

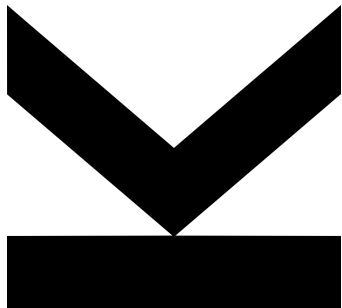
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# Circuit Modelling



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# Kurzfassung

Diese Arbeit hat zum Ziel, einen Überblick über die Methode der Modified Nodal Analysis (MNA) und deren Diskretisierung zu geben, wobei der Fokus auf einfacheren Netzwerken liegt, die nur aus Widerständen, Kondensatoren, Induktivitäten, Spannungsquellen und Stromquellen bestehen, sogenannte RLC-Netzwerken. Wir beginnen mit der Beschreibung elektrischer Schaltungen und ihrer Komponenten sowie physikalischer Gesetze. Dazu formulieren wir die Topologie der elektrischen Schaltung mathematisch, anschließend leiten wir die Formulierung der MNA her. Es handelt sich bei der Struktur des resultierenden Systems um eine Differential-Algebraische Gleichung. Diese Struktur analysieren wir weiter und wenden die Ergebnisse auf die MNA an. Abschließend betrachten wir Möglichkeiten, dieses System mittels *Mehrschrittmethoden* zu diskretisieren und diskutieren deren Konvergenzraten.

# Abstract

In this talk we give an overview of the Modified Nodal Analysis Method (MNA) and its discretization by focusing on simpler networks only consisting of resistors, capacitors, inductors, voltage sources and current sources, so-called RLC-networks. We start by describing electrical circuits and their components as well as the relevant physical laws. To this end, we first describe the topology of the electrical circuit mathematically and then derive the MNA formulation. The structure of the resulting system is a Differential Algebraic Equation, which we continue to analyze further. These results are then applied to the MNA formulation. Finally, we discuss the discretization of such systems using *Multistep Methods*, examine their convergence rates, and provide illustrative examples to support the theoretical results.

-numerische resultate um convergence rates zu illustraten

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# 1 Introduction

Circuit modelling plays an essential part the design process of modern devices. Beginning in the last century and still ongoing research has made many advancements in understanding electrical networks and in developing more sophisticated methods of analyzing and discretizing electrical networks. In the late 20th century, this lead to the introduction of the so called *Modified Nodal Analysis* which overcame the previously wide spread *Nodal Methods* limitations and was in turn widely used. The resulting system of equations of this Modified Nodal Analysis are of the form of so called *Differential Algebraic Equations*. These equations introduce new challenges when it comes to the numerical solution.

This thesis aims to give an overview of the Modified Nodal Analysis Method and its discretization by focusing on simpler networks only consisting of resistors, capacitors, inductors, voltage sources and current sources (so called RLC-networks). It is organized as follows. In Section 2 we start by describing electrical circuits and their components as well as physical laws. We first describe the topology of the electrical circuit mathematically, then we derive the Modified Nodal Analysis formulation. The structure of the resulting system is a Differential Algebraic Equation. Section 3 analyses this structure further. In Section 4 we then apply the results of the previous section to the Modified Nodal Analysis. Finally in Section 5 we discuss the discretization of this system by means of *Multistep Methods* and discuss their convergence rates.

## 2 Formulating a Mathematical Model

To accurately represent a physical system in a mathematical model we first have to think about **how** to best formulate this mathematical model. This chapter aims to elaborate on the Modified Nodal Analysis (MNA) of an electrical circuit, which is the common choice for these kinds of models. For this we have to introduce the notions of network topology as well as some basic electrical components and physical laws. Most of the following notions such as voltage, current and electrical potentials can depend on time. For better readability we will leave out the time argument  $t$  in some cases. This chapter is based on [2].

### 2.1 Network Topology

An electrical circuit is usually considered as a graph  $(\mathcal{N}, \mathcal{E})$  where  $\mathcal{N} = (n_0, n_1, n_2, \dots, n_k)$  denotes the set of nodes and  $\mathcal{E} = \{e_j : j = 1, \dots, l\}$  is the set of edges where  $|\mathcal{N}| = k$  is the number of nodes and  $|\mathcal{E}| = l$  the number of edges. If for some  $i$  and some  $j$  the nodes  $n_i$  and  $n_j$  are connected, then there exists an edge connecting those two nodes. We can store this information in an *incidence matrix*  $\tilde{A} = (\tilde{a}_{ij}) \in \mathbb{R}^{k \times l}$  which is defined by

$$\tilde{a}_{ij} = \begin{cases} 1 & \text{edge } j \text{ starts at node } i, \\ -1 & \text{edge } j \text{ ends at node } i, \\ 0 & \text{else.} \end{cases}$$

We call  $u = (u_0, u_1, u_2, \dots)$  the corresponding *electrical potentials* (or just *potentials*) to the nodes  $\mathcal{N}$ . The difference between the potentials at two connected nodes is called the *voltage* at the respective edge. To fix the absolute values of these potentials we have to set one node to a fixed value. We will do that by “grounding” the node  $n_0$ , which means we



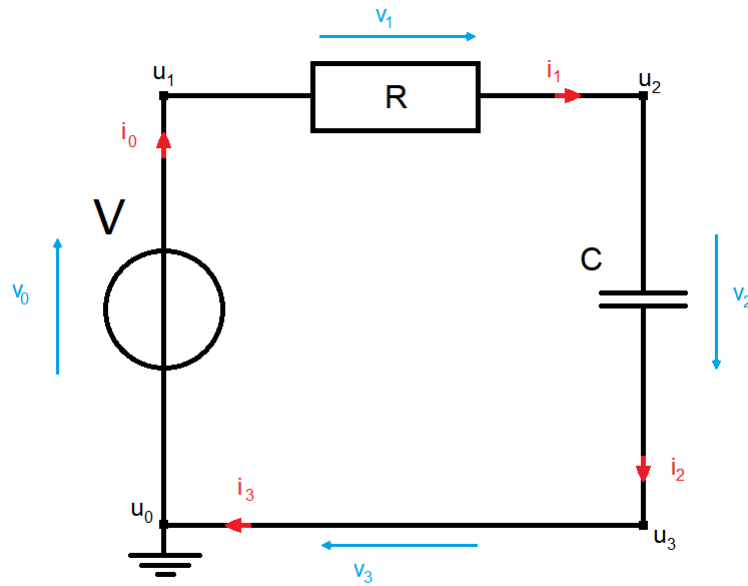
## 2 Formulating a Mathematical Model

set the potential  $u_0 = 0$ . The grounding of a node allows us to remove the corresponding row from the incidence matrix to get the *reduced incidence matrix*  $A$ . The vector  $v = (v_{ij})_{ij}$  represents the voltages at the edges. For some indices  $i$  and  $j$ , the voltage at edge  $ij$  is defined as  $v_{ij} = u_i - u_j$ .

We will later see that the components of an electrical circuit, which will be installed along the edges, describe a relationship between the current along the edge and the corresponding voltage. Thus a current vector  $i = (i_1, i_2, i_3, \dots)$  containing the currents along the edges is required.

### Example 1.1 (Charging of a Capacitor)

As an example we consider the charging of a capacitor. The circuit consists of a capacitor which is to be charged by a voltage source. This charging is also influenced by a resistor which is placed in series to the other components. This is depicted in Figure 2.1.



**Figure 2.1:** charging capacitor with series resistor and voltage source

## 2 Formulating a Mathematical Model

With the node potentials, the voltages and the currents collected in the vectors

$$u = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad v = \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad i = \begin{pmatrix} i_0 \\ i_1 \\ i_2 \\ i_3 \end{pmatrix},$$

the incidence matrix of this circuit has the form

$$\tilde{A} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

The rows of this matrix correspond to the nodes of the circuit and the columns correspond to the edges and tell us if the edge either starts (1) or end (−1) at a node. By grounding node  $u_0$  this matrix reduces to

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

In this circuit edge 3 is not populated with any components. This means that the voltage along this edge does not change, or in terms of potentials

$$u_0 - u_3 = 0.$$

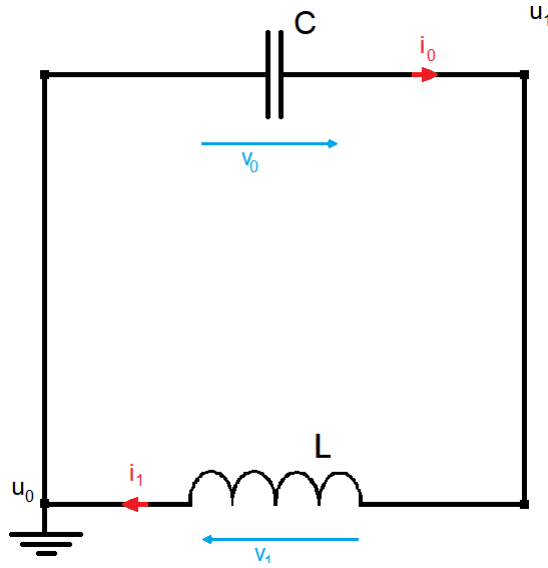
Thus we can consider this circuit with node 0 and node 3 merged. This leads to a slightly different incidence and reduced incidence matrix, i.e.

$$\tilde{A} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad \text{respectively.}$$

### Example 2.1 (LC-Circuit)

Our second example is a circuit that only contains a capacitor and an inductor, see Figure 2.2. This circuit will oscillate, if given appropriate initial conditions.

## 2 Formulating a Mathematical Model



**Figure 2.2:** capacitor and inductor oscillator

We again collect the node potentials, the voltages and the currents in the vectors

$$u = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \quad v = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}, \quad i = \begin{pmatrix} i_0 \\ i_1 \end{pmatrix}.$$

The incidence matrix of this circuit has the form

$$\tilde{A} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

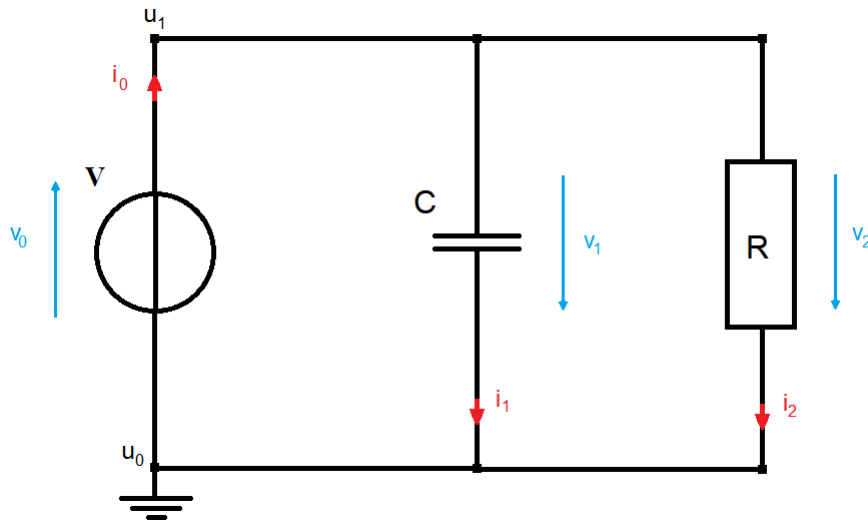
By grounding node  $u_0$  this can again be reduced and we obtain the reduced incidence matrix

$$A = \begin{pmatrix} -1 & 1 \end{pmatrix}.$$

### **Example 3.1** (Parallel Voltage Source, Capacitor and Resistor)

As a third example we consider a similar circuit to the circuit presented in Example 1.1, the only difference is that we connect the resistor parallel to the capacitor instead of connecting it in series.

