Circuit Modelling

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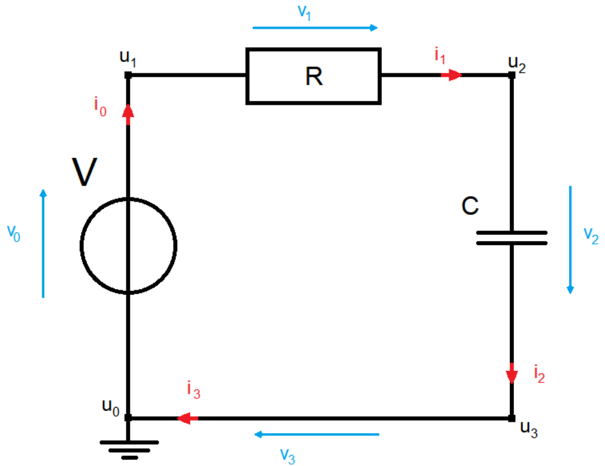
Introducing example: Charging of a capacitor:



Formulating a Mathematical Model

Network Topology

Define the incidence matrix $A = (a_{ij}) \in \mathbb{R}^{k \times l}$:

$$\tilde{a}_{ij} = \begin{cases} 1 & \text{edge } j \text{ starts at node } i, \\ -1 & \text{edge } j \text{ ends at node } i, \\ 0 & \text{else.} \end{cases}$$

With

$$\begin{aligned} \mathcal{N} &= (n_0, n_1, n_2, ..., n_k) \quad \cdots \quad \text{nodes,} \\ \mathcal{E} &= \{e_j : j = 1, ..., l\} \quad \cdots \quad \text{edges,} \end{aligned}$$

furthermore

$$u = (u_0, u_1, u_2, ...) \quad \cdots \quad \text{corresponding electrical potentials at the nodes.}$$

By grounding node 0, i.e. $u_0 = 0$ we obtain the reduced incidence matrix.

Formulating a Mathematical Model

Energy Conservation Laws

- **Kirchhoff's voltage law (KVL):**
  The sum of voltages along each loop of the network must equal to zero.

$$\rightarrow A^\top u = v. \tag{1}$$

- **Kirchhoff's current law (KCL):**
  For any node, the sum of currents flowing into the node is equal to the sum of currents flowing out of the node.

$$\rightarrow Ai = 0. \tag{2}$$

Formulating a Mathematical Model

Electrical Components and their Relations

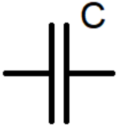
- **Resistor**

$$v = R\,i \quad \text{or} \quad i = G\,u. \tag{3}$$

Figure: resistor symbol

- **Capacitor**

$$Q = C\,v \quad \text{and by derivation in t} \quad I = C\,\frac{d}{dt}v = C\,v'. \tag{4}$$



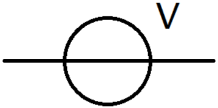
- **Inductor (Coil)**

$$\Phi = L\, i \quad \text{and by derivation in t} \quad v = L\, i'. \tag{5}$$

Figure: inductor symbol

- **Voltage Source**

$$v = v_{src} \tag{6}$$



- **Current Source**

$$i = i_{src} \tag{7}$$

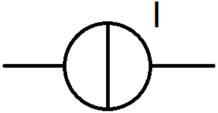


Figure: current source symbol

Formulating a Mathematical Model

Modified Nodal Analysis - MNA

Rearrange the columns of the reduced incidence matrix A into

$$A = (A_R A_C A_L A_V A_I)$$

A_R, A_C, A_L, A_V and A_I ... columns related to components
Represent voltages:

$$v = A^\top u$$

$\rightarrow$ rearrange v into $v = (v_R, v_C, v_L, v_{src}, v_I)$ and i into $i = (i_R, i_C, i_L, i_V, i_s rc)$. Rewrite component relations:

$$i_R = G\, v_R = G\, A_R^\top u,$$
$$i_C = C\, v_C' = C\, A_C^\top u'.$$

Kirchhoffs current law:

$$A_C i_C + A_R i_R + A_L i_L + A_V i_V = -A_I i_{src}.$$

Combine:

$$A_C C A_C^\top u' + A_R G A_R^\top u + A_L i_L + A_V i_V = -A_I i_{src}$$

Together with component law for inductors (5) and potential-voltage relation for voltage sources (6):

$$
\begin{aligned}
A_C C A_C^\top u' + A_R G A_R^\top u + A_L i_L + A_V i_V &= -A_I i_{src}, \\
L i_L' - A_L^\top u &= 0, \\
-A_V^\top u &= -v_{src}.
\end{aligned}
$$

