

Circuit Modelling



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Formulating a Mathematical Model



Formulating a Mathematical Model



Network Topology

Incidence Matrix $A = (a_{ij}) \in \mathbb{R}^{k \times l}$:

$$\tilde{a}_{ij} = \begin{cases} 1 & \text{edge } j \text{ starts at node } i, \\ -1 & \text{edge } j \text{ ends at node } i, \\ 0 & \text{else.} \end{cases}$$

With $N = (n_0, n_1, n_2, \dots, n_k)$ nodes and $E = \{e_j : j = 1, \dots, l\}$ edges, where $|N| = k$ is the number of nodes and $|E| = l$

$u = (u_0, u_1, u_2, \dots)$ the corresponding electrical potentials to the nodes.

→ reduced incidence matrix

Formulating a Mathematical Model



Energy Conservation Laws

- **Kirchhoff's voltage law (KVL):**

The sum of voltages along each loop of the network must equal to zero. Using the incidence matrix A this law can be formulated as

$$A^T u = v. \quad (1)$$

- **Kirchhoff's current law (KCL):**

For any node, the sum of currents flowing into the node is equal to the sum of currents flowing out of the node. Using the incidence matrix A again, this law can be formulated as

$$A i = 0. \quad (2)$$

Formulating a Mathematical Model



Electrical Components and their Relations

- **Resistor**

$$v = R i \quad \text{or} \quad i = G u. \quad (3)$$

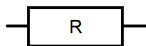
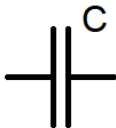


Figure: resistor symbol

- **Capacitor**

$$Q = C v \quad \text{and by derivation in t} \quad I = C \frac{d}{dt} v = C v'. \quad (4)$$



- Inductor (Coil)

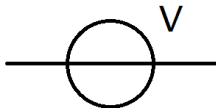
$$\Phi = L i \quad \text{and by derivation in t} \quad v = L i'. \quad (5)$$



Figure: inductor symbol

- Voltage Source

$$v = v_{src} \quad (6)$$



- **Current Source**

$$i = i_{src} \quad (7)$$

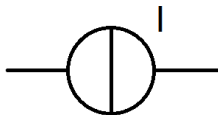


Figure: current source symbol

Formulating a Mathematical Model



Modified Nodal Analysis - MNA

$$\begin{pmatrix} A_C C A_C^T & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{pmatrix} * \begin{pmatrix} u' \\ i'_L \\ i'_V \end{pmatrix} + \begin{pmatrix} A_R G A_R^T & A_L & A_V \\ -A_L^T & 0 & 0 \\ -A_V^T & 0 & 0 \end{pmatrix} * \begin{pmatrix} u \\ i_L \\ i_V \end{pmatrix} = \begin{pmatrix} -A_I i_{src} \\ 0 \\ -v_{src} \end{pmatrix}, \quad (8)$$

Differential Algebraic Equations



Differential Algebraic Equations



Types of DAEs

In the most general form a DAE can be written as: Find $y : \mathbb{R} \rightarrow \mathbb{R}^n$ such that

$$F(t, y(t), y'(t)) = 0, \quad \forall t \in I \quad (9)$$

with $F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ sufficiently smooth and I the time-interval.

- **Linear systems with constant coefficients**

are systems of the form: find y such that

$$Ay'(t) + By(t) = f(t), \quad (10)$$

with $A, B \in \mathbb{R}^{n \times n}$, A singular, B regular and $f: \mathbb{R} \rightarrow \mathbb{R}^n$ a function in time.

- **Linear time dependent systems** are systems of the form: find y such that

$$A(t)y'(t) + B(t)y(t) = f(t),$$

with $A, B: \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$, $f: \mathbb{R} \rightarrow \mathbb{R}^n$ functions, such that for every $t \in \mathbb{R}$ the matrix $A(t)$ is singular and the matrix $B(t)$ regular.

- **Structured (non-linear) systems**

are semi-explicit systems of the form: find (y, z) such that

$$y'(t) = f(t, y(t), z(t)), \quad (11)$$

$$0 = g(t, y(t), z(t)), \quad (12)$$

with $f: \mathbb{R} \rightarrow \mathbb{R}^n$ and $g: \mathbb{R} \rightarrow \mathbb{R}^d$ functions.

