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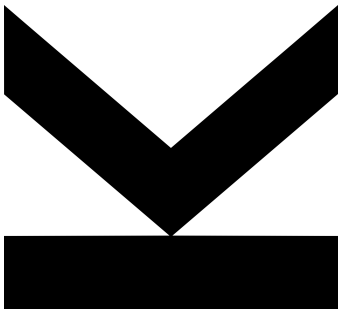
Circuit Modelling



Bachelor Thesis

to obtain the academic degree of

Bachelor of Science

in the Bachelor's Program

Technische Mathematik

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Kurzfassung

Kapitel zwei gibt Überblick über die Physikalischen Zusammenhänge und deren Mathematische Beschreibung. Anschließend wird ein Mathematisches Modell nach der Methode der ”Modified Nodal Analysis“ erstellt. Im dritten Kapitel analysieren wir die Systeme, welche durch diese Analyse entstehen, wobei das vierte Kapitel eine wichtige Eigenschaft dieser Systeme genauer untersucht. Das letzte Kapitel beschäftigt sich mit der Numerischen Lösung dieser Systeme.

→ je etwas mehr Text zu den Kapiteln (Struktur der Kapitel)

Abstract

Abstract in English.

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1 Introduction

This chapter should include information about what circuit modelling wants to achieve as well as giving an overview of what this bachelor-thesis is about.

What is this thesis about?
Modelling and numerically solving systems that arrise from electrical circuits with RLC elements. Furthermore it will briefly discuss on expanding this baseline with more complicated electrical components. What is the goal of this thesis?
The goal of this thesis is to give insight into the state of the art of circuit modelling. It aims to elaborate on the underlying concepts of MNA as well as on the most commonly used numerical methods.

2 Formulating a Mathematical Model

To accurately represent a physical system in a mathematical model we first have to think about **how** to best formulate this mathematical model. This chapter aims to elaborate on the Modified Nodal Analysis (MNA) of an electrical circuit, which is the common choice for these kinds of models. For this we have to introduce the notions of network topology as well as some basic electrical components and physical laws. Most of the following notions such as voltage, current and electrical potentials can depend on time. For better readability we will leave out the time argument t in some cases. This chapter is based on [2].

2.1 Network Topology

An electrical circuit is usually considered as a graph $(\mathcal{N}, \mathcal{E})$ where $\mathcal{N} = (n_0, n_1, n_2, ..., n_k)$ denotes nodes and $\mathcal{E} = \{e_j : j = 1, ..., l\}$ is the set of edges where $|\mathcal{N}| = k$ is the number of nodes and $|\mathcal{E}| = l$ the number of edges. If for some i and some j the nodes n_i and n_j are connected, then there exists an edge connecting those two nodes. We can store this information in an *incidence matrix* $\tilde{A} = (\tilde{a}_{ij}) \in \mathbb{R}^{k \times l}$ which is defined by

$$\tilde{a}_{ij} = \begin{cases} 1 & \text{edge } j \text{ starts at node } i, \\ -1 & \text{edge } j \text{ ends at node } i, \\ 0 & \text{else.} \end{cases}$$

We call $u = (u_0, u_1, u_2, ...)$ the corresponding *electrical potentials* (or just *potentials*) to the nodes $\mathcal{N}$. The difference between the potentials at two connected nodes is called the *voltage* at the respective edge. To fix the absolute values of these potentials we have to set one node to a fixed value. We will do that by "grounding" the node n_0, which means we

set the potential $u_0 = 0$. The grounding of a node allows us to remove the corresponding row from the incidence matrix to get the *reduced incidence matrix* A. The vector $v = (v_{ij})_{ij}$ represents the voltages at the edges. For some indices i and j, the voltage at edge ij is defined as $v_{ij} = u_i - u_j$.

We will later see that the components of an electrical circuit, which will be installed along the edges, describe a relationship between the current along the edge and the corresponding voltage. Thus a current vector $i = (i_1, i_2, i_3, ...)$ containing the currents along the edges is required.

Example 1.1 (Charging of a Capacitor)
As an example we consider the charging of a capacitor. The circuit consists of a capacitor which is to be charged by a voltage source. This charging is also influenced by a resistor which is placed in series to the other components. This is depicted in Figure 2.1.

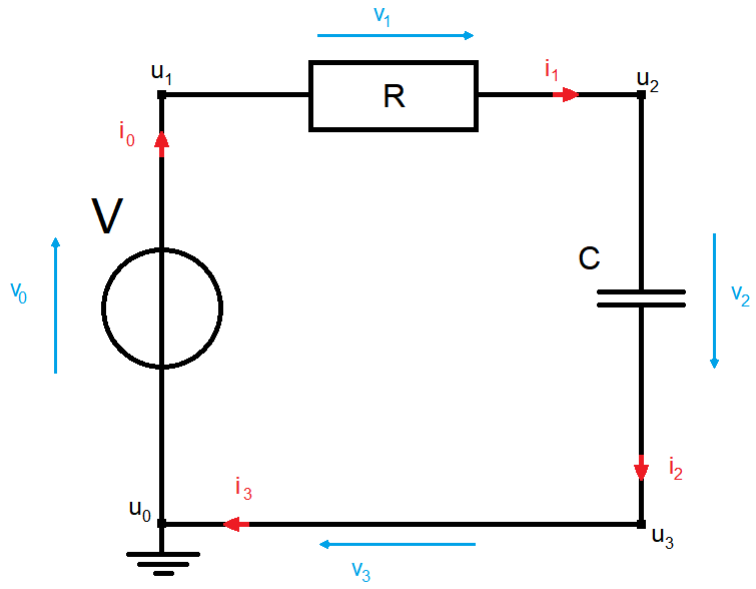

Figure 2.1: charging capacitor with series resistor and voltage source

With the node-potentials, the voltages and the currents collected in the vectors

$$u = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad v = \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad i = \begin{pmatrix} i_0 \\ i_1 \\ i_2 \\ i_3 \end{pmatrix},$$

the incidence matrix of this circuit has the form

$$\tilde{A} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

The rows of this matric correspond to the nodes of the circuit and the columns correspond to the edges and tell us if the edge either starts (1) or end (-1) at a node. By grounding node u_0 this matrix reduces to

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

In this circuit edge 3 is not populated with any components. This means that the voltage along this edge does not change, or in terms of potentials

$$u_0 - u_3 = 0.$$

Thus we can consider this circuit with node 0 and node 3 merged. This leads to a slightly different incidence and reduced incidence matrix, i.e.

$$\tilde{A} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad \text{respectively.}$$

Example 2.1 (LC-Circuit)
Our second example is a circuit that only contains a capacitor and an inductor, see Figure 2.2. This circuit will oscillate, if given appropriate initial conditions.

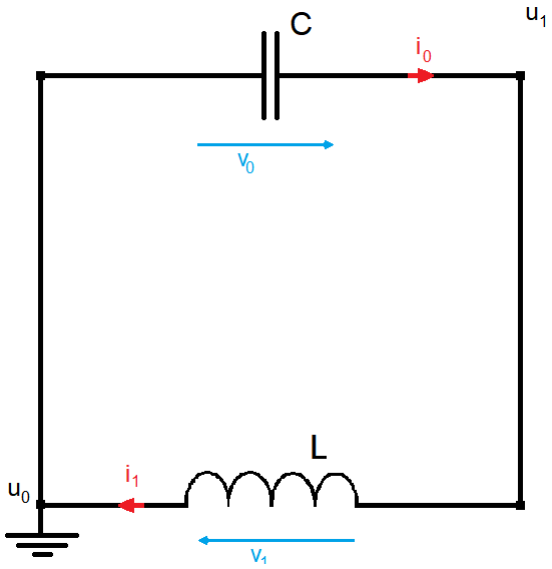

Figure 2.2: capacitor and inductor oscillator

We again collect the node-potentials, the voltages and the currents in the vectors

$$u = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \quad v = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}, \quad i = \begin{pmatrix} i_0 \\ i_1 \end{pmatrix}.$$

The incidence matrix of this circuit has the form

$$\tilde{A} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

By grounding node u_0 this can again be reduced, we obtain the reduced incidence matrix

$$A = \begin{pmatrix} -1 & 1 \end{pmatrix}.$$

Example 3.1 (Parallel Voltage Source, Capacitor and Resistor)
As a third example we consider a similar circuit to the circuit presented in Example 1.1, the only difference is that we connected the resistor parallel to the capacitor instead of connecting it in series.

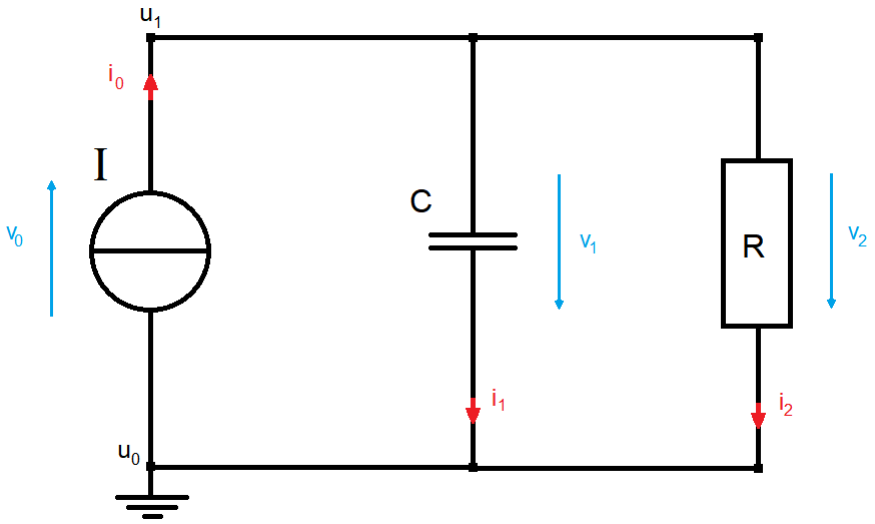

Figure 2.3: Capacitor with resistor and voltage source

We collect the node-potentials, the voltages and the currents in the vectors

$$u = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \quad v = \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix}, \quad i = \begin{pmatrix} i_0 \\ i_1 \\ i_2 \end{pmatrix}.$$

The incidence matrix of this circuit has the form

$$\tilde{A} = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \end{pmatrix},$$

this leads to the reduced incidence matrix

$$A = \begin{pmatrix} -1 & 1 & 1 \end{pmatrix}.$$

2.2 Energy Conservation Laws

To fully fix all the variables that arise in the model of an electrical circuit we will need some *conservation laws*. In this context, conservation laws are a description of physical properties

that can be observed in electrical circuits. This means that they have to be fulfilled non-theless, thus including them into the system also makes sense from a physical perspective. They were proposed by Kirchhoff in 1845, see [6].