In matrix form:

$$
\begin{pmatrix} A_C C A_C^\top & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{pmatrix} * \begin{pmatrix} u' \\ i_L' \\ i_V' \end{pmatrix} + \begin{pmatrix} A_R G A_R^\top & A_L & A_V \\ -A_L^\top & 0 & 0 \\ -A_V^\top & 0 & 0 \end{pmatrix} * \begin{pmatrix} u \\ i_L \\ i_V \end{pmatrix} = \begin{pmatrix} -A_I i_{src} \\ 0 \\ -v_{src} \end{pmatrix}. \tag{8}
$$

Differential Algebraic Equations

Types of DAEs

In the most general form a DAE can be written as: Find $y : \mathbb{R} \to \mathbb{R}^n$ such that

$$F(t, y(t), y'(t)) = 0, \qquad \forall t \in I \tag{9}$$

with $F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ sufficiently smooth and I the time-interval.

- **Linear systems with constant coeffiecients**
  find y such that
  $$Ay'(t) + By(t) = f(t), \tag{10}$$
  with $A, B \in \mathbb{R}^{n \times n}$, A singular, B regular and $f : \mathbb{R} \to \mathbb{R}^n$ a function in time.
- **Linear time dependent systems** are systems of the form: find y such that
  $$A(t)y'(t) + B(t)y(t) = f(t),$$
  with $A, B : \mathbb{R} \to \mathbb{R}^{n \times n}$, $f : \mathbb{R} \to \mathbb{R}^n$ functions, $\forall t \in \mathbb{R}$: $A(t)$ is singular and $B(t)$ regular.
- **Structured (non-linear) systems**
  are semi-explicit systems of the form: find (y, z) such that
  $$y'(t) = f(t, y(t), z(t)), \tag{11}$$
  $$0 = g(t, y(t), z(t)), \tag{12}$$
  with $f : \mathbb{R} \to \mathbb{R}^n$ and $g : \mathbb{R} \to \mathbb{R}^d$ functions.

Differential Algebraic Equations

Weierstrass-Kronecker normalform

prerequisites:

Definition

The matrix pencil $\{A, B\}$ is called *regular* if there exists some $c \in \mathbb{R}$, such that $(cA + B)$ is regular (i.e. $\det(cA + B) \neq 0$), otherwise it is called singular.

Theorem (Jordan Normalform)

For every matrix $Q \in \mathbb{R}^{n\times n}$ *there exists a regular matrix* $T \in \mathbb{C}^{n\times n}$*, such that*

$$T^{-1}QT = J = \mathrm{diag}(J_1, ..., J_r) \quad \textit{with} \quad J_i = \begin{pmatrix} \lambda_i & 1 & & 0 \\ 0 & \lambda_i & \ddots & \vdots \\ & \ddots & \ddots & 1 \\ 0 & \dots & 0 & \lambda_i \end{pmatrix} \in \mathbb{C}^{m_i\times m_i}$$

and $n = m_1 + ... + m_r$.

Theorem (Weierstrass-Kronecker normalform)

Let $\{A, B\}$ *be a regular matrix pencil. There exist* $P, Q \in \mathbb{C}^{n\times n}$ *such that*

$$PAQ = \begin{pmatrix} I_d & 0 \\ 0 & N \end{pmatrix}, \quad PBQ = \begin{pmatrix} R & 0 \\ 0 & I_{n-d} \end{pmatrix}$$

where

$$N = \mathrm{diag}(N_1, \ldots, N_r) \quad \textit{with} \quad N_i = \begin{pmatrix} 0 & 1 & & 0 & \\ & \ddots & \ddots & & \\ & & & 0 & 1 \\ 0 & & & 0 & \end{pmatrix} \in \mathbb{R}^{n_i \times n_i}$$

and R has Jordan Normalform. By I_k *we denote the identity matrix of size* $k \times k$*.*

using these findings: Using the findings above we are able to transform the initial DAE (10) using the matrix P from Theorem 3. By multiplying with P from the left, we obtain

$$PAy'(t) + PBy(t) = Pf(t).$$

Setting

$$y(t) = Q \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}, \quad Pf(t) = \begin{pmatrix} s(t) \\ q(t) \end{pmatrix},$$

with $u(t), s(t) : \mathbb{R} \to \mathbb{R}^d$ and $q(t), v(t) : \mathbb{R} \to \mathbb{R}^{n-d}$.