## 2 Formulating a Mathematical Model



**Figure 2.3:** Capacitor with resistor and voltage source

We collect the node potentials, the voltages and the currents in the vectors

$$u = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \quad v = \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix}, \quad i = \begin{pmatrix} i_0 \\ i_1 \\ i_2 \end{pmatrix}.$$

The incidence matrix of this circuit has the form

$$\tilde{A} = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \end{pmatrix},$$

this leads to the reduced incidence matrix

$$A = \begin{pmatrix} -1 & 1 & 1 \end{pmatrix}.$$

## 2.2 Energy Conservation Laws

To fully fix all the variables that arise in the model of an electrical circuit we will need some *conservation laws*. In this context, conservation laws are a description of physical properties

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that can be observed in electrical circuits. This means that they have to be fulfilled non-the-less, thus including them into the system also makes sense from a physical perspective. They were proposed by Kirchhoff in 1845, see [6].

- **Kirchhoff's voltage law (KVL):**

The sum of voltages along each loop of the network must equal to zero. Using the incidence matrix  $A$  and the vector of potentials  $u$  as well as the voltage vector  $v$ , this law can be formulated as

$$A^\top u = v. \quad (2.1)$$

- **Kirchhoff's current law (KCL):**

For any node, the sum of currents flowing into the node is equal to the sum of currents flowing out of the node. Using again the incidence matrix  $A$  and the vector of currents  $i$ , this law can be formulated as

$$Ai = 0. \quad (2.2)$$

### 2.3 Electrical Components and their relations

Electrical components are described by equations relating their edge voltage  $v$  to their edge current  $i$ . We will mainly focus on so-called RLC-networks (short for Resistor, Inductor and Capacitor). These networks consist only of *resistors*, *capacitors*, *inductors*, *voltage sources* and *current sources*. Diodes and transistors as well as other electrical components can be described in a similar way, although these lead to a more difficult analysis of the system, see e.g. [2] and [1]. Using the notation introduced in Section 2.1, the relations read as:

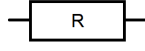
- **Resistor**

Resistors "resist" the flow of current, which causes voltage to drop. This behaviour is described by the *resistance*  $R \in \mathbb{R}^+ := \{x \in \mathbb{R} : x > 0\}$  which is given in *Ohm* ( $\Omega$ ) and its reciprocal, the *conductance*  $G \in \mathbb{R}^+$ , which is given in *Siemens* ( $S = \frac{1}{\Omega}$ ).

$$v = R i \quad \text{or} \quad i = G u. \quad (2.3)$$

The electrical schematic symbol for resistors is depicted in Figure 2.4.

## 2 Formulating a Mathematical Model



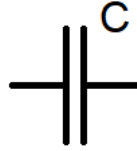
**Figure 2.4:** resistor symbol

- **Capacitor**

Capacitors “store” electrical energy by accumulating electrical charge. Their characteristic equations can be described directly using the stored charge  $Q \in \mathbb{R}_0^+ := \{x \in \mathbb{R} : x \geq 0\}$  or indirectly using the change in charge, which is nothing other than the current  $I$ . The *capacitance*  $C \in \mathbb{R}^+$  is given in *Farads* (F).

$$Q = C v \quad \text{and by derivation in t} \quad I = \frac{d}{dt}Q = C \frac{d}{dt}v = C v'. \quad (2.4)$$

The electrical schematic symbol for capacitors is depicted in Figure 2.5.



**Figure 2.5:** capacitor symbol

- **Inductor**

An electric current flowing through a conductor generates a magnetic field  $\Phi \in \mathbb{R}^{n \times n}$  surrounding it. This magnetic field causes a voltage drop dependent on the change in current. The *inductance*  $L \in \mathbb{R}^+$  is given in *Henry* (H).

$$\Phi = L i \quad \text{and by derivation in t} \quad v = \Phi' = L i'. \quad (2.5)$$

The electrical schematic symbol for inductors is depicted in Figure 2.6.



**Figure 2.6:** inductor symbol

- **Voltage Source**

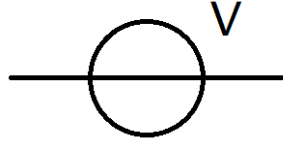
A voltage source supplies the system with a voltage. It can either supply varying

## 2 Formulating a Mathematical Model

amounts of voltage (with the special case of alternating current AC) or a fixed amount of voltage. The unit of voltage is *Volts* (V).

$$v = v_{src} \quad (2.6)$$

The electrical schematic symbol for voltage sources is depicted in Figure 2.7.



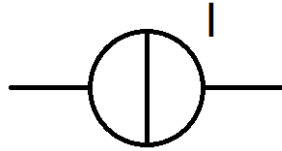
**Figure 2.7:** voltage source symbol

- **Current Source**

A current source supplies the system with electrical current. It can either supply varying amounts of current (with the special case of alternating current AC) or a fixed amount of current. The unit of current is *Ampere* (A).

$$i = i_{src} \quad (2.7)$$

The electrical schematic symbol for current sources is depicted in Figure 2.8.



**Figure 2.8:** current source symbol

The resistance  $R$ , conductance  $G$ , capacitance  $C$  and inductance  $L$  are respectively described as a scalar constant that relates the edge current to the edge voltage. If there are more components of the same kind in one circuit their corresponding constants will usually be collected into a matrix which is also called the same and also denoted by the same letter, respectively. These matrices are then positive definite diagonal matrices.

## 2.4 Modified Nodal Analysis - MNA

Based on [2] and [5] we will now derive the Modified Nodal Analysis formulation. To analyse the network further we will rearrange the columns of the reduced incidence matrix  $A$  such that it has the block form

$$A = \begin{pmatrix} A_R & A_C & A_L & A_V & A_I \end{pmatrix}$$

where  $A_R$ ,  $A_C$ ,  $A_L$ ,  $A_V$  and  $A_I$  include the columns that are related to the resistors, capacitors, coils, voltage sources and current sources, respectively.

To mathematically describe the circuit, we use *modified nodal analysis* (or short MNA), see the paper by Ho, Ruehli and Brennan [4]. MNA uses the node voltages as well as the currents of the coils and the voltage sources as unknowns. It is based on the conservation laws (2.2) and (2.1), as well as on the voltage-current relations of the electrical components discussed in Section 2.3. The voltages can be represented using the node potentials and the incidence matrix  $A$ , i.e.

$$v = A^\top u.$$

The vector  $v$  can thus be rearranged into  $v = (v_R, v_C, v_L, v_{src}, v_I)$ . Analogously we also rearrange the current vector into  $i = (i_R, i_C, i_L, i_V, i_{src})$ . Using the sorted incidence matrix blocks, we can rewrite the resistor current relation as

$$i_R = G v_R = G A_R^\top u.$$

Analogously, we rewrite the capacitor relation as

$$i_C = C v'_C = C A_C^\top u'.$$

From Kirchhoffs current law (2.2) we deduce that

$$A_C i_C + A_R i_R + A_L i_L + A_V i_V = -A_I i_{src}.$$



## 2 Formulating a Mathematical Model

We plug in the component relations derived above and obtain

$$A_C C A_C^\top u' + A_R G A_R^\top u + A_L i_L + A_V i_V = -A_I i_{src}.$$

Combining this with the component law for inductors (2.5) and the potential-voltage relation for voltage sources (2.6) we finally get the MNA equations

$$\begin{aligned} A_C C A_C^\top u' + A_R G A_R^\top u + A_L i_L + A_V i_V &= -A_I i_{src}, \\ L i_L' - A_L^\top u &= 0, \\ -A_V^\top u &= -v_{src}. \end{aligned}$$

In matrix form these read

$$\begin{pmatrix} A_C C A_C^\top & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{pmatrix} * \begin{pmatrix} u' \\ i_L' \\ i_V' \end{pmatrix} + \begin{pmatrix} A_R G A_R^\top & A_L & A_V \\ -A_L^\top & 0 & 0 \\ -A_V^\top & 0 & 0 \end{pmatrix} * \begin{pmatrix} u \\ i_L \\ i_V \end{pmatrix} = \begin{pmatrix} -A_I i_{src} \\ 0 \\ -v_{src} \end{pmatrix}, \quad (2.8)$$

where the diagonal matrices  $C$ ,  $G$  and  $L$  contain the capacities, conductivities and inductors, respectively.

The resulting systems are *stiff* systems. This means that for their numerical solution special care has to be put into which methods are suitable for solving these systems in a stable manner. For analyzing nonlinear components, other formulations can be used, e.g. the charge flux oriented formulation, for details see [2].

### Example 1.2 (Charging of a Capacitor)

We again consider the charging of a capacitor. The circuit is again given as in Example 1.1, where we have already derived the reduced incidence matrix

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

This matrix can be split into the three submatrices  $A_V$ ,  $A_R$  and  $A_C$  containing the columns of the matrix  $A$ . Because there are no inductors as well as no current sources present,

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the matrices  $A_L$  and  $A_V$  are empty. The diagonal matrices containing the components constants are

$$C = (C), \quad L = (), \quad G = \left(\frac{1}{R}\right).$$

Plugging this into the formula (2.8) we obtain the system

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{pmatrix} * \begin{pmatrix} u'_1 \\ u'_2 \\ i'_0 \end{pmatrix} + \begin{pmatrix} \frac{1}{R} & -\frac{1}{R} & -1 \\ -\frac{1}{R} & \frac{1}{R} & 0 \\ 1 & 0 & 0 \end{pmatrix} * \begin{pmatrix} u_1 \\ u_2 \\ i_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -v_{src} \end{pmatrix}. \quad (2.9)$$

Note that because  $L$  is empty, the system does not contain any equations related to inductances.

### Example 2.2 (LC-Circuit)

Similarly, we also derived the incidence matrix for this system in Example 2.1, i.e.

$$A = \begin{pmatrix} -1 & 1 \end{pmatrix}.$$

This matrix can be split into the two submatrices  $A_C$  and  $A_L$  containing the columns of the matrix  $A$ . Because we do not have any resistors, voltage sources or current sources, the matrices  $A_R$ ,  $A_V$  and  $A_I$  are empty. The diagonal matrices containing the components constants are

$$C = (C), \quad L = (L), \quad G = ().$$

Plugging this into the formula (2.8) we obtain the system

$$\begin{pmatrix} C & 0 \\ 0 & L \end{pmatrix} * \begin{pmatrix} u'_1 \\ i'_L \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} * \begin{pmatrix} u_1 \\ i_L \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

## 2 Formulating a Mathematical Model

### **Example 3.2** (Parallel Voltage Source, Capacitor and Resistor)

We consider the circuit introduced in Example 3.1, with the corresponding incidence matrix

$$A = \begin{pmatrix} -1 & 1 & 1 \end{pmatrix}.$$

This matrix can be split into the three submatrices  $A_I$ ,  $A_R$  and  $A_C$  containing the columns of the matrix  $A$ . As there are no inductors and no voltage sources present, the matrices  $A_L$  and  $A_V$  are empty. The diagonal matrices containing the components constants are

$$C = (C), \quad L = (), \quad G = \left(\frac{1}{R}\right).$$

Plugging this into the formula (2.8) we obtain the system

$$\begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} * \begin{pmatrix} u'_1 \\ i'_V \end{pmatrix} + \begin{pmatrix} \frac{1}{R} & -1 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} u_1 \\ i_V \end{pmatrix} = \begin{pmatrix} 0 \\ -v_{src} \end{pmatrix}. \quad (2.10)$$

## 3 Differential Algebraic Equations

The MNA of electrical circuits leads to a special form of differential equations, namely *differential algebraic equations* (DAE). Additionally to differential equations they also contain algebraic equations, or are equivalent to a system of differential-algebraic equations. To understand the general solvability of those systems, and to find good numerical methods we first have to take a look at the theory of DAEs. For this we build upon the results presented in [5].

