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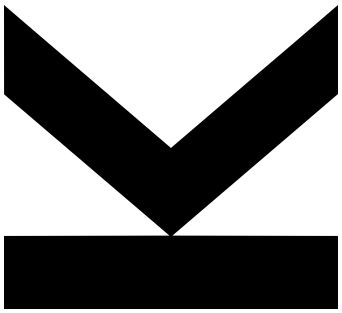
Circuit Modelling



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Kurzfassung

Kurzfassung auf Deutsch.

Abstract

Abstract in English.

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1 Introduction

This chapter should include information about what circuit modelling wants to achieve as well as giving an overview of what this bachelor-thesis is about.

What is this thesis about?
Modelling and numerically solving systems that arrise from electrical circuits with RLC elements. Furthermore it will briefly discuss on expanding this baseline with more complicated electrical components. What is the goal of this thesis?
The goal of this thesis is to give insight into industrial standards concerning circuit modelling. It aims to elaborate on the underlying concepts of MNA as well as on the most commonly used numerical methods.

2 Formulating a Mathematical Model

based on DAE lecture and [1]

To accurately represent a physical system in a mathematical model we first have to think about **how** to best formulate a mathematical model that formulates this system accurately enough. This chapter aims to elaborate on the Modified Nodal Analysis (MNA) of an electrical circuit, which is the industry standard for these kinds of models. For this we have to introduce the notions of network topology as well as some basic electrical components and laws. Most of the following notions such as voltage, current and electrical potentials are time dependent. For better readability we will leave out the time argument t in some cases.

2.1 Network Topology

An electrical circuit is usually considered as a graph $(\mathcal{N}, \mathcal{E})$ where $\mathcal{N} = (n_0, n_1, n_2, ..., n_k)$ denotes nodes and $\mathcal{E} = \{e_{ij} : i, j = 1, ..., l\}$ is the set of edges where k is the number of nodes and l the number of edges. For some i and some j we have that $e_{ij} = (n_i, n_j)$ is the edge from node i to node j. We can store this information in an *incidence matrix* $\tilde{A} = (\tilde{a}_{ij}) \in \mathbb{R}^{k \times l}$ which is defined by

$$\tilde{a}_{ij} = \begin{cases} 1 & \text{edge } j \text{ starts at node } i, \\ -1 & \text{edge } j \text{ ends at node } i, \\ 0 & \text{else.} \end{cases}$$

We call $u = (u_0, u_1, u_2, ...)$ the corresponding electrical potentials (or just potentials) to the nodes N. The difference between those potentials at two connected nodes is called the voltage at the respective edge. To fix the absolute values of these potentials we have

to set one node to a fixed potential. We will do that by "grounding" the node n_0, which means we set the potential $u_0 := 0$. The grounding of a node allows us to remove the corresponding row from the incidence matrix to get the *reduced incidence matrix* A. The vector $v = (v_{ij})_{ij}$ represents the voltages at the edges. For some i and some j the voltage at edge ij is $v_{ij} = u_i - u_j$.

We will later see, that the components of an electrical crircuit, which will be installed along the edges, describe a relationship between the edges current and its voltage. Thus a current vector $i = (i_1, i_2, i_3, ...)$ containing the currents along the edges is required.

Example
As an example we consider the charging of a capacitor. The circuit is given as

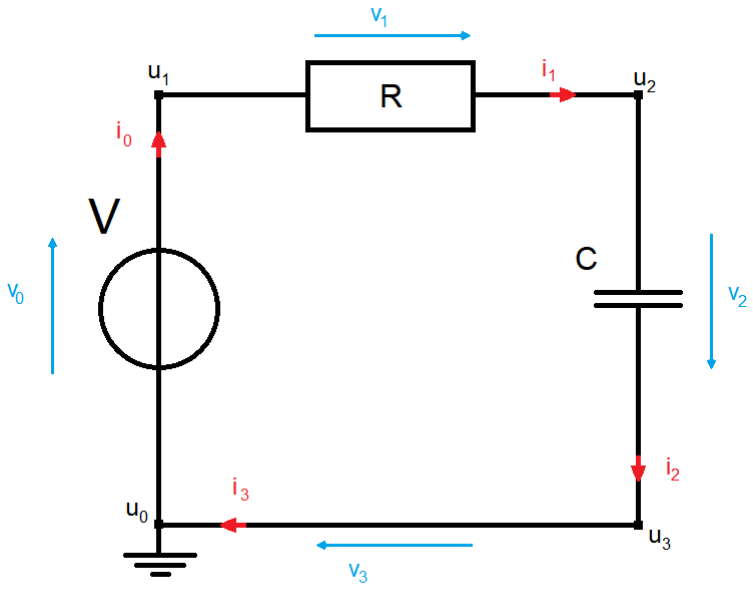

Figure 2.1: charging capacitor with series resistor and voltage source

with the node-potentials, the voltages and the currents collected in the vectors

$$u = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad v = \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad i = \begin{pmatrix} i_0 \\ i_1 \\ i_2 \\ i_3 \end{pmatrix}.$$

The incidence matrix of this circuit has the form

$$\tilde{A} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

The rows of this matric correspond with the nodes of the circuit and the columns correspond with the edges. By grounding node 0 this reduces to

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

In this circuit edge 3 is not populated with any components. This means that the voltage along this edge does not change, or in formula

$$u_0 - u_3 = 0.$$

Thus we can consider this circuit with node 0 and node 3 merged. This leads to a slightly different incidence and reduced incidence matrix.

$$\tilde{A} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

2.2 Energy Conservation Laws

To fully fix all the variables that arrise in the model of an electrical circuit we will need some *conservation laws*:

- **Kirchhoff's voltage law (KVL):**
  The sum of voltages along each loop of the network must equal to zero. Using the incidence matrix A this law can be formulated as

$$A^\top u = v. \tag{2.1}$$

- **Kirchhoff's current law (KCL):**
  For any node, the sum of currents flowing into the node is equal to the sum of currents flowing out of the node. Using the incidence matrix A again, this law can be formulated as

$$Ai = 0. \tag{2.2}$$

2.3 Electrical Components and their relations

Electrical components are described by equations relating their edge voltage v to their edge current i. We will mainly focus on so-called RLC-networks which consist of resistors, capacitors, inductances, voltage sources and current sources. Diodes and Transistors as well as other electrical components can be described in a similar way, although these lead to a more difficult analysis of the system.