We get a system of the form

$$\begin{aligned} u'(t) + Ru(t) &= s(t), \\ Nv'(t) + v(t) &= q(t), \end{aligned} \tag{13}$$

where $PAQ = \begin{pmatrix} I & \\ & N \end{pmatrix}$ and $PBQ = \begin{pmatrix} R & \\ & I \end{pmatrix}$.

$$
\begin{aligned}
v(t) &= q(t) - Nv'(t) = q(t) - N(\underbrace{q(t) - Nv'(t)}_{=v(t)})' = q - Nq' + N^2 v'' \\
&= q - Nq' + N^2(q - Nv')'' = q - Nq' + N^2 q'' - N^3 v''' \\
&\vdots \\
&= q - Nq' + \ldots + (-1)^{k-1} N^{k-1} \underbrace{\frac{d^k}{dt^k} q}_{:= q^{(k-1)}} + (-1) \underbrace{N^k v^{(k)}}_{=0} \\
&= \sum_{i=0}^{k-1} (-1)^i N^i q^{(i)}(t)
\end{aligned} \tag{14}
$$

where k is the nilpotency index of N.

Definition

The nilpotency index k of the matrix N from the Weierstraß-Kronecker Normalform of a matrix pencil $\{A, B\}$ with A singular is called the *Kronecker-Index* of $\{A, B\}$, which we denote by $\mathrm{ind}\{A, B\}$. Note that for A regular we set $\mathrm{ind}\{A, B\} = 0$.

Differential Algebraic Equations

Index of a Differential Algebraic Equation

Definition

Consider the differential algebraic equation (9) to be uniquely locally solvable and F sufficiently smooth. For a given $m \in \mathbb{N}$ consider the equations

$$\begin{aligned} F(t, y, y') &= 0, \\ \frac{dF(t, y, y')}{dt} &= 0, \\ &\vdots \\ \frac{d^m F(t, y, y')}{dt^m} &= 0. \end{aligned}$$

The smallest natural number m for which the above system results in an explicit system of ordinary differential equations (ODEs), i.e. it has the form

$$y' = \phi(t, y),$$

Definition

Let $y(t)$ be the exact solution of (9) and $\tilde{y}(t)$ be the solution of the perturbed system $F(t, \tilde{y}, \tilde{y}') = \delta(t)$. The smallest number $k \in \mathbb{N}$ such that

$$\|y(t) - \tilde{y}(t)\| \leq C \left(\|y(t_0) - \tilde{y}(t_0)\| + \sum_{j=0}^{k} \max_{t_0 \leq \xi \leq T} \left\| \int_{t_0}^{\xi} \frac{d^j \delta}{d\tau^j}(\tau) d\tau \right\| \right)$$

for all $\tilde{y}(t)$, is called the **perturbation index** of this system.

Differential Algebraic Equations

Consistent Initial Values

index $\nu = 0$.

Case: Index $\nu = 1$.

By rewriting our system into the form

$$\begin{aligned} y'(t) &= f(t, y(t), z(t)), \\ 0 &= g(t, y(t), z(t)). \end{aligned}$$

we are able to give conditions for consistent initial values. Namely y_0 and z_0 are consistent initial values for this system, if $g(t_0, y_0, z_0) = 0$ holds.

Case: Index $\nu = 2$.
For index-2 systems we rewrite our system into

$$\begin{aligned} y' &= f(t, y(t), z(t)), \\ 0 &= g(t, y(t)). \end{aligned}$$

Consistent initial values y_0, z_0 for this case not only have to fulfill $g(t_0, y_0) = 0$ but also the *hidden constraint* $g_t(t_0, y_0) + g_y(t_0, y_0) f(t_0, y_0, z_0)$. By g_t and g_y we denote the derivative of g with respect to t or y, respectively.

