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# Circuit Modelling



Subtitle / Untertitel

# Kurzfassung

Kapitel zwei gibt Überblick über die Physikalischen Zusammenhänge und deren Mathematische Beschreibung. Anschließend wird ein Mathematisches Modell nach der Methode der "Modified Nodal Analysis" erstellt. Im dritten Kapitel analysieren wir die Systeme, welche durch diese Analyse entstehen, wobei das vierte Kapitel eine wichtige Eigenschaft dieser Systeme genauer untersucht. Das letzte Kapitel beschäftigt sich mit der Numerischen Lösung dieser Systeme.

→ je etwas mehr Text zu den Kapiteln (Struktur der Kapitel)

# **Abstract**

Abstract in English.

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# 1 Introduction

This chapter should include information about what circuit modelling wants to achieve as well as giving an overview of what this bachelor-thesis is about.

What is this thesis about?

Modelling and numerically solving systems that arise from electrical circuits with RLC elements. Furthermore it will briefly discuss on expanding this baseline with more complicated electrical components. What is the goal of this thesis?

The goal of this thesis is to give insight into the state of the art of circuit modelling. It aims to elaborate on the underlying concepts of MNA as well as on the most commonly used numerical methods.

## 2 Formulating a Mathematical Model

based on DAE lecture and [2]

To accurately represent a physical system in a mathematical model we first have to think about **how** to best formulate a mathematical model that formulates this system precise enough. This chapter aims to elaborate on the Modified Nodal Analysis (MNA) of an electrical circuit, which is the a common choice for these kinds of models. For this we have to introduce the notions of network topology as well as some basic electrical components and physical laws. Most of the following notions such as voltage, current and electrical potentials can depend on time. For better readability we will leave out the time argument  $t$  in some cases.

### 2.1 Network Topology

An electrical circuit is usually considered as a graph  $(\mathcal{N}, \mathcal{E})$  where  $\mathcal{N} = (n_0, n_1, n_2, \dots, n_k)$  denotes nodes and  $\mathcal{E} = \{e_j : j = 1, \dots, l\}$  is the set of edges where  $|\mathcal{N}| = k$  is the number of nodes and  $|\mathcal{E}| = l$  the number of edges. If for some  $i$  and some  $j$  the nodes  $n_i$  and  $n_j$  are connected, then there exists an edge connecting those two nodes. We can store this information in an *incidence matrix*  $\tilde{A} = (\tilde{a}_{ij}) \in \mathbb{R}^{k \times l}$  which is defined by

$$\tilde{a}_{ij} = \begin{cases} 1 & \text{edge } j \text{ starts at node } i, \\ -1 & \text{edge } j \text{ ends at node } i, \\ 0 & \text{else.} \end{cases}$$

We call  $u = (u_0, u_1, u_2, \dots)$  the corresponding electrical potentials (or just potentials) to the nodes  $\mathcal{N}$ . The difference between the potentials at two connected nodes is called the voltage at the respective edge. To fix the absolute values of these potentials we have



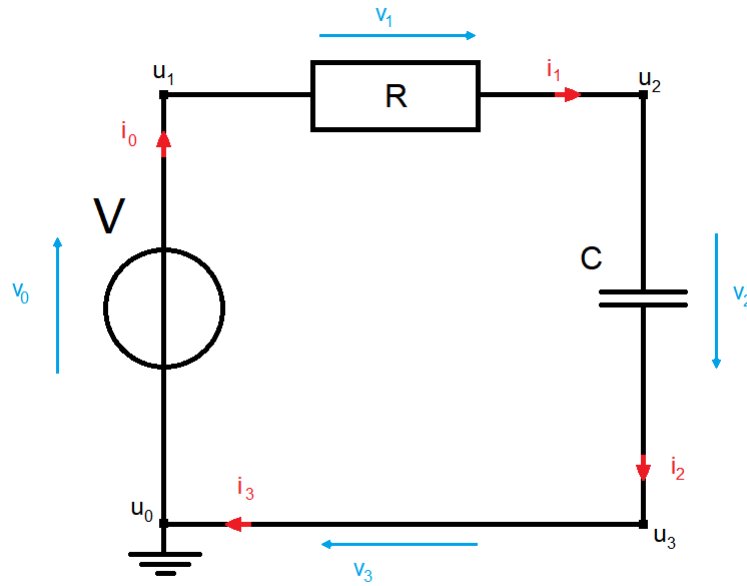
## 2 Formulating a Mathematical Model

to set one node to a fixed value. We will do that by “grounding” the node  $n_0$ , which means we set the potential  $u_0 = 0$ . The grounding of a node allows us to remove the corresponding row from the incidence matrix to get the *reduced incidence matrix*  $A$ . The vector  $v = (v_{ij})_{ij}$  represents the voltages at the edges. For some  $i$  and some  $j$  the voltage at edge  $ij$  is  $v_{ij} = u_i - u_j$ .

We will later see, that the components of an electrical circuit, which will be installed along the edges, describe a relationship between the current along the edge the corresponding voltage. Thus a current vector  $i = (i_1, i_2, i_3, \dots)$  containing the currents along the edges is required.

### Example 1

As an example we consider the charging of a capacitor. The circuit is given in Figure 2.1.



**Figure 2.1:** charging capacitor with series resistor and voltage source

## 2 Formulating a Mathematical Model

With the node-potentials, the voltages and the currents collected in the vectors

$$u = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad v = \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad i = \begin{pmatrix} i_0 \\ i_1 \\ i_2 \\ i_3 \end{pmatrix},$$

the incidence matrix of this circuit has the form

$$\tilde{A} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

The rows of this matrix correspond to the nodes of the circuit and the columns correspond to the edges and tell us if the edge either starts (1) or end (−1) at a node. By grounding node  $u_0$  this matrix reduces to

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

In this circuit edge 3 is not populated with any components. This means that the voltage along this edge does not change, or in terms of potentials

$$u_0 - u_3 = 0.$$

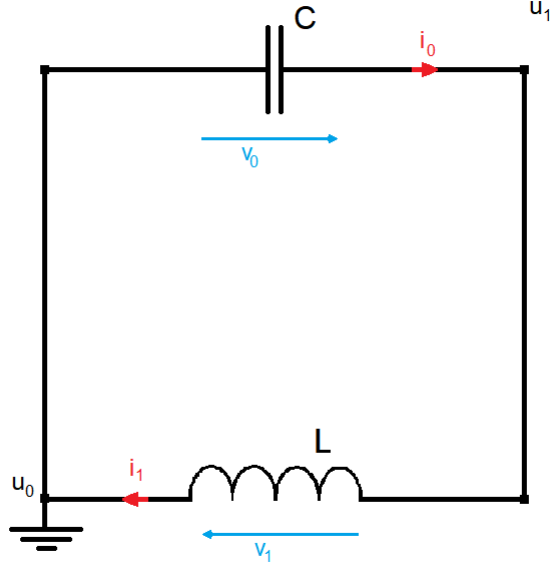
Thus we can consider this circuit with node 0 and node 3 merged. This leads to a slightly different incidence and reduced incidence matrix, i.e.

$$\tilde{A} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad \text{respectively.}$$

### Example 2

Another example is the a circuit that only contains a capacitor and an inductor, see Figure ???. This circuit will oscillate, if given appropriate initial conditions.

## 2 Formulating a Mathematical Model



**Figure 2.2:** capacitor and inductor oscillator

We again collect the node-potentials, the voltages and the currents in the vectors

$$u = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \quad v = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}, \quad i = \begin{pmatrix} i_0 \\ i_1 \end{pmatrix}.$$

The incidence matrix of this circuit has the form

$$\tilde{A} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

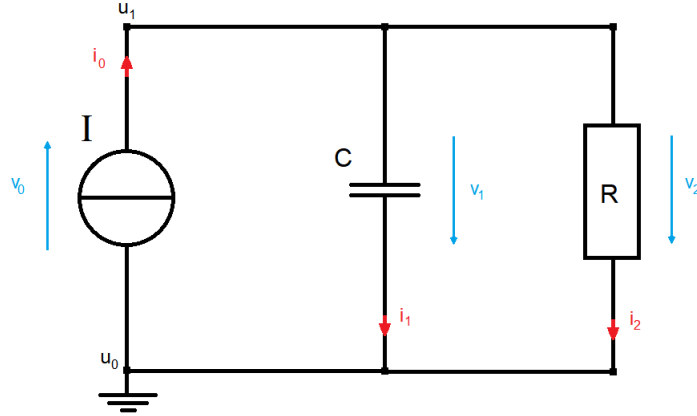
By grounding node  $u_0$  this can again be reduced, we obtain the reduced incidence matrix

$$A = \begin{pmatrix} -1 & 1 \end{pmatrix}.$$

### Example 3

As a third example we consider a similar circuit to the circuit presented in **Example 1**, the only difference is that we connected the resistor parallel to the capacitor instead of in series.

## 2 Formulating a Mathematical Model



**Figure 2.3:** Capacitor with resistor and voltage source

We collect the node-potentials, the voltages and the currents collected in the vectors

$$u = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \quad v = \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix}, \quad i = \begin{pmatrix} i_0 \\ i_1 \\ i_2 \end{pmatrix}.$$

The incidence matrix of this circuit has the form

$$\tilde{A} = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \end{pmatrix},$$

this leads to the reduced incidence matrix

$$A = \begin{pmatrix} -1 & 1 & 1 \end{pmatrix}.$$

## 2.2 Energy Conservation Laws

To fully fix all the variables that arise in the model of an electrical circuit we will need some *conservation laws*. In this context, conservation laws are a description of physical properties

## 2 Formulating a Mathematical Model

that can be observed in electrical circuits. This means that they have to be fulfilled non the less, thus including them into the system also makes sense from a physical perspective.