Differential Algebraic Equations



Weierstrass-Kronecker Normalform

prerequisites

Theorem

Let $\{A, B\}$ be a regular matrix pencil. There exist $P, Q \in \mathbb{C}^{n \times n}$ such that

$$PAQ = \begin{pmatrix} I_d & 0 \\ 0 & N \end{pmatrix}, \quad PBQ = \begin{pmatrix} R & 0 \\ 0 & I_{n-d} \end{pmatrix}$$

where

$$N = \text{diag}(N_1, \dots, N_r) \quad \text{with} \quad N_i = \begin{pmatrix} 0 & 1 & & 0 \\ & \boxed{?} & \boxed{?} & \\ & & & 0 \\ 0 & & & 0 \end{pmatrix} \in \mathbb{R}^{n_i \times n_i}$$

and R has Jordan Normalform. By I_k we denote the identity matrix of size $k \times k$.

Proof on blackboard

Differential Algebraic Equations



Index of a Differential Algebraic Equation

Definition

Consider the differential algebraic equation (9) to be uniquely locally solvable and F sufficiently smooth. For a given $m \in \mathbb{N}$ consider the equations

$$\begin{aligned} F(t, y, y') &= 0, \\ \frac{dF(t, y, y')}{dt} &= 0, \\ &\vdots \\ \frac{d^m F(t, y, y')}{dt^m} &= 0. \end{aligned}$$

The smallest natural number m for which the above system results in an explicit system of ordinary differential equations (ODEs), i.e. it has the form

$$y' = \phi(t, y),$$

Definition

Let $y(t)$ be the exact solution of *Abstract-DAE!!!!!!!* and $\tilde{y}(t)$ be the solution of the perturbed system $F(t, \tilde{y}, \tilde{y}') = \delta(t)$. The smallest number $k \in \mathbb{N}$ such that

$$\|y(t) - \tilde{y}(t)\| \leq C \left(\|y(t_0) - \tilde{y}(t_0)\| + \sum_{j=0}^k \max_{t_0 \leq \xi \leq T} \left\| \int_{t_0}^{\xi} \frac{d^j \delta}{d\tau^j}(\tau) d\tau \right\| \right)$$

for all $\tilde{y}(t)$, is called the **perturbation index** of this system.

Differential Algebraic Equations



Consistent Initial Values

index $v = 0$.

Case: Index $v = 1$.

By rewriting our system into the form

$$\begin{aligned}y'(t) &= f(t, y(t), z(t)), \\ 0 &= g(t, y(t), z(t)).\end{aligned}$$

we are able to give conditions for consistent initial values. Namely y_0 and z_0 are consistent initial values for this system, if $g(t_0, y_0, z_0) = 0$ holds.

Case: Index $v = 2$.

For index-2 systems we rewrite our system into

$$\begin{aligned}y' &= f(t, y(t), z(t)), \\ 0 &= g(t, y(t)).\end{aligned}$$

Consistent initial values y_0, z_0 for this case not only have to fulfill $g(t_0, y_0) = 0$ but also the *hidden constraint* $g_t(t_0, y_0) + g_y(t_0, y_0)f(t_0, y_0, z_0) = 0$. By g_t and g_y we denote the derivative of g with respect to t or y , respectively.

Index Analysis of the Modified Nodal Analysis



Index Analysis of the Modified Nodal Analysis



General Index Analysis

content...

Index Analysis of the Modified Nodal Analysis



Topological Conditions

Theorem (Index conditions [shashkov_tuprints27452])

Let the matrices of the capacitances, inductances and resistances be positive definite.

- If

$$\ker([A_R, A_C, A_V, A_L]^{\mathbb{Q}}) = \{0\} \quad \text{and} \quad \ker(A_V) = \{0\} \quad (13)$$

holds, then the MNA (8) leads to a system with index $\nu \leq 2$.