Index Analysis of the Modified Nodal Analysis

General Index Analysis

content...

Index Analysis of the Modified Nodal Analysis

Topological Conditions

Theorem (Index conditions)

Let the matrices of the capacitances, inductances and resistances be positive definite.

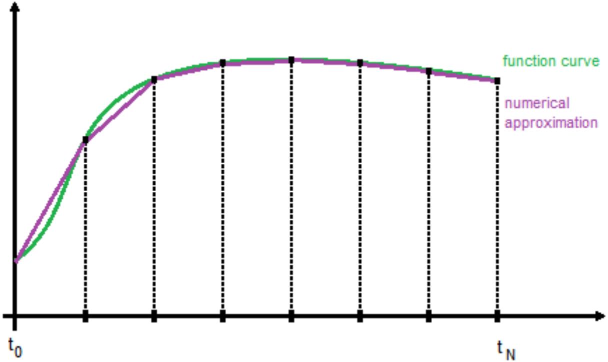
- *If*
$$\ker([A_R, A_C, A_V, A_L]^\top) = \{0\} \quad \textit{and} \quad \ker(A_V) = \{0\} \tag{15}$$
holds, then the MNA (8) *leads to a system with index* $\nu \leq 2$.
- *If additionally*
$$\ker([A_R, A_C, A_V]^\top) = \{0\} \quad \textit{and} \quad \ker([A_C, A_V]) = \{0\} \tag{16}$$
holds, then the system is of index $\nu \leq 1$
- *If further*
$$\ker(A_C^\top) = \{0\} \quad \textit{and} \quad \dim(v_{src}) = 0 \tag{17}$$
holds, then the system has index $\nu = 0$.

- Condition (15) can be interpreted, as the circuit neither containing loops of voltage sources nor cutsets of current sources.
- Condition (16) can be interpreted, as the circuit containing neither loops of capacitors and/or voltage sources nor cutsets of inductors and/or current sources.
- Condition (17) can be interpreted, as every node in the circuit being connected to the reference node (ground) through a path containing only the capacitors.

general initial value problem Find y, such that

$$y'(t) = f(t, y), \quad t \in [t_0, t_l], \tag{18}$$

$$y(t_0) = y_0. \tag{19}$$



Numerical Solutions

Single-Step Methods

Definition

A numerical method to approximate a differential equation 18 on a time-grid $t_0, ..., t_l$ with the intermediate values $y_0, ..., y_l$ is called a single-step method if it is of the form

$$y_{j+1} = y_j + h_j \phi(t_j, y_j, y_{j+1}, h_j). \tag{20}$$

We call ϕ the *procedural function*. If ϕ does not depend on y_{j+1}, then the method is called *explicit*, otherwise it is called *implicit*.

Numerical Solutions

Single-Step Methods

Consistency, Stability and Convergence

Definition

Let y_{m+1} be the result of one step of a single step method (20) with the exact start-vector $y_m = y(t_m)$ then

$$\delta_{m+1} = \delta(t_m + h) = y(t_{m+1}) - \tilde{y}_{m+1}, \quad m = 0, ..., N-1 \tag{21}$$

is called the *local discretization error* of the single step method at the point t_{m+1}.

Definition

A single-step method is called *consistent* if for all initial value problems (18)

$$\lim_{h \to 0} \frac{\|\delta(t+h)\|}{h} = 0 \quad \text{for} \quad t_0 \le t \le t_l \tag{22}$$

holds.
It is called *consistent of order p*, if for a sufficiently smooth function f

$$\|\delta(t+h)\| \le Ch^{p+1} \quad \text{for all} \quad h \in (0, H] \quad \text{and} \quad t_0 \le t \le t_l - h \tag{23}$$

holds with C independent of h.

Definition

A single-step method is called *convergent*, if for all initial value problems (18) for the *global discretization error*

$$e_m = y(t_m) - y_m$$

holds that

$$\max_m \|e_m\| \to 0 \quad \text{for} \quad h_{max} \to 0.$$

The single-step method is called to have the *convergence order* p, if

$$\max_m \|e_m\| \leq Ch_{max}^p \quad \text{for} \quad h_{max} \in (0, H] \quad \text{with} \quad t_0 \leq t_m \leq t_l$$

with the constant C not dependent on the step size h.

