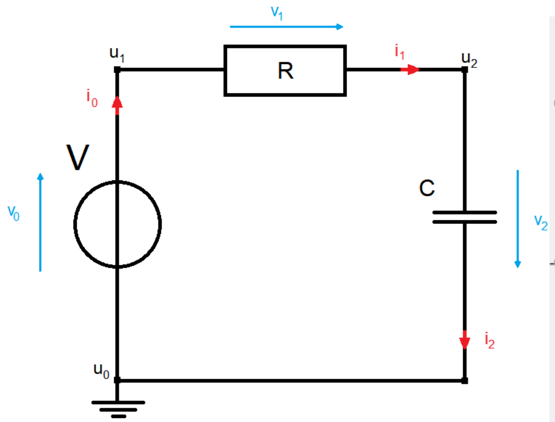


Figure: charging capacitor with series resistor and voltage source

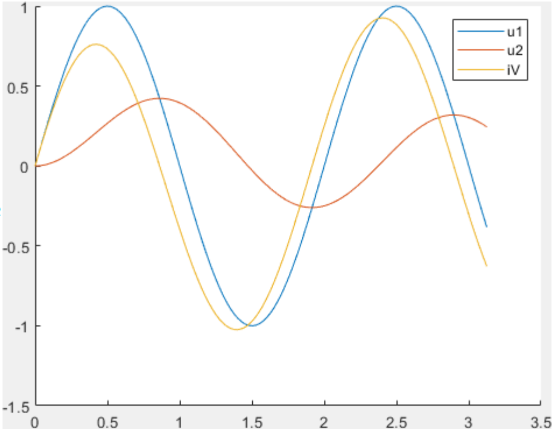


Figure: Exact solution for example 1.

h	k = 1		k = 2		k = 3		trapezoidal	
	$\mathrm{err}(u_2)$	$\mathrm{err}(i_V)$	$\mathrm{err}(u_2)$	$\mathrm{err}(i_V)$	$\mathrm{err}(u_2)$	$\mathrm{err}(i_V)$	$\mathrm{err}(u_2)$	$\mathrm{err}(i_V)$
0.1	4.620×10^{-2}	4.620×10^{-2}	9.567×10^{-3}	9.567×10^{-3}	2.852×10^{-3}	2.852×10^{-3}	3.344×10^{-3}	3.344×10^{-3}
0.05	2.339×10^{-2}	2.339×10^{-2}	2.454×10^{-3}	2.454×10^{-3}	3.645×10^{-4}	3.645×10^{-4}	8.367×10^{-4}	8.367×10^{-4}
0.025	1.178×10^{-2}	1.178×10^{-2}	6.264×10^{-4}	6.264×10^{-4}	4.928×10^{-5}	4.928×10^{-5}	2.092×10^{-4}	2.092×10^{-4}

Table: Resulting errors for the BDF-k methods and the trapezoidal rule.

Numerical Solutions

Numerical Examples

Gauss

Gauss-1 method or implicit midpoint rule:

$$y_{n+1} = y_n + hf(t_n + \frac{h}{2}, \frac{y_n + y_{n+1}}{2})$$

different convergence rates for algebraic and differential variables:

index $\nu = 1$		index $\nu = 2$	
differential	algebraic	differential	algebraic
2	2	2	0

Table: Convergence order for Gauss method.

h	example 1			example 2		example 3	
	$\mathrm{err}(u_1)$ (alg)	$\mathrm{err}(u_2)$ (diff)	$\mathrm{err}(i_0)$ (alg)	$\mathrm{err}(u_1)$ (diff)	$\mathrm{err}(i_L)$ (diff)	$\mathrm{err}(u_1)$ (diff)	$\mathrm{err}(i_V)$ (alg)
0.1	1.247×10^{-2}	2.141×10^{-3}	1.299×10^{-2}	6.589×10^{-3}	7.885×10^{-3}	1.247×10^{-2}	1.318×10^{-1}
0.05	3.092×10^{-3}	5.343×10^{-4}	3.252×10^{-3}	1.649×10^{-3}	1.974×10^{-3}	3.092×10^{-3}	3.270×10^{-2}
0.025	7.716×10^{-4}	1.335×10^{-4}	8.115×10^{-4}	4.123×10^{-4}	4.936×10^{-4}	7.716×10^{-4}	8.153×10^{-3}

Table: Resulting errors for the Gauss method with one stage.

Definition

Let $y(t)$ be the exact solution of (4) and $\tilde{y}(t)$ be the solution of the perturbed system $F(t, \tilde{y}, \tilde{y}') = \delta(t)$. The smallest number $k \in \mathbb{N}$ such that

$$\|y(t) - \tilde{y}(t)\| \leq C \left(\|y(t_0) - \tilde{y}(t_0)\| + \sum_{j=0}^{k} \max_{t_0 \leq \xi \leq T} \left\| \int_{t_0}^{\xi} \frac{d^j \delta}{d\tau^j}(\tau) d\tau \right\| \right)$$

for all $\tilde{y}(t)$, is called the **perturbation index** of this system.