- **Kirchhoff's voltage law (KVL):**

The sum of voltages along each loop of the network must equal to zero. Using the incidence matrix  $A$  this law can be formulated as

$$A^T u = v. \quad (2.1)$$

- **Kirchhoff's current law (KCL):**

For any node, the sum of currents flowing into the node is equal to the sum of currents flowing out of the node. Using the incidence matrix  $A$  again, this law can be formulated as

$$Ai = 0. \quad (2.2)$$

### 2.3 Electrical Components and their relations

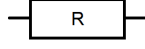
Electrical components are described by equations relating their edge voltage  $v$  to their edge current  $i$ . We will mainly focus on so-called RLC-networks (short for Resistor, Inductor, Capacitor), which consist of resistors, capacitors, inductors, voltage sources and current sources. Diodes and transistors as well as other electrical components can be described in a similar way, although these lead to a more difficult analysis of the system, see e.g. [2] and [1]. Using the notation introduced in Section 2.1, the relations read as:

- **Resistor**

Resistors "resist" the flow of current, which causes voltage to drop. This behaviour is described by the *resistance*  $R \in \mathbb{R}^+ := \{x \in \mathbb{R} : x > 0\}$  which is given in *Ohm* ( $\Omega$ ) and its reciprocal, the *conductance*  $G \in \mathbb{R}^+$ , which is given in *Siemens* ( $S = \frac{1}{\Omega}$ ).

$$v = R i \quad \text{or} \quad i = G u. \quad (2.3)$$

## 2 Formulating a Mathematical Model

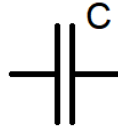


**Figure 2.4:** resistor symbol

- **Capacitor**

Capacitors “store” electrical energy by accumulating electrical charge. Their characteristic equations can be described directly using the stored charge  $Q \in \mathbb{R}_0^+ := \{x \in \mathbb{R} : x \geq 0\}$  or indirectly using the change in charge, which is nothing other than the current  $I$ . The *capacitance*  $C \in \mathbb{R}^+$  is given in *Farads* (F).

$$Q = C v \quad \text{and by derivation in } t \quad I = C \frac{d}{dt} v = C v'. \quad (2.4)$$



**Figure 2.5:** capacitor symbol

- **Inductor (Coil)**

An electric current flowing through a conductor generates a magnetic field  $\Phi \in \mathbb{R}^{n \times n}$  surrounding it. This magnetic field causes a voltage drop dependant on the change in current. The *inductance*  $L \in \mathbb{R}^+$  is given in *Henry* (H).

$$\Phi = L i \quad \text{and by derivation in } t \quad v = L i'. \quad (2.5)$$



**Figure 2.6:** inductor symbol

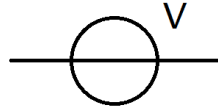
- **Voltage Source**

A voltage source supplies the system with a voltage. It can either supply varying

## 2 Formulating a Mathematical Model

amounts of voltage (with the special case of alternating current AC) or a fixed amount of voltage. The unit of voltage is *Volts* (V).

$$v = v_{src} \quad (2.6)$$

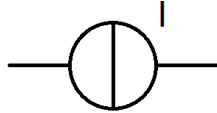


**Figure 2.7:** voltage source symbol

- **Current Source**

A current source supplies the system with current. It can either supply varying amounts of current (with the special case of alternating current AC) or a fixed amount of current. The unit of current is *Ampere* (A).

$$i = i_{src} \quad (2.7)$$



**Figure 2.8:** current source symbol

The resistance  $R$ , conductance  $G$ , capacitance  $C$  and inductance  $L$  are respectively described as a scalar constant that relates the edge-current to the edge-voltage. If there are more components of the same kind in one circuit their corresponding constants will usually be collected into a matrix which is also called the same and also denoted by the same letter, respectively. These matrices are then positive definite diagonal matrices.

## 2.4 Modified Nodal Analysis - MNA

[2] and [4]

To analyse the network further we will rearrange the columns of the reduced incidence matrix  $A$  such that it has the block form

$$A = (A_R A_C A_L A_V A_I)$$

where  $A_R$ ,  $A_C$ ,  $A_L$ ,  $A_V$  and  $A_I$  include the columns that are related to the resistors, capacitors, coils, voltage sources and current sources, respectively.

To mathematically describe the circuit we will use *modified nodal analysis* (or short MNA), see [3].

MNA uses the node voltages as well as the currents of the coils and the voltage sources as unknowns. It is based on the conservation laws (2.2) and (2.1) as well as on the voltage-current relations of the electrical components, discussed in Section ?? . The voltages can be represented using the node-potentials

$$v = A^\top u$$

The vector  $v$  can thus be rearranged into  $v = (v_R, v_C, v_L, v_{src}, v_I)$ . In a similar way we also rearrange the current vector into  $i = (i_R, i_C, i_L, i_V, i_{src})$ . Using the sorted incidence matrix blocks we can rewrite the resistor current relation as

$$i_R = G v_R = G A_R^\top u.$$

Analogously, we rewrite the capacitor relation as

$$i_C = C v'_C = C A_C^\top u'.$$

Kirchhoffs current law (2.2) gives us that

$$A_C i_C + A_R i_R + A_L i_L + A_V i_V = -A_I i_{src}.$$



## 2 Formulating a Mathematical Model

We plug in the component relations derived above and obtain

$$A_C C A_C^\top u' + A_R G A_R^\top u + A_L i_L + A_V i_V = -A_I i_{src}.$$

Combining this with the component law for inductors (2.5) and the potential-voltage relation for voltage sources (2.6) we finally get the modified nodal analysis equations

$$\begin{aligned} A_C C A_C^\top u' + A_R G A_R^\top u + A_L i_L + A_V i_V &= -A_I i_{src}, \\ L i_L' - A_L^\top u &= 0, \\ -A_V^\top u &= -v_{src}. \end{aligned}$$

In matrix form these read

$$\begin{pmatrix} A_C C A_C^\top & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{pmatrix} * \begin{pmatrix} u' \\ i_L' \\ i_V' \end{pmatrix} + \begin{pmatrix} A_R G A_R^\top & A_L & A_V \\ -A_L^\top & 0 & 0 \\ -A_V^\top & 0 & 0 \end{pmatrix} * \begin{pmatrix} u \\ i_L \\ i_V \end{pmatrix} = \begin{pmatrix} -A_I i_{src} \\ 0 \\ -v_{src} \end{pmatrix}, \quad (2.8)$$

where the diagonal matrices  $C$ ,  $G$  and  $L$  contain the capacities, conductivities and inductivities.

The resulting systems are *stiff* systems. This means that for their numerical solution special care has to be put into which methods are suitable for solving these systems in a stable manner.

### Example 1

We again consider the charging of a capacitor. The circuit is again given as in **Example 1**, 2.1, where we have already derived the reduced incidence matrix

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

This matrix can be split into the three submatrices  $A_V$ ,  $A_R$  and  $A_C$  containing the columns of the matrix  $A$ . The matrices  $A_L$  and  $A_V$  are empty in this case. The diagonal matrices containing the components constants are

$$C = (C), \quad L = (), \quad G = \left(\frac{1}{R}\right).$$

## 2 Formulating a Mathematical Model

Plugging this into the formula (2.8) we obtain the system

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{pmatrix} * \begin{pmatrix} u'_1 \\ u'_2 \\ i'_0 \end{pmatrix} + \begin{pmatrix} \frac{1}{R} & -\frac{1}{R} & 0 \\ -\frac{1}{R} & \frac{1}{R} & -1 \\ 0 & -1 & 0 \end{pmatrix} * \begin{pmatrix} u_1 \\ u_2 \\ i_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -v_{src} \end{pmatrix}.$$

Note that because  $L$  is empty, the system does not contain any related equations to inductances.

### Example 2

Similarly we also derived the incidence Matrix of ?? in **Example 2**, (2.1).

$$A = \begin{pmatrix} -1 & 1 \end{pmatrix}.$$

This matrix can be split into the two submatrices  $A_C$  and  $A_L$  containing the columns of the matrix  $A$ . The matrices  $A_R$ ,  $A_V$  and  $A_I$  are empty in this case. The diagonal matrices containing the components constants are

$$C = (C), \quad L = (L), \quad G = ().$$

Plugging this into the formula (2.8) we obtain the system

$$\begin{pmatrix} C & 0 \\ 0 & L \end{pmatrix} * \begin{pmatrix} u'_1 \\ i'_L \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} * \begin{pmatrix} u_1 \\ i_L \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

### Example 3

Consider the circuit of **Example 3**, 2.1. We have already found the incidence matrix for this circuit.

$$A = \begin{pmatrix} -1 & 1 & 1 \end{pmatrix}.$$

This matrix can be split into the three submatrices  $A_I$ ,  $A_R$  and  $A_C$  containing the columns of the matrix  $A$ . The matrices  $A_L$  and  $A_V$  are empty in this case. The diagonal matrices containing the components constants are

$$C = (C), \quad L = (), \quad G = \left(\frac{1}{R}\right).$$

## 2 Formulating a Mathematical Model

Plugging this into the formula (2.8) we obtain the system

$$\begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} * \begin{pmatrix} u'_1 \\ i'_V \end{pmatrix} + \begin{pmatrix} \frac{1}{R} & -1 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} u_1 \\ i_V \end{pmatrix} = \begin{pmatrix} 0 \\ -v_{src} \end{pmatrix}.$$

## 3 Differential Algebraic Equations

von [4] und DAE lecture

The resulting MNA of electrical circuits leads to a special form of differential equations, namely *differential algebraic equations* (DAE). Additionally to differential equations they also contain algebraic equations or are equivalent to such a system. They are distinct from ordinary differential equations in that the Jacobian matrix for a DAE system is singular. To understand the general solvability of those systems and to find good numerical methods we first have to take a look at the theory of such DAEs.

