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Structural analysis of electric circuits and consequences for MNA

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SUMMARY

The development of integrated circuits requires powerful numerical simulation programs. Naturally, there is no method that treats all the different kinds of circuits successfully. The numerical simulation tools provide reliable results only if the circuit model meets the assumptions that guarantee a successful application of the integration software. Owing to the large dimension of many circuits (about 10^7 circuit elements) it is often difficult to find the circuit configurations that lead to numerical difficulties. In this paper, we analyse electric circuits with respect to their structural properties in order to give circuit designers some help for fixing modelling problems if the numerical simulation fails. We consider one of the most frequently used modelling techniques, the modified nodal analysis (MNA), and discuss the index of the differential algebraic equations (DAEs) obtained by this kind of modelling. Copyright © 2000 John Wiley & Sons, Ltd.

KEY WORDS: circuit simulation; differential-algebraic equation; DAE; index; modified nodal analysis; MNA; structural properties; modelling; state-space form; state-variable approach; normal form; consistent initial values; topological criteria

1. INTRODUCTION

The modern simulation of electric networks is based on modelling techniques that allow an automatic generation of the model equations. One of the most used technique is the modified nodal analysis (MNA). It leads to differential–algebraic equations (DAEs) of the type

$$f(x'(t), x(t), t) = 0. \tag{1}$$

Powerful numerical methods like the backward difference formulae (BDF) method can be applied directly to such DAEs. They are often used successfully in simulation packages. However they may fail. We know from investigations of general DAEs (cf. References [1–3]) that this is usually the case if the DAE has a higher index.† Therefore, we are interested in easy checkable modelling

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† The exact definition of the index of a DAE is given later in Section 3.

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criteria that guarantee a low index. This question is closely related to two major problems in the state-variable analysis:

1. The uniqueness of solutions.
2. The determination of a minimal set of variables whose instantaneous values are sufficient to determine the state of the network completely. In other words, the formulation of the modelling equations in normal form, i.e. a reduction of Equation (1) to an explicit ODE.

The corresponding variables are commonly denoted as state variables and they are relevant to fix consistent initial values, i.e. to fix initial values for which a solution of Equation (1) exists.

Referring to this, first results were presented in Reference [4] for linear passive RLC networks. The state approach is here based on the construction of a normal tree, i.e. a tree that contains all independent voltage sources, no independent current sources, a maximal number of capacitive branches and a minimal number of inductive branches. Although the term 'index' is not used in this paper, one can easily conclude that the index for such systems is 1 or 2, depending on the topology of the network. Afterwards, these results have been widely generalized for linear networks containing mutual inductances, gyrators, ideal transformers or controlled sources. In Reference [5], we find a detailed discussion of these generalizations and a current extension of the method based on the normal tree. Noteworthy here that all extensions are based on strategies using the construction of normal trees or (later on) matroids.

Numerous authors have also studied non-linear networks with respect to the uniquely solvability and the determination of state variables (cf. e.g. References [6–17]). Typically, the results are based on the existence of specific trees or on certain positive definite and regular matrices, respectively. We will discuss these results in detail in Section 4. In Reference [14] topological criteria for unique solutions of nonlinear resistive circuits containing linear controlled sources have already been discussed. The authors apply a set of operations to the graph of the circuit and check a certain oriented structure, a cactus graph or a generalization thereof.

Recently, some results in circuit theory, considering the mathematical Ansatz from the DAE point of view, have been established (cf. Reference [18–24]). By doing so, solvability criteria and results about the numerical solution of DAEs can be applied. Most of these results depend strongly on the index of a DAE. Roughly speaking, the index is a measure for the deviation of a DAE from an explicit ODE. DAEs of higher index ($\geqslant 2$) are ill-posed in the sense that small perturbations in the initial data may cause arbitrarily large changes in the solution data.

We focused our investigations onto the determination of the index for circuit systems. Under consideration of the large dimension (often 10^7) of the network systems, we are interested in topological criteria that can be checked very fast. Furthermore, the criteria should be based on local assumptions, i.e. we want to provide the opportunity to localize critical element modellings. In Reference [25], it has already been shown that, for non-linear time-independent networks without controlled sources, the index of the MNA equations does not exceed 2 and can be determined by topological criteria, assuming the positive definiteness of the Jacobians of the element-characterizing functions. Afterwards, in Reference [26] it was proved that the standard circuit equations of linear as well as nonlinear RLCTG-networks, i.e. networks composed of independent sources, resistive, capacitive an inductive subnetworks, ideal transformers and gyrators, do not exceed 2, again under positive-definiteness assumptions.

In the present article, the results from Reference [25] are generalized for non-linear, time-dependent networks that may contain a wide class of controlled sources. Furthermore, many mathematically important structural properties of these differential–algebraic equations are elaborated. For instance, the presented results can be employed to compute consistent initial conditions that has been performed in References [27, 28]. Further, the structural properties reveal stability properties of numerical integration methods if we apply the results from Reference [29]. We would like to point out that the required assumptions are not necessary to guarantee unique solvability, but they are sufficient to ensure a general description of structural properties. As already reported in Reference [14], it is far from hopeless to formulate a general existence and uniqueness theorem for non-linear circuits based only on topological conditions. In spite of that, graph-theoretic results are extremely useful since they are easy to check.

The article is organized as follows. In Section 2, we introduce general structural properties of the networks and the equations arising from MNA. Section 3 describes briefly the index concept, in particular the differential and the tractability index. In Section 4, the main results of this article are presented, whereas the proofs are outlined in Sections 5 and 6.

2. STRUCTURAL ANALYSIS

We consider lumped electric circuits containing non-linear, time-variant resistances, capacitances, inductances, voltage sources and current sources. Usually, circuit simulation tools are based on these kinds of network elements. For two-terminal (one-port) lumped elements, the current through the element and the voltage across it are well-defined quantities. For lumped elements with more than two terminals, the current entering any terminal and the voltage across any pair of terminals are well defined at all times (cf. Reference [30]). Hence, general n-terminal elements (see Figure 1) are completely described by $(n-1)$ currents entering the $(n-1)$ terminals and the $(n-1)$ branch voltages across each of these $(n-1)$ terminals and the reference terminal n.

In particular, n-terminal resistances can be modelled by an equation system of the form

$$j_k = r_k^e(u_1, \dots, u_{n-1}, t) \quad \text{for} \quad k = 1, \dots, n-1$$

if j_k represents the current entering the terminal k and u_l describes the voltage across the pair of terminals $\{l, n\}$ (for $k, l = 1, \dots, n-1$). The Kirchhoff's current law implies the current entering the terminal n to be given by $j_n = -\sum_{k=1}^{n-1} j_k$. The conductance matrix $G^e(u_1, \dots, u_{n-1}, t)$ is then defined by the Jacobian

$$G^e(u_1, \dots, u_{n-1}, t) := \begin{pmatrix} \dfrac{\partial r_1^e}{\partial u_1} & \cdots & \dfrac{\partial r_1^e}{\partial u_{n-1}} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial r_{n-1}^e}{\partial u_1} & \cdots & \dfrac{\partial r_{n-1}^e}{\partial u_{n-1}} \end{pmatrix}.$$

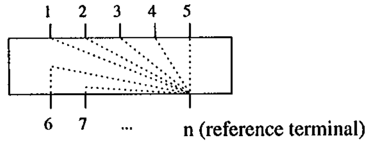

Figure 1. n-Terminal circuit element.

The index e shall specify the correlation to a special element of a circuit. Later on we will introduce the conductance matrix $G(u, t)$ describing all resistances of a circuit. Correspondingly, the capacitance matrix $C^e(u_1, \dots, u_{n-1}, t)$ of a general n-terminal capacitance is given by

$$C^e(u_1, \dots, u_{n-1}, t) := \begin{pmatrix} \dfrac{\partial q_1^e}{\partial u_1} & \cdots & \dfrac{\partial q_1^e}{\partial u_{n-1}} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial q_{n-1}^e}{\partial u_1} & \cdots & \dfrac{\partial q_{n-1}^e}{\partial u_{n-1}} \end{pmatrix}$$

if the voltage–current relation is defined by means of charges by

$$j_k = \frac{\mathrm{d}}{\mathrm{d}t} q_k^e(u_1, \dots, u_{n-1}, t) \quad \text{for} \quad k = 1, \dots, n-1.$$

In order to illustrate what the matrices C^e may look like, let us consider a MOSFET-model as an example of a common n-terminal element.

Choosing the source node S as the reference node, we have the reference voltages u_{GS}, u_{DS}, and u_{BS}. For the currents we obtain

$$\begin{aligned} j_{\mathrm{G}} &= C_{\mathrm{GS}}\dot{u}_{\mathrm{GS}} + C_{\mathrm{GD}}(\dot{u}_{\mathrm{GS}} - \dot{u}_{\mathrm{DS}}), \\ j_{\mathrm{D}} &= -C_{\mathrm{GD}}(\dot{u}_{\mathrm{GS}} - \dot{u}_{\mathrm{DS}}) - C_{\mathrm{BD}}(\dot{u}_{\mathrm{BS}} - \dot{u}_{\mathrm{DS}}) \\ &\quad + d(u_{\mathrm{BS}} - u_{\mathrm{DS}}) + i(u_{\mathrm{GS}}, u_{\mathrm{DS}}, u_{\mathrm{BS}}) + \frac{1}{R} u_{\mathrm{DS}} \\ j_{\mathrm{B}} &= C_{\mathrm{BS}}\dot{u}_{\mathrm{BS}} + C_{\mathrm{BD}}(\dot{u}_{\mathrm{BS}} - \dot{u}_{\mathrm{DS}}) - d(u_{\mathrm{BS}}) - d(u_{\mathrm{BS}} - u_{\mathrm{DS}}). \end{aligned}$$

Note that j_{S} is given by the formula $j_{\mathrm{S}} = -j_{\mathrm{G}} - j_{\mathrm{D}} - j_{\mathrm{B}}$ due to Kirchoff's Current Law. Now it is easy to verify that

$$C^e(u_{\mathrm{GS}}, u_{\mathrm{DS}}, u_{\mathrm{BS}}) = \begin{pmatrix} C_{\mathrm{GS}} + C_{\mathrm{GD}} & -C_{\mathrm{GD}} & 0 \\ -C_{\mathrm{GD}} & C_{\mathrm{GD}} + C_{\mathrm{BD}} & -C_{\mathrm{BD}} \\ 0 & -C_{\mathrm{BD}} & C_{\mathrm{BS}} + C_{\mathrm{BD}} \end{pmatrix}$$

for the MOSFET-model from Reference [31] (see Figure 2).

Inductances can be modelled by means of fluxes by

$$u_k = \frac{\mathrm{d}}{\mathrm{d}t} \phi_k^e(j_1, \dots, j_{n-1}, t) \quad \text{for} \quad k = 1, \dots, n-1.$$

Then, the inductance matrix $L^e(j_1, \dots, j_{n-1}, t)$ of a general n-terminal inductance is given by the Jacobian

$$L^e(j_1, \dots, j_{n-1}, t) := \begin{pmatrix} \dfrac{\partial \phi_1^e}{\partial j_1} & \cdots & \dfrac{\partial \phi_1^e}{\partial j_{n-1}} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial \phi_{n-1}^e}{\partial j_1} & \cdots & \dfrac{\partial \phi_{n-1}^e}{\partial j_{n-1}} \end{pmatrix}.$$

A commonly used method, for network analysis in circuit simulation packages like SPICE[‡] and TITAN,[§] is the modified nodal analysis (MNA). It represents a systematic treatment of general circuits and is important when computers perform the analysis of networks automatically. The scheme to set up the MNA equations is:

1. Write node equations by applying Kirchhoff's current law (KCL) to each node except for the datum node:
$$Aj = 0. \tag{2}$$
The vector j represents the branch current vector. The matrix A is called the (reduced) incidence matrix and describes the network graph, the branch-node relations.
2. Replace the currents j_k of voltage-controlled elements by the voltage-current relations of these elements in Equation (2).
3. Add the current–voltage relations for all current-controlled elements.

Note that, in case of multi-terminal elements with n terminals, we speak of branches if they represent a pair of terminals $\{l, n\}$ with $1 \leqslant l \leqslant n-1$.