- If additionally

$$\ker([A_R, A_C, A_V]^{\mathbb{Q}}) = \{0\} \quad \text{and} \quad \ker([A_C, A_V]) = \{0\} \quad (14)$$

holds, then the system is of index $\nu \leq 1$

- If further

$$\ker(A_C^{\mathbb{Q}}) = \{0\} \quad \text{and} \quad \dim(v_{src}) = 0 \quad (15)$$

holds, then the system has index $\nu = 0$.

- Condition (13) can be interpreted, as the circuit neither containing loops of voltage sources nor cutsets of current sources.
- Condition (14) can be interpreted, as the circuit containing neither loops of capacitors and/or voltage sources nor cutsets of inductors and/or current sources.
- Condition (15) can be interpreted, as every node in the circuit being connected to the reference node (ground) through a path containing only the capacitors.

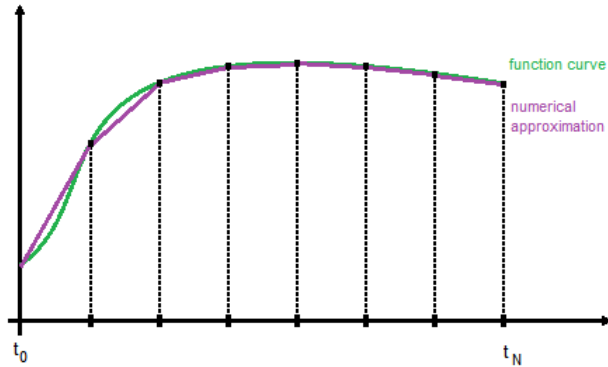
Numerical Solutions



general initial value problem Find y , such that

$$y'(t) = f(t, y), \quad t \in [t_0, t_l], \quad (16)$$

$$y(t_0) = y_0. \quad (17)$$



Numerical Solutions



Single-Step Methods

Definition

A numerical method to approximate a differential equation 16 on a time-grid t_0, \dots, t_l with the intermediate values y_0, \dots, y_l is called a single-step method if it is of the form

$$y_{j+1} = y_j + h_j \phi(t_j, y_j, y_{j+1}, h_j). \quad (18)$$

We call ϕ the *procedural function*. If ϕ does not depend on y_{j+1} , then the method is called *explicit*, otherwise it is called *implicit*.

Numerical Solutions



Single-Step Methods

Consistency, Stability and Convergence

Definition

Let y_{m+1} be the result of one step of a single step method (18) with the exact start-vector $y_m = y(t_m)$ then

$$\delta_{m+1} = \delta(t_m + h) = y(t_{m+1}) - \tilde{y}_{m+1}, \quad m = 0, \dots, N-1 \quad (19)$$

is called the *local discretization error* of the single step method at the point t_{m+1} .

Definition

A single-step method is called *consistent* if for all initial value problems (16)

$$\lim_{h \rightarrow 0} \frac{\|\delta(t+h)\|}{h} = 0 \quad \text{for} \quad t_0 \leq t \leq t_l \quad (20)$$

holds.

It is called *consistent of order p* , if for a sufficiently smooth function f

$$\|\delta(t+h)\| \leq Ch^{p+1} \quad \text{for all} \quad h \in (0, H] \quad \text{and} \quad t_0 \leq t \leq t_l - h \quad (21)$$

holds with C independent of h .

Definition

A single-step method is called *convergent*, if for all initial value problems (16) for the *global discretization error*

$$e_m = y(t_m) - y_m$$

holds that

$$\max_m \|e_m\| \rightarrow 0 \quad \text{for} \quad h_{\max} \rightarrow 0.$$

The single-step method is called to have the *convergence order* p , if

$$\max_m \|e_m\| \leq Ch_{\max}^p \quad \text{for} \quad h_{\max} \in (0, H] \quad \text{with} \quad t_0 \leq t_m \leq t_l$$

with the constant C not dependent on the step size h .