In the most general form a DAE can be written as: Find  $y : \mathbb{R} \rightarrow \mathbb{R}^n$  such that

$$F(t, y(t), y'(t)) = 0, \quad \forall t \in I \quad (3.1)$$

with  $F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  sufficiently smooth and  $I$  the time-interval. As this is usually too general, one has to consider more specific types of DAEs. This chapter focuses on giving a brief overview of the different types of differential algebraic equations. We will discuss the specific form of linear systems with constant coefficients in more detail, as those are the systems that arise from modified nodal analysis of a RLC-system. Additionally this chapter also discusses the notion of the index of an MNA-system.

### 3.1 Types of DAEs

Commonly systems of either of the three types below are considered when talking about DAEs.

### 3 Differential Algebraic Equations

- **Linear systems with constant coefficients**

are systems of the form: find  $y$  such that

$$Ay'(t) + By(t) = f(t), \quad (3.2)$$

with  $A, B \in \mathbb{R}^{n \times n}$ ,  $A$  singular,  $B$  regular and  $f : \mathbb{R} \rightarrow \mathbb{R}^n$  a function in time. If  $A$  is regular this equation can be transformed into an ordinary differential equation, since

$$y'(t) + A^{-1}By(t) = A^{-1}f(t)$$

Thus it is reasonable to assume  $A$  singular.

- **Linear time dependent systems** are systems of the form: find  $y$  such that

$$A(t)y'(t) + B(t)y(t) = f(t),$$

with  $A, B : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ ,  $f : \mathbb{R} \rightarrow \mathbb{R}^n$  functions, such that for every  $t \in \mathbb{R}$  the matrix  $A(t)$  is singular and the matrix  $B(t)$  regular.

- **Structured (non-linear) systems**

are semi-explicit systems of the form: find  $(y, z)$  such that

$$y'(t) = f(t, y(t), z(t)), \quad (3.3)$$

$$0 = g(t, y(t), z(t)), \quad (3.4)$$

with  $f : \mathbb{R} \rightarrow \mathbb{R}^n$  and  $g : \mathbb{R} \rightarrow \mathbb{R}^d$  functions.

For our analysis of electrical networks we will focus on linear systems with constant coefficients. The other systems could for example occur when we also consider time-dependant component relations e.g. Temperature. A splitting of the variable  $y$  is usually common, where we differ between the *differential variable* and the *algebraic variable*. A differential variable in this sense is the part of  $y$ , which derivatives also appear in the differential algebraic equation. The algebraic variable on the other hand only appears in algebraic equations.

### 3.1.1 Weierstraß-Kronecker Normalform

To determine the solvability of a linear system with constant coefficients (3.2) we first need to introduce a normalform for the system, the *Weierstraß-Kronecker normalform*. This normalform is dependent on the family  $\{A, B\} := \{\mu A + B | \mu \in \mathbb{R}\}$ , which is called the *matrix pencil* of the DAE.

**Definition 1.** The matrix pencil  $\{A, B\}$  is called *regular* if there exists some  $c \in \mathbb{R}$ , such that  $(cA + B)$  is regular (i.e.  $\det(cA + B) \neq 0$ ), otherwise it is called *singular*.

**Theorem 1** (Jordan Normalform [4, Theorem 13.2.1]). For every matrix  $Q \in \mathbb{R}^{n \times n}$  there exists a regular matrix  $T \in \mathbb{C}^{n \times n}$ , such that

$$T^{-1}QT = J = \text{diag}(J_1, \dots, J_r) \quad \text{with} \quad J_i = \begin{pmatrix} \lambda_i & 1 & & 0 \\ 0 & \lambda_i & \ddots & \vdots \\ & \ddots & \ddots & 1 \\ 0 & \dots & 0 & \lambda_i \end{pmatrix} \in \mathbb{C}^{m_i \times m_i}$$

and  $n = m_1 + \dots + m_r$ .

The matrix  $J$  is called Jordan Normalform of  $Q$ , the matrices  $J_i$  are called Jordan Blocks, where  $\lambda_i$  are corresponding the eigenvalues of  $Q$ . The matrix  $J$  is uniquely determined by  $Q$  except for the arrangement of the diagonal blocks. If  $Q$  possesses only real eigenvalues, then  $T$  can also be chosen from the reals.

In order to separate the equation (3.2) into an algebraic and a differential part, we transform the matrices  $A$  and  $B$  according to the following theorem.

**Theorem 2** ([4, Satz 13.2.2]). Let  $\{A, B\}$  be a regular matrix pencil. There exist  $P, Q \in \mathbb{C}^{n \times n}$  such that

$$PAQ = \begin{pmatrix} I_d & 0 \\ 0 & N \end{pmatrix}, \quad PBQ = \begin{pmatrix} R & 0 \\ 0 & I_{n-d} \end{pmatrix}$$

### 3 Differential Algebraic Equations

where

$$N = \text{diag}(N_1, \dots, N_r) \quad \text{with} \quad N_i = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} \in \mathbb{R}^{n_i \times n_i}$$

and  $R$  has Jordan Normalform. By  $I_k$  we denote the identity matrix of size  $k \times k$ .

*Proof.* Because  $\{A, B\}$  is a regular matrix pencil, there exists  $c \in \mathbb{R}$  such that  $(cA + B)$  is regular. Set

$$\hat{A} := (cA + B)^{-1}A, \quad \hat{B} := (cA + B)^{-1}B.$$

Considering

$$(cA + B)^{-1}(cA + B) = I \implies (cA + B)^{-1}B + c(cA + B)^{-1}A = I,$$

we get that

$$\hat{B} = I - c\hat{A}.$$

Let  $J_{\hat{A}}$  be the Jordan Normalform of  $\hat{A}$ , this means that there exists a regular matrix  $T_1$  such that

$$T_1^{-1}AT_1 = J_{\hat{A}} = \begin{pmatrix} W & 0 \\ 0 & \tilde{N} \end{pmatrix}.$$

The matrix  $W$  contains the Jordanblocks with Eigenvalues which are nonzero, the matrix  $\tilde{N}$  contains the Jordan blocks with Eigenvalues equal to zero, thus  $\tilde{N}$  is *nilpotent*. The Jordan Normalform  $J_{\hat{B}}$  of  $\hat{B}$  is given by

$$T_1^{-1}\hat{B}T_1 = J_{\hat{B}} = \begin{pmatrix} I - cW & 0 \\ 0 & I - c\tilde{N} \end{pmatrix}.$$

### 3 Differential Algebraic Equations

The following two transformations will allow us to get the desired structure. First we will transform  $J_{\hat{A}}$  with

$$T_2 := \begin{pmatrix} W & 0 \\ 0 & I - c\tilde{N} \end{pmatrix}$$

in

$$T_2^{-1}J_{\hat{A}} = \begin{pmatrix} I & 0 \\ 0 & (I - c\tilde{N})^{-1}\tilde{N} \end{pmatrix}$$

and  $J_{\hat{B}}$  in

$$T_2^{-1}J_{\hat{B}} = \begin{pmatrix} W^{-1} - cI & 0 \\ 0 & I \end{pmatrix}.$$

Let now  $R$  be the Jordan Normalform of  $(W^{-1} - cI)$  and  $N$  be the Normalform of  $(I - c\tilde{N})^{-1}\tilde{N}$ , this means

$$T_W^{-1}(W^{-1} - cI)T_W = R \quad \text{and} \quad T_{\tilde{N}}^{-1}(I - c\tilde{N})^{-1}\tilde{N}T_{\tilde{N}} = N$$

Considering this definition together with the Neumann-series of  $(I - c\tilde{N})^{-1}$  we obtain

$$\tilde{N}(I - c\tilde{N})^{-1} = \tilde{N}\left(c \sum_{i=0}^{\infty} \tilde{N}^i\right) = \tilde{N}\left(c \sum_{i=0}^{k-1} \tilde{N}^i\right) = c \sum_{i=0}^{k-1} \tilde{N}^{i+1} = \left(c \sum_{i=0}^{k-1} \tilde{N}^i\right)\tilde{N}$$

where we used that  $\tilde{N}$  is nilpotent with nilpotency index  $k$ . This shows that  $\tilde{N}$  and  $(I - c\tilde{N})^{-1}$  commute.