- **Kirchhoff's voltage law (KVL):**
  The sum of voltages along each loop of the network must equal to zero. Using the incidence matrix A and the vector of potentials u as well as the voltage vector v, this law can be formulated as

$$A^\top u = v. \tag{2.1}$$

- **Kirchhoff's current law (KCL):**
  For any node, the sum of currents flowing into the node is equal to the sum of currents flowing out of the node. Using again the incidence matrix A and the vector of currents i, this law can be formulated as

$$Ai = 0. \tag{2.2}$$

2.3 Electrical Components and their relations

Electrical components are described by equations relating their edge voltage v to their edge current i. We will mainly focus on so-called RLC-networks (short for Resistor, Inductor and Capacitor). These networks consist only of resistors, capacitors, inductors, voltage sources and current sources. Diodes and transistors as well as other electrical components can be described in a similar way, although these lead to a more difficult analysis of the system, see e.g. [2] and [1]. Using the notation introduced in Section 2.1, the relations read as:

- **Resistor**
  Resistors "resist" the flow of current, which causes voltage to drop. This behaviour is described by the *resistance* $R \in \mathbb{R}^+ := \{x \in \mathbb{R} : x > 0\}$ which is given in *Ohm* (Ω) and its reciprocal, the *conductance* $G \in \mathbb{R}^+$, which is given in *Siemens* ($S = \frac{1}{\Omega}$).

$$v = R\, i \quad \text{or} \quad i = G\, u. \tag{2.3}$$

The electrical schematic symbol for resistors is depicted in Figure **??**.

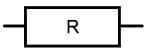

Figure 2.4: resistor symbol

- **Capacitor**
  Capacitors "store" electrical energy by accumulating electrical charge. Their characteristic equations can be described directly using the stored charge $Q \in \mathbb{R}_0^+ := \{x \in \mathbb{R} : x \geq 0\}$ or indirectly using the change in charge, which is nothing other than the current I. The *capacitance* $C \in \mathbb{R}^+$ is given in *Farads* (F).

$$Q = C\,v \quad \text{and by derivation in t} \quad I = \frac{d}{dt}Q = C\,\frac{d}{dt}v = C\,v'. \tag{2.4}$$

  The electrical schematic symbol for capacitors is depicted in Figure **??**.

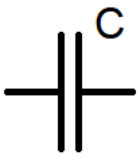

Figure 2.5: capacitor symbol

- **Inductor**
  An electric current flowing through a conductor generates a magnetic field $\Phi \in \mathbb{R}^{n\times n}$ surrounding it. This magnetic field causes a voltage drop dependent on the change in current. The *inductance* $L \in \mathbb{R}^+$ is given in *Henry* (H).

$$\Phi = L\,i \quad \text{and by derivation in t} \quad v = \Phi' = L\,i'. \tag{2.5}$$

  The electrical schematic symbol for inductors is depicted in Figure **??**.

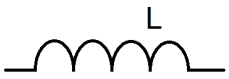

Figure 2.6: inductor symbol

- **Voltage Source**
  A voltage source supplies the system with a voltage. It can either supply varying amounts of voltage (with the special case of alternating current AC) or a fixed amount of voltage. The unit of voltage is *Volts* (V).

  $$v = v_{src} \tag{2.6}$$

  The electrical schematic symbol for voltage sources is depicted in Figure **??**.

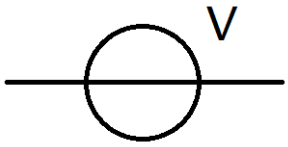

Figure 2.7: voltage source symbol

- **Current Source**
  A current source supplies the system with electrical current. It can either supply varying amounts of current (with the special case of alternating current AC) or a fixed amount of current. The unit of current is *Ampere* (A).

  $$i = i_{src} \tag{2.7}$$

  The electrical schematic symbol for current sources is depicted in Figure **??**.

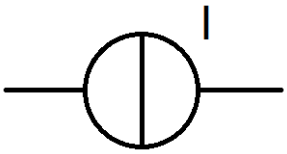

Figure 2.8: current source symbol

The resistance R, conductance G, capacitance C and inductance L are respectively described as a scalar constant that relates the edge-current to the edge-voltage. If there are more components of the same kind in one circuit their corresponding constants will usually be collected into a matrix which is also called the same and also denoted by the same letter, respectively. These matrices are then positive definite diagonal matrices.

2.4 Modified Nodal Analysis - MNA

[2] and [5]

To analyse the network further we will rearrange the columns of the reduced incidence matrix A such that it has the block form

$$A = \begin{pmatrix} A_R & A_C & A_L & A_V & A_I \end{pmatrix}$$

where A_R, A_C, A_L, A_V and A_I include the columns that are related to the resistors, capacitors, coils, voltage sources and current sources, respectively.

To mathematically describe the circuit, we use *modified nodal analysis* (or short MNA), see the paper by Ho, Ruehli and Brennan, [4]. MNA uses the node voltages as well as the currents of the coils and the voltage sources as unknowns. It is based on the conservation laws (2.2) and (2.1) as well as on the voltage-current relations of the electrical components, discussed in Section 2.3. The voltages can be represented using the node-potentials and the incidence matrix A, i.e.

$$v = A^\top u.$$

The vector v can thus be rearranged into $v = (v_R, v_C, v_L, v_{src}, v_I)$. In a similar way we also rearrange the current vector into $i = (i_R, i_C, i_L, i_V, i_{src})$. Using the sorted incidence matrix blocks, we can rewrite the resistor current relation as

$$i_R = G\, v_R = G\, A_R^\top u.$$

Analogously, we rewrite the capacitor relation as

$$i_C = C\, v_C' = C\, A_C^\top u'.$$

From Kirchhoffs current law (2.2) we deduce that

$$A_C i_C + A_R i_R + A_L i_L + A_V i_V = -A_I i_{src}.$$

We plug in the component relations derived above and obtain

$$A_C C A_C^\top u' + A_R G A_R^\top u + A_L i_L + A_V i_V = -A_I i_{src}.$$

Combining this with the component law for inductors (2.5) and the potential-voltage relation for voltage sources (2.6) we finally get the MNA equations

$$\begin{aligned} A_C C A_C^\top u' + A_R G A_R^\top u + A_L i_L + A_V i_V &= -A_I i_{src}, \\ L i_L' - A_L^\top u &= 0, \\ -A_V^\top u &= -v_{src}. \end{aligned}$$

In matrix form these read

$$\begin{pmatrix} A_C C A_C^\top & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{pmatrix} * \begin{pmatrix} u' \\ i_L' \\ i_V' \end{pmatrix} + \begin{pmatrix} A_R G A_R^\top & A_L & A_V \\ -A_L^\top & 0 & 0 \\ -A_V^\top & 0 & 0 \end{pmatrix} * \begin{pmatrix} u \\ i_L \\ i_V \end{pmatrix} = \begin{pmatrix} -A_I i_{src} \\ 0 \\ -v_{src} \end{pmatrix}, \quad (2.8)$$

where the diagonal matrices C, G and L contain the capacities, conductivities and inductors, respectively.

The resulting systems are *stiff* systems. This means that for their numserical solution special care has to be put into which methods are suitable for solving these systems in a stable manner. For analyzing nonlinear components, other formulations can be used, e.g. the charge flux oriented formulation, for details see [2].

Example 1.2 (Charging of a Capacitor)
We again consider the charging of a capacitor. The circuit is again given as in Example 1.1, where we have already derived the reduced incidence matrix

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

This matrix can be split into the three submatrices A_V, A_R and A_C containing the columns of the matrix A. Because there are no inductors as well as no current sources resent,

the matrices A_L and A_V are empty. The diagonal matrices containing the components constants are

$$C = (C), \qquad L = (), \qquad G = (\frac{1}{R}).$$

Plugging this into the formula (2.8) we obtain the system

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{pmatrix} * \begin{pmatrix} u_1' \\ u_2' \\ i_0' \end{pmatrix} + \begin{pmatrix} \frac{1}{R} & -\frac{1}{R} & -1 \\ -\frac{1}{R} & \frac{1}{R} & 0 \\ 1 & 0 & 0 \end{pmatrix} * \begin{pmatrix} u_1 \\ u_2 \\ i_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -v_{src} \end{pmatrix}. \tag{2.9}$$

Note that because L is empty, the system does not contain any related equations to inductances.

Example 2.2 (LC-Circuit)
Similarly we also derived the incidence Matrix for this system in Example 2.1, i.e.

$$A = \begin{pmatrix} -1 & 1 \end{pmatrix}.$$

This matrix can be split into the two submatrices A_C and A_L containing the columns of the matrix A. Because we do not have any resistors, voltage sources or current sources, the matrices A_R, A_V and A_I are empty. The diagonal matrices containing the components constants are

$$C = (C), \qquad L = (L), \qquad G = ().$$

Plugging this into the formula (2.8) we obtain the system

$$\begin{pmatrix} C & 0 \\ 0 & L \end{pmatrix} * \begin{pmatrix} u_1' \\ i_L' \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} * \begin{pmatrix} u_1 \\ i_L \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Example 3.2 (Parallel Voltage Source, Capacitor and Resistor)
We consider the circuit introduced in Example 3.1, with the corresponding incidence matrix

$$A = \begin{pmatrix} -1 & 1 & 1 \end{pmatrix}.$$

This matrix can be split into the three submatrices A_I, A_R and A_C containing the columns of the matrix A. As there are no inductors and no voltage sources present, the matrices A_L and A_V are empty. The diagonal matrices containing the components constants are

$$C = (C), \qquad L = (), \qquad G = (\frac{1}{R}).$$

Plugging this into the formula (2.8) we obtain the system

$$\begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} * \begin{pmatrix} u_1' \\ i_V' \end{pmatrix} + \begin{pmatrix} \frac{1}{R} & -1 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} u_1 \\ i_V \end{pmatrix} = \begin{pmatrix} 0 \\ -v_{src} \end{pmatrix}. \tag{2.10}$$

3 Differential Algebraic Equations

von [5] und DAE lecture

The resulting MNA of electrical circuits leads to a special form of differential equations, namely *differential algebraic equations* (DAE). Additionally to differential equations they also contain algebraic equations or are equivalent to such a system. They are distinct from ordinary differential equations in that the Jacobian matrix for a DAE system is singular. To understand the general solvability of those systems and to find good numerical methods we first have to take a look at the theory of such DAEs.

In the most general form a DAE can be written as: Find $y : \mathbb{R} \to \mathbb{R}^n$ such that

$$F(t, y(t), y'(t)) = 0, \qquad \forall t \in I \tag{3.1}$$

with $F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ sufficiently smooth and I the time-interval. As this is usually too general, one has to consider more specific types of DAEs. This chapter focuses on giving a brief overview of the different types of differential algebraic equations. We will discuss the specific form of linear systems with constant coefficients in more detail, as those are the systems that arise from modified nodal analysis of a RLC-system. Additionally this chapter also discusses the notion of the index of an MNA-system.

3.1 Types of DAEs

Let us consider the following three types of DAEs.