In the most general form a DAE can be written as: Find  $y : \mathbb{R} \rightarrow \mathbb{R}^n$  such that

$$F(t, y(t), y'(t)) = 0, \quad \forall t \in I \subset \mathbb{R}, \quad (3.1)$$

with  $F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  sufficiently smooth and  $I$  the time-interval. As this is usually too general, one has to consider more specific types of DAEs. This chapter focuses on giving a brief overview of the different types of differential algebraic equations. We will discuss the specific form of linear systems with constant coefficients in more detail, as those are the systems that arise from modified nodal analysis of a RLC-system. Additionally this chapter also discusses the notion of the index of an MNA-system.

### 3.1 Types of DAEs

Let us consider the following three types of DAEs.

- **Linear systems with constant coefficients**  
are systems of the form: Find  $y$  such that

$$Ay'(t) + By(t) = f(t), \quad \forall t \in I \subset \mathbb{R}, \quad (3.2)$$

### 3 Differential Algebraic Equations

with  $A, B \in \mathbb{R}^{n \times n}$ ,  $A$  singular,  $B$  regular and  $f : \mathbb{R} \rightarrow \mathbb{R}^n$  a function in time. If  $A$  is regular, this equation can be transformed into an ordinary differential equation, since

$$y'(t) + A^{-1}By(t) = A^{-1}f(t)$$

Thus it is reasonable to assume  $A$  singular.

- **Linear time dependent systems** are systems of the form: Find  $y$  such that

$$A(t)y'(t) + B(t)y(t) = f(t), \quad \forall t \in I \subset \mathbb{R},$$

with  $A, B : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ ,  $f : \mathbb{R} \rightarrow \mathbb{R}^n$  functions, such that for every  $t \in \mathbb{R}$  the matrix  $A(t)$  is singular and the matrix  $B(t)$  regular.

- **Structured (non-linear) systems**  
are semi-explicit systems of the form: Find  $(y, z)$  such that

$$\begin{aligned} y'(t) &= f(t, y(t), z(t)), & \forall t \in I \subset \mathbb{R}, \\ 0 &= g(t, y(t), z(t)), \end{aligned}$$

with  $f : \mathbb{R} \rightarrow \mathbb{R}^n$  and  $g : \mathbb{R} \rightarrow \mathbb{R}^d$  functions.

For our analysis of electrical networks, we will focus on linear systems with constant coefficients (3.2). The other systems could for example occur when we additionally consider time-dependent component relations e.g. temperature. A splitting of the variable  $y$  is usually common, where we differ between the *differential variable* and the *algebraic variable*. A differential variable in this sense is the part of  $y$ , whose derivatives also appear in the differential algebraic equation. The algebraic variable on the other hand only appears in algebraic equations.

#### 3.1.1 Weierstraß-Kronecker normal form

To determine the solvability of a linear system with constant coefficients (3.2), we first need to introduce a normal form for the system, the *Weierstraß-Kronecker normal form*. This

### 3 Differential Algebraic Equations

normal form is dependent on the family  $\{A, B\} := \{\mu A + B | \mu \in \mathbb{R}\}$ , which is called the *matrix pencil* of the DAE.

**Definition 1.** The matrix pencil  $\{A, B\}$  is called *regular* if there exists some  $c \in \mathbb{R}$ , such that  $(cA + B)$  is regular (i.e.  $\det(cA + B) \neq 0$ ), otherwise it is called *singular*.

**Theorem 1** (Jordan normal form [5, Theorem 13.2.1]). For every matrix  $Q \in \mathbb{R}^{n \times n}$  there exists a regular matrix  $T \in \mathbb{C}^{n \times n}$ , such that

$$T^{-1}QT = J = \text{diag}(J_1, \dots, J_r) \quad \text{with} \quad J_i = \begin{pmatrix} \lambda_i & 1 & & 0 \\ 0 & \lambda_i & \ddots & \vdots \\ & \ddots & \ddots & 1 \\ 0 & \dots & 0 & \lambda_i \end{pmatrix} \in \mathbb{C}^{m_i \times m_i}$$

and  $n = m_1 + \dots + m_r$ .

The matrix  $J$  is called Jordan normal form of  $Q$ , the matrices  $J_i$  are called Jordan Blocks, where  $\lambda_i$  are the corresponding eigenvalues of  $Q$ . The matrix  $J$  is uniquely determined by  $Q$  except for the arrangement of the diagonal blocks. If  $Q$  possesses only real eigenvalues, then  $T$  can also be chosen from the reals.

### 3 Differential Algebraic Equations

In order to separate the equation (3.2) into an algebraic and a differential part, we transform the matrices  $A$  and  $B$  according to the following theorem.

**Theorem 2** ([5, Satz 13.2.2]). *Let  $\{A, B\}$  be a regular matrix pencil. There exist  $P, Q \in \mathbb{C}^{n \times n}$  such that*

$$PAQ = \begin{pmatrix} I_d & 0 \\ 0 & N \end{pmatrix}, \quad PBQ = \begin{pmatrix} R & 0 \\ 0 & I_{n-d} \end{pmatrix}$$

where

$$N = \text{diag}(N_1, \dots, N_r) \quad \text{with} \quad N_i = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & 0 & 1 \\ 0 & & & 0 \end{pmatrix} \in \mathbb{R}^{n_i \times n_i}$$

and  $R$  has Jordan normal form. By  $I_k$  we denote the identity matrix of size  $k \times k$ .

*Proof.* Because  $\{A, B\}$  is a regular matrix pencil, there exists  $c \in \mathbb{R}$  such that  $(cA + B)$  is regular. Set

$$\hat{A} := (cA + B)^{-1}A, \quad \hat{B} := (cA + B)^{-1}B.$$

Considering

$$(cA + B)^{-1}(cA + B) = I \implies (cA + B)^{-1}B + c(cA + B)^{-1}A = I,$$

we get that

$$\hat{B} = I - c\hat{A}.$$

Let  $J_{\hat{A}}$  be the Jordan normal form of  $\hat{A}$ , this means that there exists a regular matrix  $T_1$  such that

$$T_1^{-1}AT_1 = J_{\hat{A}} = \begin{pmatrix} W & 0 \\ 0 & \tilde{N} \end{pmatrix}.$$

### 3 Differential Algebraic Equations

The matrix  $W$  contains the Jordan blocks with eigenvalues which are nonzero, the matrix  $\tilde{N}$  contains the Jordan blocks with eigenvalues equal to zero, thus  $\tilde{N}$  is *nilpotent*. The Jordan normal form  $J_{\hat{B}}$  of  $\hat{B}$  is given by

$$T_1^{-1} \hat{B} T_1 = J_{\hat{B}} = \begin{pmatrix} I - cW & 0 \\ 0 & I - c\tilde{N} \end{pmatrix}.$$

The following two transformations will allow us to obtain the desired structure. First we will transform  $J_{\hat{A}}$  with

$$T_2 := \begin{pmatrix} W & 0 \\ 0 & I - c\tilde{N} \end{pmatrix}$$

in

$$T_2^{-1} J_{\hat{A}} = \begin{pmatrix} I & 0 \\ 0 & (I - c\tilde{N})^{-1} \tilde{N} \end{pmatrix}$$

and  $J_{\hat{B}}$  in

$$T_2^{-1} J_{\hat{B}} = \begin{pmatrix} W^{-1} - cI & 0 \\ 0 & I \end{pmatrix}.$$

Let now  $R$  be the Jordan normal form of  $(W^{-1} - cI)$  and  $N$  be the normal form of  $(I - c\tilde{N})^{-1} \tilde{N}$ , this means

$$T_W^{-1} (W^{-1} - cI) T_W = R \quad \text{and} \quad T_{\tilde{N}}^{-1} (I - c\tilde{N})^{-1} \tilde{N} T_{\tilde{N}} = N$$

Considering this definition together with the Neumann series of  $(I - c\tilde{N})^{-1}$  we obtain

$$\tilde{N} (I - c\tilde{N})^{-1} = \tilde{N} \left( c \sum_{i=0}^{\infty} \tilde{N}^i \right) = \tilde{N} \left( c \sum_{i=0}^{k-1} \tilde{N}^i \right) = c \sum_{i=0}^{k-1} \tilde{N}^{i+1} = \left( c \sum_{i=0}^{k-1} \tilde{N}^i \right) \tilde{N}$$

where we used that  $\tilde{N}$  is nilpotent with nilpotency index  $k$ . This shows that  $\tilde{N}$  and  $(I - c\tilde{N})^{-1}$  commute.



### 3 Differential Algebraic Equations

From this we conclude that

$$N^k = [T_{\tilde{N}}^{-1}(I - c\tilde{N})^{-1}\tilde{N}T_{\tilde{N}}]^k = T_{\tilde{N}}^{-1}[(I - c\tilde{N})^{-1}\tilde{N}]^k T_{\tilde{N}} = T_{\tilde{N}}^{-1}(I - c\tilde{N})^{-k} \underbrace{\tilde{N}^k}_{=0} T_{\tilde{N}} = 0,$$

where we used the commutativity in the third step. The nilpotent matrix  $N$  thus also has the nilpotency index  $k$ . A transformation with

$$T_3 := \begin{pmatrix} T_W & 0 \\ 0 & T_{\tilde{N}} \end{pmatrix}$$

transforms  $T_2^{-1}J_{\hat{A}}$  into the Jordan normal form

$$J_{\hat{A}} := T_3^{-1}T_2^{-1}J_{\hat{A}}T_3 = T_3^{-1}T_2^{-1}T_1^{-1}\hat{A}T_1T_3 = \begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix}$$

and  $T_2^{-1}J_{\hat{B}}$  into

$$J_{\hat{B}} := T_3^{-1}T_2^{-1}J_{\hat{B}}T_3 = T_3^{-1}T_2^{-1}T_1^{-1}\hat{B}T_1T_3 = \begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix}.$$

Now set

$$P := T_3^{-1}T_2^{-1}T_1^{-1}(cA + B)^{-1} \quad \text{and} \quad Q = T_1T_3$$

to get the statement. □

Using the findings above we are able to transform the initial DAE (3.2) using the matrix  $P$  from Theorem 2. By multiplying with  $P$  from the left, we obtain

$$PAy'(t) + PBy(t) = Pf(t).$$

Setting

$$y(t) = Q \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}, \quad Pf(t) = \begin{pmatrix} s(t) \\ q(t) \end{pmatrix},$$

### 3 Differential Algebraic Equations

with  $u(t), s(t) : \mathbb{R} \rightarrow \mathbb{R}^d$  and  $q(t), v(t) : \mathbb{R} \rightarrow \mathbb{R}^{n-d}$ , we get a system of the form

$$\begin{aligned} u'(t) + Ru(t) &= s(t), \\ Nv'(t) + v(t) &= q(t), \end{aligned} \tag{3.3}$$

where  $PAQ = \begin{pmatrix} I & \\ & N \end{pmatrix}$  and  $PBQ = \begin{pmatrix} R & \\ & I \end{pmatrix}$ .

We call this system the *Weierstraß-Kronecker normal form* of the DAE. The first equation of (3.3) is a first order ordinary differential equation and possesses a unique solution  $u(t)$  in  $[t_0, t_l]$  for any initial values  $u_0 \in \mathbb{R}^d$ . Now let  $q(t) \in C^{k-1}([t_0, t_l])$ , where  $C^{k-1}(I)$  denotes the space of all functions with domain  $I$  that are  $k-1$  times continuously differentiable. We differentiate the second equation in (3.3) and obtain

$$\begin{aligned} v(t) &= q(t) - Nv'(t) = q(t) - N \underbrace{(q(t) - Nv'(t))'}_{=v(t)} = q - Nq' + N^2v'' \\ &= q - Nq' + N^2(q - Nv')'' = q - Nq' + N^2q'' - N^3v''' \\ &\vdots \\ &= q - Nq' + \dots + (-1)^{k-1} N^{k-1} \underbrace{\frac{d^{k-1}}{dt^{k-1}} q}_{:=q^{(k-1)}} + (-1)^k \underbrace{N^k v^{(k)}}_{=0} \\ &= \sum_{i=0}^{k-1} (-1)^i N^i q^{(i)}(t) \end{aligned} \tag{3.4}$$

where  $k$  is the nilpotency index of  $N$ . Hence we have an explicit form of the solution of  $v(t)$ . The Kronecker index  $k$  thus tells us how many differentiations are required to obtain an ordinary differential equation. Note that this way of splitting the DAE allows us to differ between algebraic and differential variables.