Definition

A single-step method is called (*discretely*) *stable* if for grid-functions y_h and \tilde{y}_h with

$$y_{i+1} = y_i + h\phi(t_i, y_i), \quad (22)$$

$$\tilde{y}_{i+1} = \tilde{y}_i + h[\phi(t_i, \tilde{y}_i) + \theta_i], \quad (23)$$

and perturbations $\theta_i = \theta_h(t_i)$ of the right side as well as a bounded perturbation in the initial-values $y_0 - \tilde{y}_0$ the error is bounded by

$$\|y_h - \tilde{y}_h\|_{\infty, h} \leq C(\|y_0 - \tilde{y}_0\|_{l^2} + \|\theta_h\|_{\infty, h})$$

with a constant C which is not dependent on h . The norm $\|\cdot\|_{\infty, h}$ denotes the maximum norm over the time-grid, i.e. for a function $b : T = t_0, \dots, t_N \rightarrow \mathbb{R}^d$ we have $\|b\|_{\infty, h} = \max_{t \in T} \|b(t)\|$, $\|b\|$ is the euclidean norm.

Numerical Solutions



Single-Step Methods
further stability properties

Dahlquist equation, i.e. find y such that

$$y' = \lambda y, \quad t > 0 \quad (24)$$

$$y(0) = y_0 \quad (25)$$

with $\lambda \in \mathbb{C}$ and y_0 fixed.

Definition

1. If a single-step method can be written in the form

$$y_{i+1} = R(z) y_i, \quad z := h\lambda \quad (26)$$

then we call $R : \mathbb{C} \rightarrow \mathbb{C}$ the *stability function* of the single-step method.

2. The set

$$S := \{z \in \mathbb{C} : |R(z)| \leq 1\} \quad (27)$$

is called the *region of stability* of the method.

3. A single-step method is called

- *0-stable*, if $0 \in S$.
- *A-stable*, if $\mathbb{C}^- \subset S$.
- *L-stable*, if $R(z) \rightarrow 0$ for $\operatorname{Re}(z) \rightarrow -\infty$.

Numerical Solutions



Multistep Methods

Definition

For given $\alpha_0, \dots, \alpha_k$ and β_0, \dots, β_k the iteration rule

$$\sum_{l=0}^k \alpha_l y_{m+l} = h \sum_{l=0}^k \beta_l f(t_{m+l}, y_{m+l}), \quad m = 0, 1, \dots, N - k \quad (28)$$

is called a *linear multistep method* (linear k-step method). It is always assumed that $\alpha_k \neq 0$ and $|\alpha_0| + |\beta_k| > 0$. If $\beta_k = 0$ holds, then the method is called explicit, otherwise implicit.

Numerical Solutions



Multistep Methods

Consistency, Convergence and Stability

Definition

Let y_{m+k} be the result of one step of the multi-step method (28) with the start-values given as the evaluations of the exact solution $y_{m+l} = y(t_{m+l})$ at $0 \leq l < k$. This means

$$\alpha_k \tilde{u}_{m+k} = \sum_{l=0}^{k-1} (h\beta_l f(t_{m+l}, y(t_{m+l})) - \alpha_l y(t_{m+l})) + h\beta_k f(t_{m+k}, y_{m+k}).$$

Then

$$\delta_{m+k} = \delta(t_{m+k}) = y(t_{m+k}) - y_{m+k}, \quad m = 0, 1, \dots, N - k$$

is called the *local discretization error* (local error) of the linear multi-step method, see Def. 28 at the point t_{m+k} .

Definition

A linear multi-step method is called *consistent*, if for all functions $y(t) \in C^2([t_0, t_l])$

$$\lim_{h \rightarrow 0} \frac{1}{h} L[y(t), h] = 0$$

holds. It has the *consistency order* p , if for all functions $y(t) \in C^{p+1}[t_0, t_l]$

$$L[y(t), h] = O(h^{p+1}) \quad \text{for } h \rightarrow 0$$

holds.