Definition

A single-step method is called *(discretely) stable* if for grid-functions y_h and $\tilde{y}_h$ with

$$y_{i+1} = y_i + h\phi(t_i, y_i), \tag{24}$$

$$\tilde{y}_{i+1} = \tilde{y}_i + h[\phi(t_i, \tilde{y}_i) + \theta_i], \tag{25}$$

and perturbations $\theta_i = \theta_h(t_i)$ of the right side as well as a bounded perturbation in the initial-values $y_0 - \tilde{y}_0$ the error is bounded by

$$\|y_h - \tilde{y}_h\|_{\infty,h} \leq C(\|y_0 - \tilde{y}_0\|_{l^2} + \|\theta_h\|_{\infty,h})$$

with a constant C which is not dependent on h. The norm $\|.\|_{\infty,h}$ denotes the maximum norm over the time-grid, i.e. for a function $b : T = t_0, ..., t_N \to \mathbb{R}^d$ we have $\|b\|_{\infty,h} = \max_{t \in T} \|b(t)\|$, $\|b\|$ is the euclidean norm.

Numerical Solutions

Single-Step Methods

further stability properties

Dahlquist equation, i.e. find y such that

$$y' = \lambda y, \quad t > 0 \tag{26}$$

$$y(0) = y_0 \tag{27}$$

with $\lambda \in \mathbb{C}$ and u_0 fixed.

Definition

1. If a single-step method can be written in the form

$$y_{i+1} = R(z)\, y_i, \quad z := h\lambda \tag{28}$$

   then we call $R : \mathbb{C} \to \mathbb{C}$ the *stability function* of the single-step method.

2. The set

$$S := \{z \in \mathbb{C} : |R(z)| \leq 1\} \tag{29}$$

   is called the *region of stability* of the method.

3. A single-step method is called
   - *0-stable*, if $0 \in S$.
   - *A-stable*, if $\mathbb{C}^- \subset S$.
   - *L-stable*, if $R(z) \to 0$ for $Re(z) \to -\infty$.

Numerical Solutions

Multistep Methods

Definition

For given $\alpha_0, ..., \alpha_k$ and $\beta_0, ..., \beta_k$ the iteration rule

$$\sum_{l=0}^{k} \alpha_l y_{m+l} = h \sum_{l=0}^{k} \beta_l f(t_{m+l}, y_{m+l}), \quad m = 0, 1, ..., N-k \tag{30}$$

is called a *linear multistep method* (linear k-step method). It is always assumed that $\alpha_k \neq 0$ and $|\alpha_0| + |\beta_k| > 0$. If $\beta_k = 0$ holds, then the method is called explicit, otherwise implicit.

Numerical Solutions

Multistep Methods

Consistency, Convergence and Stability

Definition

Let y_{m+k} be the result of one step of the multi-step method (30) with the start-values given as the evaluations of the exact solution $y_{m+l} = y(t_{m+l})$ at $0 \leq l < k$. This means

$$\alpha_k \tilde{u}_{m+k} = \sum_{l=0}^{k-1} \left(h\beta_l f(t_{m+l}, y(t_{m+l})) - \alpha_l y(t_{m+l})\right) + h\beta_k f(t_{m+k}, y_{m+k}).$$

Then

$$\delta_{m+k} = \delta(t_{m+k}) = y(t_{m+k}) - y_{m+k}, \quad m = 0, 1, ..., N-k$$

is called the *local discretization error* (local error) of the linear multi-step method, see Def. 30 at the point t_{m+k}.

Definition

A linear multi-step method is called *consistent*, if for all functions $y(t) \in C^2([t_0, t_l])$

$$\lim_{h \to 0} \frac{1}{h} L[y(t), h] = 0$$

holds. It has the *consistency order p*, if for all functions $y(t) \in C^{p+1}[t_0, t_l]$

$$L[y(t), h] = \mathcal{O}(h^{p+1}) \quad \text{for} \quad h \to 0$$

holds.