From this we can conclude that

$$N^k = [T_{\tilde{N}}^{-1}(I - c\tilde{N})^{-1}\tilde{N}T_{\tilde{N}}]^k = T_{\tilde{N}}^{-1}[(I - c\tilde{N})^{-1}\tilde{N}]^k T_{\tilde{N}} = T_{\tilde{N}}^{-1}(I - c\tilde{N})^{-k} \underbrace{\tilde{N}^k}_{=0} T_{\tilde{N}} = 0$$



### 3 Differential Algebraic Equations

We used the commutativity in the third step here. The nilpotent matrix  $N$  thus also has the nilpotency index  $k$ . A transformation with

$$T_3 := \begin{pmatrix} T_W & 0 \\ 0 & T_{\tilde{N}} \end{pmatrix}$$

transforms  $T_2^{-1}J_{\hat{A}}$  into the Jordan Normalform

$$J_{\hat{A}} := T_3^{-1}T_2^{-1}J_{\hat{A}}T_3 = T_3^{-1}T_2^{-1}T_1^{-1}\hat{A}T_1T_3 = \begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix}$$

and  $T_2^{-1}J_{\hat{B}}$  into

$$J_{\hat{B}} := T_3^{-1}T_2^{-1}J_{\hat{B}}T_3 = T_3^{-1}T_2^{-1}T_1^{-1}\hat{B}T_1T_3 = \begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix}.$$

Now set

$$P := T_3^{-1}T_2^{-1}T_1^{-1}(cA + B)^{-1} \quad \text{and} \quad Q = T_1T_3$$

to get the statement. □

Using the findings above we are able to transform the initial DAE (3.2) using the matrix  $P$  from Theorem 2. By multiplying with  $P$  from the left, we obtain

$$PAy'(t) + PBy(t) = Pf(t).$$

Setting

$$y(t) = Q \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}, \quad Pf(t) = \begin{pmatrix} s(t) \\ q(t) \end{pmatrix},$$

with  $u(t), s(t) : \mathbb{R} \rightarrow \mathbb{R}^d$  and  $q(t), v(t) : \mathbb{R} \rightarrow \mathbb{R}^{n-d}$ .

### 3 Differential Algebraic Equations

We get a system of the form

$$\begin{aligned} u'(t) + Ru(t) &= s(t), \\ Nv'(t) + v(t) &= q(t), \end{aligned} \tag{3.5}$$

where  $PAQ = \begin{pmatrix} I & \\ & N \end{pmatrix}$  and  $PBQ = \begin{pmatrix} R & \\ & I \end{pmatrix}$ .

We call this system the *Weierstraß-Kronecker normalform* of the DAE. The first equation of (3.5) is a first order ordinary differential equation and possesses a unique solution  $u(t)$  in  $[t_0, t_l]$  for any initial values  $u_0 \in \mathbb{R}^d$ . Now let  $q(t) \in C^{k-1}([t_0, t_l])$ , where  $C^{k-1}(I)$  denotes the space of all functions with domain  $I$  that are  $k - 1$  times differentiable. We differentiate the second equation in (3.5) and obtain

$$\begin{aligned} v(t) &= q(t) - Nv'(t) = q(t) - N \underbrace{(q(t) - Nv'(t))'}_{=v(t)} = q - Nq' + N^2v'' \\ &= q - Nq' + N^2(q - Nv')'' = q - Nq' + N^2q'' - N^3v''' \\ &\vdots \\ &= q - Nq' + \dots + (-1)^{k-1} N^{k-1} \underbrace{\frac{d^k}{dt^k} q}_{:=q^{(k-1)}} + (-1) \underbrace{N^k v^{(k)}}_{=0} \\ &= \sum_{i=0}^{k-1} (-1)^i N^i q^{(i)}(t) \end{aligned} \tag{3.6}$$

where  $k$  is the nilpotency index of  $N$ . Hence we have an explicit form of the solution of  $v(t)$ . The Kronecker index  $k$  thus tells us how many differentiations are required to obtain an ordinary differential equation. Note that this way of splitting the DAE allows us to differ between algebraic and differential variables.

We can also give a general definition of a DAE in Weierstraß-Kronecker normalform.

### 3 Differential Algebraic Equations

**Definition 2** ([4, Definition 13.2.4]). A linear differential equation (3.2) is said to be in Weierstraß-Kronecker normalform if

$$A = \begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix}, \quad B = \begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix},$$

where  $N$  is a nilpotent Jordan-block matrix.

**Definition 3.** The nilpotency index  $k$  of the matrix  $N$  from the Weierstraß-Kronecker Normalform of a matrix pencil  $\{A, B\}$  with  $A$  singular is called the Kronecker-Index of  $\{A, B\}$ , which we denote by  $\text{ind}\{A, B\}$ . Note that for  $A$  regular we set  $\text{ind}\{A, B\} = 0$ .

**Lemma 1.** The Kronecker-Index  $\text{ind}\{A, B\}$  is independent of the choice of the matrices  $P$  and  $Q$ .

*Proof.* See Lemma 13.2.1 in [4, p.403]. □

## 3.2 Index of a Differential Algebraic Equation

The index of a DAE gives us insight about its numerical properties and in general about the solvability where the intuition is “The higher the index, the harder to solve”.

We will consider two types of index concepts, the differentiation index and the perturbation index.

Since numerical differentiation is an unstable procedure the differentiation index aims to give a measure for the numerical problems to be expected when solving such systems.

### 3 Differential Algebraic Equations

**Definition 4** (differentiation index [4, Definition 13.3.1]). Consider the differential algebraic equation (3.1) to be uniquely locally solvable and  $F$  sufficiently smooth. For a given  $m \in \mathbb{N}$  consider the equations

$$\begin{aligned} F(t, y, y') &= 0, \\ \frac{dF(t, y, y')}{dt} &= 0, \\ &\vdots \\ \frac{d^m F(t, y, y')}{dt^m} &= 0. \end{aligned}$$

The smallest natural number  $m$  for which the above system results in an explicit system of ordinary differential equations (ODEs), i.e. it has the form

$$y' = \phi(t, y),$$

is called **differentiation index**.

In the previous chapter we have already discussed, that for a DAE with constant coefficients (3.2) and a regular matrix pencil  $\{A, B\}$  we need  $k = \text{ind}\{A, B\}$  differentiations to receive an ordinary differential equation. Hence for a DAE with constant coefficients the Kronecker index  $k$  is equal to the differentiation index.

**Definition 5** (perturbation index [4, Definition 13.3.3]). Let  $y(t)$  be the exact solution of (3.1) and  $\tilde{y}(t)$  be the solution of the perturbed system  $F(t, \tilde{y}, \tilde{y}') = \delta(t)$ . The smallest number  $k \in \mathbb{N}$  such that

$$\|y(t) - \tilde{y}(t)\| \leq C \left( \|y(t_0) - \tilde{y}(t_0)\| + \sum_{j=0}^k \max_{t_0 \leq \xi \leq T} \left\| \int_{t_0}^{\xi} \frac{d^j \delta}{d\tau^j}(\tau) d\tau \right\| \right)$$

for all  $\tilde{y}(t)$ , is called the **perturbation index** of this system.

Considering the case of a perturbed linear DAE with constant coefficients and a regular matrix pencil  $\{A, B\}$

$$A\tilde{y}'(t) + B\tilde{y}(t) = -\delta(t).$$

### 3 Differential Algebraic Equations

Then for the difference of the unperturbed system with the perturbed system we obtain

$$A(y'(t) - \tilde{y}'(t)) + B(y(t) - \tilde{y}(t)) = \delta(t)$$

Due to 2 we can transform this system into a system of the form (3.5)

$$\begin{aligned} u'(t) - \tilde{u}'(t) + \hat{R}(u(t) - \tilde{u}(t)) &= \delta_1(t), \\ N(v'(t) - \tilde{v}'(t)) &= \delta_2(t). \end{aligned}$$

Where  $u(t)$  holds the first  $d$  entries of  $y(t) : \mathbb{R} \rightarrow \mathbb{R}^n$  and  $v(t)$  holds the remaining  $n - d$  entries. According to (3.6), the solution of the algebraic variable has the form

$$v(t) - \tilde{v}(t) = \sum_{i=0}^{k-1} (-1)^i N^i \delta_2^{(i)}(t)$$

Thus the perturbation index is the same as the Kronecker-index  $\text{ind}\{A, B\}$  [4]. More over we observe that the error depends on the derivatives of the pertubation.

## 4 Index Analysis of the modified nodal analysis

[2]

In the previous chapter we have seen two different kinds of index concepts for differential algebraic equations. We have also seen that these two, even though they describe rather different structural aspects of the equation, are identical for our use-cases. This leads to the question, what indices we expect obtain for our MNA system.

This chapter answers this question for the case of linear RLC-circuits. As already discussed in Section 2.3, RLC circuits contain only resistors, inductors and capacities as well as current and voltage sources (RLC-circuits). Linear means that all circuit components are described by linear positive functions. Thus the matrices  $C$ ,  $R$ , and  $L$  are positive definite and symmetric.

Recall that we consider the equations resulting from the analysis in Section 2.4. These equations are of the form (3.2): find  $y$  such that

$$Ay'(t) + By(t) = f(t).$$

In particular we obtain the equation

$$\begin{pmatrix} A_C C A_C^\top & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{pmatrix} * \begin{pmatrix} u' \\ i'_L \\ i'_V \end{pmatrix} + \begin{pmatrix} A_R G A_R^\top & A_L & A_V \\ -A_L^\top & 0 & 0 \\ -A_V^\top & 0 & 0 \end{pmatrix} * \begin{pmatrix} u \\ i_L \\ i_V \end{pmatrix} = \begin{pmatrix} -A_I i_{src} \\ 0 \\ -v_{src} \end{pmatrix}.$$

## 4.1 General Index analysis

Assuming the system only contains linear elements, or is linearized at an operating point in order to investigate the system behaviour, then the corresponding network equation represents a DAE with constant coefficients (3.2). We collect all unknowns in  $x(t) = (u(t), i_L(t), i_V(t))^T$ . The structure of the system is reliant on the matrix  $B$ , thus we consider two cases.