- **Linear systems with constant coeffiecients**
  are systems of the form: find y such that

  $$Ay'(t) + By(t) = f(t), \tag{3.2}$$

  with $A, B \in \mathbb{R}^{n\times n}$, A singular, B regular and $f : \mathbb{R} \to \mathbb{R}^n$ a function in time. If A is regular this equation can be transformed into an ordinary differental equation, since

  $$y'(t) + A^{-1}By(t) = A^{-1}f(t)$$

  Thus it is reasonable to assume A singular.

- **Linear time dependent systems** are systems of the form: find y such that

  $$A(t)y'(t) + B(t)y(t) = f(t),$$

  with $A, B : \mathbb{R} \to \mathbb{R}^{n\times n}$, $f : \mathbb{R} \to \mathbb{R}^n$ functions, such that for every $t \in \mathbb{R}$ the matrix $A(t)$ is singular and the matrix $B(t)$ regular.

- **Structured (non-linear) systems**
  are semi-explicit systems of the form: find (y, z) such that

  $$\begin{aligned} y'(t) &= f(t, y(t), z(t)), \\ 0 &= g(t, y(t), z(t)), \end{aligned}$$

  with $f : \mathbb{R} \to \mathbb{R}^n$ and $g : \mathbb{R} \to \mathbb{R}^d$ functions.

For our analysis of electrical networks, we will focus on linear systems with constant coefficients (3.2). The other systems could for example occur when we additionally consider time-dependent component relations e.g. temperature. A splitting of the variable y is usually common, where we differ between the *differential variable* and the *algebraic variable*. A differential variable in this sense is the part of y, whose derivatives also appear in the differential algebraic equation. The algebraic variable on the other hand only appears in algebraic equations.

3.1.1 Weierstraß-Kronecker normal form

To determine the solvability of a linear system with constant coefficients (3.2), we first need to introduce a normal form for the system, the *Weierstraß-Kronecker normal form*. This normal form is dependent on the family $\{A, B\} := \{\mu A + B | \mu \in \mathbb{R}\}$, which is called the *matrix pencil* of the DAE.

Definition 1. *The matrix pencil $\{A, B\}$ is called* regular *if there exists some $c \in \mathbb{R}$, such that $(cA + B)$ is regular (i.e. $det(cA + B) \neq 0$), otherwise it is called singular.*

Theorem 1 (Jordan normal form [5, Theorem 13.2.1])**.** *For every matrix $Q \in \mathbb{R}^{n\times n}$ there exists a regular matrix $T \in \mathbb{C}^{n\times n}$, such that*

$$T^{-1}QT = J = diag(J_1, ..., J_r) \quad with \quad J_i = \begin{pmatrix} \lambda_i & 1 & & 0 \\ 0 & \lambda_i & \ddots & \vdots \\ & \ddots & \ddots & 1 \\ 0 & \dots & 0 & \lambda_i \end{pmatrix} \in \mathbb{C}^{m_i \times m_i}$$

and $n = m_1 + ... + m_r$.

The matrix J is called Jordan normal form of Q, the matrices J_i are called Jordan Blocks, where λ_i are the corresponding eigenvalues of Q. The matrix J is uniquely determined by Q except for the arrangement of the diagonal blocks. If Q posesses only real eigenvalues, then T can also be choosen from the reals.
In order to seperate the equation (3.2) into an algebraic and a differential part, we transform the matrices A and B according to the following theorem.

Theorem 2 ([5, Satz 13.2.2])**.** *Let $\{A, B\}$ be a regular matrix pencil. There exist $P, Q \in \mathbb{C}^{n\times n}$ such that*

$$PAQ = \begin{pmatrix} I_d & 0 \\ 0 & N \end{pmatrix}, \quad PBQ = \begin{pmatrix} R & 0 \\ 0 & I_{n-d} \end{pmatrix}$$

where

$$N = diag(N_1, ..., N_r) \quad with \quad N_i = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 0 \quad 1 \\ 0 & & & 0 \end{pmatrix} \in \mathbb{R}^{n_i \times n_i}$$

and R has Jordan normal form. By I_k we denote the identity matrix of size $k \times k$.

Proof. Because $\{A, B\}$ is a regular matrix pencil, there exists $c \in \mathbb{R}$ such that $(cA + B)$ is regular. Set

$$\hat{A} := (cA + B)^{-1}A, \quad \hat{B} := (cA + B)^{-1}B.$$

Considering

$$(cA + B)^{-1}(cA + B) = I \implies (cA + B)^{-1}B + c(cA + B)^{-1}A = I,$$

we get that

$$\hat{B} = I - c\hat{A}.$$

Let $J_{\hat{A}}$ be the Jordan normal form of $\hat{A}$, this means that there exists a regular matrix T_1 such that

$$T_1^{-1}AT_1 = J_{\hat{A}} = \begin{pmatrix} W & 0 \\ 0 & \tilde{N} \end{pmatrix}.$$

The matrix W contains the Jordan blocks with eigenvalues which are nonzero, the matrix $\tilde{N}$ contains the Jordan blocks with eigenvalues equal to zero, thus $\tilde{N}$ is *nilpotent*. The Jordan normal form $J_{\hat{B}}$ of $\hat{B}$ is given by

$$T_1^{-1}\hat{B}T_1 = J_{\hat{B}} = \begin{pmatrix} I - cW & 0 \\ 0 & I - c\tilde{N} \end{pmatrix}.$$

The following two transformations will allow us to get the desired structure. First we will transform $J_{\hat{A}}$ with

$$T_2 := \begin{pmatrix} W & 0 \\ 0 & I - c\tilde{N} \end{pmatrix}$$

in

$$T_2^{-1} J_{\hat{A}} = \begin{pmatrix} I & 0 \\ 0 & (I - c\tilde{N})^{-1}\tilde{N} \end{pmatrix}$$

and $J_{\hat{B}}$ in

$$T_2^{-1} J_{\hat{B}} = \begin{pmatrix} W^{-1} - cI & 0 \\ 0 & I \end{pmatrix}.$$

Let now R be the Jordan normal form of $(W^{-1} - cI)$ and N be the normal form of $(I - c\tilde{N})^{-1}\tilde{N}$, this means

$$T_W^{-1}(W^{-1} - cI)T_W = R \quad \text{and} \quad T_{\tilde{N}}^{-1}(I - c\tilde{N})^{-1}\tilde{N}T_{\tilde{N}} = N$$

Considering this definition together with the Neumann series of $(I - c\tilde{N})^{-1}$ we obtain

$$\tilde{N}(I - c\tilde{N})^{-1} = \tilde{N}(c\sum_{i=0}^{\infty}\tilde{N}^i) = \tilde{N}(c\sum_{i=0}^{k-1}\tilde{N}^i) = c\sum_{i=0}^{k-1}\tilde{N}^{i+1} = (c\sum_{i=0}^{k-1}\tilde{N}^i)\tilde{N}$$

where we used that $\tilde{N}$ is nilpotent with nilpotency index k. This shows that $\tilde{N}$ and $(I - c\tilde{N})^{-1}$ commute.

From this we can conclude that

$$N^k = [T_{\tilde{N}}^{-1}(I - c\tilde{N})^{-1}\tilde{N}T_{\tilde{N}}]^k = T_{\tilde{N}}^{-1}[(I - c\tilde{N})^{-1}\tilde{N}]^k T_{\tilde{N}} = T_{\tilde{N}}^{-1}(I - c\tilde{N})^{-k}\underbrace{\tilde{N}^k}_{=0}T_{\tilde{N}} = 0$$

We used the commutativity in the third step here. The nilpotent matrix N thus also has the nilpotency index k. A transformation with

$$T_3 := \begin{pmatrix} T_W & 0 \\ 0 & T_{\tilde{N}} \end{pmatrix}$$

transforms $T_2^{-1} J_{\hat{A}}$ into the Jordan normal form

$$J_{\tilde{A}} := T_3^{-1} T_2^{-1} J_{\hat{A}} T_3 = T_3^{-1} T_2^{-1} T_1^{-1} \hat{A} T_1 T_3 = \begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix}$$

and $T_2^{-1} J_{\hat{B}}$ into

$$J_{\tilde{B}} := T_3^{-1} T_2^{-1} J_{\hat{B}} T_3 = T_3^{-1} T_2^{-1} T_1^{-1} \hat{B} T_1 T_3 = \begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix}.$$

Now set

$$P := T_3^{-1} T_2^{-1} T_1^{-1} (cA + B)^{-1} \quad \text{and} \quad Q = T_1 T_3$$

to get the statement. □

Using the findings above we are able to transform the initial DAE (3.2) using the matrix P from Theorem 2. By multiplying with P from the left, we obtain

$$PAy'(t) + PBy(t) = Pf(t).$$

Setting

$$y(t) = Q \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}, \quad Pf(t) = \begin{pmatrix} s(t) \\ q(t) \end{pmatrix},$$

with $u(t), s(t) : \mathbb{R} \to \mathbb{R}^d$ and $q(t), v(t) : \mathbb{R} \to \mathbb{R}^{n-d}$, we get a system of the form

$$\begin{aligned} u'(t) + Ru(t) &= s(t), \\ Nv'(t) + v(t) &= q(t), \end{aligned} \tag{3.3}$$

where $PAQ = \begin{pmatrix} I & \\ & N \end{pmatrix}$ and $PBQ = \begin{pmatrix} R & \\ & I \end{pmatrix}$.

We call this system the *Weierstraß-Kronecker normal form* of the DAE. The first equation of (3.3) is a first order ordinary differential equation and posesses a unique solution $u(t)$ in $[t_0, t_l]$ for any initial values $u_0 \in \mathbb{R}^d$. Now let $q(t) \in C^{k-1}([t_0, t_l])$, where $C^{k-1}(I)$ denotes the space of all functions with domain I that are $k-1$ times continuously differentiable. We differentiate the second equation in (3.3) and obtain

$$\begin{aligned} v(t) &= q(t) - Nv'(t) = q(t) - N(\underbrace{q(t) - Nv'(t)}_{=v(t)})' = q - Nq' + N^2v'' \\ &= q - Nq' + N^2(q - Nv')'' = q - Nq' + N^2q'' - N^3v''' \\ &\vdots \\ &= q - Nq' + ... + (-1)^{k-1}N^{k-1} \underbrace{\frac{d^k}{dt^k}q}_{:=q^{(k-1)}} + (-1)\underbrace{N^kv^{(k)}}_{=0} \\ &= \sum_{i=0}^{k-1}(-1)^iN^iq^{(i)}(t) \end{aligned} \tag{3.4}$$

where k is the nilpotency index of N. Hence we have an explicit form of the solution of $v(t)$. The Kronecker index k thus tells us how many differentiations are required to obtain an ordinary differential equation. Note that this way of splitting the DAE allows us to differ between algebraic and differential variables.

We can also give a general definition of a DAE in Weierstraß-Kronecker normal form.

Definition 2 ([5, Definition 13.2.4]). *A linear differential equation* (3.2) *is said to be in Weierstraß-Kronecker normal form if*

$$A = \begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix}, \quad B = \begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix},$$

where N is a nilpotent Jordan-block matrix.