- **Resistor**
  Resistors "resist" the flow of current, which causes voltage to drop. This behaviour is described by the *resistance* $R \in \mathbb{R}^+ := \{x \in \mathbb{R} : x > 0\}$ which is given in *Ohm* (Ω) and its reciprocal, the *conductance* $G \in \mathbb{R}^+$, which is given in *Siemens* ($S = \frac{1}{\Omega}$).

$$v = R\,i \quad \text{or} \quad i = G\,u.$$

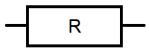

Figure 2.2: resistor symbol

- **Capacitor**
  Capacitors "store" electrical energy by accumulating electrical charge. Their characteristic equations can be described directly using the stored charge $Q \in \mathbb{R}_0^+ := \{x\ in\mathbb{R} : x \geq 0\}$ or indirectly using the change in charge, which is nothing other than the current I. The *capacitance* $C \in \mathbb{R}^+$ is given in *Farads* (F).

$$Q = C\,v \quad \text{and by derivation in t} \quad I = C\,\frac{d}{dt}v := C\,v'.$$

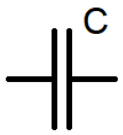

Figure 2.3: capacitor symbol

- **Inductor (Coil)**
  An electric current flowing through a conductor generates a magnetic field $\Phi \in \mathbb{R}$ surrounding it. This magnetic field causes a voltage drop dependant on the change in current. The *inductance* $L \in \mathbb{R}^+$ is given in *Henry* (H).

  $$\Phi = L\, i \quad \text{and by derivation in t} \quad v = L\, i'.$$

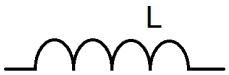

Figure 2.4: inductor symbol

- **Voltage Source**
  A voltage source supplies the system with a voltage. It can either supply varying amounts of voltage (with the special case of alternating current AC) or a fixed amount of voltage. The unit of voltage is *Volts* (V).

  $$v = v_{src}$$

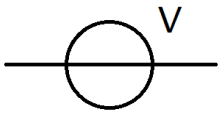

Figure 2.5: voltage source symbol

- **Current Source**
  A current source supplies the system with current. It can either supply varying

amounts of current (with the special case of alternating current AC) or a fixed amount of current. The unit of current is *Ampere* (A).

$$i = i_{src}$$

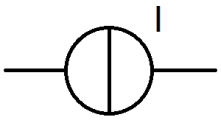

Figure 2.6: current source symbol

The resistance R, conductance G, capacitance C and inductance L are respectively described as a scalar constant that relates the edge-current to the edge-voltage. If there are more components of the same kind in one circuit their corresponding constants will usually be collected into a matrix which is also called the same and also denoted by the same letter, respectively. These matrices are then positive definite diagonal matrices.

2.4 Modified Nodal Analysis - MNA

[1] and [2]

To analyse the network further we will sort the reduced incidence matrix A such that is has the block form

$$A = (A_R A_C A_L A_V A_I)$$

where A_R, A_C, A_L, A_V and A_I include the columns that are related to the resistors, capacitors, coils, voltage sources and current sources, respectively.

To mathematically describe the circuit we will use *modified nodal analysis* (or short MNA). MNA uses the node voltages as well as the currents of the coils and the voltage sources as

unknowns. It is based on the conservation laws (2.2) and (2.1) as well as on the voltage-current relations of the electrical components. The voltages can be represented using the node-potentials

$$v = A^\top u$$

The vector v can thus be divided into $v = (v_R, v_C, v_L, v_s rc, v_I)$. In a similar way we also divide the current vector into $i = (i_R, i_C, i_L, i_V, i_s rc)$. Using the sorted incidence matrix blocks we can write the voltage-current relations of the components as follows.

- The resistor current relation results in

$$i_R = G\, v_R = G,$$

- the capacitor relation results in

$$i_C = C\, v_C' = C\, A_C^\top u'$$

Kirchhoffs current law (2.2) gives us that

$$A_C i_C + A_R i_R + A_L i_L + A_V i_V = -A_I i_{src}.$$

Now plugging in the component relations we obtain

$$A_C C A_C^\top u' + A_R G A_R^\top + A_L i_L + A_V i_V = -A_I i_{src}$$

Combining this with the component law for inductors and the potential-voltage relation for voltage sources we finally get the modified nodal analysis equations

$$\begin{aligned} A_C C A_C^\top u' + A_R G A_R^\top u + A_L i_L + A_V i_V &= -A_I i_{src}, \\ L i_L' - A_L^\top u &= 0, \\ -A_V^\top &= -v_{src}. \end{aligned}$$

In matrix form these read as

$$\begin{pmatrix} A_C C A_C^\top & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{pmatrix} * \begin{pmatrix} u' \\ i_L' \\ i_V' \end{pmatrix} + \begin{pmatrix} A_R G A_R^\top & A_L & A_V \\ -A_L^\top & 0 & 0 \\ -A_V^\top & 0 & 0 \end{pmatrix} * \begin{pmatrix} u \\ i_L \\ i_V \end{pmatrix} = \begin{pmatrix} -A_I i_{src} \\ 0 \\ -v_{src} \end{pmatrix}, \quad (2.3)$$

where the diagonal matrices C, G and L contain the capacities, conductivities and inductivities.

The resulting systems are *stiff* systems. This means that for their numerical solution special care has to be put into which methods are suitable for a solving these sytsems in a stable manner.

Example
We again consider the charging of a capacitor. The circuit is again given as in 2.1, where we have also already found the reduced incidence matrix

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

This matrix can be split into the three submatrices A_V, A_R and A_C containing the columns of the matrix A. The matrices A_L and A_V are empty in this case and the diagonal matrices containing the components constants are

$$C = (C), \qquad L = (), \qquad G = (\frac{1}{R}).$$

Plugging this into the formula (2.3) we obtain the system

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{pmatrix} * \begin{pmatrix} u_,' \\ u_2' \\ i_0' \end{pmatrix} + \begin{pmatrix} \frac{1}{R} & \frac{1}{R} & 0 \\ \frac{1}{R} & \frac{1}{R} & -1 \\ 0 & -1 & 0 \end{pmatrix} * \begin{pmatrix} u_1 \\ u_2 \\ i_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -v_{src} \end{pmatrix}.$$

2.5 Charge/Flux oriented formulation of MNA

This time we use the charge and flux based formulations. Again using KCL (2.2) and the component equations we formulate a system of equations. This means that instead

of directly using current and voltage we use the flux of the magnetic field (used in the description of inductances) and the charge (used in the description of capacitors). We obtain

$$A_C q' + A_R r(A_R^\top u, t) + A_L i_L + A_V i_V + A_I i(A^\top u, q', i_L, i_V, t) = 0, \tag{2.4}$$

$$\phi' - A_L^\top u = 0, \tag{2.5}$$

$$v(A^\top u, q', i_L, i_V, t) - A_V^\top u = 0, \tag{2.6}$$

$$q - q_C(A_C^\top u) = 0, \tag{2.7}$$

$$\phi - \phi_L(i_L) = 0. \tag{2.8}$$

Using node potentials u, branch currents through voltage and flux controlled elements i_V and i_L, charges and fluxes q and ϕ, voltage dependent resistors r, voltage and current dependent charge and flux sources q_C and ϕ_L, controlled current and voltage sources i_{src} and v_{src}.

We call this formulation the *charge-flux oriented modified nodal analysis*.

3 Differential Algebraic Equations

von [2] und DAE lecture

The resulting MNA of electrical circuits leads to a special form of differential equations, namely *differential algebraic equations* (DAE). To understand the general solvability of those systems and to find good numerical methods we first have to take a look at the theory of such DAEs.

In the most general form a DAE can be written as: Find $y : \mathbb{R} \to \mathbb{R}^n$ such that

$$F(t, y(t), y'(t)) = 0, \qquad \forall t \in I \tag{3.1}$$

with $F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ sufficiently smooth and I the time-interval. As this is usually too general one has to consider more specific types of DAEs. This chapter focuses on giving a brief overview of the different types of differential algebraic equations. We will discuss the specific form of linear systems with constant coefficients in more detail as those are the systems that arise from modified nodal analysis. Additionally this chapter also discusses the notion of the index of an MNA-system.

3.1 Types of DAEs

Commonly systems of either of the three types below are considered when talking about DAEs.