Definition

We say that a linear multi-step method is convergent if for a solution y of the problem a solution vector created by an LMSM y_j for $j \in 0, \dots, k$ we have that

$$\lim_{h \rightarrow \infty} \max_{0 \leq j \leq k} \|y(t_j) - y_j\| = 0.$$

Definition

A linear multi-step method is called (discretely) stable, if for solutions y_h and \tilde{y}_h of

$$\sum_{l=0}^k \alpha_l y_{m+l} = h \sum_{l=0}^k \beta_l f(t_{m+l}, y_{m+l}), \quad (29)$$

$$\sum_{l=0}^k \alpha_l \tilde{y}_{m+l} = h \sum_{l=0}^k \beta_l f(t_{m+l}, \tilde{y}_{m+l}) + h\theta_n \quad (30)$$

and bounded initial values $y_j - \tilde{y}_j$ for $j \in 0, \dots, k$ we have that

$$\max_{t_0 \leq t_n \leq T} \|y_n - \tilde{y}_n\| \leq C \sum_{j=0}^{k-1} \|y_j - \tilde{y}_j\| + \max_{t_0 \leq t_n \leq T} \|\theta_n\|.$$

Numerical Solutions



Multistep Methods
further stability properties

Dahlquist test problem as a model problem, find y such that

$$y' = \lambda y, \quad t > 0 \quad (31)$$

$$y(0) = y_0 \quad (32)$$

with $\lambda \in \mathbb{C}$ and y_0 fixed.

Thus the resulting linear multistep method is of the form

$$\sum_{l=0}^k \alpha_l y_{n+l} = h \sum_{l=0}^k \beta_l \lambda y_{n+l}$$

$$\iff \sum_{l=0}^k [\alpha_l - h\beta_l \lambda] y_{n+l}$$

Definition

1. The set

$$S := \{z \in \mathbb{C} : \rho(\xi) - z\sigma(\xi) = 0 \implies \xi \in \mathbb{C} \text{ and } |\xi| \leq 1. \\ \text{If } \xi \text{ has multiplicity greater than 1, then } |\xi| < 1\} \quad (33)$$

is called the region of stability of the method.

2. A linear multistep method is called

- *0-stable*, if $0 \in S$.
- *stable* in the point $z \in \mathbb{C}$, if $z \in S$.
- *$A(\alpha)$ -stable*, if it is stable in all z that lie within the set $\{z \in \mathbb{C}^- : |\arg(z) - \pi| \leq \alpha\}$ for $\alpha \in (0, \frac{\pi}{2})$.

Theorem

Let $f(t, y)$ be sufficiently smooth and the linear multi-step method be zero-stable and consistent of order p , then it is also convergent of order p .

Numerical Solutions



Implicit Linear Multistep Formulas

Numerical Solutions



Implicit Linear Multistep Formulas
BDK-k Methods

The *backward differentiation formula (BDF)* is a family of implicit linear multistep methods. They have the general form

$$\sum_{k=0}^s \alpha_k y_{n+k} = h\beta f(t_{n+s}, y_{n+s}) \quad (34)$$

The BDF or BDF-k formulas for $k = 1, \dots, 3$ have the following form

$$k = 1 : hf_{m+1} = y_{m+1} - y_m$$

$$k = 2 : hf_{m+2} = \frac{1}{2}(3y_{m+2} - 4y_{m+1} + y_m)$$

$$k = 3 : hf_{m+3} = \frac{1}{6}(11y_{m+3} - 18y_{m+2} + 9y_{m+1} - 2y_m)$$

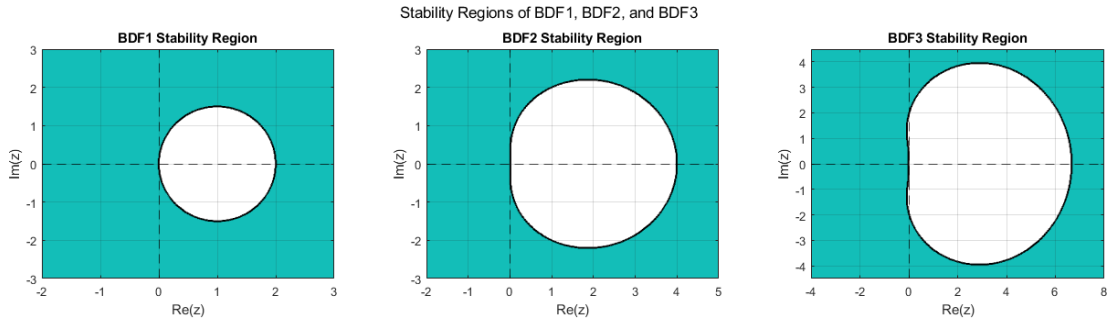


Figure: stability regions of BDF-schemes

Theorem ([NumerikGewöhnlicherDifferentialgleichungen])

The BDF- k methods have consistency order $p = k$.