Definition 3. *The nilpotency index k of the matrix N from the Weierstraß-Kronecker Normalform of a matrix pencil $\{A, B\}$ with A singular is called the* Kronecker index *of $\{A, B\}$, which we denote by $ind\{A, B\}$. Note that for A regular we set $ind\{A, B\} = 0$.*

Lemma 1. *The Kronecker-Index $ind\{A, B\}$ is independent of the choice of the matrices P and Q.*

Proof. See Lemma 13.2.1 in [5, p.403]. □

3.2 Index of a Differential Algebraic Equation

The index of a DAE gives us insight about its numerical properties and in general about the solvability of the equation. In particular, as we will also mention later, the index is relevant for choosing initial values that allow us to solve the system uniquely.

We will consider two types of index concepts, the differentiation index and the perturbation index.

Since numerical differentiation is an unstable procedure the differentiation index aims to give a measure for the numerical problems to be expected when solving such systems.

Definition 4 (Differentiation index [5, Definition 13.3.1])**.** *Consider the differential algebraic equation* (3.1) *to be uniquely locally solvable and F sufficiently smooth. For a given $m \in \mathbb{N}$ consider the equations*

$$\begin{aligned} F(t, y, y') &= 0, \\ \frac{\mathrm{d}F(t, y, y')}{\mathrm{d}t} &= 0, \\ &\vdots \\ \frac{\mathrm{d}^m F(t, y, y')}{\mathrm{d}t^m} &= 0. \end{aligned}$$

The smallest natural number m for which the above system results in an explicit system of ordinary differential equations (ODEs), i.e. it has the form

$$y' = \phi(t, y),$$

is called ***differentiation index****.*

In the previous chapter we have already discussed, that for a DAE with constant coefficients (3.2) and a regular matrix pencil $\{A, B\}$ we need $k = ind\{A, B\}$ differentiations to receive an ordinary differential equation. Hence for a DAE with constant coefficients the Kronecker index k is equal to the differentiation index.

Definition 5 (perturbation index [5, Definition 13.3.3])**.** *Let $y(t)$ be the exact solution of (3.1) and $\tilde{y}(t)$ be the solution of the perturbed system $F(t, \tilde{y}, \tilde{y}') = \delta(t)$. The smallest number $k \in \mathbb{N}$ such that*

$$\|y(t) - \tilde{y}(t)\| \leq C \left(\|y(t_0) - \tilde{y}(t_0)\| + \sum_{j=0}^{k} \max_{t_0 \leq \xi \leq T} \left\| \int_{t_0}^{\xi} \frac{\mathrm{d}^j \delta}{\mathrm{d}\tau^j}(\tau) d\tau \right\| \right)$$

for all $\tilde{y}(t)$, is called the ***perturbation index*** *of this system.*

We now consider the case of a perturbed linear DAE with constant coefficients and a regular matrix pencil $\{A, B\}$

$$A\tilde{y}'(t) + B\tilde{y}(t) = -\delta(t).$$

Then we obtain for the difference of the unperturbed system with the perturbed system

$$A(y'(t) - \tilde{y}'(t)) + B(y'(t) - \tilde{y}(t)) = \delta(t)$$

Due to Theorem 2 we can transform this system into a system of the form (3.3)

$$\begin{aligned} u'(t) - \tilde{u}'(t) + \hat{R}(u(t) - \tilde{u}(t)) &= \delta_1(t), \\ N(v'(t) - \tilde{v}'(t)) &= \delta_2(t), \end{aligned}$$

where $u(t)$ holds the first d entries of $y(t) : \mathbb{R} \to \mathbb{R}^n$ and $v(t)$ holds the remaining $n - d$ entries. According to (3.4), the solution of the algebraic variable has the form

$$v(t) - \tilde{v}(t) = \sum_{i=0}^{k-1} (-1)^i N^i \delta_2^{(i)}(t)$$

Thus the perturbation index is the same as the Kronecker-index $ind\{A, B\}$, see [5]. More over we observe that the error depends on the derivatives of the pertubation.

4 Index Analysis of the Modified Nodal Analysis

[2]

In the previous chapter we have seen two different kinds of index concepts for differential algebraic equations. We have also seen that these two, even though they describe rather different structural aspects of the equation, are identical for the problems discussed in this thesis. This leads to the question, what indices we expect obtain for our MNA system.

This chapter answers this question for the case of linear RLC-circuits. As already discussed in Section 2.3, RLC circuits contain only resistors, inductors and capacities as well as current and voltage sources. Linear means that all circuit components are described by linear positive functions. Thus the matrices C, R, and L are positive definite and symmetric.

Recall that we consider the equations resulting from the analysis in Section 2.4. These equations are of the form (3.2): find y such that

$$Ay'(t) + By(t) = f(t).$$

In particular we obtain the system of equations

$$\begin{pmatrix} A_C C A_C^\top & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{pmatrix} * \begin{pmatrix} u' \\ i_L' \\ i_V' \end{pmatrix} + \begin{pmatrix} A_R G A_R^\top & A_L & A_V \\ -A_L^\top & 0 & 0 \\ -A_V^\top & 0 & 0 \end{pmatrix} * \begin{pmatrix} u \\ i_L \\ i_V \end{pmatrix} = \begin{pmatrix} -A_I i_{src} \\ 0 \\ -v_{src} \end{pmatrix}.$$

4.1 General Index analysis

Assuming the system only contains linear elements, or is linearized at an operating point in order to investigate the system behaviour, then the corresponding network equation represents a DAE with constant coefficients (3.2). We collect all unknowns in $x(t) = (u(t), i_L(t), i_V(t))^\top$. The structure of the system is reliant on the matrix B, thus we consider two cases.

ODE-case: The matrix B in (3.2) is regular. This is the case iff the circuit contains no voltage sources and every node has a path to ground, that leads through at least one capacitor. Then the system represents a linear-implicit system of ODEs and can be transformed into the explicit ODE sytstem

$$x' = B^{-1}(-Ax + f(t)).$$

Thus we obtain an index of 0.

DAE-case: The matrix B in (3.2) is singular. This is the interesting case which we will analyse in the following.

For a singular matrix B we have already obtained a representation of the form

$$\begin{aligned} u'(t) + Ru(t) &= s(t), \\ Nv'(t) + v(t) &= q(t) \end{aligned}$$

in the previous section, see (3.3). We now consider the nilpotency index ν of the matrix N, since we already know that this index correlates with the differentiation index as well as the perturbation index. We consider two cases:

1. **The nilpotency index is 1**
   Because N is nilpotent with nipotency index $\nu = 1$, it holds that $N^1 = 0$, thus the system transforms to

   $$\begin{aligned} u'(t) + Ru(t) &= s(t), \\ v(t) &= q(t). \end{aligned}$$

   This means that the algebraic variables are given explicitly. Thus the system is in ODE form.

2. **The nilpotency index is ≥ 2**
   This case is the situation we have discussed in section 3.1.1, where we obtained the representation (3.4)

   $$v(t) = \sum_{i=0}^{\nu-1} (-1)^i N^i q^{(i)}(t). \tag{4.1}$$

   By differentiating the above equation one more time, we obtain

   $$v'(t) = \sum_{i=0}^{\nu-1} (-1)^i N^i q^{(i+1)}(t) \tag{4.2}$$

   with the assumption that q is ν times differentiable.

We observe that in both cases the solution v has to fulfill an algebraic constraint. In the case of index $\nu = 1$ this equation is given explicitly, while for index $\nu \geq 2$ it is given implicitly by (4.2).

This system is also sensitive to perturbations. Note that the solution formula (4.1) for v also contains derivatives of the input q. Thus small noise in the input can have arbitrarily large derivatives, which can influence the solution.

4.2 Topological Conditions

From analyzing the MNA some conditions to the circuit topology can be obtained. We will be considering the impact of special arrangements of components on the index of

the system. In [9] as well as in [8] and [7] they present results about the index of MNA equations.

Theorem 3 (Index conditions [8, Theorem 2.2.1])**.** *Let the matrices of the capacitances, inductances and resistances be positive definite.*

- *If*

$$ker([A_R, A_C, A_V, A_L]^\top) = \{0\} \quad and \quad ker(A_V) = \{0\} \tag{4.3}$$

  holds, then the MNA (2.8) *leads to a system with index* $\nu \leq 2$*.*

- *If additionally*

$$ker([A_R, A_C, A_V]^\top) = \{0\} \quad and \quad ker([A_C, A_V]) = \{0\} \tag{4.4}$$

  holds, then the system is of index $\nu \leq 1$

- *If further*

$$ker(A_C^\top) = \{0\} \quad and \quad dim(v_{src}) = 0 \tag{4.5}$$

  holds, then the system has index $\nu = 0$*.*

By $ker(A)$ we denote the nullspace of the matrix A and by $dim(y)$ the size of the vector y. As (4.3) - (4.5) are just conditions concerning the topology of the circuit they can also be interpreted as follows:

check again, maybe need to add "only contains" somwhere - maybe at index 1 condition

- Condition (4.3) can be interpreted as the circuit neither containing loops of voltage sources nor cutsets of current sources.
- Condition (4.4) can be interpreted as the circuit containing neither loops of capacitors and/or voltage sources nor cutsets of inductors and/or current sources.
- Condition (4.5) can be interpreted as every node in the circuit being connected to the reference node (ground) through a path containing only the capacitors.

We will apply those results to the previously considered examples:

Example 1.3 (Index analysis)
As our first example we consider again the charging of a capacitor with a series resistor as in **Example 1**.

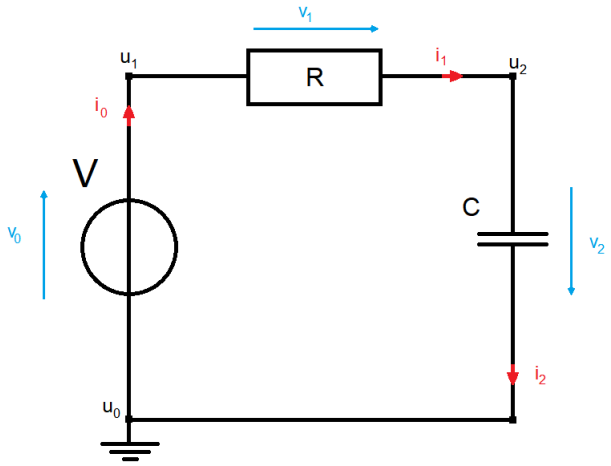

Figure 4.1: charging capacitor with series resistor and voltage source (note the difference to 2.1)

Because this circuit only contains one loop, which has a voltage source as well as a capacitor and a resistor we expect the resulting MNA-system to be of index $\nu = 1$.