In order to obtain more detailed information about the structure of the equations arising from MNA, we split the (reduced) incidence matrix A into the element-related incidence matrices

$$A = (A_C, A_L, A_R, A_V, A_I),$$

where A_C, A_L, A_R, A_V, and A_I describe the branch-current relations for capacitive branches, inductive branches, resistive branches, branches of voltage sources and branches of current sources, respectively. Denote by e the node potentials (excepting the datum node) and by j_L and j_V the current vectors of inductances and voltage sources. Defining the vector of functions for current and voltage sources by i and v, respectively, we obtain the following system from the MNA:

$$A_C \frac{\mathrm{d}q(A_C^{\mathrm{T}}e, t)}{\mathrm{d}t} + A_R r(A_R^{\mathrm{T}}e, t) + A_L j_L + A_V j_V + A_I i\left(A^{\mathrm{T}}e, \frac{\mathrm{d}q(A_C^{\mathrm{T}}e, t)}{\mathrm{d}t}, j_L, j_V, t\right) = 0, \tag{3}$$

$$\frac{\mathrm{d}\phi(j_L, t)}{\mathrm{d}t} - A_L^{\mathrm{T}}e = 0, \tag{4}$$

$$A_V^{\mathrm{T}}e - v(A^{\mathrm{T}}e, \frac{\mathrm{d}q(A_C^{\mathrm{T}}e, t)}{\mathrm{d}t}, j_L, j_V, t) = 0. \tag{5}$$

Note that the vectors $A_C^{\mathrm{T}}e$, $A_L^{\mathrm{T}}e$, $A_R^{\mathrm{T}}e$, and $A_V^{\mathrm{T}}e$ describe the branch voltages for the capacitive, inductive, resistive and voltage source branches, respectively.

Remark. Owing to the fact that the currents through resistances are functions of the branch potentials, we do not include them separately as controlling functions. Of course, if the network

[‡] Originally developed at Berkeley in the 1970s and early 1980s, and subsequently commercialized by companies including MetaSoftware (HSPICE), MicroSim (PSPICE), Intusoft (IS_SPICE), Spectrum Software (Micro-Cap V), Cadence (Analog Workbench), Analogy (Saber), and Deutsch Research (DR. SPICE, ViewSpice).
[§] Developed by SIEMENS.

does not contain controlled sources, then the source functions reduce to functions $i(t)$ and $v(t)$ which depend on time only.

Nowadays circuit simulation packages use two different approaches for solving Equations (3–5), the conventional and the charge-oriented one.

2.1. The conventional MNA

For the conventional MNA the vector of unknowns consists of all node voltages and all branch currents of current-controlled elements.

Defining

$$C(u,t) := \frac{\partial q(u,t)}{\partial u},\ q_t'(u,t) := \frac{\partial q(u,t)}{\partial t},\ L(j,t) := \frac{\partial \phi(j,t)}{\partial j},\ \phi_t'(j,t) := \frac{\partial \phi(j,t)}{\partial t}$$

we obtain[¶]

$$A_C C(A_C^{\mathrm{T}} e, t) A_C^{\mathrm{T}} \frac{\mathrm{d}e}{\mathrm{d}t} + A_C q_t'(A_C^{\mathrm{T}} e, t) + A_R r(A_R^{\mathrm{T}} e, t)$$

$$+ A_L j_L + A_V j_V + A_I i\left(A^{\mathrm{T}} e, A_C^{\mathrm{T}} \frac{\mathrm{d}e}{\mathrm{d}t}, j_L, j_V, t\right) = 0, \tag{6}$$

$$L(j_L, t) \frac{\mathrm{d}j_L}{\mathrm{d}t} + \phi_t'(j_L, t) - A_L^{\mathrm{T}} e = 0, \tag{7}$$

$$A_V^{\mathrm{T}} e - v\left(A^{\mathrm{T}} e, A_C^{\mathrm{T}} \frac{\mathrm{d}e}{\mathrm{d}t}, j_L, j_V, t\right) = 0. \tag{8}$$

Later on we will also need

$$G(u,t) := \frac{\partial r(u,t)}{\partial u}, \quad r_t'(u,t) := \frac{\partial r(u,t)}{\partial t}.$$

2.2. The charge-oriented MNA

In comparison with the conventional MNA, the vector of unknowns consists additionally of the charge of capacitances and the flux of inductances. Moreover, the original voltage–charge and current–flux equations are added to the system.

The resulting system is then of the form (cf. Reference [23])

$$A_C \frac{\mathrm{d}q}{\mathrm{d}t} + A_R r(A_R^{\mathrm{T}} e, t) + A_L j_L + A_V j_V + A_I i\left(A^{\mathrm{T}} e, \frac{\mathrm{d}q}{\mathrm{d}t}, j_L, j_V, t\right) = 0, \tag{9}$$

[¶] Note that we have

$$\frac{\mathrm{d}q(A_C^{\mathrm{T}} e, t)}{\mathrm{d}t} = C(A_C^{\mathrm{T}} e, t) A_C^{\mathrm{T}} \frac{\mathrm{d}e}{\mathrm{d}t} + q_t'(A_C^{\mathrm{T}} e, t)$$

Therefore, $i(A^{\mathrm{T}} e, (\mathrm{d}q(A_C^{\mathrm{T}} e, t)/\mathrm{d}t), j_L, j_V, t) = i_*(A^{\mathrm{T}} e, A_C^{\mathrm{T}}(\mathrm{d}e/\mathrm{d}t), j_L, j_V, t)$ for a suitable function i_*. An analogous relation is valid for the controlled voltage sources. For simplicity, we drop the index *.

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} - A_L^{\mathrm{T}} e = 0, \tag{10}$$

$$A_V^{\mathrm{T}} e - v\left(A^{\mathrm{T}} e, \frac{\mathrm{d}q}{\mathrm{d}t}, j_L, j_V, t\right) = 0, \tag{11}$$

$$q - q_C(A_C^{\mathrm{T}} e, t) = 0, \tag{12}$$

$$\phi - \phi_L(j_L, t) = 0. \tag{13}$$

2.3. Topological characterization of the splitted incidence matrix

The splitting of the incidence matrix $A = (A_C, A_L, A_R, A_V, A_I)$ corresponding to certain branches leads to useful structural information for lumped circuits. In order to describe the different parts of Equations (6)–(8) in more detail, we will introduce some useful projectors. Before doing this, let us recall the definition of a projector.

Definition 2.1. (1) A matrix $Q \in \mathbb{R}^{m \times m}$ is a projector onto R_1 if and only if $Q^2 = Q$ and $\operatorname{im} Q = R_1$.

(2) A matrix $Q \in \mathbb{R}^{m \times m}$ is a projector along R_2 if and only if $Q^2 = Q$ and $\ker Q = R_2$.

(3) For $\mathbb{R}^m = R_1 \oplus R_2$ a matrix $Q \in \mathbb{R}^{m \times m}$ is the uniquely defined projector onto R_1 along R_2 if and only if $Q^2 = Q$, $\operatorname{im} Q = R_1$, and $\ker Q = R_2$.

We denote by $Q_C, Q_{V\text{-}C}, Q_{R-CV}, Q_{\mathrm{V}}, \bar{Q}_C$, and $\bar{Q}_{V-C}$ projectors onto $\ker A_C^{\mathrm{T}}, \ker A_V^{\mathrm{T}} Q_C$, $\ker A_R^{\mathrm{T}} Q_C Q_{V\text{-}C}, \ker A_V^{\mathrm{T}}, \ker A_C$, and $\ker Q_C^{\mathrm{T}} A_V$, respectively.[‖] The complementary projectors will be denoted by $P := I - Q$, with the corresponding subindex. We observe that

$$\operatorname{im} P_C \subset \ker P_{V\text{-}C}, \quad \operatorname{im} P_{V\text{-}C} \subset \ker P_{R\text{-}CV} \quad \text{and} \quad \operatorname{im} P_C \subset \ker P_{R\text{-}CV},$$

and that thus $Q_C Q_{V\text{-}C}$ is a projector onto $\ker(A_C A_V)^{\mathrm{T}}$ and $Q_C Q_{V\text{-}C} Q_{R\text{-}CV}$ is a projector onto $\ker(A_C A_R A_V)^{\mathrm{T}}$. In order to shorten denotations, we use the abbreviation $Q_{CRV} := Q_C Q_{V\text{-}C} Q_{R\text{-}CV}$. Remark that the projector P_{CRV} does not coincide with the projector $P_{R\text{-}CV}$ in general. The following theorem describes well-known topological properties in terms of the introduced incidence matrices and projectors.

Theorem 2.2. *Given a lumped circuit with capacitances, inductances, resistances, voltage sources and current sources. Then, the following relations are satisfied for the* (*reduced*) *incidence matrix* $A = (A_C A_L A_R A_V A_I)$:

1. *Then matrix* $(A_C A_L A_R A_V)$ *has full row rank, because cutsets of current sources are forbidden.*
2. *The matrix* A_V *has full column rank, because loops of voltage sources are forbidden.*
3. *The matrix* $(A_C A_R A_V)$ *has full row rank if and only if the circuit does not contain a cutset consisting of inductances and/or current sources only.*
4. *The matrix* $Q_C^{\mathrm{T}} A_V$ *has full column rank if and only if the circuit does not contain a loop consisting of capacitances and voltage sources only.*

‖ An explicit description of such projectors is given in Reference [27].

Note that loops containing only capacitances are excluded under point 4, whereas cutsets containing only inductances are included under point 3 of Theorem 2.2.

In the following, the special cutsets and loops considered in Theorem 2.2 will be important. Therefore we define:

1. An *L–I cutset* is a cutset consisting of inductances and/or current sources only.
2. A *C–V loop* is a loop consisting of capacitances and voltage sources only.

Using the introduced projections we obtain the following corollary from Theorem 2.2.

Corollary 2.3. *Theorem* 2.2 *implies that*

1. $Q_{CRV}=0$ *if and only if the network does not contain L–I cutsets,*
2. $\bar{Q}_{V\text{-}C}=0$ *if and only if the network does not contain C–V loops.*

For a simpler description later on, we adduce two lemmata:

Lemma 2.4. *If M is a positive-definite* $m\times m$*-matrix and N is a rectangular matrix of dimension* $k\times m$, *then it holds that*

$$\ker NMN^{\mathrm{T}}=\ker N^{\mathrm{T}} \quad \text{and} \quad \operatorname{im} NMN^{\mathrm{T}}=\operatorname{im} N$$

The correctness of Lemma 2.4 follows immediately from the definition of positive-definite matrices.

Lemma 2.5. *If* $C(A_C^{\mathrm{T}}e,t)$, $L(j_L,t)$ *and* $G(A_R^{\mathrm{T}}e,t)$ *are positive definite, then the matrices*

$$H_1(A_C^{\mathrm{T}}e,t):=A_C C(A_C^{\mathrm{T}}e,t)A_C^{\mathrm{T}}+Q_C^{\mathrm{T}}Q_C$$

$$H_2:=Q_C^{\mathrm{T}}A_V A_V^{\mathrm{T}}Q_C+Q_{V\text{-}C}^{\mathrm{T}}Q_{V\text{-}C}$$

$$H_3:=A_V^{\mathrm{T}}Q_C Q_C^{\mathrm{T}}A_V+\bar{Q}_{V\text{-}C}^{\mathrm{T}}\bar{Q}_{V\text{-}C}$$

$$H_4(A_R^{\mathrm{T}}e):=Q_{V\text{-}C}^{\mathrm{T}}Q_C^{\mathrm{T}}A_{\mathrm{R}}G(A_R^{\mathrm{T}}e,t)A_R^{\mathrm{T}}Q_C Q_{V\text{-}C}+Q_{R\text{-}CV}^{\mathrm{T}}Q_{R\text{-}CV}$$

$$H_5(j_L,t):=Q_{CRV}^{\mathrm{T}}A_L L^{-1}(j_L,t)A_L^{\mathrm{T}}Q_{CRV}+P_{CRV}^{\mathrm{T}}P_{CRV}^{\mathrm{T}}$$

$$H_6(A_C^{\mathrm{T}}e,t):=\bar{Q}_{V\text{-}C}^{\mathrm{T}}A_{\mathrm{V}}^{\mathrm{T}}H_1^{-1}(A_C^{\mathrm{T}}e,t)A_{\mathrm{V}}\bar{Q}_{V\text{-}C}+\bar{P}_{V\text{-}C}^{\mathrm{T}}\bar{P}_{V\text{-}C}$$

are regular.

Proof. Using Lemma 2.4, the regularity of $H_1(A_C^{\mathrm{T}}e,t)$ is obvious since $C(A_C^{\mathrm{T}}e,t)$ is positive definite. For H_2 and for H_3 the regularity follows immediately, and for H_4 analogously if we consider that $G(A_R^{\mathrm{T}}e,t)$ is positive definite.