**ODE-case:** The matrix  $B$  in (3.2) is regular. This is the case iff the circuit contains no voltage sources and every node has a path to ground, that leads through at least one capacitor. Then the system represents a linear-implicit system of ODEs and can be transformed into the explicit ODE system

$$x' = B^{-1}(-Ax + f(t)).$$

Thus we obtain an index of 0.

**DAE-case:** The matrix  $B$  in (3.2) is singular. This is the interesting case which we will analyse in the following.

For a singular matrix  $B$  we have already obtained a representation of the form

$$\begin{aligned} u'(t) + Ru(t) &= s(t), \\ Nv'(t) + v(t) &= q(t) \end{aligned}$$

in the previous section, see (3.5). We now consider the nilpotency index  $\nu$  of the matrix  $N$ , since we already know that this index correlates with the differentiation index as well as the perturbation index. We consider two cases:

#### 4 Index Analysis of the modified nodal analysis

##### 1. The nilpotency index is 1

Because  $N$  is nilpotent with nilpotency index  $\nu = 1$ , it holds that  $N^1 = 0$ , thus the system transforms to

$$\begin{aligned} u'(t) + Ru(t) &= s(t), \\ v(t) &= q(t). \end{aligned}$$

This means that the algebraic variables are given explicitly. Thus the system is in ODE form.

##### 2. The nilpotency index is $\geq 2$

This case is the situation we have discussed in section 3.1.1, where we obtained the representation (3.6)

$$v(t) = \sum_{i=0}^{\nu-1} (-1)^i N^i q^{(i)}(t). \quad (4.1)$$

By differentiating the above equation one more time, we obtain

$$v'(t) = \sum_{i=0}^{\nu-1} (-1)^i N^i q^{(i+1)}(t) \quad (4.2)$$

with the assumption that  $q$  is  $\nu$  times differentiable.

We observe that in both cases the solution  $v$  has to fulfill an algebraic constraint. In the case of index  $\nu = 1$  this equation is given explicitly, while for index  $\nu \geq 2$  it is given implicitly by (4.2).

This system is also sensitive to perturbations. Note that the solution formula (4.1) for  $v$  also contains derivatives of the input  $q$ . Thus small noise in the input can have arbitrarily large derivatives, which can influence the solution.

## 4.2 Topological Conditions

From analyzing the MNA some conditions to the circuit topology can be obtained. We will be considering the impact of special arrangements of components on the index of



#### 4 Index Analysis of the modified nodal analysis

the system. In [7] as well as in [6] and [5] they present results about the index of MNA equations.

**Theorem 3** (Index conditions [6, Theorem 2.2.1]). *Let the matrices of the capacitances, inductances and resistances be positive definite.*

- *If*

$$\ker([A_R, A_C, A_V, A_L]^\top) = \{0\} \quad \text{and} \quad \ker(A_V) = \{0\} \quad (4.3)$$

*holds, then the MNA (2.8) leads to a system with index  $\nu \leq 2$ .*

- *If additionally*

$$\ker([A_R, A_C, A_V]^\top) = \{0\} \quad \text{and} \quad \ker([A_C, A_V]) = \{0\} \quad (4.4)$$

*holds, then the system is of index  $\nu \leq 1$*

- *If further*

$$\ker(A_C^\top) = \{0\} \quad \text{and} \quad \dim(v_{src}) = 0 \quad (4.5)$$

*holds, then the system has index  $\nu = 0$ .*

By  $\ker(A)$  we denote the nullspace of the matrix  $A$  and by  $\dim(y)$  the size of the vector  $y$ . As (4.3) - (4.5) are just conditions concerning the topology of the circuit they can also be interpreted as follows:

check again, maybe need to add "only contains" somewhere - maybe at index 1 condition

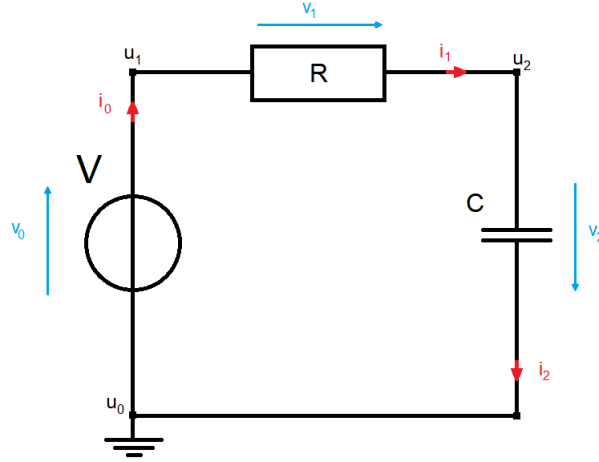
- Condition (4.3) can be interpreted, as the circuit neither containing loops of voltage sources nor cutsets of current sources.
- Condition (4.4) can be interpreted, as the circuit containing neither loops of capacitors and/or voltage sources nor cutsets of inductors and/or current sources.
- Condition (4.5) can be interpreted, as every node in the circuit being connected to the reference node (ground) through a path containing only the capacitors.

#### 4 Index Analysis of the modified nodal analysis

We will apply those results to some examples:

##### Example 1

As our first example we consider again the charging of a capacitor with a series resistor as in **Example 1**.



**Figure 4.1:** charging capacitor with series resistor and voltage source (note the difference to 2.1)

Because this circuit only contains one loop, which has a voltage source as well as a capacitor and a resistor we expect the resulting MNA-system to be of index  $\nu = 1$ .

The incidence matrix for this system is

$$A = [A_V, A_R, A_C] = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

Thus condition (4.3) gives us

$$\ker([A_R, A_C, A_V, A_L]^\top) = \ker \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} = \{0\} \quad \text{and} \quad \ker(A_V) = \ker \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \{0\}.$$

#### 4 Index Analysis of the modified nodal analysis

This condition is fulfilled, thus the index has to be smaller or equal to 2. The condition (4.4) results in

$$\ker([A_R, A_C, A_V]^\top) = \ker \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} = \{0\} \quad \text{and} \quad \ker([A_C, A_V]) = \ker \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \{0\}.$$

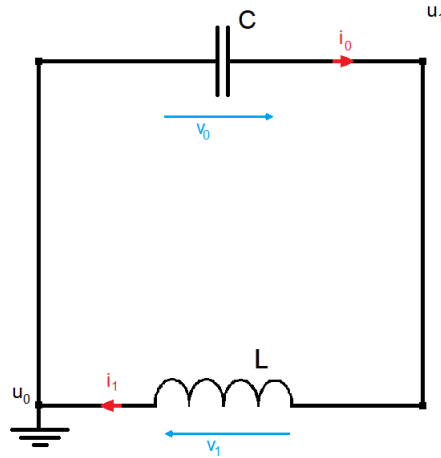
Thus the index also has to be smaller or equal to 1. Finally condition (4.5) results in

$$\ker(A_C^\top) = \ker \begin{pmatrix} 0 & 1 \end{pmatrix} = \{0\} \quad \text{and} \quad \dim(v_{src}) \neq 0.$$

This means that this system has index  $\nu = 1$ .

#### Example 2

This example depicts an energy conserving system. The electric energy stored in the capacitor is converted into the magnetic energy of the coil and vice versa.



**Figure 4.2:** LC circuit

Because this circuit only contains one capacitor and one inductance in a singular loop and also has no sources, we expect the resulting MNA-system to be of index  $\nu = 0$ . This means that the system is an ODE!

#### 4 Index Analysis of the modified nodal analysis

The incidence matrix for this system is

$$A = [A_L \ A_C] = \begin{pmatrix} -1 & 1 \end{pmatrix}.$$

Thus condition (4.3) gives us

$$\ker([A_R, A_C, A_V, A_L]^\top) = \ker \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \{0\} \quad \text{and} \quad \ker(A_V) = \{0\}.$$

This condition is fulfilled, thus the index has to be smaller or equal to 2.

Condition (4.4) results in

$$\ker([A_R, A_C, A_V]^\top) = \ker \begin{pmatrix} -1 \end{pmatrix} = \{0\} \quad \text{and} \quad \ker([A_C, A_V]) = \ker \begin{pmatrix} -1 \end{pmatrix} = \{0\}.$$

This condition is also fulfilled, thus the index has to be smaller or equal to 1.

Finally condition (4.5) results in

$$\ker(A_C^\top) = \ker \begin{pmatrix} -1 \end{pmatrix} = \{0\} \quad \text{and} \quad \dim(vsrc) = 0.$$

Thus it is fulfilled and we obtain index 0.

If we have consider the MNA directly, we also see that this system is of index  $\nu = 0$ , because the MNA has the form

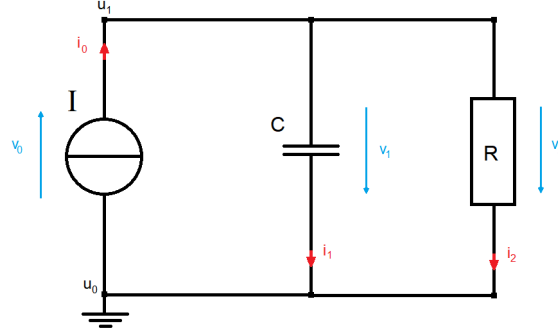
$$\begin{aligned} Cu'_1 - i_L &= 0, \\ Lu'_L + u_1 &= 0 \end{aligned}$$

which is an ODE.