- **Linear systems with constant coeffiecients**
  are systems of the form: find y such that

$$Ay'(t) + By(t) = f(t), \tag{3.2}$$

with $A, B \in \mathbb{R}^{n\times n}$, A singular, B regular and $f : \mathbb{R} \to \mathbb{R}^n$ a function in time. If A would is regular this equation could be transformed into an ordinary differental equation, since

$$y'(t) + A^{-1}By(t) = A^{-1}f(t)$$

Thus it is reasonable to assume A singular.

- **Linear time dependent systems** are systems of the form: find y such that

  $$A(t)y'(t) + B(t)y(t) = f(t),$$

  with $A, B : \mathbb{R} \to \mathbb{R}^{n\times n}$, $f : \mathbb{R} \to \mathbb{R}^n$ functions.

- **Structured (non-linear) systems**
  are semi-explicit systems of the form: find (y, z) such that

  $$y'(t) = f(t, y(t), z(t)), \tag{3.3}$$
  $$0 = g(t, y(t), z(t)), \tag{3.4}$$

  with $f : \mathbb{R} \to \mathbb{R}^n$ and $g : \mathbb{R} \to \mathbb{R}^d$ functions.

For our analysis of electrical networks we will focus on linear systems with constant coefficients. The other systems could for example occur when we also consider time-dependant component relations e.g. Temperature.

3.1.1 Weierstraß-Kronecker Normalform

To determine the solvability of a linear system with constant coefficients (3.2) we first need to introduce a Normalform for the system, the *Weierstraß-Kronecker Normalform*. This Normalform is dependent on the family $\{A, B\} := \{\mu A + B | \mu \in \mathbb{R}\}$, which is called the *matrix pencil* of the DAE.

Definition 1. *The matrix pencil $\{A, B\}$ is called* regular *if there exists some $c \in \mathbb{R}$, such that $(cA + B)$ is regular ($det(cA + B) \neq 0$), otherwise it is called singular.*

Theorem 1 (Jordan Normalform [2, Theorem 13.2.1]). *For every matrix $Q \in \mathbb{R}^{n\times n}$ there exists a regular matrix $T \in \mathbb{C}^{n\times n}$, such that*

$$T^{-1}QT = J = diag(J_1, ..., J_r) \quad with \quad J_i = \begin{pmatrix} \lambda_i & 1 & & 0 \\ 0 & \lambda_i & \ddots & \vdots \\ & \ddots & \ddots & 1 \\ 0 & \dots & 0 & \lambda_i \end{pmatrix} \in \mathbb{C}^{m_i\times m_i}$$

and $n = m_1 + ... + m_r$.

The matrix J is called Jordan Normalform of Q, the matrices J_i are called Jordan Blocks, where λ_i are corresponding the eigenvalues of Q. The matrix J is uniquely determined by Q except for the arrangement of the diagonal blocks. If Q posesses only real eigenvalues, then T can also be choosen from the reals.
A transformation from A and B in 3.2 enables a seperation into differential and algebraic variables.

Theorem 2 ([2, Satz 13.2.2]). *Let $\{A, B\}$ be a regular matrix pencil. There exist $P, Q \in \mathbb{C}^{n\times n}$ such that*

$$PAQ = \begin{pmatrix} I_d & 0 \\ 0 & N \end{pmatrix}, \quad PBQ = \begin{pmatrix} R & 0 \\ 0 & I_{n-d} \end{pmatrix}$$

where

$$N = diag(N_1, ..., N_r) \quad with \quad N_i = \begin{pmatrix} 0 & 1 & & 0 & \\ & \ddots & \ddots & & \\ & & & 0 & 1 \\ 0 & & & 0 & \end{pmatrix} \in \mathbb{R}^{n_i\times n_i}$$

and R has Jordan Normalform.

By I_k we denote the identity matrix of size $k \times k$.

Proof. Because $\{A, B\}$ is a regular matrix pencil, there exists $c \in \mathbb{R}$ such that $(cA + B)$ is regular. Set

$$\hat{A} := (cA + B)^{-1}A, \quad \hat{B} := (cA + B)^{-1}B.$$

Considering

$$(cA + B)^{-1}(cA + B) = I \implies (cA + B)^{-1}B + c(cA + B)^{-1}A = I,$$

we get that

$$\hat{B} = I - c\hat{A}.$$

Let $J_{\hat{A}}$ be the Jordan Normalform of $\hat{A}$, this means that there exists a regular matrix T_1 such that

$$T_1^{-1}AT_1 = J_{\hat{A}} = \begin{pmatrix} W & 0 \\ 0 & \tilde{N} \end{pmatrix}.$$

The matrix W contains the Jordanblocks with Eigenvalues which are nonzero, the matrix $\tilde{N}$ contains the Jordan blocks with Eigenvalues equal to zero, thus $\tilde{N}$ is *nilpotent*. The Jordan Normalform $J_{\hat{B}}$ of $\hat{B}$ is given by

$$T_1^{-1}\hat{B}T_1 = J_{\hat{B}} = \begin{pmatrix} I - cW & 0 \\ 0 & I - c\tilde{N} \end{pmatrix}.$$

The following two transformations will allow us to get the desired structure. First we will transform $J_{\hat{A}}$ with

$$T_2 := \begin{pmatrix} W & 0 \\ 0 & I - c\tilde{N} \end{pmatrix}$$

in

$$T_2^{-1}J_{\hat{A}} = \begin{pmatrix} I & 0 \\ 0 & (I - c\tilde{N})^{-1}\tilde{N} \end{pmatrix}$$

and $J_{\hat{B}}$ in

$$T_2^{-1} J_{\hat{B}} = \begin{pmatrix} W^{-1} - cI & 0 \\ 0 & I \end{pmatrix}.$$

Let now R be the Jordan Normalform of $(W^{-1} - cI)$ and N be the Normalform of $(I - c\tilde{N})^{-1}\tilde{N}$, this means

$$T_W^{-1}(W^{-1} - cI)T_W = R \quad \text{and} \quad T_{\tilde{N}}^{-1}(I - c\tilde{N})^{-1}\tilde{N}T_{\tilde{N}} = N$$

Considering this definition together with the Neumann-series of $(I - c\tilde{N})^{-1}$ we obtain

$$\tilde{N}(I - c\tilde{N})^{-1} = \tilde{N}(c\sum_{i=0}^{\infty}\tilde{N}^i) = \tilde{N}(c\sum_{i=0}^{k-1}\tilde{N}^i) = c\sum_{i=0}^{k-1}\tilde{N}^{i+1} = (c\sum_{i=0}^{k-1}\tilde{N}^i)\tilde{N}$$

where we used that $\tilde{N}$ is nilpotent with nilpotency index k. This shows that $\tilde{N}$ and $(I - c\tilde{N})^{-1}$ commute.