Dahlquist test problem as a model problem, find y such that

$$y' = \lambda y, \quad t > 0 \tag{12}$$
$$y(0) = y_0 \tag{13}$$

with $\lambda \in \mathbb{C}$ and y_0 fixed.
Thus the resulting linear multistep method is of the form

$$\sum_{l=0}^{k} \alpha_l y_{n+l} = h \sum_{l=0}^{k} \beta_l \lambda y_{n+l}$$
$$\iff \sum_{l=0}^{k} [\alpha_l - h\beta_l\lambda] y_{n+l} = 0$$

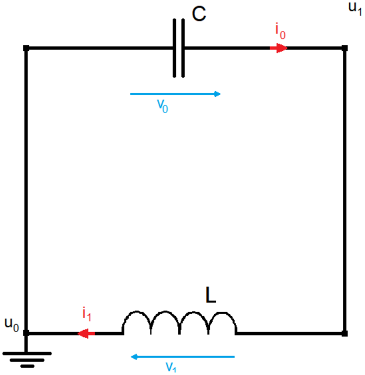


Figure: LC-circuit

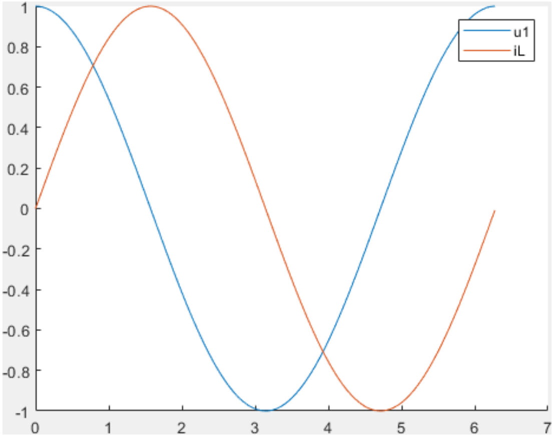


Figure: Exact solution for example 2.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} u_1' \\ i_L' \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} * \begin{pmatrix} u_1 \\ i_L \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

h	k = 1		k = 2		k = 3		trapezoidal	
	$\mathrm{err}(u_1)$	$\mathrm{err}(i_l)$	$\mathrm{err}(u_1)$	$\mathrm{err}(i_l)$	$\mathrm{err}(u_1)$	$\mathrm{err}(i_l)$	$\mathrm{err}(u_1)$	$\mathrm{err}(i_l)$
0.1	2.659×10^{-1}	2.106×10^{-1}	1.686×10^{-2}	1.917×10^{-2}	1.286×10^{-3}	1.042×10^{-3}	4.007×10^{-3}	5.140×10^{-3}
0.05	1.446×10^{-1}	1.125×10^{-1}	4.266×10^{-3}	5.067×10^{-3}	1.501×10^{-4}	1.131×10^{-4}	1.003×10^{-3}	1.301×10^{-3}
0.025	7.543×10^{-2}	5.817×10^{-2}	1.070×10^{-3}	1.294×10^{-3}	1.460×10^{-5}	9.674×10^{-6}	2.507×10^{-4}	3.268×10^{-4}

Table: Resulting errors for the BDF-k methods and the trapezoidal rule.

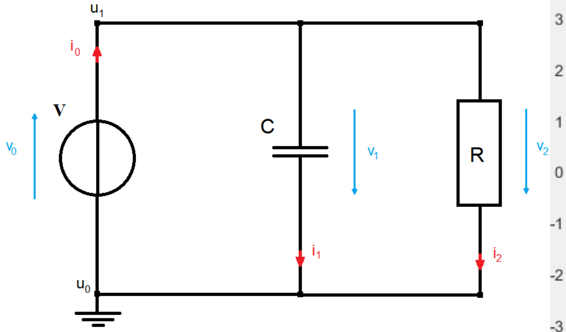


Figure: Current source with capacitor and resistor.

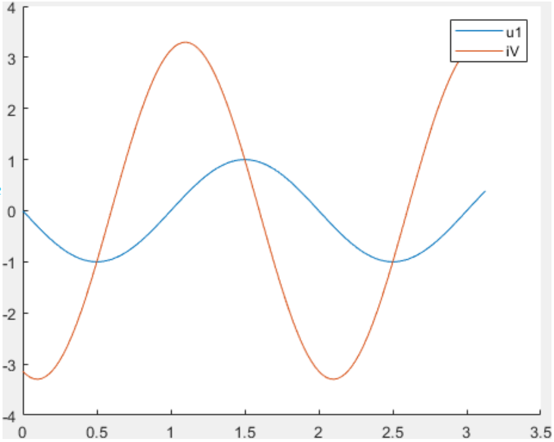


Figure: Exact solution for example 3.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} * \begin{pmatrix} u_1' \\ i_V' \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} u_1 \\ i_V \end{pmatrix} = \begin{pmatrix} 0 \\ -\sin(\pi t) \end{pmatrix}.$$

h	k = 1 $\mathrm{err}(i_V)$	k = 2 $\mathrm{err}(i_V)$	k = 3 $\mathrm{err}(i_V)$	trapezoidal $\mathrm{err}(i_V)$
0.1	4.894×10^{-1}	1.023×10^{-1}	2.403×10^{-2}	5.219×10^{-2}
0.05	2.462×10^{-1}	2.577×10^{-2}	3.034×10^{-3}	1.295×10^{-2}
0.025	1.233×10^{-1}	6.456×10^{-3}	4.029×10^{-4}	3.232×10^{-3}

Table: Resulting errors for the BDF-k methods and the trapezoidal rule.