#### Example 3

Finally, as the third example seems very similar to the first one, we might also expect them to have the same index. But this is indeed not the case as the following analysis shows.

#### 4 Index Analysis of the modified nodal analysis



**Figure 4.3:** current source with resistor and capacitor

This circuit contains two loops, the first with only a voltage source and a capacitor and the second with a capacitor and a resistor. Because of the first loop, we expect this circuit to omit a system of index  $\nu = 2$ .

The incidence matrix for this system is

$$A = [A_V \ A_C \ A_R] = \begin{pmatrix} -1 & 1 & 1 \end{pmatrix}.$$

Thus condition (4.3) gives us (first ker is indeed not 0, no? - would need to not have transposed - maybe because  $A_L$  is empty)

$$\ker([A_R, A_C, A_V, A_L]^\top) = \ker \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \{0\} \quad \text{and} \quad \ker(A_V) = \ker(-1) = \{0\}$$

This condition is fulfilled, thus the index has to be smaller or equal to 2. Condition (4.4) results in

$$\ker([A_R, A_C, A_V]^\top) = \ker \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \{0\} \quad \text{and} \quad \ker([A_C, A_V]) = \ker \begin{pmatrix} 1 & -1 \end{pmatrix} \neq \{0\}.$$

Thus this condition is not fulfilled, which means that the system indeed has index  $\nu = 2$ .

## 5 Numerical Solutions

As we are interested in finding real valued solutions of our systems we also have to look into solving them numerically. This chapter focuses on the numerical solution of the mentioned systems.

We will first focus on methods used to solve a more general initial value problem of the form:

Find  $y$ , such that

$$y'(t) = f(t, y), \quad t \in [t_0, t_l], \quad (5.1)$$

$$y(t_0) = y_0. \quad (5.2)$$

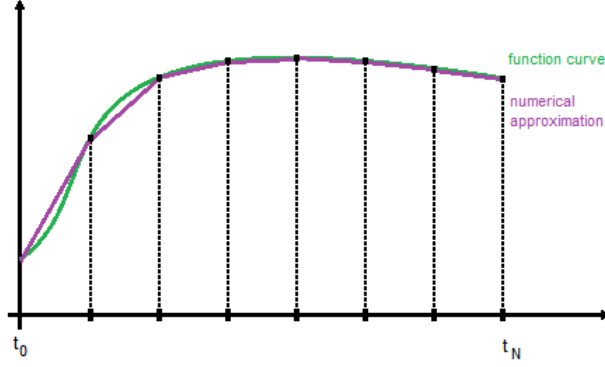
For this we presume that the function  $f(t, y)$  is continuous and Lipschitz, we can apply the theorem of *Picard-Lindelöf* [4, Satz 1.2.1], which states, that for every initial value  $y_0$ , the initial value problem (5.1) is uniquely solvable in  $[t_0, t_l]$ .

In order to obtain an approximation to this unique solution we have to discretize. We first divide the time-intervall  $[t_0, t_l]$  into

$$t_0 < t_1 < \dots < t_N \leq t_l$$

and consider approximations  $y_m \approx y(t_m)$  for  $m = 1, \dots, N$ , see ???. We call this a *time-grid* and we call the difference  $h_{ij} = t_j - t_i$  with  $0 \leq i < j \leq N$  the *step size* from  $t_i$  to  $t_j$ . We will only consider equidistant grids, i.e.  $h_{ij} = h$  for all  $0 \leq i, j \leq N$ . A function defined on such a grid is called a *grid-function*.

The grid function  $y_h : T \rightarrow A$  with the time-grid  $T = t_0, \dots, t_N$  and the approximated values  $A = y_0, \dots, y_N$  is called the *numerical approximation* (of  $y$  on the time grid).



**Figure 5.1:** approximation of a function using numerical methods

## 5.1 Single-Step-Methods

The first class of numerical methods we will consider are single-step methods. These methods use the previous approximated value  $y_j$  and (for implicit methods) also the current approximated value  $y_{j+1}$  to determine the current value  $y_{j+1}$  through a *procedural function*.

**Definition 6.** A numerical method to approximate a differential equation 5.1 on a time-grid  $t_0, \dots, t_l$  with the intermediate values  $y_0, \dots, y_l$  is called a single-step method if it is of the form

$$y_{j+1} = y_j + h_j \phi(t_j, y_j, y_{j+1}, h_j). \quad (5.3)$$

We call  $\phi$  the procedural function. If  $\phi$  does not depend on  $y_{j+1}$ , then the method is called explicit, otherwise it is called implicit.

### 5.1.1 Consistency, Stability and Convergence

In order to compare different single-step methods we have to define some notions to compare their quality. This leads to the definition of the error of the method, its consistency and its convergence. We begin with the definition of the error.

## 5 Numerical Solutions

**Definition 7.** Let  $\tilde{y}_{m+1}$  be the result of one step of a single step method (5.3) with the exact start-vector  $y_m = y(t_m)$  then

$$\delta_{m+1} = \delta(t_m + h) = y(t_{m+1}) - \tilde{y}_{m+1}, \quad m = 0, \dots, N-1 \quad (5.4)$$

is called the local discretization error of the single step method at the point  $t_{m+1}$ .

The local error quantifies the error of every step of the method with respect to the exact solution. In most applications the exact solution is not known. Next we consider the consistency.

**Definition 8.** A single-step method is called consistent if for all initial value problems (5.1)

$$\lim_{h \rightarrow 0} \frac{\|\delta(t+h)\|}{h} = 0 \quad \text{for } t_0 \leq t \leq t_l \quad (5.5)$$

holds.

It is called consistent of order  $p$ , if for a sufficiently smooth function  $f$

$$\|\delta(t+h)\| \leq Ch^{p+1} \quad \text{for all } h \in (0, H] \quad \text{and } t_0 \leq t \leq t_l - h \quad (5.6)$$

holds with  $C$  independent of  $h$ .

Consistency aims to give insight in how similar the problem (5.3) that the numerical method solves is to the real problem (5.1). Finally we consider convergence of a single step method 6.

**Definition 9.** A single-step method is called convergent, if for all initial value problems (5.1) for the global discretization error

$$e_m = y(t_m) - y_m$$

holds that

$$\max_m \|e_m\| \rightarrow 0 \quad \text{for } h_{\max} \rightarrow 0.$$



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*The single-step method is called to have the convergence order  $p$ , if*

$$\max_m \|e_m\| \leq Ch_{max}^p \quad \text{for } h_{max} \in (0, H] \quad \text{with } t_0 \leq t_m \leq t_l$$

*with the constant  $C$  not dependent on the step size  $h$ .*

As the name suggests convergence tries to quantify how far off a numerical solution is from the real solution of a system. A very interesting result follows if we additionally require the single-step method to be stable.

**Definition 10.** *A single-step method is called (discretely) stable if for grid-functions  $y_h$  and  $\tilde{y}_h$  with*

$$y_{i+1} = y_i + h\phi(t_i, y_i), \tag{5.7}$$

$$\tilde{y}_{i+1} = \tilde{y}_i + h[\phi(t_i, \tilde{y}_i) + \theta_i], \tag{5.8}$$

*and perturbations  $\theta_i = \theta_h(t_i)$  of the right side as well as a bounded perturbation in the initial-values  $y_0 - \tilde{y}_0$  the error is bounded by*

$$\|y_h - \tilde{y}_h\|_{\infty, h} \leq C(\|y_0 - \tilde{y}_0\|_{l^2} + \|\theta_h\|_{\infty, h})$$

*with a constant  $C$  which is not dependent on  $h$ . The norm  $\|\cdot\|_{\infty, h}$  denotes the maximum norm over the time-grid, i.e. for a function  $b : T = t_0, \dots, t_N \rightarrow \mathbb{R}^d$  we have  $\|b\|_{\infty, h} = \max_{t \in T} \|b(t)\|$ ,  $\|b\|$  is the euclidean norm.*

For single-step methods which are consistent and stable we obtain the following convergence theorem.

**Theorem 4 (Lax-Richtmyer).** *A consistent (with order  $p$ ) and discretely stable single-step method is convergent (with order  $p$ ).*

This theorem is due to Lax and Richtmyer. The converse of this statement is also true.

### 5.1.2 further stability properties

from numpdgl skript

In this section we consider as a model problem the Dahlquist equation, i.e. find  $u$  such that

$$u' = \lambda u, \quad t > 0 \quad (5.9)$$

$$u(0) = u_0 \quad (5.10)$$

with  $\lambda \in \mathbb{C}$  and  $u_0$  fixed.

**Definition 11.** 1. If a single-step method can be written in the form

$$u_{i+1} = R(z) u_i, \quad z := h y \quad (5.11)$$

then we call  $R : \mathbb{C} \rightarrow \mathbb{C}$  the stability function of the single-step method.

2. The set

$$S := \{z \in \mathbb{C} : |R(z)| \leq 1\} \quad (5.12)$$

is called the region of stability of the method.

3. A single-step method is called

- 0-stable, if  $0 \in S$ .
- A-stable, if  $\mathbb{C}^- \subset S$ .
- L-stable, if  $R(z) \rightarrow 0$  for  $\operatorname{Re}(z) \rightarrow -\infty$ .