Definition

We say that a linear multi-step method is convergent if for a solution y of the problem a solution vector created by an LMSM y_j for $j \in 0, ..., k$ we have that

$$\lim_{h\to\infty} \max_{0\leq j\leq k} \|y(t_j) - y_j\| = 0.$$

Definition

A linear multi-step method is called (discretely) stable, if for solutions y_h and $\tilde{y}_h$ of

$$\sum_{l=0}^{k} \alpha_l y_{m+l} = h \sum_{l=0}^{k} \beta_l f(t_{m+l}, y_{m+l}), \tag{31}$$

$$\sum_{l=0}^{k} \alpha_l \tilde{y}_{m+l} = h \sum_{l=0}^{k} \beta_l f(t_{m+l}, \tilde{y}_{m+l}) + h\theta_n \tag{32}$$

and bounded initial values $y_j - \tilde{y}_j$ for $j \in 0, ..., k$ we have that

$$\max_{t_0 \leq t_n \leq T} \|y_n - \tilde{y}_n\| \leq C \sum_{j=0}^{k-1} \|y_j - \tilde{y}_j\| + \max_{t_0 \leq t_n \leq T} \|\theta_n\|.$$

Numerical Solutions

Multistep Methods

further stability properties

Dahlquist test problem as a model problem, find y such that

$$y' = \lambda y, \quad t > 0 \tag{33}$$

$$y(0) = y_0 \tag{34}$$

with $\lambda \in \mathbb{C}$ and y_0 fixed.
Thus the resulting linear multistep method is of the form

$$\sum_{l=0}^{k} \alpha_l y_{n+l} = h \sum_{l=0}^{k} \beta_l \lambda y_{n+l}$$

$$\iff \sum_{l=0}^{k} [\alpha_l - h\beta_l \lambda] y_{n+l}$$

Definition

1. The set
$$S := \{z \in \mathbb{C} : \rho(\xi) - z\sigma(\xi) = 0 \implies \xi \in \mathbb{C} \text{ and } |\xi| \leq 1. \\ \text{If } \xi \text{ has multiplicity greater than 1, then } |\xi| < 1\} \tag{35}$$
is called the region of stability of the method.
2. A linear multistep method is called
   - *0-stable*, if $0 \in S$.
   - stable in the point $z \in \mathbb{C}$, if $z \in S$.
   - $A(\alpha)$*-stable*, if it is stable in all z that lie within the set $\{z \in \mathbb{C}^- : |arg(z) - \pi| \leq \alpha\}$ for $\alpha \in (0, \frac{\pi}{2})$.

Theorem

Let $f(t, y)$ be sufficiently smooth and the linear multi-step method be zero-stable and consistent of order p, then it is also convergent of order p.

Numerical Solutions

Implicit Linear Multistep Formulas

Numerical Solutions

Implicit Linear Multistep Formulas

BDF-k Methods

The *backward differentiation formula (BDF)* is a family of implicit linear multistep methods. They have the general form

$$\sum_{k=0}^{s} \alpha_k y_{n+k} = h\beta f(t_{n+s}, y_{n+s}) \tag{36}$$

The BDF or BDF-k formulas for $k = 1, ..., 3$ have the following form

$$\begin{aligned}
k = 1 : hf_{m+1} &= y_{m+1} - y_m \\
k = 2 : hf_{m+2} &= \frac{1}{2}(3y_{m+2} - 4y_{m+1} + y_m) \\
k = 3 : hf_{m+3} &= \frac{1}{6}(11y_{m+3} - 18y_{m+2} + 9y_{m+1} - 2y_m)
\end{aligned}$$

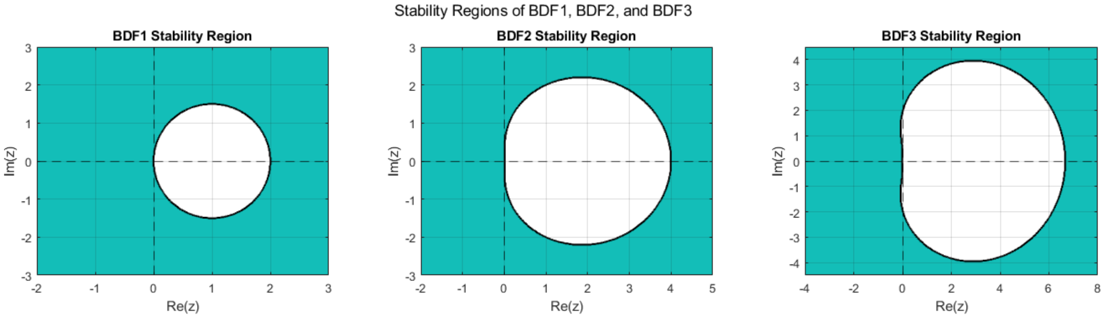


Figure: stability regions of BDF-schemes

Theorem

The BDF-k methods have consistency order $\mathrm{p} = \mathrm{k}$.