Let us prove the regularity of H_5. Let z be an element of $\ker H_5$. Then we have

$$(Q_{CRV}^{\mathrm{T}}A_L L^{-1}(j_L,t)A_L^{\mathrm{T}}Q_{CRV}+P_{CRV}^{\mathrm{T}}P_{CRV})z=0.$$

If we multiply this equation by P_{CRV}^{T}, it results that $P_{CRV}^{\mathrm{T}}P_{CRV}z=0$ and, therefore, $P_{CRV}z=0$. Hence, we obtain

$$Q_{CRV}^{\mathrm{T}}A_L L^{-1}(j_L,t)A_L^{\mathrm{T}}Q_{CRV}z=0.$$

Then, since $L^{-1}(j_L, t)$ is positive definite, $A_L^{\mathrm{T}} Q_{CRV} z = 0$ holds. Applying that $(A_C, A_R, A_V, A_L)^{\mathrm{T}}$ has full column rank, we conclude $Q_{CRV} z = 0$, i.e. $z = P_{CRV} z$ and, since $P_{CRV} z = 0$, the regularity is verified.

The regularity from $H_6(A_C^{\mathrm{T}} e, t)$ can be easily shown making use of the facts that $C(A_C^{\mathrm{T}} e, t)$ is positive definite and that A_V has full column rank. □

3. DAEs AND THEIR INDEX

In Section 2, we observed that the MNA leads to a coupled system of implicit differential equations and non-linear equations, i.e. to a differential-algebraic equation (DAE)

$$f(\dot{x}(t), x(t), t) = 0 \tag{14}$$

where the partial derivative $f'_{\dot{x}}(\dot{x}(t), x(t), t)$ is singular (cf. References [18, 21–23]). The analytical and numerical solutions of Equation (14) depend strongly on its structure and index. DAEs have, among other things, the following two important properties (see e.g. References [1–3]):

(i) DAEs do not only represent integration problems, but also differentiation problems. Some parts of a DAE must be differentiable sufficiently often.
(ii) Some components of the solution are determined algebraically. This implies that we cannot prescribe initial values for all variables.**

Let us note that numerical methods can fail in higher index cases, particularly if the index is greater than 2. Typical problems to expect are extremely small step sizes and simulation interrupts caused be exceptions in numerical routines. Therefore, we are looking for conditions (depending on the network topology) that guarantee a lower index ($\leqslant 2$).

In the non-linear case, the index concept is not unique in the literature. In this article we analyse two important index concepts, the differential index and the tractability index, with regard to circuits.

3.1. Definition of the differential index

The most general definition of the differential index of non-linear DAE systems is (cf. Reference [2]):

Definition 3.1. The differential index ν of the general non-linear, sufficiently smooth DAE

$$f(x', x, t) = 0 \tag{15}$$

** The consequences of the results presented in this article referring this have been developed in References [27, 28].

is the smallest v such that

$$f(x', x, t) = 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}t} f(x', x, t) = 0,$$

$$\vdots$$

$$\frac{\mathrm{d}^v}{\mathrm{d}t^v} f(x', x, t) = 0$$

uniquely determines the variable x' as a continuous function of (x, t).

Note that the singularity of $f'_{x'}(x', x, t)$ implies that Equation (14) contains some derivative-free equations which we will denote by explicit constraints. If the index is higher than 1, then combinations and substitutions of the original and the differentiated DAE may lead to further derivative-free equations, called hidden constraints.

Fortunately, the structure of the DAEs that results from the MNA in circuit simulation is such that it will not be necessary to derive the whole function f. As we will see, it suffices to derive the explicit constraints in index 1 case and, additionally, the hidden constraints in index 2 case.

3.2. Definition of the tractability index

The tractability index [1, 32] orientates on the linearization of a DAE. This index concept requires only weak smoothness conditions. Furthermore, solvability and stability results exist for index-1-tractable and index-2-tractable DAEs (see, e.g. References [21, 20]).

We consider non-linear DAEs

$$f(x', x, t) = 0 \tag{16}$$

for which $N := \ker f'_{x'}(x', x, t)$ is constant and f is continuously differentiable. We denote $A(x', x, t) := f'_{x'}(x', x, t)$ and $B(x', x, t) := f'_x(x', x, t)$.

Definition 3.2. DAE (16) is called index-1-tractable if the matrix $A_1(x', x, t) := A(x', x, t) + B(x', x, t)Q$ is regular for a constant projector Q onto N.

Remarks. (1) The matrix $A_1(x', x, t)$ is regular if and only if $N \cap S(x', x, t) = \{0\}$ for $S(x', x, t) := \{z \colon B(x', x, t)\, z \in \operatorname{im} A(x', x, t)\}$.

(2) The condition does not depend on the choice of the projector Q.

For a proof see, e.g. Reference [1].

Definition 3.3. DAE (16) is called index-2-tractable if

1. it is not index-1-tractable,
2. $N_1(x', x, t) := \ker A_1(x', x, t)$ is of constant rank,
3. $A_2(x', x, t) := A_1(x', x, t) + B_1(x', x, t)\, Q_1(x', x, t)$ is regular for a projector $Q_1(x', x, t)$ onto $N_1(x', x, t)$ and $B_1(x', x, t) := B(x', x, t)(I - Q)$.

Remarks. (1) The matrix $A_2(x', x, t)$ is regular if and only if $N_1(x', x, t) \cap S_1(x', x, t) = \{0\}$ for $S_1(x', x, t) := \{z: B_1(x', x, t) z \in \operatorname{im} A_1(x', x, t)\}$.

(2) The condition does not depend on the choice of the projector Q_1.

For a proof see again Reference [1].

4. THE INDEX OF DAEs RESULTING FROM THE MNA FOR ELECTRIC CIRCUITS

The investigations of numerical methods for DAEs have shown that available codes for general nonlinear DAEs provide reliable results only for DAEs of lower index ($\leqslant 2$). Therefore, we are interested in adequate conditions for electric circuits that guarantee a lower index DAE. In particular, the voltage-controlled voltage sources (VCVS), current-controlled voltage sources (CCVS), voltage-controlled current sources (VCCS), and current-controlled current sources (CCVS) have to be analysed very carefully.

We consider the two defined index concepts for the systems arising form circuits. The differential index is investigated in Section 5 and the tractability index in Section 6. The results show that both concepts lead to the same index in case of circuit simulation applying MNA. They are summarized in the following theorems.

Theorem 4.1. *Consider lumped electric circuits containing resistances, capacitances, inductances, and voltage and current sources. Let the capacitance, inductance and conductance matrices of all capacitances, inductances, and resistances, respectively, be positive definite.*†† *Furthermore, let the following conditions for the controlled sources be satisfied*:

1. *The controlled voltage sources do not form a part of any C–V loop and their controlling elements fulfil the conditions exposed in the Tables I and II.*
2. *Each controlled current source fulfils at least one of the following conditions*:
   (a) *It does not form a part of any L–I cutset and the controlling elements fulfil the conditions exposed in the Tables III and IV.*
   (b) *There exists a path formed by capacitances that connects its incidence nodes. The controlling elements fulfil the conditions exposed in Table V for CCCS, and the VCCS are controlled by an arbitrary voltage.*
   (c) *There exists a path formed by capacitances and voltage sources that connects its incidence nodes. The controlling elements fulfil the conditions exposed in Table VI for CCCS, and the VCCS are controlled by an arbitrary voltage.*

Then, the conventional MNA leads to an index-1 *DAE*‡‡ *if and only if the network contains neither L–I cutsets nor C–V loops. Otherwise, the conventional MNA leads to an index*-2 *DAE.*

Theorem 4.2. *The same conclusions as in Theorem* 4.1 *are valid under the same assumptions, if we consider the charge-oriented MNA instead of the conventional MNA.*

†† For capacitances and inductances with affine characteristics the positive definiteness implies that they are strictly locally passive (cf. Reference [17]).

‡‡ For reasons of simplicity, we do not consider the index-0 cases, which result if $f'_x(\dot{x}(t), x(t), t)$ is regular, separately.

Table I. VCVS—condition (1)

The controlling voltages of a VCVS can be voltages of:
1. Capacitances,
2. Independent voltage sources,
3. CCVSs that are controlled by:
   (a) Inductances,
   (b) Independent current sources,
   (c) Resistances or VCCSs for which the controlling nodes are connected by:
      i. Capacitances,
      ii. Independent voltage sources,
      iii. Paths containing only the elements described in (3(c)i), (3(c)ii),
   (d) Branches that form a cutset with the elements described in (3a), (3b) and (3c),
4. Branches that form a loop with the elements described in (1), (2) and (3).

Table II. CCVS—condition (1)

The controlling currents of a CCVS can be currents of:
1. Inductances,
2. Independent current sources,
3. Resistances or VCCSs for which the controlling nodes are connected by:
   (a) Capacitances,
   (b) Independent voltage sources,
   (c) VCVSs for which the nodes that incide with the controlling branch are connected by
      i. Capacitances,
      ii. Independent voltage sources,
      iii. Paths containing only the elements described in (3(c)i), (3(c)ii),
   (d) Paths containing only the elements described in (3a), (3b) and (3c),
4. Branches that form a cutset with the elements described in (1), (2) and (3).

Table III. VCCS—condition (2a)

The controlling voltages of a VCCS can be voltages of:
1. Capacitances,
2. Voltage sources,
3. Branches that form a loop with branches like those described in (1) and (2).

Remarks: 1. The presented criteria can be checked locally. It is neither necessary to find special trees nor to make additional assumptions on the functions and parameters that define the controlled sources. Usually, it is not difficult to check whether a model of a network element including controlled sources satisfies these conditions or not.

2. If a model of a network element does not satisfy the conditions, it is not difficult to fulfill them by introducing some additional capacitances, resistances or inductances into the model for this element.
3. Nevertheless, the topological assumptions made for the controlled sources are sufficient but not necessary.

Table IV. CCCS—condition (2a)

The controlling currents of a CCCS can be currents of:
1. Inductances,
2. Independent current sources,
3. Resistances or VCCSs for which the controlling nodes are connected by:
   (a) Capacitances,
   (b) Voltage sources,
   (c) Paths containing only the elements described in (3a) and (3b),
4. Branches that form a cutset with the elements described in (1), (2) or (3).

Table V. CCCS—condition (2b)

The controlling current of a CCCS can be the current of:
1. Inductances,
2. Independent current sources,
3. Resistances,
4. Voltage sources that do not form a part of a C–V loop,
5. VCCS,
6. A branch that forms a cutset with the elements described in (1), (2), (3), (4) and (5).

Table VI. CCCS—condition (2c)

The controlling current of a CCCS can be the current of:
1. Inductances,
2. Resistances,
3. Independent current sources,
4. VCCS,
5, A branch that forms a cutset with the elements described in (1), (2), (3) and (4).

4. Comparing the class of networks satisfying the assumptions of Theorem 4.1 with the class of allowable networks described in Table 10-3-1 in Reference [10], the assumptions for the controlled sources differ mainly in two aspects:
   4.1. In Reference [10], branch currents of C–V loops and branch voltages of L–I cutsets are not allowed to be controlling variables for controlled sources. However we allow that branch voltages of L–I cutsets appear as controlling variables under certain circumstances.
   4.2. We forbid controlled voltage sources in C–V loops and controlled current sources in L-I cutsets whereas they are allowed in Reference [10] if they depend only on a voltage of a tree capacitor or the current of a cotree inductor for a normal tree. In other words, controlled voltage sources in C–V loops and controlled current sources in L–I cutsets are allowed if they depend only on state variables.

Following the proof for the existence of normal-form equations in Reference [10] it is easy to verify that a differentiation is necessary for all tree inductor voltages and all cotree capacitor currents. In other words, the index of the DAE is 2 if and only if the network contains a C–V loop or an L–I cutset. Observe that, in this case, C-only loops have to be added to the class of C–V loops since the currents through C-only loops belong to the network variables whereas these currents are excluded in MNA formulations. If the network does not contain C–V loops or L–I cutsets then the index is 1, i.e. the results correspond to our results.

The approach in Reference [10] assumes additionally that the global system matrices Z (defined in Equation (10–35a)) and M (defined in Equations (10–46)) have to be non-singular. The authors say that one can always satisfy these assumptions if one perturbes some element parameters. They do not give an advice which parameters should be perturbed. But the enormous dimension of nowadays relevant systems requires a localization of those parameters which should be perturbed. That is why we assumed the element related matrices to be positive definite which is a condition that can be controlled locally.

5. Corresponding results for linear systems are already presented in Reference [8]. They base on a normal tree for which the controlled sources (in this paper only two-ports are considered) have to satisfy the following conditions:
   5.1. Elements with only impedance matrix representation (e.g. CCVS) must be cotree elements.
   5.2. Elements with only admittance matrix representation (e.g. VCCS) must be tree elements.
   5.3. Elements with only hybrid matrix representation (e.g. CCCS, VCVS) must have the admittance branch in the tree and the other branch in the cotree.
6. The assumption that the element characteristic matrices C and L are positive definite can be found in numerous articles (cf., among others, [8, 11, 13]). Moreover, in References [9, 25, 26], we find already the assumption that the matrices C, L, and G have to be positive definite. But all consider only nonlinear time-invariant networks without controlled sources.
7. More recently, several geometrical approaches (cf. References [11–13]) have been established. These results reflect exactly the geometrical index approach (cf. References [33, 34]). All papers deal with time-invariant networks and base on the transversality condition: the space of branch voltages and branch currents satisfying the constitutive relations and the Kirchhoff space are transversal. In Reference [12], only systems that do not contain C-loops and L-cutsets (i.e. index-1 systems because of the time-invariance) are considered. The index-2 case is developed in Reference [13] (without using the term 'index').
8. Similar results to those in Theorem 4.1 are reported for nonlinear networks containing multiports in Reference [35]. But, unfortunately, the assumptions on controlled sources are not specified at all.