From this we can conclude that

$$N^k = [T_{\tilde{N}}^{-1}(I - c\tilde{N})^{-1}\tilde{N}T_{\tilde{N}}]^k = T_{\tilde{N}}^{-1}[(I - c\tilde{N})^{-1}\tilde{N}]^k T_{\tilde{N}} = T_{\tilde{N}}^{-1}(I - c\tilde{N})^{-k}\underbrace{\tilde{N}^k}_{=0}T_{\tilde{N}} = 0$$

We used the commutativity in the third step here. The nilpotent matrix N thus also has the nilpotency index k. A transformation with

$$T_3 := \begin{pmatrix} T_W & 0 \\ 0 & T_{\tilde{N}} \end{pmatrix}$$

transforms $T_2^{-1} J_{\hat{A}}$ into the Jordan Normalform

$$J_{\tilde{A}} := T_3^{-1}T_2^{-1}J_{\hat{A}}T_3 = T_3^{-1}T_2^{-1}T_1^{-1}\hat{A}T_1T_3 = \begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix}$$

and $T_2^{-1} J_{\hat{B}}$ into

$$J_{\tilde{B}} := T_3^{-1} T_2^{-1} J_{\hat{B}} T_3 = T_3^{-1} T_2^{-1} T_1^{-1} \hat{B} T_1 T_3 = \begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix}.$$

Now set

$$P := T_3^{-1} T_2^{-1} T_1^{-1} (cA + B)^{-1} \quad \text{and} \quad Q = T_1 T_3$$

to get the statement. □

Using the findings above we are able to Transform the initial DAE (3.2) using the matrix P from Theorem 2. By multiplying with P from the left, we obtain

$$PAy'(t) + PBy(t) = Pf(t).$$

Setting

$$y(t) = Q \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}, \quad Pf(t) = \begin{pmatrix} s(t) \\ q(t) \end{pmatrix} \quad \text{with} \quad u(t), s(t) : \mathbb{R} \to \mathbb{R}^d,$$

we get a system of the form

$$\begin{aligned} u'(t) + Ru(t) &= s(t), \\ Nv'(t) + v(t) &= q(t). \end{aligned} \tag{3.5}$$

We call this system the *Weierstraß-Kronecker normalform* of the DAE. The first equation of (3.5) is a first order ordinary differential equation and posesses a unique solution $u(t)$ in $[t_0, t_l]$ for any initial values $u_0 \in \mathbb{R}^d$. Now let $q(t) \in C^{k-1}([t_0, t_l])$ (where we denote by

$C^{k-1}([I])$ the room of all functions with domain I that is $k-1$ times differentiable) and differentiate the second equation in 3.5 to obtain

$$\begin{aligned} v(t) &= q(t) - Nv'(t) = q(t) - N(\underbrace{q(t) - Nv'(t)}_{=v(t)})' = q - Nq' + N^2v'' \\ &= q - Nq' + N^2(q - Nv')'' = q - Nq' + N^2q'' - N^3v''' \\ &\vdots \\ &= q - Nq' + ... + (-1)^{k-1}N^{k-1} \underbrace{\frac{d^k}{dt^k}q}_{:=q^{(k-1)}} + (-1)\underbrace{N^kv^{(k)}}_{=0} \\ &= \sum_{i=0}^{k-1}(-1)^iN^iq^{(i)}(t) \end{aligned} \tag{3.6}$$

where k is the nilpotency index of N. Hence we have an explicit form of the solution of $v(t)$. The Kronecker index k thus tells us how many differentiations are required to obtain an ordinary differential equation.

We can also give a general definition of a DAE in Weierstraß-Kronecker normalform.

Definition 2 ([2, Definition 13.2.4])**.** *A linear differential equation* (3.2) *is said to be in Weierstraß-Kronecker normalform if*

$$A = \begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix}, \quad B = \begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix},$$

where N is a nilpotent Jordan-block matrix.

Definition 3. *The nilpotency index k of the matrix N from the Weierstraß-Kronecker Normalform of a matrix pencil $\{A, B\}$ with A singular is called the* Kronecker-Index *of $\{A, B\}$, denote $ind\{A, B\}$. Note that for A regular we set $ind\{A, B\} = 0$.*

Lemma 1. *The Kronecker-Index ind$\{A, B\}$ is independant of the choice of the matrices P and Q.*

For the proof of Lemma 1 see [2, p.403].

3.2 Index of a Differential Algebraic Equation

The Index of a DAE gives us insight about its numerical properties and in general about the solvability where the intuition is "The higher the index, the harder to solve".

We will consider two types of index concepts, the differentiation index and the perturbation index.

Since numerical differentiation is an unstable procedure the differentiation index aims to gives a measure for the numerical problems to be expected when solving such systems.

Definition 4 (differentiation index [2, Definition 13.3.1])**.** *Consider the differential algebraic equation 3.1 to be uniquely locally solvable and F suffieciently smooth. For a given $m \in \mathbb{N}$ consider the equations*

$$\begin{aligned} F(t, y, y') &= 0, \\ \frac{\mathrm{d}F(t, y, y')}{\mathrm{d}t} &= 0, \\ &\vdots \\ \frac{\mathrm{d}^m F(t, y, y')}{\mathrm{d}t^m} &= 0. \end{aligned}$$

The smallest natural number m for which the above System results in an explicit system of ordinary differential equations (ODEs), i.e. has the form

$$y' = \phi(t, y)$$

is called ***differentiation index****.*

above definition from [2] seite 407

In the previous chapter we have already discussed, that for a DAE with constant coefficients (3.2) and a regular matrix pencil $\{A, B\}$ we need $k = ind\{A, B\}$ differentiations to receive an ordinary differential equation. This means, that the Kronecker index k is equal to the differentiation index (in the case of a DAE with constant coefficients).

Definition 5 (perturbation index [2, Definition 13.3.3])**.** *Let $y(t)$ be the exact solution of (3.1) and $\tilde{y}(t)$ be the solution of the perturbed system $F(t, \tilde{y}, \tilde{y}') = \delta(t)$. The smallest number $k \in \mathbb{N}$ such that*

$$||y(t) - \tilde{y}(t)|| \leq C \left(||y(t_0) - \tilde{y}(t_0)|| + \sum_{j=0}^{k} \max_{t_0 \leq \xi \leq T} \left\| \int_{t_0}^{\xi} \frac{\mathrm{d}^j \delta}{\mathrm{d}\tau^j}(\tau) d\tau \right\| \right)$$

*for all $\tilde{y}(t)$, is called the **perturbation index** of this system.*

In the case of linear DAEs with constant coefficients and a regular matrix pencil $\{A, B\}$, we can transform the DAE

$$A(y'(t) - \tilde{y}'(t)) + B(y'(t) - \tilde{y}(t)) = \delta(t)$$

into

$$\begin{aligned} u'(t) - \tilde{u}'(t) + \hat{R}(u(t) - \tilde{u}(t)) &= \delta_1(t), \\ N(v'(t) - \tilde{v}'(t)) &= \delta_2(t). \end{aligned}$$

Where $u(t)$ holds the first d entries of $y(t) : \mathbb{R} \to \mathbb{R}^n$ and $v(t$ holds the remaining $n - d$ entries. According to (3.6) the solution of the algebraic variable has the form

$$v(t) - \tilde{v}(t) = \sum_{i=0}^{k-1} (-1)^i N^i \delta_2^{(i)}(t)$$

Thus the perturbation index is the same as $ind\{A, B\}$ [2]. We see that the error depends on the derivatives of the pertubation.

4 Index Analysis of the modified nodal analysis

[1]

In the previous chapter we have seen two different kinds of Index concepts for differential algebraic equations. We have also seen that these two, even though they describe rather different structural aspects of the equation, are identical for our use-cases. This leads to the question, what indices we expect obtain for our MNA system.

This chapter aims to answer that question for the linear case of circuits only containing of resistors, inductors and capacities as well as current and voltage sources (RLC-circuits). Linear means that the RLC components are described by linear functions with positive capacitances, inductances and resistances. Thus the matrices C, R, and L are positive definite and symmetric. The generalization of these results to the nonlinear case still relies on positive definiteness.