The incidence matrix for this system is

$$A = [A_V, A_R, A_C] = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

Thus condition (4.3) gives us

$$ker([A_R, A_C, A_V, A_L]^\top) = ker \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} = \{0\} \quad \text{and} \quad ker(A_V) = ker \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \{0\}.$$

This condition is fulfilled, thus the index has to be smaller or equal to 2. The condition (4.4) results in

$$ker([A_R, A_C, A_V]^\top) = ker \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} = \{0\} \quad \text{and} \quad ker([A_C, A_V]) = ker \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \{0\}.$$

Thus the index also has to be smaller or equal to 1. Finally condition (4.5) results in

$$ker(A_C^\top) = ker \begin{pmatrix} 0 & 1 \end{pmatrix} = \{0\} \quad \text{and} \quad dim(v_s rc) \neq 0.$$

This means that this system has index $\nu = 1$.

Example 2.3 (Index analysis)
This example depicts an energy conserving system. The electric energy stored in the capacitor is converted into the the magnetic energy of the coil and vice versa.

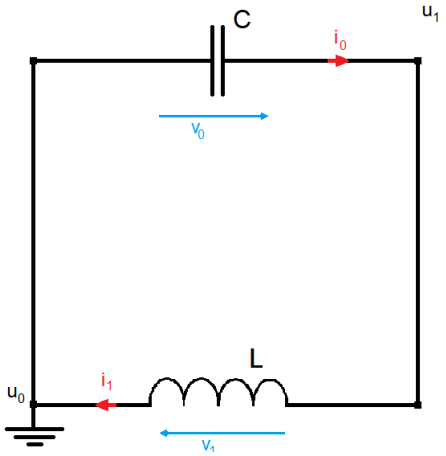

Figure 4.2: LC circuit

Because this circuit only contains one capacitor and one inductance in a singular loop and also has no sources, we expect the resulting MNA-system to be of index $\nu = 0$. This means that the system is an ODE.

The incidence matrix for this system is

$$A = [A_L\ A_C] = \begin{pmatrix} -1 & 1 \end{pmatrix}.$$

Thus condition (4.3) gives us

$$ker([A_R, A_C, A_V, A_L]^\top) = ker \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \{0\} \quad \text{and} \quad ker(A_V) = \{0\}.$$

This condition is fulfilled, thus the index has to be smaller or equal to 2.
Condition (4.4) results in

$$ker([A_R, A_C, A_V]^\top) = ker \begin{pmatrix} -1 \end{pmatrix} = \{0\} \quad \text{and} \quad ker([A_C, A_V]) = ker \begin{pmatrix} -1 \end{pmatrix} = \{0\}.$$

This condition is also fulfilled, thus the index has to be smaller or equal to 1.
Finally condition (4.5) results in

$$ker(A_C^\top) = ker \begin{pmatrix} -1 \end{pmatrix} = \{0\} \quad \text{and} \quad dim(v_s rc) = 0.$$

Thus it is fulfilled and we obtain index 0.

If we have consider the MNA directly, we also see that this system is of index $\nu = 0$, because the MNA has the form

$$\begin{aligned} Cu_1' - i_L &= 0, \\ Lu_L' + u_1 &= 0 \end{aligned}$$

which is an ODE.

Example 3.3 (Index analysis)
Finally, as the third example seems very similar to the first one, we might also expect them to have the same index. But this is indeed not the case as the following analysis shows.

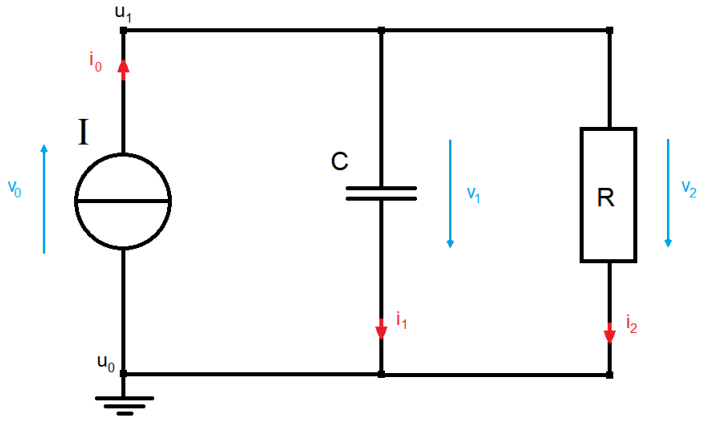

Figure 4.3: current source with resistor and capacitor

This circuit contains two loops, the first with only a voltage source and a capacitor and the second with a capacitor and a resistor. Because of the first loop, we expect this circuit to emit a system of index $\nu = 2$.

The incidence matrix for this system is

$$A = [A_V \; A_C \; A_R] = \begin{pmatrix} -1 & 1 & 1 \end{pmatrix}.$$

Thus condition (4.3) gives us

$$ker([A_R, A_C, A_V, A_L]^\top) = ker \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \{0\} \quad \text{and} \quad ker(A_V) = ker \begin{pmatrix} -1 \end{pmatrix} = \{0\}$$

This condition is fulfilled, thus the index has to be smaller or equal to 2. Condition (4.4) results in

$$ker([A_R, A_C, A_V]^\top) = ker \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \{0\} \quad \text{and} \quad ker([A_C, A_V]) = ker \begin{pmatrix} 1 & -1 \end{pmatrix} \neq \{0\}.$$

Thus this condition is not fulfilled, which means that the system indeed has index $\nu = 2$.

4.3 Consistent Initial Values

When solving a problem numerically, we usually want this problem to poses a (unique) solution. If it would not have a solution, then the numerical method might converge to some arbitrary value that has nothing to do with a solution. When dealing with real-world problems there might be more than one choices of initial values, to ensure (unique) solvability. This is the reason why we want to find conditions to construct *consistent* initial values. More details can be found in [5, chapter 13.3.1]

Case Index $\nu = 0$. As already discussed previously, the case $\nu = 0$ leads to an ODE, and due to the Picard-Lindelöff theorem [5, Satz 1.2.1] we know that every initial condition is consistent (for ordinary differential equations).

Example 2.4 (Consistent initial vaues)
We have already seen, that this system is of index $\nu = 0$. This means that we can prescribe arbitrary initial values. As the initial values still have a physical interpretation, still not every consistent initial value also makes sense! We will assume that at time $t = 0$ the capacitor holds no charge, i.e. $u_1(0) = 1$ and that there is no energy stored in the inductance, i.e. $i_L(0) = 0$.

Case: Index $\nu = 1$. By rewriting our system into the form

$$\begin{aligned} y'(t) &= f(t, y(t), z(t)), \\ 0 &= g(t, y(t), z(t)). \end{aligned}$$

we are able to give conditions for consistent initial values. Namely y_0 and z_0 are consistent initial values for this system, if $g(t_0, y_0, z_0) = 0$ holds.

Example 1.4 (Consistent initial vaues)
We have already seen, that this system is of index $\nu = 1$. Due to the fact that u_1 and i_V are only algebraically constrained, we have to determine their initial values by solving

for them at the point $t = 0$. On the other hand u_2 and its derivative both appear, thus we can simply prescribe an initial value for u_2. Again prescribing an arbitrary initial value might make sense from a mathematical perspective, since we might only be interested in the existence of a solution. But due to the physical interpretation of u_2 as being related to the charge that might already be stored in the capacitor, it makes sense to consider the capacitor to have no initial charge, i.e. $u_2(0) = 0$. The third line of the MNA-system (2.9) together with setting $v_{src} = sin(\pi t)$ immediately gives us, that

$$u_1(0) = -sin(\pi * 0) = 0.$$

From line one follows, that

$$u_1(0) - u_2(0) - i_V(0) = 0,$$

thus $i_V(0) = 0$ is a consistent initial value for i_V.

Case: Index $\nu = 2$. For index-2 systems we rewrite our system into

$$\begin{aligned} y' &= f(t, y(t), z(t)), \\ 0 &= g(t, y(t)). \end{aligned}$$

Consistent initial values y_0, z_0 for this case not only have to fulfill $g(t_0, y_0) = 0$ but also the *hidden constraint* $g_t(t_0, y_0) + g_y(t_0, y_0) f(t_0, y_0, z_0)$. By g_t and g_y we denote the derivative of g with respect to t or y, respectively.

Example 3.4 (Consistent initial values)
Previously we found, that the index of this system, it will turn out to be $\nu = 2$. Considering (2.10), we can define

$$\begin{aligned} f(t, u_1(t), i_v(t)) &= i_v(t) - u_1(t), \\ g(t, u_1(t)) &= u_1(t) + v_{src}(t). \end{aligned}$$

Computing the first condition gives together with the choice $v_{src} = sin(\pi t)$

$$u_1(0) = -v_{src}(0) = -sin(\pi * 0) = 0.$$

While the second condition leads to

$$0 = v'_{src}(0) + 1 * (i_V(0) - u_1(0)) = \pi cos(\pi * 0) + i_v(0) - 0 = \pi + i_V(0).$$

Thus we obtain $i_V(0) = -\pi$.

5 Numerical Solutions

As we are interested in finding real valued solutions of our systems we also have to look into solving them numerically. This chapter focuses on the numerical solution of the mentioned systems.

We will first focus on methods used to solve a more general initial value problem of the form:

Find y, such that

$$\begin{aligned} &y'(t) = f(t,y), \quad t \in [t_0, t_l], \\ &y(t_0) = y_0. \end{aligned} \tag{5.1}$$

For this we presume that the function $f(t,y)$ is continuous and Lipschitz, we can apply the theorem of *Picard-Lindelöf* [5, Satz 1.2.1], which states, that for every initial value y_0, the initial value problem (5.1) is uniquely solvable in $[t_0, t_l]$.

In order to obtain an approximation to this unique solution we have to discretize. We first divide the time-intervall $[t_0, t_l]$ into

$$t_0 < t_1 < ... < t_N \leq t_l$$

and consider approximations $y_m \approx y(t_m)$ for $m = 1, ..., N$, see 5.1. We call this a *time-grid* and we call the difference $h_{ij} = t_j - t_i$ with $0 \leq i < j \leq N$ the *step size* from t_i to t_j. We will only consider equidistant grids, i.e. $h_{ij} = h$ for all $0 \leq i, j \leq N$. A function defined on such a grid is called a *grid function*.

The grid function $y_h : T \to A$ with the time grid $T = \{t_0, ..., t_N\}$ and the approximated values $A = \{y_0, ..., y_N\}$ is called the *numerical approximation* (of y on the time grid).

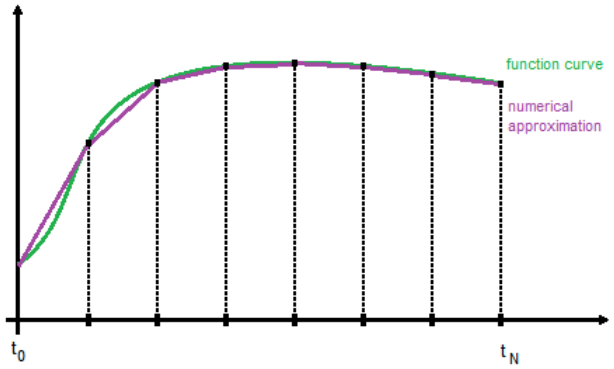

Figure 5.1: approximation of a function

5.1 Single Step Methods

The first class of numerical methods we will consider are single step methods. These methods use the previous approximated value y_j and (for implicit methods) also the current approximated value y_{j+1} to determine the current value y_{j+1} through a *procedural function*.