Example: (1) Consider again the MOSFET-model given in Figure 2. The VCCS from source to drain is controlled by the branch voltages u_{GS}, u_{DS}, and u_{BS}. For this, the conditions (2a)–(2c) are satisfied since there are capacitive ways from gate to source, from drain to source as well as from bulk to source, and there exists a capacitive way from source to drain.

(2) Consider the VCCS in Figure 3 (from Reference [10]). The considered CCVS does not form a part of a C–V loop and it is controlled by the current of a branch that forms a cutset with inductances. Therefore, it meets the condition (1) of Theorem 4.1.

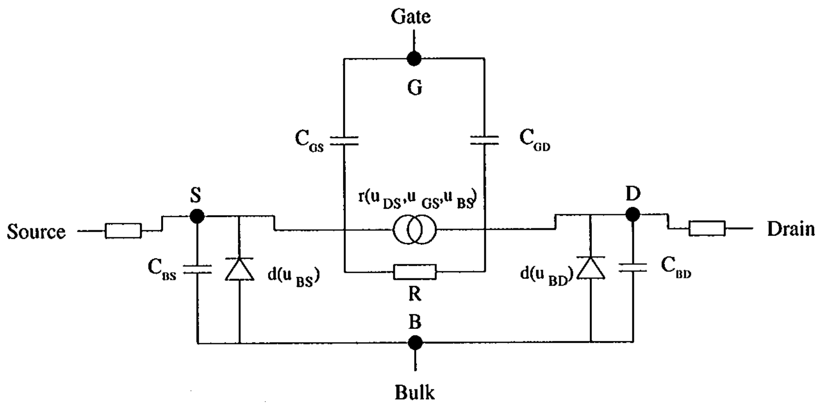

Figure 2. MOSFET-model.

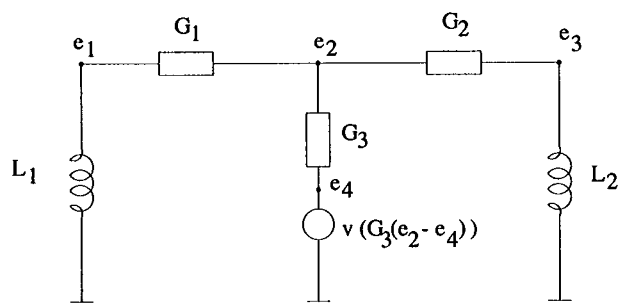

Figure 3. Circuit with CCVS.

Corollary 4.3. *The assumption of Theorem* 4.1 *on the resistances can be slightly reduced. In fact, only the positive definiteness of the conductance matrix corresponding to those resistances that do not form a loop with capacitances and/or voltage sources is required.*

This statement follows immediately from Theorem 4.1 if we consider the resistances as VCCS. In order to obtain a description of assumption (1) by means of projectors, we split the incidence matrix A_V into $(A_{Vt} A_{V\text{co}})$ for independent and controlled sources, respectively.

Lemma 4.4. *The condition that controlled voltage sources do not form a part of a C–V loop is equivalent to*

$$\bar{Q}_{V\text{-}C} = \begin{pmatrix} (\bar{Q}_{V\text{-}C})_t \\ 0 \end{pmatrix}.$$

Here, $(\bar{Q}_{V\text{-}C})_t$ *denotes the upper part of* $\bar{Q}_{V\text{-}C}$ *corresponding to* A_{Vt}.

Proof. A controlled voltage source forms a part of a C–V loop if and only if the column a_s of $A_{V\text{co}}$ corresponding to this source depends linearly on the columns of $(A_C\bar{A}_V)$, where $\bar{A}_V$ denotes the matrix A_V reduced by the column a_s, i.e. there is a vector v such that

$$(A_C A_V)v = 0 \quad \text{and} \quad v_s \neq 0$$

for the sth component of v corresponding to the controlled source considered. That means, there is a vector v such that

$$Q_C^{\mathrm{T}} A_V v = 0 \quad \text{and} \quad v_s \neq 0,$$

i.e. the sth row of $\bar{Q}_{V\text{-}C}$ has a non-zero entry. This is equivalent to

$$\bar{Q}_{V\text{-}C} \neq \begin{pmatrix} (\bar{Q}_{V\text{-}C})_t \\ 0 \end{pmatrix}. \qquad \square$$

Hence, assumption (1) of Theorem 4.1 implies that

$$\bar{Q}_{V\text{-}C}^{\mathrm{T}} v\left(A^{\mathrm{T}}e, \frac{\mathrm{d}q(A_C^{\mathrm{T}}e, t)}{\mathrm{d}t}, j_L, j_V, t\right) = \bar{Q}_{V\text{-}C}^{\mathrm{T}} v_t(t), \tag{17}$$

$$v\left(A^{\mathrm{T}}e, \frac{\mathrm{d}q(A_C^{\mathrm{T}}e, t)}{\mathrm{d}t}, j_L, j_V, t\right) = v_*(A_C^{\mathrm{T}}e, j_L, t), \tag{18}$$

for a suitable function v_* and for a vector $v_t(t)$ that contains the functions of independent voltage sources and zeros instead of the functions of controlled voltage sources. In the following we will drop the index *.

In order to transcribe the assumptions made for controlled current sources, we split the incidence matrix A_I into $(A_{It}, A_{Ia}, A_{Ib}, A_{Ic})$ and the current vector i correspondingly, for the independent current sources and the controlled current sources that fulfill Equations (2a)–(2c), respectively. If a controlled current source fulfills more than one of conditions (2a)–(2c), the corresponding column of A_I should be assigned to only one of the matrices A_{Ia}, A_{Ib}, and A_{Ic}.

Lemma 4.5. *The condition that controlled current sources do not form a part of an L–I cutset is equivalent to the relation* $Q_{CRV}^{\mathrm{T}} A_I = (Q_{CRV}^{\mathrm{T}} A_{It} 0)$.

Proof. A controlled current source forms a part of an L–I cutset if and only if the column a_s of (A_{Ia}, A_{Ib}, A_{Ic}) corresponding to this controlled source is linearly independent of the columns belonging to (A_C, A_R, A_V), i.e.

$$a_s \notin \operatorname{im}(A_C A_R A_V) \quad \text{and, therefore,} \quad Q_{CRV}^{\mathrm{T}} a_s \neq 0.$$

But, this is equivalent to the condition that $Q_{CRV}^{\mathrm{T}}(A_{Ia}, A_{Ib}, A_{Ic}) \neq 0$. $\square$

Thus, assumption (2a) of Theorem 4.1 implies that

$$Q_{CRV}^{\mathrm{T}} A_I i\left(A^{\mathrm{T}}e, \frac{\mathrm{d}q(A_C^{\mathrm{T}}e, t)}{\mathrm{d}t}, j_L, j_V, t\right) = Q_{CRV}^{\mathrm{T}} A_{It} i_t, \tag{19}$$

$$i\left(A^{\mathrm{T}}e, \frac{\mathrm{d}q(A_C^{\mathrm{T}}e, t)}{\mathrm{d}t}, j_L, j_V, t\right) = i_a(A_C^{\mathrm{T}}e, A_V^{\mathrm{T}}e, j_L, t), \tag{20}$$

for a suitable function i_a.

Furthermore, assumption (2b) of Theorem 4.1 implies by definition that

$$Q_C^{\mathrm{T}} A_{Ib} = 0, \tag{21}$$

$$i\left(A^{\mathrm{T}} e, \frac{\mathrm{d}q(A_C^{\mathrm{T}} e, t)}{\mathrm{d}t}, j_L, j_V, t \right) = i_b(A^{\mathrm{T}} e, j_L, \bar{P}_{V\text{-}C} j_V, t) \tag{22}$$

for a suitable function i_b.

Finally, assumption (2c) of Theorem 4.1 implies that

$$Q_{V\text{-}C}^{\mathrm{T}} Q_C^{\mathrm{T}} A_{Ic} = 0 \tag{23}$$

$$i\left(A^T e, \frac{dq(A_C^{\mathrm{T}} e, t)}{\mathrm{d}t}, j_L, j_V, t \right) = i_c(A^{\mathrm{T}} e, j_L, t) \tag{24}$$

for a suitable function i_c.

Regarding Equations (19), (21), and (23), the assumptions imply that

$$Q_{CRV}^{\mathrm{T}} A_I i\left(A^{\mathrm{T}} e, \frac{\mathrm{d}q(A_C^{\mathrm{T}} e, t)}{\mathrm{d}t}, j_L, j_V, t \right) = Q_{CRV}^{\mathrm{T}} A_{It} i_t \tag{25}$$

is always fulfilled. To shorten denotations we write

$$i(A^{\mathrm{T}} e, j_L, P_{V\text{-}C} j_V, t) \tag{26}$$

when we do not distinguish between Equations (20), (22), and (24).

The proofs of the theorems follow in the next sections.

5. THE DIFFERENTIAL INDEX OF THE DAE SYSTEMS IN CIRCUIT SIMULATION

In this section we obtain the differential index of the DAE system as well as expressions for the constraints. In the following, we assume that the required smoothness is given.

Theorem 5.1. *Consider lumped electric circuits satisfying the assumptions of Theorem* 4.1. *Then it holds*:

1. *If the network contains neither L–I cutsets nor C–V loops, then the conventional MNA leads to a DAE system with differential index*-1 *and the constraints are only the explicit ones*:

$$Q_C^{\mathrm{T}}[A_R r(A_R^{\mathrm{T}} e, t) + A_L j_L + A_V j_V + A_{Ia,c} i_{a,c}(A^{\mathrm{T}} e, j_L, t)] = 0, \tag{27}$$

$$A_V^{\mathrm{T}} e - v(A_C^{\mathrm{T}} e, j_{\mathrm{L}}, t) = 0. \tag{28}$$

2. *If the network contains L–I cutsets or C–V loops, then the conventional MNA leads to a DAE system with differential index*-2. *With regard to the constraints, we distinguish the following three possibilities.*
   (a) *If the network does not contain an L–I cutset* (*but contains C–V loops*), *then the constraints are the explicit ones, namely Equations* (27) *and* (28), *and, additionally, the hidden constraint*:

$$\bar{Q}_{V\text{-}C}^{\mathrm{T}} A_V^{\mathrm{T}} H_1^{-1}(A_C^{\mathrm{T}} e, t) P_C^{\mathrm{T}}[A_C q_t'(A_C^{\mathrm{T}} e, t) + A_R r(A_R^{\mathrm{T}} e, t) + A_L j_L + A_V j_V + A_I i(A^{\mathrm{T}} e, j_L, \bar{P}_{V\text{-}C} j_V, t)] + \bar{Q}_{V\text{-}C}^{T} \frac{\mathrm{d}v_t}{\mathrm{d}t} = 0. \tag{29}$$

(b) *If the network does not contain C–V loops, but contains L–I cutsets, the constraints are the explicit ones*, (27) *and* (28), *and, additionally, the hidden constraint*:

$$Q_{CRV}^{\mathrm{T}}\left(A_L L^{-1}(j_L,t)(A_L^{\mathrm{T}}e-\phi_t'(j_L,t))+A_{It}\frac{\mathrm{d}i_t}{\mathrm{d}t}\right)=0. \tag{30}$$

(c) *If the network contains L–I cutsets and C–V loops, then the MNA leads to a DAE system with differential index*-2. *In this case, the constraints are the explicit ones*, (27) *and* (28), *and the hidden ones* (29) *and* (30).[§§]

Remember that the functions $v_t(t)$ and $i_t(t)$ represent the functions of the independent voltage and current sources and that the matrices $H_1(\cdot)$–$H_6(\cdot)$ were defined in Lemma 2.5.