We can also give a general definition of a DAE in Weierstraß-Kronecker normal form.

### 3 Differential Algebraic Equations

**Definition 2** ([5, Definition 13.2.4]). A linear differential equation (3.2) is said to be in Weierstraß-Kronecker normal form if

$$A = \begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix}, \quad B = \begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix},$$

where  $N$  is a nilpotent Jordan-block matrix.

**Definition 3.** The nilpotency index  $k$  of the matrix  $N$  from the Weierstraß-Kronecker normalform of a matrix pencil  $\{A, B\}$  with  $A$  singular is called the Kronecker index of  $\{A, B\}$ , which we denote by  $\text{ind}\{A, B\}$ . Note that for  $A$  regular we set  $\text{ind}\{A, B\} = 0$ .

**Lemma 1.** The Kronecker index  $\text{ind}\{A, B\}$  is independent of the choice of  $P$  and  $Q$ .

*Proof.* See Lemma 13.2.1 in [5, p.403]. □

## 3.2 Index of a Differential Algebraic Equation

The index of a DAE gives us insight about its numerical properties and in general about the solvability of the equation. In particular, as we will also mention later, the index is relevant for choosing initial values that allow us to solve the system uniquely.

We will consider two types of index concepts, the differentiation index and the perturbation index.

Since numerical differentiation is an unstable procedure the differentiation index aims to give a measure for the numerical problems to be expected when solving such systems.

### 3 Differential Algebraic Equations

**Definition 4** (Differentiation index [5, Definition 13.3.1]). Consider the differential algebraic equation (3.1) to be uniquely locally solvable and  $F$  sufficiently smooth. For a given  $m \in \mathbb{N}$  consider the equations

$$\begin{aligned} F(t, y, y') &= 0, \\ \frac{dF(t, y, y')}{dt} &= 0, \\ &\vdots \\ \frac{d^m F(t, y, y')}{dt^m} &= 0. \end{aligned}$$

The smallest natural number  $m$  for which the above system results in an explicit system of ordinary differential equations (ODEs), i.e. it has the form

$$y' = \phi(t, y),$$

is called **differentiation index**.

In the previous chapter we have already discussed, that for a DAE with constant coefficients (3.2) and a regular matrix pencil  $\{A, B\}$  we need  $k = \text{ind}\{A, B\}$  differentiations to receive an ordinary differential equation. Hence for a DAE with constant coefficients the Kronecker index  $k$  is equal to the differentiation index.

**Definition 5** (Perturbation index [5, Definition 13.3.3]). Let  $y(t)$  be the exact solution of (3.1) and  $\tilde{y}(t)$  be the solution of the perturbed system  $F(t, \tilde{y}, \tilde{y}') = \delta(t)$ . The smallest number  $k \in \mathbb{N}$  such that

$$\|y(t) - \tilde{y}(t)\| \leq C \left( \|y(t_0) - \tilde{y}(t_0)\| + \sum_{j=0}^k \max_{t_0 \leq \xi \leq T} \left\| \int_{t_0}^{\xi} \frac{d^j \delta}{d\tau^j}(\tau) d\tau \right\| \right)$$

for all  $\tilde{y}(t)$ , is called the **perturbation index** of this system.

We now consider the case of a perturbed linear DAE with constant coefficients and a regular matrix pencil  $\{A, B\}$

$$A\tilde{y}'(t) + B\tilde{y}(t) = -\delta(t).$$

### 3 Differential Algebraic Equations

Then we obtain for the difference of the unperturbed system with the perturbed system

$$A(y'(t) - \tilde{y}'(t)) + B(y(t) - \tilde{y}(t)) = \delta(t)$$

Due to Theorem 2 we can transform this system into a system of the form (3.3)

$$\begin{aligned} u'(t) - \tilde{u}'(t) + \hat{R}(u(t) - \tilde{u}(t)) &= \delta_1(t), \\ N(v'(t) - \tilde{v}'(t)) &= \delta_2(t), \end{aligned}$$

where  $u(t)$  holds the first  $d$  entries of  $y(t) : \mathbb{R} \rightarrow \mathbb{R}^n$  and  $v(t)$  holds the remaining  $n - d$  entries. According to (3.4), the solution of the algebraic variable has the form

$$v(t) - \tilde{v}(t) = \sum_{i=0}^{k-1} (-1)^i N^i \delta_2^{(i)}(t)$$

Thus the perturbation index is the same as the Kronecker-index  $\text{ind}\{A, B\}$ , see [5]. Moreover we observe that the error depends on the derivatives of the perturbation.

## 4 Index Analysis of the Modified Nodal Analysis

In the previous chapter we have seen two different kinds of index concepts for differential algebraic equations. We have also seen that these two, even though they describe rather different structural aspects of the equation, are identical for the problems discussed in this thesis. This leads to the question, what indices we expect to obtain for our MNA system.

This chapter answers this question for the case of linear RLC-circuits. It builds upon results discussed in [2], [5] and [8]. As already discussed in Section 2.3, RLC circuits contain only resistors, inductors and capacities as well as current and voltage sources. Linear means that all circuit components are described by linear positive functions. Thus the matrices  $C$ ,  $R$ , and  $L$  are positive definite and symmetric.

Recall that we consider the equations resulting from the analysis in Section 2.4. These equations are of the form (3.2): Find  $y$  such that

$$Ay'(t) + By(t) = f(t).$$

In particular we obtain the system of equations

$$\begin{pmatrix} A_C C A_C^\top & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{pmatrix} * \begin{pmatrix} u' \\ i'_L \\ i'_V \end{pmatrix} + \begin{pmatrix} A_R G A_R^\top & A_L & A_V \\ -A_L^\top & 0 & 0 \\ -A_V^\top & 0 & 0 \end{pmatrix} * \begin{pmatrix} u \\ i_L \\ i_V \end{pmatrix} = \begin{pmatrix} -A_I i_{src} \\ 0 \\ -v_{src} \end{pmatrix}.$$

### 4.1 General Index analysis

Assuming the system only contains linear elements, then the corresponding network equation represents a DAE with constant coefficients (3.2). We collect all unknowns in

#### 4 Index Analysis of the Modified Nodal Analysis

$y(t) = (u(t), i_L(t), i_V(t))^T$ . The structure of the system is reliant on the matrix  $B$ , thus we consider two cases.

**ODE-case:** The matrix  $B$  in (3.2) is regular. This is the case iff the circuit contains no voltage sources and every node has a path to ground, that leads through at least one capacitor. Then the system represents a linear-implicit system of ODEs and can be transformed into the explicit ODE system

$$y(t)' = B^{-1}(-Ay(t) + f(t)).$$

Thus we obtain an index of 0.

**DAE-case:** The matrix  $B$  in (3.2) is singular. This is the interesting case which we will analyse in the following.

For a singular matrix  $B$  we have already obtained a representation of the form

$$\begin{aligned} u'(t) + Ru(t) &= s(t), \\ Nv'(t) + v(t) &= q(t) \end{aligned}$$

in the previous section, see (3.3). We now consider the nilpotency index  $\nu$  of the matrix  $N$ , since we already know that this index correlates with the differentiation index as well as the perturbation index. We consider two cases:

##### 1. The nilpotency index is 1

Because  $N$  is nilpotent with nilpotency index  $\nu = 1$ , it holds that  $N^1 = 0$ , thus the system transforms to

$$\begin{aligned} u'(t) + Ru(t) &= s(t), \\ v(t) &= q(t). \end{aligned}$$

This means that the algebraic variables are given explicitly. Thus the system is in ODE form.

## 4 Index Analysis of the Modified Nodal Analysis

### 2. The nilpotency index is $\geq 2$

This case is the situation we have discussed in section ??, where we obtained the representation (3.4)

$$v(t) = \sum_{i=0}^{\nu-1} (-1)^i N^i q^{(i)}(t). \quad (4.1)$$

By differentiating the above equation one more time, we obtain

$$v'(t) = \sum_{i=0}^{\nu-1} (-1)^i N^i q^{(i+1)}(t) \quad (4.2)$$

with the assumption that  $q$  is  $\nu$  times differentiable.

We observe that in both cases the solution  $v$  has to fulfill an algebraic constraint. In the case of index  $\nu = 1$  this equation is given explicitly, while for index  $\nu \geq 2$  it is given implicitly by (4.2).

This system is also sensitive to perturbations. Note that the solution formula (4.1) for  $v$  also contains derivatives of the input  $q$ . Thus small noise in the input can have arbitrarily large derivatives, which can influence the solution.

## 4.2 Topological Conditions

From analysing the MNA equations, some conditions to the circuit topology can be obtained. We will be considering the impact of special arrangements of components on the index of the system. General results are presented in [9] as well as in [8] and [7].



#### 4 Index Analysis of the Modified Nodal Analysis

**Theorem 3** (Index conditions [8, Theorem 2.2.1]). *Let the matrices of the capacitances, inductances and resistances be positive definite.*

- *If*

$$\ker([A_R, A_C, A_V, A_L]^\top) = \{0\} \quad \text{and} \quad \ker(A_V) = \{0\} \quad (4.3)$$

*holds, then the MNA (2.8) leads to a system with index  $\nu \leq 2$ .*

- *If additionally*

$$\ker([A_R, A_C, A_V]^\top) = \{0\} \quad \text{and} \quad \ker([A_C, A_V]) = \{0\} \quad (4.4)$$

*holds, then the system is of index  $\nu \leq 1$*

- *If further*

$$\ker(A_C^\top) = \{0\} \quad \text{and} \quad \dim(v_{src}) = 0 \quad (4.5)$$

*holds, then the system has index  $\nu = 0$ .*

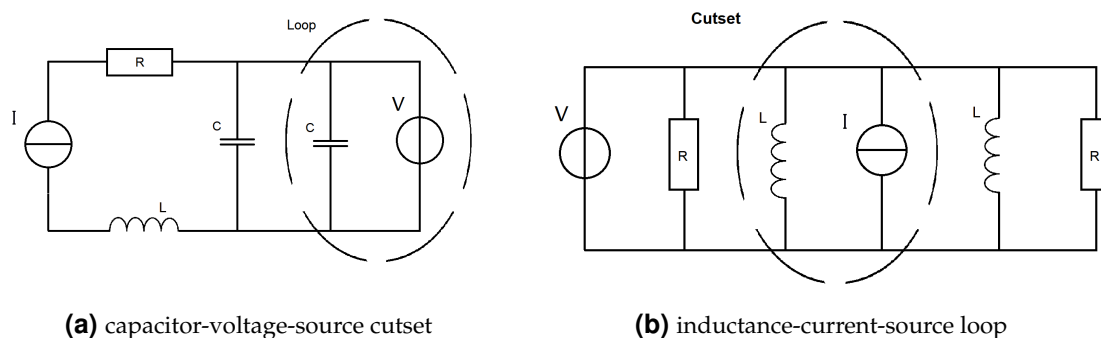
By  $\ker(A)$  we denote the nullspace of the matrix  $A$  and by  $\dim(y)$  the size of the vector  $y$ . As (4.3) - (4.5) are just conditions concerning the topology of the circuit they can also be interpreted as follows:

- Condition (4.3) means that the circuit neither contains loops of voltage sources nor cutsets of current sources.
- Condition (4.4) means that the circuit contains neither loops of capacitors and/or voltage sources nor cutsets of inductors and/or current sources.
- Condition (4.5) means that every node in the circuit is connected to the reference node (ground) through a path containing only capacitors.