Definition 6. *A numerical method to approximate a differential equation 5.1 on a time-grid $T = \{t_0, ..., t_l\}$ with the intermediate values $y_0, ..., y_l$ is called a single step method, if it is of the form*

$$y_{j+1} = y_j + h_j \phi(t_j, y_j, y_{j+1}, h_j). \tag{5.2}$$

We call ϕ the procedural function. *If ϕ does not depend on y_{j+1}, then the method is called* explicit, *otherwise it is called* implicit.

5.1.1 Consistency, Stability and Convergence

In order to compare different single-step methods we have to define the notions of consistency, stability and convergence. This leads to the definition of the error of the method, its consistency and its convergence. We begin with the definition of the error.

Definition 7. *Let y_{m+1} be the result of one step of a single step method* (5.2) *with the exact start-vector $y_m = y(t_m)$ then*

$$\delta_{m+1} = \delta(t_m + h) = y(t_{m+1}) - \tilde{y}_{m+1}, \quad m = 0, ..., N-1 \tag{5.3}$$

is called the local discretization error *of the single step method at the point t_{m+1}.*

The local error quantifies the error of every step of the method with respect to the exact solution. In most applications the exact solution is not known. Next we consider the consistency.

Definition 8. *A single-step method is called* consistent *if for all initial value problems* (5.1)

$$\lim_{h \to 0} \frac{\|\delta(t+h)\|}{h} = 0 \quad for \quad t_0 \leq t \leq t_l \tag{5.4}$$

holds.
It is called consistent of order p*, if for a sufficiently smooth function f*

$$\|\delta(t+h)\| \leq Ch^{p+1} \quad for\ all \quad h \in (0, H] \quad and \quad t_0 \leq t \leq t_l - h \tag{5.5}$$

holds with C independent of h.

Consistency aims to give insight in how similar the problem (5.2) that the numerical method solves is to the real problem (5.1). Finally we consider convergence of a single step method 6.

Definition 9. *A single-step method is called* convergent*, if for all initial value problems* (5.1) *for the* global discretization error

$$e_m = y(t_m) - y_m$$

holds that

$$\max_m \|e_m\| \to 0 \quad for \quad h_{max} \to 0.$$

The single-step method is called to have the convergence order *p, if*

$$\max_m \|e_m\| \leq Ch_{max}^p \quad for \quad h_{max} \in (0, H] \quad with \quad t_0 \leq t_m \leq t_l$$

with the constant C not dependent on the step size h.

As the name suggestes convergence tries to quantify how far off a numerical solution is from the real solution of a system. A very interesting result follows if we additionally require the single-step method to be stable.

Definition 10. *A single-step method is called* (discretely) stable *if for grid-functions y_h and $\tilde{y}_h$ with*

$$y_{i+1} = y_i + h\phi(t_i, y_i), \tag{5.6}$$
$$\tilde{y}_{i+1} = \tilde{y}_i + h[\phi(t_i, \tilde{y}_i) + \theta_i], \tag{5.7}$$

and perturbations $\theta_i = \theta_h(t_i)$ of the right side as well as a bounded perturbation in the initial-values $y_0 - \tilde{y}_0$ the error is bounded by

$$\|y_h - \tilde{y}_h\|_{\infty,h} \leq C(\|y_0 - \tilde{y}_0\|_{l^2} + \|\theta_h\|_{\infty,h})$$

with a constant C which is not dependent on h. The norm $\|.\|_{\infty,h}$ denotes the maximum norm over the time-grid, i.e. for a function $b : T = t_0, ..., t_N \to \mathbb{R}^d$ we have $\|b\|_{\infty,h} = \max_{t \in T} \|b(t)\|$, $\|b\|$ is the euclidean norm.

For single-step methods which are consistent and stable we obtain the following convergence theorem.

Theorem 4 (Lax-Richtmyer)**.** *A consistent (with order p) and discretely stable single-step method is convergent (with order p).*

This theorem is due to Lax and Richtmyer. The converse of this statement is also true.

5.1.2 Further stability properties

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In this section we consider as a model problem the Dahlquist equation, i.e. find $y : \mathbb{R} \to \mathbb{C}$ such that

$$y' = \lambda y, \quad t > 0 \tag{5.8}$$
$$y(0) = y_0 \tag{5.9}$$

with $\lambda \in \mathbb{C}$ and $y_0 \in \mathbb{C}$ fixed.

Definition 11. *1. If a single-step method can be written in the form*

$$y_{i+1} = R(z)\, y_i, \quad z := h\lambda \tag{5.10}$$

then we call $R : \mathbb{C} \to \mathbb{C}$ the stability function *of the single-step method.*

2. The set

$$S := \{z \in \mathbb{C} : |R(z)| \leq 1\} \tag{5.11}$$

is called the region of stability *of the method.*

3. A single-step method is called

- 0-stable, *if* $0 \in S$.
- A-stable, *if* $\mathbb{C}^- \subset S$, *where* $\mathbb{C}^- = \{x \in \mathbb{C} : Re(x) \leq 0\}$
- L-stable, *if* $R(z) \to 0$ *for* $Re(z) \to -\infty$.

5.2 Multistep Methods

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Linear multistep methods (LMSM) use approximations y_{m+l} along the gridpoints t_{m+l}, $\quad l = 0, 1, ..., k-1$ to calculate the new approximation y_{m+k} at t_{m+k}. We will first discuss topics

related to the order of the methods depending on its parameters, stability and convergence.

Definition 12. *For given $\alpha_0, ..., \alpha_k$ and $\beta_0, ..., \beta_k$ the iteration rule*

$$\sum_{l=0}^{k} \alpha_l y_{m+l} = h \sum_{l=0}^{k} \beta_l f(t_{m+l}, y_{m+l}), \quad m = 0, 1, ..., N-k \tag{5.12}$$

is called a linear multistep method *(linear k-step method). It is always assumed that $\alpha_k \neq 0$ and $|\alpha_0| + |\beta_k| > 0$. If $\beta_k = 0$ holds, then the method is called explicit, otherwise implicit.*

A linear multi-step method consists of two parts:

1. In the *starting-phase* approximations $y_1, ..., y_{k-1}$ for the first $k-1$ gridpoints $t_l = t_0 + l\ h, l = 1, ..., k-1$ have to be calculated. For example we can use a single-step method or a multi-step method with fewer steps.
2. In the *run-phase* the multi-step formula is used to determine new approximations u_{m+k} for the gridpoint t_{m+k}

For theoretical analysis of the multi-step methods we consider the generating polynomials of a multistep method.

$$\rho(x) := \sum_{l=0}^{k} \alpha_l x^l \qquad \text{and} \qquad \sigma(x) := \sum_{l=0}^{k} \beta_l x^l \tag{5.13}$$

5.2.1 Consistency, Stability and Convergence

[5]

Again we need to define the properties consistency, stability and convergence for this class of methods.

Definition 13. *Let y_{m+k} be the result of one step of the multi-step method* (5.12) *with the start-values given as the evaluations of the exact solution $y_{m+l} = y(t_{m+l})$ at $0 \leq l < k$. This means*

$$\alpha_k \tilde{u}_{m+k} = \sum_{l=0}^{k-1} (h\beta_l f(t_{m+l}, y(t_{m+l})) - \alpha_l y(t_{m+l})) + h\beta_k f(t_{m+k}, y_{m+k}).$$

Then

$$\delta_{m+k} = \delta(t_{m+k}) = y(t_{m+k}) - y_{m+k}, \quad m = 0, 1, ..., N-k$$

is called the local discretization error *(local error) of the linear multi-step method, see Def. 5.12 at the point t_{m+k}.*

For $k = 1$ this definition agrees with definition 7, which is the local discretization error for single-step methods. If we assign the difference operator

$$L[y(t), h] = \sum_{l=0}^{k} \left(\alpha_l y(t + lh) - h\beta_l y'(t + lh)\right) \tag{5.14}$$

to the linear mutli-step method, we gain the following definition for consistency of such a methods.

Definition 14. *A linear multi-step method is called* consistent, *if for all functions $y(t) \in C^2([t_0, t_l])$*

$$\lim_{h \to 0} \frac{1}{h} L[y(t), h] = 0$$

holds. It has the consistency order p, *if for all functions $y(t) \in C^{p+1}[t_0, t_l]$*

$$L[y(t), h] = \mathcal{O}(h^{p+1}) \quad \textit{for} \quad h \to 0$$

holds.

From the generating polynomials (5.13) we can derive simple consistency conditions, i.e.

$$\rho(1) = 0 \quad \text{and} \quad \rho'(1) = \sigma(1).$$

Next we also need convergence of such methods.

Definition 15. *We say that a linear multi-step method is convergent, if for a solution y of the problem and a solution vector created by an LMSM $(y_j)_{j=0}^k$, we have that*

$$\lim_{h\to\infty} \max_{0\le j\le k} ||y(t_j) - y_j|| = 0.$$

The discrete stability is also very similar to the single-step methods.

Definition 16. *A linear multi-step method is called (discretely) stable, if for solutions y_h and $\tilde{y}_h$ of*

$$\sum_{l=0}^{k} \alpha_l y_{m+l} = h \sum_{l=0}^{k} \beta_l f(t_{m+l}, y_{m+l}), \tag{5.15}$$

$$\sum_{l=0}^{k} \alpha_l \tilde{y}_{m+l} = h \sum_{l=0}^{k} \beta_l f(t_{m+l}, \tilde{y}_{m+l}) + h\theta_n \tag{5.16}$$

and bounded initial values $y_j - \tilde{y}_j$ for $j \in 0, ..., k$ we have that

$$\max_{t_0\le t_n\le T} ||y_n - \tilde{y}_n|| \le C \sum_{j=0}^{k-1} ||y_j - \tilde{y}_j|| + \max_{t_0\le t_n\le T} ||\theta_n||.$$

5.2.2 Further stability properties

In this section we consider again the Dahlquist test problem as a model problem, i.e. find $y : \mathbb{R} \to \mathbb{C}$ such that

$$y' = \lambda y, \quad t > 0 \tag{5.17}$$

$$y(0) = y_0 \tag{5.18}$$

with $\lambda \in \mathbb{C}$ and $y_0 \in \mathbb{C}$ fixed.

Thus the resulting linear multistep method is of the form

$$\sum_{l=0}^{k} \alpha_l y_{n+l} = h \sum_{l=0}^{k} \beta_l \lambda y_{n+l}$$
$$\iff \sum_{l=0}^{k} [\alpha_l - h\beta_l \lambda] y_{n+l} = 0$$

Using this we define the following important stability notions.