Proof. In the following, we will take advantage of the fact that the analysed system is quasi-linear and that the matrices $C(A_C^{\mathrm{T}}e,t)$, $L(j_L,t)$ and $G(A_R^{\mathrm{T}}e,t)$ are positive definite. Our aim is to obtain a representation of $\mathrm{d}e/\mathrm{d}t$, $\mathrm{d}j_L/\mathrm{d}t$, $\mathrm{d}j_V/\mathrm{d}t$ as continuous functions of e, j_L and j_V. To this purpose, we consider the following splittings:

$$\frac{\mathrm{d}e}{\mathrm{d}t}=P_C\frac{\mathrm{d}e}{\mathrm{d}t}+Q_C P_{V\text{-}C}\frac{\mathrm{d}e}{\mathrm{d}t}+Q_C Q_{V\text{-}C}P_{R\text{-}CV}\frac{\mathrm{d}e}{\mathrm{d}t}+Q_{CRV}\frac{\mathrm{d}e}{\mathrm{d}t},$$

$$\frac{\mathrm{d}j_V}{\mathrm{d}t}=\bar{P}_{V\text{-}C}\frac{\mathrm{d}j_V}{\mathrm{d}t}+\bar{Q}_{V\text{-}C}\frac{\mathrm{d}j_V}{\mathrm{d}t}.$$

First, we make a general approach and afterwards we distinguish the different cases with regard to the topological properties of the network.

Step 0: If we multiply Equation (6) by $H_1^{-1}(A_C^{\mathrm{T}},t)\,P_C^{\mathrm{T}}$ and Q_C^{T}, respectively, we obtain

$$P_C\frac{\mathrm{d}e}{\mathrm{d}t}=-H_1^{-1}(A_C^{\mathrm{T}}e,t)P_C^{\mathrm{T}}[A_C q_t'(A_C^{\mathrm{T}}e,t)+A_R r(A_R^{\mathrm{T}}e,t)+A_L j_L$$
$$+A_V j_V+A_I i(A^{\mathrm{T}}e,j_L,P_{V\text{-}C}j_V,t)] \tag{31}$$

and Equation (27). As $L(j_L,t)$ is regular, we obtain equations for $\mathrm{d}j_L/\mathrm{d}t$ directly from Equation (7)

$$\frac{\mathrm{d}j_L}{\mathrm{d}t}=L^{-1}(j_L,t)(A_L^{\mathrm{T}}e-\phi_t'(j_L,t)). \tag{32}$$

Note that the arguments of the controlled sources in Equations (31), (27), and (28) are written in accordance with Equations (18)–(26).

Step 1: Next, we differentiate Equations (28) and (27), i.e. the constraints, and split them in the following way:

Step 1*a*: Using Equation (17) we split the derivative of Equation (28) into

$$\bar{Q}_{V\text{-}C}^{\mathrm{T}}A_V^{\mathrm{T}}P_C\frac{\mathrm{d}e}{\mathrm{d}t}=\bar{Q}_{V\text{-}C}^{\mathrm{T}}\frac{\mathrm{d}v_t}{\mathrm{d}t} \tag{33}$$

[§§] The expressions for the explicit and the hidden constraints are of special interest with regard to a consistent initialization (cf. References [27, 28]).

and

$$\bar{P}^{\mathrm{T}}_{V\text{-}C} A_V^{\mathrm{T}} \left(P_C + Q_C \right) \frac{\mathrm{d}e}{\mathrm{d}t} = \bar{P}^{\mathrm{T}}_{V\text{-}C} \frac{\mathrm{d}v(A_C^{\mathrm{T}} e, j_L, t)}{\mathrm{d}t}. \tag{34}$$

If we consider Equation (33), we can realize that $P_C(\mathrm{d}e/\mathrm{d}t)$ can be substituted making use of Equation (31) to achieve the hidden constraint (29).

Step 1*b*: Using Equation (25) we split the derivative of Equation (27) into

$$Q_{CRV}^{\mathrm{T}} \left[A_L \frac{\mathrm{d}j_L}{\mathrm{d}t} + A_{It} \frac{\mathrm{d}i_t}{\mathrm{d}t} \right] = 0, \tag{35}$$

$$P_{R\text{-}CV}^{\mathrm{T}} Q_{V\text{-}C}^{\mathrm{T}} Q_C^{\mathrm{T}} \left[A_R G(A_R^{\mathrm{T}} e, t) A_R^{\mathrm{T}} \frac{\mathrm{d}e}{\mathrm{d}t} + A_R r_t'(A_R^{\mathrm{T}} e, t) + A_L \frac{\mathrm{d}j_L}{\mathrm{d}t} + A_{Ia} \frac{\mathrm{d}i_a(A_C^{\mathrm{T}} e, A_V^{\mathrm{T}} e, j_L, t)}{\mathrm{d}t} \right] = 0 \tag{36}$$

and

$$P_{V\text{-}C}^{\mathrm{T}} Q_C^{\mathrm{T}} \left[A_R G(A_R^{\mathrm{T}} e, t) A_R^{\mathrm{T}} \frac{\mathrm{d}e}{\mathrm{d}t} + A_R r_t'(A_R^{\mathrm{T}} e, t) + A_L \frac{\mathrm{d}j_L}{\mathrm{d}t} + A_V \frac{\mathrm{d}j_V}{\mathrm{d}t} + A_{Ia,c} \frac{\mathrm{d}i_{a,c}(A^{\mathrm{T}} e, j_L, t)}{\mathrm{d}t} \right] = 0. \tag{37}$$

Taking into account Equation (32), Equation (35) leads to the hidden constraint (30).

Step 2: Finally, we differentiate the two hidden constraints (30) and (29) obtained in the last step:

$$\frac{\mathrm{d}}{\mathrm{d}t} (Q_{CRV}^{\mathrm{T}} A_L L^{-1}(j_L, t) (A_L^{\mathrm{T}} e - \phi_t'(j_L, t))) + Q_{CRV}^{\mathrm{T}} A_{It} \frac{\mathrm{d}^2 i_t}{\mathrm{d}t^2} = 0 \tag{38}$$

and

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} (\bar{Q}^{\mathrm{T}}_{V\text{-}C} A_V^{\mathrm{T}} H_1^{-1}(A_C^{\mathrm{T}} e, t) P_C^{\mathrm{T}} [A_C q_t'(A_C^{\mathrm{T}} e, t) + A_R r(A_R^{\mathrm{T}} e, t) \\ &\quad + A_L j_L + A_V j_V + A_I i(A^{\mathrm{T}} e, j_L, \bar{P}_{V\text{-}C} j_V, t)]) + \bar{Q}^{\mathrm{T}}_{V\text{-}C} \frac{\mathrm{d}^2 v_t}{\mathrm{d}t^2} = 0. \end{aligned} \tag{39}$$

Step 3: Let us now take into account the different topological properties of the systems we mentioned in Theorem 4.1:

1. If the network does not contain L–I cutsets, then $Q_{CRV} = 0$ (cf. point 1 in Corollary 2.3). Thus, in this case there is no hidden constraint (30). Moreover, as we have $\mathrm{d}e/\mathrm{d}t = P_C(\mathrm{d}e/\mathrm{d}t) + Q_C P_{V\text{-}C}(\mathrm{d}e/\mathrm{d}t) + Q_C Q_{V\text{-}C} P_{R\text{-}CV}(\mathrm{d}e/\mathrm{d}t)$ then, already Step 1 leads to an expression for $\mathrm{d}e/\mathrm{d}t$. If we multiply Equation (34) by $H_2^{-1} Q_C^{\mathrm{T}} A_V$ after substituting expressions (31) for $P_C(\mathrm{d}e/\mathrm{d}t)$ and (32) for $\mathrm{d}j_L/\mathrm{d}t$, we obtain an expression for $P_{V\text{-}C}(\mathrm{d}e/\mathrm{d}t)$. Then, substituting $P_C(\mathrm{d}e/\mathrm{d}t) + Q_C P_{V\text{-}C}(\mathrm{d}e/\mathrm{d}t)$ and $\mathrm{d}j_L/\mathrm{d}t$ into Equation (36) and multiplying by $H_4^{-1}(\cdot)$, we obtain a representation for $P_{R\text{-}CV}(\mathrm{d}e/\mathrm{d}t)$. Note that these transformations are reversible by multiplication by $H_3^{-1} A_V^{\mathrm{T}} Q_C H_2$ and $H_4(\cdot)$, respectively.
2. If the network contains an L–I cutset, then $Q_{CRV} \neq 0$ (cf. point 1 in Corollary 4). Therefore, we consider

$$\frac{\mathrm{d}e}{\mathrm{d}t} = P_C \frac{\mathrm{d}e}{\mathrm{d}t} + Q_C P_{V\text{-}C} \frac{\mathrm{d}e}{\mathrm{d}t} + Q_C Q_{V\text{-}C} P_{R\text{-}CV} \frac{\mathrm{d}e}{\mathrm{d}t} + Q_{CRV} \frac{\mathrm{d}e}{\mathrm{d}t}$$

and observe that we obtain the needed expression for $Q_{CRV}(\mathrm{d}e/\mathrm{d}t)$ when multiplying Equation (38) by $H_5^{-1}(\cdot)$ after substituting the expressions for $P_C(\mathrm{d}e/\mathrm{d}t) + Q_C P_{V\text{-}C}(\mathrm{d}e/\mathrm{d}t) + Q_C Q_{V\text{-}C} P_{R\text{-}CV}(\mathrm{d}e/\mathrm{d}t)$ and $\mathrm{d}j_L/\mathrm{d}t$.

3. If the network does not contain C–V loops, then $Q_C^{\mathrm{T}} A_V$ has full column rank (cf. point 4 in Theorem 2.2). Therefore, $\bar{P}_{V\text{-}C} = I$, and we obtain an expression for $\mathrm{d}j_V/\mathrm{d}t$ when multiplying Equation (37) by $H_3^{-1} A_V^{\mathrm{T}} Q_C$ after substituting the obtained expressions for $\mathrm{d}e/\mathrm{d}t$ and $\mathrm{d}j_L/\mathrm{d}t$. This transformation is reversible as well, as can be seen by multiplication by $H_2^{-1} Q_C^{\mathrm{T}} A_V H_3$. On the other hand, as $\bar{Q}_{V\text{-}C} = 0$, there is no hidden constraint (29).
4. If the network contains a C–V loop, then $Q_C^{\mathrm{T}} A_V$ does not have full column rank (cf. point 4 in Theorem 2.2). Therefore, $\bar{Q}_{V\text{-}C} \neq 0$, and we obtain an expression for $\bar{Q}_{V\text{-}C}(\mathrm{d}j_V/\mathrm{d}t)$ from Equation (39) by multiplication by $H_6^{-1}(\cdot)$ after the substitution of $\mathrm{d}e/\mathrm{d}t, \mathrm{d}j_L/\mathrm{d}t$ and $\bar{P}_{V\text{-}C}(\mathrm{d}j_V/\mathrm{d}t)$.

Note that this is successively possible because of Equations (19)–(24) and that it is important to achieve first the complete expression for $\mathrm{d}e/\mathrm{d}t$ and afterwards those for $\mathrm{d}j_{\mathrm{V}}/\mathrm{d}t$, because of the allowed controlling elements in Equation (2c).[¶¶]

Step 4: Now we analyse the possible cases:

1. If the network contains neither L–I cutsets nor controlled C–V loops, both equations, (29) and (30), do not appear, i.e. we obtain a representation for $\mathrm{d}e/\mathrm{d}t$, $\mathrm{d}j_L/\mathrm{d}t$, $\mathrm{d}j_V/\mathrm{d}t$ as functions of e, j_L and j_V with the expressions obtained in Step 1. Thus, the differential index of the system is 1 and no hidden constraints appear.
2. In the other three cases, one more differentiation has to be carried out in order to find explicit expressions for the derivatives. Therefore, the differential index is 2.

With Step 3 the statements of the theorem follow immediately from Step 2.

Theorem 5.2. *If the differential index is* 1, *then the network contains neither C–V loops nor L–I cutsets. If the differential index is* 2, *then the network contains at least a C–V loop or an L–I cutset.*

Proof. Let us now suppose that the differential index is 1. Then the hidden constraints have to be trivial, i.e. if we regard the homogeneous system, we obtain

$$\bar{Q}_{V\text{-}C}^{\mathrm{T}} A_V^{\mathrm{T}} P_C \frac{\mathrm{d}e}{\mathrm{d}t} = 0, \tag{40}$$

$$Q_{CRV}^{\mathrm{T}} A_L \frac{\mathrm{d}j_L}{\mathrm{d}t} = 0. \tag{41}$$

Making use of the fact that A_V^{T} and (A_C, A_R, A_V, A_L) have full row rank, we obtain $\bar{Q}_{V\text{-}C} = 0$ and $Q_{CRV} = 0$ (cf. Theorem 2.2), i.e. the network does not contain C–V loops or L–I cutsets.

If the index is supposed to be 2, then at least one constraint has to appear, i.e. either $\bar{Q}_{V\text{-}C}$ or Q_{CRV} has to be nontrivial (or both). Corollary 2.3 implies that the network has to contain at least a C–V loop or an L–I cutset. □

[¶¶] This variation of the order changes the spaces associated with the DAE-system, as will be shown in Lemma 6.2.

