Numerical Methods for Differential-Algebraic Equations

Exercise Sheet 6, June 20, 2024, 10:15-11:45, Room S2 346

We study the solution of ordinariy differential equations and semi-explicit differential algebraic equations by linear multistep methods. We start start with the index-0 case, i.e., with ordinary differential equations

$$2y' = f(y), \quad t > 0, \qquad y(0) = y_0. \tag{1}$$

For the solution of (1) we consider the k-step method

$$\sum_{j=0}^{k} \alpha_k y^{n+j} = h \sum_{j=0}^{k} \beta_j f^{n+j} \tag{2}$$

with constant stepsize $h > 0$ and $f^j = f(y^j)$. We assume $\alpha^k \neq 0$ such that (2) allows to determine y^{n+k}, given y^{n+j}, $j < k$.

Exercise 1 (0-stability)**.**
For the simple differential equation $y' = 0$, the scheme (2) reduces to

$$\sum_{j=0}^{k} \alpha_j y^{n+j}. \tag{3}$$

(i) Argue that for any choice of initial values y^j, $0 \leq k < k$, this recursion defines a unique sequence $(y^j)_{j\geq 0}$.

(ii) Assume that all zeros ξ_ℓ of $\rho(\xi) := \sum_{j=0}^{k} \alpha_j \xi^k = 0$ are simple. Show that the nth element of the above sequence can be written in the form

$$y^n = \sum_{l=1}^{k} c_\ell \xi_\ell^n, \qquad n \geq 0$$

and determine c_ℓ, $\ell = 1, \ldots, k$ in dependence of the initial values y^j, $j < k$.

(iii) Argue that the sequence (y^n) stays bounded if, and only if all zeros ξ_ℓ have modulus smaler or equal to one.

(iv) Look up the literature for the generalization to the case of multiple zeros.

Exercise 2 (BDF-formulas)**.**
We consider BDF-k formulas for $k = 1, \ldots, 6$; see e.g. Wikipedia for coefficients.

(i) Verify (e.g. with Matlab's `root` function) that the roots ξ_ℓ of the first associated polynomial $\rho(\xi)$ satisfy the "root"-condition required for 0-stability.

(ii) Further verify that the BDF-k formulas satisfy are consistent of order $p = k$.

(iii) Determine numerically the set of $z \in \mathbb{C}$ where the roots ξ_ℓ of $R(\xi; z) := \rho(\xi) - z\sigma(\xi)$ have modulus $|\xi_\ell| \leq 1$. Verify the assertions about $A(\alpha$ stability made in the lecture.

Exercise 3 (Implementation).
We consider the simple linear ODE

$$My'(t) = Ay(t) + f(t), \qquad y(0) = y_0 \tag{4}$$

with $A, M \in \mathbb{R}^{n\times n}$, $y_0 \in \mathbb{R}^n$ and $f : \mathbb{R} \to \mathbb{R}^n$ given.

(i) Implement a Matlab script that allows you to solve this initial value problem numerically by the BDF-k methods above.

Hint: For the moment assume that appropriate values for the initial iterates are given. We will later discuss how to construct these.

(ii) Test your implementation for the problem

$$y_1' = y_1 + t, \qquad y_1(0) = 1, \tag{5}$$

$$y_1' + y_2' = y_1 + 2y_2 + t, \qquad y_2(0) = 1. \tag{6}$$

Note: Since the solution of this problem can be computed analytically, one can also determine appropriate starting values $y^j = y(t^j)$, $0 \le j < k$ required for the simulation with the BDF-k method.

(iii) Validate your implementation of the BDF-k method. Verify the convergence rates $\max_{t^n \le T} |y^n - y(t^n)| \le Ch^p$ with $p = k$.

Exercise 4 (Computation of starting values).
the following strategies can be used to obtain appropriate starting values y^j, $1 \le j < k$ for the BDF-k methods.

(i) Use BDF-1 method to determine y^1, then BDF-2 method to determine y^2, a.s.o.

(ii) Use a RK-method of order $p = k$.

(iii) Compute approximations for $y(t^j)$, $j < k$ by BDF-1 method with stepsize h, $h/2$, $h/4$, etc. and utilize "extrapolation"; see literature.

Experiment with some of these choices for the example of the previous exercise.

Exercise 5.
(i) Argue that your code developed in the previous exercise can be applied immediately also to the solution of linear time-invariant systems of DAEs.

(ii) Repeat the numerical tests for some examples of the first exercise sheet and verify the assertions about convergence rates made in the lecture.

Hint: For all examples, the solutions should be computable analytically.

Exercise 6. Discuss the modifications needed in your implementation to extend your code to nonlinear semi-explicit systems of DAEs of the general form

$$M(Y)Y' = F(Y), \qquad Y(0) = Y_0. \tag{7}$$

Note: This form is general enough to cover semi-explicit DAEs of index 1-3.