Recall that we consider the equations resulting from the analysis above. These equations are of the form (3.2):

$$Ay'(t) + By(t) = f(t).$$

Specifically the obtained equations from section 2.4 are

$$\begin{pmatrix} A_C C A_C^\top & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{pmatrix} * \begin{pmatrix} u' \\ i_L' \\ i_V' \end{pmatrix} + \begin{pmatrix} A_R G A_R^\top & A_L & A_V \\ -A_L^\top & 0 & 0 \\ -A_V^\top & 0 & 0 \end{pmatrix} * \begin{pmatrix} u \\ i_L \\ i_V \end{pmatrix} = \begin{pmatrix} -A_I i_{src} \\ 0 \\ -v_{src} \end{pmatrix}.$$

4.1 General Index analysis

Assuming the system only contains linear elements, or is linearized at an operating point in order to investigate the system behaviour, then the corresponding network equation represents a DAE with constant coefficients (3.2). We denote $x(t) = (u(t), i_L(t), i_V(t))^\top$. The structure of the system is reliant on the matrix B, thus we consider two cases

ODE-case: The matrix B in (3.2) is regular. This is the case iff the circuit contains no voltage sources and there are no nodes which have no path to ground via capacitors. Then the system represents a linear-implicit system of ODEs and can be transformed into the explicit ODE sytstem

$$x' = B^{-1}(-Ax + f(t)).$$

Thus we obtain an index of 0.

DAE-case: The matrix B in (3.2) is singular. This is the interesting case which we will analyse in the following.

For the singular matrix B we have already obtained a representation of the form

$$\begin{aligned} u'(t) + Ru(t) &= s(t), \\ Nv'(t) + v(t) &= q(t) \end{aligned}$$

in the previous section, see (3.5). We now consider the nilpotency index ν of the matrix N, since we already know that this index correlates with the differentiation as well as the perturbation index. We consider two cases:

1. **The nilpotency index is 1**
   Because N is nilpotent with nipotency index $\nu = 1$ it holds that $N^1 = 0$, thus the system transforms to

$$\begin{aligned} u'(t) + Ru(t) &= s(t), \\ v(t) &= q(t). \end{aligned}$$

This means that the algebraic variables are given explicitly. Thus the system is in ODE form.

2. **The nilpotency index is ≥ 2**
   This case is the situation we have seen in section 3.1.1, it led to (3.6)

$$v(t) = \sum_{i=0}^{\nu-1} (-1)^i N^i q^{(i)}(t).$$

   By differentiating the above equation one more time, we obtain

$$v'(t) = \sum_{i=0}^{\nu-1} (-1)^i N^i q^{(i+1)}(t) \tag{4.1}$$

   with the assumption that q is ν times differentiable.

We can see that the solution has to fulfill an algebraic constraint. In the case of index $\nu = 1$ this equation is given explicitly, while for index $\nu \geq 2$ it is given implicitly.

This system is also sensitive to perturbations. Since in the solution formula for v also derivatives of the input q are present. Thus small noise in the input can have arbitrarily large derivatives which can influence the solution.

4.2 Topological Conditions

From analyzing the MNA some conditions to the circuit topology can be obtained. We will be considering the impact of special arrangements of components on the index of the system. In [4] as well as in [3] they present results about the index of MNA equations.

Theorem 3 (Index conditions [3, Theorem 2.2.1])**.** *Let the matrices of the capacitances, inductances and resistances be positive definite.*

- *If*

$$ker([A_R, A_C, A_V, A_L]^\top) = 0 \quad and \quad ker(A_V) = 0 \tag{4.2}$$

holds, then the MNA (2.3) *leads to a system with index* $\nu \leq 2$.

- *If additionally*

$$ker([A_R, A_C, A_V]^\top) = 0 \quad and \quad ker([A_C, A_V]) = 0 \tag{4.3}$$

holds, then the system is of index $\nu \leq 1$

- *If further*

$$ker(A_C^\top) = 0 \quad and \quad dim(v_s rc) = 0 \tag{4.4}$$

holds, then the system has index $\nu = 0$.

By $ker(A)$ we denote the nullspace of the matrix A and by $dim(y)$ the size of the vector y. The conditions (4.2) - (**??**) can be interpreted in terms of the circuit topology as follows:

- Condition (4.2) can be interpreted, as the circuit neither containing loops of voltage sources nor cutsets of current sources.
- Condition (4.2) can be interpreted, as the circuit containing neither loops of capacitors and/or voltage sources not cutsets of inductors and/or current sources.
- Condition (4.2) can be interpreted, as every node in the circuit being connected to the reference node (ground) through a path containing only the capacitors.

We will apply those results to some examples:

Example1
As our first example we consider again the charging of a capacitor with a series resistor.

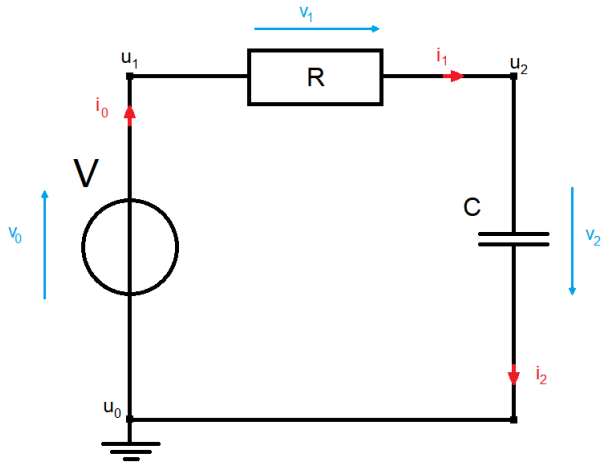

Figure 4.1: charging capacitor with series resistor and voltage source

Because this circuit only contains one loop, which has a voltage source as well as a capacitor we expect the resulting MNA-system to be of index $\nu = 2$.

The incidence matrix for this system is

$$A = [A_V\ A_R\ A_C] = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

Thus condition 1 gives us (first ker is indeed not 0, no? - would need to not have transposed - maybe because A_L is empty)

$$ker([A_R, A_C, A_V, A_L]^\top) = ker \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} = 0 \quad \text{and} \quad ker(A_V) = 0$$

This condition is fulfilled, thus the index has to be smaller or equal to 2. But condition 2 results in

$$ker([A_R, A_C, A_V]^\top) = ker \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \neq 0 \quad \text{and} \quad ker([A_C, A_V]) = ker \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 0.$$

We see that this condition is violated, thus the index can not be smaller than 2.

Example2
This example depicts an energy conserving system. The electric energy stored in the capacitor is converted into the the magnetic energy of the coil and vice versa.

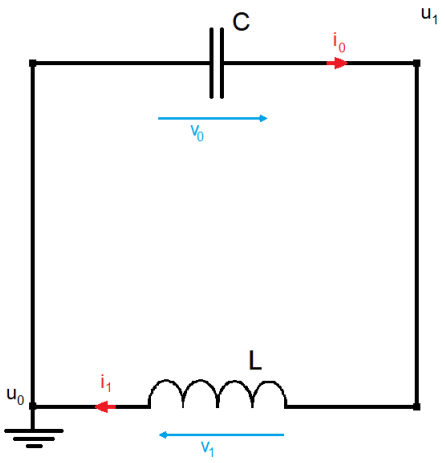

Figure 4.2: LC circuit

Because this circuit only contains only one capacitor and one inductance, the absence of sources means that we expect the resulting MNA-system to be of index $\nu = 0$. This means that the system is an ODE!