Proof of Theorem 4.2. To conclude, we observe that Theorem 5.1 holds analogously for the charge-oriented MNA. The results obtained with the differential index method are basically the same as those for the conventional MNA. To shorten denotations, we drop the arguments of the controlled sources, because they appear in the same way as in the proof of Theorem 5.1. Analogously to Step 0 from the proof of Theorem 4.1, we can split Equation (9) into

$$A_C \frac{\mathrm{d}q}{\mathrm{d}t} + P_C^{\mathrm{T}}(A_R r(A_R^{\mathrm{T}} e, t) + A_L j_L + A_V j_V + A_I i) = 0, \tag{42}$$

$$Q_C^{\mathrm{T}}(A_R r(A_R^{\mathrm{T}} e, t) + A_L j_L + A_V j_V + A_I i) = 0. \tag{43}$$

If we define $\bar{Q}_C$ as a projector onto $\ker A_C$, we can define the matrix $\bar{H}_{C1} := A_C^{\mathrm{T}} A_C + \bar{Q}_C^{\mathrm{T}} \bar{Q}_C$, which is regular, and obtain the expression

$$\bar{P}_C \frac{\mathrm{d}q}{\mathrm{d}t} = -\bar{H}_{C1}^{-1} A_C^{\mathrm{T}} P_C^{\mathrm{T}}(A_R r(A_R^{\mathrm{T}} e, t) + A_L j_L + A_V j_V + A_I i)$$

by multiplication of Equation (42) by $\bar{H}_{C1}^{-1} A_C^{\mathrm{T}}$. Note that this transformation is reversible by multiplication by $\bar{H}_{C2}^{-1} A_C A_C^{\mathrm{T}} A_C$, if $\bar{H}_{C2} := A_C A_C^{\mathrm{T}} + Q_C^{\mathrm{T}} Q_C$. As Equation (10) is already an expression for $\mathrm{d}\phi/\mathrm{d}t$, the constraints are Equations (11–13) and (43). The derivatives of Equations (12) and (13) can be splitted as follows:

$$\bar{P}_C \frac{\mathrm{d}q}{\mathrm{d}t} - \bar{P}_C C(A_C^{\mathrm{T}} e, t) A_C^{\mathrm{T}} \frac{\mathrm{d}e}{\mathrm{d}t} - \bar{P}_C q_t'(A_C^{\mathrm{T}} e, t) = 0, \tag{44}$$

$$\bar{Q}_C \frac{\mathrm{d}q}{\mathrm{d}t} - \bar{Q}_C C(A_C^{\mathrm{T}} e, t) A_C^{\mathrm{T}} \frac{\mathrm{d}e}{\mathrm{d}t} - \bar{Q}_C q_t'(A_C^{\mathrm{T}} e, t) = 0, \tag{45}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} - L(j_L, t) \frac{\mathrm{d}j_L}{\mathrm{d}t} - \phi_t'(j_L, t) = 0. \tag{46}$$

From Equation (44) we obtain the following expression for $P_C(\mathrm{d}e/\mathrm{d}t)$ and multiplication by $H_1^{-1}(A_C^{\mathrm{T}} e, t) A_C$

$$\begin{aligned} P_C \frac{\mathrm{d}e}{\mathrm{d}t} = & -H_1^{-1}(A_C^{\mathrm{T}} e, t) P_C^{\mathrm{T}}(A_R r(A_R^{\mathrm{T}} e, t) + A_L j_L + A_V j_V + A_I i) \\ & -H_1^{-1}(A_C^{\mathrm{T}} e, t) A_C q_t'(A_C^{\mathrm{T}} e, t). \end{aligned}$$

Note that this holds because of

$$H_1^{-1}(\cdot) A_C \bar{H}_{C1}^{-1} A_C^{\mathrm{T}} P_C^{\mathrm{T}} = H_1^{-1}(\cdot) \overbrace{A_C \underbrace{\bar{H}_{C1}^{-1} A_C^{\mathrm{T}} A_C}_{\bar{P}_C} A_C^{T} \bar{H}_{C2}^{-1}}^{P_C^{\mathrm{T}}} P_C^{\mathrm{T}} = H_1^{-1}(\cdot) P_C^{\mathrm{T}}.$$

Inserting this into Equation (45), we obtain an equation for $\bar{Q}_C(\mathrm{d}q/\mathrm{d}t)$. On the other hand, Equation (46) leads to the expression (32) for $\mathrm{d}j_L/\mathrm{d}t$. As the constraints (11) and (43) are the same as Equations (27) and (28) in the conventional MNA, the expressions for the remaining derivatives are identical. This implies that the index statements of Theorem 5.1 are valid for the charge-oriented MNA, too. □

Remark: Observe that only the required smoothness has to be given in each case, and that we can recognize the smoothness requirements directly in the above equations. The next chapter shows how it is possible to define an index with considerably fewer smoothness assumptions on the variables and on the input functions. This is specially relevant for circuit simulation because, in general, only low smoothness is given.

6. THE TRACTABILITY INDEX OF THE DAE SYSTEM IN CIRCUIT SIMULATION

Note that the assumption $N := \ker f'_{x'}(x', x, t)$ is constant is given for the quasi-linear DAEs (6)–(8) (cf. Equations (50)) and (9)–(13).

For shorter expressions we drop the arguments in the following section. In order to distinguish between constant and non-constant terms, we will use a dot as an argument for non-constant terms.

Theorem 6.1. *Let the assumptions of Theorem* 4.1 *be satisfied. Then it holds*:

1. *If the network contains neither L–I cutsets nor C-V loops, then the conventional MNA leads to an index*-1 *tractable DAE system.*
2. *If the network contains L–I cutsets or C–V loops, then the conventional MNA leads to an index*-2 *tractable DAE system. The canonical projector $Q_1(\cdot)$ onto $N_1(x, t)$ along S_1 is given by*

$$\begin{pmatrix} H_1^{-1}(\cdot)A_V\bar{Q}_{V\text{-}C}H_6^{-1}(\cdot)\bar{Q}_{V\text{-}C}^{\mathrm{T}}A_V^{\mathrm{T}}P_C & \begin{matrix} Q_{CRV}H_5^{-1}(\cdot)Q_{CRV}^{\mathrm{T}}A_L \\ -P_CQ_VX(\cdot)Q_{CRV}H_5^{-1}(\cdot)Q_{CRV}^{\mathrm{T}}A_L \end{matrix} & 0 \\ 0 & L^{-1}(\cdot)A_L^{\mathrm{T}}Q_{CRV}H_5^{-1}(\cdot)Q_{CRV}^{\mathrm{T}}A_L & 0 \\ -\bar{Q}_{V\text{-}C}H_6^{-1}(\cdot)\bar{Q}_{V\text{-}C}^{\mathrm{T}}A_V^{\mathrm{T}}P_C & -A_V^{\mathrm{T}}Y(\cdot)Q_{CRV}H_5^{-1}(\cdot)Q_{CRV}^{\mathrm{T}}A_L & 0 \end{pmatrix} \tag{47}$$

where the matrices $X(\cdot)$ and $Y(\cdot)$ are chosen in such a way that

$$A_{Ib,c}\frac{\mathrm{d}i_{b,c}(\cdot)}{\mathrm{d}e}Q_C = A_CC(\cdot)A_C^{\mathrm{T}}Q_VX(\cdot) + A_VA_V^{\mathrm{T}}Y(\cdot).$$

Note, the matrices $H_1(\cdot)$, $H_5(\cdot)$, and $H_6(\cdot)$ were defined in Lemma 2.5.

Remark. The existence of such matrices $X(\cdot)$ and $Y(\cdot)$ is satisfied since the relation $Q_{V\text{-}C}^{\mathrm{T}}Q_C^{\mathrm{T}}A_{Ib,c}(\mathrm{d}i_{b,c}(\cdot)/\mathrm{d}e) = 0$ is true (cf. Equations (21) and (23)).

Before we will prove this theorem, we want to consider the special structure of $A(\cdot)$, $B(\cdot)$, $Q(\cdot)$, $S(\cdot)$, $A_1(\cdot)$, $B_1(\cdot)$, $Q_1(\cdot)$, and $S_1(\cdot)$ in case of circuit simulation.

Writing system (6)–(8) as a non-linear DAE (16) with $A(x', x, t) := f'_{x'}(x', x, t)$ and $B(x', x, t) := f'_x(x', x, t)$, we obtain that

$$A(\cdot) = \begin{pmatrix} A_CC(\cdot)A_C^{\mathrm{T}} & 0 & 0 \\ 0 & L(\cdot) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{48}$$

and

$$B(\cdot)=\begin{pmatrix} A_C\bar{C}(\cdot)A_C^{\mathrm{T}}+A_{\mathrm{R}}G(\cdot)A_R^{\mathrm{T}}+A_I\dfrac{\mathrm{d}i(\cdot)}{\mathrm{d}e} & A_L+A_I\dfrac{\mathrm{d}i(\cdot)}{\mathrm{d}j_L} & A_V+A_I\dfrac{\mathrm{d}i(\cdot)}{\mathrm{d}j_V} \\ -A_L^{\mathrm{T}} & \bar{L}(\cdot) & 0 \\ A_V^{\mathrm{T}}-\dfrac{\mathrm{d}v(\cdot)}{\mathrm{d}e} & -\dfrac{\mathrm{d}v(\cdot)}{\mathrm{d}j_L} & 0 \end{pmatrix} \tag{49}$$

with

$$\bar{C}(u',u,t)=\frac{\mathrm{d}}{\mathrm{d}u}C(u,t)u'+\frac{\mathrm{d}}{\mathrm{d}u}q_t'(u,t)$$

and

$$\bar{L}(j_L',j_L,t)=\frac{\mathrm{d}}{\mathrm{d}j_L'}L(j_L,t)j_L'+\frac{\mathrm{d}}{\mathrm{d}j_L}\phi_t'(j_L,t).$$

Let us remark here that $A(\cdot)$ represents the leading coefficient matrix. It has a different meaning than the incidence matrix $A=(A_C, A_L, A_R, A_V, A_I)$. We will not use the notation A any longer. Therefore, this denotation should be acceptable.

Since $C(\cdot)$ is positive definite, we may conclude that

$$\ker A(\cdot)=\ker A_C^{\mathrm{T}}\times\{0\}\times\mathbb{R}^{n_V} \tag{50}$$

and

$$\operatorname{im} A(\cdot)=\operatorname{im} A_C\times\mathbb{R}^{n_L}\times\{0\}. \tag{51}$$

Here, n_V describes the number of voltage sources and n_L describes the number of inductances in the network. Note, the null-space of $A(\cdot)$ as well as the image-space of $A(\cdot)$ are constant in any case. The null-space of $A(\cdot)$ describes the non-dynamic components of the circuit. For further considerations let us introduce a projector Q onto $\ker A(\cdot)$ as

$$Q=\begin{pmatrix} Q_C & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{pmatrix}.$$

For the definition of Q_C see Section 2.3. The space $S(\cdot)=\{z\colon B(\cdot)z\in\operatorname{im}A(\cdot)\}$ describes all solution components for which we do not find an algebraic representation. Regarding Equation (51) it is given by

$$S(\cdot)=\Bigg\{z\colon\left(A_V^{\mathrm{T}}-\frac{\mathrm{d}v(\cdot)}{\mathrm{d}e}\right)z_e-\frac{\mathrm{d}v(\cdot)}{\mathrm{d}j_L}z_L=0,$$
$$\left(A_RG(\cdot)A_R^{\mathrm{T}}+A_I\frac{\mathrm{d}i(\cdot)}{\mathrm{d}e}\right)z_e+\left(A_L+A_I\frac{\mathrm{d}i(\cdot)}{\mathrm{d}j_L}\right)z_L+\left(A_V+A_I\frac{\mathrm{d}i(\cdot)}{\mathrm{d}j_V}\right)z_V\in\operatorname{im}A_C\Bigg\}.$$

Consider the space $N\cap S(\cdot)$. It represents all components that are determined neither by a differential equation nor by an algebraic equation. If $N\cap S(\cdot)\neq\{0\}$, then these components can be determined only by inherent differentiation instead of integration. The next lemma provides a possibility to determine from the network topology whether a differentiation problem is involved in the DAE (3.3) obtained applying MNA. This has a big influence onto numerical solving since differentiation problems are ill-posed in the sense of Hadamard, i.e. small perturbations in the input data can provide arbitrarily large perturbations in the output data.

Lemma 6.2. *Let the conditions of Theorem* 4.1 *be satisfied. Then it holds that*

$$N \cap S(\cdot) = \operatorname{im} \begin{pmatrix} Q_{CRV} & 0 & 0 \\ 0 & 0 & 0 \\ -\bar{P}_{V\text{-}C} Z(\cdot) Q_{CRV} & 0 & \bar{Q}_{V\text{-}C} \end{pmatrix}$$

is true for a matrix $Z(\cdot)$ *satisfying* $Q_C^{\mathrm{T}} A_{Ic} (\mathrm{d}i_c(\cdot)/\mathrm{d}e) = Q_C^{\mathrm{T}} A_V Z(\cdot)$.