A cutset of a circuit is a part (subcircuit) of the circuit that can be separated from the main circuit by only “cutting” edges containing no components, the cutset and the original circuit without the cutset both cannot have any open edges. A loop is a subcircuit, which can be constructed by removing edges and nodes from the main circuit until we are

#### 4 Index Analysis of the Modified Nodal Analysis

left with only one continuous “loop” of edges and nodes. Figure 4.1 illustrates the two concepts.

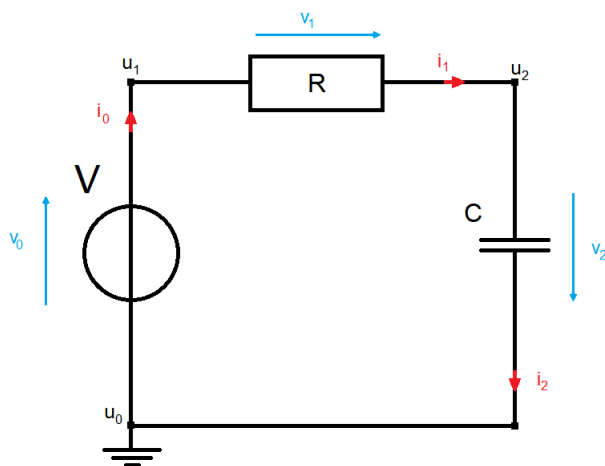


**Figure 4.1:** Illustration of a cutset and a loop.

We will apply those results to the previously considered examples:

##### Example 1.3 (Index analysis)

As our first example we consider again the charging of a capacitor with a series resistor as in **Example 1**.



**Figure 4.2:** charging capacitor with series resistor and voltage source (note the difference to 2.1)

Because this circuit only contains one loop, which has a voltage source as well as a capacitor and a resistor we expect the resulting MNA system to be of index  $\nu = 1$ .

#### 4 Index Analysis of the Modified Nodal Analysis

The incidence matrix for this system is

$$A = [A_V, A_R, A_C] = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

Thus condition (4.3) gives us

$$\ker([A_R, A_C, A_V, A_L]^\top) = \ker \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} = \{0\} \quad \text{and} \quad \ker(A_V) = \ker \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \{0\}.$$

This condition is fulfilled, thus the index has to be smaller or equal to 2. The condition (4.4) results in

$$\ker([A_R, A_C, A_V]^\top) = \ker \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} = \{0\} \quad \text{and} \quad \ker([A_C, A_V]) = \ker \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \{0\}.$$

Thus the index also has to be smaller or equal to 1. Finally condition (4.5) results in

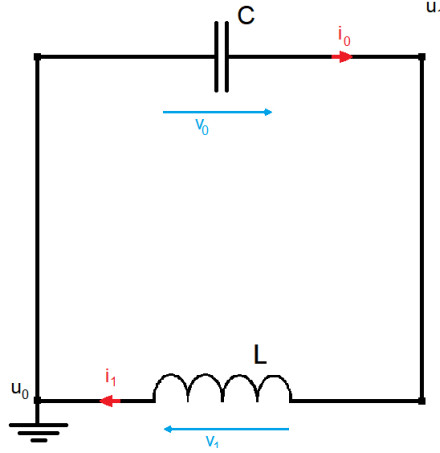
$$\ker(A_C^\top) = \ker \begin{pmatrix} 0 & 1 \end{pmatrix} = \{0\} \quad \text{and} \quad \dim(v_{src}) \neq 0.$$

This means that this system has index  $\nu = 1$ .

##### **Example 2.3** (Index analysis)

This example depicts an energy conserving system. The electric energy stored in the capacitor is converted into the the magnetic energy of the coil and vice versa.

#### 4 Index Analysis of the Modified Nodal Analysis



**Figure 4.3:** LC circuit

Because this circuit only contains one capacitor and one inductance in a singular loop and also has no sources, we expect the resulting MNA system to be of index  $\nu = 0$ . This means that the system is an ODE.

The incidence matrix for this system is

$$A = [A_L \ A_C] = \begin{pmatrix} -1 & 1 \end{pmatrix}.$$

Thus condition (4.3) gives us

$$\ker([A_R, A_C, A_V, A_L]^\top) = \ker \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \{0\} \quad \text{and} \quad \ker(A_V) = \{0\}.$$

This condition is fulfilled, thus the index has to be smaller or equal to 2.

Condition (4.4) results in

$$\ker([A_R, A_C, A_V]^\top) = \ker \begin{pmatrix} -1 \end{pmatrix} = \{0\} \quad \text{and} \quad \ker([A_C, A_V]) = \ker \begin{pmatrix} -1 \end{pmatrix} = \{0\}.$$

This condition is also fulfilled, thus the index has to be smaller or equal to 1.

Finally condition (4.5) results in

$$\ker(A_C^\top) = \ker \begin{pmatrix} -1 \end{pmatrix} = \{0\} \quad \text{and} \quad \dim(v_{src}) = 0.$$

Thus it is fulfilled and we obtain index 0.

#### 4 Index Analysis of the Modified Nodal Analysis

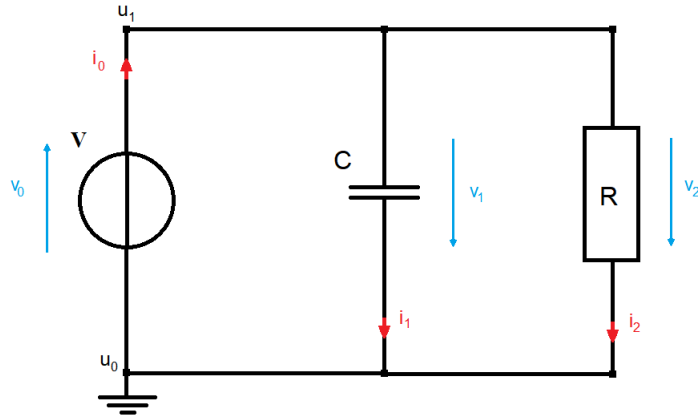
If we have consider the MNA directly, we also see that this system is of index  $\nu = 0$ , because the MNA has the form

$$\begin{aligned} Cu'_1 - i_L &= 0, \\ Lu'_L + u_1 &= 0 \end{aligned}$$

which is clearly a system of ODEs.

##### Example 3.3 (Index analysis)

Finally, as the third example seems very similar to the first one, we might also expect them to have the same index. But this is indeed not the case as the following analysis shows.



**Figure 4.4:** current source with resistor and capacitor

This circuit contains two loops, the first with only a voltage source and a capacitor and the second with a capacitor and a resistor. Because of the first loop, we expect this circuit to emit a system of index  $\nu = 2$ .

The incidence matrix for this system is

$$A = [A_V \ A_C \ A_R] = \begin{pmatrix} -1 & 1 & 1 \end{pmatrix}.$$

## 4 Index Analysis of the Modified Nodal Analysis

Thus condition (4.3) gives us

$$\ker([A_R, A_C, A_V, A_L]^\top) = \ker \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \{0\} \quad \text{and} \quad \ker(A_V) = \ker(-1) = \{0\}$$

This condition is fulfilled, thus the index has to be smaller or equal to 2. Condition (4.4) results in

$$\ker([A_R, A_C, A_V]^\top) = \ker \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \{0\} \quad \text{and} \quad \ker([A_C, A_V]) = \ker \begin{pmatrix} 1 & -1 \end{pmatrix} \neq \{0\}.$$

Thus this condition is not fulfilled, which means that the system indeed has index  $\nu = 2$ .

### 4.3 Consistent Initial Values

When solving a problem numerically, it is important to ensure that the problem has a (unique) solution. Without a solution, a numerical method could converge to an arbitrary value unrelated to any actual solution. In real-world applications, there may be multiple possible initial values. To guarantee a (unique) solution, it is crucial to choose initial values that are consistent. This is why identifying conditions for constructing *consistent* initial values is important. For further details, see [5, chapter 13.3.1].

**Case Index  $\nu = 0$ .** As already discussed previously, the case  $\nu = 0$  leads to an ODE, and due to the Picard-Lindelöf theorem [5, Satz 1.2.1] we know that every initial condition is consistent (for ordinary differential equations).

#### **Example 2.4** (Consistent initial values)

We have already seen, that this system is of index  $\nu = 0$ . This means that any initial value is consistent. As the initial values still have a physical interpretation, hence not every consistent initial value also makes sense. We will assume that at time  $t = 0$  the capacitor holds no charge, i.e.  $u_1(0) = 1$  and that there is no energy stored in the inductance, i.e.  $i_L(0) = 0$ .

#### 4 Index Analysis of the Modified Nodal Analysis

**Case: Index  $\nu = 1$ .** By rewriting our system into the form

$$\begin{aligned} y'(t) &= f(t, y(t), z(t)), \\ 0 &= g(t, y(t), z(t)). \end{aligned}$$

we obtain conditions for consistent initial values. Namely  $y_0 = y(0)$  and  $z_0 = z(0)$  are consistent initial values for this system, if  $g(t_0, y_0, z_0) = 0$  holds.

**Example 1.4** (Consistent initial values)

We have already seen that this system is of index  $\nu = 1$ . Due to the fact that  $u_1$  and  $i_V$  are only algebraically constrained, we have to determine their initial values by solving for them at the point  $t = 0$ . On the other hand  $u_2$  and its derivative both appear, thus we can simply prescribe an initial value for  $u_2$ . Again prescribing an any initial value might make sense from a mathematical perspective, since we might only be interested in the existence of a solution. But due to the physical interpretation of  $u_2$  as being related to the charge that might already be stored in the capacitor, it makes sense to consider the capacitor to have no initial charge, i.e.  $u_2(0) = 0$ . The third line of the MNA system (2.9) together with setting  $v_{src} = \sin(\pi t)$  immediately gives us, that

$$u_1(0) = -\sin(\pi * 0) = 0.$$

From line one follows, that

$$u_1(0) - u_2(0) - i_V(0) = 0,$$

thus  $i_V(0) = 0$  is a consistent initial value for  $i_V$ .

**Case: Index  $\nu = 2$ .** For index-2 systems we rewrite our system into

$$\begin{aligned} y' &= f(t, y(t), z(t)), \\ 0 &= g(t, y(t)). \end{aligned}$$

#### 4 Index Analysis of the Modified Nodal Analysis

Consistent initial values  $y_0, z_0$  for this case not only have to fulfill  $g(t_0, y_0) = 0$  but also the *hidden constraint*  $g_t(t_0, y_0) + g_y(t_0, y_0)f(t_0, y_0, z_0) = 0$ . By  $g_t$  and  $g_y$  we denote the derivative of  $g$  with respect to  $t$  or  $y$ , respectively.

**Example 3.4** (Consistent initial values)

We already found the index of this system to be  $\nu = 2$ . Considering (2.10), we can define

$$\begin{aligned} f(t, u_1(t), i_v(t)) &= i_v(t) - u_1(t), \\ g(t, u_1(t)) &= u_1(t) + v_{src}(t). \end{aligned}$$

Computing the first condition gives together with the choice  $v_{src} = \sin(\pi t)$

$$u_1(0) = -v_{src}(0) = -\sin(\pi * 0) = 0.$$

While the second condition leads to the hidden constraint

$$0 = v'_{src}(0) + 1 * (i_v(0) - u_1(0)) = \pi \cos(\pi * 0) + i_v(0) - 0 = \pi + i_v(0).$$

Thus we obtain  $i_v(0) = -\pi$ .



## 5 Numerical Solutions

This chapter focuses on the numerical solution of the mentioned systems. It is based on results presented in [5] and [3].