## 5.2 Multistep-Methods

based on chapter 4 of book num gew dgl steif nichtsteif

Linear multistep methods (LMSM) use approximations  $u_{m+l}$  along the gridpoints  $t_{m+l}$ ,  $l = 0, 1, \dots, k-1$  to calculate the new approximation  $u_{m+k}$  at  $t_{m+k}$ . We will first discuss topics

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related to the order of the methods depending on its parameters, stability and convergence.

**Definition 12.** For given  $\alpha_0, \dots, \alpha_k$  and  $\beta_0, \dots, \beta_k$  the iteration rule

$$\sum_{l=0}^k \alpha_l u_{m+l} = h \sum_{l=0}^k \beta_l f(t_{m+l}, u_{m+l}), \quad m = 0, 1, \dots, N - k \quad (5.13)$$

is called a linear multistep method (linear  $k$ -step method). It is always assumed that  $\alpha_k \neq 0$  and  $|\alpha_0| + |\beta_k| > 0$ . If  $\beta_k = 0$  holds, then the method is called explicit, otherwise implicit.

A linear multi-step method consists of two parts:

1. In the *starting-phase* approximations  $u_1, \dots, u_{k-1}$  for the first  $k - 1$  gridpoints  $t_l = t_0 + lh, l = 1, \dots, k - 1$  have to be calculated. For example we can use a single-step method or a multi-step method with fewer steps.
2. In the *run-phase* the multi-step formula is used to determine new approximations  $u_{m+k}$  for the gridpoint  $t_{m+k}$

For theoretical analysis of the multi-step methods we consider the generating polynomials of a multi-step method 12.

$$\rho(x) := \sum_{l=0}^k \alpha_l x^l \quad \text{and} \quad \sigma(x) := \sum_{l=0}^k \beta_l x^l \quad (5.14)$$

### 5.2.1 Consistency, Stability and Convergence

[4]

Again we need to define the properties consistency, stability and convergence for this method.

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**Definition 13.** Let  $\tilde{y}_{m+k}$  be the result of one step of the multi-step method (5.13) with the start-values given as the evaluations of the exact solution  $y_{m+l} = y(t_{m+l})$  at  $0 \leq l < k$ . This means

$$\alpha_k \tilde{u}_{m+k} = \sum_{l=0}^{k-1} (h\beta_l f(t_{m+l}, y(t_{m+l})) - \alpha_l y(t_{m+l})) + h\beta_k f(t_{m+k}, \tilde{u}_{m+k}).$$

Then

$$\delta_{m+k} = \delta(t_{m+k}) = y(t_{m+k}) - \tilde{u}_{m+k}, \quad m = 0, 1, \dots, N - k$$

is called the local discretization error (local error) of the linear multi-step method, see Def. 5.13 at the point  $t_{m+k}$ .

For  $k = 1$  this definition agrees with the definition of the local discretization error for single-step methods 7. If we assign the difference operator

$$L[y(t), h] = \sum_{l=0}^k (\alpha_l y(t + lh) - h\beta_l y'(t + lh)) \quad (5.15)$$

to the linear multi-step method, we gain the following definition for consistency of such a methods.

**Definition 14.** A linear multi-step method is called consistent, if for all functions  $y(t) \in C^2([t_0, t_1])$

$$\lim_{h \rightarrow 0} \frac{1}{h} L[y(t), h] = 0$$

holds. It has the consistency order  $p$ , if for all functions  $y(t) \in C^{p+1}[t_0, t_1]$

$$L[y(t), h] = \mathcal{O}(h^{p+1}) \quad \text{for } h \rightarrow 0$$

holds.

From the generating polynomials (5.14) we can derive simple consistency conditions, i.e.

$$\rho(1) = 0 \quad \text{and} \quad \rho'(1) = \sigma(1).$$

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Next we also need convergence of such methods.

**Definition 15.** We say that a linear multi-step method is convergent if for a solution  $y$  of the problem a solution vector created by an LMSM  $y_j$  for  $j \in 0, \dots, k$  we have that

$$\lim_{h \rightarrow \infty} \max_{0 \leq j \leq k} \|y(t_j) - y_j\| = 0.$$

The discrete stability is also very similar to the single-step methods.

**Definition 16.** A linear multi-step method is called (discretely) stable, if for solutions  $u_h$  and  $\tilde{u}_h$  of

$$\sum_{l=0}^k \alpha_l u_{m+l} = h \sum_{l=0}^k \beta_l f(t_{m+l}, u_{m+l}), \quad (5.16)$$

$$\sum_{l=0}^k \alpha_l \tilde{u}_{m+l} = h \sum_{l=0}^k \beta_l f(t_{m+l}, \tilde{u}_{m+l}) + h\theta_n \quad (5.17)$$

and bounded initial values  $y_j - \tilde{y}_j$  for  $j \in 0, \dots, k$  we have that

$$\max_{t_0 \leq t_n \leq T} \|y_n - \tilde{y}_n\| \leq C \sum_{j=0}^{k-1} \|y_j - \tilde{y}_j\| + \max_{t_0 \leq t_n \leq T} \|\theta_n\|.$$

### 5.2.2 further stability properties

In this section we consider again the Dahlquist test problem as a model problem

$$u' = \lambda u, \quad t > 0 \quad (5.18)$$

$$u(0) = u_0 \quad (5.19)$$

with  $\lambda \in \mathbb{C}$  and  $u_0$  fixed.

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Thus the resulting linear multistep method is of the form

$$\begin{aligned}\sum_{l=0}^k \alpha_l u_{n+l} &= h \sum_{l=0}^k \beta_l \lambda u_{n+l} \\ \iff \sum_{l=0}^k [\alpha_l - h\beta_l \lambda] u_{n+l}\end{aligned}$$

Using this we define the following important stability notions.

**Definition 17.** 1. The set

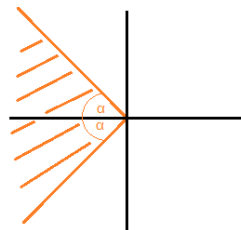
$$\begin{aligned}S := \{z \in \mathbb{C} : \rho(\zeta) - z\sigma(\zeta) = 0 \implies \zeta \in \mathbb{C} \text{ and } |\zeta| \leq 1. \\ \text{If } \zeta \text{ has multiplicity greater than 1, then } |\zeta| < 1\}\end{aligned} \quad (5.20)$$

is called the region of stability of the method.

2. A linear multistep method is called

- 0-stable, if  $0 \in S$ .
- stable in the point  $z \in \mathbb{C}$ , if  $z \in S$ .
- $A(\alpha)$ -stable, if it is stable in all  $z$  that lie within the set  $\{z \in \mathbb{C}^- : |\arg(z) - \pi| \leq \alpha\}$  for  $\alpha \in (0, \frac{\pi}{2})$ .

seite 314 im buch numerik



**Figure 5.2:**  $A(\alpha)$  stability for  $\alpha = \frac{\pi}{4}$

following from numerik buch seite 134

**Theorem 5** ([4, Satz 4.2.10]). *Let  $f(t, y)$  be sufficiently smooth and the linear multi-step method be zero-stable and consistent of order  $p$ , then it is also convergent of order  $p$ .*

equivalent to lax richtmyer in single step because for lmsm zero stable = stable

### 5.3 Implicit linear multi-step formulas

These kinds of multi-step methods are commonly used to numerically solve the systems obtained using modified nodal analysis.

What equations do we consider?

The approach to solve those systems numerically can be split into three main steps:

1. Computation of consistent initial values

The first step is to compute consistent initial values  $(x_0, y_0)$  at the initial time point  $t_0$ . In the index-1 case this can be done by solving a so called steady state problem. Steady state means that we consider a time-independent circuit (DC operating point). This means we have to solve

$$F(0, x_0, t_0) = 0$$

for  $x_0$  and then set  $y_0 = g(x_0)$ .

2. Numerical integration based on multi-step schemes

Using the consistent initial values, we obtain solutions of the network equations at discrete timepoints  $t_1, t_2, \dots$  by integration with linear multi-step methods.

In the following two subchapters we will give a short overview over the two most commonly used methods for solving these kinds of systems.

### 5.3.1 BDF-k methods

chapter 9.2 numerik book wikipedia

The most commonly used numerical methods for solving the systems that arise in electrical circuits are the BDF-scheme and the trapezoidal rule.

We will not give a deeper look into their construction but will only state their properties. BDF schemes are appealing because they save function evaluations as much as possible, since this is very costly in circuit simulation.

The *backward differentiation formula (BDF)* is a family of implicit linear multistep methods. They have the general form

$$\sum_{k=0}^s \alpha_k y_{n+k} = h\beta f(t_{n+s}, y_{n+s}) \quad (5.21)$$

Since we are interested in the unknown  $y_{n+s}$  which is used to evaluate  $f$ , this method is implicit. The coefficients  $\alpha_k$  and  $\beta$  are chosen, such that the method achieves the best possible order  $s$ .