Definition 17. *1. The set*

$$S := \{z \in \mathbb{C} : \rho(\xi) - z\sigma(\xi) = 0 \implies \xi \in \mathbb{C} \text{ and } |\xi| \leq 1. \\ \text{If } \xi \text{ has multiplicity greater than } 1, \text{ then } |\xi| < 1\} \tag{5.19}$$

is called the region of stability of the method.

2. *A linear multistep method is called*

   - 0-stable, *if* $0 \in S$.
   - *stable in the point* $z \in \mathbb{C}$, *if* $z \in S$.
   - $A(\alpha)$-stable, *if it is stable in all* z *that lie within the set* $\{z \in \mathbb{C}^- : |arg(z) - \pi| \leq \alpha\}$ *for* $\alpha \in (0, \frac{\pi}{2})$.

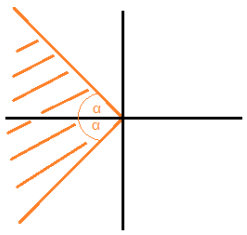

Figure 5.2: $A(\alpha)$ stability for $\alpha = \frac{\pi}{4}$

Theorem 5 ([5, Satz 4.2.10]). *Let $f(t, y)$ be sufficiently smooth and the linear multi-step method be zero-stable and consistent of order p, then it is also convergent of order p.*

5.3 Implicit linear multi-step formulas

These kinds of multi-step methods are commonly used to numerically solve the systems obtained using modified nodal analysis.

The approach to solve those systems numerically can be split into two main steps:

1. Start-Phase

   Here we calculate the first $k - 1$ steps using a method of lower order than the method we want to use ultimately. To achieve the desired convergence rates it is also important to compute these initial steps appropriately, i.e. the methods with which these initial steps are calculated have to also be of the desired order with respect to the step size h. (In applications this can mean that we calculate the initial steps with methods of lower order, but compensate by using a smaller step-size)

2. Run-Phase

   We successively apply the multi-step method to approximate at new points of the time-grid.

In the following two subchapters we will give a short overview over the two most commonly used methods for solving these kinds of systems.

5.3.1 BDF-k methods

chapter 9.2 numerik book wikipedia

The most commonly used numerical methods for solving the systems that arise in electrical circuits are the BDF-scheme and the trapezoidal rule.

We will not give a deeper look into their construction but will only state their properties. BDF schemes are appealing because they save function evaluations as much as possible.

The *backward differentiation formula (BDF)* is a family of implicit linear multistep methods. They have the general form

$$\sum_{k=0}^{s} \alpha_k y_{n+k} = h\beta f(t_{n+s}, y_{n+s}) \tag{5.20}$$

Since we are interested in the unknown y_{n+s} which is used to evaluate f, this method is implicit. The coefficients α_k and β are chosen, such that the method achieves the best possible order s.

The BDF or BDF-k formulas for $k = 1, ..., 3$ have the following form

$$\begin{aligned}
k = 1 &: hf_{m+1} = y_{m+1} - y_m \\
k = 2 &: hf_{m+2} = \frac{1}{2}(3y_{m+2} - 4y_{m+1} + y_m) \\
k = 3 &: hf_{m+3} = \frac{1}{6}(11y_{m+3} - 18y_{m+2} + 9y_{m+1} - 2y_m)
\end{aligned}$$

Methods with $k > 6$ are not zero-stable [5, page 325]. In practice, methods with order greater than 3 are rarely used because of their stability properties, as the stability-region shrinks with increasing order.

BDF schemes have consistency order $p = k$. They are methods for solving stiff equations, thus their stability is indicated by their region of absolute stability. Unfortunately not all BDF-schemes are A-stable, but their stability region still contains a large part of the complex left half-plane, see Figure 5.3.

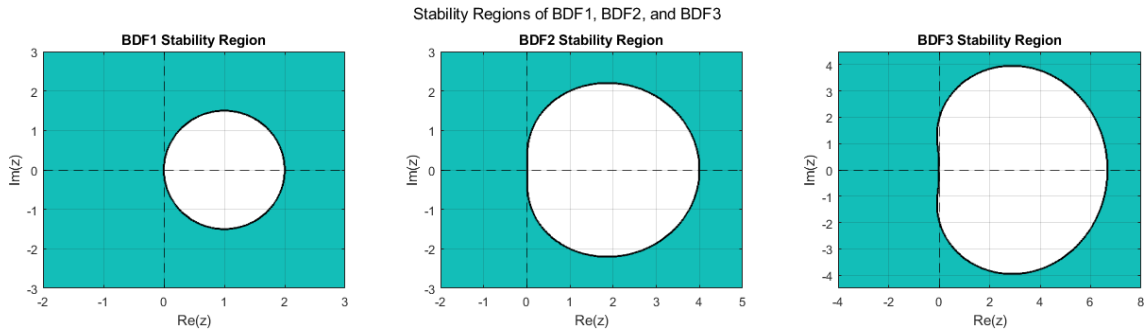

Figure 5.3: stability regions of BDF-schemes

The order of consistency of this family of multi-step methods is given by the following theorem

Theorem 6 ([5, Satz 9.2.1]). *The BDF-k methods have consistency order $p = k$.*

Corrolary 1. *The BDF-k methods with $k \leq 6$ are convergent with orderk*

Proof. Due to Theorem 5 and the fact that BDF-k methods with $k \leq 6$ are 0-stable as well as Theorem 6, they are also convergent with order k. □

5.3.2 Trapezoidal rule

The trapezoidal rule is a natural alternative to BDF2 since it is A-stable and a linear multistep method of order 2. ([2])

It works by approximating the region under the graph of the function $f(x)$ as a trapezoid, hence the name. It follows that

$$\int_a^b f(x)dx \approx (b-a)\frac{1}{2}(f(a)+f(b))$$

This procedure is repeated for small subsections of the interval $[a, b]$. Thus we obtain the iteration formula

$$u_h(t+h) = u_h(t) + \frac{h}{2}[f(t, u_h(t)) + f(t+h, u_h(t+h))].$$

Because $u_h(t+h)$ appears in f again we see that this is an implicit method. The trapezoidal rule has convergence order $p = 2$ for our applications. For details see [3], where they discuss the more general form of the trapezoidal rule, namely the Lobatto III A methods amongst other methods.

5.3.3 Gauss Method with one stage

When dealing with numerical methods for solving differential algebraic equations we have already seen that the trapezoidal rule and the BDF-k methods are relatively straight forward. But this is not always the case, many numerical methods can cause problems if one is not careful in choosing the right method. More detailed results can be found in [3] To illustrate this we will have a look at the Gauss method with one stages, also called the implicit midpoint rule, i.e.

$$y_{n+1} = y_n + hf(t_n + \frac{h}{2}, \frac{y_n + y_{n+1}}{2})$$

Applied to ordinary differential equations, this method yields the maximum possible convergence order. (source) For the special case of differential algebraic equations this does not hold true. Here we may observe lower convergence rates, especially in the algebraic variable. The convergence orders for this method are also different, if the amount of stages is even or odd. Table 5.1 displays the according convergence orders.

number of stages s	index $\nu = 1$		index $\nu = 2$	
	differential	algebraic	differential	algebraic
even	$2s$	$s+1$	$s+1$	$s-1$
odd		s	s	$s-2$

Table 5.1: Convergence order for Gauss method.

5.4 Numerical Examples

In this section we will apply the trapezoidal rule as well as the BDF-k schemes discussed in Section 5.3.1 and Section 5.3.2 to our three examples. We will also discuss the discretization error in the solution and compute appropriate consistant inital values for our problems.

Example 1.5 (Numerical computation)
We again consider the charging of a capacitor, depicted in the schematics in Figure 5.4.

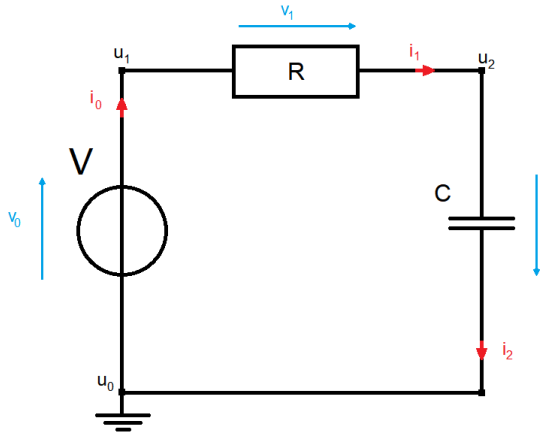

Figure 5.4: charging capacitor with series resistor and voltage source

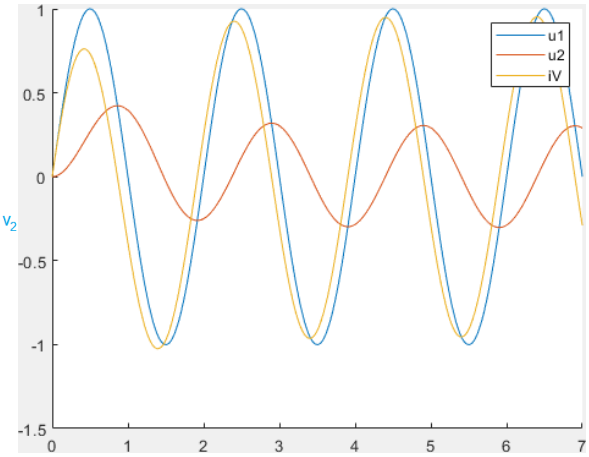

Figure 5.5: Exact solution for example 1.

For testing we set the resistance to $R = 1$ and the capacitance to $C = 1$. In our case we let the voltage source supply a voltage of the form $v_{src} = sin(\pi t)$. The resulting MNA-system for this example reads

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} * \begin{pmatrix} u_1' \\ u_2' \\ i_0' \end{pmatrix} + \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} * \begin{pmatrix} u_1 \\ u_2 \\ i_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -sin(\pi t) \end{pmatrix}. \tag{5.21}$$

We expect the resulting potentials and currents to be of the form seen in Figure 5.5.

The resulting errors for our numerical tests are displayed in table 5.2, we omit u_1 since it just equals $-v_{src}$ and thus produces no interesting error results. The reported errors are computed using $err = \max_{0 \leq n \leq N} |a_n)t_n - \tilde{y}(t_n)|$ where $y_n(t_n)$ denotes the numerical approximation and $\tilde{y}(t_n)$ denoted the exact solution at the time-step t_n.

h	k = 1		k = 2		k = 3		trapezoidal	
	u2	iV	u2	iV	u2	iV	u2	iV
0.1	4.620×10^{-2}	4.620×10^{-2}	9.567×10^{-3}	9.567×10^{-3}	3.057×10^{-3}	3.057×10^{-3}	3.344×10^{-3}	3.344×10^{-3}
0.05	2.339×10^{-2}	2.339×10^{-2}	2.454×10^{-3}	2.454×10^{-3}	6.083×10^{-4}	6.083×10^{-4}	8.367×10^{-4}	8.367×10^{-4}
0.025	1.178×10^{-2}	1.178×10^{-2}	6.264×10^{-4}	6.264×10^{-4}	1.672×10^{-4}	1.672×10^{-4}	2.092×10^{-4}	2.092×10^{-4}

Table 5.2: Resulting errors for the BDF-k methods and the trapezoidal rule.