We will first focus on methods used to solve a more general initial value problem of the form: Find  $y$ , such that

$$\begin{aligned} y'(t) &= f(t, y), \quad t \in [t_0, t_l], \\ y(t_0) &= y_0. \end{aligned} \tag{5.1}$$

For this we presume that the function  $f(t, y)$  is continuous and Lipschitz-continuous, we can apply the theorem of *Picard-Lindelöf* [5, Satz 1.2.1], which states, that for every initial value  $y_0$ , the initial value problem (5.1) is uniquely solvable in  $[t_0, t_l]$ .

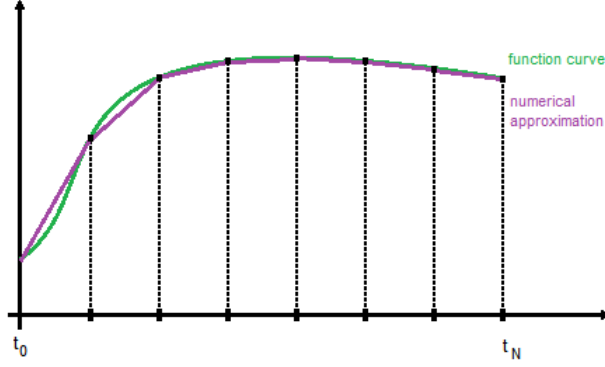
In order to obtain an approximation to this unique solution we have to discretize. We first divide the time-intervall  $[t_0, t_l]$  into smaller intervals

$$I_{i+1} = [t_i, t_{i+1}] \quad \text{with} \quad i = 0, \dots, N \text{ and } t_0 < t_1 < \dots < t_N \leq t_l$$

and consider approximations  $y_m \approx y(t_m)$  on each interval  $I_m$  with  $m = 1, \dots, N$ , see Figure 5.1 for an illustration. We call this a *time-grid* and we call the difference  $h_{ij} = t_j - t_i$  with  $0 \leq i < j \leq N$  the *step size* from  $t_i$  to  $t_j$ . We will only consider equidistant grids, i.e.  $h_{ij} = h$  for all  $0 \leq i, j \leq N$ . A function defined on such a grid is called a *grid function*.

The grid function  $y_h : T \rightarrow A$  with the time grid  $T = \{t_0, \dots, t_N\}$  and the approximated values  $A = \{y_0, \dots, y_N\}$  is called the *numerical approximation* (of  $y$  on the time grid).

## 5 Numerical Solutions



**Figure 5.1:** approximation of a function

### 5.1 Multistep Methods

In this thesis we will only consider the class of *linear multistep methods* (LMSM). This chapter is in large parts based on [5]. These methods use multiple previously approximated values  $y_{m+l}$  at the gridpoints  $t_{m+l}$ ,  $l = 0, 1, \dots, k-1$  to approximate the current value  $y_{m+k}$  at  $t_{m+k}$ . The definition for such a method is given in Definition 6

**Definition 6.** For given  $\alpha_0, \dots, \alpha_k$  and  $\beta_0, \dots, \beta_k$  the iteration rule

$$\sum_{l=0}^k \alpha_l y_{m+l} = h \sum_{l=0}^k \beta_l f(t_{m+l}, y_{m+l}), \quad m = 0, 1, \dots, N-k \quad (5.2)$$

is called a *linear multistep method* (*linear  $k$ -step method*). It is always assumed that  $\alpha_k \neq 0$  and  $|\alpha_0| + |\beta_k| > 0$ . If  $\beta_k = 0$  holds, then the method is called *explicit*, otherwise it is called *implicit*.

The calculation of approximations using a multistep method consists of two phases:

1. In the *starting phase*, approximations  $y_1, \dots, y_{k-1}$  for the first  $k-1$  gridpoints  $t_l = t_0 + l h$ ,  $l = 1, \dots, k-1$  have to be calculated. For example we can use a single-step method or a multi-step method with fewer steps.

## 5 Numerical Solutions

2. In the *run phase*, the multi-step formula is used to determine new approximations  $u_{m+k}$  for the gridpoint  $t_{m+k}$

### 5.1.1 Consistency, Stability and Convergence

In order to compare different multistep methods, we have to introduce the notions of error, consistency, stability and convergence. We begin with the definition of the error.

**Definition 7.** Let  $y_{m+k}$  be the result of one step of the multi-step method (5.2) with the start-values given as the evaluations of the exact solution  $y_{m+l} = y(t_{m+l})$  at  $0 \leq l < k$ . This means

$$\alpha_k y_{m+k} = \sum_{l=0}^{k-1} (h\beta_l f(t_{m+l}, y(t_{m+l})) - \alpha_l y(t_{m+l})) + h\beta_k f(t_{m+k}, y_{m+k}).$$

Then

$$\delta_{m+k} = y(t_{m+k}) - y_{m+k}, \quad m = 0, 1, \dots, N - k$$

is called the local discretization error (local error) of the linear multi-step method (5.2) at the point  $t_{m+k}$ .

If we assign the difference operator

$$L[y(t), h] = \sum_{l=0}^k (\alpha_l y(t + lh) - h\beta_l y'(t + lh)) \tag{5.3}$$

to the linear mutli-step method, we gain the following definition for consistency of such a method.

## 5 Numerical Solutions

**Definition 8.** A linear multi-step method is called consistent, if for all functions  $y(t) \in C^2([t_0, t_l])$

$$\lim_{h \rightarrow 0} \frac{1}{h} L[y(t), h] = 0$$

holds. It is called consistent with order  $p$ , if for all functions  $y(t) \in C^{p+1}[t_0, t_l]$

$$L[y(t), h] = \mathcal{O}(h^{p+1}) \quad \text{for } h \rightarrow 0$$

holds.

Next we introduce the convergence and the convergence order of such methods.

**Definition 9.** We say that a linear multi-step method is convergent, if for a solution  $y$  of the problem and a vector  $(y_j)_{j=0}^k$  created by an LMSM, we have that

$$\lim_{h \rightarrow 0} \max_{0 \leq j \leq k} \|y(t_j) - y_j\| = 0.$$

If for  $p \in \mathbb{N}$  and a constant  $C$  not dependent on the step size  $h$  we have

$$\max_{0 \leq j \leq k} \|y(t_j) - y_j\| \leq Ch^p,$$

then we call the LMSM convergent with order  $p$ .

The notion of stability tells us, if the output is continuously dependent on input. Thus we consider two different inputs for this definition and compare their outputs.

**Definition 10.** A linear multi-step method is called (discretely) stable, if for solution vectors  $(y_l)_{l=0}^k$  and  $(\tilde{y}_l)_{l=0}^k$  of

$$\sum_{l=0}^k \alpha_l y_{m+l} = h \sum_{l=0}^k \beta_l f(t_{m+l}, y_{m+l}), \quad (5.4)$$

$$\sum_{l=0}^k \alpha_l \tilde{y}_{m+l} = h \sum_{l=0}^k \beta_l f(t_{m+l}, \tilde{y}_{m+l}) + h \theta_m, \quad (5.5)$$

with bounded initial values  $y_j, \tilde{y}_j$  for  $j \in 0, \dots, k$ , we have that

$$\max_{t_0 \leq t_n \leq T} \|y_n - \tilde{y}_n\| \leq C \sum_{j=0}^{k-1} \|y_j - \tilde{y}_j\| + \max_{t_0 \leq t_n \leq T} \|\theta_n\|.$$

### 5.1.2 Further stability properties

In this section we consider again the Dahlquist test problem as a model problem, i.e. find  $y : \mathbb{R} \rightarrow \mathbb{C}$  such that

$$y' = \lambda y, \quad t > 0 \quad (5.6)$$

$$y(0) = y_0 \quad (5.7)$$

with  $\lambda \in \mathbb{C}$  and  $y_0 \in \mathbb{C}$  fixed.

Thus the resulting linear multistep method is of the form

$$\begin{aligned} \sum_{l=0}^k \alpha_l y_{n+l} &= h \sum_{l=0}^k \beta_l \lambda y_{n+l} \\ \iff \sum_{l=0}^k [\alpha_l - h \beta_l \lambda] y_{n+l} &= 0 \end{aligned}$$

To be able to define further stability notions we first need to introduce the so called *generating polynomials* of a multistep method. They are defined using the coefficients of the

## 5 Numerical Solutions

method and thus hold information about the method. In the following we define these polynomials.

$$\rho(x) := \sum_{l=0}^k \alpha_l x^l \quad \text{and} \quad \sigma(x) := \sum_{l=0}^k \beta_l x^l \quad (5.8)$$

For the simpler class of methods called *singlestep methods* there is a notion of A-stability, see [5]. Unfortunately for many multistep methods we do not have A-stability, thus we define the weaker  $A(\alpha)$ -stability.

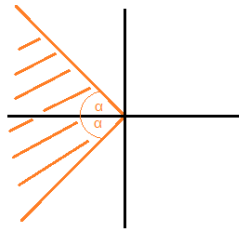
**Definition 11.** 1. The set

$$S := \{z \in \mathbb{C} : \rho(\xi) - z\sigma(\xi) = 0 \implies \xi \in \mathbb{C} \text{ and } |\xi| \leq 1. \\ \text{If } \xi \text{ has multiplicity greater than 1, then } |\xi| < 1\} \quad (5.9)$$

is called the region of stability of the method.

2. A linear multistep method is called

- 0-stable, if  $0 \in S$ .
- stable in the point  $z \in \mathbb{C}$ , if  $z \in S$ .
- $A(\alpha)$ -stable, if it is stable in all  $z$  that lie within the set  $\{z \in \mathbb{C}^- : |\arg(z) - \pi| \leq \alpha\}$  for  $\alpha \in (0, \frac{\pi}{2})$ .



**Figure 5.2:**  $A(\alpha)$  stability for  $\alpha = \frac{\pi}{4}$

**Theorem 4** ([5, Satz 4.2.10]). *Let  $f(t, y)$  be sufficiently smooth and the linear multi-step method be zero-stable and consistent of order  $p$ , then it is also convergent of order  $p$ .*

## 5.2 Implicit linear multistep formulas

In this chapter we discuss three different linear multistep methods. Whereas the trapezoidal rule and the implicit midpoint rule (or Gauss with one stage) really only use one timestep to approximate the next value and might thus be considered as *singlestep* methods, the Backward Differentiation Formulas with  $k$  stages (BDF- $k$ ) really use  $k$  steps and thus are “true” multistep methods. We will see that the BDF- $k$  methods as well as the trapezoidal rule are preferably used over the implicit midpoint rule due to their advantageous behaviour when applied to differential algebraic equations. Examples are discussed in 5.3. For more detailed analysis of the mentioned methods we refer to [5] and [3].

In Section 5.1 we briefly mentioned the starting phase and the run phase of a multistep method. However, we left out one important aspect of the starting phase. Recall that in the starting phase we calculate the first  $k - 1$  steps using a method with fewer stages than the method we want to use ultimately. This also means that this method might be of lower order than the final method. To achieve the desired convergence rates it is thus also important to compute these initial steps appropriately, i.e. the method with which these initial steps are calculated has to also be of the desired order with respect to the *final* step size  $h$ . (In applications this can mean that we calculate the initial steps with methods of lower order, but compensate by using a smaller step-size)

### 5.2.1 BDF- $k$ methods

The Backward-Differentiation-Formulas with  $k$  stages (or short *BDF- $k$  methods*) are constructed, such that they are  $A(\alpha)$ -stable with the largest possible angle  $\alpha$ . We will not go into detail about their construction, for this we again refer to [5, chapter 9.2]

We will not give a deeper look into their construction but will only state their properties. BDF schemes are appealing because they save function evaluations as much as possible.