The incidence matrix for this system is

$$A = [A_L \; A_C] = \begin{pmatrix} -1 & 1 \end{pmatrix}.$$

Thus condition 1 gives us (empty matrices, thus fulfilled)

$$ker([A_R, A_C, A_V, A_L]^\top) = ker \begin{pmatrix} 1 & -1 \end{pmatrix} = 0 \quad \text{and} \quad ker(A_V) = 0$$

This condition is fulfilled, thus the index has to be smaller or equal to 2. Condition 2 results in

$$ker([A_R, A_C, A_V]^\top) = ker \begin{pmatrix} 1 \end{pmatrix} \neq 0 \quad \text{and} \quad ker([A_C, A_V]) = ker \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 0.$$

This condtion is also fulfilled, thus the index has to be smaller or equal to 1. Condition 3 results in

$$ker(A_C^\top) = 0 \quad \text{and} \quad dim(v_s rc) = 0$$

Thus it is fulfilled and we obtain index 0.

If we have a look at the MNA we can also directly see that this system is of index 0. The MNA has the form

$$C\dot{u}_1 - i_L = 0,$$
$$L\dot{i}_L + u_1 = 0$$

which is an ODE.

5 Numerical Solutions

As we are usually interested in finding real (or complex) valued solutions of our systems we also have to look into solving them numerically. This chapter focuses on the numerical solution of the mentioned systems.

We will first focus on methods used to solve a more general problem

$$y'(t) = f(t, y), \quad t \in [t_0, t_l], \tag{5.1}$$
$$y(t_0) = y_0. \tag{5.2}$$

For this we presume that the function $f(t, y)$ is continuous and Lipschitz, thus *Picard-Lindelöf* [2, Satz 1.2.1] gives us, that for every y_0 it is uniquely solvable in $[t_0, t_l]$.

Numerical Methods work by discretization, this means we divide the time-intervall $[t_0, t_l]$ into

$$t_0 < t_1 < ... < t_N \leq t_l$$

and consider approximations $y_m \approx y(t_m)$ for $m = 1, ..., N$, see picture. We call this a *time-grid* and we call the difference $h_i j = t_j - t_i$ with $0 \leq i < j \leq N$ the step size from t_i to t_j. We Will only consider equidistant grids, which means that this step size does not differ, thus we simply denote it by h.

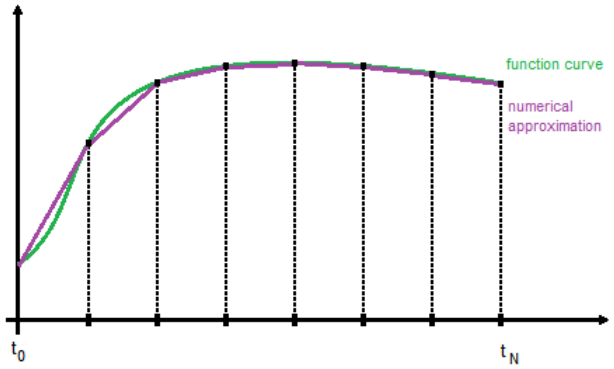

Figure 5.1: approximation of a function using numerical methods

5.1 Single-Step-Methods

The first class of numerical methods we will consider are single-step methods. These methods use the previous approximated value y_j and (for implicit methods) also the current approximated value y_{j+1} to determine the current value y_{j+1} through a *procedural function*.

Definition 6. *A numerical method to approximate a differential equation 5.1 on a time-grid $t_0, ..., t_l$ with the intermediate values $y_0, ..., y_l$ is called a single-step method if it is of the form*

$$y_{j+1} = y_j + h_j \phi(t_j, y_j, y_{j+1}, h_j). \tag{5.3}$$

We call ϕ the procedural function. *If ϕ does not depend on y_{j+1}, then the method is called* explicit, *otherwise it is called* implicit.

5.1.1 Consistency, Stability and Convergence

In order to compare different single-step methods we have to define some notions to compare their quality. This leads to the definition of the error of the mehtod, its consistency and its convergence. We begin with the definition of the error.

Definition 7. *Let $\tilde{y}_{m+1}$ be the result of one step of a single step method* (5.3) *with the exact start-vector $y_m = y(t_m)$ then*

$$\delta_{m+1} = \delta(t_m + h) = y(t_{m+1}) - \tilde{y}_{m+1}, \quad m = 0, ..., N-1 \tag{5.4}$$

is called the local discretization error *of the single step method at the point t_{m+1}.*

The local error quantifies the error of every step of the method with respect to the exact solution. In most applications the exact solution is not known. Next we consider the consistency.

Definition 8. *A single-step method is called* consistent *if for all initial value problems* (5.1)

$$\lim_{h \to 0} \frac{||\delta(t+h)||}{h} = 0 \quad for \quad t_0 \leq t \leq t_l \tag{5.5}$$

holds.
It is called consistent of order p*, if for a sufficiently smooth function f*

$$||\delta(t+h)|| \leq Ch^{p+1} \quad for\ all \quad h \in (0, H] \quad and \quad t_0 \leq t \leq t_l - h \tag{5.6}$$

holds with C independent of h.

Consistency aims to give insight in how similar the problem that the numerical methods solves it to the real problem that we want the solution from. Finally we consider the convergence.

Definition 9. *A single-step method is called* convergent*, if for all initial value problems 5.1 for the* global discretization error

$$e_m = y(t_m) - y_h(t_m)$$

holds that

$$\max_m ||e_m|| \to 0 \quad for \quad h_{max} \to 0.$$

The single-step method is called to have the convergence order *p, if*

$$\max_m ||e_m|| \leq Ch_max^p \quad for \quad h_{max} \in (0, H] \quad with \quad t_0 \leq t_m \leq t_l$$

with the constant C not dependent on the step size h.

As the name suggestes convergence tries to quantify how far off a numerical solution is from the real solution of a system. A very interesting result follows if we also require the Single-Step Method to be stable.

Definition 10. *A Single-Step Method is called* (discretely) stable *if for grid-functions y_h and $\tilde{y}_h$ with*

$$y_{i+1} = y_i + h\phi(t_i, y_i), \tag{5.7}$$
$$\tilde{y}_{i+1} = \tilde{y}_i + h[\phi(t_i, \tilde{y}_i) + \theta_i], \tag{5.8}$$

and perturbations $\theta_i = \theta_h(t_i)$ of the right side as well as a bounded perturbation in the staring-values $y_0 - \tilde{y}_0$ the Error is bounded by

$$||y_h - \tilde{y}_h||_{\infty,h} \leq C(||y_0 - \tilde{y}_0||_{l^2} + ||\theta_h||_{\infty,h}$$

with a constant C which is not dependent on h.

For Single-Step Methods which are consistent and stable we obtain the following convergence theorem.

Theorem 4 (Lax-Richtmyer)**.** *A consistent (with order p) and discretely stable Single-Step Method is convergent (with order p). (assuming smoothness of the solution y)*

This theorem is due to Lax and Richtmyer. The converse of this statement is also true.

5.1.2 further stability properties

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In this section we consider as a model problem the Dahlquist equation, i.e. find u such that

$$u' = \lambda u, \quad t > 0 \tag{5.9}$$
$$u(0) = u_0 \tag{5.10}$$

with $\lambda \in \mathbb{C}$ and u_0 fixed.