The BDF or BDF-k formulas for  $k = 1, \dots, 3$  have the following form

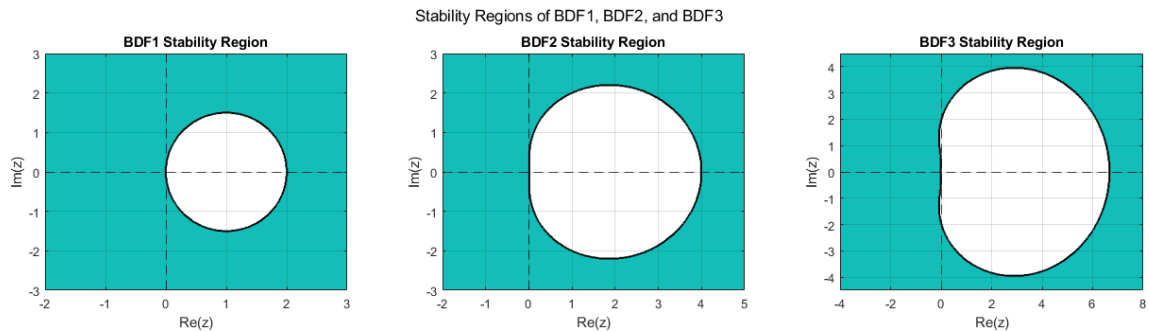
$$\begin{aligned} k = 1 : hf_{m+1} &= u_{m+1} - u_m \\ k = 2 : hf_{m+2} &= \frac{1}{2}(3u_{m+2} - 4u_{m+1} + u_m) \\ k = 3 : hf_{m+3} &= \frac{1}{6}(11u_{m+3} - 18u_{m+2} + 9u_{m+1} - 2u_m) \end{aligned}$$

Methods with  $k > 6$  are not zero-stable. Indeed in practice, methods with order greater than 3 are rarely used because of their stability properties. The stability-region shrinks with increasing order.

BDF schemes have consistency order  $p = k$ . They are methods for solving stiff equations, thus their stability is indicated by their region of absolute stability. Unfortunately not all BDF-schemes are A-stable, but their stability region still contains a large part of the complex left half-plane, see Figure 5.3 They are the most efficient linear multistep methods of this kind.



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**Figure 5.3:** stability regions of BDF-schemes

The first timestep is always performed by BDF1 (implicit Euler scheme) as a starting procedure. sure? not really

The Order of consistency of this family of multi-step methods is given by the following theorem

**Theorem 6** ([4, Satz 9.2.1]). *The BDF-k methods have consistency order  $p = k$ .*

Due to Theorem 5 and the fact that BDF-k methods are 0-stable, they are also convergent with order k.

Seite 106 numerik - schrittweite für startwerte bei bdf3 kleiner für kleineren fehler, halbieren, vierteln, achtern whatever

### 5.3.2 trapezoidal rule

The trapezoidal rule is a natural alternative to BDF2 since it is A-stable and a linear multistep method of order 2 the one with the smallest leading error coefficient. ([2])

It works by approximating the region under the graph of the function  $f(x)$  as a trapezoid, hence the name. It follows that

$$\int_a^b f(x)dx \approx (b-a)\frac{1}{2}(f(a) + f(b))$$

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This procedure is repeated for small subsections of the intervall  $[a, b]$ . Thus we obtain the iteration formula

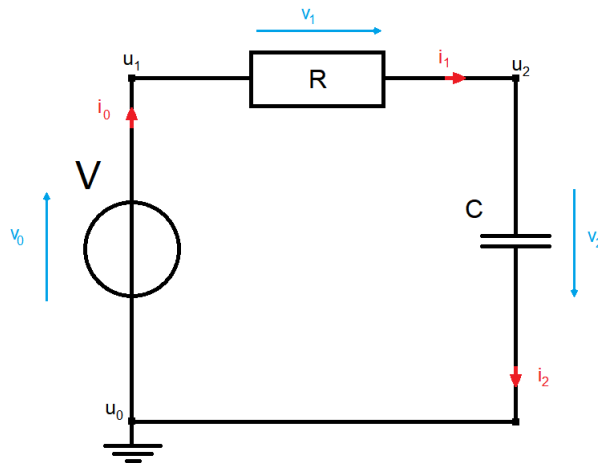
$$u_h(t+h) = u_h(t) + \frac{h}{2} [f(t, u_h(t)) + f(t+h, u_h(t+h))].$$

Because  $u_h(t+h)$  appears in  $f$  again we see that this is an implicit method.

### 5.4 Numerical Examples

In this section we will apply the Trapezoidal rule as well as the BDF-k schemes for our three examples (reference!!!!!!!!!!).

#### Example 1



**Figure 5.4:** charging capacitor with series resistor and voltage source

The MNA-system for this example reads as

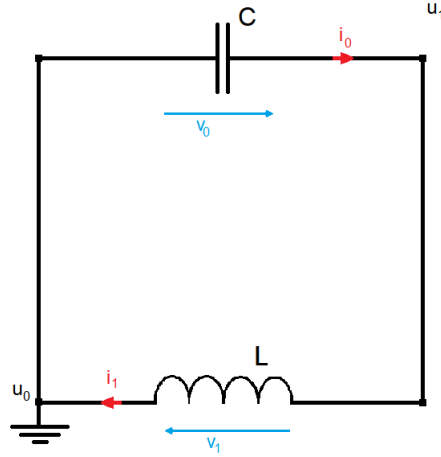
## 5 Numerical Solutions

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{pmatrix} * \begin{pmatrix} u'_1 \\ u'_2 \\ i'_0 \end{pmatrix} + \begin{pmatrix} \frac{1}{R} & \frac{1}{R} & 0 \\ \frac{1}{R} & \frac{1}{R} & -1 \\ 0 & -1 & 0 \end{pmatrix} * \begin{pmatrix} u_1 \\ u_2 \\ i_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -v_{src} \end{pmatrix}. \quad (5.22)$$

For testing we set the resistance to  $R = 1$  and the capacitance to  $C = 1$ . In our case we let the voltage source supply a voltage of the form  $v_{src} = \sin(\pi t)$ .

h	k = 1			k = 2			k = 3		
	u1	u2	iL	u1	u2	iL	u1	u2	iL
	Error			Error			Error		
1	number	number	number	number	number	number	number	number	number
0,1	number	number	number	number	number	number	number	number	number
0,01	number	number	number	number	number	number	number	number	number

### Example 2



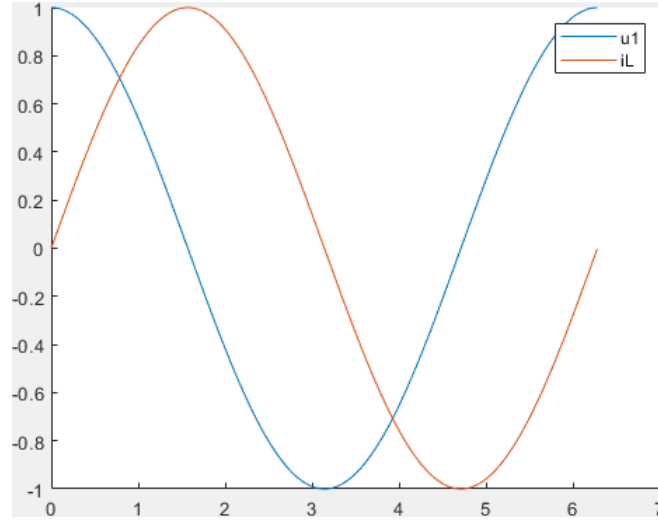
**Figure 5.5:** LC-circuit

The MNA-system for this example reads as

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For testing we set the capacitance to  $C = 1$  and the inductance to  $L = 1$ .

With the initial conditions  $u_1(0) = 1$  and  $i_L(0) = 0$ , the exact solutions for the voltage  $u_1$  and the current  $i_L$  then have the form ??.



**Figure 5.6:** Exact solution for example 2.

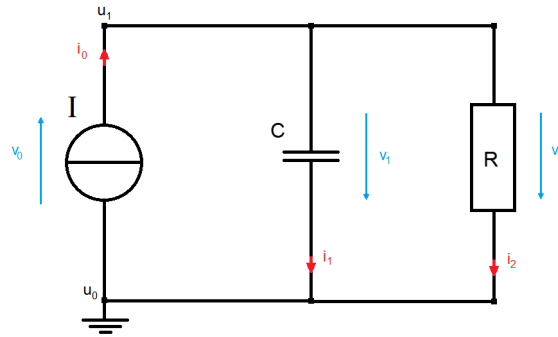
Applying the BDF-k methods 5.3.1 and the Trapezoidal rule 5.3.2 to this problem we can observe the maximal error  $err = \max_{0 \leq t \leq T} \|y(t) - \tilde{y}(t)\|$ . This error is displayed in table ??.

h	k = 1		k = 2		k = 3		Trapezoidal	
	u1	iL	u1	iL	u1	iL	u1	iL
1	1.000	1.016	1.226	1.248	3.402	3.091	1.390	1.462
0.1	$7.145 \times 10^{-1}$	$6.905 \times 10^{-1}$	$7.763 \times 10^{-2}$	$8.060 \times 10^{-2}$	$5.395 \times 10^{-3}$	$5.180 \times 10^{-3}$	$1.963 \times 10^{-2}$	$2.087 \times 10^{-2}$
0.01	$1.181 \times 10^{-1}$	$1.112 \times 10^{-1}$	$7.884 \times 10^{-4}$	$8.368 \times 10^{-4}$	$9.256 \times 10^{-6}$	$6.884 \times 10^{-6}$	$1.965 \times 10^{-4}$	$2.094 \times 10^{-4}$

By comparing the errors of the methods we can see that they indeed have the predicted convergence order. (reference appropriate theorems here)

### Example 3

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**Figure 5.7:** Current source with capacitor and resistor.

h	k = 1			k = 2			k = 3		
	u1	u2	iL	u1	u2	iL	u1	u2	iL
	Error			Error			Error		
1	number	number	number	number	number	number	number	number	number
0,1	number	number	number	number	number	number	number	number	number
0,01	number	number	number	number	number	number	number	number	number

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