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The BDF-k methods have the general form

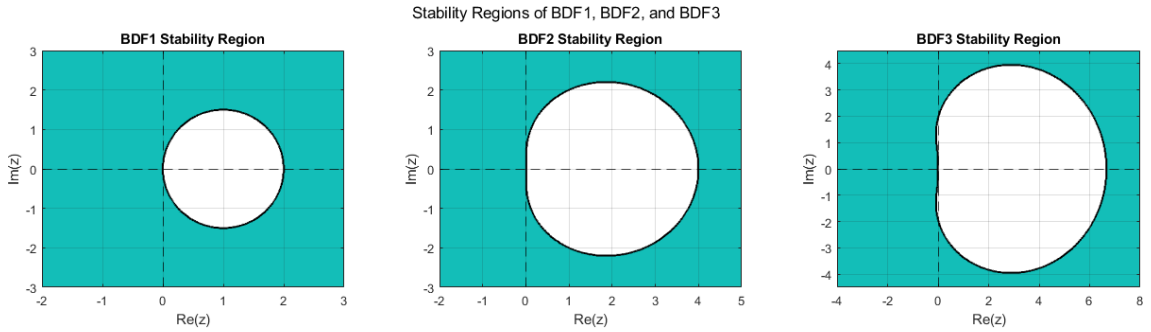
$$\sum_{k=0}^s \alpha_k y_{n+k} = h\beta f(t_{n+s}, y_{n+s}). \quad (5.10)$$

Due to the construction of the method, which is not built around conserving zero-stability, the method is not zero-stable for all  $k \in \mathbb{N}$ . It turns out that it is only zero-stable for  $k \leq 6$ , [5, page 325]. However in practice, methods with order greater than 3 are rarely used because of their stability properties, as the stability region shrinks with increasing order.

The BDF or BDF-k formulas for  $k = 1, \dots, 3$  have the following form

$$\begin{aligned} k = 1 : hf_{m+1} &= y_{m+1} - y_m \\ k = 2 : hf_{m+2} &= \frac{1}{2}(3y_{m+2} - 4y_{m+1} + y_m) \\ k = 3 : hf_{m+3} &= \frac{1}{6}(11y_{m+3} - 18y_{m+2} + 9y_{m+1} - 2y_m) \end{aligned}$$

As already mentioned BDF-k methods are not A-stable, but  $A(\alpha)$  stable. In Figure 5.3 the stability regions for  $k = 1, 2, 3$  are depicted in cyan. Note that the zero-point is contained within the stability regions.



**Figure 5.3:** stability regions of BDF-k methods for  $k = 1, 2, 3$  (cyan)

The order of consistency of this family of multi-step methods is given by the following theorem



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**Theorem 5** ([5, Satz 9.2.1]). *The BDF- $k$  methods have consistency order  $p = k$ .*

Due to Theorem 4 and the fact that BDF- $k$  methods with  $k \leq 6$  are 0-stable as well as Theorem 5 we immediately gain the following corollary.

**Corollary 1** (Convergence rate). *The BDF- $k$  methods with  $k \leq 6$  are convergent with order  $k$*

### 5.2.2 Trapezoidal rule

By approximating the area under a curve  $f(x)$  on a short interval  $[a, b]$  as a trapezoid,

$$\int_a^b f(x)dx \approx (b - a)\frac{1}{2}(f(a) + f(b)),$$

we can construct a numerical method for integrating a function. This method is called the trapezoidal rule, formally it has the iteration rule

$$y_{n+1} = y_n + \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_{n+1})].$$

The trapezoidal rule has convergence order  $p = 2$  for our applications. For details see [3], where more general the general form of the trapezoidal rule, namely the Lobatto III-A methods amongst other methods are discussed.

Due to the fact that the trapezoidal rule is A-stable [2] and of order  $p = 2$ , it can be considered as a natural alternative to the BDF-2 method.

### 5.2.3 Gauss Method with one stage

We have already seen that the results for the BDF- $k$  methods as well as for the trapezoidal rule are relatively straight forward. But this is not always the case, many numerical methods can cause problems when applied to DAEs, see also [3]. To illustrate this we will

## 5 Numerical Solutions

have a look at the Gauss method with one stage, also called the implicit midpoint rule, i.e.

$$y_{n+1} = y_n + hf\left(t_n + \frac{h}{2}, \frac{y_n + y_{n+1}}{2}\right)$$

Applied to ordinary differential equations, this method yields a convergence order of  $p = 2$  [5, chapter 8.1.2]. For the special case of differential algebraic equations this does not hold true. Here we may observe lower convergence rates, especially in the algebraic variable. The convergence orders for this method are also different, if the amount of stages is even or odd. We say that this method is *not stiffly accurate*. Table 5.1 displays the according convergence orders.

number of stages $s$	index $\nu = 1$		index $\nu = 2$	
	differential	algebraic	differential	algebraic
even	$2s$	$s + 1$	$s + 1$	$s - 1$
odd		$s$	$s$	$s - 2$

**Table 5.1:** Convergence order for Gauss method.

### 5.3 Numerical Examples

In this section we will apply the trapezoidal rule, the BDF-k method and the implicit midpoint rule, discussed in Section 5.2.1, Section 5.2.2 and Section 5.2.3 to our three examples. We will also discuss the discretization error in the solution and compute appropriate constant initial values for our problems. The code used is provided in Section 5.4.

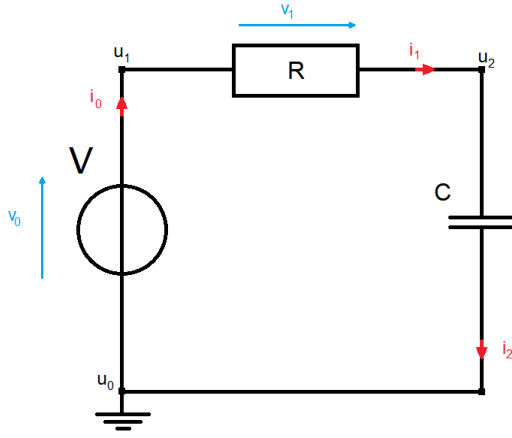
#### 5.3.1 BDF-k Method and Trapezoidal Rule

We will start by applying the “well-behaving” methods to our examples.

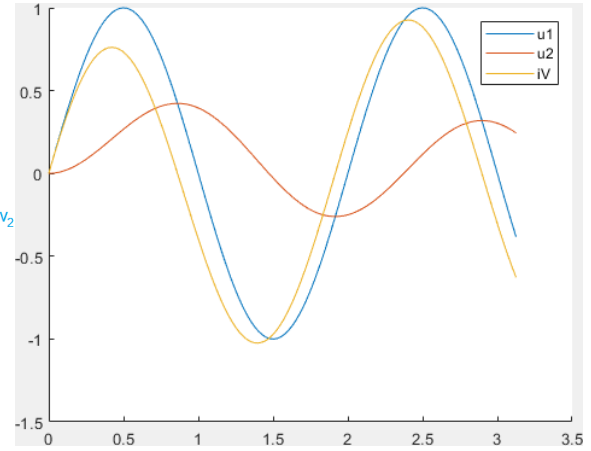
##### **Example 1.5** (Numerical computation)

We again consider the charging of a capacitor, depicted in the schematics in Figure 5.4.

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**Figure 5.4:** charging capacitor with series resistor and voltage source



**Figure 5.5:** Exact solution for Example 1.

For testing we set the resistance to  $R = 1$  and the capacitance to  $C = 1$ . In our case we let the voltage source supply a voltage of the form  $v_{src} = \sin(\pi t)$ . The resulting MNA-system for this example reads

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} * \begin{pmatrix} u'_1 \\ u'_2 \\ i'_0 \end{pmatrix} + \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} * \begin{pmatrix} u_1 \\ u_2 \\ i_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\sin(\pi t) \end{pmatrix}. \quad (5.11)$$

We expect the resulting potentials and currents to be of the form seen in Figure 5.5.

The resulting errors for our numerical tests are displayed in table 5.2, we omit  $u_1$  since it just equals  $-v_{src}$  and thus produces no interesting error results. The reported errors are computed using  $err(y) = \max_{0 \leq n \leq N} |y(t_n) - y_n|$  where  $y_n$  denotes the numerical approximation and  $y(t_n)$  denoted the exact solution at the time-step  $t_n$ .

h	k = 1		k = 2		k = 3		trapezoidal	
	$err(u_2)$	$err(i_V)$	$err(u_2)$	$err(i_V)$	$err(u_2)$	$err(i_V)$	$err(u_2)$	$err(i_V)$
0.1	$4.620 \times 10^{-2}$	$4.620 \times 10^{-2}$	$9.567 \times 10^{-3}$	$9.567 \times 10^{-3}$	$2.852 \times 10^{-3}$	$2.852 \times 10^{-3}$	$3.344 \times 10^{-3}$	$3.344 \times 10^{-3}$
0.05	$2.339 \times 10^{-2}$	$2.339 \times 10^{-2}$	$2.454 \times 10^{-3}$	$2.454 \times 10^{-3}$	$3.645 \times 10^{-4}$	$3.645 \times 10^{-4}$	$8.367 \times 10^{-4}$	$8.367 \times 10^{-4}$
0.025	$1.178 \times 10^{-2}$	$1.178 \times 10^{-2}$	$6.264 \times 10^{-4}$	$6.264 \times 10^{-4}$	$4.928 \times 10^{-5}$	$4.928 \times 10^{-5}$	$2.092 \times 10^{-4}$	$2.092 \times 10^{-4}$

**Table 5.2:** Resulting errors for the BDF-k methods and the trapezoidal rule.

## 5 Numerical Solutions

The errors presented in Table 5.2 confirm our results from Section 5.2.1 and 5.2.2, i.e. the BDF- $k$  schemes have convergence order  $p = k$  and the trapezoidal rule has convergence order  $p = 2$ .

### Example 2.5 (Numerical computation)

For this example we again consider the LC-circuit given in Figure 5.6.

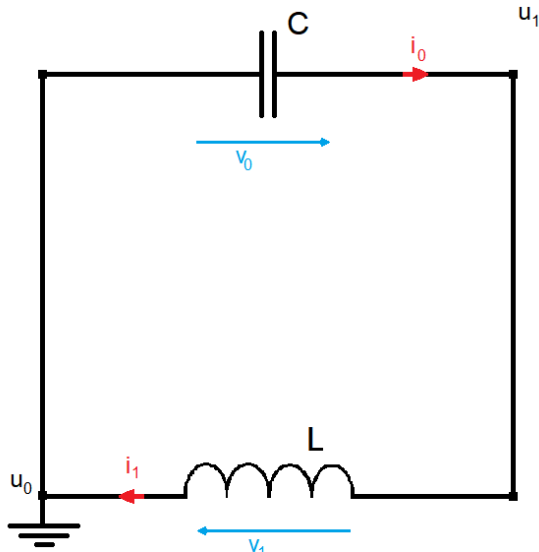


Figure 5.6: LC-circuit

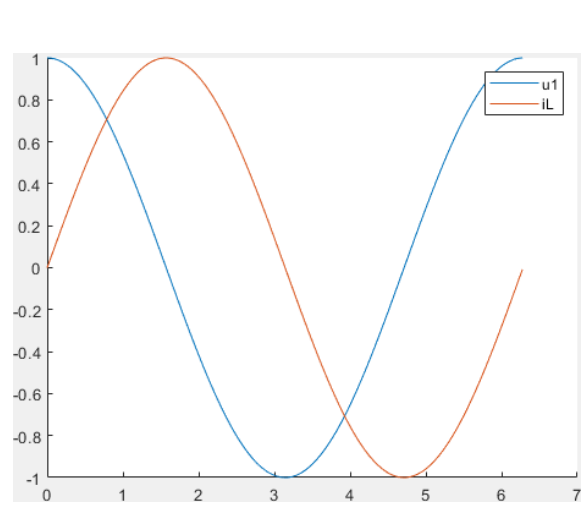


Figure 5.7: Exact solution for Example 2.