Definition 11. *1. If a single-step method can be written in the form*

$$u_{i+1} = R(z)\, u_i, \quad z := h\, y \tag{5.11}$$

then we call $R : \mathbb{C} \to \mathbb{C}$ the stability function *of the single-step method.*

2. The set

$$S := \{z \in \mathbb{C} : |R(z)| \leq 1\} \tag{5.12}$$

is called the region of stability of the method.

3. A single-step method is called

- 0-stable, *if* $0 \in S$.
- A-stable, *if* $\mathbb{C}^- \subset S$.
- L-stable, *if* $R(z) \to 0$ *for* $Re(z) \to -\infty$.

5.2 Multistep-Methods

based on chapter 4 of book num gew dgl steif nichtsteif
Linear multistep methods (LMSV) use approximations u_{m+l} along the gridpoints t_{m+l}, $\quad l = 0, 1, ..., k-1$ to calculate the new approximation u_{m+k} at t_{m+k}. We will first discuss topics

related to the order of the methods depending on its parameters, stability and convergence.

Definition 12. *For given $\alpha_0, ..., \alpha_k$ and $\beta_0, ..., \beta_k$ the iteration rule*

$$\sum_{l=0}^{k} \alpha_l u_{m+l} = h \sum_{l=0}^{k} \beta_l f(t_{m+l}, u_{m+l}), \quad m = 0, 1, ..., N-k \tag{5.13}$$

is called a linear multistep method *(linear k-step method). It is always assumed that $\alpha_k \neq 0$ and $|\alpha_0| + |\beta_k| > 0$. If $\beta_k = 0$ holds, then the method is called explicit, otherwise implicit.*

A linear multi-step method consists of two parts:

1. In the *starting-phase* approximations $u_1, ..., u_{k-1}$ for the first $k-1$ gridpoints $t_l = t_0 + th, l = 1, ..., k-1$ are calculated using a single-step method. For example using an explicit Runge-Kutta Method or a multi-step method with fewer steps.
2. In the *run-phase* the multi-step formula is used to determine new approximations u_{m+k} for the gridpoint t_{m+k}

For theoretical analysis of the multi-step methods we consider the generating polynomials

$$\rho(x) := \sum_{l=0}^{k} \alpha_l x^l \qquad \text{and} \qquad \sigma(x) := \sum_{l=0}^{k} \beta_l x^l \tag{5.14}$$

5.2.1 Consistency, Stability and Convergence

[2]

Again we need to define the properties consistency, stability and convergence for this method.

Definition 13. *Let $\tilde{y}_{m+k}$ be the result of one step of the multi-step method 5.13 with the start-values given as the exact results of the solution $y_{m+l} = y(t_m + l)$, $0 \leq l < k$. This means*

$$\alpha_k \tilde{u}_{m+k} = \sum_{l=0}^{k-1} (h\beta_l f(t_{m+l}, y(t_{m+l})) - \alpha_l y(t_{m+l})) + h\beta_k f(t_{m+k}, \tilde{u}_{m+k}).$$

Then

$$\delta_{m+k} = \delta(t_{m+k}) = y(t_{m+k}) - \tilde{u}_{m+k}, \quad m = 0, 1, ..., N - k$$

is called the local discretization error *(local error) of the linear multi-step method 5.13 at the point t_{m+k}.*

For $k = 1$ this definition agrees with the definition of the local discretization error for single-step methods 7. By assigning the linear difference operator

$$L[y(t), h] = \sum_{l=0}^{k} \left(\alpha_l y(t + lh) - h\beta_l y'(t + lh)\right) \tag{5.15}$$

to the local discretization error we gain the following definition.

Definition 14. *A linear multi-step method is called* consistent, *if for all functions $y(t) \in C^2[t_0, t_l]$*

$$\lim_{h \to 0} \frac{1}{h} L[y(t), h] = 0$$

holds. It has the consistency order p, *if for all functions $y(t) \in C^{p+1}[t_0, t_l]$*

$$L[y(t), h] = \mathcal{O}(h^{p+1}) \quad \textit{for} \quad h \to 0$$

holds.

From the generating polynomials (5.14) we can derive simple consistency conditions, i.e.

$$\rho(1) = 0 \quad \text{and} \quad \rho'(1) = \sigma(1).$$

Next we also need convergence of such methods.

Definition 15. *We say that a linear multi-step method is convergent if for a solution y of the problem a solution vector created by an LMSV y_j for $j \in 0, ..., k$ we have that*

$$\lim_{h\to\infty} \max_{0\leq j\leq k} ||y(t_j) - y_j|| = 0.$$

The discrete stability is also very similar to the single-step methods.

Definition 16. *A linear multi-step method is called (discretely) stable, if for solutions u_h and $\tilde{u}_h$ of*

$$\sum_{l=0}^{k} \alpha_l u_{m+l} = h \sum_{l=0}^{k} \beta_l f(t_{m+l}, u_{m+l}), \tag{5.16}$$

$$\sum_{l=0}^{k} \alpha_l \tilde{u}_{m+l} = h \sum_{l=0}^{k} \beta_l f(t_{m+l}, \tilde{u}_{m+l}) + h\theta_n \tag{5.17}$$

and bounded initial values $y_j - \tilde{y}_j$ for $j \in 0, ..., k$ we have that

$$\max_{t_0\leq t_n\leq T} ||y_n - \tilde{y}_n|| \leq C \sum_{j=0}^{k-1} ||y_j - \tilde{y}_j|| + \max_{t_0\leq t_n\leq T} ||\theta_n||.$$

5.2.2 further stability properties

In this section we consider again the Dahlquist test problem as a model problem

$$u' = \lambda u, \quad t > 0 \tag{5.18}$$

$$u(0) = u_0 \tag{5.19}$$

with $\lambda \in \mathbb{C}$ and u_0 fixed.

Thus the resulting linear multistep method is of the form

$$\sum_{l=0}^{k} \alpha_l u_{n+l} = h \sum_{l=0}^{k} \beta_l \lambda u_{n+l}$$
$$\iff \sum_{l=0}^{k} [\alpha_l - h\beta_l \lambda] u_{n+l}$$

Using this we define the following important stability notions.

Definition 17. *1. The set*

$$S := \{z \in \mathbb{C} : \rho(\xi) - z\sigma(\xi) = 0 \implies \xi \in \mathbb{C} \textit{fullfills root criteria}\} \tag{5.20}$$

is called the region of stability of the method.

2. *A linear multistep method is called*

   - 0-stable, *if* $0 \in S$.
   - *stable in the point* $z \in \mathbb{C}$, *if Wurzelbedingung erfüllt bei* z, *aber doch eigentlich* z *in* S *oder?*
   - $A(\alpha)$-stable, *if it is stable in all* z *that lie within the set* $\{z \in \mathbb{C}^- : |arg(z) - \pi| \leq \alpha\}$ *for* $\alpha \in (0, \frac{\pi}{2})$.

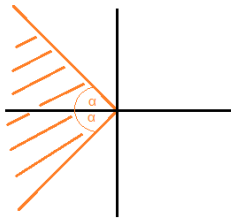

Figure 5.2: $A(\alpha)$ stability for $\alpha = \frac{\pi}{4}$

5.3 Implicit linear multi-step formulas

These kinds of multi-step methods are conventionally used to numerically solve the systems obtained using modified nodal analysis. Our assumption that we only consider network equations arising from networks consisting of RLC components as well as controlled sources which keep the index between 1 and 2 still holds.