The table 5.2 confirms our results from Section 5.3.1 and 5.3.2, that the BDF-k schemes have convergence order of $p = k$ and the trapezoidal rule has convergence order $p = 2$.

Example 2.5 (Numerical computation)
For this example we again want to consider the LC-circuit given in Figure 5.6.

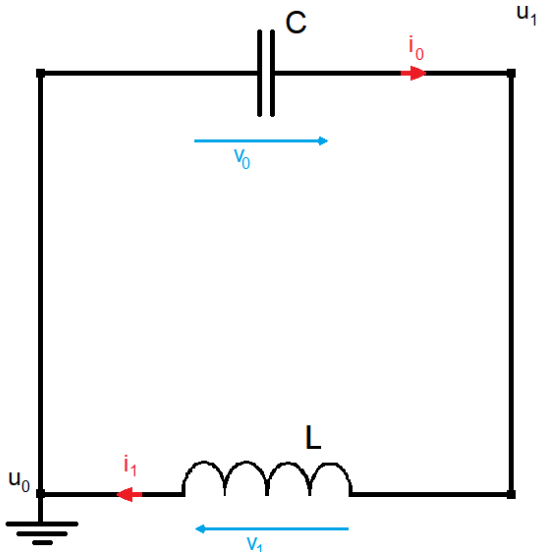

Figure 5.6: LC-circuit

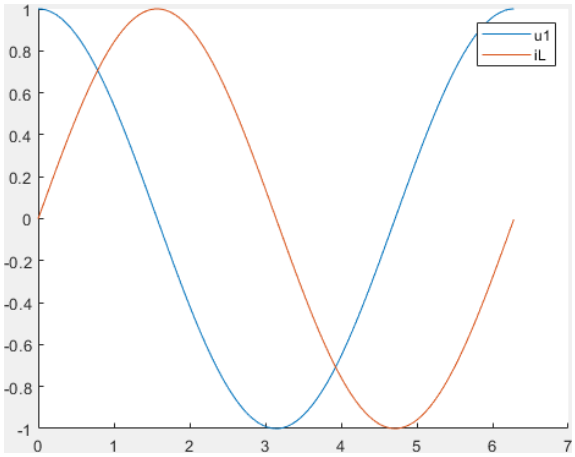

Figure 5.7: Exact solution for example 2.

We again set $L = 1$ and $C = 1$. The resulting MNA-system for this example then reads

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} u_1' \\ i_L' \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} * \begin{pmatrix} u_1 \\ i_L \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When applying the BDF-k methods **??** and the Trapezoidal rule 5.3.2 to this problem, we can observe the maximal error $err = \max_{0 \le t \le T} \|y(t) - \tilde{y}(t)\|$. This error is displayed in table 5.3.

h	k = 1		k = 2		k = 3		trapezoidal	
	u1	iL	u1	iL	u1	iL	u1	iL
0.1	7.145×10^{-1}	6.905×10^{-1}	7.763×10^{-2}	8.060×10^{-2}	5.395×10^{-3}	5.180×10^{-3}	1.963×10^{-2}	2.087×10^{-2}
0.05	4.659×10^{-1}	4.448×10^{-1}	1.964×10^{-2}	2.066×10^{-2}	5.938×10^{-4}	5.579×10^{-4}	4.912×10^{-3}	5.224×10^{-3}
0.025	2.695×10^{-1}	2.551×10^{-1}	4.924×10^{-3}	5.216×10^{-3}	5.773×10^{-5}	4.740×10^{-5}	1.228×10^{-3}	1.308×10^{-3}

Table 5.3: Resulting errors for the BDF-k methods and the trapezoidal rule.

once again we may see, that the errors reflect the predicted convergence rates.

Example 3.5 (Numerical computation)
In our last example we consider again the circuit given in Figure 5.8.

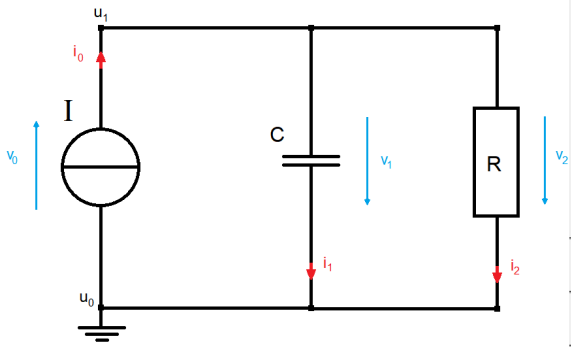

Figure 5.8: Current source with capacitor and resistor.

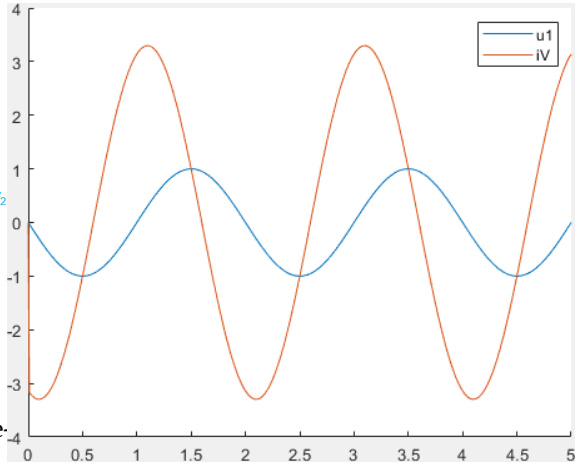

Figure 5.9: Exact solution for example 3.

Setting $R = 1, C = 1$ and $v_{src} = sin(\pi t)$ the resulting system has the form

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} * \begin{pmatrix} u_1' \\ i_V' \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} u_1 \\ i_V \end{pmatrix} = \begin{pmatrix} 0 \\ -sin(\pi t) \end{pmatrix}.$$

With this we expect the potential and current to be given as illustrated in Figure 5.9.

The numerical methods result in errors displayed in Table 5.4. Due to the fact, that u_1 is again the same as $-v_{src}$, it is not of interest for our comparison of the errors. We omit u_1.

h	k = 1 iV	k = 2 iV	k = 3 iV	trapezoidal iV
0.1	4.894×10^{-1}	1.023×10^{-1}	2.530×10^{-2}	5.219×10^{-2}
0.05	2.462×10^{-1}	2.577×10^{-2}	6.426×10^{-3}	1.295×10^{-2}
0.025	1.233×10^{-1}	6.456×10^{-3}	1.613×10^{-3}	3.232×10^{-3}

Table 5.4: Resulting errors for the BDF-k methods and the trapezoidal rule.

The errors lie again within expected convergence orders.

Code Listings

```
function y = BDFk(A, B, f, y0, tspan, h,k, yexact)
% BDF-k method for solving Ay' + By = f or Ay' = f - By or Ay' + By - f = 0
    % A: Matrix
    % B: Matrix
    % f: vector-valued function f(t)
    % y0: Initial condition - vector of size k
    % tspan: Time interval [t0, tf]
    % h: Step size
    % k: indicates the method used
    % yexact: exact solution evaluated at the initial points (# depending
    % on method used) - if given the initial values are taken directly from
    % the exact solution
    %output: vector y of size length(y0) x length(tspan(1):h:tspan(2)
    %containing the calculated timesteps of the method

    % Time vector
    t = tspan(1):h:tspan(2);
    n = length(t);

    % Preallocate solution array (containing all the time steps)
    y = zeros(length(y0), n);
    for i=1:length(y0)
        y(i, 1) = y0(i, 1);
    end

    %construct initial k values using BDF1 and BDF2 method on finer grid
    g = 2; %factor by how much the grid is finer
    if k ~= 1
        tempspan = [tspan(1),tspan(1)+h*(k-1)]; %yk is already calculated by BDFk method
        if nargin == 8
            y(:,1:k) = yexact(tspan(1):h:tspan(1) + h*(k-1));
        else
            initial = BDFk(A, B, f, y0, tempspan, h/g,k-1);
            y(:,1:k) = initial(:,1:g:end);
        end
    end

    %apply appropriate bdf-scheme
    if k == 1 %for k=1 we just implicit euler
        for i = k+1:n
            y(:,i) = (A+h*B) \ (A*y(:,i-1) + h*f(t(i)));
        end
    elseif k == 2
        for i = k+1:n
            y(:,i) = (3*A+2*h*B) \ (-A*y(:,i-2) + 4*A*y(:,i-1) + 2*h*f(t(i)));
        end
    elseif k == 3
        for i = k+1:n
            y(:,i) = (11*A+6*h*B) \ (18*A*y(:,i-1) - 9*A*y(:,i-2) + 2*A*y(:,i-3) + 6*h*f(t(i)));
        end
    end
end
```

```
function y = trapezoidal(A, B, f, y0, tspan, h)
%TRAPEZOIDAL rule for solving Ay' + By = f or Ay' = f - By or Ay' + By - f = 0
    % A: Matrix
    % B: Matrix
    % f: vector-valued function f(t)
    % y0: Initial condition - vector of size k
    % tspan: Time interval [t0, tf]
    % h: Step size
    %output: vector y of size length(y0) x length(tspan(1):h:tspan(2)
    %containing the calculated timesteps of the method

    % Time vector
    t = tspan(1):h:tspan(2);
    n = length(t);

    % Preallocate solution array (containing all the time steps)
    y = zeros(length(y0), n);
    for i=1:length(y0)
        y(i, 1) = y0(i, 1);
    end

    %time-stepping loop for Trapezoidal
    for i = 2:n
        y(:, i) = (A+(h/2)*B) \ ((A-(h/2)*B)*y(:,i-1) + (h/2)*(f(t(i)) + f(t(i-1))));  %solve the implicit equation
    end

end
```

```
function y = gauss1(A, B, f, y0, tspan, h)
%Gauss method with 1 stage for solving Ay' + By = f or Ay' = f - By or Ay' + By - f = 0
    % A: Matrix
    % B: Matrix
    % f: vector-valued function f(t)
    % y0: Initial condition - vector of size k
    % tspan: Time interval [t0, tf]
    % h: Step size
    %output: vector y of size length(y0) x length(tspan(1):h:tspan(2)
    %containing the calculated timesteps of the method

    % Time vector
    t = tspan(1):h:tspan(2);
    n = length(t);

    % Preallocate solution array (containing all the time steps)
    y = zeros(length(y0), n);
    for i=1:length(y0)
        y(i, 1) = y0(i, 1);
    end

    %time-stepping loop for Gauss
    for i = 2:n
        y(:, i) = (A+(1/2)*B) \ ((A-(1/2)*B)*y(:,i-1) + h*(f(t(i)+h/2))); %solve the implicit equation
    end

end
```

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