We again set  $L = 1$  and  $C = 1$ . The resulting MNA-system for this example then reads

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} u_1' \\ i_L' \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} * \begin{pmatrix} u_1 \\ i_L \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When applying the BDF- $k$  methods in Section 5.2.1 and the trapezoidal rule in Section 5.2.2 to this problem, we can observe the maximal error  $err(y) = \max_{0 \leq n \leq N} |y(t_n) - y_n|$ . The error is displayed in Table 5.3, where we again observe, that the errors reflect the predicted convergence rates.

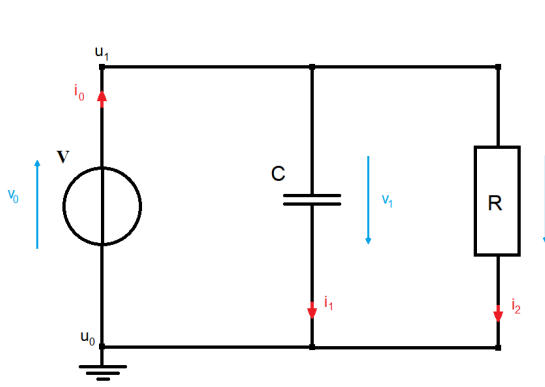
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h	k = 1		k = 2		k = 3		trapezoidal	
	$err(u_1)$	$err(i_l)$	$err(u_1)$	$err(i_l)$	$err(u_1)$	$err(i_l)$	$err(u_1)$	$err(i_l)$
0.1	$2.659 \times 10^{-1}$	$2.106 \times 10^{-1}$	$1.686 \times 10^{-2}$	$1.917 \times 10^{-2}$	$1.286 \times 10^{-3}$	$1.042 \times 10^{-3}$	$4.007 \times 10^{-3}$	$5.140 \times 10^{-3}$
0.05	$1.446 \times 10^{-1}$	$1.125 \times 10^{-1}$	$4.266 \times 10^{-3}$	$5.067 \times 10^{-3}$	$1.501 \times 10^{-4}$	$1.131 \times 10^{-4}$	$1.003 \times 10^{-3}$	$1.301 \times 10^{-3}$
0.025	$7.543 \times 10^{-2}$	$5.817 \times 10^{-2}$	$1.070 \times 10^{-3}$	$1.294 \times 10^{-3}$	$1.460 \times 10^{-5}$	$9.674 \times 10^{-6}$	$2.507 \times 10^{-4}$	$3.268 \times 10^{-4}$

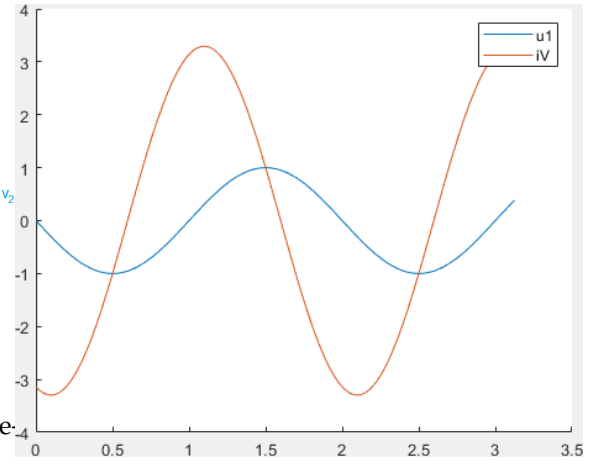
**Table 5.3:** Resulting errors for the BDF-k methods and the trapezoidal rule.

### Example 3.5 (Numerical computation)

In our last example we consider again the circuit given in Figure 5.8.



**Figure 5.8:** Current source with capacitor and resistor.



**Figure 5.9:** Exact solution for Example 3.

Setting  $R = 1$ ,  $C = 1$  and  $v_{src} = \sin(\pi t)$  the resulting system has the form

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} * \begin{pmatrix} u_1' \\ i_V' \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} u_1 \\ i_V \end{pmatrix} = \begin{pmatrix} 0 \\ -\sin(\pi t) \end{pmatrix}.$$

With this we expect the potential and current to be given as illustrated in Figure 5.9.

The numerical methods result in errors displayed in Table 5.4. Due to the fact, that  $u_1$  is again the same as  $-v_{src}$ , it is not of interest for our comparison of the errors, hence we omit  $u_1$ .

## 5 Numerical Solutions

h	k = 1 $err(i_V)$	k = 2 $err(i_V)$	k = 3 $err(i_V)$	trapezoidal $err(i_V)$
0.1	$4.894 \times 10^{-1}$	$1.023 \times 10^{-1}$	$2.403 \times 10^{-2}$	$5.219 \times 10^{-2}$
0.05	$2.462 \times 10^{-1}$	$2.577 \times 10^{-2}$	$3.034 \times 10^{-3}$	$1.295 \times 10^{-2}$
0.025	$1.233 \times 10^{-1}$	$6.456 \times 10^{-3}$	$4.029 \times 10^{-4}$	$3.232 \times 10^{-3}$

**Table 5.4:** Resulting errors for the BDF-k methods and the trapezoidal rule.

The errors rates are again within the expected range.

### 5.3.2 Results for the Gauss method

Observer that in the first example only the derivative of  $u_2$  appears, thus  $u_2$  is the differential variable while  $u_1$  and  $i_0$  are algebraic variables. For the second example both  $u_1$  and  $i_L$  are differential variables. In the third example  $u_1$  is differential and  $i_V$  is algebraic. We indicate this in Table 5.5 by “alg” and by “diff”.

h	example 1			example 2		example 3	
	$err(u_1)$ (alg)	$err(u_2)$ (diff)	$err(i_0)$ (alg)	$err(u_1)$ (diff)	$err(i_L)$ (diff)	$err(u_1)$ (diff)	$err(i_V)$ (alg)
0.1	$1.247 \times 10^{-2}$	$2.141 \times 10^{-3}$	$1.299 \times 10^{-2}$	$6.589 \times 10^{-3}$	$7.885 \times 10^{-3}$	$1.247 \times 10^{-2}$	$1.318 \times 10^{-1}$
0.05	$3.092 \times 10^{-3}$	$5.343 \times 10^{-4}$	$3.252 \times 10^{-3}$	$1.649 \times 10^{-3}$	$1.974 \times 10^{-3}$	$3.092 \times 10^{-3}$	$3.270 \times 10^{-2}$
0.025	$7.716 \times 10^{-4}$	$1.335 \times 10^{-4}$	$8.115 \times 10^{-4}$	$4.123 \times 10^{-4}$	$4.936 \times 10^{-4}$	$7.716 \times 10^{-4}$	$8.153 \times 10^{-3}$

**Table 5.5:** Resulting errors for the Gauss method with one stage.

Comparing the table of errors 5.5 with the table of convergence rates 5.1 we observe that firstly the method converges as expected with approximately order  $p = 2$  for the ODE in example 2. In example 1 and in example 3 on the other hand we can observe different rates for the algebraic and the differential variables. In particular example 3, which has index  $\nu = 2$  does not converge in its algebraic variables, in contrary, the error even grows.

## 5.4 Conclusion

The primary objective of this thesis was to explore the mathematical foundations of circuit modelling by means of the *Modified Nodal Analysis*. This was done by first formulating a mathematical model from a given circuit topology and then analyzing the resulting system of equations. This system of equations turned out to be of a *Differential Algebraic* structure. This information gave way to finding suitable numerical methods. These theoretical derivations were complemented by worked examples to illustrate the behaviour of different numerical methods when applied to Differential Algebraic systems.

From a mathematical standpoint this analysis is very useful as it can be completely automated. Also adding further components can be done within this setting, as it is described in [2].

## Code Listings

This chapter holds the code used in Section 5.3. The first listing uses the BDF-k method for  $k = 1, 2, 3$  to solve the given system. If not supplied with starting values, it calculates the starting values for higher stage methods using the BDF methods with less stages.

```
function y = BDFk(A, B, f, y0, tspan, h, k, yexact)
% BDF-k method for solving Ay' + By = f or Ay' = f - By or Ay' + By - f = 0
% A: Matrix
% B: Matrix
% f: vector-valued function f(t)
% y0: Initial condition - vector of size k
% tspan: Time interval [t0, tf]
% h: Step size
% k: indicates the method used
% yexact: exact solution as function - if given the initial values
% are taken directly from the exact solution
% output: vector y of size length(y0) x length(tspan(1):h:tspan(2))
% containing the calculated timesteps of the method

% Time vector
t = tspan(1):h:tspan(2);
n = length(t);

% Preallocate solution array (containing all the time steps)
y = zeros(length(y0), n);
for i=1:length(y0)
    y(i, 1) = y0(i, 1);
end

%construct initial k values using BDF1 and BDF2 method on finer grid
if k == 2; g = 2; elseif k == 3; g=4; end %factor by how much the grid is finer

if k ~= 1
    tempspan = [tspan(1), tspan(1)+h*(k-1)]; %yk is already calculated by BDFk method
    if nargin == 8
        y(:, 1:k) = yexact(tspan(1):h:(tspan(1) + h*(k-1)));
    else
        initial = BDFk(A, B, f, y0, tempspan, h/g, k-1);
        y(:, 1:k) = initial(:, 1:g:end);
    end
end

%apply appropriate bdf-scheme
if k == 1 %for k=1 we just implicit euler
    for i = k+1:n
        y(:, i) = (A+h*B) \ (A*y(:, i-1) + h*f(t(i)));
    end
elseif k == 2
    for i = k+1:n
        y(:, i) = (3*A+2*h*B) \ (-A*y(:, i-2) + 4*A*y(:, i-1) + 2*h*f(t(i)));
    end
elseif k == 3
    for i = k+1:n
        y(:, i) = (11*A+6*h*B) \ (18*A*y(:, i-1) - 9*A*y(:, i-2) + 2*A*y(:, i-3) + 6*h*f(t(i)));
    end
end
end
```



## 5 Numerical Solutions

The second listing implements the trapezoidal rule.

```
function y = trapezoidal(A, B, f, y0, tspan, h)
%TRAPEZOIDAL rule for solving  $Ay' + By = f$  or  $Ay' = f - By$  or  $Ay' + By - f = 0$ 
% A: Matrix
% B: Matrix
% f: vector-valued function f(t)
% y0: Initial condition - vector of size k
% tspan: Time interval [t0, tf]
% h: Step size
%output: vector y of size length(y0) x length(tspan(1):h:tspan(2))
%containing the calculated timesteps of the method

% Time vector
t = tspan(1):h:tspan(2);
n = length(t);

% Preallocate solution array (containing all the time steps)
y = zeros(length(y0), n);
for i=1:length(y0)
    y(i, 1) = y0(i, 1);
end

%time-stepping loop for Trapezoidal
for i = 2:n
    y(:, i) = (A+(h/2)*B) \ ((A-(h/2)*B)*y(:, i-1) + (h/2)*(f(t(i)) + f(t(i-1)))); %solve the implicit equation
end

end
```

## 5 Numerical Solutions

In the third listing the Gauss method with one stage is implemented.

```
function y = gauss1(A, B, f, y0, tspan, h)
%Gauss method with 1 stage for solving  $Ay' + By = f$  or  $Ay' = f - By$  or  $Ay' + By - f = 0$ 
% A: Matrix
% B: Matrix
% f: vector-valued function f(t)
% y0: Initial condition - vector of size k
% tspan: Time interval [t0, tf]
% h: Step size
%output: vector y of size length(y0) x length(tspan(1):h:tspan(2))
%containing the calculated timesteps of the method

% Time vector
t = tspan(1):h:tspan(2);
n = length(t);

% Preallocate solution array (containing all the time steps)
y = zeros(length(y0), n);
for i=1:length(y0)
    y(i, 1) = y0(i, 1);
end

%time-stepping loop for Gauss
for i = 2:n
    y(:, i) = (A+h*(1/2)*B) \ ((A-h*(1/2)*B)*y(:, i-1) + h*(f(t(i-1)+h/2))); %solve the implicit equation
end
end
```

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