We consider the equations in charge/flux oriented formulation introduced in section 2.5

$$0 = \underbrace{\begin{pmatrix} A_c & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix}}_{=:A} \underbrace{\begin{pmatrix} q' \\ \phi' \end{pmatrix}}_{=:y'} + \underbrace{\begin{pmatrix} A_R r(A_R^\top u, t) + A_L i_L + A_V i_V + A_I i(u, i_L, i_V, t) \\ -A_L^\top u \\ v(u, i_L, i_V, t) - A_V^\top u \end{pmatrix}}_{=:f(x,t)},$$

$$\underbrace{\begin{pmatrix} q \\ \phi \end{pmatrix}}_{=:y} = \underbrace{\begin{pmatrix} q_C(A_C^\top u) \\ \phi_L(i_L) \end{pmatrix}}_{=:g(x,t)},$$

or simply

$$0 = F(y'(t), x(t), t) := Ay'(t) + f(x(t), t),$$
$$0 = y(t) - g(x(t)).$$

with the unknowns $x := (u, i_L, i_V)^\top$.

The conventional approach to solve those systems numerically can be split into three main steps:

1. Computation of consistent initial values

   The first step is to compute consistant inital values (x_0, y_0) at the initial time point t_0. In the index-1 case this can be done by solving a so called steady state problem. Steady state means that we consider a not time-dependent circuit (DC operating point). This means we have to solve

   $$F(0, x_0, t_0) = 0$$

for x_0 and then set $y_0 = g(x_0)$.

2. Numerical integration based on multi-step schemes

   Using the consistent initial values we obtain solutions of the network equations at discrete timepoints $t_1, t_2, ...$ by integration with linear multi-step methods.

3. Transformation of the DAE into a nonlinear system of equations.

   The numerical solution is now reduced into solving a system of equations of the form

   $$F(\alpha_k g(x_k) + r_k, x_k, t_k) = 0$$

   which can be solved iteratively using Newton's method.

In the following two subchapters we will give a short overview over the two most commonly used methods for solving those systems.

5.3.1 BDF-schemes

chapter 9.2 numerik book wikipedia

The most commonly used numerical methods for solving the systems that arise in electrical circuits are the BDF-scheme and the trapezoidal rule.

We will not give a deeper look into their construction but will only state their properties. BDF schemes are appealing because they save function evaluation as much as possible, since this is very costly in circuit simulation.

The *backward differentiation formula (BDF)* is a family of implicit linear multistep methods. They have the general form

$$\sum_{k=0}^{s} \alpha_k y_{n+k} = h\beta f(t_{n+s}, y_{n+s}) \tag{5.21}$$

Since we are interested in the unknown y_{n+s} which is used to evaluate f, this method is implicit. The coefficients α_k and β are chosen, such that the method achieves order s which is the best possible.

The BDF or BDF-k formulas for $k = 1, ..., 3$ have the following form

$$
\begin{aligned}
k = 1 &: hf_{m+1} = u_{m+1} - u_m \\
k = 2 &: hf_{m+2} = \frac{1}{2}(3u_{m+2} - 4iu_{m+1} + u_m) \\
k = 3 &: hf_{m+3} = \frac{1}{6}(11u_{m+3} - 18u_{m+2} + 9u_{m+1} - 2u_m)
\end{aligned}
$$

Methods with $s > 6$ are not zero-stable. Indeed in reality methods with order greater than 3 are rarely used because of their low smoothness properties.

BDF schemes have consistency order $p = k$. They are methods for solving stiff equations, thus their stability is indicated by their region of absolute stability. Unfortunately not all BDF-schemes are A stable, but their stability region still contains a large part of the complex left half-plane (see figure 5.3 They are the most efficient linear multistep methods of this kind.

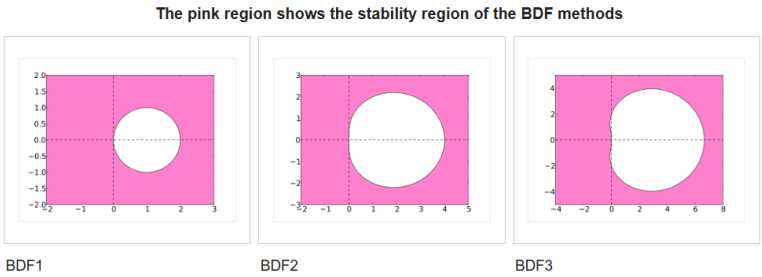

Figure 5.3: stability regions of BDF-schemes

The first timestep is always performed by BDF1 (implicit Euler scheme) as a starting procedure.

5.3.2 trapezoidal rule

The trapezoidal rule is a natural alternative to BDF2 since it is A-stable and as a linear multistep method of order 2 the one with the smallest leading error coefficient. ([1])

It works by approximating the region under the graph of the function $f(x)$ as a trapezoid, hence the name. It follows that

$$\int_a^b f(x)dx \approx (b-a)\frac{1}{2}(f(a)+f(b))$$

This procedure is repeated for small subsections of the Intervall $[a,b]$. Thus we obtain the iteration formula

$$u_h(t+h) = u_h(t) + \frac{h}{2}[f(t,u_h(t)) + f(t+h,u_h(t+h))].$$

Because $u_h(t+h)$ appears in f again we see that this is an implicit method.

5.4 Numerical Examples

In this section we will give two explicit examples of circuits being solved using both of the above mentioned methods and compare the convergence speed between the methods as well as comparing the suitability of the methods for systems of different index.

Example 1

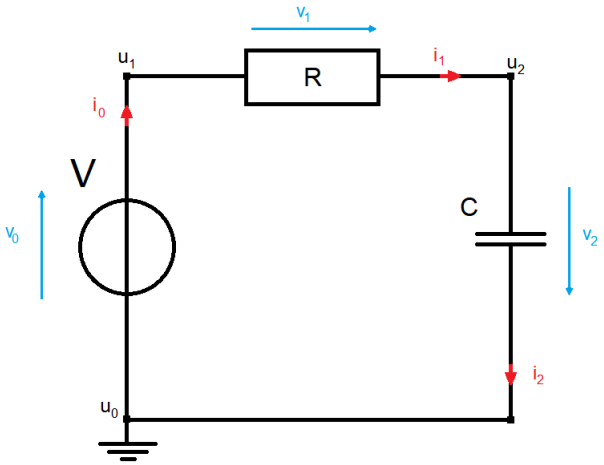

Figure 5.4: charging capacitor with series resistor and voltage source

Example 2

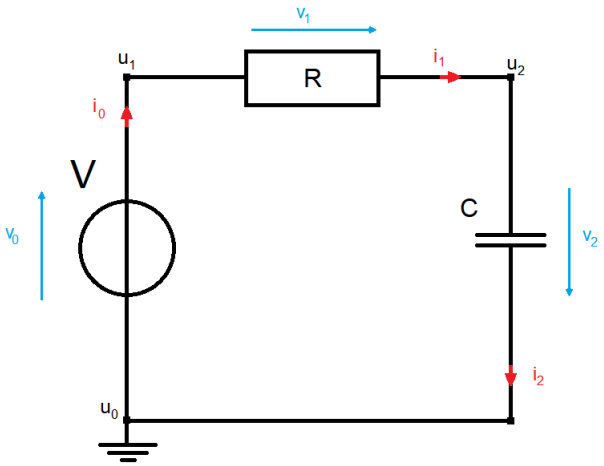

Figure 5.5: LC-circuit

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