Remarks. (1) The existence of such a matrix $Z(\cdot)$ is guaranteed by condition (2c) of Theorem 4.1 (cf. Equation (23)).

(2) Regarding the definitions of Q_{CRV} and $\bar{Q}_{V\text{-}C}$ on Section 2.3 as well as Theorem 2.2, Lemma 6.2 implies that the network equations involve a differentiation problem if and only if the network contains a C–V loop or an L–I cutset.

(3) If all controlled current sources satisfy conditions (2a) or (2b) of Theorem 4.1, then the relation

$$N \cap S(\cdot) = \operatorname{im} Q_{CRV} \times \{0\} \times \operatorname{im} \bar{Q}_{V\text{-}C}$$

is true.

(4) The different structure of the general case and the one discussed in the latter point corresponds to the alteration of the order in which we solve the system for the differential index (cf. footnote ¶). At this point it is recognizable that $N \cap S(\cdot)$ represents those components for which the differential index definition requires two differentiations to obtain the representation of their derivative as a continuous function of the variables.

Proof. Firstly, we show that the relation '$\subseteq$' is true. Assuming $z \in N \cap S(\cdot)$ we know that $z_e = Q_C z_e$, $z_L = 0$ and $z \in S(\cdot)$. Using Equations (18), (20), (21), and (24) we obtain

$$A_V^{\mathrm{T}} z_e = 0, \tag{52}$$

$$Q_C^{\mathrm{T}} A_R G(\cdot) A_R^{\mathrm{T}} Q_C z_e + Q_C^{\mathrm{T}} A_{Ic} \frac{\mathrm{d}i_c(\cdot)}{\mathrm{d}e} z_e + Q_C^{\mathrm{T}} A_V z_V = 0. \tag{53}$$

Then, Equation (52) provides additionally that $Q_{V\text{-}C}^{\mathrm{T}} Q_C^{\mathrm{T}} A_{Ic} (\mathrm{d}i_c(\cdot)/\mathrm{d}e) z_e = 0$ (cf. Equation (23)). Multiplying Equation (53) by $Q_{V\text{-}C}^{\mathrm{T}}$ and regarding $A_V^{\mathrm{T}} Q_C z_e = 0$ we obtain

$$Q_{V\text{-}C}^{\mathrm{T}} Q_C^{\mathrm{T}} A_R G(\cdot) A_R^{\mathrm{T}} Q_C Q_{V\text{-}C} z_e = 0.$$

Since $G(\cdot)$ is positive definite, this implies $A_R^{\mathrm{T}} Q_C Q_{V\text{-}C} z_e = 0$, i.e. $A_R^{\mathrm{T}} z_e = 0$ and so $z_e \in \operatorname{im} Q_{CRV}$. Now relation (53) implies that

$$Q_C^{\mathrm{T}} A_V z_V = -Q_C^{\mathrm{T}} A_{Ic} \frac{\mathrm{d}i_c(\cdot)}{\mathrm{d}e} = -Q_C^{\mathrm{T}} A_V Z(\cdot) z_e = -Q_C^{\mathrm{T}} A_V Z(\cdot) Q_{CRV} z_e,$$

i.e.

$$z_V = -\bar{P}_{V\text{-}C} Z(\cdot) Q_{CRV} z_e + \bar{Q}_{V\text{-}C} z_V.$$

Secondly, we show that the relation '$\supseteq$' is satisfied. Assume that $z_e = Q_{CRV} z_e$ and $z_L = 0$. Furthermore, we have

$$z_V = \bar{Q}_{V\text{-}C} z_V - \bar{P}_{V\text{-}C} Z(\cdot) z_e. \tag{54}$$

Then $z \in N = \ker A(\cdot)$ holds trivially and

$$\left(A_V^{\mathrm{T}} - \frac{\mathrm{d}v(\cdot)}{\mathrm{d}e}\right) z_e - \frac{\mathrm{d}v(\cdot)}{\mathrm{d}j_L} z_L = \frac{\mathrm{d}v(\cdot)}{\mathrm{d}e} Q_{CRV} z_e = 0 \tag{55}$$

is fulfilled. Using Equation (54) we obtain additionally that

$$\begin{aligned} &Q_C^{\mathrm{T}}\left[\left(A_R G(\cdot) A_R^{\mathrm{T}} + A_I \frac{\mathrm{d}i(\cdot)}{\mathrm{d}e}\right) z_e + \left(A_L + A_I \frac{\mathrm{d}i(\cdot)}{\mathrm{d}j_L}\right) z_L + \left(A_V + A_I \frac{\mathrm{d}i(\cdot)}{\mathrm{d}j_V}\right) z_V\right] \\ &\quad = Q_C^{\mathrm{T}}\left[A_{Ic} \frac{\mathrm{d}i_c(\cdot)}{\mathrm{d}e} z_e + A_V z_V\right] = Q_C^{\mathrm{T}} A_V Z(\cdot) z_e + Q_C^{\mathrm{T}} A_V z_V = 0. \quad \square \end{aligned}$$

Corollary 6.3. *Let the conditions of Theorem* 4.1 *be satisfied. The network equation system obtained applying MNA is index*-1-*tractable if and only if the network contains neither a C–V loop nor an L–I cutset.*

Let us now study the higher index case. For that we investigate $A_1(\cdot)$ and $B_1(\cdot)$ defined in Section 3.2.

$$A_1(\cdot) = \begin{pmatrix} A_C C(\cdot) A_C^{\mathrm{T}} + A_R G(\cdot) A_R^{\mathrm{T}} Q_C + A_I \dfrac{\mathrm{d}i(\cdot)}{\mathrm{d}e} Q_C & 0 & A_V + A_I \dfrac{\mathrm{d}i(\cdot)}{\mathrm{d}j_V} \\ -A_L^{\mathrm{T}} Q_C & L(\cdot) & 0 \\ A_V^{\mathrm{T}} Q_C & 0 & 0 \end{pmatrix}$$

$$B_1(\cdot) = \begin{pmatrix} A_C \bar{C}(\cdot) A_C^{\mathrm{T}} + A_R G(\cdot) A_R^{\mathrm{T}} P_C + A_I \dfrac{\mathrm{d}i(\cdot)}{\mathrm{d}e} P_C & A_L + A_I \dfrac{\mathrm{d}i(\cdot)}{\mathrm{d}j_L} & 0 \\ -A_L^{\mathrm{T}} P_C & \bar{L}(\cdot) & 0 \\ A_V^{\mathrm{T}} P_C - \dfrac{\mathrm{d}v(\cdot)}{\mathrm{d}e} P_C & 0 & 0 \end{pmatrix}$$

Lemma 6.4. *Let the conditions of Theorem* 4.1 *be satisfied. Then the relation*

$$\operatorname{im} A_1(\cdot) = \ker \begin{pmatrix} Q_{CRV}^{\mathrm{T}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \bar{Q}_{V-C}^{\mathrm{T}} \end{pmatrix}$$

is satisfied.

Proof. Firstly, $\operatorname{im} A_1(\cdot) \subseteq \ker Q_{CRV}^{\mathrm{T}} \times \mathbb{R}^{n_L} \times \ker \bar{Q}_{V-C}^{\mathrm{T}}$ holds trivially, because of

$$Q_{CRV}^{\mathrm{T}} A_{\mathrm{I}} \frac{\mathrm{d}i(\cdot)}{\mathrm{d}e} = 0 \quad \text{and} \quad Q_{CRV}^{\mathrm{T}} A_I \frac{\mathrm{d}i(\cdot)}{\mathrm{d}j_V} = 0$$

for all admitted controlled current sources (see Equations (19), (21), and (23)).

Secondly, we assume that $z \in \ker Q_{CRV}^{\mathrm{T}} \times \mathbb{R}^{n_L} \times \ker \bar{Q}_{V-C}^{\mathrm{T}}$, i.e. $Q_{CRV}^{\mathrm{T}} z_1 = 0$ and $\bar{Q}_{V-C}^{\mathrm{T}} z_3 = 0$. Then, there is an α_0 such that

$$z_3 = A_V^{\mathrm{T}} Q_C \alpha_0. \tag{56}$$

Since $Q_{CRV}^{\mathrm{T}} A_{\mathrm{I}} = 0$ (see Equations (19), (21), and (23)), the relation

$$z_1 - A_R G(\cdot) A_R^{\mathrm{T}} Q_C P_{V-C} \alpha_0 - A_I \frac{\mathrm{d}i(\cdot)}{\mathrm{d}e} Q_C P_{V-C} \alpha_0 \in \ker Q_{CRV}^{\mathrm{T}}$$

holds, i.e. there are α_1, α_2 and γ_1 such that

$$\begin{aligned} & z_1 - A_R G(\cdot) A_R^{\mathrm{T}} Q_C P_{V-C} \alpha_0 - A_I \frac{\mathrm{d}i(\cdot)}{\mathrm{d}e} Q_C P_{V-C} \alpha_0 \\ & \quad = A_C C(\cdot) A_C^{\mathrm{T}} \alpha_1 + A_R G(\cdot) A_R^{\mathrm{T}} Q_C Q_{V-C} \alpha_2 + A_V \gamma_1. \end{aligned} \tag{57}$$

This is a simple conclusion of the fact that

$$\ker Q_{CRV}^{\mathrm{T}} = \operatorname{im} (A_C C(\cdot) A_C^{\mathrm{T}}, A_R G(\cdot) A_R^{\mathrm{T}} Q_C Q_{V-C}, A_V A_V^{\mathrm{T}})$$

since $C(\cdot)$ and $G(\cdot)$ are positive definite. Regarding Equations (20), (22), and (24) we obtain that

$$A_I \frac{\mathrm{d}i(\cdot)}{\mathrm{d}j_V} = A_{Ib} \frac{\mathrm{d}i_b(\cdot)}{\mathrm{d}j_V} \bar{P}_{V-C}. \tag{58}$$

Considering Equation (23) we see that

$$\frac{\mathrm{d}i_a(\cdot)}{\mathrm{d}e} Q_C = \frac{\mathrm{d}i_a(A_C^{\mathrm{T}} e, A_V^{\mathrm{T}} e, j_L, t)}{\mathrm{d}e} Q_C = \frac{\mathrm{d}i_a(\cdot)}{\mathrm{d}e} Q_C P_{V-C}. \tag{59}$$

Regarding Equation (23) we find α_3 and γ_2 such that

$$A_{Ic} \frac{\mathrm{d}i_c(\cdot)}{\mathrm{d}e} Q_C Q_{V-C} \alpha_2 = A_C C(\cdot) A_C^{\mathrm{T}} \alpha_3 + A_V \gamma_2. \tag{60}$$

Using Equation (21) we find α_4 and α_5 such that

$$A_{Ib} \frac{\mathrm{d}i_b(\cdot)}{\mathrm{d}e} Q_C Q_{V-C} \alpha_2 = A_C C(\cdot) A_C^{\mathrm{T}} \alpha_4, \tag{61}$$

$$A_{Ib}\frac{\mathrm{d}i_b(\cdot)}{\mathrm{d}j_V}(\gamma_1-\gamma_2)=A_C C(\cdot)A_C^{\mathrm{T}}\alpha_5. \tag{62}$$

Choosing $\alpha := P_C(\alpha_1-\alpha_3-\alpha_4-\alpha_5)+Q_C P_{V\text{-}C}\alpha_0+Q_C Q_{V\text{-}C}\alpha_2$, $\beta := L^{-1}(\cdot)(z_2+A_L^{\mathrm{T}}Q_C\alpha)$, $\gamma := \gamma_1-\gamma_2$ and regarding Equations (56)–(61), we obtain that

$$z=A_1(\cdot)\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}\in \operatorname{im} A_1(\cdot). \qquad \square$$

Considering $B_1(\cdot)$ and Lemma 6.4 we obtain a simple description of the constant space S_1:

$$S_1=\ker \bar{Q}_{V\text{-}C}^{\mathrm{T}}A_V^{\mathrm{T}}P_C\times\ker Q_{CRV}^{\mathrm{T}}A_L\times R^{n_V}. \tag{63}$$

For a definition of S_1 see Section 3.2.

Lemma 6.5. *Let the conditions of Theorem* 4.1 *be satisfied. Then, the canonical projector onto* $N_1(\cdot)=\ker A_1(\cdot)$ *along* S_1 *is given by Equation* (47).

Proof.