Numerical Solutions



Implicit Linear Multistep Formulas
Trapezoidal rule

This procedure is repeated for small subsections of the interval $[a, b]$. Thus we obtain the iteration formula

$$u_h(t+h) = u_h(t) + \frac{h}{2}[f(t, u_h(t)) + f(t+h, u_h(t+h))].$$

Numerical Solutions



Numerical Examples

Numerical Solutions



Numerical Examples

Example1

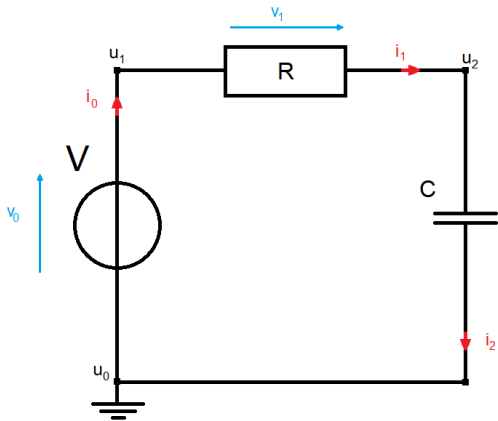


Figure: charging capacitor with series resistor and voltage source

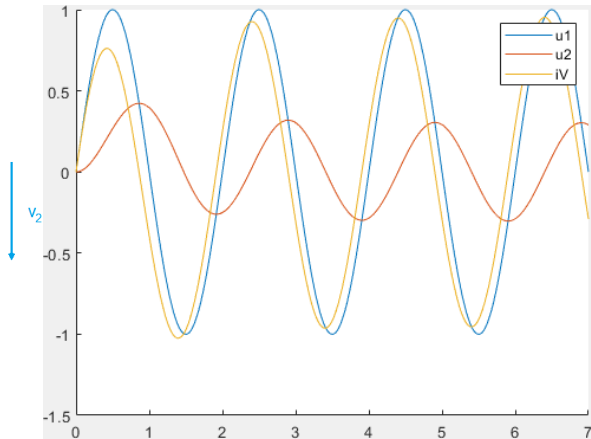


Figure: Exact solution for example 1.

h	k = 1		k = 2		k = 3		trapezoidal	
	u2	iV	u2	iV	u2	iV	u2	iV
0.1	4.620×10^{-2}	4.620×10^{-2}	9.567×10^{-3}	9.567×10^{-3}	3.057×10^{-3}	3.057×10^{-3}	3.344×10^{-3}	3.344×10^{-3}
0.05	2.339×10^{-2}	2.339×10^{-2}	2.454×10^{-3}	2.454×10^{-3}	6.083×10^{-4}	6.083×10^{-4}	8.367×10^{-4}	8.367×10^{-4}
0.025	1.178×10^{-2}	1.178×10^{-2}	6.264×10^{-4}	6.264×10^{-4}	1.672×10^{-4}	1.672×10^{-4}	2.092×10^{-4}	2.092×10^{-4}

Table: Resulting errors for the BDF-k methods and ther trapezoidal rule.