Numerical Solutions

Implicit Linear Multistep Formulas

Trapezoidal rule

This procedure is repeated for small subsections of the interval $[a, b]$. Thus we obtain the iteration formula

$$u_h(t+h) = u_h(t) + \frac{h}{2}[f(t, u_h(t)) + f(t+h, u_h(t+h))].$$

Numerical Solutions

Numerical Examples

Example1

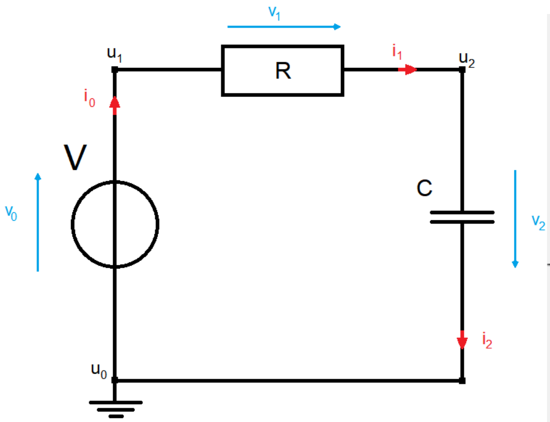


Figure: charging capacitor with series resistor and voltage source

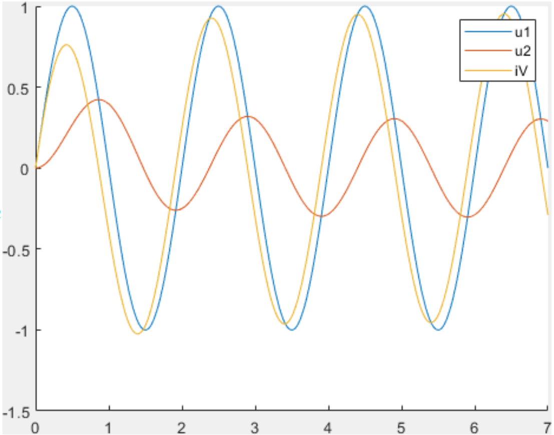


Figure: Exact solution for example 1.

h	k = 1		k = 2		k = 3		trapezoidal	
	u2	iV	u2	iV	u2	iV	u2	iV
0.1	4.620×10^{-2}	4.620×10^{-2}	9.567×10^{-3}	9.567×10^{-3}	3.057×10^{-3}	3.057×10^{-3}	3.344×10^{-3}	3.344×10^{-3}
0.05	2.339×10^{-2}	2.339×10^{-2}	2.454×10^{-3}	2.454×10^{-3}	6.083×10^{-4}	6.083×10^{-4}	8.367×10^{-4}	8.367×10^{-4}
0.025	1.178×10^{-2}	1.178×10^{-2}	6.264×10^{-4}	6.264×10^{-4}	1.672×10^{-4}	1.672×10^{-4}	2.092×10^{-4}	2.092×10^{-4}

Table: Resulting errors for the BDF-k methods and ther trapezoidal rule.