1. $Q_1(\cdot)$ is a projector, since

$$\bar{Q}_{V\text{-}C}H_6^{-1}(\cdot)=H_6^{-1}(\cdot)\bar{Q}_{V\text{-}C}^{\mathrm{T}}.$$

$$H_6^{-1}(\cdot)\bar{Q}_{V\text{-}C}^{\mathrm{T}}A_V^{\mathrm{T}}H_1^{-1}(\cdot)A_V\bar{Q}_{V\text{-}C}=\bar{Q}_{V\text{-}C},$$

$$P_C Q_{CRV}=0,\ Q_{CRV}H_5^{-1}(\cdot)=H_5^{-1}(\cdot)Q_{CRV}^{\mathrm{T}}$$

and

$$H_5^{-1}(\cdot)Q_{CRV}^{\mathrm{T}}A_L L^{-1}(\cdot)A_L^{\mathrm{T}}Q_{CRV}=Q_{CRV}.$$

These relations are simple conclusions of the definitions of $H_1(\cdot)$, $H_5(\cdot)$, and $H_6(\cdot)$.

2. We show that $\operatorname{im} Q_1(\cdot)\subseteq\ker A_1(\cdot)$. Regarding

$$A_C C(\cdot)A_C^{\mathrm{T}}H_1^{-1}(\cdot)=P_C^{\mathrm{T}},$$

$$Q_C H_1^{-1}(\cdot)=H_1^{-1}(\cdot)Q_C^{\mathrm{T}},$$

$$\frac{\mathrm{d}i(\cdot)}{\mathrm{d}j_V}=\frac{\mathrm{d}i_b(\cdot)}{\mathrm{d}j_V}\bar{P}_{V\text{-}C}$$

this holds trivially.

3. We show that $\ker A_1(\cdot)\subseteq\operatorname{im} Q_1(\cdot)$. Assume $z\in\ker A_1(\cdot)$. Then,

$$A_C C(\cdot)A_C^{\mathrm{T}}z_e+A_R G(\cdot)A_R^{\mathrm{T}}Q_C z_e+A_I\frac{\mathrm{d}i(\cdot)}{\mathrm{d}e}Q_C z_e+A_V z_V+A_I\frac{\mathrm{d}i(\cdot)}{\mathrm{d}j_V}z_V=0, \tag{64}$$

$$-A_L^{\mathrm{T}}Q_C z_e+L(\cdot)z_L=0, \tag{65}$$

$$A_V^{\mathrm{T}}Q_C z_e=0. \tag{66}$$

Considering Equation (66) we see that

$$z_e = Q_{V\text{-}C}\, z_e. \tag{67}$$

Next, we have (cf. Equations (20), (21), and (23))

$$A_{Ia}\frac{\mathrm{d}i_a(\cdot)}{\mathrm{d}e}Q_C z_e = 0, \quad A_{Ia}\frac{\mathrm{d}i_a(\cdot)}{\mathrm{d}j_V}z_V = 0, \tag{68}$$

$$Q_C^{\mathrm{T}}A_{Ib}\frac{\mathrm{d}i_b(\cdot)}{\mathrm{d}e}Q_C z_e = 0, \quad Q_C^{\mathrm{T}}A_{Ib}\frac{\mathrm{d}i_b(\cdot)}{\mathrm{d}j_v}z_V = 0, \tag{69}$$

$$Q_{V\text{-}C}^{T}Q_C^{\mathrm{T}}A_{Ic}\frac{\mathrm{d}i_c(\cdot)}{\mathrm{d}e}Q_C z_e = 0, \quad Q_{V\text{-}C}^{\mathrm{T}}Q_C^{\mathrm{T}}A_{Ic}\frac{\mathrm{d}i_e(\cdot)}{\mathrm{d}j_V}z_V = 0. \tag{70}$$

Multiplying Equation (64) by $Q_{V\text{-}C}^{\mathrm{T}}Q_C^{\mathrm{T}}$ yields

$$Q_{V\text{-}C}^{\mathrm{T}}Q_C^{\mathrm{T}}A_R G(\cdot)A_R^{\mathrm{T}}Q_C Q_{V\text{-}C}\, z_e = 0.$$

Since $G(\cdot)$ is positive definite and Equation (67) is valid, it holds that

$$Q_C z_e = Q_{CRV}\, z_e. \tag{71}$$

Relation (65) leads to

$$z_L = L^{-1}(\cdot)A_L^{\mathrm{T}}Q_C z_e = L^{-1}(\cdot)A_L^{\mathrm{T}}Q_{CRV}\, z_e. \tag{72}$$

Multiplying Equation (64) by Q_C^{T} yields now

$$Q_C^{\mathrm{T}}A_V z_V + Q_C^{\mathrm{T}}A_{Ic}\frac{\mathrm{d}i_c(\cdot)}{\mathrm{d}e}Q_C z_e = 0.$$

If we regard Equations (68)–(70), then relation (64) reduces to

$$A_C C(\cdot)A_C^{\mathrm{T}}z_e + A_{Ib,c}\frac{\mathrm{d}i_{b,c}(\cdot)}{\mathrm{d}e}Q_C z_e + A_V z_V = 0. \tag{73}$$

Since $A_{Ib,c}(\mathrm{d}i_{b,c}(\cdot)/\mathrm{d}e)Q_C = A_C C(\cdot)A_C^{\mathrm{T}}Q_V X(\cdot) + A_V A_V^{\mathrm{T}}Y(\cdot)$, we obtain that

$$Q_C^{\mathrm{T}}A_{Ib,c}\frac{\mathrm{d}i_{b,c}(\cdot)}{\mathrm{d}e} = Q_C^{\mathrm{T}}A_V A_V^{\mathrm{T}}Y(\cdot).$$

Multiplying Equation (73) by Q_C^{T} we conclude that

$$Q_C^{\mathrm{T}}A_V(A_V^{\mathrm{T}}Y(\cdot)Q_C z_e + z_V) = 0,$$

i.e.

$$A_V^{\mathrm{T}}Y(\cdot)Q_C z_e + z_V = \bar{Q}_{V\text{-}C}(A_V^{\mathrm{T}}Y(\cdot)Q_C z_e + z_V). \tag{74}$$

From Equation (73) we obtain

$$A_C C(\cdot)A_C^{\mathrm{T}}z_e + A_C C(\cdot)A_C^{\mathrm{T}}Q_V X(\cdot)Q_C z_e$$
$$+ A_V A_V^{\mathrm{T}}Y(\cdot)Q_C z_e + A_V z_V = 0,$$

i.e.

$$P_C(z_e + Q_V X(\cdot) Q_C z_e) = -H_1^{-1}(\cdot) A_V (z_V + A_V^{\mathrm{T}} Y(\cdot) Q_C z_e)$$
$$= -H_1^{-1}(\cdot) A_V \bar{Q}_{V\text{-}C}(z_V + A_V^{\mathrm{T}} Y(\cdot) Q_C z_e)$$

because of Equation (74). Thus,

$$z = Q_1(\cdot)\begin{pmatrix} z_e + Q_V X(\cdot) Q_C z_e \\ L^{-1}(\cdot) A_L^{\mathrm{T}} Q_{CRV} z_e \\ 0 \end{pmatrix} \in \operatorname{im} Q_1(\cdot).$$

4. The relation $S_1 \subseteq \ker Q_1(\cdot)$ is a simple conclusion of Equation (63).

5. We show that $\ker Q_1(\cdot) \subseteq S_1$. Assume $Q_1(\cdot) z = 0$. Then

$$L^{-1} A_L^{\mathrm{T}} Q_{CRV} H_5^{-1}(\cdot) Q_{CRV}^{\mathrm{T}} A_L z_L = 0 \tag{75}$$

and

$$\bar{Q}_{V\text{-}C} H_6^{-1}(\cdot) \bar{Q}_{V\text{-}C}^{\mathrm{T}} A_V^{\mathrm{T}} P_C z_e = 0. \tag{76}$$

Multiplying Equation (75) by $Q_{CRV}^{\mathrm{T}} A_L$ yields $Q_{CRV}^{\mathrm{T}} A_L z_L = 0$. Regarding

$$\bar{Q}_{V\text{-}C} H_6^{-1}(\cdot) = H_6^{-1}(\cdot) \bar{Q}_{V\text{-}C}^{\mathrm{T}}$$

we conclude from Equation (75) that $\bar{Q}_{V\text{-}C}^{\mathrm{T}} A_V^{\mathrm{T}} P_C z_e = 0$. Considering Equation (63) the assertion is proved. □

Remark. Lemmas 6.2 and 6.5 now imply Theorem 6.1 for the tractability index.

The validity of Theorem 4.2 follows by similar considerations as above. More precisely, it can be shown that the following relations are satisfied for the charge-oriented MNA.

(i) The analogue to Lemma 6.2 reads

$$N \cap S(\cdot) = \operatorname{im}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Q_{CRV} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\bar{P}_{V\text{-}C} Z(\cdot) Q_{CRV} & 0 & \bar{Q}_{V\text{-}C} \end{pmatrix}$$

for $Z(\cdot)$ chosen as in Lemma 6.2.

(ii) Lemma 6.4 reads

$$\operatorname{im} A_1(\cdot) = \ker \begin{pmatrix} Q_{CRV}^{\mathrm{T}} & 0 & 0 & 0 & Q_{CRV}^{\mathrm{T}} A_L L^{-1}(\cdot) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{Q}_{V-C}^{\mathrm{T}} & \bar{Q}_{V-C}^{\mathrm{T}} A_V^{\mathrm{T}} H_1^{-1}(\cdot) A_C & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(iii) Equation (63) corresponds to

$$S_1(\cdot) = \ker \bar{Q}_{V-C}^{\mathrm{T}} A_V^{\mathrm{T}} H_1^{-1}(\cdot) A_C \times \ker Q_{CRV}^{\mathrm{T}} A_L L^{-1}(\cdot) \times \mathbb{R}^{n_e} \times \mathbb{R}^{n_L} \times \mathbb{R}^{n_V}.$$

(iv) Theorem 6.1 holds analogously for the charge-oriented MNA, and the canonical projector $Q_1(\cdot)$ onto $N_1(\cdot)$ along $S_1(\cdot)$ is given by

$$\begin{pmatrix} \begin{matrix} \bar{P}_C A_C^{\mathrm{T}} \bar{H}_1^{-1} A_V \bar{Q}_{V-C} \cdot \\ H_6^{-1}(\cdot) \bar{Q}_{V-C}^{\mathrm{T}} A_V^{\mathrm{T}} H_1^{-1}(\cdot) A_C \end{matrix} & \begin{matrix} -\bar{P}_C C(\cdot) A_C^{\mathrm{T}} Q_V X(\cdot) Q_{CRV} \cdot \\ H_5^{-1}(\cdot) Q_{CRV}^{\mathrm{T}} A_L L^{-1}(\cdot) \end{matrix} & 0 & 0 & 0 \\ 0 & A_L^{\mathrm{T}} Q_{CRV} H_5^{-1}(\cdot) Q_{CRV}^{\mathrm{T}} A_L L^{-1}(\cdot) & 0 & 0 & 0 \\ 0 & Q_{CRV} H_5^{-1}(\cdot) Q_{CRV}^{\mathrm{T}} A_L L^{-1}(\cdot) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\bar{Q}_{V-C} H_6^{-1}(\cdot) \bar{Q}_{V-C}^{\mathrm{T}} A_V^{\mathrm{T}} H_1^{-1}(\cdot) A_C & -A_V^{\mathrm{T}} Y(\cdot) Q_{CRV} H_5^{-1}(\cdot) Q_{CRV}^{\mathrm{T}} A_L L^{-1}(\cdot) & 0 & 0 & 0 \end{pmatrix}$$

where $\bar{H}_1 := A_C A_C^{\mathrm{T}} + Q_C^{\mathrm{T}} Q_C$ and the matrices $X(\cdot)$ and $Y(\cdot)$ are chosen as before.

Note that if no controlled current sources that fulfil only the conditions (2b) or (2c) of Theorem 4.1 appear, then N_1 is constant.

7. CONCLUSION

The presented results provide information about the index of systems (6)–(8) and (9)–(13) by topological analysis of the network. The only assumption made on parameters is the mentioned positive definiteness.

The class of controlled sources described in the paper is precisely the one that does not seriously affect the structure of the spaces associated with the DAE-systems. Basically,[‖] these spaces are the same as for networks without controlled sources. If the class of controlled sources is not restricted then different problems arise.

[‖] One exception is the one discussed in Lemma 6.2 and footnote [¶¶]. Straightforward computation shows that also $N_1(\cdot)$ may change depending on the considered controlled sources.

On the one hand, if arbitrary controlling elements for the controlling sources are considered then the index of the network equations may depend on the parameters (cf. Reference [22]).

On the other hand, if controlled sources are allowed to form a part of L–I cutsets or C–V loops then it is possible to be confronted with higher index (>2) problems (cf. Reference [24]).

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