Numerical Solutions



Numerical Examples
Example 2

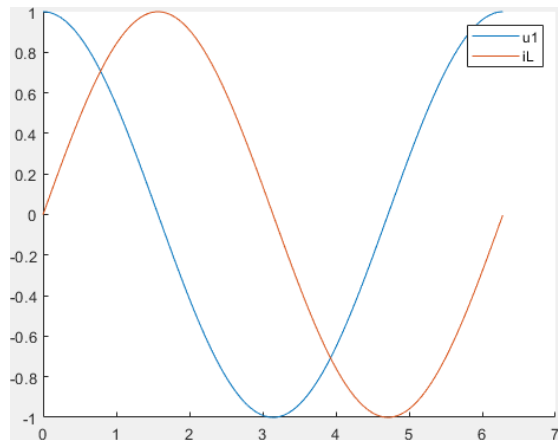
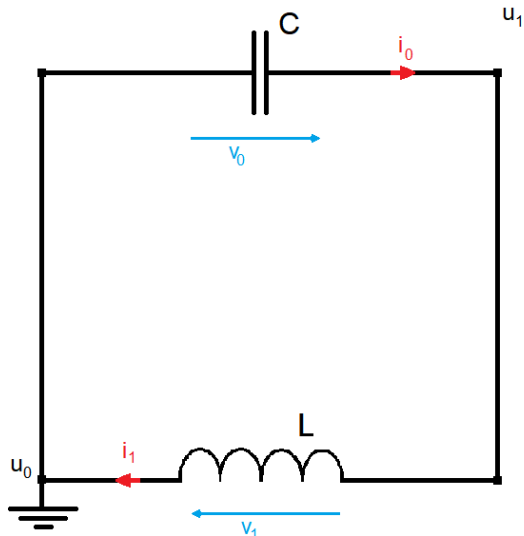


Figure: Exact solution for example 2.

h	k = 1		k = 2		k = 3		trapezoidal	
	u1	iL	u1	iL	u1	iL	u1	iL
0.1	7.145×10^{-1}	6.905×10^{-1}	7.763×10^{-2}	8.060×10^{-2}	5.395×10^{-3}	5.180×10^{-3}	1.963×10^{-2}	2.087×10^{-2}
0.05	4.659×10^{-1}	4.448×10^{-1}	1.964×10^{-2}	2.066×10^{-2}	5.938×10^{-4}	5.579×10^{-4}	4.912×10^{-3}	5.224×10^{-3}
0.025	2.695×10^{-1}	2.551×10^{-1}	4.924×10^{-3}	5.216×10^{-3}	5.773×10^{-5}	4.740×10^{-5}	1.228×10^{-3}	1.308×10^{-3}

Table: Resulting errors for the BDF-k methods and ther trapezoidal rule.

Numerical Solutions



Numerical Examples
Example 3

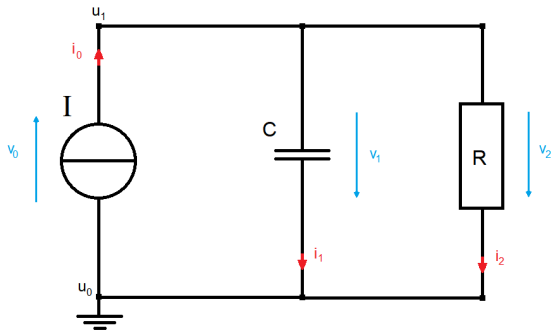


Figure: Current source with capacitor and resistor.

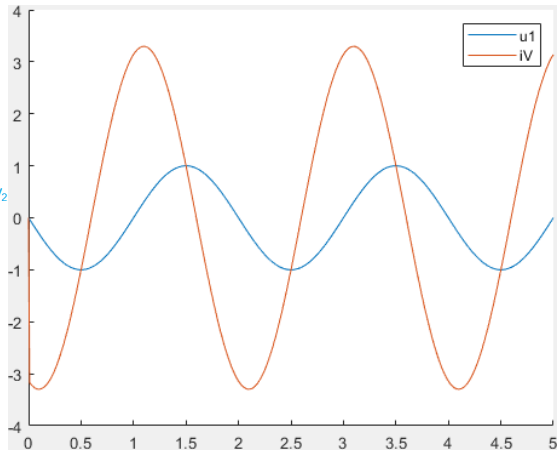


Figure: Exact solution for example 3.

h	k = 1 iV	k = 2 iV	k = 3 iV	trapezoidal iV
0.1	4.894×10^{-1}	1.023×10^{-1}	2.530×10^{-2}	5.219×10^{-2}
0.05	2.462×10^{-1}	2.577×10^{-2}	6.426×10^{-3}	1.295×10^{-2}
0.025	1.233×10^{-1}	6.456×10^{-3}	1.613×10^{-3}	3.232×10^{-3}

Table: Resulting errors for the BDF-k methods and the trapezoidal rule.