Numerical Solutions

Numerical Examples

Example 2

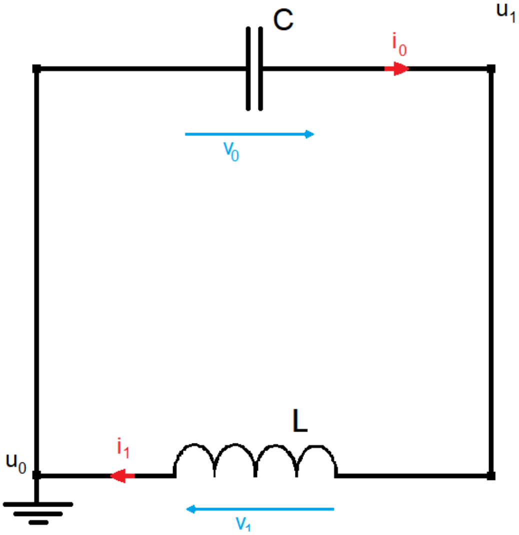


Figure: LC circuit

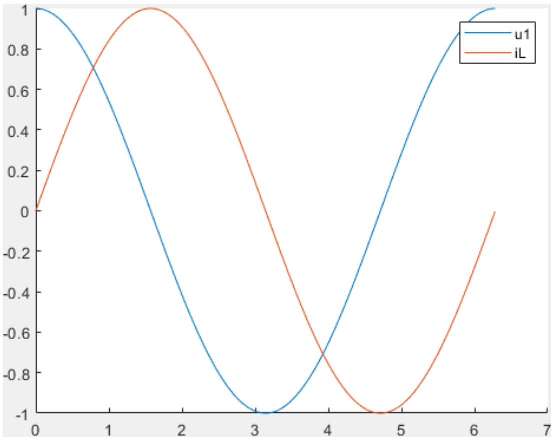


Figure: Exact solution for example 2.

h	k = 1		k = 2		k = 3		trapezoidal	
	u1	iL	u1	iL	u1	iL	u1	iL
0.1	7.145×10^{-1}	6.905×10^{-1}	7.763×10^{-2}	8.060×10^{-2}	5.395×10^{-3}	5.180×10^{-3}	1.963×10^{-2}	2.087×10^{-2}
0.05	4.659×10^{-1}	4.448×10^{-1}	1.964×10^{-2}	2.066×10^{-2}	5.938×10^{-4}	5.579×10^{-4}	4.912×10^{-3}	5.224×10^{-3}
0.025	2.695×10^{-1}	2.551×10^{-1}	4.924×10^{-3}	5.216×10^{-3}	5.773×10^{-5}	4.740×10^{-5}	1.228×10^{-3}	1.308×10^{-3}

Table: Resulting errors for the BDF-k methods and ther trapezoidal rule.

Numerical Solutions

Numerical Examples

Example 3

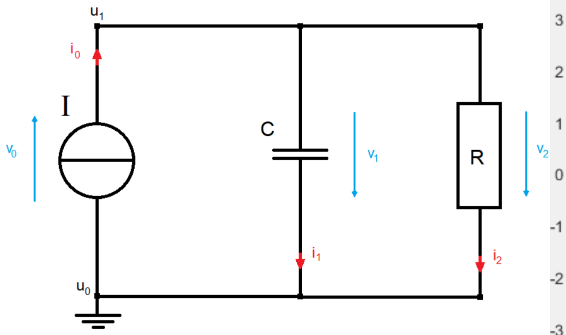


Figure: Current source with capacitor and resistor.

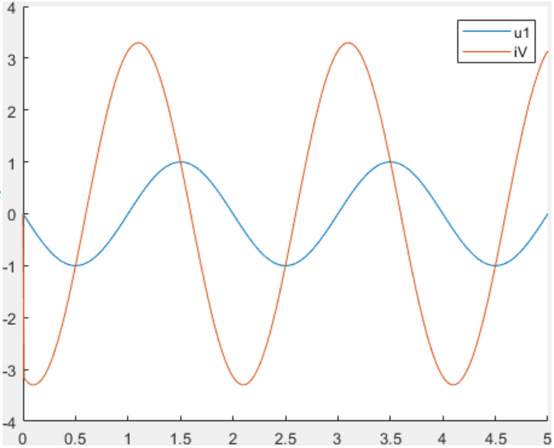


Figure: Exact solution for example 3.

h	k = 1 iV	k = 2 iV	k = 3 iV	trapezoidal iV
0.1	4.894×10^{-1}	1.023×10^{-1}	2.530×10^{-2}	5.219×10^{-2}
0.05	2.462×10^{-1}	2.577×10^{-2}	6.426×10^{-3}	1.295×10^{-2}
0.025	1.233×10^{-1}	6.456×10^{-3}	1.613×10^{-3}	3.232×10^{-3}

Table: Resulting errors for the BDF-k methods and the trapezoidal rule.